Tensor Induction As Left Kan Extension

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December 2014

Master Thesis in Topology

Acknowledgment

I express my warm thanks to my supervisor Mr Morten Brun for his support, guidance on the project and engagement through the learning process of this master thesis.

I am using the opportunity to express my gratitude to everyone who supported me throughout the courses for the master in Topology. I am thankful for their aspiring guidance, invaluable constructive criticism and friendly advice during the whole period of studying for Master in Mathematics at University of Bergen. I am sincerely grateful to them for sharing their truthful and illuminating views on number of issue related to me and my studying.

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1 Introduction

A function between sets can be extended by many different ways! If A,B and C are sets and A is non-empty, $B \hookrightarrow C$, then a function $f: B \to A$, can be extended as $f': C \to A$, by many different ways. But there is not a canonical or unique way. Besides, if A, B and C are even groups or Rings or Modules, f can be extended as many different functions. But it is not same in Category theory, if we have a functor $T: M \to A$, and M is subcategory of C and all colimits and limits exist in A, there is ways to find two canonical extension functors from M to functors L, $R: C \to A$. These extensions functors are called Left Kan Extension functor L and Right Kan Extension R. I am going to study here in my thesis the category which is all colimits exist and the Left Kan Extension between Category of R-Modules (Mod_R). I start with the category R[fin], its objects are finite sets and its hom sets are R-modules. R[fin] is full subcategory of Mod_R and the Left Kan Extension of T along the inclusion functor will be found later in the chapter 2.

In the processes of constructing the left kan extension L, some tools are necessary to use. I have found the co-equalizer, co-product, bi-product diagrams in the category Mod_R as my tools. After I define our functor L as co-equalizer digram, the universal property of co-equalizer diagram gives beautifully the unique natural transformation between two functors T and L along the full and faithful functor M to C. which is necessary to prove L is the left kan extension.

Tensor product (\otimes) is though as another parallel functor with L in here. Tensor is bilinear as defining property but it is not a linear. As of Kan extension properties, another parallel functor is not an additive functor, tensor is not linear nor additive, we need to make a long proof to find the unique natural transformation between functors L and \otimes by using universal property of co-equalizers. I could manage to prove that tensor product has a quality to use as a parallel functor of the left kan extension.

In the last part of chapter 2, the natural transformation γ between L and \otimes is proved as a unique isomorphism. It becomes $L \cong \otimes$ and it shows that Tensor product is a kind of left kan extension.

In chapter (3), I introduce two category C_G and B^{OP} , the category of the transitive G-set of finite group G and Category of finite G-sets. I construct these two categories with the maps between objects are composing three kinds of maps, the induction, restriction and transferring. I am going to use three kinds of functions when I need the finite g-sets to move between G's subgroups. Then I prove that C_G is full subcategory of B^{OP} . Being C_G is full subcategory and the left Kan extension properties construct the left induction which is a functor category. This left induction functor category gives the connection between tensoring and the Grothendicks group representation.

End of chapter three I introduce the tensor induction with our categories B_H^{OP} , B_K^{OP} , B_H and B_K . If we defining the $Tens_H^K$ to get well adjustment between the two Modules categories $Mod_{R(B_H)}$ and $Mod_{R(B_K)}$, It works and we get the commute diagram with $Tens_H^K$ as the left kan extension.

2 Tensor product

2.1 Tensor Product is not a additive functor

Definition 2.1 (Tensor products of Rmodules, \bigotimes). : Tensor product is bilinear maps. For any two Rmodules M and N, there exist a pair (T, g), Rmodules T and Rmordules morphism $g: M \times N \to T$, with the following property: Given any module P and bilinear $f: M \times N \to P$, there exists a unique morphism $f': T \to P$ such that $f = f' \circ g$. Every R-bilinear map on $M \times N$ factors through T. Moreover, (T, g) and (T', g') are two pairs with this property, then there exists unique isomorphism $j: T \to T'$ such that $j \circ g = g'$.

The modules T constructed above is called the tensor product of M an N, and is denoted by $M \otimes_R N$. It is generated as an Rmodule by the products $x \otimes_R y$. The elements $x_i \otimes_R y_j$ generate $M \otimes_R N$ if x_i and y_j are families of generators of M and N.

The tensor product is not an additive functor.

Definition 2.2 (Additive functor). A functor T from additive categories U to V with properties T(f+g) = Tf + Tg for any parallel pair of arrows $f, g : u \to u'$ in U and T send zero object to zero object of V and binary bi-product diagram in U to a bi-product diagram in V.

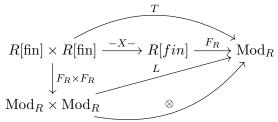
Lemma 2.3. : Tensor product is not additive functor.

Proof. Tensor product is though as a functor as follow: \otimes : Hom $(A, A') \times$ Hom(B, B') \longrightarrow Hom $(A \otimes B, A' \otimes B')$. If we consider our categories $A, A', B, B' = \mathbb{R}$, then \otimes : Hom $(\mathbb{R},\mathbb{R}) \times$ Hom $(\mathbb{R},\mathbb{R}) \longrightarrow$ Hom $(\mathbb{R} \otimes \mathbb{R}, \mathbb{R} \otimes \mathbb{R})$ is $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ since Hom $(\mathbb{R},\mathbb{R}) = \mathbb{R}$ and $\mathbb{R} \otimes \mathbb{R} = \mathbb{R}$. Let \otimes (a,b) = a.b and f(1)=a $\neq 0$ and g(1)=b $\neq 0$, (a,b) $\in \mathbb{R} \times \mathbb{R}$ and (f,g) \mapsto f \otimes g, f,g are morphisms in Hom(R,R). We consider (1, 1) in $\mathbb{R} \times \mathbb{R}$, (1,1)= (1,0)+ (0,1). (f \otimes g)(1,1)= \otimes [f(1), g(1)]= \otimes (a,b)= ab $\neq 0$. It is bilinear. But (f \otimes g)[(1,0)+ (0,1)]=a.0+0.b=0 and (f \otimes g) [(1,1)] \neq (f \otimes g) [(0,1) + (1,0)].

Tensor product does not have the property as additive functor. So, Tensor product is not an additive functor. $\hfill \Box$

2.2 Left Kan extension

In this chapter we are going to study about the left kan extension of the following diagram:



Let L : $\operatorname{Mod}_R \times \operatorname{Mod}_R \to \operatorname{Mod}_R$ be a functor together with a natural transformation η : T $\to L(F_R \times F_R)$. I am going to prove that the functor \otimes together with the natural transformation β : T $\to \otimes \circ F_R \times F_R$ is a Left Kan extension of T along $F_R \times F_R$. Let $\gamma : L \to \otimes$ is natural transformation.

I am proving that the γ such that $\beta = \gamma K \circ \eta$ is an unique natural transformation. That is as follow:

$$Nat(L, \otimes) \cong Nat(T, \otimes \circ F_R \times F_R)$$
$$\gamma \mapsto (\gamma F_R \times F_R \circ \eta) = \beta.$$

Definition 2.4 (Left Kan extension). Let $T: M \to A$ and $K: M \to C$ be functors. In the diagram,

$$\begin{array}{c} M \xrightarrow{T} A \\ \downarrow_{K} \\ C \end{array}$$

the left Kan extension of T along K is a functor $L_KT : C \to A$ together with a natural transformation $\eta : T \to L_KT$ K with the following properties: given any functor $S : C \to A$ together with the natural transformation $\beta : T \to SK$, there exist a unique natural transformation $\gamma : L_KT \to S$ such that $\beta = \gamma K \circ \eta$.

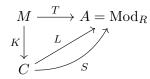
$$Nat(L, S) \cong Nat(T, S \circ K)$$

 $\gamma \mapsto (\gamma K \circ \eta) = \beta$

is bijection.

We illustrate the concept at a left Kan extension in the following diagrams category and functors:

Given two functors $T : M \to Mod_R$ and $K : M \to C$, then the left Kan extension $L_K T$ = L of T along K exists and L : C $\to Mod_R$ is characterized by a universal property.



Natural transformations η : T \rightarrow LK, β : T \rightarrow SK, and γ : L \rightarrow S with $\beta = \gamma K \circ \eta$ give the diagram.



Now I want to explain a notation $\gamma \mathbf{K}$ which I am going to use.

Definition 2.5. $\gamma K: \gamma$ is the natural transformation defined as above. γK is the morphism $\gamma_c: L(c) \rightarrow S(c)$ for each object c of C such that c=Km. $\gamma K_m: L(c=Km) \rightarrow S(c=Km)$. Note that L(Km)=(LK)(m) and S(Km)=(SK)(m). The morphisms γK_m for m in M is a natural transformation from LK to SK, which we call γK . Let $\alpha: m \rightarrow m'$ be a morphism in M and the diagram

$$\begin{array}{c} L(c) \xrightarrow{\gamma K_m} S(c) \\ \downarrow LK(\alpha) & \downarrow SK(\alpha) \\ L(c') \xrightarrow{\gamma K_{m'}} S(c') \end{array}$$

commutes because γ is the unique natural transformation $\gamma : L(c) \to S(c)$ for all $c \in C$. So, γK is natural too.

Lemma 2.6. : If $[L, \eta : T \to LK]$ and $[L', \eta' : T \to L'K]$ are left Kan Extensions, then there exists a unique isomorphism $\gamma : L \to L'$ with $\eta' = \gamma K \circ \eta$.

Proof. By the definition property of left Kan extension unique natural transformations $\gamma : L \to L'$ and $\gamma' : L' \to L$. with $\eta' = \gamma K \circ \eta$ and $\eta = \gamma' K \circ \eta'$.

Now $\gamma' \circ \gamma : L \to L$ is a natural transformation, with $(\gamma' \circ \gamma)K \circ \eta = \gamma'K \circ \gamma K \circ \eta = \gamma'K \circ \gamma K \circ \eta$ = $\gamma'K \circ \eta' = \eta$ as a natural transformation $\eta : T \to LK$. Also id : $L \to L$ is a natural transformation with $\eta = (id_L K) \circ \eta$, so by uniqueness in the defining property of Left Kan extensions we have that $\gamma' \circ \gamma = id_L$. Similarly, $(\gamma \circ \gamma')K \circ \eta' = \gamma K \circ \gamma' K \circ \eta' = \gamma K \circ \eta = \eta'$ as a natural transformation $\eta' : T \to L'K$. $id_{L'} : L' \to L'$ is natural transformation with $\eta' = (id_{L'}K) \circ \eta'$. Again uniqueness of natural transformation gives $\gamma \circ \gamma' = id_{L'}$.

 $\gamma' \circ \gamma = id_L$ and $\gamma \circ \gamma' = id_{L'}$ give that γ and γ' are bijections and one of them is the inverse of the other.

 γ is isomorphism.

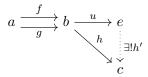
2.3 All colimit exist in Mod_R , then a left Kan extension of T along K exists.

 Mod_R is a cocomplete category by the Theorem 3.13 of the reserved paper named "Limits, colimits and how to calculate them in the category of modules over a PID" by KAIRUI WANG. The theorem states that;

Theorem 2.7 (Theorem 3.13). Cocompleteness Theorem,: A category C is cocomplete if and only if the coproduct of any set of objects in C exists and the coequalizer between any two morphisms with the same source and target exists.

Definition 2.8 (Cocomplete category). a cocomplete category is a category where colimits over diagrams F with a small source category J exist. F is an object of the category of functors C^J , J is a small category.

Definition 2.9 (Coequizer).



Given in a category a pair of maps f and g with the same domain a and codomain b, a coequalizer of [f, g] is a pair (u, e) of a morphism $u: b \to e$ and codomain e such that (1) uf=ug(2) if $h: b \to c$ has hf=hg then h = h'u for a unique $h': e \to c$.

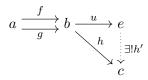
Definition 2.10 (A map of co-equilzer diagrams). A map of co-equalizer diagrams is a diagram of the form:

So that the rows are co-equalizer diagrams and

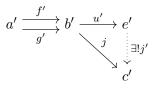
$$\beta f = f'\alpha, \beta g = g'\alpha \quad and \quad \gamma u = u'\alpha.$$

Lemma 2.11. If in a map of co- equalizers diagrams (1), the maps α and β are isomorphisms, then γ is an isomorphism.

Proof.



Given diagram, maps f and g are such that : uf=ug and if h: b \rightarrow c has hf=hg then h = h'u for a unique $h' : e \rightarrow c$.



h is the surjective map and the maps f', g' and u' are such that : u'f' = u'g' and if $j: b' \to c'$ has jf' = jg' then j = j'u' for a unique $j': e' \to c'$. We get the diagram below:

$$a \xrightarrow{f} b \xrightarrow{u} e \xrightarrow{h'} c$$

$$\downarrow^{\alpha} \downarrow^{\beta} \downarrow^{\beta} \downarrow^{\gamma}$$

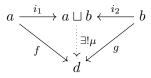
$$a' \xrightarrow{g'} b' \xrightarrow{u'} e'_{j} \xrightarrow{j'} c'$$

If α and β are isomorphisms, γ must be a isomorphism because in this diagram, we know that

$$\beta f = f'\alpha, \beta g = g'\alpha \text{ and } \gamma u = u'\alpha.$$

h = h'u and j = j'u', then $c \cong c'$.

Definition 2.12 (co-product diagram).



is a coproduct diagram. i_1 and i_2 are injectives. If there exists $d, f: a \to d$ and $g: b \to d$, then there always exists unique μ such that $f = \mu \circ i_1$ and $g = \mu \circ i_2$.

2.3.1 Prosition

Given diagram of the form,

$$\begin{array}{c} M \xrightarrow{T} A = Mod_R \\ \downarrow_K \\ C \end{array}$$

a left Kan extension of T along K exists. The functor $L : C \to A$ and natural transformation $\eta : T \to LK$ can be constructed as follows: For c, an object of C, the value L(c) of L of c is given by the coequalizer of the diagram

$$\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \xrightarrow[b]{a} (\bigoplus_{Km \to c} Tm)$$

where the upper map a takes an element

$$x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \quad \text{of} \quad \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$$

to the element $(f \circ K\alpha, t)$ of $\bigoplus_{Km\to c} Tm$, and the lower map b takes x to the element $(f, T(\alpha)(t))$ of $\bigoplus_{Km\to c} Tm$. The natural transformation $\eta : T \to LK$ takes and element t of Tm to the element in the co-equalizer LKm represented by the element

$$[id: Km \to Km, t \in Tm]$$
 of $\bigoplus_{Km_0 \to c} (Tm_0).$

Proof. First we define L. Given an object c of C, let Lc be the co-equalizer described in the statement of the proposition. Given $h : c \to c'$,

we define $Lh : Lc \to Lc'$ as follow;

an element
$$x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1)$$
 of $\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$

will be sent to x' by composing with h

$$x' = (\alpha : m_1 \to m_0, h \circ f : Km_0 \to c \to c', t \in Tm_1) \quad \text{in} \quad \bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1).$$

And the map c send x' to $(h \circ f \circ K\alpha, t)$ in $\bigoplus_{Km \to c'} Tm$.

The map a sent **x** to

$$(f \circ K\alpha, t)$$
 in $(\bigoplus_{Km \to c} Tm)$ and it is sent to $(h \circ f \circ K\alpha, t)$ in $\bigoplus_{Km \to c'} Tm$.

So,

$$(h \circ a)(x) = (c \circ h)(x)$$

We get the commute diagram for the upper maps a and c. For maps b and d.

an element
$$x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1)$$
 of $\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$

will be sent to x' by composing with h

$$x' = (\alpha : m_1 \to m_0, h \circ f : Km_0 \to c \to c', t \in Tm_1) \quad \text{in} \quad \bigoplus_{Km_0 \to c'} \bigoplus_{m_1 \to m_0} (Tm_1)$$

and the map d send x' to $(h \circ f, T(\alpha)(t)))$ in $\bigoplus_{Km \to c'} Tm$

the map b sent **x** to

$$(f, T(\alpha)(t))$$
 in $(\bigoplus_{Km \to c} Tm)$ and it is sent to $(h \circ f, T(\alpha)(t))$ in $\bigoplus_{Km \to c'} Tm$.
 $(h \circ b)(x) = (d \circ h)(x)$

So we get commute digram for both of the pairs of maps a and c and b and d. It gives the commuted diagram below and the defined properties of Lc gives the unique morphism Lh from Lc to Lc' which gives the commute diagram as $(Lh \circ \mu)(t) = (\theta \circ h)(t)$, for all follow $t \in (\bigoplus_{Km \to c} Tm)$.

L is defined for all map h in C.

We are going to show that L is a functor $L : C \to Mod_R$ and η is a natural transformation. We have proved that Lh is exist in Mod_R for all h in C. In C, there exists Id_c : $c \to c$ in C. Composing with Id_c to x and get the commute diagram below and get Id_{Lc} .

 $Id_{Lc} = L(Id_c)$ exists . If g: $c' \to c''$ in C, $g \circ h : c \to c''$ will induced a unique map Lg $\circ Lh : Lc \to Lc''$ as follow:

$$x = (\alpha : m_1 \to m_0, f : Km_0 \to c, t \in Tm_1) \quad \text{of} \quad \bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1)$$

is sent same as above by map a, b, c and d. Again, sent x' to x'' by composing with g.

$$x'' = (\alpha : m_1 \to m_0, g \circ h \circ f : Km_0 \to c'', t \in Tm_1) \quad \text{in} \quad \bigoplus_{Km_0 \to c''} \bigoplus_{m_1 \to m_0} (Tm_1).$$

We get

$$(g \circ h \circ a)(x) = (u \circ g \circ h)(x) = (g \circ h \circ f \circ K\alpha, t)$$

and

$$(g \circ h \circ b)(x) = (v \circ g \circ h)(x) = (g \circ h \circ f, T(\alpha)(t))$$

and the commute diagrams with the unique map

$$Lh: Lc' \to Lc''.$$

Again, we consider map $L(g \circ h)$, we get

This diagram works same way and get the same equations above,

$$(g \circ h \circ a)(x) = (u \circ g \circ h)(x) = (g \circ h \circ f \circ K\alpha, t)$$

and

$$(g \circ h \circ b)(x) = (v \circ g \circ h)(x) = (g \circ h \circ f, T(\alpha)(t))$$

So,

$$L(g \circ h) = Lg \circ Lh.$$

L is a functor.

Then we are going to show that η is natural transformation. The morphisms

$$\eta_m: Tm \to LKm$$

is such that:

$$t \mapsto L(id_{Km})(\eta_m t) = \eta_m t$$

and for any $t \in TM$ and morphism f ,

$$(f: Km \to c, t) \mapsto (L(f \circ K_{\alpha})(\eta m_1 t)).$$

Then the diagramas

$$\begin{array}{ccc} Tm_1 & & Tm_0 \\ & & \downarrow^{\eta m_1} & & \downarrow^{\eta m_0} \\ L(Km_1) & \xrightarrow{LK_{\alpha}} LKm_0 & \xrightarrow{Lf} Lc \end{array}$$

Any element t in Tm_1 is sent by map $(LK_{\alpha} \circ \eta m_1)$

$$(f: Km \to c, t) \mapsto (L(f \circ K_{\alpha})(\eta m_1 t))$$

t is sent by map $(T\alpha \circ \eta m_0)$

$$(f: Km \to c, t) \mapsto (L(f)\eta m_0(T(\alpha)t))$$
$$(L(f \circ K\alpha)(\eta m_1 t)) = (L(f)\eta m_0(T(\alpha)t)).$$

It makes the previous diagram commute. And η is natural.

Let S is the another functor $C \to Mod_R$ together with $\beta : T \to SK$. I am going to prove that there is a unique natural transformation $\gamma : L \to S$ such that $\beta = \gamma K \circ \eta$,

$$\begin{array}{c} M \xrightarrow{T} A = Mod_R \\ K \downarrow & \swarrow \\ C & S \end{array}$$

The morphisms β_m : Tm \rightarrow SKm induces a morphism $\bigoplus_{Km \rightarrow c} Tm \rightarrow Sc$

$$[f: Km \to c, t \in Tm)] \mapsto S(f)(\beta_m t).$$

And it gives a commute diagramas ;

$$\begin{array}{c} Tm_1 \xrightarrow{T\alpha} Tm_0 \\ \downarrow^{\beta m_1} \qquad \qquad \downarrow^{\beta m_0} \\ S(Km_1) \xrightarrow{SK_{\alpha}} SKm_0 \xrightarrow{Sf} Sc \end{array}$$

Any element t in Tm_1 is sent by map $(SK_{\alpha} \circ \beta m_1)$

$$(f: Km \to c, t) \mapsto (S(f \circ K_{\alpha})(\beta m_1 t))$$

t is sent by map $(T\alpha \circ \beta m_0)$

$$((f:Km \to c,t)) \mapsto (S(f)\beta m_0(T(\alpha)t))$$
$$(S(f \circ K\alpha)(\beta m_1 t)) = (S(f)\beta m_0(T(\alpha)t)).$$

It gives a commute diagram and the map ϕ as follow;

$$\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \xrightarrow[b]{a} (\bigoplus_{Km \to c} Tm) \xrightarrow{\phi} Sc$$

By universal property of coequalizer , we get a uniquely determined morphism $\gamma_c:Lc \to Sc$

$$\bigoplus_{Km_0 \to c} \bigoplus_{m_1 \to m_0} (Tm_1) \xrightarrow[b]{a} (\bigoplus_{Km \to c} Tm) \xrightarrow{\psi} Lc$$

for any modules c in C such that $\phi = \gamma_c \circ \psi$. Then we get the unique natural transformation $\gamma : L \to S, \forall c \in C$. It holds for any free module of finite set m in M, so we get $\gamma K_m : LKm \to SKm$. We have defined $\beta_m : Tm \to SKm$ which gives ϕ in above co-equalizerby composing with f: $Km \to c$ and $\eta_m : Tm \to LKm$ which gives ψ in above co-equalizerby composing with f: $Km \to c$. The composite of γK_m and η_m is

$$\beta_m = \gamma K_m \circ \eta_m : Tm \to SKm, \forall m \in M.$$

We can express it as

$$\beta = \gamma K \circ \eta.$$

and the diagram is ,

.



So defining functor L as coequilizer and unique natural transformation γ as above make the L is left Kan extension.

Definition 2.13. : R[fin] is the category with finite sets as objects and the hom set in R[fin](X,Y) is Rmodules generated by maps between two finite sets X and Y.

$$\sum_{i} a_i f_i, a_i \in R, f_i \in hom(X, Y).$$

Definition 2.14 (Full subcategory). : We say that S is a full subcategory of C when the inclusion functor $T: S \to C$ is full. If every function $T_{(c,c')}$: hom $(c,c') \to$ hom (Tc, Tc'), for all pair (c, c') of C, is surjective, T is full.

Definition 2.15. : Let X and Y are finite sets. F_R is a full embedding functor which makes a finite set to a free R modules.

$$F_R X = \bigoplus_{x \in X} R.$$

Every map of R[fin](X, Y) is sent the map in $map(X, F_RY)$ as follow:

$$R[fin](X,Y) \to map(X,F_RY)$$
$$(\sum_i a_i f_i) \mapsto (x \mapsto \sum_i a_i f_i(x)),$$

and every map α in $(X_0, F_R Y)$ will send to a map in $Mod_R(F_R X, F_R Y)$ as follow:

$$map(X, F_R Y) \to Mod_R(F_R X, F_R Y).$$
$$(\alpha : X \to F_R Y) \mapsto [(\sum_i \lambda_i x_i) \mapsto \sum_i \lambda_i \alpha(x_i)].$$

In this chapter I am going to prove that R[Fin] is full subcategory of Mod_R by using the left kan extension as co-equalizer.

Definition 2.16 (-X-). $:R[fin] \times R[fin] \rightarrow R[fin], -X$ - is a functor which makes pair of two finite sets to a Cartesian product of two finite sets.

$$(X,Y) \mapsto X \times Y$$

and morphisms

$$(\sum a_i f_i, \sum b_j g_j) \mapsto \sum_{i,j} a_i b_j(f_i, g_j).$$

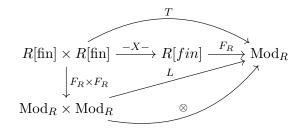
Definition 2.17 (T and η). : T is a functor of composition of two functors $F_R \circ - \times -$,

$$T(X,Y) = \bigoplus_{X \times Y} R,$$

with a natural transformation

$$\eta: T \to \otimes_R \circ F_R \times F_R.$$
$$\bigoplus_{X \times Y} R \to F_R X \otimes F_R Y$$
$$\sum_{(x,y)} c_{(x,y)}(x,y) \mapsto \sum_{(x,y)} c_{(x,y)}(x \otimes y).$$

Theorem 2.18. In the diagram (1) if there is a functor $\otimes : Mod_R \times Mod_R \to Mod_R$ and an natural transformations $\beta : T \to \otimes \circ F_R \times F_R$, then there exist a unique natural isomorphism $\gamma : L \to \otimes$ such that $\beta = (\gamma F_R \times F_R \circ \eta) : T \to S$.



(2)

Lemma 2.19. There is a natural isomorphism

$$LM \cong M$$

Proof. Let

$$\alpha: LM \to M$$

such that: we have co-equalizer diagram

$$\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1) \xrightarrow[b]{a} (\bigoplus_{FX \to M} FX) \xrightarrow{\psi} LM$$

an element $x = (g : X_1 \to X_0, f : FX_0 \to M, t \in FX_1)$ of $\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1)$

will be sent to x' by map a

$$x' = (f, F(g)(t))$$
 of $\bigoplus_{FX \to M} FX$

an element $x = (g: X_1 \to X_0, f: FX_0 \to M, t \in FX_1)$ of $\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1)$

will be sent to x'' by map b

$$x'' = (f \circ Fg, t) \quad \text{of} \quad \bigoplus_{FX \to M} FX.$$

An element

$$y = (f : Km_0 \to M, t \in FX) \quad \text{of} \quad \bigoplus_{FX \to M} FX$$

will be sent to (f(t)) in M by map ϕ in diagram below .

$$\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1) \xrightarrow[b]{a} (\bigoplus_{FX \to M} FX) \xrightarrow[\phi]{\forall} LM$$

In above diagram ψ is surjective and we get the unique α according to the universal properties of Co-equalizer. It is factor out the map ϕ such that $\phi = \alpha \circ \psi$, $(\alpha \circ \psi)(y) = f(t)$.

Case 1. If M is a free R modules : M = FY, Y is a finite set. Let

$$\alpha: LM \to M$$
$$[f: FX \to M = FY, t \in FX] \mapsto f(t),$$

and

$$\label{eq:basic} \begin{split} \beta &: M \to LM \\ m \mapsto (id: FY \to M, m \in FY), \end{split}$$

we consider

$$\alpha(\beta(m)) = \alpha(id:FY \to M = FY, m \in FY) = id(m) = m$$

$$\bigoplus_{FX_0 \to M} \bigoplus_{X_1 \to X_0} (FX_1) \xrightarrow{a}_{b} (\bigoplus_{FX \to M} FX) \xrightarrow{\psi} LM$$

$$f=\phi \qquad \exists!\alpha \qquad \beta$$

$$(M = FY) \xrightarrow{(\alpha \circ \beta) = id} (M = FY)$$

and

$$\beta(\alpha(f:FX \to M = FY, t \in FX)) = \beta(f(t)) = (id:FY \to M, f(t) \in FY)$$

we know that, in LM,

$$(f:FX \to M = FY, t \in FX)) = (id:FY \to M, f(t) \in FY)$$

because $t \in FX$ will be sent to $f(t) \in FY$ by f and $f(t) \in M=FY$ will be to itself by id_{FY} . Both elements are in the same equivalence class of LM. So, we have prove LM \cong M for M, any finitely generated FREE module.

Case 2. If

$$M = \oplus_{x \in X} R,$$

for X is an infinite set. Let

$$\alpha_M : LM \to M$$
$$[f : FX \to M, t \in FX] \mapsto f(t).$$

Let any $m \in M, m = \sum m_x[x]$, only finitely many m_x are not zero. Let $Y = [x \in X/m_x \neq 0]$. So we can express m as $m = \sum \lambda_y[y]$.

L is left adjoint functor. Then we get the diagram below,

$$LM \xrightarrow{=} L(\bigoplus_{x \in X} R) \xrightarrow{\cong} \bigoplus_{x \in X} L(R)$$

$$\downarrow^{\alpha_M} \qquad \qquad \downarrow^{(\bigoplus \alpha_R) = \cong}$$

$$M \xrightarrow{=} \bigoplus_{x \in X} R \xrightarrow{=} \bigoplus_{x \in X} R.$$

We have $\bigoplus_{x \in X} L(R)$ is isomorphic to $\bigoplus_{x \in X} R$. Then we get

$$L(\oplus_{x\in X}R)\cong \oplus_{x\in X}L(R)\cong \oplus_{x\in X}R$$

and an isomorphism

$$\alpha_M : L(\oplus_{x \in X} R) \to \oplus_{x \in X} R.$$
$$LM \cong M.$$

Case 3. If M is any Rmodule: we can write M as a co-equalizer of free Rmodules

$$K = ker(\bigoplus_{m \in M} Rm \to M)$$
$$\sum_{m \in M} a_m[m] \mapsto \sum a_m m.$$

Get a surjective map

$$\oplus_{k \in K} Rk \to K$$

We have exact sequence

$$\oplus_{k \in K} Rk \xrightarrow{\beta} \oplus_{m \in M} Rm \to M \to 0$$

Thus,

$$\oplus_{k \in K} Rk \xrightarrow[\beta]{0} \oplus_{m \in M} Rm \longrightarrow M$$
(3)

is an coequalizer sequence. We get coequalizer sequence with L too as L is left adjont.

$$L(\oplus_{k \in K} Rk) \xrightarrow[L(\beta)]{L(\beta)} L(\oplus_{m \in M} Rm) \longrightarrow LM$$
(4)

As the lemma 2.11, these two co-equalizers diagram 3 and 4 have the same universal property of co-equalizer. We get commute diagram as follow:

$$L(\bigoplus_{k \in K} Rk) \xrightarrow{L0} L(\bigoplus_{m \in M} Rm) \longrightarrow LM$$
$$\downarrow^{\alpha_{\bigoplus Rk} \cong} \qquad \downarrow^{\alpha_{\bigoplus Rk} \cong} \qquad \downarrow^{\alpha_{\bigoplus Rk} \cong} \qquad \downarrow^{\alpha_M}$$
$$\bigoplus_{k \in K} Rk \xrightarrow{0} \qquad \stackrel{\beta}{\longrightarrow} \oplus_{m \in M} Rm \longrightarrow M$$

 α_M works same as above in case2. It is an isomorphism. Therefor

$$\alpha: LM \to M$$

is isomophism for all mordules $\mathbf{M} \in \mathrm{Mod}_R$ and

$$LM \cong M.$$

2.4 Defining L, the left kan extension and a co-equalizer

Let functors - × -, product of sets. × (X,Y) = X×Y, F_R is a functor which makes free modules of finite sets, $F_R(X \times Y) = \bigoplus_{X \times Y} R$ and $L : mod_R \times mod_R \to mod_R$ be a co-equalizer functor of modules. T is a functor of composition of two functors $F_R \circ - \times -$, $T(X,Y) = \bigoplus_{X \times Y} R$.

$$R[fin] \times R[fin] \xrightarrow{-X-} R[fin] \xrightarrow{F} Mod_R$$

$$\downarrow_{F \times F} \xrightarrow{L} Mod_R \times Mod_R$$
(5)

In the diagram 5, Let L is a coequalizer such that:

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} F(X_0 \times Y_0)$$

$$\begin{array}{c}
u \downarrow \downarrow v \\
\bigoplus_{FX \to M, FY \to N} F(X \times Y) \\
\downarrow \zeta \\
L(M, N)
\end{array}$$
(6)

Lemma 2.20. There is a coequalizer diagram in 5 as follow:

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} FX_0 \times FY_0$$

$$\begin{array}{c}
u' \downarrow \downarrow v' \\
\bigoplus_{FX \to M, FY \longrightarrow N} FX \times FY \\
\downarrow \zeta' \\
M \times N
\end{array}$$
(7)

Proof. We have shown in 2.19 that LM is isomorphic to M and we have the co-equalizer diagram:

$$\bigoplus_{FX_1 \to M} \bigoplus_{X_0 \to X_1} (FX_0) \xrightarrow[b]{a} (\bigoplus_{FX \to M} FX) \xrightarrow{\psi} M$$

$$h \qquad \downarrow_{l\xi}$$

$$M' \qquad (8)$$

h works

$$(h \circ a)(x) = (h \circ b)(x), \forall x \in (\bigoplus_{FX_1 \to M} \bigoplus_{X_0 \to X_1} (FX_0)), h = \xi \circ \psi.$$

The two maps work such that : en element x in FX_0 is send to different element in FX as follow:

$$x = (\alpha : FX_1 \to M, f : FX_0 \to FX_1, t \in FX_0) \mapsto (\alpha, f(t))$$

by map a and

$$x = (\alpha : FX_1 \to M, f : FX_0 \to FX_1, t \in FX_0) \mapsto (\alpha \circ f, t)$$

by map b. But these two different elements in FX are sent to same elements f(t) of M by ψ .

There is a co-equalizer in FY too such that:

$$\bigoplus_{FY_1 \to N} \bigoplus_{FY_0 \to FY_1} (FY_0) \xrightarrow[b']{a'} (\bigoplus_{FY \to N} \to FY) \xrightarrow{\psi'} N$$

$$\downarrow \downarrow \xi'$$

$$N'$$

$$\downarrow I\xi'$$

$$N'$$

$$\downarrow I\xi'$$

$$N'$$

$$\downarrow I\xi'$$

$$\downarrow$$

h' works

$$(h' \circ a')(y) = (h' \circ b')(y), \forall y \in (\bigoplus_{FY_1 \to N} \bigoplus_{Y_0 \to Y_1} (FY_0)), h' = \xi' \circ \psi'.$$
$$y = (\beta : FY_1 \to N, g : FY_0 \to FY_1, s \in FY_0) \mapsto (\beta, g(s))$$

by map a' and

$$y = (\beta : FY_1 \to N, g : FY_0 \to FY_1, s \in FY_0) \mapsto (\beta \circ g, s)$$

by map b' . But these two different elements in FY are sent to same elements g(s) of N by $\psi'.$

The co-product of co-equalizer diagrams is a co-equalizer diagram.

In our category R[fin] there are objects which co-product of its object, finite set. So these coproduct objects will be the

coproduct of 8 and 9 gives the following equation

$$\bigoplus_{FX_1 \to M} \oplus_{FX_0 \to FX_1} (FX_0) \bigoplus \oplus_{FY_1 \to N} \oplus_{FY_0 \to FY_1} (FY_0)$$

$$u \oplus u' \bigcup v \oplus v'$$

$$(\oplus_{FX \to M} FX) \oplus (\oplus_{FY \to N} FY)$$

$$\downarrow \psi \oplus \psi'$$

$$M \oplus N$$

This is equal to

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \xrightarrow[w \times w']{} \bigoplus_{FX \to M, FY \to N} (FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)$$

Since we have co-product diagram:

and get unique map $\xi \oplus \xi'$ such that $h = (\xi \oplus \xi') \circ \psi$ and $h' = (\xi \oplus \xi') \circ \psi'$, we get co-equalizer diagram:

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \xrightarrow[v \times v']{u \times u'} \bigoplus_{FX \to M, FY \to N} (FX \times FY) \xrightarrow[\psi \times \psi']{\psi \times \psi'} (M \times N)$$

(10)

Lemma 2.21. The bilinear map

$$\hat{\phi}: FX \times FY \to F(X \times Y)$$
$$\hat{\phi}(\sum_{i} \lambda_{i} x_{i}, \sum_{j} \mu_{j} x_{j}) = \sum_{i,j} \lambda_{i} \mu_{j}(x_{i}, y_{j})$$

induces a map of coequalizer diagrams and the map

$$\phi: M \times N \to L(M, N).$$

Proof. We have defined the co-equalizer diagram 6 as follow

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (F(X_0 \times Y_0) \xrightarrow[b]{a} \bigoplus_{FX \to M, FY \to N} F(X \times Y) \xrightarrow{\zeta} L(M \times N)$$

and I have got a co-equalizer in the lemma 2.20

$$\bigoplus_{FX_1 \to M, FY_1 \to N} \bigoplus_{FX_0 \to FX_1, FY_0 \to FY_1} (FX_0 \times FY_0) \xrightarrow[v \times v']{u \times u'} \bigoplus_{FX \to M, FY \to N} (FX \times FY) \xrightarrow{\psi \times \psi'} (M \times N)$$

From the commute diagram of the two co-equalizer diagrams, get a map ϕ as follw:

In the diagram, \forall finite set X and Y,

$$\hat{\phi} : FX \times FY \to F(X \times Y)$$
$$(\sum_{i} \lambda_{i} x_{i}, \sum_{j} \mu_{j} x_{j}) \mapsto \sum_{i,j} \lambda_{i} \mu_{j}(x_{i}, y_{j}).$$

Proposition 2.22. ϕ is bilinear.

Proof. ϕ inherit bilinearlity from the bilinear $\hat{\phi}$ such that: Given x_0, x_0' and y_0 , choose x, x', y such that

$$\psi(x) = x_0, \psi(x') = x'_0, \psi'(y) = y_0.$$

We define $\hat{\phi}$ is bilinear map, then

$$\hat{\phi}(x+x',y) = \hat{\phi}(x,y) + \hat{\phi}(x',y)$$
$$(\xi)((\hat{\phi})(x,y) + \hat{\phi}(x',y)) = (\xi)(\hat{\phi})(x,y) + (\xi)(\hat{\phi})(x',y).....(*),$$

since ξ is bilinear too. In the above commute diagram

$$(\xi)(\phi)(x,y) = (\phi)(\psi)(x), (\phi)(\psi')(y) = \phi(x_0, y_0).$$

In the (*)

$$(\xi)((\hat{\phi})(x,y) + \hat{\phi}(x',y)) = (\xi)(\hat{\phi})(x,y) + (\xi)(\hat{\phi})(x',y) = \phi(x_0,y_0) + \phi(x'_0,y_0).$$

We have

$$(\xi)((\hat{\phi})(x,y) + \hat{\phi}(x',y)) = \phi((\psi(x) + \psi(x'),\psi'(y))) = \phi(x_0 + x_0, y_0)$$
$$\phi(x_0 + x'_0, y_0) = \phi(x_0, y_0) + \phi(x'_0, y_0).$$

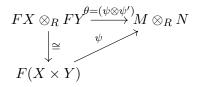
 ϕ is a bilear map.

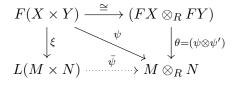
Proposition 2.23. There is a homomorphism $\overline{\phi}$: $M \otimes_R N \to L(M,N)$

Proof: Universal properties for defining tensor product (this is the unique natural morphism γ).

Lemma 2.24. The maps $FX \to M$ and $FY \to N$ induces a homomorphism $\overline{\psi} : L(M,N) \to M \otimes_R N$

proof:



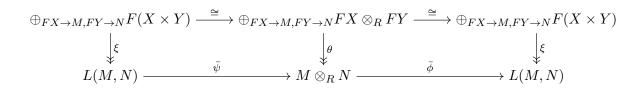


Lemma 2.25. $\bar{\psi} \circ \bar{\phi} = id[M \otimes_R N]$

Proof. Tensor is bi-linearity, so θ and ξ are mordules homomorphisms and conjugacy of upper horizontal maps give the identity map $\bar{\psi} \circ \bar{\phi}$.

$$\begin{array}{cccc} \oplus_{FX \to M, FY \to N} FX \otimes_R FY & \xrightarrow{\cong} & \oplus_{FX \to M, FY \to N} F(X \times Y) & \xrightarrow{\cong} & \oplus_{FX \to M, FY \to N} FX \otimes_R FY \\ & & \downarrow^{\theta} & & \downarrow^{\xi} & & \downarrow^{\theta} \\ & & & & \downarrow^{\theta} & & & \downarrow^{\theta} \\ & & & & M \otimes_R N & \xrightarrow{\bar{\phi}} & & & L(M, N) & \xrightarrow{\bar{\psi}} & & & M \otimes_R N \end{array}$$

Lemma 2.26. $\bar{\phi} \circ \bar{\psi} = id[L(M, N)]$ Proof.



Tensor is bi-linearity, θ and ξ are mordules homomorphisms and conjugacy of upper horizontal maps give the identity composing $\bar{\phi} \circ \bar{\psi}$.

Lemma 2.27.

$$\bar{\psi}: L(M, N) \to M \otimes_R N$$

is a natural isomorphism

Proof :Lemma 2.26 and 2.27 give that both $\bar{\phi}$ and $\bar{\psi}$ are natural isomorphism. And the unique natural morphism γ of diagram 1 is

$$\gamma = \bar{\psi} : L(M, N) \to M \otimes_R N$$

Conclusion is our two functor are isomorphic.

 $L \cong \otimes$

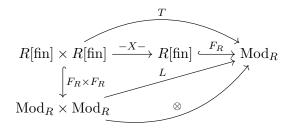
Theorem 2.28. Let R[fin] be the category of finitely generated free R-modules (2.13). Let $F_R : R[\text{fin}] \to \text{Mod}_R$ be the full embedding from (??) and $T : R[\text{fin}] \times R[\text{fin}] \to \text{Mod}_R$ is a composing of F_R and $- \times -$, $T(X, Y) = F_R(X \times Y)$, as (2.17). Let L be the left Kan Extension of T along $F_R \times F_R$,

$$L: \operatorname{Mod}_R \times \operatorname{Mod}_R \to \operatorname{Mod}_R$$

with the natural transformation $\eta: T \to L \circ F_R \times F_R$, and the another functor \otimes with the natural transformation $\beta: T \to \otimes \circ F_R \times F_R$, then there exist a unique natural isomorphism $\gamma: L \to \otimes$

such that

$$\beta = (\gamma F_R \times F_R \circ \eta).$$



3 Tensor induction

3.1 Constructing the category B^{op}

I am going to start with the category of G-sets. The category B^+ will be constructed with objects of the category of G-sets but maps are only some kinds of G-maps we need. Then I will get the *B* from B^+ by Grothendieck construction. It is an additive category. Then a contra-variant functor will give the category of B^{op} which I am going to study.

There are two different categories of "Mackey functors" but I use the original one defined by Dress.

Definition 3.1 (A Mackey functor). Mackey functor is an additive functor from an additive category B^{OP} to Ab catigory Mod_R. We work with Mackey functors over a commutative ring R. A Mackey functor over R is a functor

$$M: B^{OP} \to \operatorname{Mod}_R$$

Definition 3.2 (Additive Category). Additive category is an Ab Category which has a zero object and a bi-product for each pair of its objects.

Definition 3.3 (Bi-product diagram). *Bi-product diagram for the objects a*, $b \in A$ *is a diagram*

$$a \underbrace{\overbrace{p_1}^{i_1}}_{p_1} c \underbrace{\overbrace{p_2}^{i_2}}_{p_2} b$$

So that,

$$a \xleftarrow{p_1} c \xrightarrow{p_2} b$$

is a product diagram and

$$a \xrightarrow{i_1} c \xleftarrow{i_2} b$$

is a co-product diagram since

$$p_1i_1 = 1_a, p_2i_2 = 1_b$$
 and $i_1p_1 + i_2p_2 = 1_c$

Definition 3.4 (G-set). : Let G is a finite group. A left G-set is a set and a group homomorphism

$$f: G \times X \to X,$$
$$(g, x) \mapsto gx \in X,$$

such that the following conditions hold:

1) if $g,h \in G$ and $x \in X$, then g.(h.x) = (gh)x, 2) if 1_G is identity element of G then $1_G.x=x$.

Definition 3.5 (G-equivariant map or G-map). If G is a group and X and Y are left G-set, a morphism of G-sets from X to Y is a map $f : X \to Y$ such that f(gx) = g f(x), for any $g \in G$ and $x \in X$. Such a map is called G=equivariant map from X to Y and the set of such a map is denoted by $Hom_G(X, Y)$.

 B^+ is constructed from category of G-sets by taking all objects, G-sets, **a**, **b**, **c**, **d**..etc and G-maps which are able to be written by the composition of induction, transfer and restriction maps in representation ring. For example: a map f from a to b of B^+ we can describe such that

$$a \xleftarrow{f_1} c \xrightarrow{f_2} b$$

 f_1 get from f'_1 , a G map from **c** to **a**. f_1 is from a to c rather than c to a. f_1 and f_2 are G equavarience maps. f_1 the map with dotted arrow in B^+ , correspond to induction maps with indentity or a transfer maps in the familier makey functors like representation ring and so are called transfer. f_2 induces the restriction maps and are called restrictions. The hom set of B^+ are commutative monoids (semi group with identity).

If two maps are determine the same map in B^+ , then there is an inner isomorphism of c and d as shown in diagram;



Composition of two maps f and g is

$$a \xleftarrow{f_1} c \xrightarrow{f_2} b \xleftarrow{g_1} e \xrightarrow{g_2} d$$

and this compositions of two maps are given by the following pullback diagram:

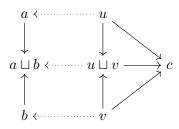
$$\begin{array}{c} h \xrightarrow{P_2} e \\ P_1 & g_1 \\ c \xrightarrow{f_2} b \\ c \xleftarrow{P_1} & h \xrightarrow{P_2} e \end{array}$$

We get the composition map a to d in B^+ is as follow:

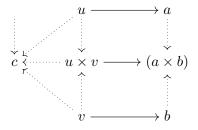
$$a \xleftarrow{(gf)_1} h \xrightarrow{(gf)_2} d$$

$$g \circ f : a \to d.$$

In B^+ zero set is the initial and terminal object. Disjoin union of sets, in the B^+ , get from the direct sum and direct produce of each map as follow':



This is a pair of maps out of $a \sqcup b$ and it is coproduct diagram in B^+ and



the above is a poduct diagram in B^+

Category B is obtained from with B^+ . They have same objects, finite G-sets but hom set are free abelian group. An abelian monoid, set of homomorphism of B^+ is quotient by the subgroup generated by the elements of the form

$$[f \sqcup g] - [f] - [g],$$

f and g are g-maps in of B^+ , [f] denotes the isomorphism class of f and $f \sqcup g$ is disjoint union of f and g. So, objects of B are finite G-sets same as B^+ objects and the morphisms of B are formal differences of maps in B^+ . That's why hom sets in B become abelian groups. There is an obvious functor from B to it's opposite category B^{OP} .

3.2 The category C_G

G is a finite group. I am going to construct the C_G from category C(G) and C(G) is constructed from category \mathcal{C} . \mathcal{C} is the same category \mathcal{C} in the book Biset Functors for Finite Groups of Serge Bous. It is the biset category of finite groups. Objects of the \mathcal{C} are finite groups and morphism from finite groups G to H are

$$Hom_{\mathcal{C}}(G,H) = B(H,G).$$

Definition 3.6. B(H,G) B(H,G), the Grothendieck group of the category (H, G) bisets, is defined as the quotient of the free abelian group on the set of isomorphism classes of finite (H,G)-bisets by the subgroup generated by the element of the form,

$$[X \sqcup Y] - [X] - [Y],$$

where X and Y are finite (H, G)-bisets, [X] is an isomorphism class of X and $X \sqcup Y$ is disjoint union of X and Y.

Definition 3.7. (H,G) biset If H and G are finite groups and X is (H, G)-biset. (H,G)-biset is a left H-set and right G-set, such that

$$\forall h \in H, \forall x \in X, \forall g \in G, (h.x).g = h.(x.g)inX.$$

In C, The every morphism between finite groups G to H can be factored as the composition of $Ind_D^H \circ Inf_{D/C}^D \circ Iso(f) \circ Def_{B/A}^B \circ Res_B^G$. f is isomorphism from B/A to D/C. B and D are sub groups of G and H, A and C are normal subgroups of B and D.

$$Hom_{\mathcal{C}}(G,H) = B(H,G)$$

We have fundamental bisets in C(G) which connected with the three types of maps we are having in the category C(G). Let H is a subgroup of G.

1. G is an (H,G)-biset for the actions given by left and right multiplication in G and it is denoted by Res_{H}^{G} .

2. G is an (G,H)-biset for the actions given by left and right multiplication in G and it is denoted by Ind_{H}^{G} .

3. If $f : B \to D$ is a group isomorphism, then the set D is an (D,B)-biset, for the left action of D by multiplication, the right action of B given by taking image by f, and then multiplying on the right in D. It is denoted by **Iso(f)**.

Category C(G) can be constructed from C by taking a fixed finite group G. C(G) has objects the group G and its subgroups H, K, A, B, C, D... ect. The morphisms in C(G)can be shown as composition of only three types of maps, induction map (Ind), inner isomorphism (Iso) and restriction map(Res). Any Map from H to K can be factored as

$$Ind_D^K \circ Iso(f) \circ Res_B^H,$$

induction maps from subgroup D to K (Ind_D^K) , inner isomorphisms from B to D (Iso(f)) and restriction maps from H to B (Res_B^H) . For any objects of C(G) H and K,

$$Hom_{C(G)}(H,K) \subsetneq Hom_{\mathcal{C}}(H,K)$$

 $Mod_{C(G)}$ is not equivalent to the category of $Mod_{B^{OP}}$ due to the Theorem A of Mackey Functors and Bisets, Hambleton, Taylor and Williams.

Then C_G the category I aim, will be constructed from C(G) by a functor.

$$F: C(G) \to C_G.$$

The paper of Hambleton, Taylor and Williams, I mention above, gives such a functor, where C_G is full subcategory of B^{OP} with objects G/H where H is a subgroup of G.

3.3 B^{OP} and C_G

Category C_G is full subcategory of B^{OP} . All maps of C_G is in B^{OP} since maps are composition of Induction, inner Isomorphism and restriction maps. There is a functor

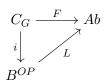
$$i: C_G \hookrightarrow B^{OP}$$

Definition 3.8 (Ab). Ab is the category which whose objects are all small (additive) abelian groups and morphisms are all homomorphisms of abelian groups.

Definition 3.9 (Left Induction ${}^{L}Ind_{C_{G}}^{B^{OP}}$). The Left Induction functor is from the book named Biset Functors for Finite Groups by Serge Bouc. Let the functor $i: C_{G} \hookrightarrow B^{OP}$,

$$G/H \mapsto iH = G/H$$

In the diagram



^L $Ind_{C_G}^{B^{OP}}$ is a functor of $\mathcal{A}b$ categories. It sends from Mod_{C_G} to $Mod_{B^{OP}}$. Let A is the \mathcal{A} b category.

$$Mod_{B^{OP}} = [B^{OP} \to A]$$
 and
 $Mod_{C_G} = [C_G \to A]$

$${}^{L}Ind_{C_{G}}^{B^{OP}}: Mod_{C_{G}} \to Mod_{B^{OP}}$$

a functor in The functor category $Mod_{C_{G}}, F: C_{G} \to Ab$

^LInd^{$$\mathcal{B}^{\mathcal{OP}}_{C_G}(F)(iG/H) = ^L Ind^{\mathcal{B}^{\mathcal{OP}}}_{C_G}(F)(G/H)$$}

^LInd^{$\mathcal{B}^{\mathcal{OP}}_{C_G}$} works as follow,

F is

$${}^{L}Ind_{C_{G}}^{\mathcal{B}^{\mathcal{OP}}}(F)(G/H) = [\bigoplus_{K \in S} Hom_{\mathcal{B}^{\mathcal{OP}}}(G/K, G/H) \bigotimes F(K)]/I.$$

S is set of representative of objects of C(G), set of subgroups of G. I is the submodule generated by the elements

$$(u \circ \alpha) \otimes f - u \otimes F(\alpha)(f),$$

For any elements J and K of S, any morphism $\alpha \in Hom_{C_G}(G/J, G/K)$, any $f \in F(G/J)$, and any u in \in Hom_{BOP} (iG/K, iG/H). J, K and H are subgroups of G.

$$i^* \circ^L Ind_{C_G}^{\mathcal{B}^{\mathcal{O}_{\mathcal{P}}}}$$
 sends $F(G/H) \mapsto \bigoplus_{K \in S} Hom_{\mathcal{B}^{\mathcal{O}_{\mathcal{P}}}}(iG/K, iG/H) \bigotimes F(G/K)/I.$

Let $f \in F(G/H)$, Then $H \in S$ and $Hom_{\mathcal{B}^{\mathcal{O}_{\mathcal{P}}}}(G/H, G/H) \bigotimes F(G/H)/I$. So, $f \mapsto [id_{G/H} \bigotimes f]$

Any map v in $Hom_{\mathcal{B}^{\mathcal{OP}}}$, v : $iG/K \rightarrow iG/J$, the map

^LInd<sup>$$\mathcal{B}^{\mathcal{OP}}_{C_G}(F)(v)$$
 : ^LInd ^{$\mathcal{B}^{\mathcal{OP}}_{C_G}(F)(iG/K) \to ^L Ind^{\mathcal{B}^{\mathcal{OP}}}_{C_G}(F)(iG/J)$}</sup>

is induced by composition on the left in $\mathcal{B}^{\mathcal{OP}}$.

Theorem 3.10. There is an equivalence of categories $Mod_{B^{OP}}$ to Mod_{C_G} .

Proof. Let every objects of B^{OP} is finite sum of objects of C_G . Claim 1. The functor $i^* : Mod_{B^{OP}} \to Mod_{C_G}$ is full and faithful. Proof for claim 1,

$$Mod_{B^{OP}} = [B^{OP} \to \mathcal{A}],$$

 \mathcal{A} is the \mathcal{A} b category, and

$$Mod_{C_G} = [C_G \to \mathcal{A}]$$

Let functor *i*: $C_G \hookrightarrow \mathcal{B}^{\mathcal{OP}}$. Every object H in C_G ,

$$i(G/H) = G/H \in ob(\mathcal{B}^{\mathcal{OP}})$$

Let isomorphism f : G/B \longrightarrow G/D in C_G $(D = gBg^{-1})$. *i* sent Iso(f) to

$$G/D \xleftarrow{f} G/B \xrightarrow{id} G/B$$

For Ind_D^K (D \subset K), $Ind_{C_G} : G/K \to G/D$ will be sent

$$G/K \xleftarrow[f_1]{} G/D \xrightarrow{id} G/D$$

For Res_B^H (B \subset H), $\operatorname{Res}_{C_G} : G/B \to G/H$ will be sent to

$$G/B \xleftarrow{id} G/B \xrightarrow{f_2} G/H$$

$$Mod_{B^{OP}} = [B^{OP} \to \mathcal{A}],$$

if we pre-compose i to any functor F' of $Mod_{B^{OP}}$, we will get the

$$i \circ F' : C_G \to \mathcal{A}$$

$$i^*: Mod_{B^{OP}} \to Mod_{C_G}.$$

If any pair of objects in $Mod_{B^{OP}}$ are exist in Mod_{C_G} , every morphism between these objects will exist in Mod_{C_G} too. So, i^* is full and faithful.

Claim 2.

$$i: C_G \hookrightarrow B^{OP}$$

i is a full and faithful functor from C_G to B^{OP} .

proof for claim 2,

Every object of C_G are exist in B^{OP} since every object in B^{OP} is the finite sum objects of C_G . Any pair of objects of C_G are exist in B^{OP} as I showed above. So, *i* is a full and faithful functor from C_G to B^{OP} . Let the functor ${}^LInd_{C_G}^{B^{OP}}: Mod_{C_G} \to Mod_{B^{OP}}$.



Both B^{OP} and \mathcal{A} are addictive categories. There exists Left Kan extension L of T along i. L is an addictive functor and pair with the natural transformation $\epsilon : T \to Li$. It is a functor of the functor category ${}^{L}Ind_{C_{G}}^{\mathcal{B}^{\mathcal{OP}}}$. I give a short name

$$i' = {}^{L} Ind_{C_{G}}^{B^{OP}}$$

According to the Corollary 3, Section X.3 of Categories for the Working Mathematician by Mac Lane, if the functor **i** is full and faithful, then the universal arrow $\eta: T \to Li$ for Functor L along **i** is a Natural Isomorphism η : T \cong Li. But I know

$$Li = i'T$$
 and $i^* \circ i'T = i'T$.

By the adjunction, there is a natural bijection map

$$(T \xrightarrow{\eta_T} i^* \circ i'T) \stackrel{bijection}{\longleftrightarrow} (i'T \longrightarrow i'T)$$

There exists
$$id \in [i'T \to i'T] \iff id \in [T \to i^* \circ i'T]$$

Then, get $i^* \circ i' \cong id_{Mod_{C_G}}$ On the other hand, the Theorem 1 of adjunction, chapter IV.1 of Saunders Mac Lane, gives a natural map

$$(i' \circ i^*L \xrightarrow{\epsilon_L} L) \stackrel{bijection}{\longleftrightarrow} (i^*L \longrightarrow i^*L)$$

There exists $id \in [i^*L \to i^*L] \iff id \in [i' \circ i^*L \to L]$ $i^* \circ i' \circ i^*L \xrightarrow{i^* \circ \epsilon} i^*L$ $i^* \circ \eta \uparrow \qquad id$ i^*L

The two isomorphisms $i^* \circ \eta$ and id are give that $i^* \circ \epsilon$ is isomorphism in the naturally commute diagram. And the following proposition 3.11 gives that ϵ is isomorphism for all L of $Mod_{B^{OP}}$.

$$i' \circ i^* L \xrightarrow{\epsilon} L$$

$$i' \circ i^* = Id_{Mod_{BOF}}$$

So, If every objects of B^{OP} is finite sum of objectives of C_G , then The functor i^* : $Mod_{B^{OP}} \to Mod_{C_G}$ is equivalence of categories.

Proposition 3.11. For every additive functor $M : \mathcal{B}^{OP} \to \mathcal{A}$, the natural map $M(a \oplus b) \to M(a) \times M(b)$ is an isomorphism.

Proof. Inmage of disjoint union of Gsets, a $\bigsqcup b$ in $\mathcal{B}^{\mathcal{OP}}$ is $M(a \bigsqcup b)$ in \mathcal{A} .

Claim. $M(a \bigsqcup b)$ is isomorphic to $M(a) \times M(b)$.

Due to definition of Additive functor 2.2, M send the bi-product diagram to a biproduct diagram in \mathcal{A} .

According to the Theorem 2 of the section VIII.2, Categories for working Mathematician of Mac Lane, for any two objects a and b in an \mathcal{A} b category \mathcal{A} , \mathcal{A} has bi-product of them if and only if \mathcal{A} has product of them.

According to the definitions of bi-product 3.3 and co-product 2.12,

$$M(a) \xrightarrow{i_1} M(a \sqcup b) \xleftarrow{i_2} M(b)$$

$$\downarrow \exists ! \alpha \qquad \qquad i_2 \qquad \qquad M(b)$$

$$\exists ! \alpha \qquad \qquad i_2 \qquad \qquad M(b)$$

there is the unique map between M(a \oplus b) and M(a) \times M(b) and the unique map α should be an isomorphism since

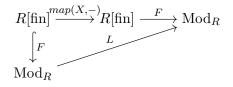
$$\alpha \circ i_1 = i_1$$
 and $\alpha \circ i_2 = i_2$
 $M(a \oplus b) \cong M(a) \times M(b)$

3.4 Tensor induction of representations

Let R is a commutative ring, then the tensor product $M \otimes_R N$ of two R-modules is itself an R-module (by functoriality). This allows us to iterate the tensor product construction. In particular, we can consider

 $\bigotimes_{x \in X} \mathbf{M} = \mathbf{M} \otimes_R M \otimes_R M \otimes_R \dots \dots \otimes_R \mathbf{M}$

This construction can also be considered as a left Kan extension : F is functor for making free modules and $L(M) = \bigotimes_{x \in X} M \in Mod_R$, L is a functor of left Kan extension. Let a finite set X is fixed. In the following diagram



$$map(X, -) : R[fin] \to R[fin]$$

 $Y \mapsto map(X, Y).$

The functor map(X, -) sends the maps $f_i : Y \to Y' \in R[fin]$ and $a_i \in R$,

$$\sum_{i=1,..n} a_i f_i \mapsto \phi = [\sum_{\underline{i}: X \to 1,..n} (\prod_{x \in X} a_{\underline{i}(x)}) f_{\underline{i}}] \ni R[\text{fin}].$$

Map

$$f_{\underline{i}}: map(X, Y) \to map(X, Y')$$

is given by the formula $f_{\underline{i}}(k)(x) = f_{\underline{i}(x)}(k)(x)$ for $k \in map(X,Y)$ and $x \in X$. We can show the previous diagram as a commute diagram as follow too,

$$\begin{array}{c} Y \in R[\mathrm{fin}] \xrightarrow{map(X,-)} R[\mathrm{fin}] \ni map(X,Y) \\ & \downarrow_{F} & \downarrow_{F} \\ FY \in \mathrm{Mod}_{R}^{M \mapsto \otimes_{x \in X} M} \mathrm{Mod}_{R} \ni F(map(X,Y)) \end{array}$$

The total number of maps in map(X,Y) is $|Y|^{|X|}$ maps and when we make the free module

$$F(map(X,Y)) \cong \bigoplus_{f \in map(X,Y)} R = R^{|y|^{|X|}}.$$

and $FY = \bigoplus_{y \in Y} R$. So,

$$\otimes_{x \in X} FY = \otimes_{x \in X} (\oplus_{y \in Y} R) = R^{|y|^{|X|}}.$$

 $F(map(X,Y)) \cong \bigotimes_{x \in X} FY.$

We define the functor L, left induction functor,

$$LM = \bigotimes_{x \in X} M = M \otimes_R M \otimes_R M \otimes_R \dots \otimes_R M$$

as a co-equalizer

$$\bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} F(map(X, Y_0)) \xrightarrow[b]{a} \bigoplus_{FY \to M} F(map(X, Y)) \xrightarrow[u]{u} LM$$

In equation 11, the element

$$x = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, t \in map(X, Y_0)) \quad \text{of} \quad \bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} F(map(X, Y_0)) = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, t \in map(X, Y_0))$$

will be sent by map a to $(\alpha \circ f, t) \in (\bigoplus_{FY \to M} F(map(X, Y)))$ and it will be sent to an element $Lm \in LM$ by u. The element x will be sent by map b to $(\alpha, f(t)) \in (\bigoplus_{FY \to M} F(map(X, Y)))$ and it will be sent to the same element $Lm \in LM$ by u.

Lemma 3.12. There is a coequalizer diagaram as follow:

$$\bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} map(X, FY_0) \xrightarrow{a'} \bigoplus_{FY \to M} map(X, FY) \xrightarrow{u'} map(X, M)$$

(11)

In equation 12

Two parallel morphisms a' and b' send on map to two different maps of $\bigoplus_{FY \to M} map(X, FY)$ but coequalizer u' make both of them send to same maps in map(X, LM) in In equation 11.

Let the element

$$x' = (\alpha : FY_1 \to M, f : FY_0 \to FY_1, a \in map(X, FY_0)) \quad \text{of} \quad \bigoplus_{FY_1 \to M} \bigoplus_{FY_0 \to FY_1} map(X, FY_0)$$

send by map a' to $(\alpha \circ f, a) \in (\bigoplus_{FY \to M} map(X, FY))$ and then we get the map $(\alpha \circ f \circ a) \in map(X, M)$ by map u'.

Let x' send by map b' to the $(\alpha, f(a) \in map(X, FY))$ and get $(\alpha \circ f \circ a) \in map(X, M)$ by u'.

Proof. The proof for this lemma is the same with case of Lemma 2.20 if the fix set X has the only two elements. If X has more than two elements we can use the induction method to prove it is right for all finite set X. I will omit this detail proof here in my thesis. \Box

Definition 3.13 (The tensor induction in the diagram). The formula for the tensor induction of representations. Let G and H be finite groups and let X be a left H, right G -set which is free as an H -set. F is functor of free modules. We define a functor $map_H(X, -)$ from HR[fin] to GR[fin] taking an object Y to $map_H(X, Y)$.

$$HR[\operatorname{fin}] \xrightarrow{\operatorname{map}_{H}(X,-)} G - Set \xrightarrow{F} R[G] - Mod$$

$$\downarrow_{F} \xrightarrow{Tens_{H}^{G}}$$

$$R[H] - Mod$$

It takes a H-morphism $f = \sum_{i=1,..n} a_i f_i, f_i : Y \to Y'$ and $a_i \in R$, to

$$\phi = [\sum_{I:H \setminus X \to [1,..n]} (\prod_{u \in H \setminus X} a_{I(u)}) f_{I \circ p}] \ni GR[\text{fin}],$$

where $p: X \to H \setminus X$ is the projection and given $J: X \to [1, ..., n]$, the map

$$f_J: map(X, Y) \to map(X, Y')$$

is given by the formula $f_J(k)(x) = f_{J(x)}(k(x))$ for $k \in map(X,Y)$ and $x \in X$. It is straight forward to check that

$$\phi(gk) = g\phi(k)$$

for every $g \in G$. Using that H acts freely on X, f is an H-morphism we can verify that if k is a H-map, then $\phi(k) \in F(map_H(X, Y'))$. This means that we have a G-morphism $\phi: map(X, Y) \to map(X, Y')$. We define

$$map(X, -)(f) := \phi.$$

Here in the diagram, $Tens_H^G M$ is the tensor induction functor, R[G]-Mod is R module with an action of G. That is $\bigotimes(M) = Tens_H^G M \in R[G]$ - Mod. There is an isomorphism of R-modules

$$Tens_H^G M \cong \bigotimes_{G/H} M.$$

3.5 Tensor induction with the category of B_G^{OP}

Let H and K are subgroups of G and $H \subset K$. We construct two categories B_H^{OP} and B_K^{OP} from H and K. Then the Mackey functors give $Mod_{R(B_H)}$ and $Mod_{R(B_K)}$ and we can have $Tens_H^K$ as a functor between two categories of modules.

$$R(B_{H}^{OP}) \xrightarrow{P_{H}^{K}} R(B_{K}^{OP}) \xrightarrow{F_{K}} Mod_{R(B_{K})}$$

$$\downarrow^{F_{H}} \xrightarrow{Tens_{H}^{K}} Mod_{R(B_{H})}$$

$$(13)$$

in the diagram $Mod_{R(B_H)}$ is the category of the functors from RB_H to Mod_R , and the functor $P_H^K = map(K, -)$. Let X, X' and Y are objects of RB_H^{OP} , P_H^K takes a map in RB_H^{OP}

$$X \xleftarrow{f_1} c \xrightarrow{f_2} Y \tag{14}$$

where X, Y and c are H-set, to

$$map(K,X) \xleftarrow{f'_1} map(K,c) \xrightarrow{f'_2} map(K,Y)$$

 F_H takes the map X to X' of $R(B_H^{OP})$

$$X \xleftarrow{f_1''} c' \xrightarrow{f_2''} X'$$

to $RB_H(X, -)$. For any Y in $ob(B^{OP})$, there is $RB_H(X, Y)$. The map (X, X') in RB_H^{OP} induces $RB_H(X', Y)$ by Yoneda embedding lemma as follow:

$$X' \xleftarrow{g_1} c' \xrightarrow{g_2} X$$

 $in RB_H$, and

$$X \xleftarrow{f_1} c \xrightarrow{f_2} Y$$

give by composing and having pull back

$$X' \xleftarrow{h_1} e \xrightarrow{h_2} Y$$

Yoneda embedding $: RB_H^{OP} \to [Mod_{RB_H} = (RB_H, Mod_R)]$

Another functor F_K is working same as F_H . If we define Tensor induction $Tens_H^K$ similar as previous section, we get the functor which makes commute the diagram 13.

References

[1]	Saunders Mac Lane, Categories for working Mathematician, secon edition, Springer.1997.
[2]	Serge Bouc, Bisets Functors for Finite Groups, SpringerLink. 2010.
[3]	A. W. M. Dress, Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications, Bielefeld, 1971.
[4]	Hambleton, Taylor and Williams, paper named Mackey Functors and Bisets, 2010.
[5]	L Gaunce Lewis, Jr, The Theory of Green Functor, Unpublished notes, 1981.
[6]	Nobuo Yoneda, Saunders Mac Lane, Categories for working Mathemati- cian, secon edition, Springer.1997.