Pseudo H-type algebras and sub-Riemannian cut locus

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Preface

The thesis is structured in three parts.
Part I is an introduction into the main topics of the thesis with a brief historical overview and a summary of the main research results and possible future research. Furthermore, we present a preliminary section which introduces the main mathematical definitions which are required to understand the results in Part II and III.
In Part II, we study the Lie algebra geometry of pseudo H-type algebras. A pseudo H-type algebra $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$ is a nilpotent Lie algebra of step two, where $\mathfrak{v}_{r,s}$ is a $\mathrm{Cl}_{r,s}$-module of minimal dimension satisfying an admissibility condition, and $\mathfrak{z}_{r,s}$ is a generating vector space of the Clifford algebra $\mathrm{Cl}_{r,s}$. In Chapter 4, we present a partial classification of the pseudo H-type algebras with minimal admissible Clifford modules. Furthermore, we prove that the subspace $\mathfrak{v}_{r,s}$ of $\mathfrak{n}_{r,s}$ is strongly bracket generating if and only if $r = 0$ or $s = 0$. Additionally, we discuss the classification of pseudo H-type algebras related to non-equivalent irreducible Clifford modules. Chapter 5 generalizes certain ideas of Eberlein [41, 42, 43]. In particular, we study standard metric Lie algebras, which arise from indefinite metric spaces. We demonstrate the main results on pseudo H-type algebras. In Chapter 6, we study the octonionic H type group and present a characterization of the critical points of the natural sub-Riemannian length functional via a differential equation, similar to the geodesic equation in Riemannian geometry.
In Part III, we study the sub-Riemannian cut locus in several different manifolds with different sub-Riemannian structures. In Chapter 7, we present a proof of the fact that the sub-Riemannian cut locus of H-type groups, starting from the origin of the group corresponds to the center of the group. In Chapter 8, we consider the Stiefel manifold $V_{n,k}$ as a principal $U(k)$-bundle over the Grassmann manifold and study the cut locus from the unit element. We give the complete description of this cut locus on $V_{n,1}$ and present a sufficient condition on the general case. At the end, we study the complement to the cut locus of $V_{2k,k}$ and give several examples in lower dimensional cases.

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Contents

Part I

Introduction

The purpose of Part I of this PhD-thesis is to give a general overview of the tools and questions needed to put the obtained results into context. We achieve this by first presenting a brief historical background, with a view towards the recent developments that have paved the way to the modern picture we have of sub-Riemannian geometry and its generalizations. Afterward, we introduce the reader to some of the preliminaries in algebra and differential geometry that are fundamental for a complete understanding of the problems dealt with in this thesis. We conclude Part I by presenting, in abbreviated form, the main results of this work, which are explained at length in the forthcoming parts.

0.1 Historical background and general ideas

The Heisenberg group has played, and still plays, a fundamental role in the development of differential geometry. Many analytic and geometric questions related to it are still unsolved and are the subject of deep research. An important part of this PhD-thesis deals with the Heisenberg type and pseudo Heisenberg-type groups, which are natural generalizations of the classical Heisenberg group. In order to understand why are these generalizations significant and useful, we first discuss the context in which they were introduced, and then relate their construction to special kinds of Clifford modules.

The Heisenberg-type groups, which are nilpotent Lie groups of step two, were first introduced by Kaplan [59], when studying the relation between compositions of positive definite quadratic forms and the fundamental solutions of certain sub-elliptic operators. In particular, he shows that this class of Lie groups have explicit fundamental solutions in elementary form for their sub-Laplacians, analogous to the known results in the case of the Heisenberg group due to Folland [45]. Kaplan continued his studies on the Riemannian geometry of Heisenberg-type groups, particularly their curvature invariants, geodesics and isometries, in the two interesting papers [60, 61].

The contribution by Kaplan which is most relevant for our work, is the fact that he established a natural connection between the Lie algebras of Heisenberg-type groups and some special actions of Clifford algebras induced by sums of squares. This was later generalized by Ciatti [37] by allowing the quadratic forms to be indefinite. These new objects are called pseudo (or generalized) H-type algebras. Some of the main references dealing with geometric, algebraic and analytic aspects of Heisenberg and pseudo H-type groups are [20, 28, 29, 46, 50, 51, 79, 80, 81].

The main source of interest in the geometry of Heisenberg-type groups comes from their natural sub-Riemannian structure. Sub-Riemannian geometry is a generalization of Riemannian geometry which has attracted much attention from the mathematical community in the last three decades. An intuitive explanation of the basic sub-Riemannian problem is that we can only measure the length of curves whose velocity vectors belong to a given set of directions, which changes from point to point. These curves are usually referred to as horizontal or admissible, and the set of allowed motions is known as the horizontal distribution. Classical examples outside of pure mathematics are related to certain problems in control theory, for example parking a car or landing a plane, and thus give evidence that sub-Riemannian geometry is a research field of interest not only in pure, but also in applied mathematics. The problem of finding shortest horizontal curves is of great interest especially in robotics, classical mechanics (nonholonomy), neurobiology, financial mathematics, quantum physics and diverse physical theories.

There exist different opinions regarding the initial moment of sub-Riemannian geometry. It is fair to say that the first theorem in sub-Riemannian geometry is due to Carathéodory and is related to Carnot's thermodynamics, but sources related to nonholonomic mechanics can be traced as far back as Hertz in the late 19th century. A generalization of Carathéodory's theorem, developed independently Chow and Rashevskiĭ [34, 78] in 1939 and 1938, respectively, introduces what is nowadays known as a bracket generating distribution. A horizontal distribution is bracket generating if at each point the space of all Lie brackets of sections of the distribution span the whole tangent space. The Chow-Rashevskiĭ theorem assures that if a distribution is bracket generating, then any two points can be connected by a horizontal path. This is of importance as it implies the existence of horizontal curves of smallest length connecting any two points. On the other hand, Strichartz [84, 85] established sub-Riemannian geometry as a mathematical area on its own right within differential geometry. He was the first who introduced the concepts of sub-Riemannian geometry, the sub-Riemannian co-metric and the sub-Riemannian Christoffel symbols. Nowadays, sub-Riemannian geometry is a fully fledged, busy and constantly developing mathematical branch. It is not possible to summarize all interesting research directions in this field. For a deeper understanding of present day research we recommend reading Montgomery, Gromov, Bellaïche and Risler: [19, 53, 70, 71]. We also would like to mention the great impact of Agrachev in this field by referring to his works [2, 3, 4, 5, 6], and his book in progress together with Barilari and Boscain [1], which gives particular attention to the Hamiltonian point of view of sub-Riemannian geometry. Finally, we would like to call to the attention of the reader the book [57] of Jurdjevic, in which control theory is presented from a geometric point of view, having the sub-Riemannian problem as a fundamental example in optimal control theory.

Chapter 1

Preliminaries

In this chapter we introduce the main mathematical objects and tools to set the context for Part II and III.

1.1 Preliminaries of H-type algebras

1.1.1 2-step nilpotent Lie algebras

One of the main objects studied in this thesis are 2-step nilpotent Lie algebras. We introduce them and state the most important properties. For more details see [62].

Definition 1.1.1. *A Lie algebra $\mathfrak{n}$ is nilpotent of step 2 if $[\mathfrak{n}\,,[\mathfrak{n}\,,\mathfrak{n}]]=\{0\}$ and $[\mathfrak{n}\,,\mathfrak{n}]\neq\{0\}$.*
A connected Lie group is called nilpotent of step 2 if its corresponding Lie algebra is nilpotent of step 2.

Remark 1.1.2. *If $\mathfrak{n}$ is a 2-step nilpotent Lie algebra, then its commutator ideal $[\mathfrak{n}\,,\mathfrak{n}]$ is a subset of the center $\mathfrak{Z}(\mathfrak{n}):=\{v\in\mathfrak{n}|[v\,,\mathfrak{n}]=\{0\}\}$ of $\mathfrak{n}$.*

We remind that if the Lie group N of the Lie algebra $\mathfrak{n}$ is simply connected and nilpotent of step 2, then the Lie group exponential map $\exp\colon\mathfrak{n}\to N$ is a diffeomorphism and we let $\log\colon N\to\mathfrak{n}$ denote its inverse.

Proposition 1.1.3 (Campbell-Baker-Hausdorff Formula)**.** *If $\mathfrak{n}$ is a 2-step nilpotent Lie algebra, then*

$$\begin{array}{lclll}\exp(v)\exp(w) & = & \exp\left(v+w+\frac{1}{2}[v\,,w]\right) & \textit{for all} & v,w\in\mathfrak{n},\\ \log(g\tilde{g}) & = & \log(g)+\log(\tilde{g})+\frac{1}{2}[\log(g)\,,\log(\tilde{g})] & \textit{for all} & g,\tilde{g}\in N.\end{array}$$

1.1.2 Standard metric 2-step nilpotent Lie algebras

In this subsection we present shortly the ideas from [41, 42], showing that any 2-step nilpotent Lie algebra can be endowed with a canonical positively definite scalar product. The choice of such kind scalar product is unique up to the Lie algebra isomorphism.

Through the present subsection we assume that a 2-step nilpotent Lie algebra $\mathfrak{g}$ has a commutator ideal $[\mathfrak{g}\,,\mathfrak{g}]$ of dimension n and its complement is of dimension m. A basis $\mathcal{B} = \{v_1, \dots, v_m, z_1, \dots, z_n\}$ of the Lie algebra $\mathfrak{g}$ is called adapted if $\{z_1, \dots, z_n\}$ is the basis of $[\mathfrak{g}\,,\mathfrak{g}]$. Define the skew-symmetric $(m \times m)$-matrices $C^1 := (C^1_{\alpha\beta})_{\alpha\beta}, \dots, C^n := (C^n_{\alpha\beta})_{\alpha\beta}$ by

$$[v_\alpha\,, v_\beta] = \sum_{k=1}^{n} C^k_{\alpha\beta} z_k.$$

Matrices C^k are elements of the group $\mathfrak{so}(m)$ and they are linearly independent in $\mathfrak{so}(m)$, see [42]. Then the n-dimensional subspace $\mathcal{C}^n = \operatorname{span}\{C^1, \dots, C^n\} \subset \mathfrak{so}(m)$ is isomorphic to $[\mathfrak{g}\,,\mathfrak{g}] = \operatorname{span}\{z_1, \dots, z_n\}$ and is called the structure space determined by the adapted basis $\mathcal{B}$. The vector space $\operatorname{span}\{v_1, \dots, v_m\} \oplus \operatorname{span}\{z_1, \dots, z_n\}$ of the 2-step nilpotent Lie algebra $\mathfrak{g}$ is isomorphic to the direct sum $\mathbb{R}^m \oplus \mathcal{C}^n$. The n-dimensional subspace $\mathcal{C}^n \subset \mathfrak{so}(m)$ depends on the choice of the adapted basis, nevertheless all possible subspaces defined by an arbitrary choice of an adapted bases form the set $\{A\mathcal{C}^n A^{\mathbf{t}} \mid A \in \mathrm{GL}(m)\}$, where $A^{\mathbf{t}}$ is the transpose matrix of A.

The spaces $\mathbb{R}^m$ and $\mathcal{C}^n \subset \mathfrak{so}(m)$ have a natural choice of inner products that will determinate the Lie algebra product on $\mathcal{G} = \mathbb{R}^m \oplus \mathcal{C}^n$. Denote by $\langle \cdot\,, \cdot \rangle_{\mathfrak{so}(m)}$ the positive definite product on $\mathfrak{so}(m)$ defined by

$$\langle Z\,, Z' \rangle_{\mathfrak{so}(m)} = -\operatorname{tr}(ZZ'),$$

and $\langle \cdot\,, \cdot \rangle_m$ the standard Euclidean inner product in $\mathbb{R}^m$. The notation $\langle \cdot\,, \cdot \rangle_{\mathfrak{so}(m)}$ is also used for the restriction of this inner product on $\mathcal{C}^n \subset \mathfrak{so}(m)$. Then the inner product $(\cdot\,, \cdot) = \langle \cdot\,, \cdot \rangle_m + \langle \cdot\,, \cdot \rangle_{\mathfrak{so}(m)}$ makes the direct sum $\mathcal{G} = \mathbb{R}^m \oplus \mathcal{C}^n$ orthogonal. Let $[\cdot\,, \cdot]$ be the unique Lie product on $\mathcal{G}$ such that $\mathcal{C}^n$ belongs to the center of $\mathcal{G}$ and

$$\langle Z(V)\,, W \rangle_m = \langle Z\,, [V\,, W] \rangle_{\mathfrak{so}(m)} \quad \text{for arbitrary} \quad V, W \in \mathbb{R}^m, \quad Z \in \mathcal{C}^n,$$

where $Z(V)$ simply denotes the action of $Z \in \mathcal{C}^n \subset \mathfrak{so}(m)$ on a vector $V \in \mathbb{R}^m$ defined by matrix multiplication. It is easy to see that $(\mathcal{G}, [\cdot\,, \cdot])$ is a 2-step nilpotent Lie algebra, such that $[\mathcal{G}\,, \mathcal{G}] = \mathcal{C}^n$ and it is called a standard metric 2-step nilpotent Lie algebra. It was shown in [41] that *any 2-step nilpotent Lie algebra $\mathfrak{g}$ is isomorphic to a standard metric 2-step nilpotent Lie algebra $\mathcal{G} = \big(\mathbb{R}^m \oplus \mathcal{C}^n, [\cdot\,, \cdot], (\cdot\,, \cdot)\big)$.*

1.1.3 Pseudo H-type algebras

Throughout this thesis we assume all scalar products to be non-degenerate unless otherwise stated. We denote by $\mathfrak{n}$ a nilpotent 2-step Lie algebra endowed with a scalar product $\langle \cdot\,, \cdot \rangle_{\mathfrak{n}}$ of signature (p, q), $p, q \in \mathbb{N}$, $p + q = \dim(\mathfrak{n})$: that means that there exists a basis $\{U_1, \dots, U_{p+q}\}$ of $\mathfrak{n}$ which satisfies

$$\langle U_i\,, U_j \rangle_{\mathfrak{n}} = \epsilon_i(p, q)\delta_{ij}, \quad \text{where} \quad \epsilon_i(p, q) = \begin{cases} 1, & \text{for } i = 1, \dots, p, \\ -1, & \text{for } i = p + 1, \dots, p + q. \end{cases}$$

Let $\mathfrak{z}$ be the center of the 2-step nilpotent Lie algebra $\mathfrak{n}$ and $\langle\cdot\,,\cdot\rangle_{\mathfrak{z}}$ the restriction of the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{n}}$ to $\mathfrak{z}$. We assume that $\langle\cdot\,,\cdot\rangle_{\mathfrak{z}}$ is non-degenerate. Then the orthogonal complement $\mathfrak{v} := \mathfrak{z}^{\perp}$ is also a non-degenerate scalar product space, where we use the symbol $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ to denote the restriction of $\langle\cdot\,,\cdot\rangle_{\mathfrak{n}}$ on $\mathfrak{v}$. Thus $\mathfrak{n} = \mathfrak{z}\oplus_{\perp}\mathfrak{v}$ is an orthogonal decomposition with respect to the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{n}} = \langle\cdot\,,\cdot\rangle_{\mathfrak{z}} + \langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$. Since $\mathfrak{z}$ is the center of $\mathfrak{n}$, the commutator is a skew-symmetric bi-linear map $[\cdot\,,\cdot]\colon\,\mathfrak{v}\times\mathfrak{v}\to\mathfrak{z}$.

Definition 1.1.4. *Let $\mathfrak{n} = (\mathfrak{z}\oplus_{\perp}\mathfrak{v}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle_{\mathfrak{n}})$ be a Lie algebra described above. We define the map $J\colon\,\mathfrak{z}\to\mathrm{End}(\mathfrak{v})$ by*

$$\langle J_Z v\,,w\rangle_{\mathfrak{v}} = \langle Z\,,[v\,,w]\rangle_{\mathfrak{z}},\quad \textit{for all } v,w\in\mathfrak{v}\,. \tag{1.1}$$

Definition 1.1.5. *[37] We call a 2-step nilpotent Lie algebra $\mathfrak{n} = (\mathfrak{z}\oplus_{\perp}\mathfrak{v}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle_{\mathfrak{n}})$ with $J\colon\,\mathfrak{z}\to\mathrm{End}(\mathfrak{v})$ from Definition 1.1.4 a pseudo H-type algebra if*

$$\langle J_Z v\,,J_Z v\rangle_{\mathfrak{v}} = \langle Z\,,Z\rangle_{\mathfrak{z}}\langle v\,,v\rangle_{\mathfrak{v}}\quad \textit{for all}\quad Z\in\mathfrak{z}\ \textit{ and } v\in\mathfrak{v}\,. \tag{1.2}$$

In the following we write $\mathfrak{n}_{r,s}$ to emphasize that $\langle\,\cdot\,,\cdot\,\rangle_{\mathfrak{z}}$ has signature (r,s).

It follows directly from Definition 1.1.4 that J_Z is skew-adjoint with respect to the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$:

$$\langle J_Z v\,,w\rangle_{\mathfrak{v}} = -\langle v\,,J_Z w\rangle_{\mathfrak{v}}\quad \text{for all}\quad Z\in\mathfrak{z},\quad v,w\in\mathfrak{v}\,. \tag{1.3}$$

Using polarization in (1.2) we obtain

$$\langle J_Z v\,,J_{Z'}v\rangle_{\mathfrak{v}} = \langle Z\,,Z'\rangle_{\mathfrak{z}}\langle v\,,v\rangle_{\mathfrak{v}},\quad \text{and}\quad \langle J_Z v\,,J_Z v'\rangle_{\mathfrak{v}} = \langle Z\,,Z\rangle_{\mathfrak{z}}\langle v\,,v'\rangle_{\mathfrak{v}}. \tag{1.4}$$

Applying the skew-adjoint property (1.3) one also obtains

$$J_Z\circ J_Z := J_Z J_Z = J_Z^2 = -\langle Z\,,Z\rangle_{\mathfrak{z}}\,\mathrm{Id}_{\mathfrak{v}},\quad \text{or}\quad J_{Z'}J_Z + J_Z J_{Z'} = -2\langle Z\,,Z'\rangle_{\mathfrak{z}}\,\mathrm{Id}_{\mathfrak{v}}, \tag{1.5}$$

for all $Z,Z'\in\mathfrak{z}$. Equalities (1.5) imply that $J\colon\,\mathfrak{z}\to\mathrm{End}(\mathfrak{v})$ defines a representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$.

We note that there exists an equivalent concept of pseudo H-type algebras, the so called general H-type algebras. Let $\mathfrak{n} = \big(\mathfrak{v}\oplus_{\perp}\mathfrak{z}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle_{\mathfrak{n}} = \langle\cdot\,,\cdot\rangle_{\mathfrak{v}} + \langle\cdot\,,\cdot\rangle_{\mathfrak{z}}\big)$ be an arbitrary 2-step nilpotent Lie algebra with center $\mathfrak{z}$ and a non-degenerate scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{n}}$. We write $\mathrm{ad}_v\colon\,\mathfrak{v}\to\mathfrak{z}$ for the linear map given by $\mathrm{ad}_v\,w = [v\,,w]$. We assume that the restriction of the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ onto the subspace $\ker(\mathrm{ad}_v)\subset\mathfrak{v}$ is non-degenerate and denote its orthogonal complement with respect to $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ by $\mathfrak{V}_v$, which is also non-degenerate. Thus the restricted map $\mathrm{ad}_v\colon\,\mathfrak{V}_v\to\mathfrak{z}$ is injective.

Definition 1.1.6. *[51] A two-step nilpotent Lie algebra $\mathfrak{n} = \big(\mathfrak{v}\oplus_{\perp}\mathfrak{z}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle_{\mathfrak{n}} = \langle\cdot\,,\cdot\rangle_{\mathfrak{v}} + \langle\cdot\,,\cdot\rangle_{\mathfrak{z}}\big)$ is of general H-type if $\mathrm{ad}_v\colon\,\mathfrak{V}_v\to\mathfrak{z}$ is a surjective isometry for all $v\in\mathfrak{v}$ with $\langle v\,,v\rangle_{\mathfrak{v}} = 1$ and a surjective anti-isometry for all $v\in\mathfrak{v}$ with $\langle v\,,v\rangle_{\mathfrak{v}} = -1$.*

We are showing in Subsection 4.6.1, that pseudo H-type algebras and general H-type algebras are equivalent.

1.1.4 Clifford algebras and representations

Definition 1.1.7. *A Clifford algebra* $\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ *generated by a scalar product space* $(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ *is the unital associative algebra generated by* $\mathfrak{z}$ *subject to the relation:*

$$Z\otimes Z=-\langle Z\,,Z\rangle_{\mathfrak{z}}\mathbb{I}_{\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})},$$

for all $Z\in\mathfrak{z}$ *and the unit* $\mathbb{I}_{\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})}$ *of the Clifford algebra.*

Thus, $\mathfrak{z}$ can be considered as a subset of $\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ and

$$Z\otimes W+W\otimes Z=-2\langle Z\,,W\rangle_{\mathfrak{z}}\mathbb{I}_{\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})}\quad\text{for all}\quad Z,W\in\mathfrak{z}.$$

Proposition 1.1.8. *[56, 64] Let* $J\colon\mathfrak{z}\to\mathcal{A}$ *be a linear map into an associative algebra* $\mathcal{A}$ *with an identity element* $\mathrm{Id}_{\mathcal{A}}$ *and product " $\cdot_A$ ", such that*

$$J(Z)\cdot_A J(Z)=-\langle Z\,,Z\rangle_{\mathfrak{z}}\,\mathrm{Id}_{\mathcal{A}}\quad\textit{for all}\quad Z\in\mathfrak{z}.$$

Then J *extends uniquely to an algebra homomorphism* $\tilde{J}\colon\ \mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})\to\mathcal{A}$*. Moreover,* $\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ *is the unique associative algebra with this property.*

Definition 1.1.9. *A representation of a Clifford algebra* $\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ *is an algebra homomorphism* $J\colon\ \mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})\to\mathrm{End}(\mathfrak{v})$ *into the algebra of linear transformations of a finite dimensional vector space* $\mathfrak{v}$*. The space* $\mathfrak{v}$ *is called a* $\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$*-module.*

Remark 1.1.10. *We note that we can refer to* $J\colon\mathfrak{z}\to\mathrm{End}(\mathfrak{v})$ *as the Clifford representation* $J\colon\ \mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})\to\mathrm{End}(\mathfrak{v})$ *by Proposition 1.1.8.*

Since J is an algebra homomorphism we obtain that

$$J_Z^2:=J_Z\circ J_Z=J_{Z\otimes Z}=J_{-\langle Z\,,Z\rangle_{\mathfrak{z}}\mathbb{I}_{\mathrm{Cl}(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})}}=-\langle Z\,,Z\rangle_{\mathfrak{z}}\,\mathrm{Id}_{\mathfrak{v}}\,.$$

We assume without loss of generality that $(\mathfrak{z},\langle\cdot\,,\cdot\rangle_{\mathfrak{z}})$ is given by $(\mathbb{R}^{r,s}:=\mathbb{R}^{r+s},\langle\cdot\,,\cdot\rangle_{r,s})$, $r+s=n$, where the scalar product $\langle\cdot\,,\cdot\rangle_{r,s}$ of signature (r,s) is defined for all $Z,W\in\mathbb{R}^{r+s}$ by $\langle Z\,,W\rangle_{r,s}:=\sum_{i=1}^{r}Z_iW_i-\sum_{j=r+1}^{r+s}Z_jW_j$. The Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s},\langle\cdot\,,\cdot\rangle_{r,s})$ is denoted by $\mathrm{Cl}_{r,s}$, where $(Z_1,\ldots,Z_{r+s})$ is the standard orthonormal basis of $\mathbb{R}^{r,s}$ with $\langle Z_i\,,Z_j\rangle_{r,s}=\epsilon_i(r,s)\delta_{ij}$.

1.1.5 Admissible Clifford modules and pseudo H-type algebras

In this subsection we explain when a representation space $\mathfrak{v}$ of a Clifford algebra can be endowed with a scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ such that the representation map J satisfies (1.3). We call a positive definite scalar product an *inner product*, and in any case we work with *only* non-degenerate scalar products.

Proposition 1.1.11. *[56] Let* $J\colon\ \mathrm{Cl}_{r,0}\to\mathrm{End}(\mathfrak{v})$ *be a representation of a Clifford algebra* $\mathrm{Cl}_{r,0}$*. Then there exists an inner product* $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}}$ *on* $\mathfrak{v}$*, such that for all* $Z\in\mathbb{R}^{r,0}$ *with* $\langle Z\,,Z\rangle_{r,0}=1$ *the following holds:*

$$\langle J_Zw\,,J_Zv\rangle_{\mathfrak{v}}=\langle w\,,v\rangle_{\mathfrak{v}}\quad\textit{for all}\quad w,v\in\mathfrak{v}.\tag{1.6}$$

Corollary 1.1.12. *Any representation $J\colon \mathrm{Cl}_{r,0} \to \mathrm{End}(\mathfrak{v})$ satisfies property* (1.3) *with respect to the inner product $\langle \cdot\,, \cdot \rangle_{\mathfrak{v}}$ from Proposition 1.1.11.*

Corollary 1.1.12 follows by replacing w by $J_Z w$ in (1.6) and applying $J_Z^2 = -\operatorname{Id}_{\mathfrak{v}}$. Thus the Clifford algebras $\mathrm{Cl}_{r,0}$ always possess an inner product on $\mathfrak{v}$ such that J_Z is skew-adjoint for all $Z \in \mathbb{R}^{r,0}$. A. Kaplan used inner products on $\mathfrak{z}$ in [59, 61] and, as a consequence, the H-type algebras correspond to $\mathrm{Cl}_{r,0}$-modules. For $\mathrm{Cl}_{r,s}$-modules with $s \geq 1$ equation (1.6) is only true for orthonormal bases and in general not true for an arbitrary element of $\mathbb{R}^{r,s}$, see [56].

Definition 1.1.13. *[37] A pair $(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$, where $\mathfrak{v}$ is a $\mathrm{Cl}_{r,s}$-module is said to be an admissible $\mathrm{Cl}_{r,s}$-module if the representation operators $J_Z\colon \mathfrak{v} \to \mathfrak{v}$ are skew-adjoint with respect to $\langle \cdot\,, \cdot \rangle_{\mathfrak{v}}$, i.e. satisfies* (1.3) *for all $Z \in \mathbb{R}^{r,s}$.*

The following proposition guarantees the existence of admissible $\mathrm{Cl}_{r,s}$-modules.

Proposition 1.1.14. *[37] For any given $\mathrm{Cl}_{r,s}$-module $\mathfrak{v}$ the vector space $\mathfrak{v}$ itself (or $\mathfrak{v} \oplus \mathfrak{v}$) can be equipped with a scalar product $\langle \cdot\,, \cdot \rangle_{\mathfrak{v}}$ (or $\langle \cdot\,, \cdot \rangle_{\mathfrak{v}\oplus\mathfrak{v}}$), such that*

$$\langle J_Z w\,, v \rangle_{\mathfrak{v}} = -\langle w\,, J_Z v \rangle_{\mathfrak{v}}, \quad (\textit{or} \quad \langle J'_Z w\,, v \rangle_{\mathfrak{v}\oplus\mathfrak{v}} = -\langle w\,, J'_Z v \rangle_{\mathfrak{v}\oplus\mathfrak{v}})$$

for all $Z \in \mathbb{R}^{r,s}$ and all $w, v \in \mathfrak{v}$ (or $w, v \in \mathfrak{v} \oplus \mathfrak{v}$, where the operator $J'\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(\mathfrak{v} \oplus \mathfrak{v})$ should be redefined correspondingly).

The relation between Lie algebras of Definition 1.1.5 and admissible Clifford modules is summarized in the following proposition.

Proposition 1.1.15. *[37] Let $\mathfrak{v}$ be a $\mathrm{Cl}_{r,s}$-module. Then $\mathfrak{n} = \mathfrak{v} \oplus \mathbb{R}^{r,s}$ can be supplied with the structure of the pseudo H-type (r, s)-algebra if and only if there exists a scalar product $\langle \cdot\,, \cdot \rangle_{\mathfrak{v}}$ making the $\mathrm{Cl}_{r,s}$-module $\mathfrak{v}$ into an admissible Clifford module $(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$. The bracket $[\cdot\,, \cdot]\colon \mathfrak{v} \times \mathfrak{v} \to \mathbb{R}^{r,s}$ on $\mathfrak{n}$ is given by Definition 1.1.4 and the scalar product is $\langle \cdot\,, \cdot \rangle_{\mathfrak{n}} := \langle \cdot\,, \cdot \rangle_{\mathfrak{v}} + \langle \cdot\,, \cdot \rangle_{r,s}$. The decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathbb{R}^{r,s}$ is orthogonal and $\mathbb{R}^{r,s}$ is the center of $\mathfrak{n}$.*

Proposition 1.1.16. *[37] Let $\mathfrak{n}$ be a pseudo H-type algebra. Then the corresponding admissible $\mathrm{Cl}_{r,s}$-module $(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$ is a neutral scalar product space for $s \geq 1$, i.e. the signature of $\langle \cdot\,, \cdot \rangle_{\mathfrak{v}}$ is (l, l), with $l \in \mathbb{N}$.*

1.1.6 Existence of the integral structure on pseudo H-type algebras

In this subsection we state some necessary facts about the latest research on pseudo H-type algebras based on the work [46]. The principal result of [46] states that pseudo H-type algebras admit a special choice of basis giving integer structure constants.

Theorem 1.1.17. *[46] Let $\mathfrak{n} = \big(\mathfrak{v} \oplus_{\perp} \mathfrak{z}, [\cdot\,, \cdot], \langle \cdot\,, \cdot \rangle_{\mathfrak{n}} = \langle \cdot\,, \cdot \rangle_{\mathfrak{v}} + \langle \cdot\,, \cdot \rangle_{\mathfrak{z}}\big)$ be a pseudo H-type algebra. Then for any orthonormal basis $\{Z_1, \ldots, Z_n\}$ for $\mathfrak{z}$ there is an orthonormal basis $\{v_1, \ldots, v_m\}$ for $\mathfrak{v}$ such that $[v_\alpha\,, v_\beta] = \sum_{k=1}^{n} C_{\alpha\beta}^k Z_k$, where $C_{\alpha\beta}^k = 0, \pm 1$.*

Corollary 1.1.18. *[46] There exists an orthonormal basis* $\{v_1, \dots, v_m, Z_1, \dots, Z_n\}$ *for any pseudo* H*-type algebra such that* $[v_\alpha, v_\beta] = \pm Z_{k_{\alpha,\beta}}$ *or* $[v_\alpha, v_\beta] = 0$. *In particular, for every* Z_k *and* v_α *in* $\mathcal{B}$ *there exists exactly one* $\beta \in \{1, \dots, m\}$ *such that* $[v_\alpha, v_\beta] = \pm Z_k$.

Definition 1.1.19. *We call an orthonormal basis* $\{v_1, \dots, v_m, Z_1, \dots, Z_n\}$ *of a pseudo* H*-type algebra with the form of Corollary 1.1.18 an integral basis. The corresponding Clifford module* $\mathfrak{v} = \operatorname{span}\{v_1, \dots, v_m\}$ *from Corollary 1.1.18 is called an integral module, and if it is of minimal possible dimension we call it a minimal integral module.*

Let $\{v_1, \dots, v_m, Z_1, \dots, Z_n\}$ be an adapted orthonormal basis of $\mathfrak{n}$. Denote by $\epsilon_\alpha^{\mathfrak{v}}$ and $\epsilon_k^{\mathfrak{z}}$ the indices corresponding to scalar product spaces $(\mathfrak{v}, \langle \cdot\,, \cdot \rangle_{\mathfrak{v}})$ and $(\mathfrak{z}, \langle \cdot\,, \cdot \rangle_{\mathfrak{z}})$. The structure constants and the coefficients of the representation operator $J\colon \mathfrak{z} \to \operatorname{End}(\mathfrak{v})$ are

$$[v_\alpha, v_\beta] = \sum_{k=1}^{n} C_{\alpha\beta}^k Z_k \qquad \text{and} \qquad J_{Z_k} v_\alpha = \sum_{\beta=1}^{m} B_{\alpha\beta}^k v_\beta. \tag{1.7}$$

Then we obtain the relation

$$\epsilon_\beta^{\mathfrak{v}} B_{\alpha\beta}^k = \epsilon_k^{\mathfrak{z}} C_{\alpha\beta}^k \tag{1.8}$$

from $\langle J_{Z_k} v_\alpha, v_\beta \rangle_{\mathfrak{v}} = \langle Z_k, [v_\alpha, v_\beta] \rangle_{\mathfrak{z}}$ by [46]. Equality (1.8) allows to relate the structure constants $C_{\alpha\beta}^k$ of pseudo H-type algebras and coefficients $B_{\alpha\beta}^k$ of the representation operator. Recall that we denote the pseudo H-type algebra induced by $\mathrm{Cl}_{r,s}$ by $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$, where $\mathfrak{v}_{r,s}$ is the minimal admissible integral module $\mathfrak{v}_{r,s}$ of $\mathrm{Cl}_{r,s}$ and $\mathfrak{z}_{r,s} = \mathbb{R}^{r,s}$ the generator space of the Clifford algebra $\mathrm{Cl}_{r,s}$ and the center of the Lie algebra $\mathfrak{n}_{r,s}$.

1.1.7 Lattices on Lie groups

In Chapter 5 we discuss the question when certain two step nilpotent Lie algebras admit bases with rational structural constants. This is equivalent to the fact that the corresponding Lie groups admit lattices. We explain this relation.

Definition 1.1.20. *A subgroup* K *of* G *is called a (co-compact) lattice if* K *is discrete and the right quotient* $K\backslash G$ *is compact. The space* $K\backslash G$ *is called a compact nilmanifold or compact* 2*-step nilmanifold if* G *is a* 2*-step nilpotent Lie group.*

Theorem 1.1.21 (Mal'cev criterion [66])**.** *The group* G *admits a lattice* K *if and only if the Lie algebra* $\mathfrak{g}$ *admits a basis* $\mathcal{B} = \{b_1, \dots, b_n\}$ *with rational structural constants* $[b_i, b_j] = \sum_{k=1}^n C_{ij}^k b_k$, $C_{ij}^k \in \mathbb{Q}$.

We denote the Lie exponent and logarithm by $\exp\colon \mathfrak{g} \to G$ and $\log\colon G \to \mathfrak{g}$. Given a lattice K, one can construct the corresponding basis $\mathcal{B}$ as follows. Set $\mathfrak{g}_{\mathbb{Q}} = \operatorname{span}_{\mathbb{Q}} \log K$, which is a Lie algebra over the field $\mathbb{Q}$. Denote by $\mathcal{B}_{\mathbb{Q}}$ a $\mathbb{Q}$-basis in $\mathfrak{g}_{\mathbb{Q}}$. Then it is also an $\mathbb{R}$-basis $\mathcal{B}$ in $\mathfrak{g}$.

Reciprocally, given a basis $\mathcal{B}$ defined as in Theorem 1.1.21, let Λ be a vector lattice in $\mathfrak{g}$, such that $\Lambda \subset \operatorname{span}_{\mathbb{Q}} \mathcal{B}$. Then the lattice K is generated by elements $\exp \Lambda$ and $\operatorname{span}_{\mathbb{Q}}(\log K) = \operatorname{span}_{\mathbb{Q}} \mathcal{B}$.

1.2 Preliminaries of sub-Riemannian geometry

Sub-Riemannian geometry is an abstract setting to study geometry with non-holonomic constraints. A sub-Riemannian manifold is a triplet $(Q, \mathcal{H}, g_{\mathcal{H}})$, where Q is a smooth manifold, $\mathcal{H}$ is a smooth subbundle of the tangent bundle TQ of the manifold Q (or a smooth distribution) and $g_{\mathcal{H}}$ is a smoothly varying with respect to $q \in Q$ inner product $g_{\mathcal{H}}(q)\colon \mathcal{H}_q \times \mathcal{H}_q \to \mathbb{R}$. The topic is actively developed during the last decades and as, now classical, sources we refer to [4, 29, 65, 71, 84]. We remind the necessary definitions and propositions based on [71] if not otherwise stated.

Definition 1.2.1. *A sub-Riemannian manifold is a triplet $(Q, \mathcal{H}, \langle \cdot\,, \cdot \rangle)$, where Q is a C^∞-manifold, $\mathcal{H}$ is a vector subbundle of the tangent bundle TQ, and $\langle \cdot\,, \cdot \rangle$ is a fibre inner-product. The subbundle $\mathcal{H}$ is called horizontal and $\mathcal{H}_q$ is a horizontal space at a point $q \in Q$. The metric $\langle \cdot\,, \cdot \rangle_q\colon \mathcal{H}_q \times \mathcal{H}_q \to \mathbb{R}$, $q \in Q$ is called a sub-Riemannian metric, and the couple $(\mathcal{H}, \langle \cdot\,, \cdot \rangle)$ is a sub-Riemannian structure on Q.*

Definition 1.2.2. *An absolutely continuous curve $\gamma\colon [0, T] \to Q$ is called horizontal if $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$ almost everywhere.*

Definition 1.2.3. *We define the length $l := l(\gamma)$ of an absolutely continuous horizontal curve $\gamma\colon [0, T] \to Q$ as in Riemannian geometry:*

$$l(\gamma) := \int_0^T \|\dot{\gamma}\| dt = \int_0^T \sqrt{\langle \dot{\gamma}(t)\,, \dot{\gamma}(t) \rangle}\, dt.$$

We introduce the function $d(q_0, q)$ for $q_0, q \in Q$ by $d(q_0, q) := \inf_\gamma \{l(\gamma)\}$, where the infimum is taken over all absolutely continuous horizontal curves that connect q_0 and q. If there is no horizontal curve joining q_0 to q, then we declare $d(q_0, q) = \infty$.

Definition 1.2.4. *A horizontal subbundle $\mathcal{H}$ is called bracket generating if for every $q \in Q$ there exists $r(q) \in \mathbb{Z}^+$ such that $\mathcal{H}^{r(q)} = T_qQ$, where $\mathcal{H}^1 := \mathcal{H}$ and $\mathcal{H}^{r+1} := [\mathcal{H}^r\,, \mathcal{H}] + \mathcal{H}^r$, $r \geq 1$.*

The following proposition, known as the Chow-Rashevskiĭ theorem [34, 78], gives a sufficient condition of the existence of horizontal curves.

Proposition 1.2.5. *Let Q be a connected manifold. If the horizontal subbundle $\mathcal{H} \subset TQ$ is bracket generating, then any two points in Q can be joined by a horizontal curve.*

It follows that if $\mathcal{H}$ is bracket generating on a connected manifold, then the function d introduced in Definition 1.2.3 is finite and defines the distance between two points on the manifold, called the Carnot-Carathéodory distance.

Definition 1.2.6. *An absolutely continuous horizontal curve that realizes the distance between two points is called a minimizing geodesic.*

The existence of local and global minimizers is stated in the following sub-Riemannian analogue of the Hopf-Rinow theorem.

Proposition 1.2.7. *[19, Theorem 2.7, p. 19 and Remark 2, p. 20] Suppose a horizontal distribution on a manifold M is bracket generating. Then*

1. *Sufficiently near points can be joined by a minimizing geodesic;*
2. *If (M, d) is a complete metric space for a Carnot-Carathéodory metric d, then any two points can be joined by a minimizing geodesic. In particular, this is true for compact M.*

For example the compactness of the Stiefel manifold guarantees the existence of global minimizing geodesics, for details see Chapter 8.

In the end of this subsection we define one of the main objects of Part III.

Definition 1.2.8. *The cut locus of a point $q_0 \in Q$ in a sub-Riemannian manifold $(Q, \mathcal{H}, g_{\mathcal{H}})$ is the set*

$$K_{q_0} = \Big\{ q \in Q |\ \textit{there exist } T > 0, \gamma_1, \gamma_2 \colon [0, T] \to Q,\ \gamma_1 \neq \gamma_2, \textit{minimizing horizontal geodesics, such that } \gamma_1(0) = \gamma_2(0) = q_0 \textit{ and } \gamma_1(T) = \gamma_2(T) = q \Big\}.$$

1.2.1 Ehresmann connection and horizontal lifts

In this subsection we want to introduce principal bundles, which allows us to relate geodesics of Riemmanian geometry to sub-Riemannian geodesics. For that purpose we first define submersions.

Definition 1.2.9. *Let Q and M be two smooth manifolds. Then a smooth map $\pi \colon Q \to M$ is called a submersion if the differential $d_q\pi \colon T_qQ \to T_{\pi(q)}M$ is a surjective map at any point $q \in Q$.*

Suppose two differentiable manifolds Q, M, and the submersion $\pi \colon Q \to M$ are given. The fibre through $q \in Q$ is the set $Q_m := \pi^{-1}(m)$, $m = \pi(q)$, which is a submanifold according to the implicit function theorem. The differential $d_q\pi \colon T_qQ \to T_{\pi(q)}M$ of π defines the vertical space $\mathcal{V}_q \subset T_qQ$ which is the tangent space to the fibre $Q_{\pi(q)}$ and it is written as $\mathcal{V}_q := \ker(d_q\pi) = T_q(Q_m)$, where $\ker(d_q\pi)$ denotes the kernel of the linear map $d_q\pi$. It can be shown that $\mathcal{V} = \bigcup_{q\in Q} \mathcal{V}_q$ is a smooth subbundle of TQ which is called vertical subbundle [71].

Definition 1.2.10. *An Ehresmann connection (or connection) for a submersion $\pi \colon Q \to M$ is a subbundle $\mathcal{H} \subset TQ$ that is everywhere transverse and of complementary dimension to the vertical: $\mathcal{V}_q \oplus \mathcal{H}_q = T_qQ$. The space $\mathcal{H}_q$ is called the horizontal subspace of T_qQ.*

We describe now the model of a sub-Riemannian manifold that is used in forthcoming sections. Let $\pi \colon Q \to M$ be a submersion of a Riemannian manifold (Q, g) onto a manifold M and $\mathcal{V}_q$ a vertical space at some point $q \in Q$. We define $\mathcal{H}_q$ to be the orthogonal complement to $\mathcal{V}_q$ with respect to the Riemannian metric g. Then, the subbundle $\mathcal{H}$ is clearly the Ehresmann connection. If $\langle\cdot\,,\cdot\rangle$ denote the restriction of the metric g on the subbundle $\mathcal{H}$, then the triplet $(Q, \mathcal{H}, \langle\cdot\,,\cdot\rangle)$ is a sub-Riemannian manifold. In Chapter 8 the manifold Q will be the Stiefel manifold, M will be the Grassmann manifold and the metric will be induced by the trace metric from the groups $U(n)$ or $SO(n)$.

The induced sub-Riemannian structure has certain related properties to its underlying Riemannian structure. To state these properties we require one more definition.

Definition 1.2.11. *Let $\pi\colon Q \to M$ be a submersion with connection $\mathcal{H}$ and let $c\colon I \to M$ be a curve starting at $m \in M$. A curve $\gamma\colon I \to Q$ is called a horizontal lift of the curve c if γ is tangent to $\mathcal{H}$ and projects to c, i.e. $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$ and $\pi \circ \gamma(t) = c(t)$ for all $t \in I$.*

If a horizontal lift of c starting at a given point $q \in Q_m$ exists, the horizontal lift is unique.

Proposition 1.2.12. *The induced sub-Riemannian structure fulfills the following properties:*

- *The sub-Riemannian length of a horizontal path in Q equals the Riemannian length of its projection to M.*
- *The horizontal lift of a Riemannian geodesic in M is a sub-Riemannian geodesic in Q. If the Riemannian geodesic minimizes between its endpoints, then its horizontal lift minimizes between the corresponding fibers.*
- *The projection π is distance decreasing, i.e. for any $q_1, q_2 \in Q$*

$$d_Q(q_1\,, q_2) \geq d_M(\pi(q_1)\,, \pi(q_2)).$$

1.2.2 Metrics on Principal bundles

We are now specializing to the case of principal G-bundles. To relate Riemannian to sub-Riemannian geodesics we are interested in metrics for which the group G acts by isometries.

Definition 1.2.13. *A fibre bundle $\pi\colon Q \to M$ is a principal G-bundle if its fibre $F \subset Q$ is a Lie group G that acts freely and transitively on each fibre F, i. e.*

- *if $g \in G$ and there exists an $x \in F$ with $gx = x$, then g is the identity element,*
- *if for any $x, y \in F$ there exists a $g \in G$ such that $gx = y$.*

We assume that the group G acts on F on the right $q \mapsto qg$, $q \in F \subset Q$, $g \in G$. As a consequence of free and transitive action we can identify M with the quotient Q/G of Q by the group action of G. Furthermore, π corresponds to the canonical projection of Q to the quotient set Q/G.

Definition 1.2.14. *A connection on $\pi\colon Q \to M$ is a principal G-bundle connection if the action of G preserves the connection.*

Definition 1.2.15. *Let $\pi\colon Q \to M$ be a principal G-bundle with a connection $\mathcal{H}$. A sub-Riemannian metric on $(Q, \mathcal{H})$, which is invariant under the action of G, is called a metric of bundle type.*

A sub-Riemannian metric which is induced from a G-invariant metric on Q is an example of a metric of bundle type.

Definition 1.2.16. *A bi-invariant Riemannian metric $\langle \cdot\,,\cdot\rangle$ on a differentiable manifold Q with the Lie group G acting on it is said to be of constant bi-invariant type if its inertia tensor $\mathbb{I}_q\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ defined by $\mathbb{I}_q(\xi,\eta) := \langle \sigma_q\xi\,, \sigma_q\eta\rangle$ is independent of $q \in Q$. Here*

$$\begin{array}{rcl} \sigma_q\colon \mathfrak{g} & \to & T_qQ, \\ \xi & \mapsto & \left.\dfrac{d}{d\epsilon}\right|_{\epsilon=0} q\exp(\epsilon\xi), \end{array}$$

and $\mathfrak{g}$ is the Lie algebra of the Lie group G.

Definition 1.2.17. *Let $\pi\colon Q \to M$ be a principal G-bundle with a Riemannian metric of constant bi-invariant type and $\mathcal{H}$ be the induced connection. We define the $\mathfrak{g}$-valued connection one-form A_q uniquely as the linear operator $A_q\colon T_qQ \to \mathfrak{g}$ which satisfies the following properties:*

$$\ker(A_q) = \mathcal{H}_q, \qquad A_q \circ \sigma_q = \mathrm{Id}_{\mathfrak{g}},$$

where $\mathrm{Id}_{\mathfrak{g}}$ is the identity map on $\mathfrak{g}$.

The map $A\colon TQ \to \mathfrak{g}$ defines a $\mathfrak{g}$-valued connection one-form on Q.

Theorem 1.2.18. *[71] Let $\pi\colon Q \to M$ be a principal G-bundle with a Riemannian metric of constant bi-invariant type. Let $\mathcal{H}$ be the induced connection, with $\mathfrak{g}$-valued connection one-form A. Let $\exp_R$ be the Riemannian exponential map, so that $\gamma_R(t) = \exp_R(tv)$ is the Riemannian geodesic through q with initial velocity $v \in T_qQ$. Then any horizontal lift γ of the projection $\pi\circ\gamma_R$ is a normal sub-Riemannian geodesic and is given by*

$$\gamma_v(t) = \exp_R(tv)\exp_G(-tA(v)), \tag{1.9}$$

where $\exp_G\colon \mathfrak{g} \to G$ is the group G exponential map. Moreover, all normal sub-Riemannian geodesics can be obtained in this way.

Remark 1.2.19. *We emphasize that the constant vector $v \in T_qQ$ is not the initial vector of the sub-Riemannian geodesic $\gamma(t)$, this is the initial vector of the Riemannian geodesic $\exp_R(tv)$, which is not necessarily horizontal. Note that $v \in T_qQ$ can be decomposed into the horizontal component and the vertical one. The horizontal component is the initial vector of the sub-Riemannian geodesic γ. The image $A(v)$ of the vertical component in $\mathfrak{g}$ gives rise to the one parametric subgroup $\exp_G(-tA(v)) \subset G$ that "corrects" the Riemannian geodesic $\exp_R(tv)$ to the sub-Riemannian geodesic γ. More details concerning*

Theorem 1.2.18, exponential map for sub-Riemannian manifolds and normal geodesics can be found in [71, Chapter 11]. We continue to call the vector v the "initial vector", since it is one of the initial data to create the normal sub-Riemannian geodesic γ of the form (1.9)*, even if it does not uniquely define the sub-Riemannian geodesic.*

1.2.3 The sub-Riemannian Hamiltonian

Let Q be an n-dimensional smooth manifold and $\mathcal{H}$ be a smooth horizontal subbundle such that $\dim \mathcal{H}_q = k \leq n$ for all $q \in Q$. Considering a neighborhood $\mathcal{O}_q$ around $q \in Q$ such that the subbundle $\mathcal{H}$ is trivialized in $\mathcal{O}_q$, one can find a local orthonormal basis $X_1, \ldots, X_k$ with respect to the sub-Riemannian metric $\langle \cdot \,, \cdot \rangle$. The associated sub-Riemannian metric Hamiltonian is given by

$$H(p, \lambda) = \frac{1}{2} \sum_{m=1}^{k} \lambda(X_m(p))^2,$$

where $(p, \lambda) \in T^*\mathcal{O}_q$, with $T^*\mathcal{O}_q$ being the dual space of $T\mathcal{O}_q$. A *normal geodesic* is defined as the projection to $\mathcal{O}_q \subset Q$ of the solution to the Hamiltonian system

$$\dot{p}_i = \frac{\partial H}{\partial \lambda_i}, \qquad \dot{\lambda}_i = -\frac{\partial H}{\partial p_i},$$

where (p_i, λ_i) are the coordinates in $T^*\mathcal{O}_q$. We note that the word "normal" appears due to the fact that in sub-Riemannian geometry exists another type of geodesics, called "abnormal" arising from a different type of Hamiltonian functions. For a more detailed study of abnormal geodesics we refer to [5, 21, 22, 23, 35, 36, 54, 65]. In this thesis abnormal geodesics appear in the discussion of the cut locus of $V_{n,k}$ with $k > 1$ and even there just play a minor role.

1.2.4 Bracket generating vs. strongly bracket generating

In this subsection we want to introduce the reader to a stronger concept of a bracket generating distribution. One of the main differences between these two concepts is that strongly bracket generating distributions imply the absence of strictly abnormal minimizers.

Definition 1.2.20. *A horizontal subbundle $\mathcal{H}$ on a manifold is strongly bracket generating if for any non-zero section $\mathcal{Z}$ of $\mathcal{H}$, the tangent bundle of the manifold is generated by $\mathcal{H}$ and $[\mathcal{Z}\,, \mathcal{H}]$.*

We note that strongly bracket generating implies bracket generating, nevertheless the following statement is just true for strongly bracket generating distributions.

Proposition 1.2.21. *On a strongly bracket generating horizontal subbundle $\mathcal{H}$ all geodesics are normal.*

The present work is mostly concerned with normal geodesics. The absence of abnormal ones in the most of the objects studied in this thesis follows from Proposition 1.2.21. Furthermore, there exist certain necessary conditions for strongly bracket generating distributions.

Proposition 1.2.22. *Let Q be an m-dimensional manifold and $\mathcal{H}$ an l-dimensional strongly bracket generating distribution of co-dimension 2 or greater. Then the following conditions*

$$(1)\ \ l \quad \textit{is a multiple of} \quad 4, \qquad (2)\ \ l \geq (m-l)+1.$$

have to be fulfilled.

The contact manifolds are one of the most important examples for manifolds with a strongly bracket generating distribution.

Definition 1.2.23. *A sub-Riemannian manifold $(Q, \mathcal{H}, g_{\mathcal{H}})$ is a contact manifold if there exists a single one-form θ such that*

- $\operatorname{kern}(\theta_q) = \mathcal{H}_q$ *for all $q \in Q$,*
- *for all $X \in \mathcal{H}$: If $d\theta|(X\,,v) = 0$ for all $v \in \mathcal{H}$, then $X = 0$.*

The main reason we state contact manifolds is their appearance in Section 8.3 and their following property.

Proposition 1.2.24. *If $(Q, \mathcal{H}, g_{\mathcal{H}})$ is a contact manifold, then is the horizontal distribution $\mathcal{H}$ strongly bracket generating.*

Chapter 2

Main results

In this chapter we summarize the main results of the research Chapters 4 - 8. Each of them, except Chapter 5, represents one project, which is summarized in a submitted or published paper. Details can be found in the chapters itself.

2.1 Classification of pseudo H-type algebras

We mainly focus on the classification of pseudo H-type algebras $\mathfrak{n}_{r,s}$ given as the direct sum of a minimal admissible Clifford module $\mathfrak{v}_{r,s}$ and the generator space $\mathfrak{z}_{r,s}$ of the Clifford algebra $\mathrm{Cl}_{r,s}$. For details about the construction of $\mathfrak{n}_{r,s}$ see Subsection 1.1.5. In the following we discover that even we are just interested in the classification of Lie algebras, the metric structure of the center $\mathfrak{z}_{r,s}$ plays a critical role. In detail, a pseudo H-type algebra $\mathfrak{n}_{r,s}$ can only be isomorphic to $\mathfrak{n}_{s,r}$, i.e. this condition is based on the signature of the scalar product of the center.

As a consequence of the proof of the necessary condition we obtain that if two different pseudo H-type algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$ are isomorphic, then there exists an isomorphism $f\colon \mathfrak{n}_{r,s} \to \mathfrak{n}_{s,r}$ such that the restriction $f|_{\mathfrak{z}_{r,s}}\colon \mathfrak{z}_{r,s} \to \mathfrak{z}_{s,r}$ is an anti-isometry, i.e.

$$\langle f|_{\mathfrak{z}_{r,s}}(Z)\,, f|_{\mathfrak{z}_{r,s}}(Z)\rangle_{\mathfrak{z}_{s,r}} = -\langle Z\,, Z\rangle_{\mathfrak{z}_{r,s}}, \qquad \text{for all } Z \in \mathfrak{z}_{r,s}\,.$$

Based on this condition for the isomorphism we classify all pseudo H-type algebras of the form $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$.

Theorem. *The pseudo H-type algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ are isomorphic if and only if $r \bmod (8) \in \{0,1,2,4\}$.*

The proof for the non-isomorphisms follows directly from different dimensions of the minimal admissible Clifford modules $\mathfrak{v}_{r,0}$ and $\mathfrak{v}_{0,r}$ for $r = 3,5,6,7$ and the fact that $\dim(\mathfrak{v}_{p,0}) = 16^k \dim(\mathfrak{v}_{r,0})$ for $p = 8k + r$ for $r = 3,5,6,7$.

To prove the isomorphisms we construct the isomorphisms for the cases $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ for $r = 1,2,4,8$. Then we use a relation between the structural constants of $\mathfrak{n}_{r+8,0}$ and the structural constants of $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{8,0}$ based on the Bott-periodicity of Clifford algebras.

This relation is a cornerstone of the classification. It is based on a construction of integral bases for pseudo H-type algebras with higher dimensional centers ($\dim(\mathfrak{z}_{r,s}) > 8$) given in [46].

The classification of the pseudo H-type algebras $\mathfrak{n}_{r,s}$ with $r, s \neq 0$ is significantly more complicated. Nevertheless, we present a partial classification of them. The status quo is that there are isomorphic and non-isomorphic pseudo H type algebras $\mathfrak{n}_{r,s}$ of equal dimension. Both are presented in Chapter 4. A summary of the known isomorphic pseudo H-type algebras is given in the following theorem.

Theorem. *The H-type algebras $\mathfrak{n}_{r+8t_1+4t_2,8t_3+4t_2}$ and $\mathfrak{n}_{8t_3+4t_2,r+8t_1+4t_2}$ are integral isomorphic for $r \in \{0, 1, 2, 4\}$ and $t_1, t_2, t_3 \in \mathbb{N}_0$.*

The H-type algebras $\mathfrak{n}_{r+8t_1+4t_2,r+8t_3+4t_2}$ and $\mathfrak{n}_{r+8t_3+4t_2,r+8t_1+4t_2}$ are integral isomorphic for $r \in \{0, 1, 2\}$ and $t_1, t_2, t_3 \in \mathbb{N}_0$.

The proof is based on a decomposition of the module space $\mathfrak{v}_{r,s}$. We decompose $\mathfrak{v}_{r,s}$ into two disjoint maximal abelian metrically neutral algebras. In particular, we choose a basis of each of the two disjoint maximal abelian metrically neutral algebras and call the union of them a block type basis of $\mathfrak{v}_{r,s}$. If there exists an isomorphism between $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$, then the decomposition of the block type basis is preserved.

The example of two pseudo H-type algebras with equal dimension, which are not isomorphic, is given by the two pseudo H-type algebras $\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{2,3}$. These algebras have the seldom property that the adjoint operator $\mathrm{ad}_X\colon \mathfrak{v}_{r,s} \to \mathfrak{z}_{r,s}$ is surjective if and only if $\langle X , X \rangle_{\mathfrak{v}_{r,s}} \neq 0$ for all $(r, s) \in \{(3, 2), (2, 3)\}$.

Furthermore, we stress that the results of the previous paragraph represent a quite exceptional case. We note that the definition of the general H-type algebras is often misinterpreted and does not imply that ad_X is not surjective for any $X \in \mathfrak{v}_{r,s}$ with $\langle X , X \rangle_{\mathfrak{v}_{r,s}} = 0$. In Chapter 4, we state a couple of lemmas illustrating that in the majority of the cases there exist $X \in \mathfrak{v}_{r,s}$ with $\langle X , X \rangle_{\mathfrak{v}_{r,s}} = 0$ such that ad_X is not surjective.

As a final remark to the classification we would like to mention that the status quo is, that it is not clear how to use Bott-periodicity for non-isomorphic pseudo H-type algebras of equal dimension.

In the end of the chapter we are studying the strongly bracket generating property of the pseudo H-type algebras $\mathfrak{n}_{r,s}$.

Definition. *Let $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$ be a pseudo H-type algebra. We call a vector space $\mathfrak{v}_{r,s}$ strongly bracket generating if for any non-zero $v \in \mathfrak{v}_{r,s}$ the linear map $\mathrm{ad}_v = [v , \cdot]\colon \mathfrak{v}_{r,s} \to \mathfrak{z}_{r,s}$ is surjective, i.e. $\mathrm{span}\{\mathfrak{v}_{r,s}, [v , \mathfrak{v}_{r,s}]\} = \mathfrak{n}_{r,s}$ for all $v \in \mathfrak{v}_{r,s} \setminus\{0\}$. We say in this case that the pseudo H-type algebra $\mathfrak{n}_{r,s}$ has the strongly bracket generating property.*

Let $N_{r,s}$ be the Lie group, corresponding to the pseudo H-type algebra $\mathfrak{n}_{r,s}$ and let $\mathcal{H}$ be the left translation of the vector space $\mathfrak{v}_{r,s}$. If $\mathfrak{v}_{r,s}$ is strongly bracket generating, then the left invariant distribution $\mathcal{H}$ is strongly bracket generating in a sense that $\mathrm{span}\{\mathcal{H}, [X , \mathcal{H}]\} = TN_{r,s}$ for any smooth non-zero section X of the distribution $\mathcal{H}$, see Definiton 1.2.20. We obtain the following theorem.

Theorem. *The pseudo H-type algebras $\mathfrak{n}_{r,s}$ do not have the strongly bracket generating property if and only if $r, s \neq 0$.*

In the beginning of our classification we restricted ourselves to the pseudo H-type algebras constructed by minimal admissible modules. If we release this constraint and allow non-minimal admissible Clifford modules, then we have to deal with new challenges and obtain more cases to consider in our classification. We note that any non-minimal admissible module $\mathfrak{v}$ is equivalent to the direct sum of minimal admissible modules. If $r - s \neq 3(\mod (4))$, then there exists only one minimal admissible module up to equivalence, but if $r - s = 3(\mod (4))$, then there exist two minimal admissible modules, which are non-equivalent. This leads to the question: are pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu_1, \mu_2)$ constructed by non-equivalent modules, but with identical center isomorphic to each other or not. This question is answered.

Theorem. *Pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu_1, 0)$ and $\mathfrak{n}_{r,s}(\mu_2, 0)$ for $r - s \neq 3(\mod 4)$ are isomorphic if and only if $\mu_1 = \mu_2$.*

Two pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu_1, \nu_1)$ and $\mathfrak{n}_{r,s}(\mu_2, \nu_2)$ for $r - s = 3(\mod 4)$ are isomorphic if and only if $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$ or $\mu_1 = \nu_2$ and $\nu_1 = \mu_2$.

2.2 Pseudo-metric 2-step nilpotent Lie algebras

Although, as we have mentioned before in subsection 1.1.2, every 2-step nilpotent Lie algebra is isomorphic to a standard metric Lie algebra in the sense of Eberlein [42], it is of interest to generalize or specify these isomorphisms under certain conditions. The identification of Eberlein is interesting on its own, but looses certain interesting properties when it comes to pseudo H-type algebras. Therefore, we developed a theory under which a 2-step nilpotent Lie algebra can be isomorphically represented by an indefinite standard metric Lie algebra $G = \mathbb{R}^{p,q} \oplus_\perp \mathcal{D}$ with $\mathcal{D} \subset \mathfrak{so}(p, q)$. The general theorem is formulated as follows.

Theorem. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that $\dim([\mathfrak{g}\,, \mathfrak{g}]_\mathfrak{g}) = n$ and the complement V to $[\mathfrak{g}\,, \mathfrak{g}]_\mathfrak{g}$ has dimension m. Then there exists an n-dimensional subspace $\mathcal{D}$ of $\mathfrak{so}(p, q)$, $p + q = m$, $n \leq \frac{m(m-1)}{2}$, such that if $\mathcal{D}$ is a non-degenerate subspace of $\mathfrak{so}(p, q)$, then $\mathfrak{g}$ is isomorphic as a Lie algebra to the standard pseudo-metric 2-step nilpotent Lie algebra $\mathcal{G} = \mathbb{R}^{p,q} \oplus_\perp \mathcal{D}$.*

Further we turn our attention to the freely generated 2-step nilpotent Lie algebra $F_2(p, q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p, q)$. Consider $\mathbb{R}^{p,q}$, $p+q = m$ with the indefinite metric $\langle x\,, y\rangle_{p,q} = x^{\mathbf{t}}\eta_{p,q}y$. We construct the Lie algebra $F_2(p, q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p, q)$, where the commutator on $\mathbb{R}^{p,q}$ is defined by

$$[w\,, v]_{F_2(p,q)} = -\frac{1}{2}(wv^{\mathbf{t}} - vw^{\mathbf{t}})\eta_{p,q}.$$

For the standard basis $\{e_i\}$ of $\mathbb{R}^{p,q}$ we get $[e_i\,, e_j]_{F_2(p,q)} = -\frac{1}{2}(E_{ij} - E_{ji})\eta_{p,q}$, where E_{ij} denote the $(m \times m)$ matrix with zero entries except of 1 at the position ij. Since $F_2(p, q)$

is a 2-step nilpotent Lie algebra, we obtain that $\mathfrak{so}(p,q)$ forms the center. Particularly, if $q = 0$ we get the free Lie algebra $F_2(m)$ studied in [42]. We show that all 2-step nilpotent free algebras $F_2(p,q)$ with $p+q=m$ are isomorphic.

This result together with the previously presented theorem allows to give a more detailed understanding of isomorphic Lie algebras of step two represented by indefinite standard metric Lie algebras.

Proposition. *Let d be an integer with $1 \le d \le \dim(\mathfrak{so}(p,q))$. Let $W_1, W_2 \subset \mathfrak{so}(p,q)$ be two d-dimensional non-degenerate subspaces with respect to $\langle\cdot\,,\cdot\rangle_{\mathfrak{so}(p,q)}$. Then the following statements are equivalent:*

1) *The Lie algebra $F_2(p,q)/W_1$ is isomorphic to $F_2(p,q)/W_2$;*

2) *There exists an element $A \in \mathrm{GL}(m)$, $m = p+q$ such that $AW_1A^{\eta_{p,q}} = W_2$;*

3) *The Lie algebra $F_2(p,q)/W_1^{\perp}$ is isomorphic to $F_2(p,q)/W_2^{\perp}$.*

4) *The Lie algebra $\mathfrak{g}_1 = \mathbb{R}^{p,q} \oplus W_1$ is isomorphic to $\mathfrak{g}_2 = \mathbb{R}^{p,q} \oplus W_2$*

In the following we focus on the relation between Lie triple systems in $\mathfrak{so}(p,q)$ and pseudo H-type algebras.

Definition. *Let W be a subspace of $\mathfrak{so}(p,q)$ such that $[a\,,[b\,,c]] \in W$ for all $a,b,c \in W$. The subspace W is called a Lie triple system in $\mathfrak{so}(p,q)$.*

The Lie triple systems associated with a representation of Clifford algebras form a subalgebra of $\mathfrak{so}(l,l)$ respectively $\mathfrak{so}(2l)$, which are reductive. Let $\mathrm{Cl}_{r,s}$ be a Clifford algebra generated by $\mathbb{R}^{r,s}$ and $J\colon \mathbb{R}^{r,s} \to \mathfrak{so}(l,l) \subset \mathrm{End}(\mathbb{R}^{l,l})$ for $s \neq 0$ and $J\colon \mathbb{R}^{r,s} \to \mathfrak{so}(2l) \subset \mathrm{End}(\mathbb{R}^{2l,0})$ for $s = 0$ be its representation. Denote $W = J(\mathbb{R}^{r,s}) \subset \mathfrak{so}(l,l)$ respectively $W = J(\mathbb{R}^{r,0}) \subset \mathfrak{so}(2l)$ and $\mathcal{L} = W + [W\,,W]$. Then we obtain that the Lie algebra $\mathcal{L}$ is simple, hence $\mathcal{L}$ is reductive. Now we can relate Lie triple systems of $\mathfrak{so}(p,q)$ with rational subspaces, which are defined as follows.

Definition. *Let $\mathfrak{g}$ be a Lie algebra such that with respect to a basis $\mathcal{B}_{\mathfrak{g}}$ the Lie algebra $\mathfrak{g}$ has rational structure constants. Then the set $\mathrm{span}_{\mathbb{Q}}\{\mathcal{B}_{\mathfrak{g}}\}$ is called the rational structure of the Lie algebra $\mathfrak{g}$. A subspace U of $\mathfrak{g}$ is called rational subspace with respect to the rational structure $\mathrm{span}_{\mathbb{Q}}\{\mathcal{B}_{\mathfrak{g}}\}$ if there is a basis B_U such that $B_U \subset \mathrm{span}_{\mathbb{Q}}\{\mathcal{B}_{\mathfrak{g}}\}$.*

The final results of Chapter 5 are obtained by the use of the recent results of Kammeyer [58], who constructed an explicit basis $C_{\mathcal{L}}$ with integer structure constants for any real semisimple Lie algebra $\mathcal{L}$. This basis $C_{\mathcal{L}}$ is a real Chevalley basis. We obtain:

- If W is a Lie triple system of $\mathfrak{so}(p,q)$, and $\mathcal{L} = W \oplus [W\,,W] = [\mathcal{L}\,,\mathcal{L}]$ is semisimple, then W is a rational subspace of $\mathcal{L}$ with respect to the rational structure $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$.

- The Lie triple system W of $\mathfrak{so}(p,q)$ is a rational subspace of $\mathcal{L} = W + [W\,,W] = [\mathcal{L}\,,\mathcal{L}]$ with respect to the rational structure $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_1}\} \oplus \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}$, where $\mathcal{L}_1 = W \cap [W\,,W] \neq 0$ and $\mathcal{L}_2 := \mathcal{L}_1^{\perp}$ with respect to any ad-invariant inner product $(\cdot\,,\cdot)_{\mathcal{L}}$ on $\mathcal{L}$.

2.3 The sub-Riemannian geodesic equation in the octonionic H-type group

The octonionic Lie algebra $\mathfrak{g}$ is the algebra, which is spanned by the vector fields $X_1, \dots, X_8, Z_1, \dots, Z_7$ with a given commutator of vector fields in $\mathbb{R}^{15}$, see Table 6.1. The octonionic H-type group G is the nilpotent Lie group structure on $\mathbb{R}^{15}$ of step 2 induced by the Lie algebra $\mathfrak{g}$ via the Baker-Campbell-Hausdorff formula. An explicit expression for the product rule can be found in [28, Equation (3.7)].
We define an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}$ such that the vector fields $X_1, \dots, X_8$, $Z_1, \dots, Z_7$ form an orthonormal frame. The left-invariant distribution

$$\mathcal{H} := \operatorname{span}\{X_1, \dots, X_8\},$$

and the restriction of $\langle \cdot\,, \cdot \rangle$ to $\mathcal{H}$ give us the sub-Riemannian structure on G we study further. We define the almost complex structures $J_r \colon \mathcal{H} \to \mathcal{H}$, $r \in \{1, \dots, 7\}$, on $\mathcal{H}$ by

$$J_r(X) := 2\nabla_X Z_r, \quad r \in \{1, \dots, 7\},$$

for any section X of $\mathcal{H}$. The class of curves we are interested in are horizontal with respect to $\mathcal{H}$ and, most importantly, critical points of the natural sub-Riemannian length functional. We present a characterization of these critical points via a differential equation, similar to the geodesic equation in Riemannian geometry, which states that for critical points of the length functional the intrinsic acceleration $\nabla_{\dot\gamma}\dot\gamma$ is a linear combination with constant coefficients of some special rotations of the velocity $\dot\gamma$. This result is summarized in the following main theorem.

Theorem. *Let $\gamma\colon [a,b] \to G$ be a horizontal curve of class C^2, parametrized by arc length. Then γ is a critical point of the length functional (with respect to admissible variations) if, and only if, there exist constants $\lambda_1, \dots, \lambda_7 \in \mathbb{R}$ such that γ satisfies the second order differential equation*

$$\nabla_{\dot\gamma}\dot\gamma - 2\sum_{r=1}^{7} \lambda_r J_r(\dot\gamma) = 0.$$

This theorem can be seen as an extension of the techniques and results obtained by many authors, among others [14, 50, 71, 81, 82].

2.4 The sub-Riemannian cut locus of H-type groups

The H-type group $N_{r,0}$ is the unique (up to isomorphism) connected and simply connected Lie group with Lie algebra $\mathfrak{n}_{r,0} = \mathfrak{v}_{r,0} \oplus \mathfrak{z}_{r,0}$, where the Lie algebra structure is defined by the corresponding Clifford representation $J\colon\ \mathfrak{z}_{r,0} \to \operatorname{End}(\mathfrak{v}_{r,0})$ and the relation

$$\langle J_Z v\,, w\rangle_{\mathfrak{v}_{r,0}} = \langle Z\,, [v\,, w]\rangle_{\mathfrak{z}_{r,0}}, \qquad \text{for all } v, w \in \mathfrak{v}_{r,0}\,.$$

Observe that in this context we are not requiring that the admissible Clifford module $\mathfrak{v}_{r,0}$ is of minimal dimension, cf. Section 1.1.5. The subspace $\mathfrak{v}_{r,0}$ defines a strongly bracket generating distribution of step 2 over $N_{r,0}$ by left-translation, and the translations of the inner product $\langle \cdot , \cdot \rangle_{\mathfrak{v}_{r,0}}$ makes $N_{r,0}$ into a sub-Riemannian manifold.

Recall that for a simply connected nilpotent Lie group, the exponential map is a diffeomorphism, see for example [40], allowing to identify $N_{r,0}$ with $\mathfrak{v}_{r,0} \oplus \mathfrak{z}_{r,0}$. Under this identification, we denote by $V_{r,0}$ and $Z_{r,0}$ the image of $\mathfrak{v}_{r,0}$ and $\mathfrak{z}_{r,0}$ respectively and identify any element $p \in N_{r,0}$ uniquely with (x_p, z_p) with $x_p \in V_{r,0}$ and $z_p \in Z_{r,0}$.

It is possible to show that the normal sub-Riemannian geodesics starting from $(0,0) \in N_{r,0}$ can be written, in general, in the following form

$$\begin{aligned} x(t) &= \frac{\sin(t|\theta|)}{|\theta|}\,\dot{x}(0) + \frac{(1-\cos(t|\theta|))}{|\theta|^2}\Omega\dot{x}(0), \\ z(t) &= \frac{|\dot{x}(0)|^2}{2|\theta|^2}\left(t - \frac{\sin(t|\theta|)}{|\theta|}\right)\theta, \end{aligned}$$

which is based on results given in [51].

Furthermore, we can specify how many of these geodesics reach a given point $(x,z) \in N_{r,0}$ in time $t = 1$, by means of the following theorem.

Theorem. *We distinguish the following three cases:*

- *Given a point $(0,z) \in N_{r,0}$ with $z \neq 0$, there are infinitely many sub-Riemannian geodesics joining the origin $(0,0)$ with $(0,z)$.*
- *Given a point $(x,z) \in N_{r,0}$ with $x \neq 0$, $z \neq 0$, there are finitely many sub-Riemannian geodesics joining the origin $(0,0)$ with (x,z).*
- *Given a point $(x,0) \in N_{r,0}$ with $x \neq 0$, there is a unique sub-Riemannian geodesic joining the origin $(0,0)$ with $(x,0)$.*

By carefully sharpening the results of the previous theorem to the case of minimizing geodesics, we can deduce the main result of Chapter 7.

Theorem. *The cut locus $K_{(0,0)}$ of the H-type group $N_{r,0}$ is given by the points of the form $(0,z)$.*

2.5 The sub-Riemannian geometry of Stiefel manifolds

We consider the Stiefel manifold $V_{n,k}$, which is the set of all k-tuples $(q_1, \dots, q_k)$ of vectors $q_i \in \mathbb{C}^n$, $i \in \{1, \dots, k\}$, which are orthonormal with respect to the standard Hermitian metric. This is a compact real analytic manifold which can be equivalently defined as

$$V_{n,k} := \{X \in \mathbb{C}^{n\times k} \mid \ \bar{X}^T X = I_k\}.$$

We can also realize the Stiefel manifold as a quotient space of $U(n)$ by the closed subgroup $U(n-k)$, which is a more convenient representation for our purposes. For that we define the equivalence class

$$[q]_{V_{n,k}} = \left\{ q \begin{pmatrix} I_k & 0 \\ 0 & U_{n-k} \end{pmatrix}, \Big| U_{n-k} \in U(n-k) \right\} \in U(n)/U(n-k),$$

for $q \in U(n)$ and note that we can identify it with a point in the Stiefel manifold. So, practically, an element of $V_{n,k}$ is thought of as an element in $U(n)$ whose first k columns from the left are of interest and the last $n-k$ columns are not.

The sub-Riemannian structure on $V_{n,k}$, which will be relevant in Chapter 8, is induced by the Grassmann manifolds. The Grassmann manifold $G_{n,k}$ is defined as the collection of all k-dimensional subspaces Λ of $\mathbb{C}^n$. We are interested in the representation of $G_{n,k}$ as a quotient of $U(n)$ by some closed subgroup. As in the case of the Stiefel manifolds, we factor $U(n)$ by $U(n-k)$, but moreover, we also factor $U(n)$ by $U(k)$. Therefore, we can identify any point in the Grassmann manifold with a unique equivalence class of the form

$$[m]_{G_{n,k}} = \left\{ m \begin{pmatrix} U_k & 0 \\ 0 & U_{n-k} \end{pmatrix} \Big| \ U_k \in U(k), U_{n-k} \in U(n-k) \right\} \subset U(n), \quad m \in U(n).$$

We define a sub-Riemannian structure on the Stiefel manifold $V_{n,k}$ over the Grassmann manifold $G_{n,k}$ by means of the natural submersion. A precise formula for the normal sub-Riemannian geodesics in that case are given by Theorem 1.2.18.

We have to be aware of the fact that we do not have any formula for abnormal geodesics, so that we mainly consider cases in which the sub-Riemannian structure is strongly bracket generating, which implies the absence of strictly abnormal minimizers.

We note that we write Id for the equivalence class $[I_n]_{V_{n,k}} \in V_{n,k}$. The main theorem is stated as follows.

Theorem. *The cut locus K_{Id} on $V_{n,1}$ is given by*

$$L_{n,1} := \left\{ \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right]_{V_{n,1}} \Big| \ C \in U(1), \ D \in U(n-1) \right\} \setminus \{\mathrm{Id}\}.$$

In particular, there are infinitely many minimizing geodesics connecting Id *with any point $q \in L_{n,1}$.*

In the general case $V_{n,k}$ we cannot rule out the presence of abnormal minimizers. Nevertheless, we are able to describe the cut locus partially.

Theorem. *The set*

$$L_{n,k} = \left\{ \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right]_{V_{n,k}} \Big| \ C \in U(k), \ D \in U(n-k)\} \setminus \mathrm{Id} \right\}$$

belongs to the cut locus K_{Id} on $V_{n,k}$. In particular, there are infinitely many minimizing geodesics connecting Id *with any point $q \in L_{n,k}$.*

The cardinality of minimizing geodesics is of particular interest in the study of small time heat kernel asymptotics [12].

In the particular case $V_{2k,k}$ it is possible to specify the previous result by studying the Riemannian geodesics in the Grassmann manifold.

Lemma. *The points* $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}} \in G_{2k,k}$ *are reached by Riemannian geodesics starting from* $[I_n]_{G_{2k,k}}$ *only if the initial velocity vector* v *has the form* $v = \begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}$*,* $B \in U(k)$*. If we assume that* $\operatorname{tr}(B\bar{B}^T) = 1$ *the condition* $B \in U(k)$ *is changed to* $\sqrt{k}B \in U(k)$*.*

The horizontal lift of a Riemannian geodesic which minimizes between its endpoints minimizes between its corresponding fibers. This leads to the following sharpened result.

Theorem. *For any point* $s = \left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ *with* $C, D \in U(k)$ *there is a unique minimizing geodesic connecting* Id *with* s*.*

All the mentioned results in this section can actually be stated analogously if we consider real Stiefel and Grassmann manifolds. We would like to emphasize that we do not input different sub-Riemannian structures on the same Stiefel manifold here. We consider different Stiefel manifolds. One of them arise from $U(n)$ factorized by a subgroup of $U(n)$ and others from $SO(n)$, factorized by a subgroup of $SO(n)$. For example all manifolds $V_{n,1}$, related to $U(n)$ possess the CR structure, but $V_{n,1}$ related to $SO(n)$ does not possess any sub-Riemannian structure.

2.6 Appendix

We present a list of works, where the most of the content of this thesis can be found.

- The main results of Chapter 4 are summarized in the submitted paper [8].
- The paper presenting the main results of Chapter 5 is in preparation.
- The main results of Chapter 6 is summarized in the accepted paper [10].
- The main results of Chapter 7 are summarized in the submitted paper [11].
- The main results of Chapter 8 are summarized and published in the paper [9].

Chapter 3

Future research

The purpose of this thesis is not just to illuminate some of the dark spots in the study of H-type algebras and related sub-Riemannian cut locus topics, but also to develop basic principles and theorems for future research. We would like to note that there are many interesting and challenging open problems in the area of H-type algebras and their sub-Riemannian cut locus. In the following we would like to give some ideas for forthcoming research in the areas discussed in this thesis. Some of the suggested topics should be solvable by results and techniques obtained here, whereas others are of general curiosity in the area.

- Answer to the question of whether Bott periodicity applied to non-isomorphic pseudo H-type algebras leads to non-isomorphic algebras.
- Complete classification of pseudo H-type algebras including the cases constructed by non-minimal admissible modules.
- A new (non-constructive) proof of the existence of lattices in pseudo H-type algebras, following the ideas of Eberlein [42].
- Study of geodesic equations for normal geodesics in 3-step nilpotent Lie groups, generalizing the ideas in the manuscript [14].
- Consider the sub-Riemannian cut locus in manifolds admitting strictly abnormal minimizers. We propose to study the Engel groups and Goursat distributions as model examples.
- Determine small-time heat kernel asymptotics of second order at the sub-Riemannian cut locus of the H-type groups $N_{r,0}$.

Part II

Pseudo H-type algebras

Chapter 4

Classification of pseudo H-type algebras

4.1 Introduction

A. Kaplan introduced the Lie groups of Heisenberg type or shortly H-type groups in 1980 [59] and studied them in detail, for instance in [60, 61]. The H-type algebras of the H-type groups, constructed in [59], used the presence of an inner product on the Lie algebra. Later this approach was extended by exploiting an arbitrary indefinite non-degenerate scalar product in [37, 51] and the introduced Lie algebras received the name pseudo H-type algebras. This construction is closely related to the existence of a special scalar product on the representation space of Clifford algebras. Namely the Clifford algebras $\mathrm{Cl}_{r,0}$ generated by a positive definite scalar product space $(\mathbb{R}^{r,0}, \langle \cdot\,,\cdot \rangle_{r,0})$, which lead to the H-type algebras $\mathfrak{n}_{r,0}$ introduced by A. Kaplan and the Clifford algebras $\mathrm{Cl}_{r,s}$ generated by indefinite non-degenerate scalar product spaces $(\mathbb{R}^{r,s}, \langle \cdot\,,\cdot \rangle_{r,s})$ creating pseudo H-type algebras $\mathfrak{n}_{r,s}$.

In the present chapter we study the isomorphism properties of the Lie algebras $\mathfrak{n}_{r,s}$, that were constructed as pseudo H-type algebras. Thus, we neglect the presence of the scalar product $\langle \cdot\,,\cdot \rangle_{r,s}$ on the Lie algebra $\mathfrak{n}_{r,s}$ and study isomorphisms of Lie algebras as themselves. The isomorphism of Lie algebras defines the isomorphism of the corresponding Lie groups. We are mostly concentrated on the minimal admissible modules. We showed that the Lie algebras $\mathfrak{n}_{r,s}$ can not be isomorphic to $\mathfrak{n}_{t,u}$ unless $r = t$ and $s = u$ or $r = u$ and $s = t$. The question of existence of isomorphisms between $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$ is much more complicated. We proved that if $s = 0$, then the Lie algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ are isomorphic if the dimensions of the centers coincide. If $r, s \neq 0$, then we present examples of both cases, isomorphic pairs and non isomorphic pairs, having equal dimensions. Some of the Lie algebras, that we call of block type, allow to use the Bott periodicity of underlying Clifford algebras and obtain more isomorphic pairs, see Theorems 4.4.10 and 4.4.11.

We stress an interesting feature, that there are no direct relations between the isomorphisms of Clifford algebras and the isomorphisms of related Lie algebras. In some cases the isomorphic Clifford algebras lead to isomorphic Lie algebras, in other cases not. For instance, in spite of the isomorphism of the Clifford algebras $\mathrm{Cl}_{8,0}$, $\mathrm{Cl}_{0,8}$, and $\mathrm{Cl}_{4,4}$, the corresponding Lie algebras $\mathfrak{n}_{8,0}$, $\mathfrak{n}_{0,8}$ are isomorphic, but not isomorphic to the Lie algebra $\mathfrak{n}_{4,4}$.

The structure of the chapter is the following. After this Introduction in Section 4.2, we discuss a necessary condition for isomorphisms of pseudo H-type algebras $\mathfrak{n}_{r,s}$, which shows that the only possible algebra which is isomorphic to $\mathfrak{n}_{r,s}$, besides itself, is the pseudo H-type algebra $\mathfrak{n}_{s,r}$. Section 4.3 is devoted to the complete classification of pseudo H-type algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$. Sections 4.4 and 4.5 study two different situations revealing that the Lie algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$ with $r, s \neq 0$ can be both isomorphic and non-isomorphic. In Section 4.6, we exhibit the strongly bracket generating property of the pseudo H-type algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,s}$ and the non-existence of this property for pseudo H-type algebras $\mathfrak{n}_{r,s}$ with both $r, s \neq 0$. Furthermore, we introduce an equivalent definition for pseudo H-type algebras, that was introduced in [51], and explain the equivalence of these definitions in details in subsection 4.6.1. In Section 4.7, we briefly discuss the isomorphism of pseudo H-type algebras related to non-equivalent irreducible Clifford modules. Finally the Appendix in Section 4.8 gives the commutator tables of $\mathfrak{n}_{8,0}$, $\mathfrak{n}_{0,8}$, $\mathfrak{n}_{4,4}$ and a table of permutations for a basis of $\mathfrak{n}_{8,0}$.

4.2 Necessary condition for isomorphisms of pseudo H-type algebras

In the present section we identify the admissible module $\mathfrak{v}_{r,s}$, $s \neq 0$ with $\mathbb{R}^{l,l}$ equipped with the neutral scalar product $\langle x\,, y\rangle_{l,l} = \sum_{i=1}^{l} x_i y_i - \sum_{j=l+1}^{2l} x_j y_j$ for $x, y \in \mathbb{R}^{l,l}$. In the case $\mathfrak{v}_{r,0}$ we use the identification of $\mathfrak{v}_{r,0}$ with $\mathbb{R}^{2l}$ endowed with the inner product $\langle x\,, y\rangle_{2l} = \sum_{i=1}^{2l} x_i y_i$ for $x, y \in \mathbb{R}^{2l}$. Thus a pseudo H-type algebra $\mathfrak{n}_{r,s}$ is isometric to $\mathbb{R}^{l,l} \oplus \mathbb{R}^{r,s}$ respectively $\mathbb{R}^{2l} \oplus \mathbb{R}^{r,s}$. Let $A \in GL(\mathbb{R}^{2l})$. We denote by A^{τ} the adjoint map with respect to the neutral scalar product $\langle \cdot\,, \cdot\rangle_{l,l}$

$$\langle Aw\,, v\rangle_{l,l} = \langle w\,, A^{\tau} v\rangle_{l,l}.$$

We use the same symbol to write the adjoint map $\langle Aw\,, v\rangle_{2l,0} = \langle w\,, A^{\tau} v\rangle_{l,l}$ with respect to scalar products $\langle \cdot\,, \cdot\rangle_{2l,0}$ and $\langle \cdot\,, \cdot\rangle_{l,l}$.

The adjoint map C^{τ} for the map $C\colon \mathbb{R}^{t,u} \to \mathbb{R}^{r,s}$ with $t + u = r + s$ with respect to corresponding scalar products is given by

$$\langle C(Z)\,, \zeta\rangle_{r,s} = \langle Z\,, C^{\tau}(\zeta)\rangle_{t,u}.$$

Assume that two pseudo H-type algebras $\mathfrak{n}_{t,u}$ and $\mathfrak{n}_{r,s}$ are isomorphic and $f\colon \mathfrak{n}_{t,u} \to \mathfrak{n}_{r,s}$ is an isomorphism. Then $t + u = r + s$ and since the center of $\mathfrak{n}_{t,u}$ is mapped to the center of $\mathfrak{n}_{r,s}$, the matrix of the map f takes the form

$$M_f = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \quad A \in GL(\mathbb{R}^{2l}), \quad C \in GL(\mathbb{R}^{r+s}) \tag{4.1}$$

and B being a $\big((r+s)\times 2l\big)$-matrix. Checking the commutation relations, we get $[Aw\,,Av]=C([w\,,v])$ for all $w,v\in\mathbb{R}^{2l}$. With this notation we prove our first classification theorem.

Lemma 4.2.1. *Under the above notation for the isomorphism $f\colon\mathfrak{n}_{t,u}\to\mathfrak{n}_{r,s}$ the matrices A and C in* (4.1) *satisfy*

$$A^{\tau}\circ J_Z\circ A=J_{C^{\tau}(Z)},\qquad \textit{for all}\quad Z\in\mathbb{R}^{r,s}. \tag{4.2}$$

Proof. Formula (4.2) follows for $s\neq 0$ from the calculations

$$\begin{aligned}\langle A^{\tau}\circ J_Z\circ Aw\,,v\rangle_{l,l} &= \langle J_ZAw\,,Av\rangle_{l,l}=\langle Z\,,[Aw\,,Av]\rangle_{r,s}\\ &= \langle Z\,,C([w\,,v])\rangle_{r,s}=\langle C^{\tau}(Z)\,,[w\,,v]\rangle_{t,u}=\langle J_{C^{\tau}(Z)}w\,,v\rangle_{l,l}\end{aligned}$$

for all $Z\in\mathbb{R}^{r,s}$ and for all $w,v\in\mathbb{R}^{2l}$. If $s=0$ we use the same arguments applied for scalar products $\langle\cdot\,,\cdot\rangle_{2l,0}$ and $\langle\cdot\,,\cdot\rangle_{l,l}$. □

Theorem 4.2.2. *A pseudo H-type algebra $\mathfrak{n}_{r,0}$ can only be isomorphic to $\mathfrak{n}_{0,r}$.*

Proof. Formula (4.2) implies that the action $J_{C^{\tau}(Z)}$ is singular if and only if J_Z is singular for $Z\in\mathbb{R}^{r,s}$ and this happens only if Z is a null vector in $\mathbb{R}^{r,s}$. Since the space $\mathbb{R}^{r,0}$ has no null vectors, the isomorphic Lie algebra can only have the center isomorphic to $\mathbb{R}^{0,r}$. □

Theorem 4.2.3. *A pseudo H-type algebra $\mathfrak{n}_{r,s}$ can only be isomorphic to $\mathfrak{n}_{s,r}$ for $r,s\neq 0$.*

Proof. Let $Z_+,Z_-\in\mathbb{R}^{t,u}$ with $\langle Z_+\,,Z_+\rangle_{t,u}>0$ and $\langle Z_-\,,Z_-\rangle_{t,u}<0$. We consider the line segment $\gamma(\zeta)=(1-\zeta)Z_++\zeta Z_-$ for $\zeta\in[0,1]$. Then there exists $\zeta_0\in(0,1)$ such that $\langle\gamma(\zeta_0)\,,\gamma(\zeta_0)\rangle_{t,u}=0$ by the continuity of the scalar product and as $\langle\gamma(0)\,,\gamma(0)\rangle_{t,u}>0$, $\langle\gamma(1)\,,\gamma(1)\rangle_{t,u}<0$.

Furthermore, if we assume that $(Z_1,\dots,Z_{t+u})\in\mathbb{R}^{t,u}$ is an orthonormal system such that $\langle Z_i\,,Z_j\rangle_{t,u}=\epsilon_i(t,u)\delta_{ij}$ and we use the notation

$$\varphi_{ij}(\zeta)=\langle(1-\zeta)Z_i+\zeta Z_j\,,(1-\zeta)Z_i+\zeta Z_j\rangle_{t,u}=(1-\zeta)^2\langle Z_i\,,Z_i\rangle_{t,u}+\zeta^2\langle Z_j\,,Z_j\rangle_{t,u},$$

for $i\neq j$, then

$$\varphi_{ij}(\zeta)>0\ \text{ if }\ i,j=1,\dots,t,\quad\text{and}\quad \varphi_{ij}(\zeta)<0\ \text{ if }\ i,j=t+1,\dots,t+u \tag{4.3}$$

for all $\zeta\in[0,1]$.

Let $f\colon\mathfrak{n}_{r,s}\to\mathfrak{n}_{t,u}$ be the isomorphism represented by (4.1). Consider an orthonormal basis $\{Z_1,\dots,Z_{t+u}\}$ of $\mathbb{R}^{t,u}$ with $\langle Z_i\,,Z_j\rangle_{t,u}=\epsilon_i(t,u)\delta_{ij}$ and the image

$$\{C^{\tau}(Z_1),\dots,C^{\tau}(Z_{t+u})\}\subset\mathbb{R}^{r,s}$$

under the map C^{τ}. We note that $\langle C^{\tau}(Z_i)\,,C^{\tau}(Z_i)\rangle_{r,s}\neq 0$ by Lemma 4.2.1.

We claim that for basis vectors Z_i, $i = 1, \dots, t$, one gets $\langle C^\tau(Z_i)\,, C^\tau(Z_i)\rangle_{r,s} > 0$ or $\langle C^\tau(Z_i)\,, C^\tau(Z_i)\rangle_{r,s} < 0$ for all indices $i = 1, \dots, t$, simultaneously.

Indeed, assume that there are $C^\tau(Z_i)$ and $C^\tau(Z_j)$ for $i, j = 1, \dots, t$ such that products $\langle C^\tau(Z_i)\,, C^\tau(Z_i)\rangle_{r,s}$ and $\langle C^\tau(Z_j)\,, C^\tau(Z_j)\rangle_{r,s}$ have opposite sign. Then there exists $\zeta_0 \in (0, 1)$ such that

$$\langle (1-\zeta_0)C^\tau(Z_i) + \zeta_0 C^\tau(Z_j)\,, (1-\zeta_0)C^\tau(Z_i) + \zeta_0 C^\tau(Z_j)\rangle_{r,s} = 0,$$

which implies that $J_{(1-\zeta_0)Z_i+\zeta_0 Z_j}$ is singular by Lemma 4.2.1, which contradicts (4.3). The same arguments are valid for the basis vectors Z_i with $i = t+1, \dots, t+u$. Thus we conclude that the scalar product $\langle\cdot\,,\cdot\rangle_{r,s}$ restricted to subspaces $\operatorname{span}\{C^\tau(Z_1), \dots, C^\tau(Z_t)\}$ and $\operatorname{span}\{C^\tau(Z_{t+1}), \dots, C^\tau(Z_{t+u})\}$ is sign definite. As $\{C^\tau(Z_1), \dots, C^\tau(Z_{t+u})\}$ is a basis of $\mathbb{R}^{r,s}$ it follows that

$$r = t \quad \text{and} \quad s = u, \qquad \text{or} \qquad r = u \quad \text{and} \quad s = t.$$

This implies, that the only possible isomorphic pseudo H-type algebra for $\mathfrak{n}_{r,s}$ is $\mathfrak{n}_{s,r}$. $\square$

Theorem 4.2.4. *If $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$, $r \neq s$, are isomorphic, then there exists a Lie algebra isomorphism $\varphi_{r,s}\colon \mathfrak{n}_{r,s} \to \mathfrak{n}_{s,r}$ given by the matrix $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ with $CC^\tau = -\operatorname{Id}_{\mathbb{R}^{s,r}}$. Moreover $C^\tau C = -\operatorname{Id}_{\mathbb{R}^{r,s}}$ and C, C^τ are anti-isometries.*

Proof. Theorem 4.2.3 implies that for any $Z \in \mathbb{R}^{s,r}$: $\langle Z\,, Z\rangle_{s,r} = -\lambda\langle C^\tau(Z)\,, C^\tau(Z)\rangle_{r,s}$ for some $\lambda > 0$. To determine λ we pick an arbitrary $Z \in \mathbb{R}^{s,r}$ and calculate

$$\big(\det(A^\tau J_Z A)\big)^2 = \big(\det(A^\tau A)\big)^2\big(\langle Z\,, Z\rangle_{s,r}\big)^{2l}.$$

On the other hand

$$\big(\det(A^\tau J_Z A)\big)^2 = \big(\det(J_{C^\tau(Z)})\big)^2 = \big(\langle C^\tau(Z)\,, C^\tau(Z)\rangle_{r,s}\big)^{2l},$$

which is equivalent to $|\det(A^\tau A)|^{1/l}\langle Z\,, Z\rangle_{s,r} = -\langle C^\tau(Z)\,, C^\tau(Z)\rangle_{r,s} = -\langle Z\,, CC^\tau(Z)\rangle_{s,r}$. It follows that $CC^\tau = -|\det(A^\tau A)|^{\frac{1}{l}}\operatorname{Id}_{\mathbb{R}^{r,s}}$.

If $\varphi_{r,s}\colon \mathfrak{n}_{r,s} \to \mathfrak{n}_{s,r}$ is a Lie algebra isomorphism, then $\tilde{\varphi}_{r,s} = \begin{pmatrix} \mu A & 0 \\ B & \mu^2 C \end{pmatrix}$ for $\mu \neq 0$ is also a Lie algebra isomorphism as

$$\begin{aligned} \tilde{\varphi}_{r,s}([w\,, v]_{r,s}) &= (\mu^2 C)([w\,, v]_{r,s}) = \mu^2(C([w\,, v]_{r,s}) = \mu^2\left([Aw\,, Av]_{s,r}\right) \\ &= [\mu Aw\,, \mu Av]_{s,r} = [\tilde{\varphi}_{r,s}(w)\,, \tilde{\varphi}_{r,s}(v)]_{s,r} \end{aligned}$$

for all $w, v \in \mathfrak{v}_{r,s}$. Hence, without loss of generality, we can assume that $|\det(A^\tau A)| = 1$, which implies that $CC^\tau = -\operatorname{Id}_{\mathbb{R}^{s,r}}$.

To show that C and C^τ are anti-isometries we choose an arbitrary $Z \in \mathbb{R}^{s,r}$ and obtain

$$\langle C^\tau(Z)\,, C^\tau(Z)\rangle_{r,s} = \langle CC^\tau(Z)\,, Z\rangle_{s,r} = -\langle Z\,, Z\rangle_{s,r}.$$

As C^τ is an isomorphism for any $Y \in \mathbb{R}^{r,s}$ there exists a unique $Z_Y \in \mathbb{R}^{s,r}$ such that $C^\tau(Z_Y) = Y$. It follows that for any $Y \in \mathbb{R}^{r,s}$ we have the equality $\langle Y , Y\rangle_{r,s} = \langle C^\tau(Z_Y) , C^\tau(Z_Y)\rangle_{r,s} = -\langle Z_Y , Z_Y\rangle_{s,r}$. Thus

$$\begin{aligned}\langle C^\tau C(Y) , Y\rangle_{r,s} &= \langle C(Y) , C(Y)\rangle_{s,r} = \langle CC^\tau(Z_Y) , CC^\tau(Z_Y)\rangle_{s,r} = \langle Z_Y , Z_Y\rangle_{s,r} \\ &= -\langle Y , Y\rangle_{r,s}.\end{aligned}$$

Hence $C^\tau C = -\operatorname{Id}_{\mathbb{R}^{r,s}}$ and C is an anti-isometry. □

Theorem 4.2.5. *For any $\mathfrak{n}_{r,s}$, $r \neq s$ there exists a Lie algebra automorphism $f : \mathfrak{n}_{r,s} \to \mathfrak{n}_{r,s}$ given by the matrix $\begin{pmatrix} A & 0 \\ B & C\end{pmatrix}$ with $CC^\tau = \operatorname{Id}_{\mathbb{R}^{r,s}}$. Moreover $C^\tau C = \operatorname{Id}_{\mathbb{R}^{r,s}}$ and C, C^τ are isometries.*

Proof. The proof follows analogously to the proof of Theorem 4.2.4 for $r \neq s$. For $r = s$ one of the possible automorphisms is the identity map. □

Remark 4.2.6. *If $r = s$, then there can be cases when there exist two automorphisms: one with $CC^\tau = \operatorname{Id}_{\mathbb{R}^{r,r}}$ and one with $CC^\tau = -\operatorname{Id}_{\mathbb{R}^{r,r}}$, see for instance Theorem 4.4.7. But there also exist cases where there are only automorphisms with $CC^\tau = \operatorname{Id}_{\mathbb{R}^{r,r}}$, see for instance Theorem 4.5.6.*

4.3 Classification of $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$

Note that since the isomorphisms have to preserve the dimensions of Lie algebras and their centers, we only need to check algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{t,u}$ with $r + s = t + u$.

4.3.1 Classification of $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ with $r = 1, 2, 4, 8$

Definition 4.3.1. *Let $\{v_1, \dots , v_m, Z_1, \dots , Z_n\}$ be an integral basis of a pseudo H-type algebra $\mathfrak{n}$ and $\{\tilde v_1, \dots , \tilde v_m, \tilde Z_1, \dots , \tilde Z_n\}$ an integral basis of a pseudo H-type algebra $\tilde{\mathfrak{n}}$. If a Lie algebra isomorphism $f : \mathfrak{n} \to \tilde{\mathfrak{n}}$ satisfies $f(v_i) = \tilde v_i$ and $f(Z_l) = \tilde Z_l$, then f is called an integral isomorphism and we say that $\mathfrak{n}$ is integral isomorphic to $\tilde{\mathfrak{n}}$.*

Notation 4.3.2. *For a given orthonormal basis $\{Z_1, \dots , Z_{r+s}\}$ of the center $\mathfrak{z}_{r,s}$ of the pseudo H-type algebra $\mathfrak{n}_{r,s}$ we simplify the notation of the operator $J_{Z_i} : \mathfrak{v}_{r,s} \to \mathfrak{v}_{r,s}$ to $J_{Z_i} := J_i$ for all $Z_i \in \{Z_1, \dots , Z_{r+s}\}$.*

Theorem 4.3.3. *The Lie algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ with $r = 1, 2, 4, 8$ are integral isomorphic.*

Proof. For all cases we assume the orthonormal basis $\{Z_1, \dots , Z_{r+s}\}$ of the center $\mathfrak{z}_{r,s}$ of the pseudo H-type algebra $\mathfrak{n}_{r,s}$ with $\langle Z_k , Z_k \rangle_{\mathfrak{z}_{r,s}} = \epsilon_k(r, s)$. Furthermore, all admissible modules are assumed to be minimal and all structural constants are obtained by the use of relation (1.8). For more details on how to obtain integral bases for general pseudo H-type algebras see [46].

Isomorphism on $\mathfrak{n}_{1,0}$ *and* $\mathfrak{n}_{0,1}$. Let $(\mathfrak{v}_{1,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{1,0}})$ be a minimal admissible module and $w \in \mathfrak{v}_{1,0}$ be such that $\langle w\,, w\rangle_{\mathfrak{v}_{1,0}} = 1$. Then the basis $w_1 := w$, $w_2 := J_1 w$ is integral and $\langle w_i\,, w_i\rangle_{\mathfrak{v}_{1,0}} = 1$, $i = 1, 2$.

Let $(\mathfrak{v}_{0,1}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,1}})$ be a minimal admissible module and $\tilde{w} \in \mathfrak{v}_{0,1}$ such that $\langle \tilde{w}\,, \tilde{w}\rangle_{\mathfrak{v}_{0,1}} = 1$. Then the basis $\tilde{w}_1 := \tilde{w}$, $\tilde{w}_2 := \tilde{J}_1\tilde{w}$, is integral and $\langle \tilde{w}_i\,, \tilde{w}_i\rangle_{\mathfrak{v}_{0,1}} = \epsilon_i(1,1)$, $i = 1, 2$.

Calculating the commutators with respect to both integral bases, presented in Table 4.1, we conclude that they coincide. It follows that $\mathfrak{n}_{1,0}$ is integral isomorphic to $\mathfrak{n}_{0,1}$ under the isomorphism $\varphi_{1,0}\colon \mathfrak{n}_{1,0} \to \mathfrak{n}_{0,1}$ defined by $w_1 \mapsto \tilde{w}_1$, $w_2 \mapsto \tilde{w}_2$, $Z_1 \mapsto \tilde{Z}_1$.

Table 4.1: Commutation relations for $\mathfrak{n}_{1,0}$ and $\mathfrak{n}_{0,1}$

$[row\,, col.]$	w_1	w_2
w_1	0	Z_1
w_2	$-Z_1$	0

Isomorphism of $\mathfrak{n}_{2,0}$ *and* $\mathfrak{n}_{0,2}$. In the minimal admissible module $(\mathfrak{v}_{2,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{2,0}})$ we pick $w \in \mathfrak{v}_{2,0}$ such that $\langle w\,, w\rangle_{\mathfrak{v}_{2,0}} = 1$. Then the basis $w_1 := w$, $w_2 := J_2J_1w$, $w_3 := J_1w$, and $w_4 := J_2w$ is integral and $\langle w_i\,, w_i\rangle_{\mathfrak{v}_{2,0}} = 1$, $i = 1, \ldots, 4$.

In the minimal admissible module $(\mathfrak{v}_{0,2}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,2}})$ we choose $\tilde{w} \in \mathfrak{v}_{0,2}$ with $\langle \tilde{w}\,, \tilde{w}\rangle_{\mathfrak{v}_{0,2}} = 1$ and construct the integral basis $\bar{w}_1 := \tilde{w}$, $\bar{w}_2 := J_1J_2\tilde{w}$, $\bar{w}_3 := J_1\tilde{w}$, and $\bar{w}_4 := J_2\tilde{w}$ with $\langle \bar{w}_i\,, \bar{w}_i\rangle_{\mathfrak{v}_{0,2}} = \epsilon_i(2,2)$.

The commutation relations with respect to both bases are equal, see Table 4.2, and this leads to the integral isomorphism. $\varphi_{2,0}\colon \mathfrak{n}_{2,0} \to \mathfrak{n}_{0,2}$ defined by

$$w_1 \mapsto \bar{w}_1, \quad w_2 \mapsto \bar{w}_2, \quad w_3 \mapsto \bar{w}_3, \quad w_4 \mapsto \bar{w}_4, \quad Z_1 \mapsto \tilde{Z}_1, \quad Z_2 \mapsto \tilde{Z}_2.$$

Table 4.2: Commutation relations on $\mathfrak{n}_{2,0}$ and $\mathfrak{n}_{0,2}$

$[row\,, col.]$	w_1	w_2	w_3	w_4
w_1	0	0	Z_1	Z_2
w_2	0	0	$-Z_2$	Z_1
w_3	$-Z_1$	Z_2	0	0
w_4	$-Z_2$	$-Z_1$	0	0

Isomorphism of $\mathfrak{n}_{4,0}$ *and* $\mathfrak{n}_{0,4}$. Let $(\mathfrak{v}_{4,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{4,0}})$ be a minimal admissible module with $w \in \mathfrak{v}_{4,0}$ such that $J_1J_2J_3J_4w = w$ and $\langle w\,, w\rangle_{\mathfrak{v}_{4,0}} = 1$. Then the basis

$$\begin{array}{llll} w_1 := w, & w_2 := J_1J_2w, & w_3 := J_1J_3w, & w_4 := J_1J_4w, \\ w_5 := J_1w, & w_6 := J_2w, & w_7 := J_3w, & w_8 := J_4w, \end{array}$$

with $\langle w_i\,, w_i\rangle_{\mathfrak{v}_{4,0}} = \epsilon_i(8,0) = 1$ is integral, with commutation relations in Table 4.3.

Table 4.3: Commutation relations on $\mathfrak{n}_{4,0}$

$[row\,,col.]$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_1	0	0	0	0	Z_1	Z_2	Z_3	Z_4
w_2	0	0	0	0	Z_2	$-Z_1$	$-Z_4$	Z_3
w_3	0	0	0	0	Z_3	Z_4	$-Z_1$	$-Z_2$
w_4	0	0	0	0	Z_4	$-Z_3$	Z_2	$-Z_1$
w_5	$-Z_1$	$-Z_2$	$-Z_3$	$-Z_4$	0	0	0	0
w_6	$-Z_2$	Z_1	$-Z_4$	Z_3	0	0	0	0
w_7	$-Z_3$	Z_4	Z_1	$-Z_2$	0	0	0	0
w_8	$-Z_4$	$-Z_3$	Z_2	Z_1	0	0	0	0

Let $(\mathfrak{v}_{0,4}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,4}})$ be a minimal admissible module and $w \in \mathfrak{v}_{0,4}$ be such that $J_1J_2J_3J_4w = w$ and $\langle w\,, w\rangle_{\mathfrak{v}_{0,4}} = 1$. Then the basis

$$\begin{array}{llll} \tilde{w}_1 := w, & \tilde{w}_2 := J_1J_2w, & \tilde{w}_3 := J_1J_3w, & \tilde{w}_4 := J_1J_4w, \\ \tilde{w}_5 := J_1w, & \tilde{w}_6 := J_2w, & \tilde{w}_7 := J_3w, & \tilde{w}_8 := J_4w, \end{array}$$

with $\langle \tilde{w}_i\,, \tilde{w}_i\rangle_{\mathfrak{v}_{0,4}} = \epsilon_i(4,4)$ is integral with commutation relations listed in Table 4.4.

Table 4.4: Commutation relations on $\mathfrak{n}_{0,4}$

$[row\,,col.]$	$\tilde{w}_1$	$\tilde{w}_2$	$\tilde{w}_3$	$\tilde{w}_4$	$\tilde{w}_5$	$\tilde{w}_6$	$\tilde{w}_7$	$\tilde{w}_8$
$\tilde{w}_1$	0	0	0	0	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$	$\tilde{Z}_4$
$\tilde{w}_2$	0	0	0	0	$-\tilde{Z}_2$	$\tilde{Z}_1$	$\tilde{Z}_4$	$-\tilde{Z}_3$
$\tilde{w}_3$	0	0	0	0	$-\tilde{Z}_3$	$-\tilde{Z}_4$	$\tilde{Z}_1$	$\tilde{Z}_2$
$\tilde{w}_4$	0	0	0	0	$-\tilde{Z}_4$	$\tilde{Z}_3$	$-\tilde{Z}_2$	$\tilde{Z}_1$
$\tilde{w}_5$	$-\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$	$\tilde{Z}_4$	0	0	0	0
$\tilde{w}_6$	$-\tilde{Z}_2$	$-\tilde{Z}_1$	$\tilde{Z}_4$	$-\tilde{Z}_3$	0	0	0	0
$\tilde{w}_7$	$-\tilde{Z}_3$	$-\tilde{Z}_4$	$-\tilde{Z}_1$	$\tilde{Z}_2$	0	0	0	0
$\tilde{w}_8$	$-\tilde{Z}_4$	$\tilde{Z}_3$	$-\tilde{Z}_2$	$-\tilde{Z}_1$	0	0	0	0

We see from Tables 4.3 and 4.4 that the linear map $\varphi_{4,0}\colon \mathfrak{n}_{4,0} \to \mathfrak{n}_{0,4}$ defined by

$$\begin{array}{lclll} w_i & \mapsto & \tilde{w}_i & \text{if} & i = 1,5,6,7,8, \\ w_i & \mapsto & -\tilde{w}_i & \text{if} & i = 2,3,4, \\ Z_k & \mapsto & \tilde{Z}_k & \text{if} & k = 1,2,3,4, \end{array}$$

is an integral isomorphism.

Isomorphism of $\mathfrak{n}_{8,0}$ *and* $\mathfrak{n}_{0,8}$. Let $\{Z_1, \ldots, Z_8\}$ be an orthonormal basis for $\mathbb{R}^{8,0}$. Take a minimal admissible module $(\mathfrak{v}_{8,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{8,0}})$ and choose $w \in \mathfrak{v}_{8,0}$ with $\langle w\,, w\rangle_{\mathfrak{v}_{8,0}} = 1$ such that

$$J_1J_2J_3J_4w = J_1J_2J_5J_6w = J_2J_3J_5J_7w = J_1J_2J_7J_8w = w.$$

A method presented in [46] shows that the basis

$$\begin{array}{llll} u_1 := w, & u_2 := J_1J_2w, & u_3 := J_1J_3w, & u_4 := J_1J_4w, \\ u_5 := J_1J_5w, & u_6 := J_1J_6w, & u_7 := J_1J_7w, & u_8 := J_1J_8w, \\ u_9 := J_1w, & u_{10} := J_2w, & u_{11} := J_3w, & u_{12} := J_4w, \\ u_{13} := J_5w, & u_{14} := J_6w, & u_{15} := J_7w, & u_{16} := J_8w, \end{array} \tag{4.4}$$

is orthonormal and satisfies $\langle u_i\,, u_i\rangle_{\mathfrak{v}_{8,0}} = \epsilon_i(16,0) = 1$. Thus the minimal admissible module $(\mathfrak{v}_{8,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{8,0}})$ receives the integral basis (4.4).

Let $\{\tilde{Z}_1, \dots, \tilde{Z}_8\}$ be an orthonormal basis for $\mathbb{R}^{0,8}$. Given a minimal admissible module $(\mathfrak{v}_{0,8}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,8}})$, we choose a vector $w^1 \in \mathfrak{v}_{0,8}$ with $\langle w^1\,, w^1\rangle_{\mathfrak{v}_{0,8}} = 1$ such that

$$\tilde{J}_1\tilde{J}_2\tilde{J}_3\tilde{J}_4w^1 = \tilde{J}_1\tilde{J}_2\tilde{J}_5\tilde{J}_6w^1 = \tilde{J}_2\tilde{J}_3\tilde{J}_5\tilde{J}_7w^1 = \tilde{J}_1\tilde{J}_2\tilde{J}_7\tilde{J}_8w^1 = w^1.$$

Then the orthonormal basis

$$\begin{array}{llll} v_1 := w^1, & v_2 := \tilde{J}_1\tilde{J}_2w^1, & v_3 := \tilde{J}_1\tilde{J}_3w^1, & v_4 := \tilde{J}_1\tilde{J}_4w^1, \\ v_5 := \tilde{J}_1\tilde{J}_5w^1, & v_6 := \tilde{J}_1\tilde{J}_6w^1, & v_7 := \tilde{J}_1\tilde{J}_7w^1, & v_8 := \tilde{J}_1\tilde{J}_8w^1, \\ v_9 := \tilde{J}_1w^1, & v_{10} := \tilde{J}_2w^1, & v_{11} := \tilde{J}_3w^1, & v_{12} := \tilde{J}_4w^1, \\ v_{13} := \tilde{J}_5w^1, & v_{14} := \tilde{J}_6w^1, & v_{15} := \tilde{J}_7w^1, & v_{16} := \tilde{J}_8w^1, \end{array} \tag{4.5}$$

with $\langle v_i\,, v_i\rangle_{\mathfrak{v}_{0,8}} = \epsilon_i(8,8)$ is integral, see [46]. Tables 4.10 and 4.11 in the Appendix show the non-vanishing commutation relations on the pseudo H-type algebras $\mathfrak{n}_{8,0}$ and $\mathfrak{n}_{0,8}$. It allows us to construct the Lie algebra integral isomorphism

$$\varphi_{8,0}\colon \begin{cases} u_i \mapsto v_i & \text{if} \quad i = 1, 9, 10, \dots, 16, \\ u_i \mapsto -v_i & \text{if} \quad i = 2, \dots, 8, \\ Z_k \mapsto \tilde{Z}_k & \text{if} \quad k = 1, \dots, 8. \end{cases} \tag{4.6}$$

□

4.3.2 Structure constants for $\mathfrak{n}_{r+8,s}$, $\mathfrak{n}_{r,s+8}$ and $\mathfrak{n}_{r+4,s+4}$

This subsection is purely technical and auxiliary for the upcoming classification. A result of [46] gives an integral basis which satisfies Theorem 1.1.17 for all admissible Clifford modules $\mathfrak{v}_{r,s}$. Furthermore, we proved that it is possible to obtain any minimal admissible integral module $\mathfrak{v}_{t,u}$ by taking the tensor product of minimal admissible integral $\mathfrak{v}_{r,s}$-modules $0 \le r, s \le 8$ by the minimal admissible integral modules $\mathfrak{v}_{8,0}$, $\mathfrak{v}_{0,8}$ or $\mathfrak{v}_{4,4}$.

Proposition 4.3.4. *[46] Consider two minimal admissible integral modules $(\mathfrak{v}_{r,s}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}})$ and $(\mathfrak{v}_{0,8}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,8}})$, where the representations $J_{\tilde{Z}_j}\colon \mathfrak{v}_{0,8} \to \mathfrak{v}_{0,8}$ permute the integral basis of $\mathfrak{v}_{0,8}$ up to sign for all orthonormal generators $\tilde{Z}_j \in \mathbb{R}^{0,8}$. Then the scalar product vector space given by the tensor product $(\mathfrak{v}_{r,s} \otimes \mathfrak{v}_{0,8}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}} \cdot \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,8}})$ is a minimal admissible integral module $(\mathfrak{v}_{r,s+8}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s+8}})$.*

Remark 4.3.5. *In Proposition 4.3.4 one can change the minimal admissible integral module* $(\mathfrak{v}_{0,8}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{0,8}})$ *to the minimal admissible integral module* $(\mathfrak{v}_{8,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{8,0}})$ *or the minimal admissible integral module* $(\mathfrak{v}_{4,4}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{4,4}})$ *and, taking the tensor product, to obtain the modules*

$$(\mathfrak{v}_{r+8,s}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r+8,s}}) = (\mathfrak{v}_{r,s}\otimes\mathfrak{v}_{8,0}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}}\cdot\langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{8,0}})$$

and

$$(\mathfrak{v}_{r+4,s+4}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r+4,s+4}}) = (\mathfrak{v}_{r,s}\otimes\mathfrak{v}_{4,4}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}}\cdot\langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{4,4}})$$

respectively, which are minimal admissible integral. Details can be found in [46].

We call a pseudo H-type algebra $\mathfrak{n}_{r,s}$ *extended* if its minimal admissible integral module $\mathfrak{v}_{r,s}$ was constructed as in Proposition 4.3.4 or in Remark 4.3.5. The tensor products can be taken several times and with different spaces. Before we show how the structure constants for the Lie algebras $\mathfrak{n}_{r+8,s}$, $\mathfrak{n}_{r+4,s+4}$ and $\mathfrak{n}_{r,s+8}$ depend on the structure constants of $\mathfrak{n}_{r,s}$, $\mathfrak{n}_{8,0}$, $\mathfrak{n}_{4,4}$ and $\mathfrak{n}_{0,8}$, we state the notation that will be used in the forthcoming sections. We write $\mathfrak{B}_V$ for an integral basis of the space V. Thus

$$\mathfrak{B}_{\mathfrak{z}_{r,s}} = \{Z_1^{r,s},\ldots,Z_{r+s}^{r,s}\},\quad \langle Z_k^{r+s}\,,Z_m^{r+s}\rangle_{\mathfrak{z}_{r,s}} = \epsilon_k(r,s)\delta_{km},$$

$$\mathfrak{B}_{\mathfrak{v}_{r,s}} = \{w_1,\ldots,w_{2l}\},\quad \langle w_i\,,w_j\rangle_{\mathfrak{v}_{r,s}} = \begin{cases}\epsilon_i(l,l)\delta_{ij} & \text{for } s\neq 0,\\ \delta_{ij} & \text{for } s=0,\end{cases}$$

and define $\mathfrak{B}_{\mathfrak{n}_{r,s}} := \mathfrak{B}_{\mathfrak{v}_{r,s}}\cup\mathfrak{B}_{\mathfrak{z}_{r,s}}$.

We fix the letters u, v, and y for the following bases given by (4.4), a modified version of (4.6) and (4.12)

$$\begin{array}{lll}\mathfrak{B}_{\mathfrak{v}_{8,0}} = \{u_1,\ldots,u_{16}\}, & \mathfrak{B}_{\mathfrak{v}_{0,8}} = \{v_1,\ldots,v_{16}\}, & \mathfrak{B}_{\mathfrak{v}_{4,4}} = \{y_1,\ldots,y_{16}\},\\ \mathfrak{B}_{\mathfrak{z}_{8,0}} = \{Z_1^{8,0},\ldots,Z_8^{8,0}\}, & \mathfrak{B}_{\mathfrak{z}_{0,8}} = \{Z_1^{0,8},\ldots,Z_8^{0,8}\}, & \mathfrak{B}_{\mathfrak{z}_{4,4}} = \{Z_1^{4,4},\ldots,Z_8^{4,4}\}\end{array}$$

with

$$\begin{array}{lll}\langle u_i\,,u_j\rangle_{\mathfrak{v}_{8,0}} = \delta_{ij}, & \langle v_i\,,v_j\rangle_{\mathfrak{v}_{0,8}} = \epsilon_i(8,8)\delta_{ij}, & \langle y_i\,,y_j\rangle_{\mathfrak{v}_{4,4}} = \epsilon_i(8,8)\delta_{ij},\\ \langle Z_k^{8,0}\,,Z_m^{8,0}\rangle_{\mathfrak{z}_{8,0}} = \delta_{km}, & \langle Z_k^{0,8}\,,Z_m^{0,8}\rangle_{\mathfrak{z}_{0,8}} = -\delta_{km}, & \langle Z_k^{4,4}\,,Z_m^{4,4}\rangle_{\mathfrak{z}_{4,4}} = \epsilon_k(4,4)\delta_{km},\end{array}$$

such that $\mathfrak{B}_{\mathfrak{n}_{8,0}}$ and $\mathfrak{B}_{\mathfrak{n}_{0,8}}$ have the same structural constants by Theorem 4.3.3, i.e.

$$\begin{aligned}\varphi_{8,0}(u_i) &= v_i, &&\text{for all } i=1,\ldots,16,\\ \varphi_{8,0}(Z_k^{8,0}) &= Z_k^{0,8}, &&\text{for all } k=1,\ldots,8.\end{aligned}$$

If $\mathfrak{v}_{t,u}$ is obtained by taking the tensor product of $\mathfrak{v}_{r,s}$ by one of the $\mathfrak{v}_{8,0}$, $\mathfrak{v}_{0,8}$ or $\mathfrak{v}_{4,4}$, we write for the basis

$$\mathfrak{B}_{\mathfrak{n}_{t,u}} = \{w_i\otimes\alpha_j, Z_m^{t,u}\mid i=1,\ldots,2l,\ j=1,\ldots,16,\ m=1,\ldots,r+s+8\},$$

where $w_i\in\mathfrak{B}_{\mathfrak{v}_{r,s}}$, the vectors α_j are from the corresponding integral bases $\mathfrak{B}_{\mathfrak{v}_{8,0}}$, $\mathfrak{B}_{\mathfrak{v}_{0,8}}$ or $\mathfrak{B}_{\mathfrak{v}_{4,4}}$, and $Z_m^{t,u}\in\mathfrak{B}_{\mathfrak{z}_{r+p,s+q}} = \mathfrak{B}_{\mathfrak{z}_{t,u}}$, with (p,q) equal to one of the pairs $(8,0),(0,8),(4,4)$. For practical reasons we preserve the order of the elements in $\mathfrak{B}_{\mathfrak{z}_{t,u}}$ and we write first those which have positive squares of the scalar product and then those with negative squares of the scalar product.

Lemma 4.3.6. *Let $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$ be a pseudo H-type algebra and A_{ij}^m the structure constants with respect to the integral basis $\mathfrak{B}_{\mathfrak{n}_{r,s}}$. Let $\mathfrak{n}_{8,0} = \mathfrak{v}_{8,0} \oplus \mathfrak{z}_{8,0}$ has the integral basis $\mathfrak{B}_{\mathfrak{n}_{8,0}}$ with the corresponding structure constants $\bar{A}_{ij}^m$. Then the Lie algebra $\mathfrak{n}_{r+8,s} = (\mathfrak{v}_{r,s} \otimes \mathfrak{v}_{8,0}) \oplus \mathfrak{z}_{r+8,s}$ has the following structure constants $\tilde{A}_{ij,pq}^m$ with respect to the integral basis $\mathfrak{B}_{\mathfrak{n}_{r+8,s}}$.*
If $s = 0$, then

$$\tilde{A}_{ij,pq}^m = \begin{cases} -A_{ip}^m & \text{if} \quad m = 1, \dots, r & \text{and} \quad j = q = 1, \dots, 8, \\ A_{ip}^m & \text{if} \quad m = 1, \dots, r & \text{and} \quad j = q = 9, \dots, 16, \\ \bar{A}_{jq}^{m-r} & \text{if} \quad m = r+1, \dots, r+8 & \text{and} \quad i = p = 1, \dots, 2l, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

If $s > 0$, then

$$\tilde{A}_{ij,pq}^m = \begin{cases} -A_{ip}^m & \text{if } m = 1, \dots, r & \text{and } j = q = 1, \dots, 8, \\ -A_{ip}^{m-8} & \text{if } m = r+8+1, \dots, r+8+s & \text{and } j = q = 1, \dots, 8, \\ A_{ip}^m & \text{if } m = 1, \dots, r & \text{and } j = q = 9, \dots, 16, \\ A_{ip}^{m-8} & \text{if } m = r+8+1, \dots, r+8+s & \text{and } j = q = 9, \dots, 16, \\ \bar{A}_{jq}^{m-r} & \text{if } m = r+1, \dots, r+8 & \text{and } i = p = 1, \dots, l, \\ -\bar{A}_{jq}^{m-r} & \text{if } m = r+1, \dots, r+8 & \text{and } i = p = l+1, \dots, 2l, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Proof. We recall that the scalar product $\langle \cdot \,, \cdot \rangle_{\mathfrak{v}_{r+8,s}}$ of $\mathfrak{v}_{r+8,s}$ is given by the product $\langle \cdot \,, \cdot \rangle_{\mathfrak{v}_{r,s}} \cdot \langle \cdot \,, \cdot \rangle_{\mathfrak{v}_{8,0}}$. To shorten the notation we write

$$J_{Z_m^{r,s}} = J_{Z_m} \colon \mathfrak{v}_{r,s} \to \mathfrak{v}_{r,s}, \quad J_{Z_m^{8,0}} = \bar{J}_{\bar{Z}_m} \colon \mathfrak{v}_{8,0} \to \mathfrak{v}_{8,0}, \quad J_{Z_m^{r+8,s}} = \tilde{J}_{\tilde{Z}_m} \colon \mathfrak{v}_{r+8,s} \to \mathfrak{v}_{r+8,s}$$

and the operator $E := \bar{J}_{\bar{Z}_1} \cdots \bar{J}_{\bar{Z}_8} \colon \mathfrak{v}_{8,0} \to \mathfrak{v}_{8,0}$ with the properties

$$E^2 = \mathrm{Id}_{\mathfrak{v}_{8,0}}, \quad E\bar{J}_{\bar{Z}_j} = -\bar{J}_{\bar{Z}_j}E, \ j = 1, \dots, 8, \quad \langle Eu \,, u^* \rangle_{\mathfrak{v}_{8,0}} = \langle u \,, Eu^* \rangle_{\mathfrak{v}_{8,0}}, \ u, u^* \in \mathfrak{v}_{8,0}\,.$$

It leads to the following equalities:

$$\begin{aligned} \langle \tilde{J}_{\tilde{Z}_m} \tilde{w}_{i,j} \,, \tilde{w}_{p,q} \rangle_{\mathfrak{v}_{r+8,s}} &= \langle J_{Z_m} w_i \,, w_p \rangle_{\mathfrak{v}_{r,s}} \langle Eu_j \,, u_q \rangle_{\mathfrak{v}_{8,0}}, & m &= 1, \dots, r, \\ \langle \tilde{J}_{\tilde{Z}_m} \tilde{w}_{i,j} \,, \tilde{w}_{p,q} \rangle_{\mathfrak{v}_{r+8,s}} &= \langle w_i \,, w_p \rangle_{\mathfrak{v}_{r,s}} \langle \bar{J}_{\bar{Z}_{m-r}} u_j \,, u_q \rangle_{\mathfrak{v}_{8,0}}, & m &= r+1, \dots, r+8, \\ \langle \tilde{J}_{\tilde{Z}_m} \tilde{w}_{i,j} \,, \tilde{w}_{p,q} \rangle_{\mathfrak{v}_{r+8,s}} &= \langle J_{Z_{m-8}} w_i \,, w_p \rangle_{\mathfrak{v}_{r,s}} \langle Eu_j \,, u_q \rangle_{\mathfrak{v}_{8,0}}, & m &= r+9, \dots, r+s+8, \end{aligned} \tag{4.9}$$

with the integral basis $\{\tilde{w}_{i,j} = w_i \otimes u_j, \tilde{Z}_m | i = 1, \dots, 2l, j = 1, \dots, 16, m = 1, \dots, r+s+8\}$ by [46]. Similar equations for the cases $\mathfrak{n}_{0,8}$ and $\mathfrak{n}_{4,4}$ can be found in [46].

Let $m = 1, \dots, r$ and note that the mapping E acts on the integral basis $\mathfrak{B}_{\mathfrak{v}_{8,0}}$ by $Eu_j = -u_j \epsilon_j(8,8)$, which follows from the permutation Table 4.13 in the Appendix. That leads to $\langle Eu_j \,, u_q \rangle_{\mathfrak{v}_{8,0}} = -\epsilon_j(8,8)\delta_{jq}$. Then the first equation in (4.9) gives

$$\tilde{B}_{ij,pq}^m \epsilon_{pq}^{\mathfrak{v}_{r+8,s}} = -B_{i,p}^m \epsilon_p^{\mathfrak{v}_{r,s}} \epsilon_j(8,8)\delta_{jq}$$

by (1.7). Making use of formula (1.8) we obtain the following equations for structure constants

$$\tilde{A}^m_{ij,pq}\big(\epsilon^{\mathfrak{v}_{r+8,s}}_{pq}\big)^2\epsilon^{\mathfrak{z}_{r+8,s}}_m = -A^m_{i,p}\big(\epsilon^{\mathfrak{v}_{r,s}}_p\big)^2\epsilon^{\mathfrak{z}_{r,s}}_m\epsilon_j(8,8)\delta_{jq}$$

that yields to the first two lines in formula (4.7) and corresponding lines in (4.8). Arguing in a similar way for the rest of the formulas in (4.9) we obtain

$$\begin{array}{lcll}
\epsilon^{\mathfrak{z}_{r+8,s}}_m\tilde{A}^m_{ij,pq} & = & -A^m_{i,p}\epsilon^{\mathfrak{z}_{r,s}}_m\epsilon_j(8,8)\delta_{jq}, & m=1,\dots,r,\\
\epsilon^{\mathfrak{z}_{r+8,s}}_m\tilde{A}^m_{ij,pq} & = & \bar{A}^{m-r}_{jq}\,\epsilon^{\mathfrak{z}_{8,0}}_{m-r}\epsilon^{\mathfrak{v}_{r,s}}_i\delta_{ip}, & m=r+1,\dots,r+8,\\
\epsilon^{\mathfrak{z}_{r+8,s}}_m\tilde{A}^m_{ij,pq} & = & -A^{m-8}_{ip}\epsilon^{\mathfrak{z}_{r,s}}_{m-8}\epsilon_j(8,8)\delta_{jq}, & m=r+9,\dots,r+s+8.
\end{array}$$

This implies (4.7) and (4.8). □

Lemma 4.3.7. *Let $\mathfrak{n}_{r,s}=\mathfrak{v}_{r,s}\oplus\mathfrak{z}_{r,s}$ be a pseudo H-type algebra with the structure constants A^m_{ij} written with respect to $\mathfrak{B}_{\mathfrak{n}_{r,s}}$ and $\mathfrak{n}_{0,8}=\mathfrak{v}_{0,8}\oplus\mathfrak{z}_{0,8}$ with the integral basis $\mathfrak{B}_{\mathfrak{n}_{0,8}}$, and the corresponding structure constants $\bar{A}^m_{ij}$. Then the Lie algebra $\mathfrak{n}_{r,s+8}=(\mathfrak{v}_{r,s}\otimes\mathfrak{v}_{0,8})\oplus\mathfrak{z}_{r,s+8}$ has the following structure constants $\tilde{A}^m_{ij,pq}$ with respect to the integral basis $\mathfrak{B}_{\mathfrak{n}_{r,s+8}}$.*

If $s=0$, then

$$\tilde{A}^m_{ij,pq}=\begin{cases}-A^m_{ip} & \text{if}\quad m=1,\dots,r \quad \text{and}\quad j=q=1,\dots,16,\\ \bar{A}^{m-r}_{jq} & \text{if}\quad m=r+1,\dots,r+8\quad \text{and}\quad i=p=1,\dots,2l,\\ 0 & \text{otherwise.}\end{cases}\tag{4.10}$$

If $s>0$, then

$$\tilde{A}^m_{ij,pq}=\begin{cases}-A^m_{ip} & \text{if}\ \ m=1,\dots,r+s \quad \text{and}\ j=q=1,\dots,16,\\ \bar{A}^{m-r-s}_{jq} & \text{if}\ \ m=r+s+1,\dots,r+s+8\ \ \text{and}\ i=p=1,\dots,l,\\ -\bar{A}^{m-r-s}_{jq} & \text{if}\ \ m=r+s+1,\dots,r+s+8\ \ \text{and}\ i=p=l+1,\dots,2l,\\ 0 & \text{otherwise.}\end{cases}\tag{4.11}$$

The proof of Lemma 4.3.7 is analogous to the proof of Lemma 4.3.6.

Before we present the structure constants for the Lie algebra $\mathfrak{n}_{r+4,s+4}$ we write the integral basis for the pseudo H-type algebra $\mathfrak{n}_{4,4}$.

$$\begin{array}{llll}
y_1=w, & y_2=J_1w, & y_3=J_2w, & y_4=J_3w,\\
y_5=J_4w, & y_6=J_1J_2w, & y_7=J_1J_3w, & y_8=J_1J_4w,\\
y_9=J_5w, & y_{10}=J_6w, & y_{11}=J_7w, & y_{12}=J_8w,\\
y_{13}=J_1J_5w, & y_{14}=J_1J_6w, & y_{15}=J_1J_7w, & y_{16}=J_1J_8w,
\end{array}\tag{4.12}$$

for $J_1J_2J_3J_4w=J_1J_2J_5J_6w=J_2J_3J_5J_7w=J_1J_2J_7J_8w=w$ with

$$\langle w_i\,,w_i\rangle_{\mathfrak{v}_{4,4}}=\epsilon_i(8,8),\qquad \langle\,Z_k\,,Z_k\,\rangle_{\mathfrak{z}_{4,4}}=\epsilon_k(4,4).$$

Lemma 4.3.8. *Let $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$ has the structure constants A_{ij}^m with respect to $\mathfrak{B}_{\mathfrak{n}_{r,s}}$ and $\mathfrak{n}_{4,4} = \mathfrak{v}_{4,4} \oplus \mathfrak{z}_{4,4}$ has the structure constants $\bar{A}_{ij}^m$ with respect to the integral basis $\mathfrak{B}_{\mathfrak{n}_{4,4}}$. Then the Lie algebra $\mathfrak{n}_{r+4,s+4} = (\mathfrak{v}_{r,s} \otimes \mathfrak{v}_{4,4}) \oplus \mathfrak{z}_{r+4,s+4}$ has the following structure constants $\tilde{A}_{ij,pq}^m$ with respect to the integral basis $\mathfrak{B}_{\mathfrak{n}_{r+4,s+4}}$.*

If $s = 0$, then

$$\tilde{A}_{ij,pq}^m = \begin{cases} A_{ip}^m & \text{if } m = 1, \dots, r \quad \text{and } j = q = 2, \dots, 5, 13, \dots, 16, \\ -A_{ip}^m & \text{if } m = 1, \dots, r \quad \text{and } j = q = 1, 6, \dots, 12, \\ \bar{A}_{jq}^{m-r} & \text{if } m = r+1, \dots, r+8 \quad \text{and } i = p = 1, \dots, 2l, \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

If $s > 0$, then

$$\tilde{A}_{ij,pq}^m = \begin{cases} A_{ip}^m & \text{if } m = 1, \dots, r \quad \text{and } j = q = 2, \dots, 5, 13, \dots, 16, \\ -A_{ip}^m & \text{if } m = 1, \dots, r \quad \text{and } j = q = 1, 6, \dots, 12, \\ A_{ip}^{m-8} & \text{if } m = r+9, \dots, r+s+8 \text{ and } j = q = 2, \dots, 5, 13, \dots, 16, \\ -A_{ip}^{m-8} & \text{if } m = r+5, \dots, r+s+4 \text{ and } j = q = 1, 6, \dots, 12, \\ \bar{A}_{jq}^{m-r} & \text{if } m = r+1, \dots, r+4 \quad \text{and } i = p = 1, \dots, l, \\ \bar{A}_{jq}^{m-r} & \text{if } m = r+5, \dots, r+8 \quad \text{and } i = p = 1, \dots, l, \\ -\bar{A}_{jq}^{m-r} & \text{if } m = r+1, \dots, r+4 \quad \text{and } i = p = l+1, \dots, 2l, \\ -\bar{A}_{jq}^{m-r} & \text{if } m = r+5, \dots, r+8 \quad \text{and } i = p = l+1, \dots, 2l, \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

The proof of Lemma 4.3.8 is analogous to the proof of Lemma 4.3.6.

4.3.3 Classification of $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ for $r > 8$

We observe an interesting property of some of the H-type algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ that will be used in the proof of Theorem 4.3.11.

Definition 4.3.9. *We say that an orthonormal basis $\{w_1, \dots, w_{2l}, Z_1, \dots, Z_n\}$ of a pseudo H-type algebra is of block-type if $[w_i\,, w_j] = 0$ for both indices $i, j = 1, \dots, l$ and $i, j = l+1, \dots, 2l$. We call a pseudo H-type algebra of block-type if it has a block-type basis.*

Lemma 4.3.10. *The pseudo H-type algebras $\mathfrak{n}_{r,0}$ with $r \mod (8) \in \{0, 1, 2, 4\}$ and $\mathfrak{n}_{0,s}$ with $s \in \mathbb{N}$ are of block-type.*

Proof. We prove by induction that $\mathfrak{n}_{r,0}$ with $r \mod (8) \in \{0, 1, 2, 4\}$ is of block-type. The base of induction follows from Tables 4.1, 4.2, 4.3, 4.10 of non-vanishing commutators on $\mathfrak{n}_{1,0}$, $\mathfrak{n}_{2,0}$, $\mathfrak{n}_{4,0}$ and $\mathfrak{n}_{8,0}$. For the induction step we assume that $\mathfrak{n}_{r,0}$ with $r \mod (8) \in \{0, 1, 2, 4\}$ has a block-type basis $\{w_1, \dots, w_{2l}, Z_1^{r,0}, \dots, Z_r^{r,0}\}$ with $[w_i\,, w_j] =$

0 for both indices $i, j = 1, \ldots, l$ and $i, j = l+1, \ldots, 2l$. By extension we construct the Lie algebra $\mathfrak{n}_{r+8,0}$ of dimension $32l + r + 8$ with the basis $\{w_1 \otimes u_1, \ldots, w_{2l} \otimes u_{16}, Z_1^{r+8,0}, \ldots, Z_{r+8}^{r+8,0}\}$. Equations (4.9) imply that $[w_i \otimes u_j, w_p \otimes u_q] = 0$ for the following cases:

- $i = p$ and both $j, q = 1, \ldots, 8$ and $j, q = 9, \ldots, 16$,
- $j = q$ and both $i, p = 1, \ldots, l$ and $i, p = l+1, \ldots, 2l$,
- $i \neq p$ and $j \neq q$ or $i = p$ and $j = q$.

We define a decomposition of the basis vectors of $\mathfrak{v}_{r+8,0}$ by

$$\begin{array}{lcll} \mathbf{A}^1_{r,0} & := & \{w_i \otimes u_j \mid i = 1, \ldots, l, & j = 1, \ldots, 8\}, \\ \mathbf{A}^2_{r,0} & := & \{w_i \otimes u_j \mid i = l+1, \ldots, 2l, & j = 9, \ldots, 16\}, \\ \mathbf{B}^1_{r,0} & := & \{w_i \otimes u_j \mid i = 1, \ldots, l, & j = 9, \ldots, 16\}, \\ \mathbf{B}^2_{r,0} & := & \{w_i \otimes u_j \mid i = l+1, \ldots, 2l, & j = 1, \ldots, 8\}, \\ \mathbf{A}_{r,0} & := & \mathbf{A}^1_{r,0} \cup \mathbf{A}^2_{r,0}, \qquad \mathbf{B}_{r,0} := \mathbf{B}^1_{r,0} \cup \mathbf{B}^2_{r,0}\,. & \end{array} \tag{4.15}$$

It follows that for any $\tilde{w}, \tilde{v} \in A_{r,0}$ and for any $\tilde{x}, \tilde{y} \in B_{r,0}$ we obtain that $[\tilde{w}, \tilde{v}] = 0 = [\tilde{x}, \tilde{y}]$. As the cardinality of the basis of $\mathfrak{v}_{r+8,s}$ is $32l$ and the cardinality of each of the sets $A_{r,0}$ and $B_{r,0}$ is $16l$ we proved that $\mathfrak{n}_{r,0}$ with $r \mod (8) \in \{0,1,2,4\}$ is of block-type.

Now we consider the H-type algebras $\mathfrak{n}_{0,s}$. The space $\mathfrak{v}_{0,s}$ is neutral with an integral basis $\{w_1, \ldots, w_{2l}\}$ satisfying $\langle w_i, w_j\rangle_{l,l} = \epsilon_i(l,l)\delta_{ij}$. Let $\mathfrak{B}_{\mathfrak{z}_{0,s}} = \{Z_1^{0,s}, \ldots, Z_s^{0,s}\}$. The map $J_{Z_k^{0,s}}$ is an anti-isometry and permutes the basis $\{w_1, \ldots, w_{2l}\}$ for all $k = 1, \ldots, s$. It follows that $B^k_{ij} = 0$ for $i, j = 1, \ldots, l$ and $i, j = l+1, \ldots, 2l$. Then by $\epsilon^{\mathfrak{v}}_\beta B^k_{\alpha\beta} = \epsilon^{\mathfrak{z}}_k A^k_{\alpha\beta}$ we obtain that $A^k_{ij} = 0$ when both indices $i, j = 1, \ldots, l$ and $i, j = l+1, \ldots, 2l$. Hence $\mathfrak{n}_{0,s}$ is a block-type Lie algebra for all $s \in \mathbb{N}$. $\square$

Theorem 4.3.11. *The Lie algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ are integral isomorphic if and only if $r \mod (8) \in \{0,1,2,4\}$.*

Proof. First we claim that the pseudo H-type algebra $\mathfrak{n}_{r+8t,0}$ is not isomorphic to $\mathfrak{n}_{0,r+8t}$ for $r = 3,5,6,7$ and a non-negative integer t.

We prove the claim by counting the dimensions of the Lie algebras. The dimensions of the minimal admissible modules are given by

$$\begin{array}{rcll} \dim(\mathfrak{v}_{3+8t,0}) & = & 4 \cdot 16^t \neq 8 \cdot 16^t = \dim(\mathfrak{v}_{0,3+8t}), & \\ \dim(\mathfrak{v}_{r+8t,0}) & = & 8 \cdot 16^t \neq 16 \cdot 16^t = \dim(\mathfrak{v}_{0,r+8t}), & \text{for } r = 5,6,7. \end{array}$$

The H-type algebras $\mathfrak{n}_{r,0}$ are integral isomorphic to $\mathfrak{n}_{0,r}$ for $r = 1,2,4,8$ by Theorem 4.3.3. Thus it remains to show that the Lie algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ are integral isomorphic if $r \mod (8) \in \{0,1,2,4\}$.

By induction we assume that $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$ are integral isomorphic for $r \mod (8) \in \{0,1,2,4\}$ with the integral block-type bases

$$\mathfrak{B}_{\mathfrak{n}_{r,0}} = \{w_1, \ldots, w_{2l}, Z_1^{r,0}, \ldots, Z_r^{r,0}\}, \quad \mathfrak{B}_{\mathfrak{n}_{0,r}} = \{x_1, \ldots, x_{2l}, Z_1^{0,r}, \ldots, Z_r^{0,r}\},$$

with equal structure constants A_{ij}^m, i.e. $\varphi_{r,0}(w_i) = x_i$ for all $i = 1, \dots, 2l$ and $\varphi_{r,0}(Z_k^{r,0}) = Z_k^{0,r}$ for all $k = 1, \dots, r$. Furthermore, we recall that in the integral block-type bases $\mathfrak{B}_{\mathfrak{n}_{8,0}}$ and $\mathfrak{B}_{\mathfrak{n}_{0,8}}$ the structure constants denoted by $\bar{A}_{ij}^m$ are equal for both Lie algebras, i.e. $\varphi_{8,0}(u_i) = v_i$ for all $i = 1, \dots, 16$ and $\varphi_{8,0}(Z_k^{8,0}) = Z_k^{0,8}$ for all $k = 1, \dots, 8$.

We exploit Proposition 4.3.4 and Remark 4.3.5 and obtain the integral bases

$$\mathfrak{B}_{\mathfrak{n}_{r+8,0}} = \{w_i \otimes u_j, Z_m^{r+8,0}\}, \quad \mathfrak{B}_{\mathfrak{n}_{0,r+8}} = \{x_i \otimes v_j, Z_m^{0,r+8}\}.$$

We define the bijective linear map $\varphi_{r+8,0} \colon \mathfrak{n}_{r+8,0} \to \mathfrak{n}_{0,r+8}$ by

$$\begin{array}{ccclll} w_i \otimes u_j & \mapsto & x_i \otimes v_j & \text{if} & i \in \{1, \dots, l\}, & j \in \{1, \dots, 16\}, \\ w_i \otimes u_j & \mapsto & x_i \otimes v_j & \text{if} & i \in \{l+1, \dots, 2l\}, & j \in \{1, \dots, 8\}, \\ w_i \otimes u_j & \mapsto & -x_i \otimes v_j & \text{if} & i \in \{l+1, \dots, 2l\}, & j \in \{9, \dots, 16\}, \\ Z_m^{r+8,0} & \mapsto & Z_m^{0,r+8} & \text{if} & m \in \{1, \dots, r+8\}. & \end{array}$$

It remains to prove that $\varphi_{r+8,0}$ is a Lie algebra isomorphism, i.e.

$$\varphi_{r+8,0}([w_i \otimes u_j\,, w_p \otimes u_q]) = [\varphi_{r+8,0}(w_i \otimes u_j)\,, \varphi_{r+8,0}(w_p \otimes u_q)].$$

We know that the structural constants $\tilde{A}_{ij,pq}^m$ of $[w_i \otimes u_j\,, w_p \otimes u_q]$ are given by formula (4.7) and that the structural constants $C_{ij,pq}^m$ for $[x_i \otimes v_j\,, x_p \otimes v_q]$ are given by formula (4.11), where we have to put the index r instead of $r+s$. It follows that if $i \neq p$ and $j \neq q$ or $i = p$ and $j = q$, the commutators vanish:

$$\varphi_{r+8,0}([w_i \otimes u_j\,, w_p \otimes u_q]) = \varphi_{r+8,0}(0) = 0 = \pm[x_i \otimes v_j\,, x_p \otimes v_q].$$

Let us consider the case $i = p$ and $j \neq q$.

- if $i = p = 1, \dots, l$, then:

$$\varphi_{r+8,0}([w_i \otimes u_j\,, w_i \otimes u_q]) = \bar{A}_{jq}^{m-r} \varphi_{r+8,0}(Z_m^{r+8,0}) = \bar{A}_{jq}^{m-r} Z_m^{0,r+8},$$

$$[\varphi_{r+8,0}(w_i \otimes u_j)\,, \varphi_{r+8,0}(w_i \otimes u_q)] = [x_i \otimes v_j\,, x_i \otimes v_q] = \bar{A}_{jq}^{m-r} Z_m^{0,r+8}$$

with $m = r+1, \dots, r+8$ by formulas (4.7) and (4.11).

- if $i = p = l+1, \dots, 2l$, then we use Lemma 4.3.10.

$$\varphi_{r+8,0}([w_i \otimes u_j\,, w_i \otimes u_q]) = \bar{A}_{jq}^{m-r} \varphi_{r+8,0}(Z_m^{r+8,0}) = \bar{A}_{jq}^{m-r} Z_m^{0,r+8},$$

$$\begin{aligned} & [\varphi_{r+8,0}(w_i \otimes u_j)\,, \varphi_{r+8,0}(w_i \otimes u_q)] \\ = & \begin{cases} [x_i \otimes v_j\,, x_i \otimes v_q] & \text{if} \quad j, q = 1, \dots, 8 \ \text{ or } \ j, q = 9, \dots, 16, \\ -[x_i \otimes v_j\,, x_i \otimes v_q] & \text{otherwise}, \end{cases} \\ = & \begin{cases} -\bar{A}_{jq}^{m-r} Z_m^{0,r+8} = 0 & \text{if} \quad j, q = 1, \dots, 8 \ \text{ or } \ j, q = 9, \dots, 16, \\ \bar{A}_{jq}^{m-r} Z_m^{0,r+8} & \text{otherwise}, \end{cases} \end{aligned}$$

with $m = r+1, \dots, r+8$ by formulas (4.7), (4.11), and the definition of $\varphi_{r+8,0}$. We observe that $\bar{A}^{m-r}_{jq} = 0$ when for both indices j, q simultaneously either $j, q = 1, \dots, 8$ or $j, q = 9, \dots, 16$, since the Lie algebras $\mathfrak{n}_{8,0}$ and $\mathfrak{n}_{0,8}$ are of block type, see Table 4.10. Thus the map $\varphi_{r+8,0}$ satisfies the Lie algebra isomorphism properties in this case.

We turn to consider the case $i \neq p$ and $j = q$

- if $j = q = 1, \dots, 8$, then

$$\begin{aligned}
\varphi_{r+8,0}([w_i \otimes u_j \,, w_p \otimes u_j]) &= -A^m_{ip}\varphi_{r+8,0}(Z^{r+8,0}_m) = -A^m_{ip} Z^{0,r+8}_m, \\
[\varphi_{r+8,0}(w_i \otimes u_j) \,, \varphi_{r+8,0}(w_p \otimes u_j)] &= [x_i \otimes v_j \,, x_p \otimes v_j] = -A^m_{ip} Z^{0,r+8}_m,
\end{aligned}$$

with $m = 1, \dots, r$ by formulas (4.7) and (4.11).

- if $j = q = 9, \dots, 16$ then we use the block form of Lie algebras $\mathfrak{n}_{r,0}$ and $\mathfrak{n}_{0,r}$. We calculate as above

$$\varphi_{r+8,0}([w_i \otimes u_j \,, w_p \otimes u_j]) = A^m_{ip}\varphi_{r+8,0}(Z^{r+8,0}_m) = A^m_{ip} Z^{0,r+8}_m.$$

On the other side

$$\begin{aligned}
&[\varphi_{r+8,0}(w_i \otimes u_j) \,, \varphi_{r+8,0}(w_p \otimes u_j)] \\
= &\begin{cases} [x_i \otimes v_j \,, x_p \otimes v_q] & \text{if} \quad i, p = 1, \dots, l \quad \text{or} \quad i, p = l+1, \dots, 2l, \\ -[x_i \otimes v_j \,, x_p \otimes v_q] & \text{otherwise}, \end{cases} \\
= &\begin{cases} -A^m_{ip} Z^{0,r+8}_m = 0 & \text{if} \quad i, p = 1, \dots, l \quad \text{or} \quad i, p = l+1, \dots, 2l, \\ A^m_{ip} Z^{0,r+8}_m & \text{otherwise}, \end{cases}
\end{aligned}$$

with $m = 1, \dots, r$ by formulas (4.7) and (4.11). This finishes the proof of the theorem. □

4.4 Isomorphism of Lie algebras $\mathfrak{n}_{r,s}$ with $r, s \neq 0$

In this section we show, making use of the ideas developed in the previous section, that the Bott-periodicity is inherited in isomorphism properties of Lie algebras.

4.4.1 Decompositions of integral bases $r, s \neq 0$

We recall the notation of the bases $\mathfrak{B}_{\mathfrak{n}_{r,s}}$ and state the result that extends the notion of the block type algebras.

Lemma 4.4.1. *Let us assume that the integral basis $\mathfrak{B}_{\mathfrak{v}_{r,s}}$, $r, s \neq 0$, for the pseudo*

H-type algebra $\mathfrak{n}_{r,s}$ with $\dim(\mathfrak{v}_{r,s}) = 2l$ satisfies the following decomposition

$$\begin{aligned}
&\mathfrak{B}_{\mathfrak{v}_{r,s}} = \mathbf{A}_{r,s} \cup \mathbf{B}_{r,s}, \quad \operatorname{card}(\mathbf{A}_{r,s}) = \operatorname{card}(\mathbf{B}_{r,s}) = l,\\
&[\mathbf{A}_{r,s}\,, \mathbf{A}_{r,s}] = [\mathbf{B}_{r,s}\,, \mathbf{B}_{r,s}] = 0,\\
&\mathbf{A}_{r,s} = \mathbf{A}^{+}_{r,s} \cup \mathbf{A}^{-}_{r,s}, \quad \mathbf{B}_{r,s} = \mathbf{B}^{+}_{r,s} \cup \mathbf{B}^{-}_{r,s}, \quad \operatorname{card}(\mathbf{A}^{\pm}_{r,s}) = \operatorname{card}(\mathbf{B}^{\pm}_{r,s}) = \frac{l}{2},\\
&\textit{where} \quad \langle\, w_i\,, w_i\,\rangle_{\mathfrak{v}_{r,s}} = \begin{cases} 1 & \textit{if} \quad w_i \in \mathbf{A}^{+}_{r,s} \cup \mathbf{B}^{+}_{r,s},\\ -1 & \textit{if} \quad w_i \in \mathbf{A}^{-}_{r,s} \cup \mathbf{B}^{-}_{r,s}\,. \end{cases}
\end{aligned} \tag{4.16}$$

Then the extended pseudo H-type algebras $\mathfrak{n}_{r+8,s}$, $\mathfrak{n}_{r,s+8}$, and $\mathfrak{n}_{r+4,s+4}$ admit a decomposition of type (4.16)*.*

Remark 4.4.2. *We would like to stress that for $r,s \neq 0$ the dimensions of the minimal admissible modules $\mathfrak{v}_{r,s}$ are a multiple of 4, such that the fraction $\frac{l}{2}$ of the previous lemma is an integer.*

Proof. Define the following sets

$$\begin{aligned}
\mathbf{A}_{8,0} = \{u_1,\dots,u_8\}, \quad \mathbf{B}_{8,0} = \{u_9,\dots,u_{16}\}, \quad \text{for} \quad u_i \in \mathfrak{B}_{\mathfrak{v}_{8,0}}\\
\mathbf{A}_{0,8} = \{v_1,\dots,v_8\}, \quad \mathbf{B}_{0,8} = \{v_9,\dots,v_{16}\}, \quad \text{for} \quad v_i \in \mathfrak{B}_{\mathfrak{v}_{0,8}}
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\mathbf{A}^{+}_{4,4} = \{y_1,y_6,y_7,y_8\}, \quad \mathbf{A}^{-}_{4,4} = \{y_{13},y_{14},y_{15},y_{16}\},\\
\mathbf{B}^{+}_{4,4} = \{y_2,y_3,y_4,y_5\}, \quad \mathbf{B}^{-}_{4,4} = \{y_9,y_{10},y_{11},y_{12}\}.
\end{aligned} \tag{4.18}$$

for $y_i \in \mathfrak{B}_{\mathfrak{v}_{4,4}}$ given by (4.12).

Now, making use of $\mathbf{A}_{r,s}$, $\mathbf{B}_{r,s}$ and (4.17), (4.18), we define the decompositions for the bases of the extended algebras.

$$\begin{aligned}
\mathbf{A}^{+}_{r+8,s} &:= \mathbf{A}^{+}_{r,s} \otimes \mathbf{A}_{8,0} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{B}_{8,0}, \quad \mathbf{A}^{-}_{r+8,s} := \mathbf{A}^{-}_{r,s} \otimes \mathbf{A}_{8,0} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{B}_{8,0},\\
\mathbf{B}^{+}_{r+8,s} &:= \mathbf{A}^{+}_{r,s} \otimes \mathbf{B}_{8,0} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{A}_{8,0}, \quad \mathbf{B}^{-}_{r+8,s} := \mathbf{A}^{-}_{r,s} \otimes \mathbf{B}_{8,0} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{A}_{8,0},
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^{+}_{r,s+8} &:= \mathbf{A}^{+}_{r,s} \otimes \mathbf{A}_{0,8} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{B}_{0,8}, \quad \mathbf{A}^{-}_{r,s+8} := \mathbf{A}^{-}_{r,s} \otimes \mathbf{A}_{0,8} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{B}_{0,8},\\
\mathbf{B}^{+}_{r,s+8} &:= \mathbf{A}^{-}_{r,s} \otimes \mathbf{B}_{0,8} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{A}_{0,8}, \quad \mathbf{B}^{-}_{r,s+8} := \mathbf{A}^{+}_{r,s} \otimes \mathbf{B}_{0,8} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{A}_{0,8},
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^{+}_{r+4,s+4} &:= \mathbf{B}^{+}_{r,s} \otimes \mathbf{B}^{+}_{4,4} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{B}^{-}_{4,4} \cup \mathbf{A}^{+}_{r,s} \otimes \mathbf{A}^{+}_{4,4} \cup \mathbf{A}^{-}_{r,s} \otimes \mathbf{A}^{-}_{4,4},\\
\mathbf{A}^{-}_{r+4,s+4} &:= \mathbf{B}^{-}_{r,s} \otimes \mathbf{B}^{+}_{4,4} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{B}^{-}_{4,4} \cup \mathbf{A}^{-}_{r,s} \otimes \mathbf{A}^{+}_{4,4} \cup \mathbf{A}^{+}_{r,s} \otimes \mathbf{A}^{-}_{4,4},\\
\mathbf{B}^{+}_{r+4,s+4} &:= \mathbf{A}^{+}_{r,s} \otimes \mathbf{B}^{+}_{4,4} \cup \mathbf{A}^{-}_{r,s} \otimes \mathbf{B}^{-}_{4,4} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{A}^{+}_{4,4} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{A}^{-}_{4,4},\\
\mathbf{B}^{-}_{r+4,s+4} &:= \mathbf{A}^{+}_{r,s} \otimes \mathbf{B}^{-}_{4,4} \cup \mathbf{A}^{-}_{r,s} \otimes \mathbf{B}^{+}_{4,4} \cup \mathbf{B}^{-}_{r,s} \otimes \mathbf{A}^{+}_{4,4} \cup \mathbf{B}^{+}_{r,s} \otimes \mathbf{A}^{-}_{4,4}\,.
\end{aligned}$$

All the necessary properties follows directly from the definition of the basis for the extended Lie algebras and Lemmas 4.3.6, 4.3.7, and 4.3.8. We only illustrate the proof of the following property $[\mathbf{A}_{r+8,s}\,, \mathbf{A}_{r+8,s}] = 0$, considering several cases. To show that $[\mathbf{A}^+_{r,s} \otimes \mathbf{A}_{8,0}\,, \mathbf{A}^+_{r,s} \otimes \mathbf{A}_{8,0}] = 0$ we choose $w_i, w_j \in \mathbf{A}^+_{r,s}$ and $u_p, u_q \in \mathbf{A}_{8,0}$ Then

$$[w_i \otimes u_p\,, w_j \otimes u_q] = \begin{cases} 0 & \text{if} \quad i \neq j,\ p \neq q,\ \text{or } i = j,\ p = q, \\ [u_p\,, u_q] = 0 & \text{if} \quad i = j,\ p \neq q,\ \text{since } [u_p\,, u_q] \in [\mathbf{A}_{8,0}\,, \mathbf{A}_{8,0}] = 0, \\ -[w_i\,, w_j] = 0 & \text{if} \quad i \neq j,\ p = q,\ \text{since } [w_i\,, w_j] \in [\mathbf{A}^+_{r,s}\,, \mathbf{A}^+_{r,s}] = 0. \end{cases}$$

Analogously we show

$$[\mathbf{A}^\pm_{r,s} \otimes \mathbf{A}_{8,0}\,, \mathbf{A}^\pm_{r,s} \otimes \mathbf{A}_{8,0}] = [\mathbf{B}^\pm_{r,s} \otimes \mathbf{B}_{8,0}\,, \mathbf{B}^\pm_{r,s} \otimes \mathbf{B}_{8,0}] = 0$$

for any combinations of $+$ and $-$. Any term of the type $[\mathbf{A}^\pm_{r,s} \otimes \mathbf{A}_{8,0}\,, \mathbf{B}^\pm_{r,s} \otimes \mathbf{B}_{8,0}]$ vanishes since $\mathbf{A}^\pm_{r,s} \cap \mathbf{B}^\pm_{r,s} = \emptyset$ and $\mathbf{A}_{8,0} \cap \mathbf{B}_{8,0} = \emptyset$ and as if both $i \neq j$, $p \neq q$ we obtain that $[w_i \otimes u_p\,, w_j \otimes u_q] = 0$ for any $w_i \in \mathbf{A}^\pm_{r,s}$, $w_j \in \mathbf{B}^\pm_{r,s}$, $u_p \in \mathbf{A}_{8,0}$, $u_q \in \mathbf{B}_{8,0}$. $\square$

In the following lemma we present a list of pseudo H-type algebras $\mathfrak{n}_{r,s}$ satisfying (4.16), which can be used as a base for the successive extensions.

Lemma 4.4.3. *The pseudo H-type algebras $\mathfrak{n}_{r,8}$, $\mathfrak{n}_{8,r}$, $\mathfrak{n}_{r+4,4}$, $\mathfrak{n}_{4,r+4}$ for $r \mod (8) \in \{0,1,2,4\}$ and $\mathfrak{n}_{11}$, $\mathfrak{n}_{2,2}$, $\mathfrak{n}_{4,4}$ admit decomposition* (4.16) *of their bases.*

Proof. Decompositions for $\mathfrak{n}_{r,8}$, $\mathfrak{n}_{8,r}$, $r \mod (8) \in \{0,1,2,4\}$.

$$\begin{aligned} \mathbf{A}^+_{r,8} &:= \{w_i \otimes v_j \mid i = 1,\dots,l, && j = 1,\dots,8\}, \\ \mathbf{A}^-_{r,8} &:= \{w_i \otimes v_j \mid i = l+1,\dots,2l, && j = 9,\dots,16\}, \\ \mathbf{B}^-_{r,8} &:= \{w_i \otimes v_j \mid i = 1,\dots,l, && j = 9,\dots,16\}, \\ \mathbf{B}^+_{r,8} &:= \{w_i \otimes v_j \mid i = l+1,\dots,2l, && j = 1,\dots,8\}, \end{aligned}$$

for $w_i \in \mathfrak{B}_{\mathfrak{v}_{r,0}}$, $v_j \in \mathfrak{B}_{\mathfrak{v}_{0,8}}$.

$$\begin{aligned} \mathbf{A}^+_{8,r} &:= \{w_i \otimes u_j \mid i = 1,\dots,l, && j = 1,\dots,8\}, \\ \mathbf{A}^-_{8,r} &:= \{w_i \otimes u_j \mid i = l+1,\dots,2l, && j = 9,\dots,16\}, \\ \mathbf{B}^+_{8,r} &:= \{w_i \otimes u_j \mid i = 1,\dots,l, && j = 9,\dots,16\}, \\ \mathbf{B}^-_{8,r} &:= \{w_i \otimes u_j \mid i = l+1,\dots,2l, && j = 1,\dots,8\}, \end{aligned}$$

for $w_i \in \mathfrak{B}_{\mathfrak{v}_{0,r}}$, $u_j \in \mathfrak{B}_{\mathfrak{v}_{8,0}}$. For the proof we use Tables (4.1)-(4.4), Tables (4.10), (4.11) and the block structure of the corresponding algebras.

Decompositions for $\mathfrak{n}_{r+4,4}$, $\mathfrak{n}_{4,r+4}$ for $r \mod (8) \in \{0,1,2,4\}$. Recall decompositions (4.15) and (4.18) and define

$$\begin{aligned} \mathbf{A}^+_{r+4,4} &:= \mathbf{B}_{r,0} \otimes \mathbf{B}^+_{4,4} \cup \mathbf{A}_{r,0} \otimes \mathbf{A}^+_{4,4}, & \mathbf{A}^-_{r+4,4} &:= \mathbf{B}_{r,0} \otimes \mathbf{B}^-_{4,4} \cup \mathbf{A}_{r,0} \otimes \mathbf{A}^-_{4,4}, \\ \mathbf{B}^+_{r+4,4} &:= \mathbf{B}_{r,0} \otimes \mathbf{A}^+_{4,4} \cup \mathbf{A}_{r,0} \otimes \mathbf{B}^+_{4,4}, & \mathbf{B}^-_{r+4,4} &:= \mathbf{B}_{r,0} \otimes \mathbf{A}^-_{4,4} \cup \mathbf{A}_{r,0} \otimes \mathbf{B}^-_{4,4}, \\ \mathbf{A}^+_{4,r+4} &:= \mathbf{B}_{0,r} \otimes \mathbf{B}^-_{4,4} \cup \mathbf{A}_{0,r} \otimes \mathbf{A}^+_{4,4}, & \mathbf{A}^-_{4,r+4} &:= \mathbf{B}_{0,r} \otimes \mathbf{B}^+_{4,4} \cup \mathbf{A}_{0,r} \otimes \mathbf{A}^-_{4,4}, \\ \mathbf{B}^+_{4,r+4} &:= \mathbf{B}_{0,r} \otimes \mathbf{A}^-_{4,4} \cup \mathbf{A}_{0,r} \otimes \mathbf{B}^+_{4,4}, & \mathbf{B}^-_{4,r+4} &:= \mathbf{B}_{0,r} \otimes \mathbf{A}^+_{4,4} \cup \mathbf{A}_{0,r} \otimes \mathbf{B}^-_{4,4}. \end{aligned}$$

For the proof we use Tables (4.1)-(4.4), Tables (4.10), (4.11), (4.12), the block structure of the corresponding algebras, and Lemma 4.3.8.

DECOMPOSITIONS FOR $\mathfrak{n}_{1,1}$, $\mathfrak{n}_{2,2}$, AND $\mathfrak{n}_{4,4}$.
We define an orthonormal basis of $\mathfrak{n}_{1,1}$ by

$$\mathfrak{B}_{\mathfrak{v}_{1,1}} = \{w_1 := w,\ w_2 := J_1 w,\ w_3 := J_2 w,\ w_4 := J_2 J_1 w\}, \quad \mathfrak{B}_{\mathfrak{z}_{1,1}} = \{Z_1, Z_2\}, \tag{4.19}$$

with $\langle w_i\, , w_i\rangle_{\mathfrak{v}_{1,1}} = \epsilon_i(2,2)$, $\langle\, Z_k\, , Z_k\,\rangle_{\mathfrak{z}_{1,1}} = \epsilon_k(1,1)$. The commutators are given by Table 4.5.

Table 4.5: Commutation relations on $\mathfrak{n}_{1,1}$

$[row\, , col.]$	w_1	w_4	w_2	w_3
w_1	0	0	Z_1	Z_2
w_4	0	0	$-Z_2$	$-Z_1$
w_2	$-Z_1$	Z_2	0	0
w_3	$-Z_2$	Z_1	0	0

The sets $\mathbf{A}_{1,1}$ and $\mathbf{B}_{1,1}$ are given by

$$\mathbf{A}_{1,1} = \mathbf{A}^+_{1,1} \cup \mathbf{A}^-_{1,1} = \{w_1\} \cup \{w_4\}, \quad \mathbf{B}_{1,1} = \mathbf{B}^+_{1,1} \cup \mathbf{B}^-_{1,1} = \{w_2\} \cup \{w_3\}.$$

We define an orthonormal basis of $\mathfrak{B}_{\mathfrak{z}_{2,2}} = \{Z_1, Z_2, Z_3, Z_4\}$ and

$$\mathfrak{B}_{\mathfrak{v}_{2,2}} = \left\{ \begin{array}{llll} w_1 := w, & w_2 := J_1 w, & w_3 := J_2 w, & w_4 := J_1 J_2 w, \\ w_5 := J_3 w, & w_6 := J_4 w, & w_7 := J_1 J_3 w, & w_8 := J_1 J_4 w, \end{array} \right\}, \tag{4.20}$$

for $J_1 J_2 J_3 J_4 w = w$ with $\langle w_i\, , w_i\rangle_{\mathfrak{v}_{2,2}} = \epsilon_i(4,4)$, $\langle\, Z_k\, , Z_k\,\rangle_{\mathfrak{z}_{2,2}} = \epsilon_k(2,2)$. The sets $\mathbf{A}_{2,2}$ and $\mathbf{B}_{2,2}$ are given by

$$\mathbf{A}_{2,2} = \mathbf{A}^+_{2,2} \cup \mathbf{A}^-_{2,2} = \{w_1, w_4\} \cup \{w_7, w_8\}, \quad \mathbf{B}_{2,2} = \mathbf{B}^+_{2,2} \cup \mathbf{B}^-_{2,2} = \{w_2, w_3\} \cup \{w_5, w_6\}$$

according to Table 4.6.

The integral basis $\mathfrak{B}_{\mathfrak{n}_{4,4}}$ of $\mathfrak{n}_{4,4}$ is given in (4.12) and the decomposition is given in (4.18) according to Table 4.12.

□

4.4.2 Inductive construction of isomorphisms of the Lie algebras of block type

In this subsection we prove that if two pseudo H-type algebras possess decomposition (4.16) and they are isomorphic under a map satisfying some special conditions, then the extensions of them are also isomorphic and the corresponding isomorphism map satisfies the same properties. It allows us to perform an induction proof. Before we state the base of induction we formulate the properties we require from the isomorphism.

Table 4.6: Commutation relations on $\mathfrak{n}_{2,2}$

$[row\,, col.]$	w_1	w_4	w_7	w_8	w_2	w_3	w_5	w_6
w_1	0	0	0	0	Z_1	Z_2	Z_3	Z_4
w_4	0	0	0	0	Z_2	$-Z_1$	Z_4	$-Z_3$
w_7	0	0	0	0	Z_3	$-Z_4$	Z_1	$-Z_2$
w_8	0	0	0	0	Z_4	Z_3	Z_2	Z_1
w_2	$-Z_1$	$-Z_2$	$-Z_3$	$-Z_4$	0	0	0	0
w_3	$-Z_2$	Z_1	Z_4	$-Z_3$	0	0	0	0
w_5	$-Z_3$	$-Z_4$	$-Z_1$	$-Z_2$	0	0	0	0
w_6	$-Z_4$	Z_3	Z_2	$-Z_1$	0	0	0	0

Remark 4.4.4. PROPERTIES OF THE ISOMORPHISM $\varphi_{r,s}\colon \mathfrak{n}_{r,s} \to \mathfrak{n}_{s,r}$. *Let the integral bases $\mathfrak{B}_{\mathfrak{n}_{r,s}}$, $\mathfrak{B}_{\mathfrak{n}_{s,r}}$ of the pseudo H-type algebras $\mathfrak{n}_{r,s}$, $\mathfrak{n}_{s,r}$ admit decomposition* (4.16). *Assume there exists a Lie algebra isomorphism $\varphi_{r,s}\colon \mathfrak{n}_{r,s} \to \mathfrak{n}_{s,r}$ such that*

$$\varphi_{r,s}(\mathbf{A}^{\pm}_{r,s}) = \mathbf{A}^{\pm}_{s,r} \quad \textit{and} \quad \varphi_{r,s}(\mathbf{B}^{\pm}_{r,s}) = \mathbf{B}^{\mp}_{s,r}\,.$$

Furthermore, the restriction $\varphi_{r,s}|_{\mathfrak{z}_{r,s}}$ is an anti-isometry and is a permutation of the set $\{Z_1, \dots, Z_{r+s}\}$, i.e. $\varphi_{r,s}(Z_k) = Z_{\pi_{r,s}(k)}$ with the permutation $\pi_{r,s}\colon \{1, \dots, r+s\} \to \{1, \dots, r+s\}$ such that $\pi_{r,s}(\{1, \dots, r\}) = \{s+1, \dots, s+r\}$ and $\pi_{r,s}(\{r+1, \dots, r+s\}) = \{1, \dots, s\}$.

Theorem 4.4.5. *The Lie algebras $\mathfrak{n}_{r,8}$ and $\mathfrak{n}_{8,r}$ are integral isomorphic if and only if $r \mod (8) \in \{0, 1, 2, 4\}$ and the Lie algebra isomorphism $\varphi_{r,8}\colon \mathfrak{n}_{r,8} \to \mathfrak{n}_{8,r}$ satisfies Remark 4.4.4 with $s = 8$.*

Proof. The H-type algebras $\mathfrak{n}_{r,0}$ are integral isomorphic to $\mathfrak{n}_{0,r}$ for $r \mod (8) \in \{0, 1, 2, 4\}$ by Theorem 4.3.11. Recall that we used the following integral block-type bases

$$\mathfrak{B}_{\mathfrak{n}_{r,0}} = \{w_1, \dots, w_{2l}, Z_1^{r,0}, \dots, Z_r^{r,0}\} \quad \mathfrak{B}_{\mathfrak{n}_{0,r}} = \{x_1, \dots, x_{2l}, Z_1^{0,r}, \dots, Z_r^{0,r}\},$$

with $\langle w_i\,, w_i\rangle_{\mathfrak{v}_{r,0}} = 1$ for $i = 1, \dots, 2l$, $\langle x_i\,, x_i\rangle_{\mathfrak{v}_{0,r}} = \epsilon_i(l,l)$, $\langle Z_k^{r,0}\,, Z_k^{r,0}\rangle_{\mathfrak{z}_{r,0}} = 1$, and $\langle Z_k^{0,r}\,, Z_k^{0,r}\rangle_{\mathfrak{z}_{r,0}} = -1$ for all $k = 1, \dots, r$, where $\varphi_{r,0}(w_i) = x_i$ for all $i = 1, \dots, 2l$ and $\varphi_{r,0}(Z_k^{r,0}) = Z_k^{0,r}$ for all $k = 1, \dots, r$. The equal structure constants are denoted by A_{ij}^m. We write $\varphi_{8,0}(u_i) = v_i$, for $u_i \in \mathfrak{B}_{\mathfrak{v}_{8,0}}$, $v_i \in \mathfrak{B}_{\mathfrak{v}_{0,8}}$, $i = 1, \dots, 16$ and $\varphi_{8,0}(Z_k^{8,0}) = Z_k^{0,8}$, $k = 1, \dots, 8$. The equal structure constants are denoted by $\bar{A}_{ij}^m$ for both Lie algebras.

We exploit Proposition 4.3.4 and Remark 4.3.5 to obtain the integral bases

$$\{w_1 \otimes v_1, \dots, w_{2l} \otimes v_{16}, Z_1^{r,8}, \dots, Z_{r+8}^{r,8}\} \quad \text{for} \quad \mathfrak{n}_{r,8},$$

$$\{x_1 \otimes u_1, \dots, x_{2l} \otimes u_{16}, Z_1^{8,r}, \dots, Z_{r+8}^{8,r}\} \quad \text{for} \quad \mathfrak{n}_{8,r}.$$

We define the bijective linear map $\varphi_{r,8}\colon \mathfrak{n}_{r,8} \to \mathfrak{n}_{8,r}$ by

$$\begin{array}{rclllll}
w_i \otimes v_j & \mapsto & x_i \otimes u_j & \text{if} & i \in \{1, \dots, l\}, & & j \in \{1, \dots, 16\}, \\
w_i \otimes v_j & \mapsto & x_i \otimes u_j & \text{if} & i \in \{l+1, \dots, 2l\}, & & j \in \{1, \dots, 8\}, \\
w_i \otimes v_j & \mapsto & -x_i \otimes u_j & \text{if} & i \in \{l+1, \dots, 2l\}, & & j \in \{9, \dots, 16\}, \\
Z^{r,8}_m & \mapsto & Z^{8,r}_{m+8} & \text{if} & m \in \{1, \dots, r\}, & & \\
Z^{r,8}_m & \mapsto & Z^{8,r}_{m-r} & \text{if} & m \in \{r+1, \dots, r+8\}. & &
\end{array}$$

It remains to prove that $\varphi_{r,8}$ is a Lie algebra isomorphism, i.e.

$$\varphi_{r,8}([w_i \otimes v_j \, , w_p \otimes v_q]) = [\varphi_{r,8}(w_i \otimes v_j) \, , \varphi_{r,8}(w_p \otimes v_q)].$$

The structural constants $\tilde{A}^m_{ij,pq}$ of the commutators $[w_i \otimes v_j \, , w_p \otimes v_q]$ of $\mathfrak{n}_{r,8}$ are given by formula (4.10), and the structural constants $C^m_{ij,pq}$ for $[x_i \otimes u_j \, , x_p \otimes u_q]$ of the Lie algebra $\mathfrak{n}_{8,r}$ are given by formula (4.8). It follows that if $i \neq p$ and $j \neq q$ or $i = p$ and $j = q$ the commutators vanish:

$$\varphi_{r,8}([w_i \otimes v_j \, , w_p \otimes v_q]) = \varphi_{r,8}(0) = 0 = \pm[x_i \otimes u_j \, , x_p \otimes u_q].$$

It is left to consider the following two remaining cases.

Case $i = p$ and $j \neq q$. If, additionally, both indices simultaneously satisfy either $j, q = 1, \dots, 8$ or $j, q = 9, \dots, 16$, then

$$\varphi_{r,8}([w_i \otimes v_j \, , w_i \otimes v_q]) = \varphi_{r,8}(0) = 0 = \pm[x_i \otimes u_j \, , x_i \otimes u_q],$$

because of the block form of the Lie algebras $\mathfrak{n}_{8,0}$, $\mathfrak{n}_{0,8}$. Thus, we can assume without loss of generality that $j = 1, \dots, 8$ and $q = 9, \dots, 16$.

- if $i = p = 1, \dots, l$, $j = 1, \dots, 8$ and $q = 9, \dots, 16$, then:

$$\varphi_{r,8}([w_i \otimes v_j \, , w_i \otimes v_q]) = \bar{A}^{m-r}_{jq} \varphi_{r,8}(Z^{r,8}_m) = \bar{A}^{m-r}_{jq} Z^{8,r}_{m-r},$$

$$[\varphi_{r,8}(w_i \otimes v_j) \, , \varphi_{r,8}(w_i \otimes v_q)] = [x_i \otimes u_j \, , x_i \otimes u_q] = \bar{A}^{m-r}_{jq} Z^{8,r}_{m-r}$$

with $m = r+1, \dots, r+8$ by formulas (4.10) and (4.8).

- if $i = p = l+1, \dots, 2l$, $j = 1, \dots, 8$ and $q = 9, \dots, 16$, then

$$\varphi_{r,8}([w_i \otimes v_j \, , w_i \otimes v_q]) = \bar{A}^{m-r}_{jq} \varphi_{r,8}(Z^{r,8}_m) = \bar{A}^{m-r}_{jq} Z^{8,r}_{m-r},$$

$$[\varphi_{r,8}(w_i \otimes v_j) \, , \varphi_{r,8}(w_i \otimes v_q)] = [x_i \otimes u_j \, , -x_i \otimes u_q] = -(-\bar{A}^{m-r}_{jq} Z^{8,r}_{m-r})$$

with $m = r+1, \dots, r+8$ by formulas (4.10) and (4.8). We see that the map $\varphi_{r,8}$ satisfies the Lie algebra isomorphism properties in this case.

Case $i \neq p$ and $j = q$. If in addition $i, p = 1, \dots, l$ or $i, p = l+1, \dots, 2l$, then the block form of the Lie algebras $\mathfrak{n}_{r,0}$, $\mathfrak{n}_{0,r}$ implies

$$\varphi_{r,8}([w_i \otimes v_j \, , w_p \otimes v_j]) = \varphi_{r,8}(0) = 0 = \pm[x_i \otimes u_j \, , x_p \otimes u_j],$$

such that we can assume that $i = 1, \dots, l$ and $p = l+1, \dots, 2l$.

• if $j = q = 1, \dots, 8$, $i = 1, \dots, l$ and $p = l+1, \dots, 2l$, then

$$\begin{aligned} \varphi_{r,8}([w_i \otimes v_j\,, w_p \otimes v_j]) &= -A_{ip}^m \varphi_{r,8}(Z_m^{r,8}) = -A_{ip}^m Z_{m+8}^{8,r}, \\ [\varphi_{r,8}(w_i \otimes v_j)\,, \varphi_{r,8}(w_p \otimes v_j)] &= [x_i \otimes u_j\,, x_p \otimes u_j] = -A_{ip}^m Z_{m+8}^{8,r}. \end{aligned}$$

• if $j = q = 9, \dots, 16$, $i = 1, \dots, l$ and $p = l+1, \dots, 2l$ then

$$\begin{aligned} \varphi_{r,8}([w_i \otimes v_j\,, w_p \otimes v_j]) &= -A_{ip}^m \varphi_{r,8}(Z_m^{r,8}) = -A_{ip}^m Z_{m+8}^{8,r}, \\ [\varphi_{r,8}(w_i \otimes v_j)\,, \varphi_{r,8}(w_p \otimes v_j)] &= [x_i \otimes u_j\,, -x_p \otimes u_j] = -A_{ip}^m Z_{m+8}^{8,r}, \end{aligned}$$

with $m = 1, \dots, r$ by formulas (4.10) and (4.8). This shows that $\varphi_{r,8}$ is a Lie algebra isomorphism. The map $\varphi_{r,8}$ satisfies Remark 4.4.4 by its definition. □

Theorem 4.4.6. *Assume that the Lie algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$, $r,s \neq 0$, satisfy Remark 4.4.4. Then there exists a Lie algebra isomorphism $\varphi_{r+8,s}\colon \mathfrak{n}_{r+8,s} \to \mathfrak{n}_{s,r+8}$ and two integral bases $\mathfrak{B}_{r+8,s}$ and $\mathfrak{B}_{r,s+8}$ satisfying Remark 4.4.4.*

Proof. Let $\varphi_{r,s}\colon \mathfrak{n}_{r,s} \to \mathfrak{n}_{s,r}$ be the assumed Lie algebra isomorphism. By extension we construct the Lie algebra $\mathfrak{n}_{r+8,s}$ of dimension $32l+r+s+8$ with the basis $\mathfrak{B}_{r+8,s} = \{x_1 \otimes u_1, \dots, x_{2l} \otimes u_{16}, Z_1^{r+8,s}, \dots, Z_{r+s+8}^{r+8,s}\}$. The assumptions imply that $[x_i \otimes u_j\,, x_p \otimes u_q] = 0$ for the following cases:

- $x_i = x_p$ and both $u_j, u_q \in \mathbf{A}_{8,0}$ or $u_j, u_q \in \mathbf{B}_{8,0}$,
- $u_j = u_q$ and both $x_i, x_p \in \mathbf{A}_{r,s}$ or $x_i, x_p \in \mathbf{B}_{r,s}$,
- $x_i \neq x_p$ and $u_j \neq u_q$ or $x_i = x_p$ and $u_j = u_q$,

by formula (4.8), where $\mathbf{A}_{8,0}, \mathbf{B}_{8,0}$ are defined in (4.17). We also recall $\mathbf{A}_{0,8}, \mathbf{B}_{0,8}$ from the same formula.

Then we define the bijective linear map $\varphi_{r+8,s}\colon \mathfrak{n}_{r+8,s} \to \mathfrak{n}_{s,r+8}$ by

$$\begin{array}{rcllll} x_i \otimes u_\alpha & \mapsto & -\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_\alpha) & \text{if} & x_i \in \mathbf{B}_{r,s}, \quad \text{and} \quad u_\alpha \in \mathbf{B}_{8,0}, \\ x_i \otimes u_\alpha & \mapsto & \varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_\alpha) & \text{if} & \text{otherwise}\ , \\ Z_m^{r+8,s} & \mapsto & Z_{\pi_{r,s}(m)}^{s,r+8} & \text{if} & m \in \{1, \dots, r\}, \\ Z_m^{r+8,s} & \mapsto & Z_{\pi_{8,0}(m-r)+r+s}^{s,r+8} & \text{if} & m \in \{r+1, \dots, r+8\}, \\ Z_m^{r+8,s} & \mapsto & Z_{\pi_{r,s}(m-8)}^{s,r+8} & \text{if} & m \in \{r+9, \dots, r+s+8\}, \end{array}$$

where $\varphi_{8,0}\colon \mathfrak{n}_{8,0} \to \mathfrak{n}_{0,8}$ is the Lie algebra isomorphism given by (4.6). We see that the restriction of $\varphi_{r+8,s}$ to $\mathfrak{z}_{r+8,s}$ is an anti-isometry, so it remains to prove that $\varphi_{r+8,s}$ is a Lie algebra homomorphism.

Before we continue, we draw the readers attention to the following. By Lemma 4.3.6 we know that $[x_i \otimes u_j\,, x_i \otimes u_q] = \pm[u_j\,, u_q]_{r+8,s} \in \operatorname{span}\{Z_k^{r+8,s} | k = r+1, \dots, r+8\}$. Since the index k belongs to the set $\{r+1, \dots, r+8\}$, the structure constants $[u_j\,, u_q]$ in $\mathfrak{n}_{r+8,s}$ coincide with the structure constants $[u_j\,, u_q]$ in $\mathfrak{n}_{8,0}$. Analogously we write

$[x_i \otimes u_j\,, x_p \otimes u_j] = \pm[x_i\,, x_p]_{r+8,s} \in \operatorname{span}\{Z_k | k = 1, \dots, r, r+9, \dots, r+s+8\}$ and observe that $[x_i\,, x_p]_{r+8,s} = [x_i\,, x_p]_{r,s}$. Thus

$$\varphi_{8,0}([u_j\,, u_q]_{r+8,s}) = [\varphi_{8,0}(u_j)\,, \varphi_{8,0}(u_q)]_{r+8,s},\ \varphi_{r,s}([x_i\,, x_p]_{r+8,s}) = [\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+8,s}$$

as $\varphi_{8,0}$ and $\varphi_{r,s}$ are Lie algebra isomorphisms. Now, we consider the following cases by using formulas (4.8) and (4.11).

- If $x_i = x_p \in \mathbf{B}^+_{r,s}$, $u_j \in \mathbf{A}_{8,0}$ and $u_q \in \mathbf{B}_{8,0}$, then:

$$\varphi_{r+8,s}([x_i \otimes u_j\,, x_i \otimes u_q]) = \varphi_{r+8,s}([u_j\,, u_q]_{r+8,s}) = \varphi_{8,0}([u_j\,, u_q]_{r+8,s}),$$

$$\begin{aligned}[\varphi_{r+8,s}(x_i \otimes u_j)\,, \varphi_{r+8,s}(x_i \otimes u_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_j)\,, -\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_q)] \\ &= [\varphi_{8,0}(u_j)\,, \varphi_{8,0}(u_q)]_{r+8,s}\end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\,\mathbf{B}^-_{s,r}$.

- If $x_i = x_p \in \mathbf{B}^-_{r,s}$, $u_j \in \mathbf{A}_{8,0}$ and $u_q \in \mathbf{B}_{8,0}$, then:

$$\varphi_{r+8,s}([x_i \otimes u_j\,, x_i \otimes u_q]) = \varphi_{r+8,s}(-[u_j\,, u_q]_{r+8,s}) = -\varphi_{8,0}([u_j\,, u_q]_{r+8,s}),$$

$$\begin{aligned}[\varphi_{r+8,s}(x_i \otimes u_j)\,, \varphi_{r+8,s}(x_i \otimes u_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_j)\,, -\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_q)] \\ &= -[\varphi_{8,0}(u_j)\,, \varphi_{8,0}(u_q)]_{r+8,s}\end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\,\mathbf{B}^+_{s,r}$.

- If $x_i = x_p \in \mathbf{A}^+_{r,s}$, $u_j \in \mathbf{A}_{8,0}$ and $u_q \in \mathbf{B}_{8,0}$, then:

$$\varphi_{r+8,s}([x_i \otimes u_j\,, x_i \otimes u_q]) = \varphi_{r+8,s}([u_j\,, u_q]_{r+8,s}) = \varphi_{8,0}([u_j\,, u_q]_{r+8,s}),$$

$$\begin{aligned}[\varphi_{r+8,s}(x_i \otimes u_j)\,, \varphi_{r+8,s}(x_i \otimes u_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_j)\,, \varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_q)] \\ &= [\varphi_{8,0}(u_j)\,, \varphi_{8,0}(u_q)]_{r+8,s}\end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\,\mathbf{A}^+_{s,r}$.

- If $x_i = x_p \in \mathbf{A}^-_{r,s}$, $u_j \in \mathbf{A}_{8,0}$ and $u_q \in \mathbf{B}_{8,0}$, then:

$$\varphi_{r+8,s}([x_i \otimes u_j\,, x_i \otimes u_q]) = \varphi_{r+8,s}(-[u_j\,, u_q]_{r+8,s}) = -\varphi_{8,0}([u_j\,, u_q]_{r+8,s}),$$

$$\begin{aligned}[\varphi_{r+8,s}(x_i \otimes u_j)\,, \varphi_{r+8,s}(x_i \otimes u_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_j)\,, \varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_q)] \\ &= -[\varphi_{8,0}(u_j)\,, \varphi_{8,0}(u_q)]_{r+8,s}\end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\,\mathbf{A}^-_{s,r}$.

- If $u_j = u_q \in \mathbf{B}_{8,0}$, $x_i \in \mathbf{A}_{r,s}$ and $x_p \in \mathbf{B}_{r,s}$, then:

$$\varphi_{r+8,s}([x_i \otimes u_j\,, x_p \otimes u_j]) = \varphi_{r+8,s}([x_i\,, x_p]_{r+8,s}) = \varphi_{r,s}([x_i\,, x_p]_{r+8,s}),$$

$$\begin{aligned}
[\varphi_{r+8,s}(x_i \otimes u_j)\,, \varphi_{r+8,s}(x_p \otimes u_j)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_j)\,, -\varphi_{r,s}(x_p) \otimes \varphi_{8,0}(u_j)] \\
&= [\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+8,s}
\end{aligned}$$

as $\varphi_{8,0}(u_j) \in \pm\,\mathbf{B}_{0,8}$.

- If $u_j = u_q \in \mathbf{A}_{8,0}$, $x_i \in \mathbf{A}_{r,s}$ and $x_p \in \mathbf{B}_{r,s}$, then:

$$\varphi_{r+8,s}([x_i \otimes u_j\,, x_p \otimes u_j]) = \varphi_{r+8,s}(-[x_i\,, x_p]_{r+8,s}) = -\varphi_{r,s}([x_i\,, x_p]_{r+8,s}),$$

$$\begin{aligned}
[\varphi_{r+8,s}(x_i \otimes u_j)\,, \varphi_{r+8,s}(x_p \otimes u_j)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{8,0}(u_j)\,, \varphi_{r,s}(x_p) \otimes \varphi_{8,0}(u_j)] \\
&= -[\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+8,s}
\end{aligned}$$

as $\varphi_{8,0}(u_j) \in \pm\,\mathbf{A}_{0,8}$.

Hence $\varphi_{r+8,s}$ is a Lie algebra isomorphism satisfying Remark 4.4.4. □

Now we turn to consider the extension obtained by making use of the tensor product with $\mathfrak{v}_{4,4}$.

Theorem 4.4.7. *For any $\mathfrak{n}_{r,r}$ with $r = 1, 2, 4$ there exists an automorphism $\varphi_{r,r}\colon \mathfrak{n}_{r,r} \to \mathfrak{n}_{r,r}$ and an integral basis $\mathfrak{B}_{r,r}$ satisfying Remark 4.4.4.*

Proof. In this proof we explicitly state the automorphisms.

The basis of $\mathfrak{n}_{1,1}$ is given in (4.19) and the commutations in Table 4.5. The automorphism $\varphi_{1,1}\colon \mathfrak{n}_{1,1} \to \mathfrak{n}_{1,1}$ with anti-isometry on the center is given by

$$\begin{array}{llll}
Z_1^{1,1} \mapsto Z_2^{1,1}, & Z_2^{1,1} \mapsto Z_1^{1,1}, & & \\
w_1 \mapsto w_1, & w_2 \mapsto w_3, & w_3 \mapsto w_2, & w_4 \mapsto w_4.
\end{array} \tag{4.21}$$

We defined an orthonormal basis of $\mathfrak{n}_{2,2}$ in (4.20) with commutators in Table 4.6. The automorphism $\varphi_{2,2}\colon \mathfrak{n}_{2,2} \to \mathfrak{n}_{2,2}$ with anti-isometry on the center is given by

$$\begin{array}{llll}
Z_1^{2,2} \mapsto Z_3^{2,2}, & Z_2^{2,2} \mapsto Z_4^{2,2}, & Z_3^{2,2} \mapsto Z_1^{2,2}, & Z_4^{2,2} \mapsto Z_2^{2,2}, \\
w_1 \mapsto w_1, & w_2 \mapsto w_5, & w_3 \mapsto w_6, & w_4 \mapsto w_4, \\
w_5 \mapsto w_2, & w_6 \mapsto w_3, & w_7 \mapsto w_7, & w_8 \mapsto w_8.
\end{array} \tag{4.22}$$

Recalling the basis (4.12) and Table 4.12 we define the automorphism $\varphi_{4,4}\colon \mathfrak{n}_{4,4} \to \mathfrak{n}_{4,4}$ with anti-isometry on the center by

$$\begin{array}{llll}
Z_1^{4,4} \mapsto Z_5^{4,4}, & Z_2^{4,4} \mapsto Z_6^{4,4}, & Z_3^{4,4} \mapsto Z_8^{4,4}, & Z_4^{4,4} \mapsto Z_7^{4,4}, \\
Z_5^{4,4} \mapsto Z_1^{4,4}, & Z_6^{4,4} \mapsto Z_2^{4,4}, & Z_7^{4,4} \mapsto Z_4^{4,4}, & Z_8^{4,4} \mapsto Z_3^{4,4}, \\
y_1 \mapsto y_1, & y_2 \mapsto y_9, & y_3 \mapsto y_{10}, & y_4 \mapsto y_{12}, \\
y_5 \mapsto y_{11}, & y_6 \mapsto y_6, & y_7 \mapsto y_7, & y_8 \mapsto -y_8, \\
y_9 \mapsto y_2, & y_{10} \mapsto y_3, & y_{11} \mapsto y_5, & y_{12} \mapsto y_4, \\
y_{13} \mapsto y_{13}, & y_{14} \mapsto y_{14}, & y_{15} \mapsto -y_{15}, & y_{16} \mapsto y_{16}.
\end{array} \tag{4.23}$$

□

Theorem 4.4.8. *The Lie algebras $\mathfrak{n}_{r+4,4}$ and $\mathfrak{n}_{4,r+4}$ are integral isomorphic if and only if $r \mod (8) \in \{0,1,2,4\}$. Furthermore, for $r \mod (8) \in \{0,1,2,4\}$ there exists a Lie algebra isomorphism $\varphi_{r+4,4}\colon \mathfrak{n}_{r+4,4} \to \mathfrak{n}_{4,r+4}$ and two integral bases $\mathfrak{B}_{r+4,4}$ and $\mathfrak{B}_{4,r+4}$ satisfying Remark 4.4.4.*

Proof. By extension we construct the Lie algebra $\mathfrak{n}_{r+4,4}$ of dimension $32l+r+8$ with the integral basis $\{x_1 \otimes y_1, \dots , x_{2l} \otimes y_{16}, Z_1^{r+4,4}, \dots , Z_{r+8}^{r+4,4}\}$, where $\{x_1, \dots , x_{2l}\} = \mathfrak{B}_{\mathfrak{v}_{r,0}}$. Lemma 4.3.8 implies that $[x_i \otimes y_j\, , x_p \otimes y_q] = 0$ for the following cases:

- $x_i = x_p$ and both $y_j, y_q \in \mathbf{A}_{4,4}$ or $y_j, y_q \in \mathbf{B}_{4,4}$,
- $y_j = y_q$ and both $x_i, x_p \in \mathbf{A}_{r,0}$ or $x_i, x_p \in \mathbf{B}_{r,0}$,
- $x_i \neq x_p$ and $y_j \neq y_q$ or $x_i = x_p$ and $y_j = y_q$,

where $\mathbf{A}_{r,0}, \mathbf{B}_{r,0}$ are defined in (4.15) and $\mathbf{A}_{4,4}, \mathbf{B}_{4,4}$ are defined in (4.18). We define the bijective linear map $\varphi_{r+4,4}\colon \mathfrak{n}_{r+4,4} \to \mathfrak{n}_{4,r+4}$ by

$$\begin{array}{ccccl} x_i \otimes y_\alpha & \mapsto & -\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_\alpha) & \text{if} & x_i \in \mathbf{B}_{r,0}, \quad \text{and} \quad y_\alpha \in \mathbf{B}_{4,4}, \\ x_i \otimes y_\alpha & \mapsto & \varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_\alpha) & \text{if} & \text{otherwise}\ , \\ Z_m^{r+4,4} & \mapsto & Z^{4,r+4}_{\pi_{r,0}(m)+8} & \text{if} & m \in \{1, \dots , r\}, \\ Z_m^{r+4,4} & \mapsto & Z^{4,r+4}_{\pi_{4,4}(m-r)} & \text{if} & m \in \{r+1, \dots , r+8\}. \end{array}$$

We see that the restriction of $\varphi_{r+4,4}$ to $\mathfrak{z}_{r+4,4}$ is an anti-isometry, such that it remains to prove that $\varphi_{r+4,4}$ is a Lie algebra homomorphism.

As in Theorem 4.4.6 we make the following observation. By Lemma 4.3.8 we know that $[x_i \otimes y_j\, , x_i \otimes y_q] = \pm[y_j\, , y_q]_{r+4,4} \in \operatorname{span}\{Z_k | k = r+1, \dots , r+8\}$ and therefore $[y_j\, , y_q]_{r+4,4} = [y_j\, , y_q]_{4,4}$. Analogously, because of $[x_i \otimes y_j\, , x_p \otimes y_j] = \pm[x_i\, , x_p]_{r+4,4} \in \operatorname{span}\{Z_k | k = 1, \dots , r\}$ we obtain $[x_i\, , x_p]_{r+4,4} = [x_i\, , x_p]_{r,0}$. Thus $\varphi_{4,4}([y_j\, , y_q]_{r+4,4}) = [\varphi_{4,4}(y_j)\, , \varphi_{4,4}(y_q)]_{r+4,4}$ and $\varphi_{r,0}([x_i\, , x_p]_{r+4,4}) = [\varphi_{r,0}(x_i)\, , \varphi_{r,0}(x_p)]_{r+4,4}$, respectively, as $\varphi_{4,4}$ and $\varphi_{r,0}$ are Lie algebra isomorphisms. We turn to consider several cases, where we use formulas (4.13) and (4.14).

- If $x_i = x_p \in \mathbf{B}_{r,0}$, $y_j \in \mathbf{A}_{4,4}$ and $y_q \in \mathbf{B}_{4,4}$, then:

$$\varphi_{r+4,4}([x_i \otimes y_j\, , x_i \otimes y_q]) = \varphi_{r+4,4}([y_j\, , y_q]_{r+4,4}) = \varphi_{4,4}([y_j\, , y_q]_{r+4,4}),$$

$$\begin{aligned} [\varphi_{r+4,4}(x_i \otimes y_j)\, , \varphi_{r+4,4}(x_i \otimes y_q)] &= [\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_j)\, , -\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_q)] \\ &= [\varphi_{4,4}(y_j)\, , \varphi_{4,4}(y_q)]_{r+4,4} \end{aligned}$$

as $\varphi_{r,0}(x_i) \in \pm\, \mathbf{B}_{0,r}$.

- If $x_i = x_p \in \mathbf{A}_{r,0}$, $y_j \in \mathbf{A}_{4,4}$ and $y_q \in \mathbf{B}_{4,4}$, then:

$$\varphi_{r+4,4}([x_i \otimes y_j\, , x_i \otimes y_q]) = \varphi_{r+4,4}([y_j\, , y_q]_{r+4,s+4}) = \varphi_{4,4}([y_j\, , y_q]_{r+4,4}),$$

$$\begin{aligned} [\varphi_{r+4,4}(x_i \otimes y_j)\, , \varphi_{r+4,4}(x_i \otimes y_q)] &= [\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_j)\, , \varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_q)] \\ &= [\varphi_{4,4}(y_j)\, , \varphi_{4,4}(y_q)]_{r+4,4} \end{aligned}$$

as $\varphi_{r,0}(x_i) \in \pm\, \mathbf{A}_{0,r}$.

• If $y_j = y_q \in \mathbf{B}^+_{4,4}$, $x_i \in \mathbf{A}_{r,0}$ and $x_p \in \mathbf{B}_{r,0}$, then:

$$\varphi_{r+4,4}([x_i \otimes y_j \,, x_p \otimes y_j]) = \varphi_{r+4,4}([x_i \,, x_p]_{r+4,4}) = \varphi_{r,0}([x_i \,, x_p]_{r+4,4}),$$

$$\begin{aligned}[][\varphi_{r+4,4}(x_i \otimes y_j) \,, \varphi_{r+4,4}(x_p \otimes y_j)] &= [\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_j) \,, -\varphi_{r,0}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= [\varphi_{r,0}(x_i) \,, \varphi_{r,0}(x_p)]_{r+4,4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{B}^-_{4,4}$.

• If $y_j = y_q \in \mathbf{B}^-_{4,4}$, $x_i \in \mathbf{A}_{r,0}$ and $x_p \in \mathbf{B}_{r,0}$, then:

$$\varphi_{r+4,4}([x_i \otimes y_j \,, x_p \otimes y_j]) = \varphi_{r+4,4}(-[x_i \,, x_p]_{r+4,4}) = -\varphi_{r,0}([x_i \,, x_p]_{r+4,4}),$$

$$\begin{aligned}[][\varphi_{r+4,4}(x_i \otimes y_j) \,, \varphi_{r+4,4}(x_p \otimes y_j)] &= [\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_j) \,, -\varphi_{r,0}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= -[\varphi_{r,0}(x_i) \,, \varphi_{r,0}(x_p)]_{r+4,4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{B}^+_{4,4}$.

• If $y_j = y_q \in \mathbf{A}^+_{4,4}$, $x_i \in \mathbf{A}_{r,0}$ and $x_p \in \mathbf{B}_{r,0}$, then:

$$\varphi_{r+4,4}([x_i \otimes y_j \,, x_p \otimes y_j]) = \varphi_{r+4,4}(-[x_i \,, x_p]_{r+4,4}) = -\varphi_{r,0}([x_i \,, x_p]_{r+4,4}),$$

$$\begin{aligned}[][\varphi_{r+4,4}(x_i \otimes y_j) \,, \varphi_{r+4,4}(x_p \otimes y_j)] &= [\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_j) \,, \varphi_{r,0}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= -[\varphi_{r,0}(x_i) \,, \varphi_{r,0}(x_p)]_{r+4,4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{A}^+_{4,4}$.

• If $y_j = y_q \in \mathbf{A}^-_{4,4}$, $x_i \in \mathbf{A}_{r,0}$ and $x_p \in \mathbf{B}_{r,0}$, then:

$$\varphi_{r+4,4}([x_i \otimes y_j \,, x_p \otimes y_j]) = \varphi_{r+4,4}([x_i \,, x_p]_{r+4,4}) = \varphi_{r,0}([x_i \,, x_p]_{r+4,4}),$$

$$\begin{aligned}[][\varphi_{r+4,4}(x_i \otimes y_j) \,, \varphi_{r+4,4}(x_p \otimes y_j)] &= [\varphi_{r,0}(x_i) \otimes \varphi_{4,4}(y_j) \,, \varphi_{r,0}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= [\varphi_{r,0}(x_i) \,, \varphi_{r,0}(x_p)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{A}^-_{4,4}$. Hence $\varphi_{r+4,4}$ is a Lie algebra isomorphism satisfying Remark 4.4.4. $\square$

Before we start with the generalization of these results we recall the technical Lemma 4.3.8, which plays a key role in the upcoming classification.

Theorem 4.4.9. *Assume that the pseudo H-type algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$, $r, s \neq 0$ are isomorphic and satisfy Remark 4.4.4. Then there exists a Lie algebra isomorphism $\varphi_{r+4,s+4} \colon \mathfrak{n}_{r+4,s+4} \to \mathfrak{n}_{s+4,r+4}$ satisfying Remark 4.4.4.*

Proof. By extension we construct the Lie algebra $\mathfrak{n}_{r+4,s+4}$ of dimension $32l + r + s + 8$ with the basis $\{x_1 \otimes y_1, \ldots, x_{2l} \otimes y_{16}, Z_1^{r+4,s+4}, \ldots, Z_{r+s+8}^{r+4,s+4}\}$, where $\{x_1, \ldots, x_{2l}\} = \mathfrak{B}_{\mathfrak{v}_{r,s}}$. The assumptions imply that $[x_i \otimes y_j\,, x_p \otimes y_q] = 0$ for the following cases:

- $x_i = x_p$ and both $y_j, y_q \in \mathbf{A}_{4,4}$ or $y_j, y_q \in \mathbf{B}_{4,4}$,
- $y_j = y_q$ and both $x_i, x_p \in \mathbf{A}_{r,s}$ or $x_i, x_p \in \mathbf{B}_{r,s}$,
- $x_i \neq x_p$ and $y_j \neq y_q$ or $x_i = x_p$ and $y_j = y_q$,

by formula (4.14). We define the bijective linear map $\varphi_{r+4,s+4}\colon \mathfrak{n}_{r+4,s+4} \to \mathfrak{n}_{s+4,r+4}$ by

$$\begin{array}{rclll} x_i \otimes y_\alpha & \mapsto & -\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_\alpha) & \text{if} & x_i \in \mathbf{B}_{r,s}, \quad \text{and} \quad y_\alpha \in \mathbf{B}_{4,4}, \\ x_i \otimes y_\alpha & \mapsto & \varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_\alpha) & \text{if} & \text{otherwise}\ , \\ Z_m^{r+4,s+4} & \mapsto & Z_{\pi_{r,s}(m)+8}^{r+4,s+4} & \text{if} & m \in \{1, \ldots, r\}, \\ Z_m^{r+4,s+4} & \mapsto & Z_{\pi_{4,4}(m-r)+s}^{r+4,s+4} & \text{if} & m \in \{r+1, \ldots, r+8\}, \\ Z_m^{r+4,s+4} & \mapsto & Z_{\pi_{r,s}(m-8)}^{r+4,s+4} & \text{if} & m \in \{r+9, \ldots, r+s+8\}. \end{array}$$

We see that the restriction of $\varphi_{r+4,s+4}$ to $\mathfrak{z}_{r+4,s+4}$ is an anti-isometry. Let us show that $\varphi_{r+4,s+4}$ is a Lie algebra homomorphism.

Observe that Lemma 4.3.8 implyis that $[x_i \otimes y_j\,, x_i \otimes y_q] = \pm[y_j\,, y_q]_{r+4,s+4} \in \operatorname{span}\{Z_k | k = r+1, \ldots, r+8\}$ and $[x_i \otimes y_j\,, x_p \otimes y_j] = \pm[x_i\,, x_p]_{r+4,s+4} \in \operatorname{span}\{Z_k | k = 1, \ldots r, r+9, \ldots, r+s+8\}$. Thus $[y_j\,, y_q]_{r+4,s+4} = [y_j\,, y_q]_{4,4}$ and $[x_i\,, x_p]_{r+4,s+4} = [x_i\,, x_p]_{r,s}$. Therefore,

$$\varphi_{4,4}([y_j\,, y_q]_{r+4,s+4}) = [\varphi_{4,4}(y_j)\,, \varphi_{4,4}(y_q)]_{r+4,s+4},$$

$$\varphi_{r,s}([x_i\,, x_p]_{r+4,s+4}) = [\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+4,s+4},$$

respectively, as $\varphi_{4,4}$ and $\varphi_{r,s}$ are Lie algebra isomorphisms. The remaining cases follow from formula (4.14).

- If $x_i = x_p \in \mathbf{B}_{r,s}^{+}$, $y_j \in \mathbf{A}_{4,4}$ and $y_q \in \mathbf{B}_{4,4}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_i \otimes y_q]) = \varphi_{r+4,s+4}([y_j\,, y_q]_{r+4,s+4}) = \varphi_{4,4}([y_j\,, y_q]_{r+4,s+4}),$$

$$\begin{aligned} [\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_i \otimes y_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, -\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_q)] \\ &= [\varphi_{4,4}(y_j)\,, \varphi_{4,4}(y_q)]_{r+4,s+4} \end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\mathbf{B}_{s,r}^{-}$.

- If $x_i = x_p \in \mathbf{B}_{r,s}^{-}$, $y_j \in \mathbf{A}_{4,4}$ and $y_q \in \mathbf{B}_{4,4}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_i \otimes y_q]) = \varphi_{r+4,s+4}(-[y_j\,, y_q]_{r+4,s+4}) = -\varphi_{4,4}([y_j\,, y_q]_{r+4,s+4}),$$

$$\begin{aligned} [\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_i \otimes y_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, -\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_q)] \\ &= -[\varphi_{4,4}(y_j)\,, \varphi_{4,4}(y_q)]_{r+4,s+4} \end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\, \mathbf{B}^+_{s,r}$.

- If $x_i = x_p \in \mathbf{A}^+_{r,s}$, $y_j \in \mathbf{A}_{4,4}$ and $y_q \in \mathbf{B}_{4,4}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_i \otimes y_q]) = \varphi_{r+4,s+4}([y_j\,, y_q]_{r+4,s+4}) = \varphi_{4,4}([y_j\,, y_q]_{r+4,s+4}),$$

$$\begin{aligned}[\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_i \otimes y_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, \varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_q)] \\ &= [\varphi_{4,4}(y_j)\,, \varphi_{4,4}(y_q)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\, \mathbf{A}^+_{s,r}$.

- If $x_i = x_p \in \mathbf{A}^-_{r,s}$, $y_j \in \mathbf{A}_{4,4}$ and $y_q \in \mathbf{B}_{4,4}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_i \otimes y_q]) = \varphi_{r+4,s+4}(-[y_j\,, y_q]_{r+4,s+4}) = -\varphi_{4,4}([y_j\,, y_q]_{r+4,s+4}),$$

$$\begin{aligned}[\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_i \otimes y_q)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, \varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_q)] \\ &= -[\varphi_{4,4}(y_j)\,, \varphi_{4,4}(y_q)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{r,s}(x_i) \in \pm\, \mathbf{A}^-_{s,r}$.

- If $y_j = y_q \in \mathbf{B}^+_{4,4}$, $x_i \in \mathbf{A}_{r,s}$ and $x_p \in \mathbf{B}_{r,s}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_p \otimes y_j]) = \varphi_{r+4,s+4}([x_i\,, x_p]_{r+4,s+4}) = \varphi_{r,s}([x_i\,, x_p]_{r+4,s+4}),$$

$$\begin{aligned}[\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_p \otimes y_j)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, -\varphi_{r,s}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= [\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{B}^-_{4,4}$.

- If $y_j = y_q \in \mathbf{B}^-_{4,4}$, $x_i \in \mathbf{A}_{r,s}$ and $x_p \in \mathbf{B}_{r,s}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_p \otimes y_j]) = \varphi_{r+4,s+4}(-[x_i\,, x_p]_{r+4,s+4}) = -\varphi_{r,s}([x_i\,, x_p]_{r+4,s+4}),$$

$$\begin{aligned}[\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_p \otimes y_j)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, -\varphi_{r,s}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= -[\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{B}^+_{4,4}$.

- If $y_j = y_q \in \mathbf{A}^+_{4,4}$, $x_i \in \mathbf{A}_{r,s}$ and $x_p \in \mathbf{B}_{r,s}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_p \otimes y_j]) = \varphi_{r+4,s+4}([x_i\,, x_p]_{r+4,s+4}) = \varphi_{r,s}([x_i\,, x_p]_{r+4,s+4}),$$

$$\begin{aligned}[\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_p \otimes y_j)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, \varphi_{r,s}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= [\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{A}^+_{4,4}$.

- If $y_j = y_q \in \mathbf{A}^-_{4,4}$, $x_i \in \mathbf{A}_{r,s}$ and $x_p \in \mathbf{B}_{r,s}$, then:

$$\varphi_{r+4,s+4}([x_i \otimes y_j\,, x_p \otimes y_j]) = \varphi_{r+4,s+4}(-[x_i\,, x_p]_{r+4,s+4}) = -\varphi_{r,s}([x_i\,, x_p]_{r+4,s+4}),$$

$$\begin{aligned}[\varphi_{r+4,s+4}(x_i \otimes y_j)\,, \varphi_{r+4,s+4}(x_p \otimes y_j)] &= [\varphi_{r,s}(x_i) \otimes \varphi_{4,4}(y_j)\,, \varphi_{r,s}(x_p) \otimes \varphi_{4,4}(y_j)] \\ &= -[\varphi_{r,s}(x_i)\,, \varphi_{r,s}(x_p)]_{r+4,s+4}\end{aligned}$$

as $\varphi_{4,4}(y_j) \in \pm\, \mathbf{A}^-_{4,4}$.

Hence $\varphi_{r+4,s+4}$ is a Lie algebra isomorphism which satisfies Remark 4.4.4. $\square$

4.4.3 Main results of Section 4.4

Theorem 4.4.10. *The H-type algebras $\mathfrak{n}_{r+8t_1+4t_2,8t_3+4t_2}$ and $\mathfrak{n}_{8t_3+4t_2,r+8t_1+4t_2}$ are integral isomorphic for $r \in \{0,1,2,4\}$ and $t_1, t_2, t_3 \in \mathbb{N}_0$.*

Proof. We prove by induction. The beginning of the induction is stated in Theorem 4.4.5 and Theorem 4.4.8. Then the induction step is given by Theorem 4.4.6 and Theorem 4.4.9. □

Theorem 4.4.11. *The H-type algebras $\mathfrak{n}_{r+8t_1+4t_2,r+8t_3+4t_2}$ and $\mathfrak{n}_{r+8t_3+4t_2,r+8t_1+4t_2}$ are integral isomorphic for $r \in \{0,1,2\}$ and $t_1, t_2, t_3 \in \mathbb{N}_0$.*

Proof. We prove by induction. The beginning of the induction is stated in Theorem 4.4.7. Then the induction step is given by Theorem 4.4.6 and Theorem 4.4.9. □

4.5 Some non-isomorphic Lie algebras $\mathfrak{n}_{r,s}$ and $\mathfrak{n}_{s,r}$

The behaviour of the Lie algebras of block type, or those that admit the decomposition (4.16) are very special, and as we saw in the previous section that it is preserved under extension. The situation is much less predictable if the Lie algebra is not of block type, then different situations can occur. In the present section we show one example of non isomorphic algebras and show also that, in contrast to algebras $\mathfrak{n}_{r,r}$, $r = 1,2,4$ mod 4, the pseudo H-type algebra $\mathfrak{n}_{3,3}$ does not admit an automorphism which restriction to the center is an anti-isometry. We also observe that our method does not allow to give any outlook if for instance, $\mathfrak{n}_{7,7}$, with $\mathfrak{v}_{7,7} = \mathfrak{v}_{3,3} \otimes \mathfrak{v}_{4,4}$, admit or do not admit an automorphism which restriction to the center is an anti-isometry.

4.5.1 Non-isomorphism of $\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{2,3}$

First we introduce the integral basis of $\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{2,3}$ which is essential for the proof of Theorem 4.5.4.

We define an orthonormal basis of $\mathfrak{n}_{3,2}$ by $\mathfrak{B}_{\mathfrak{z}_{3,2}} = \{Z_0, Z_1, Z_2, Z_3, Z_4\}$ and

$$\mathfrak{B}_{\mathfrak{v}_{3,2}} = \left\{ \begin{array}{llll} w_1 := w, & w_2 := J_1 w, & w_3 := J_2 w, & w_4 := J_1 J_2 w, \\ w_5 := J_3 w, & w_6 := J_4 w, & w_7 := J_1 J_3 w, & w_8 := J_1 J_4 w, \end{array} \right\}$$

for $J_1J_2J_3J_4 w = w$ and $J_0 J_1 J_2 w = w$ with $\langle w_i \,, w_i \rangle_{\mathfrak{v}_{3,2}} = \epsilon_i(4,4)$, $\langle\, Z_k \,, Z_k \,\rangle_{\mathfrak{z}_{3,2}} = \epsilon_{k+1}(3,2)$.

We define an orthonormal basis of $\mathfrak{n}_{2,3}$ by $\mathfrak{B}_{\mathfrak{z}_{2,3}} = \{\bar{Z}_1, \bar{Z}_2, \bar{Z}_3, \bar{Z}_4, \bar{Z}_5\}$ and

$$\mathfrak{B}_{\mathfrak{v}_{2,3}} = \left\{ \begin{array}{llll} \bar{w}_1 := \bar{w}, & \bar{w}_2 := J_1 \bar{w}, & \bar{w}_3 := J_2 \bar{w}, & \bar{w}_4 := J_1 J_2 \bar{w}, \\ \bar{w}_5 := J_3 \bar{w}, & \bar{w}_6 := J_4 \bar{w}, & \bar{w}_7 := J_1 J_3 \bar{w}, & \bar{w}_8 := J_1 J_4 \bar{w}, \end{array} \right\}$$

for $J_1J_2J_3J_4\bar{w} = \bar{w}$ and $J_1J_4J_5\bar{w} = \bar{w}$ with $\langle \bar{w}_i \,, \bar{w}_i \rangle_{\mathfrak{v}_{2,3}} = \epsilon_i(4,4)$, $\langle\, \bar{Z}_k \,, \bar{Z}_k \,\rangle_{\mathfrak{z}_{2,3}} = \epsilon_k(2,3)$.

Table 4.7: Commutation relations on $\mathfrak{n}_{3,2}$

$[row\,,col.]$	w_1	w_4	w_7	w_8	w_2	w_3	w_5	w_6
w_1	0	$-Z_0$	0	0	Z_1	Z_2	Z_3	Z_4
w_4	Z_0	0	0	0	Z_2	$-Z_1$	Z_4	$-Z_3$
w_7	0	0	0	$-Z_0$	Z_3	$-Z_4$	Z_1	$-Z_2$
w_8	0	0	Z_0	0	Z_4	Z_3	Z_2	Z_1
w_2	$-Z_1$	$-Z_2$	$-Z_3$	$-Z_4$	0	$-Z_0$	0	0
w_3	$-Z_2$	Z_1	Z_4	$-Z_3$	Z_0	0	0	0
w_5	$-Z_3$	$-Z_4$	$-Z_1$	$-Z_2$	0	0	0	Z_0
w_6	$-Z_4$	Z_3	Z_2	$-Z_1$	0	0	$-Z_0$	0

Table 4.8: Commutation relations on $\mathfrak{n}_{2,3}$

$[row\,,col.]$	$\bar{w}_1$	$\bar{w}_4$	$\bar{w}_7$	$\bar{w}_8$	$\bar{w}_2$	$\bar{w}_3$	$\bar{w}_5$	$\bar{w}_6$
$\bar{w}_1$	0	0	0	$\bar{Z}_5$	$\bar{Z}_1$	$\bar{Z}_2$	$\bar{Z}_3$	$\bar{Z}_4$
$\bar{w}_4$	0	0	$-\bar{Z}_5$	0	$\bar{Z}_2$	$-\bar{Z}_1$	$\bar{Z}_4$	$-\bar{Z}_3$
$\bar{w}_7$	0	$\bar{Z}_5$	0	0	$\bar{Z}_3$	$-\bar{Z}_4$	$\bar{Z}_1$	$-\bar{Z}_2$
$\bar{w}_8$	$-\bar{Z}_5$	0	0	0	$\bar{Z}_4$	$\bar{Z}_3$	$\bar{Z}_2$	$\bar{Z}_1$
$\bar{w}_2$	$-\bar{Z}_1$	$-\bar{Z}_2$	$-\bar{Z}_3$	$-\bar{Z}_4$	0	0	0	$\bar{Z}_5$
$\bar{w}_3$	$-\bar{Z}_2$	$\bar{Z}_1$	$\bar{Z}_4$	$-\bar{Z}_3$	0	0	$\bar{Z}_5$	0
$\bar{w}_5$	$-\bar{Z}_3$	$-\bar{Z}_4$	$-\bar{Z}_1$	$-\bar{Z}_2$	0	$-\bar{Z}_5$	0	0
$\bar{w}_6$	$-\bar{Z}_4$	$\bar{Z}_3$	$\bar{Z}_2$	$-\bar{Z}_1$	$-\bar{Z}_5$	0	0	0

Proposition 4.5.1. *The following is true.*

- *The linear map* $\mathrm{ad}_X\colon \mathfrak{v}_{3,2} \to \mathfrak{z}_{3,2}$ *is surjective if and only if* $\langle X\,,X\rangle_{\mathfrak{v}_{3,2}} \neq 0$ *for* $X \in \mathfrak{v}_{3,2}$.
- *The linear map* $\mathrm{ad}_X\colon \mathfrak{v}_{2,3} \to \mathfrak{z}_{2,3}$ *is surjective if and only if* $\langle X\,,X\rangle_{\mathfrak{v}_{2,3}} \neq 0$ *for* $X \in \mathfrak{v}_{2,3}$.

Proof. First we note that ad_X is surjective for all $X \in \mathfrak{v}_{r,s}$ with $\langle X\,,X\rangle_{\mathfrak{v}_{r,s}} \neq 0$ by Definition 1.1.6, so it suffices to prove that for $\langle X\,,X\rangle_{\mathfrak{v}_{3,2}} = 0$, $\langle X\,,X\rangle_{\mathfrak{v}_{2,3}} = 0$, respectively, the map ad_X is not surjective.

We write $X = \sum_{i=1}^{8} \lambda_i w_i$ for $X \in \mathfrak{v}_{3,2}$ and define the representation matrix M_X of ad_X with respect to the orthonormal basis $\mathfrak{B}_{\mathfrak{n}_{3,2}}$ by

$$\begin{pmatrix} V_1^X & V_4^X & V_7^X & V_8^X & V_2^X & V_3^X & V_5^X & V_6^X \end{pmatrix},$$

where V_i^X is the vector representation $\begin{pmatrix} \mu_{0i}^X \\ \vdots \\ \mu_{4i}^X \end{pmatrix}$ of $[X\,,w_i] = \sum_{k=0}^{4} \mu_{ki}^X Z_k$. The matrix M_X for $\mathfrak{n}_{3,2}$ is given by

$$\begin{pmatrix} \lambda_4 & -\lambda_1 & \lambda_8 & -\lambda_7 & \lambda_3 & -\lambda_2 & -\lambda_6 & \lambda_5 \\ -\lambda_2 & \lambda_3 & -\lambda_5 & -\lambda_6 & \lambda_1 & -\lambda_4 & \lambda_7 & \lambda_8 \\ -\lambda_3 & -\lambda_2 & \lambda_6 & -\lambda_5 & \lambda_4 & \lambda_1 & \lambda_8 & -\lambda_7 \\ -\lambda_5 & \lambda_6 & -\lambda_2 & -\lambda_3 & \lambda_7 & \lambda_8 & \lambda_1 & -\lambda_4 \\ -\lambda_6 & -\lambda_5 & \lambda_3 & -\lambda_2 & \lambda_8 & -\lambda_7 & \lambda_4 & \lambda_1 \end{pmatrix}.$$

Note that M_X is surjective if and only if $\det(M_X M_X^T) \neq 0$ as $\mathrm{rank(M_X M_X^T) = rank(M_X)}$. The determinant of $M_X M_X^T$ is

$$\begin{aligned} & (\lambda_1^2+\lambda_4^2-\lambda_7^2-\lambda_8^2+\lambda_2^2+\lambda_3^2-\lambda_5^2-\lambda_6^2)^2(\lambda_1^2+\lambda_4^2+\lambda_7^2+\lambda_8^2+\lambda_2^2+\lambda_3^2+\lambda_5^2+\lambda_6^2) \\ \times & \Big[\lambda_1^4+\lambda_4^4+\lambda_7^4+2\lambda_7^2\lambda_8^2+\lambda_8^4+2\lambda_7^2\lambda_2^2+2\lambda_8^2\lambda_2^2+\lambda_2^4+2\lambda_7^2\lambda_3^2+2\lambda_8^2\lambda_3^2+2\lambda_2^2\lambda_3^2 \\ + & \lambda_3^4+2\lambda_7^2\lambda_5^2+2\lambda_8^2\lambda_5^2-2\lambda_2^2\lambda_5^2-2\lambda_3^2\lambda_5^2+\lambda_5^4+2(\lambda_7^2+\lambda_8^2-\lambda_2^2-\lambda_3^2+\lambda_5^2)\lambda_6^2 \\ + & \lambda_6^4-8\lambda_1(\lambda_7\lambda_2\lambda_5+\lambda_8\lambda_3\lambda_5+\lambda_8\lambda_2\lambda_6-\lambda_7\lambda_3\lambda_6)+8\lambda_4(-\lambda_8\lambda_2\lambda_5+\lambda_7\lambda_3\lambda_5 \\ + & \lambda_7\lambda_2\lambda_6+\lambda_8\lambda_3\lambda_6)+2\lambda_4^2(-\lambda_7^2-\lambda_8^2+\lambda_2^2+\lambda_3^2+\lambda_5^2+\lambda_6^2) \\ + & 2\lambda_1^2(\lambda_4^2-\lambda_7^2-\lambda_8^2+\lambda_2^2+\lambda_3^2+\lambda_5^2+\lambda_6^2)\Big]. \end{aligned}$$

It follows that for all $X \in \mathfrak{v}_{3,2}$ with

$$\langle X, X \rangle_{\mathfrak{v}_{3,2}} = \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2-\lambda_5^2-\lambda_6^2-\lambda_7^2-\lambda_8^2 = 0$$

the determinant $\det(M_X M_X^T)$ vanishes. This finishes the proof for $\mathfrak{n}_{3,2}$.

For $\mathfrak{n}_{2,3}$ the matrix M_X is given by

$$\begin{pmatrix} -\lambda_2 & \lambda_3 & -\lambda_5 & -\lambda_6 & \lambda_1 & -\lambda_4 & \lambda_7 & \lambda_8 \\ -\lambda_3 & -\lambda_2 & \lambda_6 & -\lambda_5 & \lambda_4 & \lambda_1 & \lambda_8 & -\lambda_7 \\ -\lambda_5 & \lambda_6 & -\lambda_2 & -\lambda_3 & \lambda_7 & \lambda_8 & \lambda_1 & -\lambda_4 \\ -\lambda_6 & -\lambda_5 & \lambda_3 & -\lambda_2 & \lambda_8 & -\lambda_7 & \lambda_4 & \lambda_1 \\ -\lambda_8 & \lambda_7 & -\lambda_4 & \lambda_1 & -\lambda_6 & -\lambda_5 & \lambda_3 & \lambda_2 \end{pmatrix},$$

and the determinant of $M_X M_X^T$ is given by

$$\begin{aligned} & (\lambda_1^2+\lambda_4^2-\lambda_7^2-\lambda_8^2+\lambda_2^2+\lambda_3^2-\lambda_5^2-\lambda_6^2)^2(\lambda_1^2+\lambda_4^2+\lambda_7^2+\lambda_8^2+\lambda_2^2+\lambda_3^2+\lambda_5^2+\lambda_6^2) \\ \times & \Big[\lambda_1^4+\lambda_2^4+\lambda_3^4+2\lambda_3^2\lambda_4^2+\lambda_4^4-2\lambda_3^2\lambda_5^2+2\lambda_4^2\lambda_5^2+\lambda_5^4-2\lambda_3^2\lambda_6^2+2\lambda_4^2\lambda_6^2+2\lambda_5^2\lambda_6^2+\lambda_6^4 \\ - & 8\lambda_3\lambda_4\lambda_5\lambda_7+2\lambda_3^2\lambda_7^2-2\lambda_4^2\lambda_7^2+2\lambda_5^2\lambda_7^2+2\lambda_6^2\lambda_7^2+\lambda_7^4+8\lambda_3\lambda_4\lambda_6\lambda_8+2\lambda_3^2\lambda_8^2-2\lambda_4^2\lambda_8^2 \\ + & 2\lambda_5^2\lambda_8^2+2\lambda_6^2\lambda_8^2+2\lambda_7^2\lambda_8^2+\lambda_8^4-8\lambda_2\lambda_4(\lambda_6\lambda_7+\lambda_5\lambda_8) \\ + & 2\lambda_1^1(\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2\lambda_6^2-\lambda_7^2-\lambda_8^2)+2\lambda_2^2(\lambda_3^2+\lambda_4^2-\lambda_5^2-\lambda_6^2+\lambda_7^2+\lambda_8^2) \\ + & 8\lambda_1(\lambda_3(\lambda_6\lambda_7+\lambda_6\lambda_8)+\lambda_2(-\lambda_5\lambda_7+\lambda_6\lambda_8))\Big]. \end{aligned}$$

Thus if $X \in \mathfrak{v}_{2,3}$ and $\langle X, X \rangle_{\mathfrak{v}_{2,3}} = \lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2-\lambda_5^2-\lambda_6^2-\lambda_7^2-\lambda_8^2 = 0$, then $\det(M_X M_X^T) = 0$. This finishes the proof for $\mathfrak{n}_{2,3}$. □

We stress that the results of Proposition 4.5.1 and Proposition 4.5.5 represent quite exceptional cases. Definition 1.1.6 does not imply that ad_X is not surjective for any $X \in \mathfrak{v}_{r,s}$ with $\langle X , X \rangle_{\mathfrak{v}_{r,s}} = 0$. In the following we state a couple of lemmas illustrating this possibility.

Lemma 4.5.2. *For any of the pseudo H-type algebras $\mathfrak{n}_{11,2}$, $\mathfrak{n}_{7,6}$,$\mathfrak{n}_{6,7}$ and $\mathfrak{n}_{2,11}$ there exists X in the corresponding space $\mathfrak{v}_{r,s}$ such that $\langle X , X \rangle_{\mathfrak{v}_{r,s}} = 0$ but, nevertheless, the map ad_X is surjective.*

Proof. Recall that the pseudo H-type algebras $\mathfrak{n}_{11,2}$, $\mathfrak{n}_{7,6}$,$\mathfrak{n}_{6,7}$ and $\mathfrak{n}_{2,11}$ are obtained from $\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{2,3}$ by extensions. We define $X = w_1 \otimes u_1 + w_7 \otimes u_2 \in \mathfrak{n}_{11,2}$ where $w_1, w_7 \in \mathfrak{B}_{\mathfrak{v}_{3,2}}$ and $u_1, u_2 \in \mathfrak{B}_{\mathfrak{v}_{8,0}}$ and note that $\langle X , X \rangle_{\mathfrak{v}_{11,2}} = 0$. Then

$$
\begin{aligned}
[X , w_i \otimes u_1] &= [w_1 \otimes u_1 , w_i \otimes u_1] + [w_7 \otimes u_2 , w_i \otimes u_1] \\
&= \begin{cases} -[w_1 , w_i]_{\mathfrak{n}_{11,2}} & \text{for} \quad i = 1, \dots , 6, 8, \\ -[w_1 , w_i]_{\mathfrak{n}_{11,2}} - [u_2 , u_1]_{\mathfrak{n}_{11,2}} & \text{for} \quad i = 7, \end{cases} \\
&= -[w_1 , w_i]_{\mathfrak{n}_{11,2}}, \qquad\qquad\qquad \text{for } i = 1, \dots , 8.
\end{aligned}
$$

Hence $\mathrm{span}\{Z_1, Z_2, Z_3, Z_{12}, Z_{13}\} \subset \mathrm{Image}(\mathrm{ad}_X)$. Furthermore,

$$
\begin{aligned}
[X , w_1 \otimes u_j] &= [w_1 \otimes u_1 , w_1 \otimes u_j] + [w_7 \otimes u_2 , w_1 \otimes u_j] \\
&= \begin{cases} [u_1 , u_j]_{\mathfrak{n}_{11,2}} & \text{for} \quad j = 1, 3, \dots , 16, \\ [u_1 , u_j]_{\mathfrak{n}_{11,2}} - [w_7 , w_1]_{\mathfrak{n}_{11,2}} & \text{for} \quad j = 2, \end{cases} \\
&= [u_1 , u_j]_{\mathfrak{n}_{11,2}}, \qquad\qquad\qquad \text{for } j = 1, \dots , 16.
\end{aligned}
$$

Hence $\mathrm{span}\{Z_4, \dots , Z_{11}\} \subset \mathrm{Image}(\mathrm{ad}_X)$, i.e. the map ad_X is surjective.

The proof for $\mathfrak{n}_{7,6}$ and $\mathfrak{n}_{6,7}$ is obtained analogously by replacing u_1 and u_2 by y_1 and y_6. For the proof for $\mathfrak{n}_{2,11}$ we replace u_1 and u_2 by v_1 and v_2, respectively. □

Lemma 4.5.3. *For any pseudo H-type algebra $\mathfrak{n}_{r,s}$ with $r, s \neq 0$, satisfying* (4.16)*, there exists at least one $X \in \mathfrak{v}_{r,s}$ with $\langle X , X \rangle_{\mathfrak{v}_{r,s}} = 0$ such that the map ad_X is surjective.*

Proof. We choose the basis vectors $w_i \in \mathbf{A}^+_{r,s}$ and $w_j \in \mathbf{B}^-_{r,s}$ and define $X = w_i + w_j$ such that $\langle X , X \rangle_{\mathfrak{v}_{r,s}} = 0$. We note that the map $\mathrm{ad}_{w_i} : \mathfrak{V}_{w_i} \to \mathfrak{z}_{r,s}$ and $\mathrm{ad}_{w_j} : \mathfrak{V}_{w_j} \to \mathfrak{z}_{r,s}$ are surjective, where we denote by $\mathfrak{V}_{w_i}$ the orthogonal complement to the kernel of ad_{w_i}. Therefore $\mathfrak{V}_{w_i} \subset \mathrm{span}\{\mathbf{B}_{r,s}\}$ and $\mathfrak{V}_{w_j} \subset \mathrm{span}\{\mathbf{A}_{r,s}\}$ as $[w_i , \mathbf{A}_{r,s}] = 0$ and $[w_j , \mathbf{B}_{r,s}] = 0$. It follows that

$$
\begin{aligned}
\mathrm{span}\{[X , \mathbf{A}_{r,s}]\} &= \mathrm{span}\{[w_i , \mathbf{A}_{r,s}] + [w_j , \mathbf{A}_{r,s}]\} = \mathrm{span}\{[w_j , \mathbf{A}_{r,s}]\} \supset [w_j , \mathfrak{V}_{w_j}] = \mathfrak{z}_{r,s}, \\
\mathrm{span}\{[X , \mathbf{B}_{r,s}]\} &= \mathrm{span}\{[w_i , \mathbf{B}_{r,s}] + [w_j , \mathbf{B}_{r,s}]\} = \mathrm{span}\{[w_i , \mathbf{B}_{r,s}]\} \supset [w_i , \mathfrak{V}_{w_i}] = \mathfrak{z}_{r,s} .
\end{aligned}
$$

Hence the map ad_X is surjective. □

Theorem 4.5.4. *The H-type algebras $\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{2,3}$ are not isomorphic.*

Proof. We assume that there exists an isomorphism $\varphi_{3,2}\colon \mathfrak{n}_{3,2} \to \mathfrak{n}_{2,3}$ where the restriction $\varphi_{3,2}|_{\mathfrak{z}_{3,2}}\colon \mathfrak{z}_{3,2} \to \mathfrak{z}_{2,3}$ is an anti-isometry. The adjoint operator $\mathrm{ad}_{w_i}\colon \mathfrak{V}_{w_i} \to \mathfrak{z}_{3,2}$ is an isometry or anti-isometry by Definition 1.1.6 for any $w_i \in \mathfrak{B}_{\mathfrak{v}_{3,2}}$, i.e.

$$\langle\, \mathrm{ad}_{w_i}(X)\, , \mathrm{ad}_{w_i}(X)\, \rangle_{\mathfrak{z}_{3,2}} = \langle\, w_i\, , w_i\, \rangle_{\mathfrak{v}_{3,2}} \langle\, X\, , X\, \rangle_{\mathfrak{v}_{3,2}}$$

for all $X \in \mathfrak{V}_{w_i}$, where $\mathfrak{V}_{w_i}$ is the orthogonal complement to the kernel of ad_{w_i}. As the map $\varphi_{3,2}|_{\mathfrak{z}_{3,2}}$ is an anti-isometry, it follows that the composition $\varphi_{3,2} \circ \mathrm{ad}_{w_i}\colon \mathfrak{V}_{w_i} \to \mathfrak{z}_{2,3}$ is an anti-isometry for $\langle\, w_i\, , w_i\, \rangle_{\mathfrak{v}_{3,2}} = 1$ and is an isometry for $\langle\, w_i\, , w_i\, \rangle_{\mathfrak{v}_{3,2}} = -1$, hence

$$\begin{aligned} -\langle\, w_i\, , w_i\, \rangle_{\mathfrak{v}_{3,2}} \langle\, w_j\, , w_j\, \rangle_{\mathfrak{v}_{3,2}} &= \langle\, \varphi_{3,2} \circ \mathrm{ad}_{w_i}(w_j)\, , \varphi_{3,2} \circ \mathrm{ad}_{w_i}(w_j)\, \rangle_{\mathfrak{z}_{2,3}} \\ &= \langle [\varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_j)]\, , [\varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_j)]\, \rangle_{\mathfrak{z}_{2,3}} . \end{aligned}$$

As the map $\varphi_{3,2} \circ \mathrm{ad}_{w_i}$ is surjective and $\varphi_{3,2} \circ \mathrm{ad}_{w_i}(w_j) = [\varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_j)]$ it follows by Proposition 4.5.1 that $\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}} \neq 0$ for all $i = 1, \dots , 8$.

We recall that from Definition 1.1.6 it follows that for all $X \in \mathfrak{v}_{r,s}$ with $\langle\, X\, , X\, \rangle_{\mathfrak{v}_{r,s}} \neq 0$ and $Y \in \mathfrak{V}_X$:

$$\langle\, \mathrm{ad}_X(Y)\, , \mathrm{ad}_X(Y)\, \rangle_{\mathfrak{z}_{r,s}} = \langle\, X\, , X\, \rangle_{\mathfrak{v}_{r,s}} \langle\, Y\, , Y\, \rangle_{\mathfrak{v}_{r,s}},$$

hence

$$\begin{aligned} &- \langle\, w_i\, , w_i\, \rangle_{\mathfrak{v}_{3,2}} \langle\, w_j\, , w_j\, \rangle_{\mathfrak{v}_{3,2}} \\ &= \langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}} \langle\, \varphi_{3,2}(w_j)\, , \varphi_{3,2}(w_j)\, \rangle_{\mathfrak{v}_{2,3}} . \end{aligned} \tag{4.24}$$

We obtain the following relations for w_1 and w_4:

$$\begin{aligned} \mathrm{sign}(\langle\, \varphi_{3,2}(w_1)\, , \varphi_{3,2}(w_1)\, \rangle_{\mathfrak{v}_{2,3}}) &= -\,\mathrm{sign}(\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}}), && \text{for } i = 2, 3, 4, \\ \mathrm{sign}(\langle\, \varphi_{3,2}(w_1)\, , \varphi_{3,2}(w_1)\, \rangle_{\mathfrak{v}_{2,3}}) &= \mathrm{sign}(\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}}), && \text{for } i = 5, 6, \\ \mathrm{sign}(\langle\, \varphi_{3,2}(w_4)\, , \varphi_{3,2}(w_4)\, \rangle_{\mathfrak{v}_{2,3}}) &= -\,\mathrm{sign}(\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}}), && \text{for } i = 1, 2, 3, \\ \mathrm{sign}(\langle\, \varphi_{3,2}(w_4)\, , \varphi_{3,2}(w_4)\, \rangle_{\mathfrak{v}_{2,3}}) &= \mathrm{sign}(\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}}), && \text{for } i = 5, 6. \end{aligned}$$

It implies that for $i = 2, 3$

$$\begin{aligned} -\,\mathrm{sign}(\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}}) &= \mathrm{sign}(\langle\, \varphi_{3,2}(w_1)\, , \varphi_{3,2}(w_1)\, \rangle_{\mathfrak{v}_{2,3}}) \\ &= -\,\mathrm{sign}(\langle\, \varphi_{3,2}(w_4)\, , \varphi_{3,2}(w_4)\, \rangle_{\mathfrak{v}_{2,3}}) \\ &= \mathrm{sign}(\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}}). \end{aligned}$$

Hence $\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}} = 0$ for $i = 2, 3$. This contradicts

$$\langle\, \varphi_{3,2}(w_i)\, , \varphi_{3,2}(w_i)\, \rangle_{\mathfrak{v}_{2,3}} \neq 0, \qquad \text{for } i = 1, \dots , 8,$$

as the map $\mathrm{ad}_{\varphi_{3,2}(w_i)}$ is surjective.

Hence $\mathfrak{n}_{3,2}$ is not isomorphic to $\mathfrak{n}_{2,3}$. □

4.5.2 Special features of the Lie algebra $\mathfrak{n}_{3,3}$

We introduce an integral basis of $\mathfrak{n}_{3,3}$ by $\mathfrak{B}_{\mathfrak{z}_{3,3}} = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$ and

$$\mathfrak{B}_{\mathfrak{v}_{3,3}} = \left\{ \begin{array}{llll} w_1 := w, & w_2 := J_1 w, & w_3 := J_2 w, & w_4 := J_3 w, \\ w_5 := J_1 J_6 w, & w_6 := J_6 w, & w_7 := J_4 w, & w_8 := J_5 w, \end{array} \right\},$$

$$\text{for} \quad J_2 J_3 J_4 J_5 w = J_1 J_2 J_5 J_6 w = J_1 J_2 J_3 w = w,$$
$$\text{and} \quad \langle w_i \,, w_i \rangle_{\mathfrak{v}_{3,3}} = \epsilon_i(4,4), \qquad \langle\, Z_k \,, Z_k \,\rangle_{\mathfrak{z}_{3,3}} = \epsilon_k(3,3).$$

Table 4.9: Commutation relations on $\mathfrak{n}_{3,3}$

$[row\,, col.]$	w_1	w_2	w_5	w_6	w_3	w_4	w_7	w_8
w_1	0	Z_1	0	Z_6	Z_2	Z_3	Z_4	Z_5
w_2	$-Z_1$	0	$-Z_6$	0	$-Z_3$	Z_2	$-Z_5$	Z_4
w_5	0	Z_6	0	Z_1	Z_5	Z_4	Z_3	Z_2
w_6	$-Z_6$	0	$-Z_1$	0	Z_4	$-Z_5$	Z_2	$-Z_3$
w_3	$-Z_2$	Z_3	$-Z_5$	$-Z_4$	0	$-Z_1$	Z_6	0
w_4	$-Z_3$	$-Z_2$	$-Z_4$	Z_5	Z_1	0	0	$-Z_6$
w_7	$-Z_4$	Z_5	$-Z_3$	$-Z_2$	$-Z_6$	0	0	Z_1
w_8	$-Z_5$	$-Z_4$	$-Z_2$	Z_3	0	Z_6	$-Z_1$	0

Proposition 4.5.5. *The linear map* $\mathrm{ad}_X\colon\ \mathfrak{v}_{3,3} \to \mathfrak{z}_{3,3}$ *is surjective if and only if* $\langle\, X \,, X \,\rangle_{\mathfrak{v}_{3,3}} \neq 0$ *for* $X \in \mathfrak{v}_{3,3}$.

Proof. We use similar arguments as in the proof of Proposition 4.5.1. The matrix M_X that we calculate by using Table 4.9 is given by

$$\begin{pmatrix} -\lambda_2 & \lambda_1 & -\lambda_6 & \lambda_5 & \lambda_4 & -\lambda_3 & -\lambda_8 & \lambda_7 \\ -\lambda_3 & -\lambda_4 & -\lambda_8 & -\lambda_7 & \lambda_1 & \lambda_2 & \lambda_6 & \lambda_5 \\ -\lambda_4 & \lambda_3 & -\lambda_7 & \lambda_8 & -\lambda_2 & \lambda_1 & \lambda_5 & -\lambda_6 \\ -\lambda_7 & -\lambda_8 & -\lambda_4 & -\lambda_3 & \lambda_6 & \lambda_5 & \lambda_1 & \lambda_2 \\ -\lambda_8 & \lambda_7 & -\lambda_3 & \lambda_4 & \lambda_5 & -\lambda_6 & -\lambda_2 & \lambda_1 \\ -\lambda_6 & \lambda_5 & -\lambda_2 & \lambda_1 & -\lambda_7 & \lambda_8 & \lambda_3 & -\lambda_4 \end{pmatrix}.$$

The determinant of $M_X M_X^T$ has the form

$$\begin{aligned} & (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 - \lambda_5^2 - \lambda_6^2 - \lambda_7^2 - \lambda_8^2)^4 \\ \times \quad & ((\lambda_1 - \lambda_5)^2 + (\lambda_2 - \lambda_6)^2 + (\lambda_4 - \lambda_7)^2 + (\lambda_3 - \lambda_8)^2) \\ \times \quad & ((\lambda_1 + \lambda_5)^2 + (\lambda_2 + \lambda_6)^2 + (\lambda_4 + \lambda_7)^2 + (\lambda_3 + \lambda_8)^2). \end{aligned}$$

Thus the map ad_X is surjective if and only if $\langle\, X \,, X \,\rangle_{\mathfrak{v}_{3,3}} \neq 0$. □

Theorem 4.5.6. *There does not exist an automorphism* $\varphi_{3,3}$ *of* $\mathfrak{n}_{3,3}$ *such that the restriction to the center* $\varphi|_{\mathfrak{z}_{3,3}}$ *is an anti-isometry.*

Proof. By repeating the arguments of the proof of Theorem 4.5.4, we obtain equation (4.24). This implies the relations

$$\begin{array}{rcll}
\operatorname{sign}(\langle \varphi_{3,3}(w_1), \varphi_{3,3}(w_1)\rangle_{\mathfrak{v}_{3,3}}) &=& -\operatorname{sign}(\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}}), & \text{for } i = 2,3,4,\\
\operatorname{sign}(\langle \varphi_{3,3}(w_1), \varphi_{3,3}(w_1)\rangle_{\mathfrak{v}_{3,3}}) &=& \operatorname{sign}(\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}}), & \text{for } i = 6,7,8,\\
\operatorname{sign}(\langle \varphi_{3,3}(w_2), \varphi_{3,3}(w_2)\rangle_{\mathfrak{v}_{3,3}}) &=& -\operatorname{sign}(\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}}), & \text{for } i = 1,3,4,\\
\operatorname{sign}(\langle \varphi_{3,3}(w_2), \varphi_{3,3}(w_2)\rangle_{\mathfrak{v}_{3,3}}) &=& \operatorname{sign}(\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}}), & \text{for } i = 5,7,8.
\end{array}$$

Thus, for $i = 3, 4$

$$\begin{aligned}
-\operatorname{sign}(\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}}) &= \operatorname{sign}(\langle \varphi_{3,3}(w_1), \varphi_{3,3}(w_1)\rangle_{\mathfrak{v}_{3,3}})\\
&= -\operatorname{sign}(\langle \varphi_{3,3}(w_2), \varphi_{3,3}(w_2)\rangle_{\mathfrak{v}_{3,3}})\\
&= \operatorname{sign}(\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}}).
\end{aligned}$$

Hence $\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}} = 0$ for $i = 3, 4$. This contradicts to the fact that $\langle \varphi_{3,3}(w_i), \varphi_{3,3}(w_i)\rangle_{\mathfrak{v}_{3,3}} \neq 0$ for $i = 1, \ldots, 8$, as the map $\operatorname{ad}_{\varphi_{3,3}(w_i)}$ is surjective. Hence there does not exist an automorphism $\varphi_{3,3}$ of $\mathfrak{n}_{3,3}$ such that $\varphi|_{\mathfrak{z}_{3,3}}$ is an anti-isometry. $\square$

Proposition 4.5.7. *For the pseudo H-type algebras $\mathfrak{n}_{11,3}$, $\mathfrak{n}_{7,7}$ and $\mathfrak{n}_{3,11}$ there exists X in respective $\mathfrak{v}_{r,s}$ such that ad_X is surjective.*

Proof. We repeat the proof of Lemma 4.5.2 by replacing $w_1, w_7 \in \mathfrak{v}_{2,3}$ by $w_1, w_5 \in \mathfrak{v}_{3,3}$. $\square$

4.6 Strongly bracket generating property

In this section we study the bracket generating property of the pseudo H-type algebras. For that purpose we use the equivalent Definition 1.1.6 of the pseudo H-type algebras $\mathfrak{n}_{r,s}$, which is related to the definition of the strongly bracket generating property.

Definition 4.6.1. *Let $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$ be a pseudo H-type algebra. We call a vector space $\mathfrak{v}_{r,s}$ strongly bracket generating if for any non-zero $v \in \mathfrak{v}_{r,s}$ the linear map $\operatorname{ad}_v = [v, \cdot]\colon \mathfrak{v}_{r,s} \to \mathfrak{z}_{r,s}$ is surjective, i.e. $\operatorname{span}\{\mathfrak{v}_{r,s}, [v, \mathfrak{v}_{r,s}]\} = \mathfrak{n}_{r,s}$ for all $v \in \mathfrak{v}_{r,s} \setminus \{0\}$. We say in this case that the pseudo H-type algebra $\mathfrak{n}_{r,s}$ has the strongly bracket generating property.*

Let $N_{r,s}$ be the Lie group, corresponding to the pseudo H-type algebra $\mathfrak{n}_{r,s}$ and let $\mathcal{H}$ be the left translation of the vector space $\mathfrak{v}_{r,s}$. If $\mathfrak{v}_{r,s}$ is strongly bracket generating, then the left invariant distribution $\mathcal{H}$ is strongly bracket generating in a sense that $\operatorname{span}\{\mathcal{H}, [X, \mathcal{H}]\} = TN_{r,s}$ for any smooth non-zero section X of the distribution $\mathcal{H}$.

Even the strongly bracket generating property seems to be just of interest from a geometrical point of view, it actually has a close relation to the equivalent Definition 1.1.6 of pseudo H-type algebras, which can be seen in the following subsection.

4.6.1 An equivalent definition of pseudo H-type algebras

In this subsection we prove that the general H-type algebras [51] are equivalent to the pseudo H-type algebras. But first we discuss the related topic of the composition of scalar products.

Let $(U, \langle \cdot\,, \cdot\rangle_U)$, $(V, \langle \cdot\,, \cdot\rangle_V)$ be two vector spaces with corresponding non-degenerate quadratic forms, written as bi-linear symmetric forms, or scalar products.

Definition 4.6.2. *A bilinear map $\mu\colon U \times V \to V$ is called a composition of the scalar products $\langle \cdot\,, \cdot\rangle_U$ of U and $\langle \cdot\,, \cdot\rangle_V$ of V if the equality*

$$\langle \mu(u, v)\,, \mu(u, v)\rangle_V = \langle u\,, u\rangle_U \langle v\,, v\rangle_V \tag{4.25}$$

holds for any $u \in U$ and $v \in V$.

We assume that there is $u_0 \in U$ such that $\langle u_0\,, u_0\rangle_U = 1$ and $\mu(u_0, v) = v$. This can always be done by normalization procedure of quadratic forms, see [63]. Let us denote by $\mathcal{Z}$ the orthogonal complement of the non-degenerate space $\operatorname{span}\{u_0\}$ and by J the restriction of μ to $\mathcal{Z}$, thus $J\colon \mathcal{Z} \times V \to V$. The map J is skew-adjoint in the sense that $\langle J(Z, v)\,, v'\rangle_V = -\langle v\,, J(Z, v')\rangle_V$ for any $Z \in \mathcal{Z}$ and $v, v' \in V$. Therefore, the map J can be used to define a Lie algebra structure on $\mathfrak{n} = \mathcal{Z} \oplus V$ by $\langle J(Z, v)\,, v'\rangle_V = \langle Z\,, [v\,, v']\rangle_{\mathcal{Z}}$. The obtained Lie algebra is a general H-type algebra, see [51, Theorem 1]. Now, rephrasing Definition 1.1.5 of a pseudo H-type algebra, we can say that a two-step nilpotent Lie algebra is a pseudo H-type algebra if the map J defined by (1.1.4) is the restriction to the center $\mathcal{Z} = \mathfrak{z}$ of a composition of corresponding quadratic forms for vector spaces $V = \mathfrak{v}$, $U = \operatorname{span}\{u_0\} \oplus_\perp \mathcal{Z}$.

Theorem 4.6.3. *Definitions 1.1.5 and 1.1.6 are equivalent.*

Proof. Let us prove that Definition 1.1.6 implies Definition 1.1.5. It was shown in [51, Theorem 1] that any general H-type algebra $\mathfrak{n} = \big(\mathfrak{v} \oplus_\perp \mathfrak{z}, [\cdot\,, \cdot], \langle \cdot\,, \cdot\rangle_{\mathfrak{n}} = \langle \cdot\,, \cdot\rangle_{\mathfrak{v}} + \langle \cdot\,, \cdot\rangle_{\mathfrak{z}}\big)$ defines a composition of the quadratic form $\langle \cdot\,, \cdot\rangle_{\mathfrak{v}}$ and another quadratic form whose restriction to $\mathfrak{z}$ coincides with $\langle \cdot\,, \cdot\rangle_{\mathfrak{z}}$. Particularly, it implies (1.2) and therefore a general H-type algebra is a pseudo H-type algebra.

Now we assume that we are given a pseudo H-type algebra $\mathfrak{n} = \big(\mathfrak{v} \oplus_\perp \mathfrak{z}, [\cdot\,, \cdot], \langle \cdot\,, \cdot\rangle_{\mathfrak{n}} = \langle \cdot\,, \cdot\rangle_{\mathfrak{v}} + \langle \cdot\,, \cdot\rangle_{\mathfrak{z}}\big)$ with center $\mathfrak{z}$. Let us fix $v \in \mathfrak{v}$ with $\langle v\,, v\rangle_{\mathfrak{v}} = \pm 1$. We need to show that $\operatorname{ad}_v\colon \mathfrak{v} \to \mathfrak{z}$ is a surjective (anti-)isometry. The following equation is true

$$\langle Z\,, \operatorname{ad}_v(J_{Z'}v)\rangle_{\mathfrak{z}} = \langle Z\,, [v\,, J_{Z'}v]\rangle_{\mathfrak{z}} = \langle J_Z v\,, J_{Z'}v\rangle_{\mathfrak{v}} = \langle Z\,, Z'\rangle_{\mathfrak{z}} \langle v\,, v\rangle_{\mathfrak{v}} = \pm\langle Z\,, Z'\rangle_{\mathfrak{z}},$$

for all $Z, Z' \in \mathfrak{z}$ by formula (1.4). We use both notation $J(Z, v)$ and $J_Z v$. This implies $\operatorname{ad}_v(J_{Z'}v) = \pm Z'$ for all $Z' \in \mathfrak{z}$ by the non-degenerate property of the scalar product $\langle \cdot\,, \cdot\rangle_{\mathfrak{z}}$. Since $\langle J_{Z'}v\,, w\rangle_{\mathfrak{v}} = \langle Z'\,, [v\,, w]\rangle_{\mathfrak{z}} = \langle Z'\,, 0\rangle_{\mathfrak{z}} = 0$, for all $w \in \ker(\operatorname{ad}_v)$, it follows that $J_{Z'}v \in \mathfrak{V}_v = (\ker(\operatorname{ad}_v))^\perp$. We showed that ad_v is surjective.

To prove that ad_v is an isometry for $\langle v\,, v\rangle_{\mathfrak{v}} = 1$ and an anti-isometry for $\langle v\,, v\rangle_{\mathfrak{v}} = -1$, we exhibit that the maps $\operatorname{ad}_v\colon \mathfrak{V}_v \to \mathfrak{z}$ and $J_{(\cdot)}v\colon \mathfrak{z} \to \mathfrak{V}_v$ are inverse and then equality (1.2) implies the isometry and anti-isometry properties. Let us assume that $\langle v\,, v\rangle_{\mathfrak{v}} = 1$. We proved that $\mathrm{ad}_v\colon \mathfrak{V}_v \to \mathfrak{z}$ is bijective, thus the image of $J_{(\cdot)}v$ belongs to $\mathfrak{V}_v$, and $\mathrm{ad}_v(J_{(\cdot)}v) = \mathrm{Id}_{\mathfrak{z}}$, where $\mathrm{Id}_{\mathfrak{z}}$ is the identity map on $\mathfrak{z}$.

We claim that the map $J_{(\cdot)}v\colon \mathfrak{z} \to \mathfrak{V}_v$ is bijective. Indeed if we assume that $J_{(\cdot)}v$ is not surjective, then there is $w \in \mathfrak{V}_v$ which is not in the image of $J_{(\cdot)}v$. Let $\mathrm{ad}_v(w) = Z \in \mathfrak{z}$, then $\mathrm{ad}_v(J_Z v) = Z$ which implies $w = J_Z v$ by injectivity of ad_v and leads to contradiction.

If we now assume that $J_{(\cdot)}v$ is not injective, then we find $Z', Z'' \in \mathfrak{z}$, $Z' \neq Z''$, such that $J(Z', v) = J(Z'', v)$. But in this case $Z' = \mathrm{ad}_X(J_{Z'}v) = \mathrm{ad}_v(J_{Z''}v) = Z''$ by bijectivity of ad_v and we again get a contradiction. The proof for $\langle v\,, v\rangle_{\mathfrak{v}} = -1$ is analogous and we conclude that ad_v and $J_{(\cdot)}v$ are inverse maps to each other. The equality (1.2) becomes

$$\langle J_Z v\,, J_Z v\rangle_{\mathfrak{v}} = \langle Z\,, Z\rangle_{\mathfrak{z}} \text{ for } \langle v\,, v\rangle_{\mathfrak{v}} = 1, \text{ and } \langle J_Z v\,, J_Z v\rangle_{\mathfrak{v}} = -\langle Z\,, Z\rangle_{\mathfrak{z}} \text{ for } \langle v\,, v\rangle_{\mathfrak{v}} = -1,$$

which shows the (anti-)isometry property of the map $J_{(\cdot)}v\colon \mathfrak{z} \to \mathfrak{V}_v$ and its inverse $\mathrm{ad}_v\colon \mathfrak{V}_v \to \mathfrak{z}$. □

4.6.2 Bracket generating property of pseudo H-type algebras

Theorem 4.6.4. *The pseudo H-type algebras $\mathfrak{n}_{r,s}$ with $r = 0$ or $s = 0$ have the strongly bracket generating property.*

Proof. Let $s = 0$. This implies that $\langle v\,, v\rangle_{\mathfrak{v}_{r,0}} > 0$ for all $v \in \mathfrak{v}_{r,0}$ with $v \neq 0$. Definition 1.1.6 yields that ad_v is surjective, i.e. $\mathfrak{v}_{r,0}$ is strongly bracket generating.

Let $r = 0$. Recall that $\mathfrak{v}_{0,s}$ is a neutral space, i.e. $\langle \cdot\,, \cdot\rangle_{\mathfrak{v}_{0,s}}$ has index (l, l) and we can identify $\mathfrak{v}_{0,s}$ with $\mathbb{R}^{l,l}$. This implies that there exists elements $v \in \mathfrak{v}_{0,s}$, $v \neq 0$, with $\langle v\,, v\rangle_{l,l} = 0$. According to Definition 1.1.6 we only need to show that $\mathrm{ad}_v\colon \mathfrak{v}_{r,s} \to \mathfrak{z}_{r,s}$ is surjective for vectors with $\langle v\,, v\rangle_{l,l} = 0$, since for all other vectors the adjoint map is surjective.

We define the orthonormal basis $\{w_1, \dots, w_{2l}\}$ of $\mathfrak{v}_{0,s}$ with $\langle\, w_i\,, w_i\,\rangle_{\mathfrak{v}_{0,s}} = \epsilon_i(l, l)$ and fix an arbitrary $v \in \mathfrak{v}_{0,s}$ with $\langle v\,, v\rangle_{l,l} = 0$ and $v = \sum_{i=1}^{2l} \lambda_i w_i$. We split v in the form $v = v^+ + v^-$, with $v^+ = \sum_{i=1}^{l} \lambda_i w_i$, $v^- = \sum_{i=l+1}^{2l} \lambda_i w_i$ and $\langle v^+\,, v^+\rangle_{l,l} = -\langle v^-\,, v^-\rangle_{l,l} > 0$ and $\langle v^+\,, v^-\rangle_{l,l} = 0$. We note that $[w_i\,, w_j] = 0$ if $i, j = 1, \dots, l$ or $i, j = l+1, \dots, 2l$ as

$$\langle [w_i\,, w_j]\,, [w_i\,, w_j]\,\rangle_{\mathfrak{z}_{0,s}} \geq 0, \quad \text{for } i, j = 1, \dots, l, \quad \text{or} \quad i, j = l+1, \dots, 2l.$$

Hence $\mathfrak{z}_{0,s} = \mathrm{ad}_{w_i}(\mathfrak{v}_{0,s}) = \mathrm{ad}_{w_i}(\mathrm{span}\{w_1, \dots, w_l\})$ for $i = l+1, \dots, 2l$. It follows that

$$[v\,, \mathrm{span}\{w_1, \dots, w_l\}] = [v^-\,, \mathrm{span}\{w_1, \dots, w_l\}] = \mathfrak{z}_{0,s}\,.$$

Hence ad_v is surjective, i.e. the pseudo H-type algebras $\mathfrak{n}_{0,s} = \mathfrak{v}_{0,s} \oplus \mathfrak{z}_{0,s}$, where $s > 0$, have the strongly bracket generating property.

□

Theorem 4.6.5. *The pseudo H-type algebras $\mathfrak{n}_{r,s}$ with $r, s \neq 0$ do not have the strongly bracket generating property.*

Proof. We assume that $\mathfrak{n}_{r,s} = \mathfrak{v}_{r,s} \oplus \mathfrak{z}_{r,s}$ with $r, s \neq 0$ has the strongly bracket generating property, i.e. for all $v \in \mathfrak{v}_{r,s}$: $[v\,, \mathfrak{v}_{r,s}] = \mathfrak{z}_{r,s}$ and we show that it contradicts the presence of nullvectors in the scalar product space $(\mathfrak{z}_{r,s}, \langle\,\cdot\,,\cdot\,\rangle_{r,s})$. The non-degenerate property of the indefinite scalar-product $\langle\,\cdot\,,\cdot\,\rangle_{r,s}$ implies that for all $v \in \mathfrak{v}_{r,s}$ and for all $Z \in \mathfrak{z}_{r,s}$ there exists $v_Z \in \mathfrak{v}_{r,s}$ such that

$$\langle [v\,, v_Z]\,, Z\,\rangle_{r,s} \neq 0 \qquad \stackrel{(1.1)}{\Longleftrightarrow} \qquad \langle\, J_Z(v)\,, v_Z\,\rangle_{\mathfrak{v}_{r,s}} \neq 0.$$

It follows that $J_Z(v) \neq 0$ for all $v \in \mathfrak{v}_{r,s}$ and for all $Z \in \mathfrak{z}_{r,s}$, i.e. $\ker\{J_Z\} = \{0\}$ for all $Z \in \mathfrak{z}_{r,s}$. But there exist elements $Z_0 \in \mathfrak{z}_{r,s}$ such that $\langle\, Z_0\,, Z_0\,\rangle_{r,s} = 0$ as $r, s \neq 0$. This implies that $J_{Z_0}^2 = 0$ which is equivalent to $\ker\{J_{Z_0}\} \neq \{0\}$. This is a contradiction, hence the pseudo H-type algebras $\mathfrak{n}_{r,s}$ with $r, s \neq 0$ do not have the strongly bracket generating property. $\square$

We shortly want to degrade the widely misconception that 2-step nilpotent Lie algebras are in general strongly bracket generating by giving one more counterexample besides Theorem 4.6.5. The example is known in the community of sub-Riemannian geometry.

Example 4.6.6. *We consider the free Lie algebra $F_{r,2}$ of step 2, i.e.*

$$F_{r,2} = \mathrm{span}\{X_i, [X_k\,, X_l] \quad | \quad 1 \leq i \leq r, \quad 1 \leq k < l \leq r\},$$

where X_i and $[X_k\,, X_l]$ are linear independent for all $1 \leq i \leq r$, $1 \leq k < l \leq r$. The dimension of $F_{r,2}$ is $\binom{r}{2} + r$.

We claim that the subbundle $\mathrm{span}\{X_i | 1 \leq i \leq r\}$ is not strongly bracket generating for $r \geq 3$.

We know that the rank of ad_{X_1} is $r-1$. It follows that if

$$r - 1 < \binom{r}{2} = \frac{r(r-1)}{2} \Leftrightarrow 1 < \frac{r}{2},$$

then the subbundle $\mathrm{span}\{X_i | 1 \leq i \leq r\}$ cannot be strongly bracket generating. This inequality is fulfilled for $r \geq 3$.

4.7 Non-isomorphism properties for pseudo H-type groups in general position

In this section we discuss the possible extension of our results to pseudo H-type algebras, constructed from non-minimal admissible Clifford modules. Here we need to distinguish two essentially different situations. The first one when the irreducible module is unique (up to equivalence) and other one when there are two non-equivalent irreducible modules. We introduce some new notation.

1. Let the Clifford algebras $\mathrm{Cl}_{r,s}$ admit only one (up to equivalence) irreducible module and we write $\mathfrak{v}_{r,s}$ for the minimal admissible module, that could be a direct sum of two irreducible modules. This situation occurs when $r-s \neq 3 \mod (4)$. Any non-minimal admissible $\mathrm{Cl}_{r,s}$-module $\mathfrak{v}$ is isomorphic (and isometric) to the direct sum of minimal admissible modules $\mathfrak{v}_{r,s}$, see [46, 64]:

$$\mathfrak{v} = \mathfrak{v}_{r,s}(\mu) \cong \oplus^{\mu}\, \mathfrak{v}_{r,s}\,.$$

Here and further on we use the notation $\mathfrak{v}_{r,s}(\mu)$ for the μ-fold direct sum of minimal admissible modules $\mathfrak{v}_{r,s}$. Thus the argument μ shows how many equivalent (in the sense of representation theory) minimal admissible modules contains the sum. The lower index, as previously, indicates the index of the metric of the generating space for the Clifford algebra.

2. If $r-s = 3 \mod (4)$, then the Clifford algebra $\mathrm{Cl}_{r,s}$ admits two non-equivalent Clifford modules. We write $\mathfrak{v}^1_{r,s}$ and $\mathfrak{v}^2_{r,s}$ for the minimal admissible modules. Recall, that in this case each of the admissible modules $\mathfrak{v}^l_{r,s}$, $l = 1,2$ is either irreducible, or the direct sum of two equivalent irreducible modules, where the representation map is changed appropriately [37]. We emphasize that a minimal admissible module $\mathfrak{v}^k_{r,s}$ can not be a direct sum of two non-equivalent irreducible modules. In this case a non-minimal admissible $\mathrm{Cl}_{r,s}$-module $\mathfrak{v}$ is isomorphic to

$$\mathfrak{v} = \mathfrak{v}_{r,s}(\mu,\nu) \cong (\oplus^{\mu}\, \mathfrak{v}^1_{r,s}) \bigoplus (\oplus^{\nu}\, \mathfrak{v}^2_{r,s})$$

for some positive integers μ,ν which show the number of equivalent and non-equivalent minimal admissible modules contained in the admissible module $\mathfrak{v}_{r,s}(\mu,\nu)$. To unify the notation we always write $\mathfrak{v}_{r,s}(\mu,\nu)$, where $\nu = 0$ if $r-s \neq 3 \mod (4)$ and ν can be different from zero in the case $r-s = 3 \mod (4)$. According to this new notation we also write $\mathfrak{n}_{r,s}(\mu,\nu)$ for a pseudo H-type algebra in the case that it is isomorphic to the direct sum $\mathfrak{v}_{r,s}(\mu,\nu) \oplus \mathfrak{z}_{r,s}$.

Results of [46] imply that a non-minimal admissible module $(\mathfrak{v}_{r,s}(\mu,\nu), \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}(\mu,\nu)})$ of the Clifford algebra $\mathrm{Cl}_{r,s}$ is given as an orthogonal sum of n-dimensional minimal admissible modules $\mathfrak{v}_{r,s} = (\mathfrak{v}_{r,s}, \langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}})$, where each scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}}$ is the restriction of $\langle\cdot\,,\cdot\rangle_{\mathfrak{v}_{r,s}(\mu,\nu)}$ on the corresponding copy of the vector space $\mathfrak{v}_{r,s}$. To describe the Lie bracket on $\mathfrak{n}_{r,s}(\mu,\nu)$ we proceed as follows. Let $\{Z_1,\dots,Z_m\}$ be an orthonormal basis of $\mathfrak{z}_{r,s}$. We denote a basis of the j-term in the sum $\oplus_{j=1}^{\mu}\mathfrak{v}^l_{r,s}$, $l = 1,2$ by $\{v^l_{1j},\dots,v^l_{nj}\}$ with structure constants $(A^k_{ip})^l_j$. For the sum $\mathfrak{n}_{r,s}(\mu,\nu) = \big((\oplus_{j=1}^{\mu}(\mathfrak{v}^1_{r,s})_j) \bigoplus (\oplus_{q=1}^{\nu}(\mathfrak{v}^2_{r,s})_q)\big) \oplus \mathfrak{z}_{r,s}$ we choose the basis

$$\{v^1_{ij}, v^2_{pq}, Z_k \,|\, i,p = 1,\dots,n,\ k = 1,\dots,r+s,\ j = 1,\dots,\mu,\ q = 1,\dots,\nu\}. \tag{4.26}$$

The Lie bracket on $\mathfrak{n}_{r,s}(\mu,\nu)$ with respect to this basis is given by

$$[w^{l_1}_{ij}\,, w^{l_2}_{pq}] = \delta_{l_1 l_2}\delta_{jq}\sum_{k=1}^{r+s}(A^k_{ip})^{l_t}_j Z_k, \quad t = 1,2. \tag{4.27}$$

The bilinear maps $J^l_j\colon \mathfrak{z}_{r,s} \times (\mathfrak{v}^l_{r,s})_j \to (\mathfrak{v}^l_{r,s})_j$, $l = 1, 2$ are defined by a representation of $\mathrm{Cl}_{r,s}$ over $(\mathfrak{v}^l_{r,s})_j$ and are extended to $\tilde{J}\colon \mathfrak{z}_{r,s} \times \mathfrak{v}_{r,s}(\mu,\nu) \to \mathfrak{v}_{r,s}(\mu,\nu)$ by

$$\tilde{J} := \begin{pmatrix} J^1_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J^2_\nu \end{pmatrix}.$$

Then the operator $\tilde{J}\colon \mathfrak{z}_{r,s} \to \mathrm{End}(\mathfrak{v}_{r,s}(\mu,\nu))$ satisfies $\langle \tilde{J}_Z v , w \rangle_{\mathfrak{v}_{r,s}(\mu,\nu)} = \langle Z , [v , w] \rangle_{\mathfrak{z}_{r,s}}$, for all $Z \in \mathfrak{z}_{r,s}$, $v, w \in \mathfrak{v}_{r,s}(\mu,\nu)$ and can be extended to the representation of $\mathrm{Cl}_{r,s}$ over $\mathfrak{v}_{r,s}(\mu,\nu)$. We can assume, by a change of coordinates, without loss of generality, that $J^1_1 = \ldots = J^1_\mu$ and $J^2_1 = \ldots = J^2_\nu$. If a Clifford algebra $\mathrm{Cl}_{r,s}$ admits only one irreducible representation, then the notation simplifies due to the absence of upper indices l_t.

We start from general observations where the first one follows easily from the dimension argument.

Proposition 4.7.1. *Two pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu_1, 0)$ and $\mathfrak{n}_{r,s}(\mu_2, 0)$ for $r - s \neq 3 \mod (4)$ are isomorphic if and only if $\mu_1 = \mu_2$.*

Theorem 4.7.2. *Two pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu_1, \nu_1)$ and $\mathfrak{n}_{r,s}(\mu_2, \nu_2)$ for $r - s = 3 \mod (4)$ are isomorphic if and only if $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$ or $\mu_1 = \nu_2$ and $\nu_1 = \mu_2$.*

Proof. In the first step we show that $\mathfrak{n}_{r,s}(\mu, 0)$ and $\mathfrak{n}_{r,s}(0, \mu)$ are isomorphic for $r - s = 3 \mod (4)$. Let $J^1\colon \mathfrak{v}^1_{r,s} \oplus \mathfrak{z}_{r,s} \to \mathfrak{v}^1_{r,s}$ and $J^2\colon \mathfrak{v}^2_{r,s} \oplus \mathfrak{z}_{r,s} \to \mathfrak{v}^2_{r,s}$ be two non-equivalent representations over two minimal admissible modules. Let $\mathfrak{n}_{r,s}(1, 0) = (\mathfrak{v}^1_{r,s} \oplus \mathfrak{z}_{r,s}, [\cdot\,,\cdot]^1)$ and $\mathfrak{n}_{r,s}(0, 1) = (\mathfrak{v}^2_{r,s} \oplus \mathfrak{z}_{r,s}, [\cdot\,,\cdot]^2)$ be the pseudo H-type algebras, where we used the maps J^1 and J^2 to define the corresponding brackets by (1.1.4). We can assume that the vector spaces $\mathfrak{v}^1_{r,s}$ and $\mathfrak{v}^2_{r,s}$ are isomorphic under an isomorphism $A\colon \mathfrak{v}^1_{r,s} \to \mathfrak{v}^2_{r,s}$. We define a map $C\colon \mathfrak{z}_{r,s} \to \mathfrak{z}_{r,s}$ by

$$J^1(v, C(Z)) = A^\tau \circ J^2(A(v), Z), \quad \text{for any} \quad v \in \mathfrak{v}^1_{r,s}, \quad Z \in \mathfrak{z}_{r,s}, \tag{4.28}$$

where $\langle Av , u\rangle_{\mathfrak{v}^2_{r,s}} = \langle v , A^\tau u\rangle_{\mathfrak{v}^1_{r,s}}$. We claim that the map $F = A \oplus C^\tau\colon \mathfrak{v}^1_{r,s} \oplus \mathfrak{z}_{r,s} \to \mathfrak{v}^2_{r,s} \oplus \mathfrak{z}_{r,s}$ is a Lie algebra isomorphism $F\colon \mathfrak{n}_{r,s}(1, 0) \to \mathfrak{n}_{r,s}(0, 1)$, where C^τ is the adjoint map to C with respect to the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{z}_{r,s}}$. Indeed, the chain of equalities

$$\begin{aligned} \langle Z , C^\tau([v , w]^1)\rangle_{\mathfrak{z}_{r,s}} &= \langle C(Z) , [v , w]^1\rangle_{\mathfrak{z}_{r,s}} = \langle J^1_{C(Z)} v , w\rangle_{\mathfrak{v}^1_{r,s}} = \langle A^\tau \circ J^2_Z(Av) , w\rangle_{\mathfrak{v}^1_{r,s}} \\ &= \langle J^2_Z(Av) , Aw\rangle_{\mathfrak{v}^2_{r,s}} = \langle Z , [Av , Aw]^2\rangle_{\mathfrak{z}_{r,s}} \end{aligned}$$

for any $v, w \in \mathfrak{v}^1_{r,s}$, $Z \in \mathfrak{z}_{r,s}$ shows that $F([v , w]^1) = C^\tau([v , w]^1) = [Av , Aw]^2 = [Fv , Fw]^2$.

To show that the Lie algebras $\mathfrak{n}_{r,s}(\mu, \nu)$ and $\mathfrak{n}_{r,s}(\nu, \mu)$ are isomorphic, we choose the map $A\colon \mathfrak{v}^1_{r,s} \to \mathfrak{v}^2_{r,s}$ to be not only the isomorphism of vector spaces, but also an isometry between the admissible modules. It, particularly, implies that $A^\tau = A^{-1}$. The corresponding map $C\colon \mathfrak{z}_{r,s} \to \mathfrak{z}_{r,s}$ will also be an isometry by Theorem 4.2.5. We fix an orthonormal basis $Z_1, \ldots, Z_{r+s}$ of $\mathfrak{z}_{r,s}$, then the set $C(Z_1), \ldots, C(Z_{r+s})$ also forms an orthonormal basis. We construct an integral basis $v^2_{11}, \ldots, v^2_{n1}$ of $\mathfrak{v}^2_{r,s}$ by using the map $J^2\colon \mathfrak{z}_{r,s} \oplus \mathfrak{v}^2_{r,s} \to \mathfrak{v}^2_{r,s}$ and the orthonormal basis $Z_1, \ldots, Z_{r+s}$ as it was done in [46]. Then, by making use of the same method, we obtain the integral basis $v^1_{11}, \ldots, v^1_{n1}$ constructed from the orthonormal basis $C(Z_1), \ldots, C(Z_{r+s})$ and the map $J^1\colon \mathfrak{z}_{r,s} \oplus \mathfrak{v}^1_{r,s} \to \mathfrak{v}^1_{r,s}$. By the choice of the map A we get $\prod_{k=1}^{l} J^1_{C(Z_{i_k})} = A^{-1} \circ \Big(\prod_{k=1}^{l} J^2_{Z_{i_k}} \Big) \circ A$ for any choice of orthonormal generators $Z_{i_1}, \ldots, Z_{i_l}$ in $\mathfrak{z}_{r,s}$. It guarantees that there is a vector $v \in \mathfrak{v}^1_{r,s}$ such that $\langle v\,, v\rangle_{\mathfrak{v}^1_{r,s}} = \langle Av\,, Av\rangle_{\mathfrak{v}^2_{r,s}}$ and $\prod_{k=1}^{l} J^1_{C(Z_{i_k})} v = v$ implies $\prod_{k=1}^{l} J^2_{Z_{i_k}}(Av) = Av$. The method of the construction of the integral basis in [46] implies that $v^2_{ij} = Av^1_{ij}$. Hence the structural constants with respect to the basis $\{v^2_{11}, \ldots, v^2_{n1}, Z_1, \ldots, Z_{r+s}\}$ are identical to the structural constants with respect to the basis $\{v^1_{11}, \ldots, v^1_{1n}, C(Z_1), \ldots, C(Z_{r+s})\}$. More precise, if we write $(A^k_{ip})^1_1 = (A^k_{ip})^2_1 = A^k_{ip}$ in the notation (4.27), then

$$[v^1_{i1}\,, v^1_{p1}] = \sum_{k=1}^{r+s} A^k_{ip} C(Z_k) \quad \text{and} \quad [v^2_{i1}\,, v^2_{p1}] = \sum_{k=1}^{r+s} A^k_{ip} Z_k.$$

We can find the exact form of the map $C\colon \mathfrak{z}_{r,s} \to \mathfrak{z}_{r,s}$. Let $\mathcal{Z} = \{Z_1, \ldots, Z_{r+s}\}$ be an orthonormal basis for $\mathfrak{z}_{r,s}$. Then the volume elements have different actions on their modules, namely $\omega^1(\mathcal{Z}) = \prod_{k=1}^{r+s} J^1_{Z_k} = \mathrm{Id}$ on $\mathfrak{v}^1_{rs}$ and $\omega^2(\mathcal{Z}) = \prod_{k=1}^{r+s} J^2_{Z_k} = -\mathrm{Id}$ on $\mathfrak{v}^2_{rs}$, see [64]. Let $A\colon \mathfrak{v}^1_{r,s} \to \mathfrak{v}^2_{r,s}$ be an isometry and $C\colon \mathfrak{z}_{r,s} \to \mathfrak{z}_{r,s}$ be the mapping induced by (4.28). Then

$$\omega^1(C(\mathcal{Z}))v = \prod_{k=1}^{r+s} J^1_{C(Z_k)} v = A^{-1} \circ \prod_{k=1}^{r+s} J^2_{Z_k} \circ Av = A^{-1}\omega^2(\mathcal{Z})Av = A^{-1}(-Av) = -v$$

for $v \in \mathfrak{v}^1_{r,s}$. Since for $r - s = 3 \mod (4)$ we have $r + s = 2(s + 2k + 1) + 1$, $k \in \mathbb{Z}$, we conclude that $r + s$ is an odd number. Then from

$$\prod_{k=1}^{r+s} J^1_{Z_k} = \omega^1(\mathcal{Z}) = \mathrm{Id}_{\mathfrak{v}^1_{r,s}} = -\omega^1(C(\mathcal{Z})) = -\prod_{k=1}^{r+s} J^1_{C(Z_k)} = \prod_{k=1}^{r+s} J^1_{-C(Z_k)}$$

we can assume that the map $C\colon \mathfrak{z}_{r,s} \to \mathfrak{z}_{r,s}$ maps the basis $\mathcal{Z} = \{Z_1, \ldots, Z_{r+s}\}$ to the basis $-\mathcal{Z} = \{-Z_1, \ldots, -Z_{r+s}\}$. We write

$$\{v^1_{ij},\, v^2_{iq},\, Z_k | i = 1, \ldots, n, \quad j = 1, \ldots, \mu, \quad q = 1, \ldots, \nu, \quad k = 1, \ldots, r+s\}$$

for an integral basis of $\mathfrak{n}_{r,s}(\mu, \nu)$ where v^1_{ij} is the i-th coordinate in the j's copy of the module $\mathfrak{v}^1_{r,s}$ and v^2_{iq} is the i-th coordinate in the q's copy of the n-dimensional admissible module $\mathfrak{v}^2_{r,s}$. Analogously,

$$\{v^2_{iq},\, v^1_{ij},\, Z_k | i = 1, \ldots, n, \quad q = 1, \ldots, \mu, \quad j = 1, \ldots, \nu, \quad k = 1, \ldots, r+s\}$$

is an integral basis of $\mathfrak{n}_{r,s}(\nu,\mu)$. Recall that in both Lie algebras $\mathfrak{n}_{r,s}(\mu,\nu)$ and $\mathfrak{n}_{r,s}(\nu,\mu)$ the following relations hold: $[v^1_{ij}\,,v^1_{pq}]=[v^1_{ij}\,,v^2_{pr}]=[v^2_{ij}\,,v^2_{pq}]=0$ for $j\neq q$ and for any i,p,r. Moreover

$$[v^1_{ij}\,,v^1_{pj}]=[v^1_{i1}\,,v^1_{p1}]=-\sum_{k=1}^{r+s}A^k_{ip}Z_k,\qquad [v^2_{ij}\,,v^2_{pj}]=[v^2_{i1}\,,v^2_{p1}]=\sum_{k=1}^{r+s}A^k_{ip}Z_k \tag{4.29}$$

for the above chosen $C\colon \mathfrak{z}_{r,s}\to\mathfrak{z}_{r,s}$. The bijective linear map $f\colon \mathfrak{n}_{r,s}(\mu,\nu)\to\mathfrak{n}_{r,s}(\nu,\mu)$ defined by

$$\begin{array}{lllll} v^1_{ij} & \mapsto & v^2_{ij}=A(v^1_{ij}), & \text{for} & j=1,\dots,\mu,\\ v^2_{ip} & \mapsto & v^1_{ip}=A^{-1}(v^2_{ip}), & \text{for} & p=1,\dots,\nu,\\ Z_k & \mapsto & -Z_k, & \text{for} & k=1,\dots,r+s, \end{array}$$

and $i,p=1,\dots,n$ induces a Lie algebra homomorphism. Indeed, by (4.29)

$$\begin{aligned} f([v^1_{ij}\,,v^1_{pj}]) &= f([v^1_{i1}\,,v^1_{p1}])=f(-\sum_{k=1}^{r+s}A^k_{ip}Z_k)=-\sum_{k=1}^{r+s}A^k_{ip}f(Z_k)\\ &= \sum_{k=1}^{r+s}A^k_{ip}Z_k=[v^2_{ij}\,,v^2_{pj}]=[A(v^1_{ij})\,,A(v^1_{pj})]=[f(v^1_{ij})\,,f(v^1_{pj})], \end{aligned}$$

and, analogously,

$$\begin{aligned} f([v^2_{ij}\,,v^2_{pj}]) &= f([v^2_{i1}\,,v^2_{p1}])=f(\sum_{k=1}^{r+s}A^k_{ip}Z_k)=\sum_{k=1}^{r+s}A^k_{ip}f(Z_k)\\ &= -\sum_{k=1}^{r+s}A^k_{ip}Z_k=[v^1_{ij}\,,v^1_{pj}]=[A^{-1}(v^2_{ij})\,,A^{-1}(v^2_{pj})]=[f(v^2_{ij})\,,f(v^2_{pj})]. \end{aligned}$$

To show the reverse statement we assume that Lie algebras $\mathfrak{n}_{r,s}(\mu_1,\nu_1)$ and $\mathfrak{n}_{r,s}(\mu_2,\nu_2)$ are isomorphic for some $\mu_1>\mu_2$ and $\mu_1>\nu_2$. Then there are bijective maps $A_{12}\colon\ \mathfrak{v}^1_{r,s}\to\mathfrak{v}^2_{r,s}$ and $A_{11}\colon\ \mathfrak{v}^1_{r,s}\to\mathfrak{v}^1_{r,s}$ of minimal dimensional modules where the map A_{12} induces C by (4.28) and C induces A_{11} by (4.2). Then we obtain $J^1_{C(Z)}=A^\tau_{12}\circ J^2_Z\circ A_{12}=A^\tau_{11}\circ J^1_Z\circ A_{11}$, that contradicts to the assumption that modules $\mathfrak{v}^1_{r,s}$ and $\mathfrak{v}^2_{r,s}$ are non equivalent. □

4.7.1 Open problems on classification of H-type algebras $\mathfrak{n}_{r,s}(\mu,\nu)$.

The problem of the isomorphism of the pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu,\nu)$ with different signatures (r,s) turns out to be not so simple. Increasing the dimension of admissible modules allows more freedom for action of the representation maps and some isomorphic Lie algebras can appear. For instance, it is possible to show, in the above notation, that the Lie algebras $\mathfrak{n}_{2,1}$ and $\mathfrak{n}_{1,2}(1,1)$ are isomorphic, but the Lie algebras $\mathfrak{n}_{2,1}$ and $\mathfrak{n}_{1,2}(2,0)$ are not isomorphic. Thus we leave the full description of classification of pseudo H-type algebras for forthcoming papers.

4.7.2 Bracket generating properties

Theorem 4.7.3. *The pseudo H-type algebras $\mathfrak{n}_{r,s}(\mu,\nu)$ possesses the strongly bracket generating property if only if $r=0$ or $s=0$.*

Proof. First we prove that $\mathfrak{n}_{r,0}(\mu,\nu)$ is strongly bracket generating, i.e. $[w\,,\mathfrak{v}_{r,0}(\mu,\nu)]=\mathfrak{z}_{r,0}$ for all $w\in\mathfrak{n}_{r,0}(\mu,\nu)$, $w\neq 0$. We recall that $\mathfrak{v}_{r,0}(\mu,\nu)=\oplus_{j=1}^{\mu}(\mathfrak{v}_{r,0}^{1})_j\oplus_{j=1}^{\nu}(\mathfrak{v}_{r,0}^{2})_j$. Recall that $\mathfrak{n}_{r,0}$ has the strongly bracket generating property for any $r\in\mathbb{N}$ by Theorem 4.6.4. Thus, we obtain that $[v\,,(\mathfrak{v}_{r,0}^{l})_j]=\mathfrak{z}_{r,0}$ for all $v\in(\mathfrak{v}_{r,0}^{l})_j\setminus\{0\}$, $l=1,2$, $j=1,\dots,\mu+\nu$.

Let $w\in\mathfrak{v}_{r,0}(\mu,\nu)$, $w\neq 0$. There is an index $j\in\{1,\dots,\mu+\nu\}$ such that the orthogonal projection of w to $(\mathfrak{v}_{r,0}^{l})_j=:\mathfrak{v}$, $l=1$ or $l=2$ is not vanishing. We obtain

$$\mathfrak{z}_{r,0}\supset[w\,,\mathfrak{v}_{r,0}(\mu,\nu)]\supset[w\,,\mathfrak{v}]=\mathfrak{z}_{r,0}\,.$$

Hence $[w\,,\mathfrak{v}_{r,0}(\mu,\nu)]=\mathfrak{z}_{r,0}$, i.e. $\mathfrak{n}_{r,0}(\mu,\nu)$ is strongly bracket generating.

The proof for $\mathfrak{n}_{0,r}(\mu,\nu)$ follows analogously.

We consider the case $r,s\neq 0$ and recall that $\mathfrak{n}_{r,s}$ does not have the strongly bracket generating property by Theorem 4.6.5, i.e. there is $v\in\mathfrak{v}_{r,s}^{1}\setminus\{0\}$ such that $[v\,,\mathfrak{v}_{r,s}]\subsetneq\mathfrak{z}_{r,s}$. Then the vector $w:=v\oplus\underbrace{0\oplus\cdots\oplus 0}_{\mu+\nu-1\text{ times}}\in\mathfrak{v}_{r,0}(\mu,\nu)$ satisfies $[w\,,\mathfrak{v}_{r,0}(\mu,\nu)]=[v\,,(\mathfrak{v}_{r,s}^{1})_1]\subsetneq\mathfrak{z}_{r,s}$. Hence $\mathfrak{n}_{r,0}(\mu,\nu)$ do not have the strongly bracket generating property. □

4.8 Appendix

In the tables we indicate by $[r\,,c]$ that the commutators are calculated as $[row\,,column]$.

Table 4.10: Commutation relations on $\mathfrak{n}_{8,0}$

$[r\,,c]$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
u_1	0	0	0	0	0	0	0	0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
u_2	0	0	0	0	0	0	0	0	Z_2	$-Z_1$	$-Z_4$	Z_3	$-Z_6$	Z_5	$-Z_8$	Z_7
u_3	0	0	0	0	0	0	0	0	Z_3	Z_4	$-Z_1$	$-Z_2$	Z_8	Z_7	$-Z_6$	$-Z_5$
u_4	0	0	0	0	0	0	0	0	Z_4	$-Z_3$	Z_2	$-Z_1$	Z_7	$-Z_8$	$-Z_5$	Z_6
u_5	0	0	0	0	0	0	0	0	Z_5	Z_6	$-Z_8$	$-Z_7$	$-Z_1$	$-Z_2$	Z_4	Z_3
u_6	0	0	0	0	0	0	0	0	Z_6	$-Z_5$	$-Z_7$	Z_8	Z_2	$-Z_1$	Z_3	$-Z_4$
u_7	0	0	0	0	0	0	0	0	Z_7	Z_8	Z_6	Z_5	$-Z_4$	$-Z_3$	$-Z_1$	$-Z_2$
u_8	0	0	0	0	0	0	0	0	Z_8	$-Z_7$	Z_5	$-Z_6$	$-Z_3$	Z_4	Z_2	$-Z_1$
u_9	$-Z_1$	$-Z_2$	$-Z_3$	$-Z_4$	$-Z_5$	$-Z_6$	$-Z_7$	$-Z_8$	0	0	0	0	0	0	0	0
u_{10}	$-Z_2$	Z_1	$-Z_4$	Z_3	$-Z_6$	Z_5	$-Z_8$	Z_7	0	0	0	0	0	0	0	0
u_{11}	$-Z_3$	Z_4	Z_1	$-Z_2$	Z_8	Z_7	$-Z_6$	$-Z_5$	0	0	0	0	0	0	0	0
u_{12}	$-Z_4$	$-Z_3$	Z_2	Z_1	Z_7	$-Z_8$	$-Z_5$	Z_6	0	0	0	0	0	0	0	0
u_{13}	$-Z_5$	Z_6	$-Z_8$	$-Z_7$	Z_1	$-Z_2$	Z_4	Z_3	0	0	0	0	0	0	0	0
u_{14}	$-Z_6$	$-Z_5$	$-Z_7$	Z_8	Z_2	Z_1	Z_3	$-Z_4$	0	0	0	0	0	0	0	0
u_{15}	$-Z_7$	Z_8	Z_6	Z_5	$-Z_4$	$-Z_3$	Z_1	$-Z_2$	0	0	0	0	0	0	0	0
u_{16}	$-Z_8$	$-Z_7$	Z_5	$-Z_6$	$-Z_3$	Z_4	Z_2	Z_1	0	0	0	0	0	0	0	0

Table 4.11: Commutation relations on $\mathfrak{n}_{0,8}$

$[r,c]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
v_1	0	0	0	0	0	0	0	0	$\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$	$\tilde{Z}_4$	$\tilde{Z}_5$	$\tilde{Z}_6$	$\tilde{Z}_7$	$\tilde{Z}_8$
v_2	0	0	0	0	0	0	0	0	$-\tilde{Z}_2$	$\tilde{Z}_1$	$\tilde{Z}_4$	$-\tilde{Z}_3$	$\tilde{Z}_6$	$-\tilde{Z}_5$	$\tilde{Z}_8$	$-\tilde{Z}_7$
v_3	0	0	0	0	0	0	0	0	$-\tilde{Z}_3$	$-\tilde{Z}_4$	$\tilde{Z}_1$	$\tilde{Z}_2$	$-\tilde{Z}_8$	$-\tilde{Z}_7$	$\tilde{Z}_6$	$\tilde{Z}_5$
v_4	0	0	0	0	0	0	0	0	$-\tilde{Z}_4$	$\tilde{Z}_3$	$-\tilde{Z}_2$	$\tilde{Z}_1$	$-\tilde{Z}_7$	$\tilde{Z}_8$	$\tilde{Z}_5$	$-\tilde{Z}_6$
v_5	0	0	0	0	0	0	0	0	$-\tilde{Z}_5$	$-\tilde{Z}_6$	$\tilde{Z}_8$	$\tilde{Z}_7$	$\tilde{Z}_1$	$\tilde{Z}_2$	$-\tilde{Z}_4$	$-\tilde{Z}_3$
v_6	0	0	0	0	0	0	0	0	$-\tilde{Z}_6$	$\tilde{Z}_5$	$\tilde{Z}_7$	$-\tilde{Z}_8$	$-\tilde{Z}_2$	$\tilde{Z}_1$	$-\tilde{Z}_3$	$\tilde{Z}_4$
v_7	0	0	0	0	0	0	0	0	$-\tilde{Z}_7$	$-\tilde{Z}_8$	$-\tilde{Z}_6$	$-\tilde{Z}_5$	$\tilde{Z}_4$	$\tilde{Z}_3$	$\tilde{Z}_1$	$\tilde{Z}_2$
v_8	0	0	0	0	0	0	0	0	$-\tilde{Z}_8$	$\tilde{Z}_7$	$-\tilde{Z}_5$	$\tilde{Z}_6$	$\tilde{Z}_3$	$-\tilde{Z}_4$	$-\tilde{Z}_2$	$\tilde{Z}_1$
v_9	$-\tilde{Z}_1$	$\tilde{Z}_2$	$\tilde{Z}_3$	$\tilde{Z}_4$	$\tilde{Z}_5$	$\tilde{Z}_6$	$\tilde{Z}_7$	$\tilde{Z}_8$	0	0	0	0	0	0	0	0
v_{10}	$-\tilde{Z}_2$	$-\tilde{Z}_1$	$\tilde{Z}_4$	$-\tilde{Z}_3$	$\tilde{Z}_6$	$-\tilde{Z}_5$	$\tilde{Z}_8$	$-\tilde{Z}_7$	0	0	0	0	0	0	0	0
v_{11}	$-\tilde{Z}_3$	$-\tilde{Z}_4$	$-\tilde{Z}_1$	$\tilde{Z}_2$	$-\tilde{Z}_8$	$-\tilde{Z}_7$	$\tilde{Z}_6$	$\tilde{Z}_5$	0	0	0	0	0	0	0	0
v_{12}	$-\tilde{Z}_4$	$\tilde{Z}_3$	$-\tilde{Z}_2$	$-\tilde{Z}_1$	$-\tilde{Z}_7$	$\tilde{Z}_8$	$\tilde{Z}_5$	$-\tilde{Z}_6$	0	0	0	0	0	0	0	0
v_{13}	$-\tilde{Z}_5$	$-\tilde{Z}_6$	$\tilde{Z}_8$	$\tilde{Z}_7$	$-\tilde{Z}_1$	$\tilde{Z}_2$	$-\tilde{Z}_4$	$-\tilde{Z}_3$	0	0	0	0	0	0	0	0
v_{14}	$-\tilde{Z}_6$	$\tilde{Z}_5$	$\tilde{Z}_7$	$-\tilde{Z}_8$	$-\tilde{Z}_2$	$-\tilde{Z}_1$	$-\tilde{Z}_3$	$\tilde{Z}_4$	0	0	0	0	0	0	0	0
v_{15}	$-\tilde{Z}_7$	$-\tilde{Z}_8$	$-\tilde{Z}_6$	$-\tilde{Z}_5$	$\tilde{Z}_4$	$\tilde{Z}_3$	$-\tilde{Z}_1$	$\tilde{Z}_2$	0	0	0	0	0	0	0	0
v_{16}	$-\tilde{Z}_8$	$\tilde{Z}_7$	$-\tilde{Z}_5$	$\tilde{Z}_6$	$\tilde{Z}_3$	$-\tilde{Z}_4$	$-\tilde{Z}_2$	$-\tilde{Z}_1$	0	0	0	0	0	0	0	0

Table 4.12: Commutation relations on $\mathfrak{n}_{4,4}$

$[r,c]$	y_1	y_6	y_7	y_8	y_{13}	y_{14}	y_{15}	y_{16}	y_2	y_3	y_4	y_5	y_9	y_{10}	y_{11}	y_{12}
y_1	0	0	0	0	0	0	0	0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
y_6	0	0	0	0	0	0	0	0	Z_2	$-Z_1$	$-Z_4$	Z_3	Z_6	$-Z_5$	Z_8	$-Z_7$
y_7	0	0	0	0	0	0	0	0	Z_3	Z_4	$-Z_1$	$-Z_2$	Z_8	Z_7	$-Z_6$	$-Z_5$
y_8	0	0	0	0	0	0	0	0	Z_4	$-Z_3$	Z_2	$-Z_1$	$-Z_7$	Z_8	Z_5	$-Z_6$
y_{13}	0	0	0	0	0	0	0	0	Z_5	$-Z_6$	$-Z_8$	Z_7	Z_1	$-Z_2$	Z_4	$-Z_3$
y_{14}	0	0	0	0	0	0	0	0	Z_6	Z_5	$-Z_7$	$-Z_8$	Z_2	Z_1	$-Z_3$	$-Z_4$
y_{15}	0	0	0	0	0	0	0	0	Z_7	$-Z_8$	Z_6	$-Z_5$	$-Z_4$	Z_3	Z_1	$-Z_2$
y_{16}	0	0	0	0	0	0	0	0	Z_8	Z_7	Z_5	Z_6	Z_3	Z_4	Z_2	Z_1
y_2	$-Z_1$	$-Z_2$	$-Z_3$	$-Z_4$	$-Z_5$	$-Z_6$	$-Z_7$	$-Z_8$	0	0	0	0	0	0	0	0
y_3	$-Z_2$	Z_1	$-Z_4$	Z_3	Z_6	$-Z_5$	Z_8	$-Z_7$	0	0	0	0	0	0	0	0
y_4	$-Z_3$	Z_4	Z_1	$-Z_2$	Z_8	Z_7	$-Z_6$	$-Z_5$	0	0	0	0	0	0	0	0
y_5	$-Z_4$	$-Z_3$	Z_2	Z_1	$-Z_7$	Z_8	Z_5	$-Z_6$	0	0	0	0	0	0	0	0
y_9	$-Z_5$	$-Z_6$	$-Z_8$	Z_7	$-Z_1$	$-Z_2$	Z_4	$-Z_3$	0	0	0	0	0	0	0	0
y_{10}	$-Z_6$	Z_5	$-Z_7$	$-Z_8$	Z_2	$-Z_1$	$-Z_3$	$-Z_4$	0	0	0	0	0	0	0	0
y_{11}	$-Z_7$	$-Z_8$	Z_6	$-Z_5$	$-Z_4$	Z_3	$-Z_1$	$-Z_2$	0	0	0	0	0	0	0	0
y_{12}	$-Z_8$	Z_7	Z_5	Z_6	Z_3	Z_4	Z_2	$-Z_1$	0	0	0	0	0	0	0	0

Table 4.13: Permutations of the basis of $\mathfrak{n}_{8,0}$ by J_i

$J_i u_j$	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8
u_1	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}
u_2	$-u_{10}$	u_9	u_{12}	$-u_{11}$	u_{14}	$-u_{13}$	u_{16}	$-u_{15}$
u_3	$-u_{11}$	$-u_{12}$	u_9	u_{10}	$-u_{16}$	$-u_{15}$	u_{14}	u_{13}
u_4	$-u_{12}$	u_{11}	$-u_{10}$	u_9	$-u_{15}$	u_{16}	u_{13}	$-u_{14}$
u_5	$-u_{13}$	$-u_{14}$	u_{16}	u_{15}	u_9	u_{10}	$-u_{12}$	$-u_{11}$
u_6	$-u_{14}$	u_{13}	u_{15}	$-u_{16}$	$-u_{10}$	u_9	$-u_{11}$	u_{12}
u_7	$-u_{15}$	$-u_{16}$	$-u_{14}$	$-u_{13}$	u_{12}	u_{11}	u_9	u_{10}
u_8	$-u_{16}$	u_{15}	$-u_{13}$	u_{14}	u_{11}	$-u_{12}$	$-u_{10}$	u_9
u_9	$-u_1$	$-u_2$	$-u_3$	$-u_4$	$-u_5$	$-u_6$	$-u_7$	$-u_8$
u_{10}	u_2	$-u_1$	u_4	$-u_3$	u_6	$-u_5$	u_8	$-u_7$
u_{11}	u_3	$-u_4$	$-u_1$	u_2	$-u_8$	$-u_7$	u_6	u_5
u_{12}	u_4	u_3	$-u_2$	$-u_1$	$-u_7$	u_8	u_5	$-u_6$
u_{13}	u_5	$-u_6$	u_8	u_7	$-u_1$	u_2	$-u_4$	$-u_3$
u_{14}	u_6	u_5	u_7	$-u_8$	$-u_2$	$-u_1$	$-u_3$	u_4
u_{15}	u_7	$-u_8$	$-u_6$	$-u_5$	u_4	u_3	$-u_1$	u_2
u_{16}	u_8	u_7	$-u_5$	u_6	u_3	$-u_4$	$-u_2$	$-u_1$

Chapter 5

Pseudo-metric 2-step nilpotent Lie algebras

5.1 Introduction

The 2-step nilpotent Lie algebras and its diffeomorphic equivalent the 2-step nilpotent Lie group are of general interest in the area of sub-Riemannian geometry. Eberlein introduced and studied an isomorphic standard metric form $\mathbb{R}^n \oplus W$ with $W \subset \mathfrak{so}(m)$ for these particular interesting Lie groups [41, 42]. Furthermore, he showed the existence of lattices in simply connected, 2-step nilpotent Lie groups N that arise from Lie triple systems with compact center in $\mathfrak{so}(n)$. These results are closely related to Clifford algebras and modules. In particular, finite dimensional Clifford modules imply subspaces of $\mathfrak{so}(m, \mathbb{R})$ which are in the compact center of Lie triple systems. The center of the Lie triple system is trivial for representations of Clifford algebras or semisimple Lie groups.

In the present chapter we concentrate in Section 5.2 on pseudo-metric 2-step nilpotent Lie algebras, i.e. we prove that any 2-step nilpotent Lie algebra can be identified with a standard pseudo-metric Lie algebra $\mathbb{R}^{p,q} \oplus W$ with $W \subset \mathfrak{so}(p, q)$, which arises from a non-positive definite metric on $\mathbb{R}^{p,q}$. Furthermore, we study these results further to accomplish a deeper understanding of the newly constructed algebras and the problems in indefinite spaces. The main results are illustrated on the example of pseudo H-type algebras, which are induced by Clifford algebras, representations and modules. In Section 5.3, we prove that all indefinite free algebras $F_2(p, q)$ with $p + q = m$ are isomorphic. Furthermore, we combine the theory of the in Section 5.2 introduced standard pseudo-metric Lie algebras with indefinite free algebras. Additionally, we introduce Lie triple systems and study them in detail closely related to pseudo H-type algebras in the space $\mathfrak{so}(l, l)$. For further interest are the Lie algebras constructed by Lie triple systems, which are a topic in the end of this chapter.

This chapter is the result of a productive cooperation between Prof. Furutani, Prof. Vasiliev, Prof. Markina and me and is planned to be published as a paper.

5.2 Pseudo-metric on 2-step nilpotent Lie algebras

In this section we continue to develop the approach proposed in Subsection 1.1.2, i.e. any two step nilpotent Lie algebra is isomorphic to a standard metric Lie algebra $\mathbb{R}^m \oplus W$, with $W \subset \mathfrak{so}(m)$. The choice of the Euclidean product in $\mathbb{R}^m$ is very natural, but it is also possible to choose a metric of arbitrary index (p, q), $p + q = m$, as for example $\langle V\,, W\rangle_{p,q} = \sum_{i=1}^{p} V_iW_i - \sum_{i=p+1}^{p+q} V_iW_i$. It leads to a change of the structural space $\mathcal{C} \in \mathfrak{so}(m)$ to the space $\mathcal{D} \subset \mathfrak{so}(p, q)$ and the positive definite metrics to indefinite metrics. The main motivation is that the pseudo H-type algebras, introduced in Subsection 1.1.3 are much more natural to consider as standard pseudo-metric Lie algebras with indefinite metric than with positive definite metric. We also aim to show that any 2-step nilpotent Lie algebra is isomorphic to some standard pseudo-metric Lie algebra with an indefinite metric.

5.2.1 Pseudo-orthogonal groups

We start by recalling the structure of the pseudo-orthogonal group and its Lie algebra. We use the notation $\eta_{p,q} = \mathrm{diag}(I_p, -I_q)$ for the diagonal $(m \times m)$-matrix, $m = p + q$, having the first p entries on the main diagonal 1 and the last q equal to -1. Further we continue to use I_p to denote the $(p \times p)$ unit matrix. Let $\langle \cdot\,, \cdot\rangle_{p,q}$ be a scalar product in $\mathbb{R}^m$, defined by the matrix $\eta_{p,q}$, i.e. $\langle x\,, y\rangle_{p,q} = x^{\mathbf{t}}\eta_{p,q}y$ for $x, y \in \mathbb{R}^m$, where $x^{\mathbf{t}}$ is the transpose vector of x. We use the following notation established in Subsection 1.1.3

$$\epsilon_i = \epsilon_i(p, q) = \begin{cases} 1, & \text{if} \quad 1 \le i \le p, \\ -1, & \text{if} \quad p + 1 \le i \le p + q = m, \end{cases} \tag{5.1}$$

to indicate the sign in the scalar product of the vectors from an orthonormal basis for $\mathbb{R}^{p,q}$. A vector $x \in \mathbb{R}^{p,q}$ is called

- spacelike if $\langle x\,, x\rangle_{p,q} > 0$ or $x = 0$,
- timelike if $\langle x\,, x\rangle_{p,q} < 0$,
- null if $x \neq 0$ and $\langle x\,, x\rangle_{p,q} = 0$.

We denote by $\mathrm{O}(p, q)$ the pseudo-orthogonal group

$$\mathrm{O}(p, q) = \{X \in GL(m)|\ X^{\mathbf{t}}\eta_{p,q}X = \eta_{p,q}\},$$

where $X^{\mathbf{t}}$ is the matrix transposed to X. The pseudo-orthogonal group preserves the scalar product $\langle \cdot\,, \cdot\rangle_{p,q}$ in the following sense

$$\langle Xx\,, Xy\rangle_{p,q} = x^{\mathbf{t}}X^{\mathbf{t}}\eta_{p,q}Xy = x^{\mathbf{t}}\eta_{p,q}y = \langle x\,, y\rangle_{p,q}.$$

The inverse of X is given by $X^{-1} = \eta_{p,q}X^{\mathbf{t}}\eta_{p,q}$. For any matrix A define the matrix $A^{\eta_{p,q}}$ by $A^{\eta_{p,q}} := \eta_{p,q}A^{\mathbf{t}}\eta_{p,q}$. Thus, if $X \in O(p, q)$, then $X^{\eta_{p,q}}X = XX^{\eta_{p,q}} = I_m$.

If we replace $\eta_{p,q}$ by any symmetric matrix $\tilde{\eta}$ with p positive and q negative eigenvalues, then we get a group isomorphic to $\mathrm{O}(p,q)$. Diagonalizing the matrix $\tilde{\eta}$ gives a conjugation of this group with the standard group $\mathrm{O}(p,q)$. It follows from the definition that all matrices in $\mathrm{O}(p,q)$ have determinant equal to ± 1. A matrix $X \in \mathrm{O}(p,q)$ can be written in block form as

$$X = \left[\begin{array}{c|c} X_S & B \\ \hline C & X_T \end{array}\right],$$

where X_S and X_T are invertible $(p \times p)$ and $(q \times q)$ matrices, respectively. An element $X \in \mathrm{O}(p,q)$ preserves (reverses) time orientation provided that $\det(X_T) > 0$ (< 0), and preserves (reverses) space orientation provided that $\det(X_S) > 0$ (< 0). $\mathrm{O}(p,q)$ can then be split into four disjoint sets $\mathrm{O}^{++}(p,q)$, $\mathrm{O}^{+-}(p,q)$, $\mathrm{O}^{-+}(p,q)$, and $\mathrm{O}^{--}(p,q)$, indexed by the signs of the determinants of X_S and X_T, in this order. The following three disconnected subgroups of $\mathrm{O}(p,q)$ define the orientation on $\mathbb{R}^{p,q}$:

$$\mathrm{O}^{++}(p,q) \cup \mathrm{O}^{--}(p,q), \quad \mathrm{O}^{++}(p,q) \cup \mathrm{O}^{+-}(p,q), \quad \mathrm{O}^{++}(p,q) \cup \mathrm{O}^{-+}(p,q). \tag{5.2}$$

According to [75, p. 237], we call the first group *orientation* preserving, the second one *space-orientation* preserving and the last one *time-orientation* preserving. The connected component $\mathrm{O}^{++}(p,q)$ contains the identity, preserves time orientation, space orientation, and the orientation of $\mathbb{R}^{p,q}$. The component $\mathrm{O}^{++}(p,q)$ is, in some sense, an analogue of the special orthogonal subgroup $\mathrm{SO}(m)$ of the orthogonal group $\mathrm{O}(m)$ and therefore we use the notation $\mathrm{SO}(p,q) = \mathrm{O}^{++}(p,q)$. The group $\mathrm{O}(p,q)$ is not compact, but contains the compact subgroups $\mathrm{O}(p)$ and $\mathrm{O}(q)$ acting on the subspaces on which the scalar product $\langle \cdot\,,\cdot \rangle_{p,q}$ is sign definite. In fact, $\mathrm{O}(p) \times \mathrm{O}(q)$ is a maximal compact subgroup of $\mathrm{O}(p,q)$, while $S(\mathrm{O}(p) \times \mathrm{O}(q))$ is a maximal compact subgroup of $\mathrm{O}^{++}(p,q) \cup \mathrm{O}^{--}(p,q)$. Likewise, $\mathrm{SO}(p) \times \mathrm{SO}(q)$ is a maximal compact subgroup of the component $\mathrm{SO}(p,q)$. Thus up to homotopy, the spaces $S(\mathrm{O}(p) \times \mathrm{O}(q))$ and $\mathrm{SO}(p) \times \mathrm{SO}(q)$ are products of (special) orthogonal groups, from which algebra-topological invariants can be computed.

The Lie algebra of $\mathrm{O}(p,q)$ and thus of $\mathrm{SO}(p,q)$, equipped with the Lie bracket defined by the commutator $[\mathcal{A}\,,\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$, is the set

$$\mathfrak{so}(p,q) = \{\mathcal{A} \in \mathfrak{gl}(m) \mid \eta_{p,q}\mathcal{A}^{\mathbf{t}}\eta_{p,q} = -\mathcal{A}\}.$$

Thus, an element $\mathcal{X} \in \mathfrak{so}(p,q)$ satisfies $\mathcal{X}^{\eta_{p,q}} = -\mathcal{X}$ and one has $\mathcal{X}^{\eta_{p,q}}\mathcal{X} = \mathcal{X}\mathcal{X}^{\eta_{p,q}} = -\mathcal{X}^2$. In general for an arbitrary $\mathcal{A} \in \mathfrak{gl}(m)$ the following is true: $(\mathcal{A}^{\eta_{p,q}})^{\eta_{p,q}} = \mathcal{A}$ and $(\mathcal{A}\mathcal{B})^{\eta_{p,q}} = \mathcal{B}^{\eta_{p,q}}\mathcal{A}^{\eta_{p,q}}$.

The Lie algebra $\mathfrak{so}(p,q)$ can be equipped with the scalar product $\langle\,\cdot\,,\cdot\,\rangle_{\mathfrak{so}(p,q)}$ defined by $\langle\,\mathcal{X},\mathcal{Y}\,\rangle_{\mathfrak{so}(p,q)} = \mathrm{tr}\,(\mathcal{X}^{\eta_{p,q}}\mathcal{Y}) = -\,\mathrm{tr}(\mathcal{X}\mathcal{Y})$. The scalar product is positive definite only for $q = 0$. Analogously to the causal structure in $\mathbb{R}^{p,q}$, we say that a non-zero element $\mathcal{X} \in \mathfrak{so}(p,q)$ is timelike if $\langle\,\mathcal{X},\mathcal{X}\,\rangle_{\mathfrak{so}(p,q)} < 0$, it is spacelike if $\langle\,\mathcal{X},\mathcal{X}\,\rangle_{\mathfrak{so}(p,q)} > 0$ and it is null if $\langle\,\mathcal{X},\mathcal{X}\,\rangle_{\mathfrak{so}(p,q)} = 0$. The zero element is declared to be spacelike. Matrices in $\mathfrak{so}(p,q)$ can be written as

$$\mathcal{X} = \begin{pmatrix} a_p & b \\ b^{\mathbf{t}} & a_q \end{pmatrix}, \qquad a_p \in \mathfrak{so}(p), \quad a_q \in \mathfrak{so}(q), \quad b \in \mathbb{R}^{p\times q}.$$

So, for $\mathcal{X} \in \mathfrak{so}(p,q)$ one has

$$\langle \mathcal{X}, \mathcal{X} \rangle_{\mathfrak{so}(p,q)} = \operatorname{tr}(\mathcal{X}^{\eta_{p,q}} \mathcal{X}) = -\operatorname{tr}(\mathcal{X}^2) = -\operatorname{tr}(a_p^2 + a_q^2) - 2\operatorname{tr}(bb^{\mathbf{t}}).$$

As we see, the first term involving the skew-symmetric matrices a_p and a_q is always positive and represents the spacelike part. The matrix b is responsible for the timelike character of elements of the Lie algebra. The metric defined by the trace has index $\left(\frac{p(p-1)+q(q-1)}{2}, pq\right)$ as one can see from the dimensions of $\mathfrak{so}(p)$ and $\mathfrak{so}(q)$.

Notice that if $\mathcal{X} \in \mathfrak{so}(p,q)$ and $x, y \in \mathbb{R}^m$, $p+q=m$, then

$$\langle \mathcal{X}x \,, y\rangle_{p,q} = x^{\mathbf{t}} \mathcal{X}^{\mathbf{t}} \eta_{p,q} y = -x^{\mathbf{t}} \eta_{p,q} \mathcal{X} y = -\langle x \,, \mathcal{X} y\rangle_{p,q}.$$

Thus matrices from $\mathfrak{so}(p,q)$ are skew-symmetric with respect to the scalar product $\langle \cdot \,, \cdot \rangle_{p,q}$.

At the end of the section we consider a generalization of the above constructions. Let $(V, \langle \cdot \,, \cdot \rangle_V)$ be a scalar product vector space. We denote by $\mathfrak{o}(V, \langle \cdot \,, \cdot \rangle_V)$ or shortly $\mathfrak{o}(V)$ the subspace of the space $\operatorname{End}(V)$ of linear maps $\mathcal{A} \colon V \to V$ such that

$$\langle \mathcal{A}v \,, w\rangle_V = -\langle v \,, \mathcal{A}w\rangle_V. \tag{5.3}$$

We call $\mathfrak{o}(V)$ the space of *skew-symmetric* (*with respect to* $\langle \cdot \,, \cdot \rangle_V$) *maps* and note that it coincides with $\mathfrak{so}(p,q)$ when $V = \mathbb{R}^{p,q}$ and $\langle \cdot \,, \cdot \rangle_V = \langle \cdot \,, \cdot \rangle_{p,q}$. In general, it can be shown that $\mathfrak{o}(V)$ for any m-dimensional scalar product space $(V, \langle \cdot \,, \cdot \rangle_V)$ with a scalar product of index (p,q), $p+q=m$, is isomorphic to the space $\mathfrak{so}(p,q)$. We can endow the space $\mathfrak{o}(V)$ by the following scalar product

$$\langle \mathcal{A} \,, \mathcal{B}\rangle_{\mathfrak{o}(V)} = -\operatorname{tr}(\mathcal{A}\mathcal{B}).$$

One can prove that the index of $\langle \cdot \,, \cdot \rangle_{\mathfrak{o}(V)}$ is $\left(\frac{p(p-1)+q(q-1)}{2}, pq\right)$ by the isomorphism property with $\mathfrak{so}(p,q)$.

5.2.2 Lie product and compatible scalar product

In Subsection 1.1.5 the relation between skew-symmetric representations of Clifford algebras and some class of 2-step nilpotent Lie algebras, namely, pseudo H-type algebras was described. This relation is actually more general and can be given for arbitrary skew-symmetric maps and 2-step nilpotent Lie algebras endowed with some scalar product.

From Lie algebras to skew-symmetric maps. Let $(\mathfrak{g}, [\cdot \,, \cdot], \langle \cdot \,, \cdot \rangle_{\mathfrak{g}})$ be a 2-step Lie algebra with center U and a scalar product $\langle \cdot \,, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. We write $\mathfrak{g} = V \oplus_{\perp} U$ where the decomposition is orthogonal with respect to $\langle \cdot \,, \cdot \rangle_{\mathfrak{g}}$ and assume that the restriction $\langle \cdot \,, \cdot \rangle_V$ of $\langle \cdot \,, \cdot \rangle_{\mathfrak{g}}$ on V is non-degenerate. This leads to the non-degeneracy of the space U with respect to the restriction $\langle \cdot \,, \cdot \rangle_U$ of $\langle \cdot \,, \cdot \rangle_{\mathfrak{g}}$ on U. As it was mentioned before, the Lie product on $\mathfrak{g}$ together with every $z \in U$ defines a map $J_z \colon V \to V$ by

$$\langle J_z v \,, w\rangle_V = \langle z \,, [v \,, w]\rangle_U \quad \text{for all} \quad v, w \in V. \tag{5.4}$$

It is clear that J_z satisfies (5.3) and is linear with respect to both variables: $z \in U$ and $v \in V$. Therefore, we conclude that a scalar product and a Lie product together define a linear skew-symmetric map $J\colon U \to \mathfrak{o}(V)$.

From skew-symmetric maps to Lie algebras. Let $(V, \langle \cdot\,,\cdot\rangle_V)$ and $(U, \langle \cdot\,,\cdot\rangle_U)$ be two scalar product spaces and $J\colon U \to \mathfrak{o}(V)$. Then the sum $\mathfrak{g} = V \oplus U$ is orthogonal with respect to the non-degenerate scalar product $\langle \cdot\,,\cdot\rangle_{\mathfrak{g}} = \langle \cdot\,,\cdot\rangle_V + \langle \cdot\,,\cdot\rangle_U$ and we are able to define the Lie bracket for $\mathfrak{g}$ by making use of (5.4). Then $\mathfrak{g} = \big(V \oplus U, [\cdot\,,\cdot], \langle \cdot\,,\cdot\rangle_{\mathfrak{g}}\big)$ becomes a Lie algebra endowed with a non-degenerate scalar product, where U belongs to the center.

The discussions above rise the following question. Given two finite dimensional vector spaces U and V and a linear map $J\colon U \to \mathrm{End}(V)$. When can one find a scalar product on $\langle \cdot\,,\cdot\rangle_V$ such that J_z satisfies (5.3) for all $z \in U$, i.e. $J\colon U \to \mathfrak{o}(V)$? If such a scalar product $\langle \cdot\,,\cdot\rangle_V$ exists, we call it *W-invariant*, where $W = J(U) \subset \mathfrak{o}(V) \subset \mathrm{End}(V)$. If moreover, a non-degenerate scalar product $\langle \cdot\,,\cdot\rangle_U$ on U is given, then the decomposition $V \oplus U$ is orthogonal with respect to $\langle \cdot\,,\cdot\rangle_{\mathfrak{g}} = \langle \cdot\,,\cdot\rangle_V + \langle \cdot\,,\cdot\rangle_U$ and we are able to define a Lie algebra structure on $V \oplus U$ by means of (5.4) as was described above.

5.2.3 Uniqueness properties

In this subsection we study the uniqueness of the choice of an invariant scalar product. We start from a simple proposition.

Proposition 5.2.1. *Let $(V, \langle \cdot\,,\cdot\rangle_V)$ and $(U, \langle \cdot\,,\cdot\rangle_U)$ be scalar product spaces and $J\colon U \to \mathfrak{o}(V)$. The multiplication of both scalar products $\langle \cdot\,,\cdot\rangle_V$ and $\langle \cdot\,,\cdot\rangle_U$ by a non-zero number c does not change the brackets defined by $\langle J_z v\,, w\rangle_V = \langle z\,, [v\,, w]\rangle_U$ for all $v, w \in V$ and all $z \in U$.*

Proof. We observe that the defining relation of the brackets is equivalent to original one given by (5.4):

$$c\langle J_z v\,, w\rangle_V = c\langle z\,, [v\,, w]\rangle_U \quad \Leftrightarrow \quad \langle J_z v\,, w\rangle_V = \langle z\,, [v\,, w]\rangle_U.$$

□

Lemma 5.2.2. *Let V and U be finite dimensional vector spaces and $\langle \cdot\,,\cdot\rangle_U$ a non-degenerate scalar product on U. Let $\langle \cdot\,,\cdot\rangle_V^1$ and $\langle \cdot\,,\cdot\rangle_V^2$ be two W-invariant scalar products for a map $J\colon U \to \mathrm{End}(V)$, $W = J(U)$. Suppose that the scalar products $\langle \cdot\,,\cdot\rangle_V^1$ and $\langle \cdot\,,\cdot\rangle_V^2$ have equal index and that the sets of spacelike (timelike and correspondingly null) vectors coincide. Assume that $[\cdot\,,\cdot]^1$ and $[\cdot\,,\cdot]^2$ are Lie products defined by (5.4) with respect to scalar products $\langle \cdot\,,\cdot\rangle_V^1$ and $\langle \cdot\,,\cdot\rangle_V^2$ on $\mathfrak{g} = V \oplus_{\perp} U$. Then the Lie algebras $(\mathfrak{g}, [\cdot\,,\cdot]^1)$ and $(\mathfrak{g}, [\cdot\,,\cdot]^2)$ are isomorphic.*

Proof. We define the linear map $S\colon V \to V$ by

$$\langle v\,, w\rangle_V^2 = \langle Sv\,, w\rangle_V^1. \tag{5.5}$$

We claim that S is injective. We prove by contradiction and assume that there exists $v \in V$, $v \neq 0$, such that $Sv = 0$. Then we get that $\langle v\,, w\rangle_V^2 = 0$ by (5.5) for any $w \in V$, which implies that $v = 0$ by the non-degeneracy of the scalar product, that contradicts the assumption. Hence S is injective.

The map S is symmetric with respect to both scalar products. Indeed

$$\begin{aligned} \langle Sv\,, w\rangle_V^1 = \langle v\,, w\rangle_V^2 = \langle w\,, v\rangle_V^2 = \langle Sw\,, v\rangle_V^1, \\ \langle Sv\,, w\rangle_V^2 = \langle w\,, Sv\rangle_V^2 = \langle Sw\,, Sv\rangle_V^1 = \langle Sv\,, Sw\rangle_V^1 = \langle v\,, Sw\rangle_V^2. \end{aligned} \tag{5.6}$$

We claim that S has only positive eigenvalues. First we note that since S is injective, it has only non-zero eigenvalues. Assume $Su = \lambda u$. We have to distinguish the two cases $\langle u\,, u\rangle_V^i \neq 0$, $i = 1, 2$, and $\langle u\,, u\rangle_V^i = 0$, $i = 1, 2$.

First let $\langle u\,, u\rangle_V^i \neq 0$, $i = 1, 2$, then

$$\lambda\langle u\,, u\rangle_V^1 = \langle Su\,, u\rangle_V^1 = \langle u\,, u\rangle_V^2.$$

Since $\langle u\,, u\rangle_V^1$ and $\langle u\,, u\rangle_V^2$ always have the same sign by the assumptions of the lemma, we conclude $\lambda > 0$.

If $\langle u\,, u\rangle_V^i = 0$, $i = 1, 2$, then we change the arguments. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis with respect to $\langle \cdot\,, \cdot\rangle_V^1$, that always exists since the scalar product is non-degenerate. Choose one basis vector e_k such that $\langle e_k\,, u\rangle_V^1 \neq 0$. Such a vector e_k exists, since otherwise u would be the zero vector which contradicts the requirement that u is an eigenvector. Then $\langle ce_k - u\,, ce_k - u\rangle_V^1 = 0$ for $c = 2\langle e_k\,, e_k\rangle_V^1 \langle e_k\,, u\rangle_V^1$. Write $v = ce_k$, then $\langle v - u\,, v - u\rangle_V^i = 0$ for i=1,2. This implies

$$0 = \langle v - u\,, v - u\rangle_V^i = \langle v\,, v\rangle_V^i - 2\langle v\,, u\rangle_V^i,$$

and we conclude that non-vanishing values $\langle v\,, u\rangle_V^i$ have the same sign in both vector spaces. Thus

$$\lambda\langle u\,, v\rangle_V^1 = \langle Su\,, v\rangle_V^1 = \langle u\,, v\rangle_V^2$$

leads to the conclusion that $\lambda > 0$.

The map S commutes with J_z for any $z \in U$ by

$$\langle J_z Sv\,, w\rangle_V^1 = -\langle Sv\,, J_z w\rangle_V^1 = -\langle v\,, J_z w\rangle_V^2 = \langle J_z v\,, w\rangle_V^2 = \langle SJ_z v\,, w\rangle_V^1. \tag{5.7}$$

Let $V_1, \dots, V_N$ be eigenspaces of the map S corresponding to different eigenvalues, which we denote by $\lambda_1^2, \dots, \lambda_N^2$. Then the vector spaces $V_1, \dots, V_N$ are mutually orthogonal with respect to both scalar products since the map S is symmetric with respect to both scalar products. We write $V \ni v = \sum_{k=1}^N v_k$ and $V \ni w = \sum_{k=1}^N w_k$, where $v_k, w_k \in V_k$, $k = 1, \dots, N$. We claim that

$$[v_k\,, v_j]^i = 0 \quad \text{for} \quad v_k \in V_k, \quad v_j \in V_j, \quad k \neq j, \quad i = 1, 2.$$

First we observe that the subspaces V_k, $k = 1, \dots, N$ are invariant under J_z for any $z \in U$ since $SJ_z = J_z S$. We calculate

$$\langle z\,, [v_k\,, v_j]^i\rangle_U = \langle J_z v_k\,, v_j\rangle_V^i = \langle v_k'\,, v_j\rangle_V^i = 0, \quad i = 1, 2, \quad v_k' \in V_k$$

for any $z \in U$. The scalar product $\langle \cdot\,,\cdot\rangle_U$ is non-degenerate and we conclude that $[v_k\,, v_j]^i = 0$.

We are ready to define the Lie algebra isomorphism $(V \oplus U, [\cdot\,,\cdot]^2) \to (V \oplus U, [\cdot\,,\cdot]^1)$. Set $\varphi\colon V \oplus U \to V \oplus U$:

$$\varphi = \begin{cases} \lambda_k \operatorname{Id}_{V_k}, & k = 1, \dots, N, \\ \operatorname{Id}_U . \end{cases}$$

It is left to prove that $\varphi\left([v\,,w]^2\right) = [\varphi(v)\,,\varphi(w)]^1$. We obtain from one side

$$\begin{aligned} \langle z\,,\varphi([v\,,w]^2)\rangle_U &= \langle z\,,[v\,,w]^2\rangle_U = \sum_{k=1}^{N}\langle z\,,[v_k\,,w_k]^2\rangle_U = \sum_{k=1}^{N}\langle J_z v_k\,,w_k\rangle_V^2 \\ &= \sum_{k=1}^{N}\lambda_k^2\langle J_z v_k\,,w_k\rangle_V^1, \end{aligned}$$

since $\langle\cdot\,,\cdot\rangle_{V_k}^2 = \lambda_k^2\langle\cdot\,,\cdot\rangle_{V_k}^1$. From the other side

$$\langle z\,,[\varphi(v)\,,\varphi(w)]^1\rangle_U = \sum_{k=1}^{N}\lambda_k^2\langle z\,,[v_k\,,w_k]^1\rangle_U = \sum_{k=1}^{N}\lambda_k^2\langle J_z v_k\,,w_k\rangle_V^1,$$

that finishes the proof. □

2-step nilpotent Lie algebras with trivial abelian factor

The map $J\colon U \to \mathfrak{o}(V)$ is not necessarily injective. Nevertheless, if it is so, the corresponding 2-step nilpotent Lie algebra has convenient properties. Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra, then in general the commutative ideal $[\mathfrak{g}\,,\mathfrak{g}]$ and the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{g}$ are related by $[\mathfrak{g}\,,\mathfrak{g}] \subset \mathfrak{z}$. The case $[\mathfrak{g}\,,\mathfrak{g}] = \mathfrak{z}$ corresponds to the injective map $J\colon \mathfrak{z} \to \mathfrak{o}(V)$. We recall some results for arbitrary 2-step nilpotent Lie algebras in this direction.

Proposition 5.2.3. *[42] Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with center $\mathfrak{z}$. Then there is an ideal $\mathfrak{g}^*$ and an abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{a} \subseteq \mathfrak{z}$ such that*

1. $\mathfrak{g} = \mathfrak{g}^* \oplus \mathfrak{a}$ *and* $\mathfrak{z} = [\mathfrak{g}\,,\mathfrak{g}] \oplus \mathfrak{a}$*;*

2. $\mathfrak{g}^*$ *is a 2-step nilpotent Lie algebra such that* $[\mathfrak{g}\,,\mathfrak{g}] = [\mathfrak{g}^*\,,\mathfrak{g}^*] = \mathfrak{z}^*$*, where* $\mathfrak{z}^*$ *is the center of* $\mathfrak{g}^*$*;*

3. *The ideals* $\mathfrak{g}^*$ *and* $\mathfrak{a}$ *are uniquely defined up to isomorphism by item 1.*

4. *If* $\mathfrak{g}$ *has a basis* $\mathcal{B}$ *with rational structure constants, then* $\mathfrak{g}^*$ *has a basis* $\mathcal{B}^*$ *with integer structure constants.*

The factor $\mathfrak{a}$ in Proposition 5.2.3 is called the *abelian factor*. The proposition has the following useful corollary.

Corollary 5.2.4. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with center $\mathfrak{z}$. Then $\mathfrak{g}$ has a trivial abelian factor if and only if $[\mathfrak{g}\,,\mathfrak{g}] = \mathfrak{z}$.*

Lemma 5.2.5. *Let $(\mathfrak{g}, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ be a 2-step nilpotent Lie algebra with center $\mathfrak{z}$ and a scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ such that its restrictions to $\mathfrak{z}$ and $[\mathfrak{g}\,,\mathfrak{g}]$ are non degenerate. Let $V = \mathfrak{z}^{\perp}$ and $J\colon \mathfrak{z} \to \mathfrak{o}(V)$ be the linear map defined by (5.4). Then the following statements are equivalent:*

1. *The commutative ideal $[\mathfrak{g}\,,\mathfrak{g}]$ has co-dimension $d \geq 0$ in $\mathfrak{z}$;*

2. *The kernel of J has dimension d.*

Proof. We write $\mathfrak{z} = [\mathfrak{g}\,,\mathfrak{g}] \oplus [\mathfrak{g}\,,\mathfrak{g}]^{\perp}$. Then $\langle J_z v\,, w\rangle_V = \langle z\,, [v\,,w]\rangle_{\mathfrak{z}}$ and non-degeneracy of scalar products imply that $J_z = 0$ if and only if $z \in [\mathfrak{g}\,,\mathfrak{g}]^{\perp}$, that proves the equivalence of items 1 and 2. □

Lemma 5.2.5 directly implies the following corollary.

Corollary 5.2.6. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with center $\mathfrak{z}$. Then the following statements are equivalent.*

1. *The Lie algebra $\mathfrak{g}$ has a trivial abelian factor.*

2. *If there is a non-degenerate scalar product on $\mathfrak{g}$ such that the restriction to $\mathfrak{z}$ is non-degenerate, then the linear map $J\colon \mathfrak{z} \to \mathfrak{o}(V)$ for $V = \mathfrak{z}^{\perp}$ defined by (5.4) is injective.*

5.2.4 Examples

Now we give several examples of skew-symmetric maps and the Lie algebras related to them.

Example 5.2.7. *Consider $\mathbb{R}^{p,q}$, $p + q = m$ with the metric $\langle x\,,y\rangle_{p,q} = x^{\mathbf{t}}\eta_{p,q}y$. Let W be a non-zero subspace of $\mathfrak{so}(p,q)$. The inclusion map $\iota\colon W \to \mathfrak{so}(p,q)$ defines a skew-symmetric map in the following sense: if $z \in W$ and $\iota_z = \iota(z) = Z \in \mathfrak{so}(p,q)$ then*

$$\langle\, \iota_z x, y\,\rangle_{p,q} = \langle\, Zx\,, y\,\rangle_{p,q} = -\,\langle\, x\,, Zy\,\rangle_{p,q} = -\,\langle\, x\,, \iota_z y\,\rangle_{p,q}\,.$$

If the restriction of the metric, which is defined by the trace, from $\mathfrak{so}(p,q)$ to W is non-degenerate, then we can define a Lie algebra structure on $\mathbb{R}^{p,q} \oplus W$. If $W = \mathfrak{so}(p,q)$, then the constructed Lie algebra on $\mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$ is a free 2-step nilpotent Lie algebra, which we denote by $F_2(p,q)$. Thus $F_2(p,q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$, where the commutator on $\mathbb{R}^{p,q}$ is defined by

$$[w\,,v]_{F_2(p,q)} = -\frac{1}{2}(wv^{\mathbf{t}} - vw^{\mathbf{t}})\eta_{p,q}. \tag{5.8}$$

For the standard basis $\{e_i\}$ of $\mathbb{R}^{p,q}$ we get $[e_i\,,e_j]_{F_2(p,q)} = -\frac{1}{2}(E_{ij} - E_{ji})\eta_{p,q}$, where E_{ij} denote the $(m\times m)$ matrix with zero entries except of 1 at the position ij. Since $F_2(p,q)$ is a 2-step nilpotent Lie algebra, $\mathfrak{so}(p,q)$ forms the center. Particularly, if $q = 0$ we get the free Lie algebra $F_2(m)$ studied in [42].

The next example is closely related to Subsection 1.1.5.

Example 5.2.8. *Let $\mathfrak{g}$ be a pseudo H-type algebra. Then the linear map defined by (5.4) is skew-symmetric and defines the representation of a Clifford algebra. Conversely, given a representation $J\colon \mathrm{Cl}(U,\langle\cdot\,,\cdot\rangle_U) \to V$ that is also skew-symmetric with respect to a scalar product on V, we can construct a 2-step nilpotent Lie algebra that is a general H-type algebra. All details are described in Subsection 1.1.5.*

5.2.5 Standard pseudo-metric 2-step nilpotent Lie algebras

We describe the construction of 2-step nilpotent Lie algebras with some standard choice of metrics.

Let $(V,\langle\cdot\,,\cdot\rangle_V)$ be an m-dimensional scalar product vector space and $\mathfrak{o}(V)$ a space of skew-symmetric maps with respect to $\langle\cdot\,,\cdot\rangle_V$. Equip the space $\mathfrak{o}(V)$ with the metric $\langle z\,,z'\rangle_{\mathfrak{o}(V)} = -\operatorname{tr}(zz')$, $z\in\mathfrak{o}(V)$. Observe that if the scalar product $\langle\cdot\,,\cdot\rangle_V$ has index (p,q), $p+q=m$, then the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{o}(V)}$ has index $\big(\frac{p(p-1)+q(q-1)}{2}, pq\big)$. Since the Lie algebra $\mathfrak{o}(V)$ is simple, then any symmetric bi-linear form is a multiple of the Killing form.

Let W be an n-dimensional subspace of $\mathfrak{o}(V)$ such that the restriction of $\langle\cdot\,,\cdot\rangle_{\mathfrak{o}(V)}$ to W is non-degenerate. Let $\mathcal{G} = V\oplus W$ and $\langle\cdot\,,\cdot\rangle_{\mathcal{G}} = \langle\cdot\,,\cdot\rangle_V + \langle\cdot\,,\cdot\rangle_{\mathfrak{o}(V)}$. The direct sum $\mathcal{G} = V\oplus W$ is orthogonal with respect to $\langle\cdot\,,\cdot\rangle_{\mathcal{G}}$. Let $[\cdot\,,\cdot]_{\mathcal{G}}$ be the Lie product on $\mathcal{G}$ defined as follows. If $v,w\in V$, then $[v\,,w]_{\mathcal{G}}$ is the unique element of W such that

$$\langle [v\,,w]_{\mathcal{G}}\,,z\rangle_{\mathfrak{o}(V)} = \langle z(v)\,,w\rangle_V \tag{5.9}$$

for every $z\in W$.

Definition 5.2.9. *We call the Lie algebra $\mathcal{G}$ constructed above standard pseudo-metric 2-step nilpotent Lie algebra and write $\mathcal{G} = (V\oplus_{\perp} W, [\cdot\,,\cdot]_{\mathcal{G}}, \langle\cdot\,,\cdot\rangle_{\mathcal{G}})$.*

If $V=\mathbb{R}^{p,q}$ and $\langle\cdot\,,\cdot\rangle_V = \langle\cdot\,,\cdot\rangle_{p,q}$ is the scalar product defined by the matrix $\eta_{p,q} = \operatorname{diag}(I_p,-I_q)$, then we write $\mathfrak{so}(p,q)$ for skew-symmetric maps and the standard pseudo-metric 2-step nilpotent Lie algebra is $\mathcal{G} = (\mathbb{R}^{p,q}\oplus_{\perp} W, [\cdot\,,\cdot], \langle\cdot\,,\cdot\rangle_{\mathcal{G}})$ with $\langle\cdot\,,\cdot\rangle_{\mathcal{G}} = \langle\cdot\,,\cdot\rangle_{p,q} + \langle\cdot\,,\cdot\rangle_{\mathfrak{so}(p,q)}$.

We also say that the standard pseudo-metric 2-step nilpotent Lie algebra is *involutive*, if W is a subalgebra in $\mathfrak{o}(V,\langle\cdot\,,\cdot\rangle_V)$. It is easy to see that $[\mathcal{G}\,,\mathcal{G}] = W$ and $W=\mathfrak{z}$ is the center of $\mathcal{G}$ if and only if for any $v\neq 0$, $v\in V$ there is $z\in W$ such that $z(v)\neq 0$.

Example 5.2.10. *Free standard pseudo-metric Lie algebra. Let us equip the 2-step free Lie algebra $F_2(p,q) = \mathbb{R}^{p,q}\oplus\mathfrak{so}(p,q)$ with the scalar product $\langle\,\cdot\,,\cdot\,\rangle = \langle\,\cdot\,,\cdot\,\rangle_{\mathfrak{so}(p,q)} + \langle\,\cdot\,,\cdot\,\rangle_{p,q}$. Then*

$$\langle [w\,,v]_{F_2(p,q)}\,,Z\,\rangle_{\mathfrak{so}(p,q)} = \langle\, Zw\,,v\,\rangle_{p,q}$$

for all $w, v \in \mathbb{R}^{p,q}$ *and all* $Z \in \mathfrak{so}(p,q)$*, where the Lie brackets are introduced in* (5.8)*. First we calculate* $\langle Zw\,, v\rangle_{p,q}$ *and receive*

$$\langle Zw\,, v\rangle_{p,q} = w^{\mathbf{t}} Z^{\mathbf{t}} \eta_{p,q} v = -w^{\mathbf{t}} \eta_{p,q} Z v = -\operatorname{tr}(w^{\mathbf{t}} \eta_{p,q} Z v) = -\operatorname{tr}(v w^{\mathbf{t}} \eta_{p,q} Z),$$

where $\operatorname{tr}(w^{\mathbf{t}} Z \eta_{p,q} v) = w^{\mathbf{t}} Z \eta_{p,q} v$ *as* $w^{\mathbf{t}} Z \eta_{p,q} v \in \mathbb{R}$ *and we used* $Z^{\mathbf{t}} \eta_{p,q} = -\eta_{p,q} Z$ *for all* $Z \in \mathfrak{so}(p,q)$*. Moreover, since* $Z \in \mathfrak{so}(p,q)$ *we also get*

$$\langle Zw\,, v\rangle_{p,q} = -\langle w\,, Zv\rangle_{p,q} = \operatorname{tr}(w v^{\mathbf{t}} \eta_{p,q} Z).$$

With these relations we calculate $\langle [w\,,v]\,, Z\rangle_{\mathfrak{so}(p,q)}$ *and obtain the desired equality*

$$\begin{aligned}\langle [w\,,v]\,, Z\rangle_{\mathfrak{so}(p,q)} &= -\operatorname{tr}\left(-\frac{1}{2}(w v^{\mathbf{t}} - v w^{\mathbf{t}})\eta_{p,q} Z\right) = \frac{1}{2}\left(\operatorname{tr}(w v^{\mathbf{t}} \eta_{p,q} Z) - \operatorname{tr}(v w^{\mathbf{t}} \eta_{p,q} Z)\right)\\ &= \langle Zw\,, v\rangle_{p,q}\,.\end{aligned}$$

Example 5.2.11. *Representation of Clifford algebras. Let* $(\mathbb{R}^{r,s}, \langle\cdot\,,\cdot\rangle_{r,s})$ *and let* $\mathrm{Cl}_{r,s}$ *denote the Clifford algebra generated by* $\mathbb{R}^{r,s}$*. Let* $J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(V)$ *be a Clifford algebra representation on the finite dimensional vector space* V*. We identify* V *(or* $V \oplus V$ *if it is necessary) with* $\mathbb{R}^{p,p}$*,* $2p = m$*, equipped with the scalar product* $\langle\cdot\,,\cdot\rangle_{p,p}$*, such that* $W = J(\mathbb{R}^{r,s}) \subseteq \mathfrak{so}(p,p)$ *if* $s > 0$*. If* $s = 0$*, then we identify* V *with the Euclidean space* $\mathbb{R}^m$*, and in this case* $W = J(\mathbb{R}^{r,0}) \subseteq \mathfrak{so}(m)$*. As it was observed in Proposition 1.1.16 the scalar product on* V *should be neutral in the case* $s > 0$*, that defines the choice of the scalar product* $\langle\cdot\,,\cdot\rangle_{p,p}$ *and the inclusion of* $W = J(\mathbb{R}^{r,s})$ *into the space* $\mathfrak{so}(p,p)$*.*

5.2.6 Reduction of a 2-step nilpotent Lie algebra to the standard pseudo-metric form

We start with the following observation relating elements in $\mathfrak{so}(m)$ and $\mathfrak{so}(p,q)$, where $p+q=m$. Let $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$, $p+q=m$ and recall the definition of $\epsilon_i = \epsilon_i(p,q)$ given by (5.1). Then for any $(m\times m)$ matrix $A = (a_{ij})_{i,j=1,\dots,m}$ we have

$$(A\eta_{p,q})_{ij} = a_{ij}\epsilon_j, \quad (\eta_{p,q}A)_{ij} = a_{ij}\epsilon_i.$$

Let $C \in \mathfrak{so}(m)$ and define D by $D = C\eta_{p,q}$ (or equivalently $D_{ij} = \epsilon_j C_{ij}$) and claim that $D \in \mathfrak{so}(p,q)$. Indeed,

$$\eta_{p,q} D^{\mathbf{t}} \eta_{p,q} = \eta_{p,q}(C\eta_{p,q})^{\mathbf{t}}\eta_{p,q} = \eta_{p,q}\eta_{p,q}^{\mathbf{t}} C^{\mathbf{t}} \eta_{p,q} = -C\eta_{p,q} = -D.$$

Analogously we can show that $\widetilde{D} = \eta_{p,q}C \in \mathfrak{so}(p,q)$ if $C \in \mathfrak{so}(m)$, $p+q=m$. We prove the following technical lemma.

Lemma 5.2.12. *Let* $\mathfrak{g}$ *be a 2-step nilpotent Lie algebra such that* $\dim([\mathfrak{g}\,,\mathfrak{g}]) = n$ *and the complement* V *to* $[\mathfrak{g}\,,\mathfrak{g}]$ *has dimension* m*. Denote by* $z_1, \dots, z_n$ *a basis of* $[\mathfrak{g}\,,\mathfrak{g}]$ *and* $v_1, \dots, v_m$ *a basis of* V*. Let* $[v_i\,, v_j] = \sum_{k=1}^n C_{ij}^k z_k$ *for* $1 \le i,j \le m$*. Then the matrices* $D^k = C^k\eta_{p,q}$ *are linearly independent in any* $\mathfrak{so}(p,q)$*,* $p+q=m$*.*

Proof. It was proved in [42] that $C^1, \dots, C^n$ are linearly independent in $\mathfrak{so}(m)$. Thus for any real numbers $\alpha_1, \dots, \alpha_n$ we have

$$\sum_{k=1}^{n} \alpha_k C^k = 0 \quad \Longleftrightarrow \quad \alpha_k = 0,\ k = 1, \dots, n.$$

Then

$$0 = \left(\sum_{k=1}^{n} \alpha_k C^k \right) \eta_{p,q} = \sum_{k=1}^{n} \alpha_k C^k \eta_{p,q} = \sum_{k=1}^{n} \alpha_k D^k \quad \Longleftrightarrow \quad \alpha_k = 0,\ k = 1, \dots, n.$$

□

Any 2-step nilpotent Lie algebra $\mathfrak{g}$ defines a subspace $\mathcal{C} \subset \mathfrak{so}(m)$ and moreover this subspace is a non-degenerate vector space in $\mathfrak{so}(m)$. This fact allowed the construction of the isomorphism between $\mathfrak{g}$ and the corresponding standard metric Lie algebra with positive definite scalar product, see [42].

The space $\mathcal{C}$ also generates a space $\mathcal{D}$ in each $\mathfrak{so}(p, q)$. Moreover, if $\mathcal{D} \subset \mathfrak{so}(p, q)$ is non-degenerate with respect to the restriction of the indefinite trace metric in $\mathfrak{so}(p, q)$, then there exists a standard pseudo-metric Lie algebra, which is isomorphic to $\mathfrak{g}$, as it is shown in the following theorem.

Theorem 5.2.13. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that* $\dim([\mathfrak{g}\,,\mathfrak{g}]_{\mathfrak{g}}) = n$ *and the complement V to $[\mathfrak{g}\,,\mathfrak{g}]_{\mathfrak{g}}$ has dimension m. Then there exists an n-dimensional subspace $\mathcal{D}$ of $\mathfrak{so}(p, q)$, $p+q = m$, $n \leq \frac{m(m-1)}{2}$ such that if $\mathcal{D}$ is a non-degenerate subspace of $\mathfrak{so}(p, q)$, then $\mathfrak{g}$ is isomorphic as a Lie algebra to the standard pseudo-metric 2-step nilpotent Lie algebra $\mathcal{G} = \mathbb{R}^{p,q} \oplus_{\perp} \mathcal{D}$.*

Proof. Let $\mathfrak{g} = V \oplus [\mathfrak{g}\,,\mathfrak{g}]_{\mathfrak{g}}$, $v_1, \dots, v_m$, be a basis of V, and $z_1, \dots, z_n$ a basis of $[\mathfrak{g}\,,\mathfrak{g}]_{\mathfrak{g}}$. Let $e_1, \dots, e_{p+q}$ be the standard orthonormal basis in $\mathbb{R}^{p,q}$ and $\langle\,\cdot\,,\cdot\,\rangle_{p,q}$ the scalar product.

Let $[v_i\,, v_j]_{\mathfrak{g}} = \sum_{k=1}^{n} C_{ij}^k z_k$ for $1 \leq i, j \leq m$ and $D^k = \eta_{p,q} C^k$. Choose a pair $p, q \in \mathbb{N}$, $p + q = m$, such that the space $\mathcal{D} = \operatorname{span}\{D^1, \dots, D^n\} \subset \mathfrak{so}(p, q)$ is non-degenerate with respect to the metric $\langle\cdot\,,\cdot\rangle_{\mathfrak{so}(p,q)}$. Let $\rho_1, \dots, \rho_n$ be a basis of $\mathcal{D}$ such that $\langle \rho_k\,, D^l \rangle_{\mathfrak{so}(p,q)} = \delta_{kl}$ for $1 \leq k, l \leq n$.

Define the linear isomorphism $T\colon\ \mathfrak{g} \to \mathcal{G}$ by

$$T(v_i) = e_i, \quad i = 1, \dots m, \qquad T(z_k) = -\rho_k, \quad k = 1, \dots, n.$$

We claim that T is a Lie algebra isomorphism and for that it suffices to show that

$$T([v_i\,, v_j]_{\mathfrak{g}}) = [T(v_i)\,, T(v_j)]_{\mathcal{G}}.$$

Note that

$$\begin{aligned}
\langle [T(v_i)\,, T(v_j)]_{\mathcal{G}}\,, D^k \rangle_{\mathfrak{so}(p,q)} &= \langle [e_i\,, e_j]_{\mathcal{G}}\,, D^k \rangle_{\mathfrak{so}(p,q)} = \langle D^k(e_i)\,, e_j \rangle_{p,q} \\
&= (e_i)^{\mathbf{t}} (D^k)^{\mathbf{t}} \eta_{p,q} e_j = ((D^k)^{\mathbf{t}} \eta_{p,q})_{ij} \\
&= ((C^k)^{\mathbf{t}})_{ij} = -C_{ij}^k = C_{ji}^k.
\end{aligned}$$

From the other side

$$\begin{aligned}\langle T([v_i\,,v_j]_{\mathfrak{g}})\,,D^k\rangle_{\mathfrak{so}(p,q)} &= \langle\sum_{r=1}^{n} C_{ij}^r T(z_r)\,,D^k\rangle_{\mathfrak{so}(p,q)} = -\sum_{r=1}^{n} C_{ij}^r\langle\rho_r\,,D^k\rangle_{\mathfrak{so}(p,q)}\\ &= -\sum_{r=1}^{n} C_{ij}^r\delta_{rk} = -C_{ij}^k = C_{ji}^k.\end{aligned}$$

□

Theorem 5.2.14. *If $\mathfrak{g}$ admits a basis with rational structure constants, then we may choose $\mathcal{D}$ to have a basis whose matrices only have entries in $\mathbb{Z}$, relative to the standard basis $e_1,\dots,e_m$ of $\mathbb{R}^{p,q}$.*

Proof. We assume that there exists a basis $\mathcal{B}=\{v_1,\dots,v_m,z_1,\dots,z_n\}$ of $\mathfrak{g}=V\oplus_\perp[\mathfrak{g}\,,\mathfrak{g}]$, with $v_1,\dots,v_m$ being a basis of V and $z_1,\dots,z_n$ a basis of $[\mathfrak{g}\,,\mathfrak{g}]$ such that the structural constants C_{ij}^k with respect to $\mathcal{B}$ are in $\mathbb{Q}$. We write $C_{ij}^k=\frac{a_{ij}^k}{b_{ij}^k}$ with $a_{ij}^k\in\mathbb{Z}$ and $b_{ij}^k\in\mathbb{N}\setminus\{0\}$. We define the natural number d as the least common multiple of the collection $\{b_{ij}^k|i,j=1,\dots,m,\quad k=1,\dots,n\}$ and define the basis $\mathcal{B}_d=\{\sqrt{d}v_1,\dots,\sqrt{d}v_m,z_1,\dots,z_n\}$. It follows that the structural constants $\tilde{C}_{ij}^k$ with respect of $\mathcal{B}_d$ are given by dC_{ij}^k as

$$\sum_{k=1}^{n}\tilde{C}_{ij}^k z_k=[\sqrt{d}v_i\,,\sqrt{d}v_j]=d[v_i\,,v_j]=d\sum_{k=1}^{n}C_{ij}^k z_k=\sum_{k=1}^{n}dC_{ij}^k z_k.$$

Hence $\tilde{C}_{ij}^k$ are natural numbers such that the matrix $\tilde{C}^k=dC^k$ only has entries in $\mathbb{Z}$. As we know from the first part of this Theorem there exists $p,q\in\mathbb{N}$, $p+q=m$ such that the n-dimensional subspace $\mathcal{D}=\operatorname{span}\{\eta_{p,q}C^1,\dots,\eta_{p,q}C^n\}$ is a non-degenerate subspace of $\mathfrak{so}(p,q)$ such that $\mathfrak{g}\cong\mathbb{R}^{p,q}\oplus\mathcal{D}$. As $\eta_{p,q}\tilde{C}^k=d\eta_{p,q}C^k\in\mathcal{D}$ and the entries of $\eta_{p,q}\tilde{C}^k$ lie obviously in $\mathbb{Z}$, it follows that there exists a basis of $\mathcal{D}$ whose matrices only have entries in $\mathbb{Z}$, relative to the standard basis $e_1,\dots,e_m$ of $\mathbb{R}^{p,q}$. □

5.2.7 Examples of standard pseudo-metric algebras

Let us consider three pseudo H-type algebras $\mathfrak{n}_{2,0}$, $\mathfrak{n}_{1,1}$, and $\mathfrak{n}_{0,2}$ and show that they can be realized in a standard pseudo-metric algebra form for some choice of $\mathfrak{so}(p,q)$.

Lie algebra $\mathfrak{n}_{2,0}$ The center of $\mathfrak{n}_{2,0}$ is isomorphic to $\mathbb{R}^2$ with standard Euclidean metric and the complement to the center is isomorphic to $\mathbb{R}^4$ with the standard Euclidean metric. Let (z_1,z_2) be the standard basis of $\mathbb{R}^2$ and let J_{z_1}, $J_{z_2}\in\mathfrak{so}(4)$ be such that

$$J_{z_1}^2=J_{z_2}^2=-\operatorname{Id}_{\mathbb{R}^4},\quad J_{z_1}J_{z_2}=-J_{z_2}J_{z_1}.$$

We choose the following orthonormal basis in $\mathbb{R}^4$

$$v_1=e_1,\quad v_2=J_{z_2}J_{z_1}v_1,\quad v_3=J_{z_1}v_1,\quad v_4=J_{z_2}v_1.$$

In the standard basis (e_1, e_2, e_3, e_4) the basis (v_1, v_2, v_3, v_4) and the matrices of maps J_{z_1}, J_{z_2} take the following form:

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = e_3, \quad v_4 = e_4,$$

$$J_{z_1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Maps J_{z_i} permute the basis of $\mathbb{R}^4$ by the following rule:

$$\begin{array}{llll} J_{z_1}v_1 = v_3, & J_{z_1}v_2 = v_4, & J_{z_1}v_3 = -v_1, & J_{z_1}v_4 = -v_2, \\ J_{z_2}v_1 = v_4, & J_{z_2}v_2 = -v_3, & J_{z_2}v_3 = v_2, & J_{z_2}v_4 = -v_1. \end{array}$$

According to the rule $\langle [v_\alpha\,, v_\beta], z_i\rangle_{2,0} = \langle J_{z_i}v_\alpha\,, v_\beta\rangle_{4,0}$ we calculate the structural constants in $[v_\alpha\,, v_\beta] = C^1_{\alpha\beta}z_1 + C^2_{\alpha\beta}z_2$ as follows

$$C^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.10}$$

We see that $C^i = -J_{z_i}$. This also follows from the choice of the orthonormal basis by

$$C^i_{\alpha\beta} = \langle [v_\alpha\,, v_\beta]\,, z_i\rangle_{2,0} = \langle J_{z_i}v_\alpha\,, v_\beta\rangle_{4,0} = v^{\mathbf{t}}_\alpha J^{\mathbf{t}}_{z_i}v_\beta = (J^{\mathbf{t}}_{z_i})_{\alpha\beta} = -(J_{z_i})_{\alpha\beta}.$$

Lie algebra $\mathfrak{n}_{1,1} \sim \mathbb{R}^{1,1} \oplus \mathbb{R}^{2,2}$. We start with the standard basis (z_1, z_2) of the center isomorphic to $\mathbb{R}^{1,1}$ and two skew-symmetric maps $J_{z_1},\ J_{z_2} \in \mathfrak{so}(2,2)$, such that

$$J^2_{z_1} = -\operatorname{Id}_{\mathbb{R}^{2,2}}, \quad J^2_{z_2} = \operatorname{Id}_{\mathbb{R}^{2,2}}, \quad J_{z_1}J_{z_2} = -J_{z_2}J_{z_1}.$$

We choose the following orthonormal basis in $\mathbb{R}^{2,2}$

$$v_1 = e_1, \quad v_2 = J_{z_1}v_1, \quad v_3 = J_{z_2}v_1, \quad v_4 = J_{z_2}J_{z_1}v_1.$$

In the standard basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^{2,2}$ the basis (v_1, v_2, v_3, v_4) and maps J_{z_1}, J_{z_2} take the following form:

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = e_3, \quad v_4 = e_4,$$

$$J_{z_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and

$$\begin{array}{llll} J_{z_1}v_1 = v_2, & J_{z_1}v_2 = -v_1, & J_{z_1}v_3 = -v_4, & J_{z_1}v_4 = v_3, \\ J_{z_2}v_1 = v_3, & J_{z_2}v_2 = v_4, & J_{z_2}v_3 = v_1, & J_{z_2}v_4 = v_2. \end{array}$$

According to the rule $\langle [v_\alpha\,, v_\beta]\,, z_i\rangle_{1,1} = \langle J_{z_i}v_\alpha\,, v_\beta\rangle_{2,2}$ we calculate the structural constants in $[v_\alpha\,, v_\beta] = C^1_{\alpha\beta}z_1 + C^2_{\alpha\beta}z_2$ as follows

$$C^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.11}$$

We see that $C^1 = -\eta_{2,2}J_{z_1}$ and $C^2 = \eta_{2,2}J_{z_2}$. This is also defined by the choice of an orthonormal basis as follows

$$\epsilon_i(1,1)C^i_{\alpha\beta} = \langle [v_\alpha\,, v_\beta]\,, z_i\rangle_{1,1} = \langle J_{z_i}v_\alpha\,, v_\beta\rangle_{2,2} = -v_\alpha^{\mathbf{t}}\eta_{2,2}J_{z_i}v_\beta = -(\eta_{2,2}J_{z_i})_{\alpha\beta}.$$

We recall the notation (5.1).

Lie algebra $\mathfrak{n}_{0,2} \sim \mathbb{R}^{0,2} \oplus \mathbb{R}^{2,2}$. We start from the standard basis (z_1, z_2) of the center isomorphic to $\mathbb{R}^{0,2}$ and two skew-symmetric maps $J_{z_1},\ J_{z_2} \in \mathfrak{so}(2,2)$:

$$J^2_{z_1} = J^2_{z_2} = \mathrm{Id}_{\mathbb{R}^{2,2}}, \quad J_{z_1}J_{z_2} = -J_{z_2}J_{z_1}.$$

We choose the following orthonormal basis in $\mathbb{R}^{2,2}$

$$v_1 = e_1, \quad v_2 = J_{z_1}J_{z_2}v_1, \quad v_3 = J_{z_1}v_1, \quad v_4 = J_{z_2}v_1.$$

In the standard basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^{2,2}$ the basis (v_1, v_2, v_3, v_4) and maps J_{z_1}, J_{z_2} take the following form:

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = e_3, \quad v_4 = e_4,$$

$$J_{z_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad J_{z_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{array}{llll} J_{z_1}v_1 = v_3, & J_{z_1}v_2 = v_4, & J_{z_1}v_3 = v_1, & J_{z_1}v_4 = v_2, \\ J_{z_2}v_1 = v_4, & J_{z_2}v_2 = -v_3, & J_{z_2}v_3 = -v_2, & J_{z_2}v_4 = v_1. \end{array}$$

According to the rule $\langle [v_\alpha\,, v_\beta]\,, z_i\rangle_{0,2} = \langle J_{z_i}v_\alpha\,, v_\beta\rangle_{2,2}$ we calculate the structural constants in $[v_\alpha\,, v_\beta] = C^1_{\alpha\beta}z_1 + C^2_{\alpha,\beta}z_2$ as follows

$$C^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad C^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that $C^i = \eta_{2,2}J_{z_i}$. This also follows from the choice of an orthonormal basis by

$$\epsilon_i(0,2)C^i_{\alpha\beta} = \langle [v_\alpha\,, v_\beta]\,, z_i\rangle_{0,2} = -\langle v_\alpha\,, J_{z_i}v_\beta\rangle_{2,2} = -v_\alpha^{\mathbf{t}}\eta_{2,2}J_{z_i}v_\beta = -(\eta_{2,2}J_{z_i})_{\alpha\beta}.$$

Since $\epsilon_i(0,2) = -1$ for $i = 1, 2$, we get $C^i = \eta_{2,2} J_{z_i}$. We recall the notation (5.1).

We see that algebras $\mathfrak{n}_{2,0}$ and $\mathfrak{n}_{0,2}$ coincide as Lie algebras. It can be interpreted as the following illustration of Theorem 5.2.13. The Lie algebra $\mathfrak{n}_{2,0}$ is isomorphic to the standard metric Lie algebra $\mathfrak{m} = \mathbb{R}^4 \oplus \mathcal{C}$ with $\mathcal{C} = \operatorname{span}\{C^1, C^2\} \subset \mathfrak{so}(4)$ and C^1, C^2 given by (5.10). This standard metric Lie algebra is the H-type algebra since the skew-symmetric maps $J_{z_1} = -C^1$ and $J_{z_2} = -C^2$ satisfies the additional conditions $J_{z_i}^2 = -\operatorname{Id}_{\mathbb{R}^4}$ and $J_{z_1} J_{z_2} = -J_{z_2} J_{z_1}$. Let us see if the Lie algebra $\mathfrak{n}_{2,0}$ can be isomorphic to a standard Lie algebra generated by another choice of $\mathbb{R}^{p,q}$, $p + q = 4$.

Cases $\mathbb{R}^{3,1}$ and $\mathbb{R}^{1,3}$. We calculate D^1 and D^2 by using $\eta_{3,1}$

$$D^1 = C^1\eta_{3,1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad D^2 = C^2\eta_{3,1} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\langle D^i, D^j \rangle_{\mathfrak{so}(3,1)} = \operatorname{tr}(\eta_{3,1}(D^i)^{\mathbf{t}}\eta_{3,1}D^j) = 0$ the subspace $\mathcal{D} = \operatorname{span}\{D^1, D^2\} \subset \mathfrak{so}(3,1)$ is degenerate and, actually, the scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{so}(3,1)}$ vanishes on $\mathcal{D}$ and therefore the Lie algebra $\mathfrak{n}_{2,0}$ can not be realized as a standard pseudo-metric Lie algebra in $\mathbb{R}^{3,1} \oplus \mathcal{D}$. Recall, that the index of the space $\mathfrak{so}(3,1)$ with respect to the trace metric $\langle \cdot, \cdot \rangle_{\mathfrak{so}(3,1)}$ is $(3,3)$. The same calculations are valid for the case of $\mathbb{R}^{1,3}$ and we conclude that the Lie algebra $\mathfrak{n}_{2,0}$ can not be realized as a standard pseudo-metric Lie algebra neither can $\mathbb{R}^{3,1} \oplus \mathcal{D}$, $\mathcal{D} \subset \mathfrak{so}(3,1)$ nor $\mathbb{R}^{1,3} \oplus \tilde{\mathcal{D}}$, $\tilde{\mathcal{D}} \subset \mathfrak{so}(1,3)$.

Case $\mathbb{R}^{2,2}$. In this case we use the matrix $\eta_{2,2}$ and get from the matrices in (5.10) the following two matrices

$$D^1 = C^1\eta_{2,2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad D^2 = C^2\eta_{2,2} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

In this case $\langle D^1, D^1 \rangle_{\mathfrak{so}(2,2)} = -4$, $\langle D^2, D^2 \rangle_{\mathfrak{so}(2,2)} = -4$, $\langle D^1, D^2 \rangle_{\mathfrak{so}(2,2)} = 0$. The subspace $\mathcal{D} = \operatorname{span} = \{D^1, D^2\} \subset \mathfrak{so}(2,2)$ is non-degenerate and has index $(r,s) = (0,2)$. Therefore, the Lie algebra $\mathfrak{n}_{2,0}$ can be realized as a standard metric Lie algebra $\mathbb{R}^{2,2} \oplus \mathcal{D}$, $\mathcal{D} \subset \mathfrak{so}(2,2)$, and it gives the pseudo H-type algebra $\mathfrak{n}_{0,2}$ constructed above. The last statement is valid due to the relations $J_{z_i}^2 = \operatorname{Id}_{\mathbb{R}^{2,2}}$ and $J_{z_1} J_{z_2} = -J_{z_2} J_{z_1}$.

Now we turn to the Lie algebra $\mathfrak{n}_{1,1}$. Analogue calculations show that this Lie algebra can be realized in $\mathbb{R}^4 \oplus \mathcal{C}$ with $\mathcal{C} = \operatorname{span}\{C^1, C^2\} \subset \mathfrak{so}(4)$ where C^1, C^2 are from (5.11), but this is not an H-type algebra (with positive definite scalar product). The Lie algebra cannot be realized neither in $\mathfrak{so}(3,1)$ nor in $\mathfrak{so}(1,3)$ due to the degeneracy of the corresponding spaces $\mathcal{D}$. In the case $\mathfrak{so}(2,2)$ the matrices

$$D^1 = C^1\eta_{2,2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D^2 = C^2\eta_{2,2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

are satisfying $\langle D^1\,, D^1\rangle_{\mathfrak{so}(2,2)} = 4$, $\langle D^2\,, D^2\rangle_{\mathfrak{so}(2,2)} = -4$, and $\langle D^1\,, D^2\rangle_{\mathfrak{so}(2,2)} = 0$ span a two dimensional non-degenerate space of index $(r,s) = (1,1)$ in $\mathfrak{so}(2,2)$. Recall, that the index of the space $\mathfrak{so}(2,2)$ is $(2,4)$. The standard metric Lie algebra $\mathbb{R}^{2,2} \oplus \mathcal{D}$, $\mathcal{D} \subset \mathfrak{so}(2,2)$, in this case is the pseudo H-type algebra $\mathfrak{n}_{1,1}$.

Finally we observe that $D^k = C^k \eta_{2,2} = -\eta_{2,2}\epsilon^k(1,1)J_{z_k}\eta_{2,2}$. Thus, we also have that $(D^1)^{\mathbf{t}} = -D^1$, $(D^2)^{\mathbf{t}} = D^2$ and $\mathcal{D}$ is closed under transposition.

5.3 Isomorphism properties

Isomorphism properties defined by skew-symmetric maps

Given a scalar product space $(V, \langle\cdot\,,\cdot\rangle_V)$ and the space $\mathfrak{o}(V)$ of skew-symmetric maps with a scalar product given by the trace. Let $J\colon U \to \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$ be an injective map and the space $J(U)$ be a non-degenerate subspace in $\mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$. Then we can pull back the trace metric from $\mathfrak{o}(V)$ to U. We write for any $c \neq 0$

$$\langle z\,, z'\rangle_{U,c} = -c^2 \operatorname{tr}(J_z J_{z'}), \quad \text{for any} \quad z, z' \in U. \tag{5.12}$$

This scalar product has a signature, which we denote by (r,s), and it depends on the choice of the map $J\colon U \to \mathfrak{o}(V)$. The scalar product space $(U, \langle\cdot\,,\cdot\rangle_{U,c})$ is degenerate if $J(U)$ is degenerate with respect to the trace metric. Let us assume that $\big(U, \langle\cdot\,,\cdot\rangle_{U,c}\big)$ is a non-degenerate scalar product space and let $[\cdot\,,\cdot]_c$ be the 2-step nilpotent Lie algebra structure on $\mathfrak{g} = V \oplus_\perp U$ defined by the map $J\colon U \to \mathfrak{o}(V)$ by means of (5.4). The spaces V and U are orthogonal with respect to the scalar product $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}} = \langle\cdot\,,\cdot\rangle_V + \langle\cdot\,,\cdot\rangle_{U,c}$.

Definition 5.3.1. *The Lie algebra $\mathfrak{g} = (V \oplus_\perp U, [\cdot\,,\cdot]_c, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}} = \langle\cdot\,,\cdot\rangle_V + \langle\cdot\,,\cdot\rangle_{U,c})$ described above is called the standard pseudo-metric 2-step nilpotent Lie algebra induced by the map $J\colon U \to \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$.*

By diagonalizing the matrix of the scalar product $\langle\cdot\,,\cdot\rangle_V$, we get the matrix $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$ defining the canonical scalar product $\langle u\,, v\rangle_{p,q} = \sum_{i=1}^{p} u_i v_i - \sum_{i=p+1}^{p+q} u_i v_i$ for $u = (u_1, \ldots, u_m)$, $v = (v_1, \ldots, v_m)$, $m = p+q$. The matrix of the skew-symmetric map J_z will satisfy $\eta_{p,q} J_z^{\mathbf{t}} \eta_{p,q} = -J_z$. Since the trace does not depend on the choice of coordinates we get a symmetric bi-linear function defining a scalar product on U. This can also be written as follows $\langle z\,, z'\rangle_{U,c} = c^2 \operatorname{tr}(\eta_{p,q} J_z^{\mathbf{t}} \eta_{p,q} J_{z'}) = -c^2 \operatorname{tr}(J_z J_{z'})$.

Lemma 5.3.2. *In the notation above if the scalar product $\langle z\,, z'\rangle_{U,c}$ is non-degenerate, then the standard pseudo-metric Lie algebra $\mathfrak{g}$ induced by J has no abelian factor. If two scalar products $\langle\cdot\,,\cdot\rangle_V^1$ and $\langle\cdot\,,\cdot\rangle_V^2$ on V have equal signature and the sets of spacelike (timelike and null) vectors are the same, then the commutator $[\cdot\,,\cdot]_c$ does not depend on the choice of scalar product $\langle\cdot\,,\cdot\rangle_V^i$ on V.*

Proof. If the scalar product $\langle z\,, z'\rangle_{U,c}$ is non-degenerate and the map $J\colon U \to \mathfrak{o}(V)$ is injective, then the Lie algebra structure $(\mathfrak{g}, [\cdot\,,\cdot]_c)$ is unique up to isomorphism and $\mathfrak{g}$ has trivial abelian factor by Lemma 5.2.2 and Corollary 5.2.6. □

Lemma 5.3.3. *Let $(V, \langle\cdot\,,\cdot\rangle_V)$ be a scalar product space, U_1, U_2 two finite dimensional vector spaces, and $J_1\colon U_1 \to \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$, $J_2\colon U_2 \to \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$ two injective skew-symmetric linear maps such that $J_1(U_1) = J_2(U_2) = W \subseteq \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$. Let $\mathfrak{g}_1 = (V \oplus U_1, [\cdot\,,\cdot]_1)$ and $\mathfrak{g}_2 = (V \oplus U_2, [\cdot\,,\cdot]_2)$ be two pseudo-metric Lie algebras induced by the maps J_1 and J_2. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as Lie algebras.*

Proof. It suffices to construct an isomorphism between Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ for the case when $J_1(U_1) = W = U_2$ and $J_2 = \iota\colon W \hookrightarrow \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V)$ is the inclusion map.

Define scalar products on U_1 and U_2 by

$$\langle \zeta\,, \zeta'\rangle_{U_1} = -\operatorname{tr}(J_1(\zeta)J_1(\zeta')), \quad \zeta, \zeta' \in U_1$$

$$\langle z\,, z'\rangle_{U_2} = -\operatorname{tr}\big(J_2(z)J_2(z')\big) = -\operatorname{tr}(zz'), \quad z, z' \in U_2 = W \subseteq \mathfrak{o}(V, \langle\cdot\,,\cdot\rangle_V).$$

Denote by $[\cdot\,,\cdot]_1$, $[\cdot\,,\cdot]_2$ the commutators constructed by means of these scalar products, correspondingly. Define the map $\varphi\colon V \oplus U_1 \to V \oplus U_2 = V \oplus W$ by

$$\varphi = \begin{cases} \mathrm{Id}_V & \text{on} \quad V, \\ J_1 & \text{on} \quad U_1. \end{cases}$$

Then we need to show that $\varphi([v\,,w]^1) = [\varphi(v)\,,\varphi(w)]^2$. Let $v, w \in V$, $z \in W$ be arbitrary and let $\zeta_0 \in U_1$ be the unique element such that $J_1(\zeta_0) = z = J_2(z)$. Then

$$\begin{aligned} \langle \varphi([v\,,w]^1)\,, z\rangle_{U_2} &= \langle J_1([v\,,w]^1)\,, J_1(\zeta_0)\rangle_{U_2} = -\operatorname{tr}(J_1([v\,,w]^1)J_1(\zeta_0)) \\ &= \langle [v\,,w]^1\,, \zeta_0\rangle_{U_1} = \langle J_1(\zeta_0)v\,, w\rangle_V \\ &= \langle J_2(z)v\,, w\rangle_V = \langle [v\,,w]^2\,, z\rangle_{U_2} = \langle [\varphi(v)\,,\varphi(w)]^2\,, z\rangle_{U_2}, \end{aligned}$$

because $\varphi = \mathrm{Id}_V$. This finishes the proof since the scalar product is non-degenerate. □

5.3.1 Action of $GL(p+q)$ and $\mathfrak{gl}(p+q)$ on the Lie algebra $\mathfrak{so}(p,q)$

Let $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$ and $A \in GL(m)$. Define the action ρ of A on $\mathfrak{so}(p,q)$ by

$$Z \mapsto \rho(A)Z = AZA^{\eta_{p,q}}, \quad \text{where} \quad A^{\eta_{p,q}} = \eta_{p,q}A^{\mathfrak{t}}\eta_{p,q}, \quad Z \in \mathfrak{so}(p,q).$$

Indeed, if $Z^{\eta_{p,q}} = -Z$, then $(AZA^{\eta_{p,q}})^{\eta_{p,q}} = AZ^{\eta_{p,q}}A^{\eta_{p,q}} = -AZA^{\eta_{p,q}}$. Recall that the operation $A^{\eta_{p,q}}$ gives us the transpose matrix to A with respect to the scalar product $\langle\cdot\,,\cdot\rangle_{p,q}$. The action ρ is the left action on $\mathfrak{so}(p,q)$. The map $\rho(A)$ is invertible and its inverse is given by $(\rho(A))^{-1} = \rho(A^{-1})$ that shows that $\rho(A) \in \operatorname{Aut}(\mathfrak{so}(p,q))$. Thus the map

$$\rho\colon GL(m) \to \operatorname{Aut}(\mathfrak{so}(p,q))$$

defines a group homomorphism with kernel $\pm\mathrm{Id}_{\mathbb{R}^m}$ for $m \geq 3$.

The differential $d\rho$ of the map ρ is the Lie algebra homomorphism

$$d\rho\colon \mathfrak{gl}(m) \to \operatorname{End}(\mathfrak{so}(p,q))$$

defined by $\mathcal{A} \mapsto d\rho(\mathcal{A})Z = \mathcal{A}Z + Z\mathcal{A}^{\eta_{p,q}}$, with $\mathcal{A} \in \mathfrak{gl}(m)$, $Z \in \mathfrak{so}(p,q)$. It is an injective map for $m \geq 3$. We prove some properties of the maps ρ and $d\rho$.

Lemma 5.3.4. *Let $A \in GL(m)$ and $\mathcal{A} \in \mathfrak{gl}(m)$ be any elements. Then*

$$\begin{aligned} \langle \rho(A)Z\,,Z'\rangle_{\mathfrak{so}(p,q)} &= \langle Z\,,\rho(A^{\eta_{p,q}})Z'\rangle_{\mathfrak{so}(p,q)}, \\ \langle d\rho(\mathcal{A})Z\,,Z'\rangle_{\mathfrak{so}(p,q)} &= \langle Z\,,d\rho(\mathcal{A}^{\eta_{p,q}})Z'\rangle_{\mathfrak{so}(p,q)} \end{aligned} \tag{5.13}$$

for any $Z, Z' \in \mathfrak{so}(p,q)$. We can reformulate (5.13) *by the following*

$$\big(\rho(A)\big)^{\eta_{p,q}} = \rho(A^{\eta_{p,q}}), \quad \big(d\rho(\mathcal{A})\big)^{\eta_{p,q}} = d\rho(\mathcal{A}^{\eta_{p,q}}).$$

Proof. We calculate

$$\langle \rho(A)Z\,,Z'\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(AZA^{\eta_{p,q}}Z') = -\operatorname{tr}(ZA^{\eta_{p,q}}Z'A) = \langle Z\,,\rho(A^{\eta_{p,q}})Z'\rangle_{\mathfrak{so}(p,q)}$$

by the property of the trace of interchanging products. The other equality is obtained similarly. $\square$

Lemma 5.3.5. *All 2-step nilpotent free algebras $F_2(p,q)$ with $p+q=m$ are isomorphic.*

Proof. To prove Lemma 5.3.5 we show that any 2-step nilpotent Lie algebra $F_2(p,q) = \mathbb{R}^{p,q} \oplus \mathfrak{so}(p,q)$ with $p+q=m$ is isomorphic to $F_2(m) = \mathbb{R}^m \oplus \mathfrak{so}(m)$. Let $v_{ij} = -\frac{1}{2}(E_{ij}-E_{ji})$, $i \le j = 1,\dots,m$, be the standard basis of the group $\mathfrak{so}(m)$. Here E_{ij} is the $(m\times m)$-matrix having 1 on the position (ij) and 0 everywhere else. Then the matrices $\phi_{ij} = -\frac{1}{2}(F_{ij}-F_{ji}) = -\frac{1}{2}(E_{ij}-E_{ji})\eta_{p,q}$, $i \le j = 1,\dots,m$, form a basis of the space $\mathfrak{so}(p,q)$. We define the isomorphism $f\colon \mathfrak{so}(m) \to \mathfrak{so}(p,q)$ by $f(v_{ji}) = \phi_{ji}$. Then we extend this isomorphism to the isomorphism $F_2(m) \to F_2(p,q)$ by

$$e_k \mapsto e_k, \qquad v_{ij} \mapsto \phi_{ij}, \qquad \text{for} \quad 0<k<m,\ 0<i\le j\le m=p+q.$$

It follows that

$$\begin{aligned} f([v_{jk}\,,e_i+v_{lr}]_{F_2(m)}) &= 0 = [\phi_{jk}\,,e_i+\phi_{lr}]_{F_2(p,q)} = [f(v_{jk})\,,f(e_i+v_{lr})]_{F_2(p,q)}, \\ f([e_i\,,e_j]_{F_2(m)}) &= f(v_{ij}) = \phi_{ij} = -\frac{1}{2}(E_{ij}-E_{ji})\eta_{p,q} = [e_i\,,e_j]_{F_2(p,q)} \\ &= [f(e_i)\,,f(e_j)]_{F_2(p,q)}. \end{aligned}$$

Hence f is a Lie algebra isomorphism.

At the end of the proof we observe that the orthogonal basis of $F_2(m)$ is mapped to the orthogonal basis of $F_2(p,q)$, $p+q=m$ under the isomorphism f. The equalities

$$\langle E_{ij}\,,E_{\alpha\beta}\rangle_{\mathfrak{so}(m)} = -\operatorname{tr}(E_{ij}E_{\alpha\beta}) = \delta_{i\alpha}\delta_{j\beta},$$

show that the basis $-\frac{1}{2}(E_{ij}-E_{ji})$ is orthonormal with respect to the trace metric. And the basis $\phi_{ij} = -\frac{1}{2}(F_{ij}-F_{ji})$ of the space $\mathfrak{so}(p,q)$ satisfies

$$\langle \phi_{ij}\,,\phi_{\alpha\beta}\rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}\big(\phi_{ji}\phi_{\alpha\beta}\big) = \epsilon_{ij}\delta_{i\alpha}\delta_{j\beta},$$

where

$$\epsilon_{ij} = \begin{cases} 1, & \text{if} \quad i<j\le p \text{ or } i>p, \\ -1 & \text{if} \quad j>p \text{ and } i\le p. \end{cases}$$

$\square$

Let us denote by $\mathrm{Aut}(F_2(p,q))$ the group of automorphisms of $F_2(p,q)$.

Lemma 5.3.6. *For any $\phi \in \mathrm{Aut}(F_2(p,q))$ there exists a unique element $A \in \mathrm{GL}(m)$, $m = p+q$ and $S \in \mathrm{Hom}(\mathbb{R}^{p,q}, \mathfrak{so}(p,q))$ such that*

a) $\phi(x) = Ax + S(x)$ *for all $x \in \mathbb{R}^{p,q}$,*

b) $\phi(Z) = AZA^{\eta_{p,q}}$ *for all $Z \in \mathfrak{so}(p,q)$.*

Conversely, given $(A,S) \in \mathrm{GL}(m) \times \mathrm{Hom}(\mathbb{R}^{p,q}, \mathfrak{so}(p,q))$, $m = p+q$, there is a unique automorphism $\phi \in \mathrm{Aut}(F_2(p,q))$ that satisfies a) and b).

Proof. Eberlein proved Lemma 5.3.6 in [43] if we replace $F_2(p,q)$ with $F_2(m)$. Let f be an isomorphism of $F_2(m)$ and $F_2(p,q)$, $m = p+q$ which exists by Lemma 5.3.5. Then for any $\varphi \in \mathrm{Aut}(F_2(m))$ the superposition $\phi = f \circ \varphi \circ f^{-1}$ is an automorphism of $F_2(p,q)$. Thus for every automorphism $\phi \in \mathrm{Aut}(F_2(p,q))$ there exists unique $\varphi \in \mathrm{Aut}(F_2(m))$, $m = p+q$, with $\phi = f \circ \varphi \circ f^{-1}$ and moreover unique $A \in \mathrm{GL}(m)$, $S' \in \mathrm{Hom}(\mathbb{R}^m, \mathfrak{so}(m))$ such that the properties a) and b) are satisfied with $S := f \circ S' \in \mathrm{Hom}(\mathbb{R}^{p,q}, \mathfrak{so}(p,q))$. The converse statement follows easily. □

Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra with $\dim([\mathfrak{g}\,,\mathfrak{g}]) = n$ and m-dimensional complement V, such that $\mathfrak{g} = V \oplus [\mathfrak{g}\,,\mathfrak{g}]$. A basis $\{w_1,\dots,w_m,Z_1,\dots,Z_n\}$, where $V = \mathrm{span}\{w_1,\dots,w_m\}$, $[\mathfrak{g}\,,\mathfrak{g}] = \mathrm{span}\{Z_1,\dots,Z_n\}$ is called *adapted*. If $[w_i\,,w_j] = \sum_{k=1}^n C_{ij}^k Z_k$, then we call the space $\mathcal{C} = \mathrm{span}\{C^1,\dots,C^n\} \subset \mathfrak{so}(m)$, with $C^k := (C_{ij}^k)$, the structure space and the spaces $\mathcal{D}_{p,q} = \mathrm{span}\{C^1\eta_{p,q},\dots,C^n\eta_{p,q}\} \subset \mathfrak{so}(p,q)$ are called structure $\eta_{p,q}$-spaces. We aim to show in the following propositions that structure $\eta_{p,q}$-spaces of a 2-step nilpotent Lie algebra $\mathfrak{g}$ are orbits in the Grassmann manifold.

Proposition 5.3.7. *Let $\{w_1,\dots,w_m,Z_1,\dots,Z_n\}$ and $\{\hat{w}_1,\dots,\hat{w}_m,\hat{Z}_1,\dots,\hat{Z}_n\}$ be two adapted bases of a 2-step nilpotent Lie algebra $\mathfrak{g}$ with corresponding structure $\eta_{p,q}$-spaces $\mathcal{D}_{p,q} = \mathrm{span}\{C^1\eta_{p,q},\dots,C^p\eta_{p,q}\}$ and $\hat{\mathcal{D}}_{p,q} = \mathrm{span}\{\hat{C}^1\eta_{p,q},\dots,\hat{C}^p\eta_{p,q}\}$. Let $A \in \mathrm{GL}(m)$, $m = p+q$ be such that $\hat{w}_i = \sum_{j=1}^m A_{ij}w_j$, then $A\mathcal{D}_{p,q}A^{\eta_{p,q}} = \hat{\mathcal{D}}_{p,q}$.*

Proof. The proposition follows from the definition of the action of $\mathrm{GL}(m)$ on $\mathfrak{so}(p,q)$ and the fact that under the given assumptions $A\mathcal{C}A^{\eta_{p,q}} = \hat{\mathcal{C}}$ by [43]. □

Proposition 5.3.8. *Let d be an integer with $1 \leq d \leq \dim(\mathfrak{so}(p,q))$. Let $W_1, W_2 \subset \mathfrak{so}(p,q)$ be two d-dimensional non-degenerate with respect to $\langle\,\cdot\,,\cdot\,\rangle_{\mathfrak{so}(p,q)}$ subspaces. Then the following statements are equivalent:*

1) The Lie algebra $F_2(p,q)/W_1$ is isomorphic to $F_2(p,q)/W_2$.

2) There exists an element $A \in \mathrm{GL}(m)$, $m = p+q$ such that $AW_1A^{\eta_{p,q}} = W_2$.

3) The Lie algebra $F_2(p,q)/W_1^{\perp}$ is isomorphic to $F_2(p,q)/W_2^{\perp}$.

Proof. First we show that items 1) and 2) are equivalent. Recall that for any pair (p, q) with $p + q = m$ and $W_1, W_2 \subset \mathfrak{so}(p, q)$ we have $W_1\eta_{p,q}$, $W_2\eta_{p,q} \subset \mathfrak{so}(m)$. It was shown in [43] that the Lie algebras $F_2(m)/(W_1\eta_{p,q})$ and $F_2(m)/(W_2\eta_{p,q})$ are isomorphic if and only if there exists $A \in \mathrm{GL}(m)$ such that $AW_1\eta_{p,q}A^{\mathsf{t}} = W_2\eta_{p,q}$. The last equality can be written as $AW_1A^{\eta_{p,q}} = W_2$. Let f be an isomorphism between $F_2(m)$ and $F_2(p, q)$. Hence $W_i = f(W_i\eta_{p,q})$ and $F_2(p, q)/W_i = f(F_2(m)/(W_i\eta_{p,q}))$ for $i = 1, 2$. This implies that $F_2(m)/(W_1\eta_{p,q})$ and $F_2(m)/(W_2\eta_{p,q})$ are isomorphic if and only if $F_2(p, q)/W_1$ is isomorphic to $F_2(p, q)/W_2$.

Now we show that items 1) and 3) are equivalent. The arguments above illustrates that $F_2(p, q)/W_1$ is isomorphic to $F_2(p, q)/W_2$ if and only if $F_2(m)/(W_1\eta_{p,q})$ is isomorphic to $F_2(m)/(W_2\eta_{p,q})$. This is equivalent to the statement that $F_2(m)/(W_1\eta_{p,q})^\perp$ is isomorphic to $F_2(m)/(W_2\eta_{p,q})^\perp$ by [43]. Define the Lie algebra isomorphism $f^*\colon F_2(m) \to F_2(p, q)$ by

$$e_i \mapsto \begin{cases} e_i & \text{for } 1 \le i \le p, \\ -e_i & \text{for } p+1 \le i \le p+q, \end{cases} \qquad -\frac{1}{2}(E_{ij} - E_{ji}) \mapsto \eta_{p,q}\left(-\frac{1}{2}(E_{ij} - E_{ji})\right).$$

Then $F_2(m)/(W_1\eta_{p,q})^\perp$ is isomorphic to the quotient $F_2(m)/(W_2\eta_{p,q})^\perp$ if and only if $F_2(p, q)/\eta_{p,q}(W_1\eta_{p,q})^\perp$ is isomorphic to $F_2(p, q)/\eta_{p,q}(W_2\eta_{p,q})^\perp$.

It only remains to prove that W_i, $i = 1, 2$ is orthogonal to $\eta_{p,q}(W_i\eta_{p,q})^\perp$ with respect to the metric $\langle \cdot , \cdot \rangle_{\mathfrak{so}(p,q)}$. For any $w \in W_i$ and any $v \in (W_i\eta_{p,q})^\perp$ it follows that

$$\langle w , \eta_{p,q}v \rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(w\eta_{p,q}v) = \langle w\eta_{p,q} , v \rangle_{\mathfrak{so}(m)} = 0,$$

as $w\eta_{p,q} \in W_i\eta_{p,q}$ and $v \in (W_i\eta_{p,q})^\perp$. Since $\dim(\eta_{p,q}(W_i\eta_{p,q})^\perp) = \dim(\mathfrak{so}(p, q)) - \dim(W_i)$ and W_i non-degenerate, it follows that $\eta_{p,q}(W_i\eta_{p,q})^\perp = W_i^\perp$. □

Proposition 5.3.9. *Let $\{w_1, \dots , w_m, Z_1, \dots , Z_n\}$ be an adapted basis for a 2-step nilpotent Lie algebra $\mathfrak{g}$ with structure space $\mathcal{C} = \operatorname{span}\{C^1, \dots , C^n\} \subset \mathfrak{so}(m)$. Let $\rho\colon F_2(p, q) \to \mathfrak{g}$, $p + q = m$, be the unique Lie algebra homomorphism defined by $\rho(e_i) = w_i$ for $i = 1, \dots , m$.*
Then ρ is surjective and if $\mathcal{C}\eta_{p,q} \subset \mathfrak{so}(p, q)$ is non-degenerate, then $\ker(\rho) = (\mathcal{C}\eta_{p,q})^\perp$ is the orthogonal complement of $\mathcal{C}\eta_{p,q}$ in $\mathfrak{so}(p, q)$ with respect to $\langle \cdot , \cdot \rangle_{\mathfrak{so}(p,q)}$.

Proof. It is known that the Lie algebra homomorphism $\rho_1\colon F_2(m) \to \mathfrak{g}$ with $\rho_1(e_i) = w_i$ for $i = 1, \dots , m$ is surjective and $\ker(\rho_1)$ is the orthogonal complement to $\mathcal{C}$ in $\mathfrak{so}(m)$ with respect to $\langle \cdot , \cdot \rangle_{\mathfrak{so}(m)}$, see for instance [43]. Then we define the surjective linear map $\rho = \rho_1 \circ (f^*)^{-1}\colon F_2(p, q) \to \mathfrak{g}$ with f^* being the isomorphism between $F_2(m)$ and $F_2(p, q)$ from the proof of Proposition 5.3.8. The same proof shows that if $\mathcal{C}\eta_{p,q}$ is non-degenerate in $\mathfrak{so}(p, q)$, then $(\mathcal{C}\eta_{p,q})^\perp = \eta_{p,q}(\mathcal{C}^\perp)$. Since

$$(f^*)^{-1}((\mathcal{C}\eta_{p,q})^\perp) = (f^*)^{-1}(\eta_{p,q}(\mathcal{C}^\perp)) = \eta_{p,q}^2\mathcal{C}^\perp = \mathcal{C}^\perp = \ker(\rho_1),$$

it follows that $\ker(\rho) = (\mathcal{C}\eta_{p,q})^\perp$. □

Corollary 5.3.10. *Let W_1 and W_2 be two non-degenerate d-dimensional subspaces of $\mathfrak{so}(p,q)$, and let $\mathfrak{g}_1 = \mathbb{R}^{p,q} \oplus W_1$ and $\mathfrak{g}_2 = \mathbb{R}^{p,q} \oplus W_2$ be the corresponding standard pseudo-metric 2-step nilpotent Lie algebras. Then the following statements are equivalent:*

- *The Lie algebra $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2$.*
- *There exists $A \in \mathrm{GL}(m)$ such that $AW_1A^{\eta_{p,q}} = W_2$, $p+q=m$.*

Proof. The Lie algebras $\mathfrak{g}_i$ are isomorphic to $F_2(p,q)/W_i^{\perp}$ for $i = 1,2$ by Proposition 5.3.9. The statement of the corollary follows by using Proposition 5.3.8. □

Assume that $\mathfrak{g}$ is a 2-step nilpotent Lie algebra with a 1-dimensional commutator ideal $[\mathfrak{g}\,,\mathfrak{g}]$ and there exist positive integers p,q and a non-degenerate one dimensional subspace W in $\mathfrak{so}(p,q)$ such that $\mathfrak{g}$ is isomorphic to $\mathbb{R}^{p,q} \oplus W$ with $m = p+q \geq 2$. We define the set $\mathcal{A}_{p,q} = \{Z \in \mathfrak{so}(p,q) |$ rank Z is maximal$\}$.

Corollary 5.3.11. *The group $O(m)$ acts transitively by $\eta_{p,q}$-conjugation on $\mathcal{A}_{p,q}$, where $m = p+q$.*

Proof. We define the set $\mathcal{A}_m = \{Z \in \mathfrak{so}(m) |$ rank Z is maximal$\}$ which is Zariski open in $\mathfrak{so}(m)$. The group $O(m)$ acts transitively on it by conjugation, see [43]. Notice that $\mathcal{A}_m\eta_{p,q} = \mathcal{A}_{p,q}$. For every $Z, Y \in \mathcal{A}_m$ there exists an $A \in O(m)$ such that $Z = AYA^{-1} = AYA^{\mathbf{t}}$. Then

$$Z\eta_{p,q} = AY\eta_{p,q}^2A^{-1}\eta_{p,q} = AY\eta_{p,q}^2A^t\eta_{p,q} = AY\eta_{p,q}A^{\eta_{p,q}}$$

with $Z\eta_{p,q}, Y\eta_{p,q} \in \mathcal{A}_{p,q}$. This finishes the proof. □

5.4 Some useful facts about Lie triple systems

Definition 5.4.1. *Let W be a subspace of $\mathfrak{so}(p,q)$ such that $[a\,,[b\,,c]] \in W$ for all $a,b,c \in W$. The subspace W is called a Lie triple system in $\mathfrak{so}(p,q)$.*

Define the set

$$\mathfrak{Z}(W) = \{a \in W \mid [a\,,b] = 0 \text{ for all } b \in W\} \tag{5.14}$$

and call it center of W and we say that $\mathfrak{Z}(W)$ is compact if $\exp(\mathfrak{Z}(W))$ is a compact subgroup of $\mathrm{SO}(p,q)$. Remind that $\mathrm{SO}(p) \times SO(q)$ is a maximal compact subgroup of $\mathrm{SO}(p,q)$.

Theorem 5.4.2. *The set $\exp(\mathfrak{Z}(W))$ is a connected abelian subgroup of $SO(p,q)$ for any Lie triple system $W \subset \mathfrak{so}(p,q)$.*

Proof. First we prove that $\exp(\mathfrak{Z}(W))$ is a subgroup of $SO(p,q)$. Obviously, Id is an element of $\exp(\mathfrak{Z}(W))$ as $0 \in \mathfrak{Z}(W)$. If $a \in \mathfrak{Z}(W)$, then $-a \in \mathfrak{Z}(W)$, such that for any element $\exp(a)$ in $\mathfrak{Z}(W)$, its inverse $\exp(-a)$ also lies in $\mathfrak{Z}(W)$. Now we want to prove that $\exp(a)\exp(b) \in \mathfrak{Z}(W)$ and $\exp(a)\exp(b) = \exp(b)\exp(a)$ for any $a, b \in \mathfrak{Z}(W)$. As the elements of $\mathfrak{Z}(W)$ commute, i.e. $[a\,, b] = 0$, we conclude that $\exp(a + b) = \exp(a)\exp(b) = \exp(b)\exp(a)$. We note that by linearity of the Lie bracket we receive $[t_1 a + t_2 b\,, c] = t_1[a\,, c] + t_2[b\,, c] = 0$ for any $a, b, c \in \mathfrak{Z}(W)$ and any $t_1, t_2 \in \mathbb{R}$. Thus $a + b \in \mathfrak{Z}(W)$ and so $\exp(a + b) \in \exp(\mathfrak{Z}(W))$.

We claim that the set $\exp(\mathfrak{Z}(W))$ is path connected. For any $a, b \in \mathfrak{Z}(W)$ we define the path $\gamma_{a,b}\colon [0, 1] \to \exp(\mathfrak{Z}(W))$ from $\exp(a)$ to $\exp(b)$ by $\gamma_{a,b}(t) = \exp(tb + (1-t)a)$. This is a path as $\gamma_{a,b}(0) = \exp(a)$, $\gamma_{a,b}(1) = \exp(b)$ and as we proved before $tb+(1-t)a \in \mathfrak{Z}(W)$ for any $t \in [0, 1]$. □

Example 5.4.3. *Recall Example 5.2.11, where the subspace $W \subset \mathfrak{so}(l,l)$ was defined by the Clifford algebra representations on $\mathbb{R}^{l,l}$. The case $s = 0$ was studied in [42]. We state as an example of Lie triple systems the following proposition.*

Proposition 5.4.4. *The space W is a Lie triple system of $\mathfrak{so}(l,l)$ with trivial center.*

Proof. First we show that the vector space W is a Lie triple system. For any $X_1, X_2, X_3 \in W$, with $X_i = \sum_{j=1}^{r+s} \lambda_{ij} J(Z_j)$ with $\lambda_{ij} \in \mathbb{R}$, where $\{Z_1, \dots, Z_{r+s}\}$ is an orthonormal basis of $\mathbb{R}^{r,s}$ with $\langle\, Z_i\,, Z_j\,\rangle_{\mathbb{R}^{r,s}} = \epsilon_i(r,s)\delta_{ij}$. It follows that

$$[X_1\,, [X_2\,, X_3]] = \sum_{j,k,l=1}^{r+s} \lambda_{1j}\lambda_{2k}\lambda_{3l}[J(Z_j)\,, [J(Z_k)\,, J(Z_l)]]. \tag{5.15}$$

If we prove that $[J(Z_j)\,, [J(Z_k)\,, J(Z_l)]] \in W$ for all $j, k, l \in \{1, \dots, r+s\}$, then it follows that $[X_1\,, [X_2\,, X_3]] \in W$. We recall here that $J(Z_j)J(Z_k) = -J(Z_k)J(Z_j)$ for all $j \neq k$ and $j, k \in \{1, \dots, r+s\}$. If all indices j, k, l are different, then we get

$$\begin{aligned}
[J(Z_j)\,, [J(Z_k)\,, J(Z_l)]] &= [J(Z_j)\,, J(Z_k)J(Z_l)] - [J(Z_j)\,, J(Z_l)J(Z_k)] \\
&= J(Z_j)J(Z_k)J(Z_l) - J(Z_k)J(Z_l)J(Z_j) \\
&- J(Z_j)J(Z_l)J(Z_k) + J(Z_l)J(Z_k)J(Z_j) \\
&= J(Z_j)J(Z_k)J(Z_l) - J(Z_j)J(Z_k)J(Z_l) \\
&+ J(Z_j)J(Z_k)J(Z_l) - J(Z_j)J(Z_k)J(Z_l) = 0 \in W.
\end{aligned}$$

If $j = k$, then $[J(Z_j)\,, [J(Z_j)\,, J(Z_l)]] = -4\langle Z_j\,, Z_j\rangle_U J(Z_l) \in W$. If $k = l$ or $j = k = l$, then $[J(Z_j)\,, [J(Z_k)\,, J(Z_k)]] = 0 \in W$. We conclude that $W = J(\mathbb{R}^{r,s})$ is a Lie triple system.

Let us show that the center of W defined by 5.14 is trivial. For any $Z, Z' \in \mathbb{R}^{r,s}$ we obtain

$$\begin{aligned}
[J(Z)\,, J(Z')] &= J(Z)J(Z') - J(Z')J(Z) \\
&= \begin{cases} 2J(Z)J(Z') & \text{if } \langle Z\,, Z'\rangle_{r,s} = 0, \\ -2(J(Z')J(Z) + \langle Z\,, Z'\rangle_{r,s}\,\mathrm{Id}_V) & \text{if } \langle Z\,, Z'\rangle_{r,s} \neq 0. \end{cases}
\end{aligned}$$

Let us assume that the center is non-trivial and there is $Z \in \mathfrak{Z}(W)$, $Z \neq 0$, i. e. $[J(Z), J(Z^{'})] = 0$ for all $Z^{'} \in W$. There are two possible cases: $\langle Z, Z\rangle_{r,s} \neq 0$ and $\langle Z, Z\rangle_{r,s} = 0$.

Case $\langle Z, Z\rangle_{r,s} \neq 0$. Then $J(Z)^2 = -\langle Z, Z\rangle_{r,s} \operatorname{Id}_V$, which implies that $J(Z)$ is invertible. The orthogonal complement to $\operatorname{span}\{Z\}$ is a non-degenerate scalar product space and there is $Z^{'} \in W \cap (\operatorname{span}\{Z\})^{\perp}$ such that $\langle Z^{'}, Z^{'}\rangle_{r,s} \neq 0$ and $\langle Z, Z^{'}\rangle_{r,s} = 0$. Then $J(Z^{'})$ is also invertible and so is $J(Z)J(Z^{'})$, that yields $J(Z)J(Z^{'}) \neq 0$. It follows that $[J(Z), J(Z^{'})] = 2J(Z)J(Z^{'}) \neq 0$, which is a contradiction to the assumption that $Z \in \mathfrak{Z}(W)$ with $Z \neq 0$.

Case $\langle Z, Z\rangle_{r,s} = 0$. First we note that $J(Z)^2 = 0$ and therefore, $J(Z)$ can not be invertible. Let $Z^{'}$ be an element of W such that $\langle Z, Z^{'}\rangle_{r,s} \neq 0$, which exists because $\langle \cdot, \cdot\rangle_{r,s}$ is non-degenerate. Then, since $Z \in \mathfrak{Z}(W)$, we obtain

$$[J(Z), J(Z^{'})] = -2(J(Z^{'})J(Z) + \langle Z, Z^{'}\rangle_{r,s} \operatorname{Id}_V) = 0,$$

which is equivalent to $J(Z^{'})J(Z) = -\langle Z, Z^{'}\rangle_{r,s} \operatorname{Id}_V$. But this implies that $J(Z)$ is invertible with the inverse $-(\langle Z, Z^{'}\rangle_{r,s})^{-1} J(Z^{'})$. We obtain a contradiction. □

Proposition 5.4.5. *Let* $(\mathfrak{g}, [\cdot, \cdot])$ *be a Lie algebra and* W *be its Lie triple system. Then* $[W, W]$ *and* $W + [W, W]$ *are subalgebras of* $\mathfrak{g}$.

Proof. To show that $[W, W]$ is a subalgebra, we need to check

$$\big[[W, W], [W, W]\big] \subset [W, W].$$

Let $w_1, w_2, w_1', w_2' \in W$, then with the notation $[w_1', w_2'] = u$ we get by Jacobi identity

$$\big[[w_1, w_2], [w_1', w_2']\big] = \big[[w_1, w_2], u\big] = -[[w_2, u], w_1] - [[u, w_1], w_2] \in [W, W]$$

since $[w_2, u], [u, w_1] \in W$ by the definition of the Lie triple system: $[W, [W, W]] \subset W$.

To prove the second statement we choose arbitrary $a, b, c, x, y, z \in W$ and obtain

$$\big[a + [b, c], x + [y, z]\big] = [a, x] + [a, [y, z]] + [[b, c], x] + [[b, c], [y, z]] \in W + [W, W]$$

by the first statement and the definition of the Lie triple system. □

Remark 5.4.6. *Let us denote the Lie algebras in Proposition 5.4.5 by* $\mathfrak{p} = W$, $\mathfrak{k} = [W, W]$, *and* $\mathfrak{L} = W + [W, W]$. *Then Proposition 5.4.5 implies that the Lie algebra* $\mathfrak{L}$ *admits the decomposition* $\mathfrak{L} = \mathfrak{k} + \mathfrak{p}$ *with Cartan pair* $\mathfrak{k}, \mathfrak{p}$ *satisfying the following properties:*

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \tag{5.16}$$

Note that if a Lie algebra $\mathfrak{h}$ *admits a decomposition* $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$ *satisfying* (5.16), *then there is an involution* $\theta\colon \mathfrak{h} \to \mathfrak{h}$ $(\theta^2 = \operatorname{Id}_{\mathfrak{h}})$ *possessing the following properties:*

$$\mathfrak{k} \subset \mathfrak{h} \quad \textit{is such that} \quad \theta(t) = t, \quad \forall\, t \in \mathfrak{k},$$

$$\mathfrak{p} \subset \mathfrak{h} \quad \textit{is such that} \quad \theta(p) = -p, \quad \forall\, p \in \mathfrak{p}.$$

Given a Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot])$, we denote by $\mathrm{ad}_v\colon \mathfrak{g} \to \mathfrak{g}$ the linear map defined by $\mathrm{ad}_v(u) = [v\,,u]$. The map $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a Lie algebra homomorphism, named the adjoint representation of the Lie algebra $\mathfrak{g}$. The kernel of the adjoint map ad is the center of the Lie algebra.

Definition 5.4.7. *Let $\mathfrak{g}$ be a Lie algebra. A scalar product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{g}$ is called* ad-*invariant if*

$$\langle \mathrm{ad}_v(u)\,, w\rangle = -\langle u\,, \mathrm{ad}_v(w)\rangle. \tag{5.17}$$

Equivalently, it can be stated that the map $\mathrm{ad}_v\colon \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to the scalar product $\langle\cdot\,,\cdot\rangle$.

Proposition 5.4.8. *Let $(\mathfrak{g}, [\cdot\,,\cdot])$ be a Lie algebra, W be its Lie triple system, and $\mathcal{L} = W + [W\,,W]$. Let $\mathfrak{Z}(W)$ be the center of W and $(\cdot\,,\cdot)_{\mathcal{L}}$ be an* ad-*invariant inner product on $\mathcal{L}$. Then the following is true.*

1. *Denote by $\mathfrak{Z}(\mathcal{L})$ the center of $\mathcal{L}$. Then the Lie algebra $\mathcal{L}$ is decomposed into the direct sum of two ideals $\mathcal{L} = \mathfrak{Z}(\mathcal{L}) \oplus_\perp [\mathcal{L}\,,\mathcal{L}]$, where the decomposition is orthogonal with respect to $(\cdot\,,\cdot)_{\mathcal{L}}$.*
2. *$\mathfrak{Z}(W) \subseteq \mathfrak{Z}(\mathcal{L})$.*
3. *The center $\mathfrak{Z}([\mathcal{L}\,,\mathcal{L}])$ of $[\mathcal{L}\,,\mathcal{L}]$ is trivial.*
4. *If $\mathfrak{Z}(\mathcal{L}) \neq 0$, then $\mathfrak{Z}(W) \neq 0$.*
5. *If $\mathfrak{Z}(W) = 0$, then $\mathcal{L} = [\mathcal{L}\,,\mathcal{L}]$.*

Proof. Proof of 1. Let $\mathfrak{Z}(\mathcal{L})$ be the center of the algebra $\mathcal{L}$. Let us show that

$$\mathfrak{Z}(\mathcal{L}) = [\mathcal{L}\,,\mathcal{L}]^\perp \tag{5.18}$$

with respect to the ad-invariant inner product $(\cdot\,,\cdot)_{\mathcal{L}}$. Let $z \in [\mathcal{L}\,,\mathcal{L}]^\perp$ and $u, v \in \mathcal{L}$ be arbitrary. Then

$$([u\,,z]\,,v)_{\mathcal{L}} = -(z\,,[u\,,v])_{\mathcal{L}} = 0$$

since the inner product is ad-invariant. It shows that $[u\,,z] = 0$ and therefore $z \in \mathfrak{Z}(\mathcal{L})$, that implies $\mathfrak{Z}(\mathcal{L}) \supset [\mathcal{L}\,,\mathcal{L}]^\perp$. Reversing the arguments we show the inverse inclusion and conclude that $\mathcal{L} = \mathfrak{Z}(\mathcal{L}) \oplus_\perp [\mathcal{L}\,,\mathcal{L}]$ by (5.18).

Proof of 2. Let $z \in \mathfrak{Z}(W)$ and $u, v, w \in W$ be arbitrary. Then $\big[z\,,u + [v\,,w]\big] = [z\,,u] - [w\,,[z\,,v]] - [v\,,[w\,,z]] = 0$ by the Jacobi identity. Thus $z \in \mathfrak{Z}(\mathcal{L})$.

Proof of 3. Let $z \in \mathfrak{Z}([\mathcal{L}\,,\mathcal{L}])$. Then for any $u \in \mathcal{L}$ and $a \in [\mathcal{L}\,,\mathcal{L}]$ we have

$$0 = (u\,,[z\,,a])_{\mathcal{L}} = (z\,,[u\,,a])_{\mathcal{L}}.$$

Therefore, $z \in [\mathcal{L}\,,\mathcal{L}]^\perp = \mathfrak{Z}(\mathcal{L})$ as $[\mathcal{L}\,,\mathcal{L}] = [\mathcal{L}\,,[\mathcal{L}\,,\mathcal{L}]]$ by item 1. Simultaneously $z \in \mathfrak{Z}([\mathcal{L}\,,\mathcal{L}]) \subset [\mathcal{L}\,,\mathcal{L}]$. We conclude that $z = 0$ by item 1.

PROOF OF 4. Let $z \in \mathfrak{Z}(\mathcal{L})$ and $z \neq 0$. Then

$$[\mathcal{L}\,,\mathcal{L}] \subsetneq \mathcal{L} = W + [W\,,W] \quad ([\mathcal{L}\,,\mathcal{L}] \quad \text{is a proper subset of} \quad \mathcal{L} \quad \text{by item 1}).$$

Since $[\mathcal{L}\,,\mathcal{L}] = [W\,,W] + [W\,,[W\,,W]]$, we conclude that

$$[W\,,[W\,,W]] \subsetneq W \quad ([W\,,[W\,,W]] \quad \text{is a proper subset of} \quad W).$$

Let $[W\,,[W\,,W]]^{\perp}$ be the orthogonal complement to $[W\,,[W\,,W]]$ in $\mathcal{L}$ with respect to $(\cdot\,,\cdot)_{\mathcal{L}}$. Then $A = W \cap [W\,,[W\,,W]]^{\perp} \neq \emptyset$. We claim that $A \subset \mathfrak{Z}(W)$. Let $a, b, c \in W$ and $y \in A$, $y \neq 0$ be arbitrary. Then $[c\,,[a\,,b]] \subset [W\,,[W\,,W]]$ and therefore

$$0 = (y\,,[c\,,[a\,,b]])_{\mathcal{L}} = ([y\,,c]\,,[a\,,b])_{\mathcal{L}} \quad \Longrightarrow \quad [y\,,W] \subset [W\,,W]^{\perp}.$$

From the other side $[y\,,W] \subset [W\,,W]$, which implies $[y\,,W] = 0$ and thus $y \in \mathfrak{Z}(W)$. We conclude that $\mathfrak{Z}(W) \neq 0$.

PROOF OF 5. If $\mathfrak{Z}(W) = 0$, then we conclude that $\mathfrak{Z}(\mathcal{L}) = 0$ by item 4 and so $\mathcal{L} = [\mathcal{L}\,,\mathcal{L}]$ by item 1. $\square$

Our next step is to study irreducible Lie triple systems in $\mathfrak{g}$. We recall some definitions and properties.

Definition 5.4.9. *The Killing form $B_{\mathfrak{g}}$ on a Lie algebra $\mathfrak{g}$ is the map $B_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by*

$$B_{\mathfrak{g}}(u, v) := \operatorname{tr}(\operatorname{ad}_u \circ \operatorname{ad}_v).$$

The kernel of the Killing form $B_{\mathfrak{g}}$ on a Lie algebra $\mathfrak{g}$ is defined as

$$\ker(B_{\mathfrak{g}}) = \{x \in \mathfrak{g} \mid \ B_{\mathfrak{g}}(x, u) = 0 \text{ for all } u \in \mathfrak{g}\}.$$

Notice that the kernel of a Killing form is always an ideal of $\mathfrak{g}$ due to the adjoint invariance of the Killing form, i.e. $B_{\mathfrak{g}}([x\,,y], z) = B_{\mathfrak{g}}(x, [y\,,z])$ for any $x, y, z \in \mathfrak{g}$. Indeed if $x \in \ker(B_{\mathfrak{g}})$, then for any $u, v \in \mathfrak{g}$

$$B_{\mathfrak{g}}([x\,,v]\,,u) = B_{\mathfrak{g}}(x\,,[v\,,u]) = 0 \quad \Longrightarrow \quad [\ker(B_{\mathfrak{g}})\,,\mathfrak{g}] \subset \ker(B_{\mathfrak{g}}).$$

According to the Cartan criterion, a Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form $B_{\mathfrak{g}}$ is non-degenerate on $\mathfrak{g}$, or equivalently the kernel $\ker(B_{\mathfrak{g}})$ is trivial. Particularly, since the Lie algebra $\mathfrak{so}(p, q)$ is simple the Killing form $B_{\mathfrak{so}(p,q)}$ is non-degenerate.

Definition 5.4.10. *Let $\mathfrak{g}$ be a Lie algebra. A Lie triple system W of $\mathfrak{g}$ is called irreducible if there are no Lie triple systems W_1 and W_2 of $\mathfrak{g}$ such that*

$$W = W_1 \oplus W_2, \qquad [W_1\,,W_2] = \{0\}.$$

Proposition 5.4.11. *Let W be a nonabelian Lie triple system of $\mathfrak{g}$, $\mathfrak{Z}(W)$ its center, $\mathcal{L} = W + [W\,,W]$ and $(\cdot\,,\cdot)_{\mathcal{L}}$ an ad-invariant inner product on $\mathcal{L}$. Then the following properties hold:*

1. *If $\mathfrak{z}(W) \neq 0$ and $W_1 = \mathfrak{z}(W)^\perp$ is the orthogonal complement in W with respect to $(\cdot\,,\cdot)_{\mathcal{L}}$, then W_1 is a nonabelian Lie triple system and $W = \mathfrak{z}(W) \oplus_\perp W_1$.*

2. *There are nonabelian irreducible Lie triple systems W_j with $[W_i\,, W_j] = 0$, $i \neq j$ such that $W = \mathfrak{z}(W) \oplus \big(\oplus_{j=1}^N W_j \big)$.*

3. *If $\mathcal{L} = W + [W\,, W]$ and $W \cap [W\,, W] \neq 0$, then W is reducible and $W = W_1 \oplus_\perp W_2$, $[W_1\,, W_2] = 0$, where $W_1 = W \cap [W\,, W]$, W_2 is the orthogonal complement of W_1 in W with respect to $(\cdot\,,\cdot)_{\mathcal{L}}$.*

If moreover W is an irreducible nonabelian Lie triple system of $\mathfrak{g}$, then

4. *$\mathcal{L} = W = [W\,, W]$ or $W \cap [W\,, W] = 0$ and $\mathcal{L} = W \oplus [W\,, W]$; furthermore, the Lie algebra $\mathcal{L}$ has trivial center.*

5. *If $\mathcal{L} = W \oplus [W\,, W]$, then $B_{\mathcal{L}}(W, [W\,, W]) = 0$. Thus the decomposition into the direct sum is orthogonal with respect to the Killing form $B_{\mathcal{L}}$.*

Proof. Proof of 1. Let $a, b, c \in W_1$ and $z \in \mathfrak{z}(W)$ be arbitrary. Then

$$([a\,, [b\,, c]]\,, z)_{\mathcal{L}} = -([a\,, z]\,, [b\,, c])_{\mathcal{L}} = 0 \quad \Longrightarrow \quad [W_1\,, [W_1\,, W_1]] \subset \mathfrak{z}(W)^\perp = W_1$$

and we conclude that W_1 is a Lie triple system.

Proof of 2. Since W is nonabelian it follows that $\mathfrak{z}(W) \neq W$ and we can write $W = \mathfrak{z}(W) \oplus_\perp W_1$, where W_1 is the orthogonal complement of $\mathfrak{z}(W)$ in W with respect to $(\cdot\,,\cdot)_{\mathcal{L}}$. The set $\mathfrak{z}(W)$ is obviously a Lie triple system. The set W_1 is also a Lie triple system by arguments used in the proof of item 1.

If W_1 is irreducible, then we are done. If W_1 is reducible, then there exists a decomposition of W_1 by Definition 5.4.10 such that $W_1 = \oplus_{j=2}^N W_j$, where W_j is a irreducible Lie triple system such that $[W_i\,, W_j] = 0$, $i \neq j$. To show that W_j for $j = 2, \ldots, N$ is nonabelian, we prove by contradiction. Let $W_j \subset W_1$ be abelian, i.e. $[W_j\,, W_j] = 0$. Then together with $[W_i\,, W_j] = 0$, $i \neq j$ and $[W_i\,, \mathfrak{z}(W)] = 0$, it follows that W_j is a subset of the center $\mathfrak{z}(W)$, which contradicts $W_j \subset W_1 = \mathfrak{z}(W)^\perp$.

Proof of 3. We need to prove that W_1 and W_2 are Lie triple systems of $\mathfrak{g}$ such that $[W_1\,, W_2] = 0$.

Claim 1: *W_1 is an ideal of $\mathcal{L}$.* Let $a, b, c \in W$ and $x \in W_1 = W \cap [W\,, W]$ be arbitrary. Then $[a + [b\,, c], x] = [a\,, x] + [[b\,, c]\,, x]$, and $[a\,, x] \in W \cap [W\,, W]$, since $x \in W \cap [W\,, W]$. Thus $[a\,, W_1] \subset W_1$. Analogously $[[b\,, c]\,, x] \in [[W\,, W]\,, W] \subseteq W$ by $x \in W$ and $[[b\,, c]\,, x] \in [[W\,, W]\,, [W\,, W]] \subset [W\,, W]$ since $x \in [W\,, W]$ and therefore $[[b\,, c]\,, W_1] \subset W_1$. This shows that $[\mathcal{L}\,, W_1] \subset W_1$.

Claim 2: *W_1 and W_2 are $\mathrm{ad}_{[W\,,W]}$ invariant.* In the first claim we particularly received that $[[W\,, W]\,, W_1] \subset W_1$, that is $\mathrm{ad}_{[W\,,W]}$ invariance of W_1. We claim that

$\mathrm{ad}_{[W,W]}(W_2) \subset W_2$ and proof by contradiction. Assume that there exists $x, y \in W$, $w_1 \in W_1$ and $w_2 \in W_2$ such that $([[x,y],w_2],w_1)_{\mathcal{L}} \neq 0$. Then

$$0 = (w_2,[[x,y],w_1])_{\mathcal{L}} = ([[x,y],w_2],w_1)_{\mathcal{L}} \neq 0$$

as $[[W,W],W_1] \subset W_1$. This is a contradiction and hence $\mathrm{ad}_{[W,W]}(W_2) \perp W_1$, which implies $\mathrm{ad}_{[W,W]}(W_2) \subset W_2$.

Claim 3: *W_1 and W_2 are Lie triple systems.* Note that $[W_1,W_1] \subset [W,W]$, since $W_1 = W \cap [W,W]$ and $[W_2,W_2] \subset [W,W]$ by $W_2 \subset W$. Then

$$[[W_1,W_1],W_1] \subseteq [[W,W],W_1] \subseteq W_1$$

because W_1 is $\mathrm{ad}_{[W,W]}$ invariant. The same argument works for W_2.

Claim 4: $[W_1,W_2] = 0$. Notice that

$$([W,W],[W_1,W_2])_{\mathcal{L}} = (\underbrace{[[W,W],W_1]}_{\subset W_1},W_2)_{\mathcal{L}} = 0 \quad \text{by} \quad W_1 = W_2^{\perp}.$$

Thus $[W_1,W_2] \in [W,W]^{\perp}$ and from the other side $[W_1,W_2] \subset [W,W]$, since both W_1, W_2 are subsets of W. We conclude that $[W_1,W_2] = 0$.

Proof of 4. Let $W_1 = W \cap [W,W]$ and W_2 be the orthogonal complement to W_1 in W with respect to the inner product $(\cdot,\cdot)_{\mathcal{L}}$. Then the consideration is reduced to two cases

$$(a) \;\; W_1 = 0 \qquad \text{or} \qquad (b) \;\; W_1 \neq 0.$$

In the case (a) we get $\mathcal{L} = W \oplus [W,W]$. In the second case (b) we obtain $W = W_1 \oplus_{\perp} W_2$ and by the assumption of the irreducibility we conclude that $W_2 = \{0\}$. Thus

$$W_1 = W \quad \Longrightarrow \quad W = W_1 = W \cap [W,W] \subseteq [W,W].$$

By taking ad_W from both sides, we obtain $[W,W] \subseteq [W,[W,W]] \subseteq W$. We conclude that $W = [W,W] = \mathcal{L}$.

We show that the center $\mathfrak{Z}(\mathcal{L})$ is trivial. If $\mathfrak{Z}(\mathcal{L}) \neq 0$, then $\mathfrak{Z}(W) \neq 0$ by item 4 of Proposition 5.4.8. If $\mathfrak{Z}(W) \neq 0$, then W is reducible by the proofs of items 1 and 2 of Proposition 5.4.11. Thus the center $\mathfrak{Z}(\mathcal{L})$ is trivial.

Proof of 5. Let $\mathcal{L} = W \oplus [W,W]$ and $x \in W$, $y \in [W,W]$ be arbitrary. Then

$$\mathrm{ad}_y \, \mathrm{ad}_x([W,W]) \subset \mathrm{ad}_y([W,[W,W]]) \subset \mathrm{ad}_y(W) \subset [[W,W],W] \subset W$$

and

$$\mathrm{ad}_y \, \mathrm{ad}_x(W) \subset \mathrm{ad}_y([W,W]) \subset [[W,W],[W,W]] = [W,W].$$

Thus the operator $\mathrm{ad}_y \, \mathrm{ad}_x$ acts on $\mathcal{L} = W \oplus [W,W]$ by interchanging the spaces W and $[W,W]$ in the direct sum $W \oplus [W,W]$, i.e. $\mathrm{ad}_y \, \mathrm{ad}_x(W \oplus [W,W]) = [W,W] \oplus W$ and therefore

$$0 = \mathrm{tr}(\mathrm{ad}_y \, \mathrm{ad}_x) = B_{\mathcal{L}}(x,y) \quad \Longrightarrow \quad B_{\mathcal{L}}(W,[W,W]) = 0.$$

□

Proposition 5.4.12. *Let W be a Lie triple system of $\mathfrak{g}$, $\mathfrak{Z}(W)$ be the center of W and $\mathcal{L} = W + [W\,, W]$. Then for any* ad*-invariant inner product $(\cdot\,,\cdot)_{\mathcal{L}}$ we have*

1. *$\mathfrak{Z}(W) = \mathfrak{Z}(\mathcal{L})$ and $\mathcal{L} = \mathfrak{Z}(W) \oplus_{\perp} [\mathcal{L}\,, \mathcal{L}]$;*
2. *Let W_1 denote the orthogonal complement to $\mathfrak{Z}(W)$ in W with respect to the inner product $(\cdot\,,\cdot)_{\mathcal{L}}$. Then W_1 is a Lie triple system of $\mathfrak{g}$ and the ideal $[\mathcal{L}\,, \mathcal{L}]$ of $\mathcal{L}$ can be written as $[\mathcal{L}\,, \mathcal{L}] = W_1 + [W_1\,, W_1]$.*

Proof. Proof of 1. If W is abelian, then it is nothing to prove. Let W be a nonabelian Lie triple system of $\mathfrak{g}$. Then we can write

$$W = \mathfrak{Z}(W) \oplus \big(\bigoplus_{j=1}^{N} W_j\big),$$

where the W_j's are nonabelian irreducible Lie triple systems such that $[W_i\,, W_j] = 0$, for $i \neq j$, by item 2 of Proposition 5.4.11. We denote by $\mathcal{L}_j = W_j + [W_j\,, W_j]$, $j = 1, \ldots, N$ Lie subalgebras of $\mathfrak{g}$. The algebras $\mathcal{L}_j$ have trivial centers by item 4 of Proposition 5.4.11. Moreover, $[\mathcal{L}_i\,, \mathcal{L}_j] = 0$ for $i \neq j$ by the Jacobi identity and by $[W_i\,, W_j] = 0$, for $i \neq j$. Thus $\mathcal{L} = W + [W\,, W] = \mathfrak{Z}(W) \oplus \big(\bigoplus_{j=1}^{N} \mathcal{L}_j\big)$. The Lie algebra $\mathcal{L}_0 = \oplus_{j=1}^{N} \mathcal{L}_j$ has trivial center $\mathfrak{Z}(\mathcal{L}_0)$ because each of the Lie algebras $\mathcal{L}_j$ has trivial center and they mutually commute. Since we have $\mathfrak{Z}(W) \subseteq \mathfrak{Z}(\mathcal{L})$ by item 2 of Proposition 5.4.8 we conclude $\mathfrak{Z}(W) = \mathfrak{Z}(\mathcal{L})$. Indeed, if we assume that there is an $x \in \mathfrak{Z}(\mathcal{L})$ and $x \notin \mathfrak{Z}(W)$, then $x \in \mathcal{L}_0$ due to the decomposition $\mathcal{L} = \mathfrak{Z}(W) \oplus \mathcal{L}_0$. But then $[x\,, y] = 0$ for any $y \in \mathcal{L}$ and particularly $[x\,, y_0] = 0$ for any $y_0 \in \mathcal{L}_0 \subset \mathcal{L}$. It follows that $x \in \mathfrak{Z}(\mathcal{L}_0)$ and since $\mathfrak{Z}(\mathcal{L}_0) = \{0\}$ we conclude that $x = 0$.

Now we show that the decomposition $\mathcal{L} = \mathfrak{Z}(W) \oplus [\mathcal{L}\,, \mathcal{L}]$ is orthogonal with respect to the inner product $(\cdot\,,\cdot)_{\mathcal{L}}$. From

$$\mathcal{L} = \mathfrak{Z}(W) \oplus \mathcal{L}_0 = \mathfrak{Z}(\mathcal{L}) \oplus \mathcal{L}_0$$

we deduce that $[\mathcal{L}\,, \mathcal{L}] = [\mathcal{L}_0\,, \mathcal{L}_0] = \mathcal{L}_0$, since the Lie algebra $\mathcal{L}_0$ has trivial center. It is also clear that $\mathcal{L}_0 = [\mathcal{L}\,, \mathcal{L}]$ is an ideal of $\mathcal{L}$. Thus, the decomposition $\mathcal{L} = \mathfrak{Z}(W) \oplus [\mathcal{L}\,, \mathcal{L}]$ is orthogonal with respect to any ad-invariant inner product by item 1 of Proposition 5.4.8.

Proof 2. If $\mathfrak{Z}(W) = 0$, then there is nothing to prove. If $\mathfrak{Z}(W) \neq 0$, then the orthogonal complement W_1 to $\mathfrak{Z}(W)$ in W with respect to the inner product is a Lie triple system by item 1 of Proposition 5.4.11. We only need to show that $\mathcal{L}_0 = [\mathcal{L}\,, \mathcal{L}] = W_1 + [W_1\,, W_1]$. Denote $\mathcal{L}_0^* = W_1 + [W_1\,, W_1]$. Since $W = \mathfrak{Z}(W) \oplus_{\perp} W_1$ we have

$$\mathcal{L} = W + [W\,, W] = \mathfrak{Z}(W) + W_1 + [W_1\,, W_1] = \mathfrak{Z}(W) + \mathcal{L}_0^*.$$

Claim 1: $\mathcal{L} = \mathfrak{Z}(W) \oplus_{\perp} \mathcal{L}_0^*$. Because $[\mathfrak{Z}(W)\,, W_1] = 0$ we get

$$(\mathfrak{Z}(W)\,, [W_1\,, W_1])_{\mathcal{L}} = (W_1\,, [W_1\,, \mathfrak{Z}(W)])_{\mathcal{L}} = 0.$$

Together with $(\mathfrak{Z}(W), W_1)_{\mathcal{L}} = 0$ the latter equalities imply Claim 1.

Claim 2: $[\mathfrak{Z}(W)\,,\mathcal{L}_0^*]=0$. This follows from $[\mathfrak{Z}(W)\,,W_1]=0$ and the Jacobi identity. Now the chain of inclusions

$$\mathcal{L}_0=[\mathcal{L}\,,\mathcal{L}]=[(\mathfrak{Z}(W)+\mathcal{L}_0^*)\,,(\mathfrak{Z}(W)+\mathcal{L}_0^*)]=[\mathcal{L}_0^*\,,\mathcal{L}_0^*]\subseteq\mathcal{L}_0^*\quad\Longrightarrow\quad\mathcal{L}_0\subseteq\mathcal{L}_0^*$$

follows from item 1 of Proposition 5.4.12, claim 2 and the fact that $\mathcal{L}_0^*$ is a Lie algebra constructed from a Lie triple system, see Proposition 5.4.5. Finally, we conclude

$$\mathfrak{Z}(W)\oplus_\perp\mathcal{L}_0^*=\mathcal{L}=\mathfrak{Z}(W)\oplus_\perp\mathcal{L}_0\subseteq\mathfrak{Z}(W)\oplus_\perp\mathcal{L}_0^*$$

by making use of Claim 1. This implies $\mathcal{L}_0^*=\mathcal{L}_0$, that finishes the proof. □

Corollary 5.4.13. *Let W be a Lie triple system of $\mathfrak{o}(V)$ defined by representations of the Clifford algebras. Then*

$$\mathcal{L}=W+[W\,,W]=[\mathcal{L}\,,\mathcal{L}].$$

Proof. It was shown in Proposition 5.4.4 of Example 5.4.3 that the center of W is trivial. Then by applying item 1 of Proposition 5.4.12 we finish the proof. □

Definition 5.4.14. *We say that a Lie algebra $\mathfrak{g}$ is reductive if to each ideal $\mathfrak{a}$ in $\mathfrak{g}$ corresponds an ideal $\mathfrak{b}$ in $\mathfrak{g}$ with $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{b}$.*

Recall the following statement: *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}=\mathfrak{a}_1\oplus\ldots\oplus\mathfrak{a}_j$ with a_j ideals that are each simple Lie algebras. In this case the decomposition is unique, and the only ideals of $\mathfrak{g}$ are the sum of various $\mathfrak{a}_j$*, see [62, Theorem 1.54]. Thus if a Lie algebra $\mathfrak{g}$ is a direct sum of a semisimple Lie algebra and an abelian Lie algebra, then $\mathfrak{g}$ is reductive. The following proposition shows that there are no other reductive Lie algebras.

Proposition 5.4.15. *[62, Corollary 1.56] If $\mathfrak{g}$ is reductive, then $\mathfrak{g}=\mathfrak{a}(\mathfrak{g})\oplus[\mathfrak{g}\,,\mathfrak{g}]$ with $[\mathfrak{g}\,,\mathfrak{g}]$ semisimple and $\mathfrak{a}(\mathfrak{g})$ abelian.*

An important example of reductive Lie algebras is given in the following statement.

Proposition 5.4.16. *[62, Proposition 1.59] Let $\mathfrak{g}$ be a real Lie algebra of matrices over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ that is closed under the operation conjugate transpose, then $\mathfrak{g}$ is reductive.*

Corollary 5.4.17. *If W is a Lie triple system of $\mathfrak{so}(m)$, then the Lie algebra $\mathcal{L}=W+[W\,,W]$ is reductive.*

Proof. Since $C^{\mathbf{t}}=-C$ for any $C\in\mathfrak{so}(m)$, we conclude that $C\in\mathcal{L}$ implies $C^{\mathbf{t}}=-C\in\mathcal{L}$ and therefore the Lie algebra $\mathcal{L}$ is reductive. □

Working with a subalgebra $\mathcal{L}$ of $\mathfrak{so}(p,q)$ we use the following definition of the transpose: $D^{\mathbf{t}}=-\eta_{p,q}D\eta_{p,q}$, $\eta_{p,q}=\mathrm{diag}(I_p,-I_q)$. It is not true in general that if $D\in\mathcal{L}$, then $D^{\mathbf{t}}\in\mathcal{L}$. Any vector subspace $\mathcal{C}\subset\mathfrak{so}(m)$ is closed under transposition, since if $C\in\mathcal{C}$, then $C^{\mathbf{t}}=-C\in\mathcal{C}$. This is not generally true for vector subspaces of $\mathfrak{so}(p,q)$. In general they are only closed under $\eta_{p,q}$-transposition: $\mathcal{D}^{\eta_{p,q}}=\eta_{p,q}\mathcal{D}^{\mathbf{t}}\eta_{p,q}=-\mathcal{D}$.

Proposition 5.4.18. *Let $\mathcal{C} \subset \mathfrak{o}(m)$ and $\eta_{p,q} = \operatorname{diag}(I_p, -I_q)$. Define*

$$\mathcal{D}_1 = \mathcal{C}\eta_{p,q} = \{C\eta_{p,q} \mid \ C \in \mathcal{C}\} \subset \mathfrak{so}(p,q),$$

$$\mathcal{D}_2 = \eta_{p,q}\mathcal{C} = \{\eta_{p,q}C \mid \ C \in \mathcal{C}\} \subset \mathfrak{so}(p,q).$$

Then if the indefinite scalar product $\langle \cdot \, , \cdot \rangle_{\mathfrak{so}(p,q)}$ is non-degenerates on $\mathcal{D}_1$, then it is non-degenerate on $\mathcal{D}_2$ and on $\mathcal{D}_1 + \mathcal{D}_2$. Moreover the space $\mathcal{D}_1 + \mathcal{D}_2$ is closed under transposition and it is invariant under the involution

$$\begin{array}{rccc} \theta\colon & \mathfrak{so}(p,q) & \to & \mathfrak{so}(p,q) \\ & X & \mapsto & \eta_{p,q}X\eta_{p,q} \end{array}$$

Proof. We can show that the vectors $D_i = \eta_{p,q}C_i \in \mathcal{D}_2$, are linearly independent if the vectors $C_i \in \mathcal{C}$ are linearly independent by the same arguments as in Lemma 5.2.12. Note that

$$\theta(\mathcal{D}_1) = \eta_{p,q}\mathcal{D}_1\eta_{p,q} = \eta_{p,q}\mathcal{C}\eta_{p,q}^2 = \eta_{p,q}\mathcal{C} = \mathcal{D}_2$$

which implies $\mathcal{D}_1^{\mathbf{t}} = -\theta(\mathcal{D}_1) = -\mathcal{D}_2$. The space $\mathcal{D}_1 + \mathcal{D}_2$ is closed under transposition and is invariant under the involution θ since

$$(\mathcal{D}_1 + \mathcal{D}_2)^{\mathbf{t}} = -(\mathcal{D}_1 + \mathcal{D}_2), \qquad \theta(\mathcal{D}_1 + \mathcal{D}_2) = \mathcal{D}_1 + \mathcal{D}_2.$$

If the metric $\langle \cdot \, , \cdot \rangle_{\mathfrak{so}(p,q)}$ is non-degenerate on $\mathcal{D}_1$, then for any $X \in \mathcal{D}_1$ there is $Y \in \mathcal{D}_1$ such that

$$\langle X \, , Y \rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(XY) \neq 0.$$

Then

$$\langle \eta_{p,q}X\eta_{p,q} \, , \eta_{p,q}Y\eta_{p,q} \rangle_{\mathfrak{so}(p,q)} = -\operatorname{tr}(\eta_{p,q}XY\eta_{p,q}) = -\operatorname{tr}(XY) \neq 0$$

and $\langle \cdot \, , \cdot \rangle_{\mathfrak{so}(p,q)}$ is non-degenerate on $\mathcal{D}_2$. □

Corollary 5.4.19. *Under the assumption of Proposition 5.4.18 the subspaces $\mathcal{D}_1$ and $\mathcal{D}_2$ are isometric.*

Proof. Since $\theta(\mathcal{D}_1) = \mathcal{D}_2$, then for $D, D' \in \mathcal{D}_1$

$$-\operatorname{tr}(DD') = -\operatorname{tr}(\eta_{p,q}D\eta_{p,q}\eta_{p,q}D'\eta_{p,q}) = -\operatorname{tr}(\theta(D)\theta(D')).$$

□

Nevertheless, the Lie triple systems associated with a representation of Clifford algebras form a subalgebra of $\mathfrak{so}(l,l)$ respectively $\mathfrak{so}(2l)$, which are reductive. Let $\mathrm{Cl}_{r,s}$ be a Clifford algebra generated by $\mathbb{R}^{r,s}$ and $J\colon \mathbb{R}^{r,s} \to \mathfrak{so}(l,l) \subset \operatorname{End}(\mathbb{R}^{l,l})$ for $s \neq 0$ and $J\colon \mathbb{R}^{r,s} \to \mathfrak{so}(2l) \subset \operatorname{End}(\mathbb{R}^{2l,0})$ for $s = 0$. Denote $W = J(\mathbb{R}^{r,s}) \subset \mathfrak{so}(l,l)$ respectively $W = J(\mathbb{R}^{r,0}) \subset \mathfrak{so}(2l)$ and $\mathcal{L} = W + [W \, , W]$.

Theorem 5.4.20. *The Lie algebra $\mathcal{L}$ is simple.*

Proof. Recall that the representation maps of a Clifford algebra satisfy the relation

$$J_Z J_{Z'} + J_{Z'} J_Z = -2\langle Z\,, Z'\rangle_{r,s} \operatorname{Id}_{\mathbb{R}^{2l}}, \quad Z, Z' \in \mathbb{R}^{r,s}.$$

Let $\{Z_1, \dots, Z_n\}$, $n = r + s$, be an orthonormal basis of $\mathbb{R}^{r,s}$. Then the following commutation relations hold

$$[J_{Z_i}\,, J_{Z_j}] = 2J_{Z_i}J_{Z_j}, \quad [J_{Z_i}\,, [J_{Z_i}\,, J_{Z_j}]] = -4\langle Z_i\,, Z_i\rangle_{r,s} J_{Z_j}.$$

Thus

$$\mathcal{L} = W + [W\,, W] = \operatorname{span}\{J_{Z_k}, J_{Z_i}J_{Z_j},\ i, j, k = 1, \dots, n\}.$$

Let us assume that $\mathfrak{h} \subset \mathcal{L}$ is an ideal: $[\mathfrak{h}\,, \mathcal{L}] \subset \mathfrak{h}$. We aim to show that the only possible ideals are the trivial and $\mathcal{L}$ itself. We consider several cases.

Case 1. Suppose that $J_Z \in \mathfrak{h}$, with $Z \neq 0$ and $\langle Z\,, Z\rangle_{r,s} \neq 0$. We assume without loss of generality that Z is normalized. Then we can assume that there exists an orthonormal basis $\{Z_1, \dots, Z_n\}$ with $Z = Z_1$. Thus

$$\begin{array}{rcll} [J_{Z_1}\,, J_{Z_j}] & = & 2J_{Z_1}J_{Z_j} \in \mathfrak{h}, & j = 2, \dots, n, \\ [J_{Z_1}\,, [J_{Z_1}\,, J_{Z_j}]] & = & -4\langle Z_1\,, Z_1\rangle_{r,s} J_{Z_j} \in \mathfrak{h}, & j = 2, \dots, n, \\ [J_{Z_j}\,, J_{Z_i}] & = & 2J_{Z_j}J_{Z_i} \in \mathfrak{h} & i, j = 1, \dots, n,\ i \neq j. \end{array}$$

We see that all the generators of $\mathcal{L}$ are contained in $\mathfrak{h}$, which implies $\mathfrak{h} = \mathcal{L}$.

Case 2. We assume now that $J_Z \in \mathfrak{h}$, with $Z \neq 0$ and $\langle Z\,, Z\rangle_{r,s} = 0$. We choose an orthonormal basis $\{Z_1, \dots, Z_n\}$ such that $Z = \lambda_1 Z_1 + Y$ with $\lambda_1 \neq 0$, where we also can find Y such that $\langle Z_1\,, Y\rangle_{r,s} = 0$. We write $Y = \sum_{k=2}^{n} \lambda_k Z_k$ and note that $\langle Y\,, Y\rangle_{r,s} \neq 0$. Then we calculate

$$\mathfrak{h} \ni \frac{1}{2}[J_Z\,, J_{Z_1}] = \frac{1}{2}[J_Y\,, J_{Z_1}] = \sum_{k=2}^{n} \lambda_k J_{Z_k} J_{Z_1} = J_Y J_{Z_1}.$$

It follows that

$$\mathfrak{h} \ni \frac{1}{2}[J_Y J_{Z_1}\,, J_{Z_1}] = J_{-\langle Z_1\,, Z_1\rangle_{r,s} Y}.$$

Thus $J_{-\langle Z_1\,, Z_1\rangle_{r,s} Y} \in \mathfrak{h}$ and we reduce the problem to the previous case, concluding that $\mathfrak{h} = \mathcal{L}$.

In the rest of the proof we assume that the ideal $\mathfrak{h}$ contains the product $J_Z J_{Z'}$ with different type of non-zero vectors Z and Z', We calculate

$$\begin{array}{rcl} \frac{1}{2}[J_Z\,, J_Z J_{Z'}] & = & (J_Z J_Z J_{Z'} - J_Z J_{Z'} J_Z) \\ & = & (-\langle Z\,, Z\rangle_{\mathfrak{z}} J_{Z'} - (-J_{Z'} J_Z - 2\langle Z\,, Z'\rangle_{\mathfrak{z}} \operatorname{Id}_{\mathbb{R}^{2l}}) J_Z) \\ & = & (-\langle Z\,, Z\rangle_{\mathfrak{z}} J_{Z'} - \langle Z\,, Z\rangle_{\mathfrak{z}} J_{Z'} + 2\langle Z\,, Z'\rangle_{\mathfrak{z}} J_Z) \\ & = & (-\langle Z\,, Z\rangle_{\mathfrak{z}} J_{Z'} + \langle Z\,, Z'\rangle_{\mathfrak{z}} J_Z) = J_{X_1}, \end{array}$$

where $X_1 = -\langle Z, Z\rangle_{\mathfrak{z}} Z' + \langle Z, Z'\rangle_{\mathfrak{z}} Z$ and, equivalently,

$$\begin{aligned}
\frac{1}{2}[J_{Z'}, J_Z J_{Z'}] &= (J_{Z'} J_Z J_{Z'} - J_Z J_{Z'} J_{Z'}) \\
&= (\langle Z', Z'\rangle_{\mathfrak{z}} J_Z + (-J_Z J_{Z'} - 2\langle Z, Z'\rangle_{\mathfrak{z}} \mathrm{Id}) J_{Z'}) \\
&= (\langle Z', Z'\rangle_{\mathfrak{z}} J_Z + \langle Z', Z'\rangle_{\mathfrak{z}} J_Z - 2\langle Z, Z'\rangle_{\mathfrak{z}} J_{Z'}) \\
&= (\langle Z', Z'\rangle_{\mathfrak{z}} J_Z - \langle Z, Z'\rangle_{\mathfrak{z}} J_{Z'}) = J_{X_2},
\end{aligned}$$

where $X_2 = \langle Z', Z'\rangle_{\mathfrak{z}} Z - \langle Z, Z'\rangle_{\mathfrak{z}} Z'$.

If one of the vectors X_1 or X_2 differs from zero, then we apply Case 1. or 2. and conclude that $\mathfrak{h} = \mathcal{L}$. We consider the remaining case: $X_1 = X_2 = 0$ in which

$$\langle Z, Z\rangle_{\mathfrak{z}} = \langle Z', Z'\rangle_{\mathfrak{z}} = \langle Z, Z'\rangle_{\mathfrak{z}} = 0.$$

Let us make the following observation: if $Z = aZ'$ for some $a \neq 0$, then $J_Z J_{Z'} = aJ_Z^2 = a\langle Z, Z\rangle_{r,s} \mathrm{Id}_{\mathbb{R}^{2l}} = 0$ and $[J_Z J_{Z'}, \mathcal{L}] = [0, \mathcal{L}] = 0$, which implies $\mathfrak{h} = 0$. Thus we asume that Z and Z' are not proportional. Choose an orthonormal basis $\{Z_1, \dots, Z_n\}$ such that $Z = \lambda Z_1 + Y_0$ with $\lambda \neq 0$, $Z' = \mu Z_1 + Y_1$ with some μ and $Y_0, Y_1 \in \mathrm{span}\{Z_2, \dots, Z_n\}$, i.e. $\langle Z_1, Y_i\rangle_{\mathfrak{z}} = 0$ for $i = 0, 1$. We calculate

$$\begin{aligned}
J_Z J_{Z'} &= (\lambda J_{Z_1} + J_{Y_0})(\mu J_{Z_1} + J_{Y_1}) \\
&= -\lambda\mu\langle Z_1, Z_1\rangle_{\mathfrak{z}} \mathrm{Id} + \lambda J_{Z_1} J_{Y_1} + \mu J_{Y_0} J_{Z_1} + J_{Y_0} J_{Y_1},
\end{aligned}$$

such that

$$\begin{aligned}
[J_Z J_{Z'}, J_{Z_1}] &= [-\lambda\mu\langle Z_1, Z_1\rangle_{\mathfrak{z}} \mathrm{Id} + \lambda J_{Z_1} J_{Y_1} + \mu J_{Y_0} J_{Z_1} + J_{Y_0} J_{Y_1}, J_{Z_1}] \\
&= 2\langle Z_1, Z_1\rangle_{\mathfrak{z}}(\lambda J_{Y_1} - \mu J_{Y_0}) = 2\langle Z_1, Z_1\rangle_{\mathfrak{z}}(J_{\lambda Y_1 - \mu Y_0}).
\end{aligned}$$

We claim that $\lambda Y_1 - \mu Y_0 \neq 0$. We prove by contradiction and assume that $\lambda Y_1 - \mu Y_0 = 0$, which can happen if and only if $Y_1 = \frac{\mu}{\lambda} Y_0$. If $\mu = 0$, then $Y_1 = 0$ and so $Z' = 0$, which contradicts our assumption. Hence $\mu \neq 0$. It follows that

$$\frac{\mu}{\lambda} Z = \frac{\mu}{\lambda}(\lambda Z_1 + Y_0) = \mu Z_1 + \frac{\mu}{\lambda} Y_0 = \mu Z_1 + Y_1 = Z',$$

such that $Z \in \mathrm{span}\{Z'\}$, which is again a contradiction. Hence $\lambda Y_1 - \mu Y_0 \neq 0$ and we can apply Cases 1. or 2. to show that $\mathfrak{h} = \mathcal{L}$.

In all these cases the ideal $\mathfrak{h}$ coincides with $\mathcal{L}$, or is trivial and we conclude that the Lie algebra $\mathcal{L}$ is simple. □

Corollary 5.4.21. *If W is a Lie triple system of a general H-type algebra $\mathfrak{g}$, then $\mathcal{L} = W \oplus [W, W]$ is reductive.*

Proof. The proof follows directly by Theorem 5.4.20. □

As a corollary we also obtain a new proof of Corollary 5.4.13.

Definition 5.4.22. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $B_{\mathfrak{g}}$ be its Killing form. An involution θ ($\theta^2 = \mathrm{Id}_{\mathfrak{g}}$) is called a Cartan involution on $\mathfrak{g}$ if the form $C_\theta(X, Y) := -B_{\mathfrak{g}}(X, \theta(Y))$ is a positive definite bilinear form.*

As it was observed before the bilinear form

$$\langle X , Y \rangle_{\mathfrak{so}(p,q)} = \mathrm{tr}(X^{\eta_{p,q}} Y) = -\,\mathrm{tr}(XY), \quad X^{\eta_{p,q}} = \eta_{p,q} X^{\mathbf{t}} \eta_{p,q} = -X,$$

on $\mathfrak{so}(p, q)$ is a (positive) scalar multiple of the Killing form $B_{\mathfrak{so}(p,q)}$ since the Lie algebra $\mathfrak{so}(p, q)$ is simple. Define the involution θ on $\mathfrak{so}(p, q)$ by

$$X \mapsto \theta(X) = \eta_{p,q} X \eta_{p,q}. \tag{5.19}$$

We claim that θ is the Cartan involution on $\mathfrak{so}(p, q)$. Indeed, if $X \in \mathfrak{so}(p, q)$ and $X \neq 0$, then

$$\begin{aligned} C_\theta(X, X) \quad &:= \quad B_{\mathfrak{so}(p,q)}(X, \theta(X)) = c \,\langle X , \theta(X) \rangle_{\mathfrak{so}(p,q)} = -c\ \mathrm{tr}(X \eta_{p,q} X \eta_{p,q}) \\ &= \quad -c\ \mathrm{tr}\left((X\eta_{p,q})^2\right) > 0 \end{aligned}$$

because if $X \in \mathfrak{so}(p, q)$, then $X\eta_{p,q} \in \mathfrak{so}(m)$, $p + q = m$ with $\mathrm{tr}\left((X\eta_{p,q})^2\right) < 0$.

Previously we started with a Lie algebra $\mathfrak{g}$ (for example $\mathfrak{g} = \mathfrak{so}(p, q)$), its Lie triple system W and studied properties of the Lie subalgebra $\mathcal{L} = W + [W , W]$. Now we ask the opposite question: given a subalgebra $\mathcal{L} \subset \mathfrak{so}(p, q)$, can we find a Lie triple system $\mathbb{W}$ of $\mathfrak{so}(p, q)$ such that $\mathcal{L} = \mathbb{W} \oplus [\mathbb{W} , \mathbb{W}]$. Unfortunately this is not always the case.

Proposition 5.4.23. *Let $\mathcal{L}$ be a reductive Lie subalgebra of $\mathfrak{so}(p, q)$. Then there is a Lie triple system $\mathbb{W}$ of $\mathfrak{so}(p, q)$ such that $\mathcal{L} \supset \mathbb{W} \oplus [\mathbb{W} , \mathbb{W}]$.*

Proof. We write $\mathcal{L} = \mathfrak{z}(\mathcal{L}) \oplus [\mathcal{L} , \mathcal{L}] = \mathfrak{z}(\mathcal{L}) \oplus \mathcal{L}_0$, where $\mathcal{L}_0$ is semisimple. Thus the Killing form $B_{\mathcal{L}_0}$ is non-degenerate and there exists a Cartan involution $\theta\colon \mathcal{L}_0 \to \mathcal{L}_0$. We set

$$\mathfrak{p} \subset \mathcal{L}_0 \quad \text{such that} \quad \theta(p) = -p, \ \ \forall\, p \in \mathfrak{p},$$
$$\mathfrak{t} \subset \mathcal{L}_0 \quad \text{such that} \quad \theta(t) = t, \ \ \forall\, t \in \mathfrak{t}.$$

The Killing form $B_{\mathcal{L}_0}$ is negative definite on $\mathfrak{t}$ and is positive definite on $\mathfrak{p}$. Now we establish several properties.

$\mathfrak{p}$ IS A LIE TRIPLE SYSTEM OF $\mathfrak{so}(p, q)$. Let $u, v, w \in \mathfrak{p}$ be arbitrary, then for $y = [u , [v , w]] \in [\mathfrak{p} , [\mathfrak{p} , \mathfrak{p}]]$ we obtain

$$\theta(y) = \theta[u , [v , w]] = [\theta(u) , [\theta(v) , \theta(w)]] = -[u , [v , w]] = -y$$

which shows that $y \in \mathfrak{p}$. Furthermore, we remind that $\mathfrak{t} \supset [\mathfrak{p} , \mathfrak{p}]$.

IT IS OBVIOUS THAT $[\mathcal{L} , \mathcal{L}] = \mathfrak{p} \oplus \mathfrak{t}$.

THE SET $\mathbb{W} = \mathfrak{z}(\mathcal{L}) \oplus \mathfrak{p}$ IS A LIE TRIPLE SYSTEM OF $\mathfrak{so}(p, q)$. We deduce $[\mathbb{W} , \mathbb{W}] = [\mathfrak{p} , \mathfrak{p}] \subset \mathfrak{t}$ and

$$[\mathbb{W} , [\mathbb{W} , \mathbb{W}]] = [\mathfrak{p} , \mathfrak{t}] = [\mathfrak{p} , [\mathfrak{p} , \mathfrak{p}]] \subseteq \mathfrak{p} \subseteq \mathbb{W}.$$

Moreover, $\mathbb{W} \oplus [\mathbb{W} , \mathbb{W}] = \mathfrak{z}(\mathcal{L}) \oplus \mathfrak{p} \oplus [\mathfrak{p} , \mathfrak{p}] \subset \mathfrak{z}(\mathcal{L}) \oplus [\mathcal{L} , \mathcal{L}] = \mathcal{L}$. □

5.5 The Lie triple system W of $\mathfrak{so}(p,q)$ is a rational subspace of $\mathcal{L}$ in the case of trivial center $\mathfrak{Z}(W)$

Let us assume that $\mathcal{L} = W + [W\,, W]$ is reductive. Thus if $\mathfrak{Z}(W) = 0$, then $\mathcal{L} = [\mathcal{L}\,, \mathcal{L}]$ is a semisimple Lie algebra. As we saw in the previous section it is the case when the Lie triple system is defined by the Clifford algebra representations.

For any real semisimple Lie algebra $\mathfrak{g}^0$ it is possible to explicitly construct a basis with integer structure constants by [58]. In the following we give an idea of the construction, for more details see [58].

Let $\mathfrak{g}^0$ be a real semisimple real algebra with Cartan decomposition $\mathfrak{g}^0 = \mathfrak{k} \oplus \mathfrak{p}$ determined by a Cartan involution θ. We choose a maximal abelian subalgebra $\mathfrak{a} := \mathfrak{h}^0 \cap \mathfrak{p}$, where $\mathfrak{h}^0$ is a maximal abelian θ-stable subalgebra of $\mathfrak{g}^0$. Recall that the choice of $\mathfrak{h}^0$ is unique up to conjugation.

Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}^0$ and $\mathfrak{h} = \mathfrak{h}^0_{\mathbb{C}}$ a Cartan subalgebra. This determines the set of roots $\Phi(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$. The following diagram gives an overview over all relevant inclusions

$$\begin{tikzcd}
 & \mathfrak{a} \arrow[r] \arrow[dl, "k"'] \arrow[d, "i"] & \mathfrak{g}^0 \arrow[d, "l"] \\
\mathfrak{h}_{\mathbb{R}} \arrow[r, "j"] & \mathfrak{h} \arrow[r] & \mathfrak{g}\,.
\end{tikzcd}$$

We denote by σ and τ the complex conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}^0$ respectively the real compact form $\mathfrak{u} \subset \mathfrak{g}$. Apparently $\theta = l^*(\sigma\tau)$ such that $\sigma\tau$ is the unique complex linear extension of θ from $\mathfrak{g}^0$ to $\mathfrak{g}$ which we want to denote by θ as well. The set of simple roots will be denoted by $\Delta(\mathfrak{g}, \mathfrak{h}) \subset \Phi^+(\mathfrak{g}, \mathfrak{h})$, where $\Phi^+(\mathfrak{g}, \mathfrak{h})$ is the set of positive roots such that $i^*\Phi^+(\mathfrak{g}, \mathfrak{h}) = \Phi^+(\mathfrak{g}^0, \mathfrak{h}) \cup \{0\}$. For $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ let $h_\alpha = \frac{2t_\alpha}{B(t_\alpha, t_\alpha)}$, where B is the Killing form of $\mathfrak{g}$ and t_α a root vector. Furthermore, set $h_i := h_{\alpha_i}$ for the simple roots $\alpha_i \in \Delta(\mathfrak{g}, \mathfrak{h})$. We define the roots $\alpha^\sigma, \alpha^\tau, \alpha^\theta$ by $\alpha^\sigma(h) = \overline{\alpha(\sigma(h))}$, $\alpha^\tau(h) = \overline{\alpha(\tau(h))}$ and $\alpha^\theta(h) = \alpha(\theta(h))$ where $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$, $h \in \mathfrak{h}$. We establish the terminology and call a root $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$ real if it is fixed by σ, imaginary if it is fixed by θ and complex in all remaining cases. For intuition we note that a real root vanishes on $\mathfrak{h}^0 \cap \mathfrak{t}$, thus takes only real values on $\mathfrak{h}^0$, an imaginary root vanishes on $\mathfrak{a}$, thus takes purely imaginary values on $\mathfrak{h}^0$ and a complex root takes mixed complex values on $\mathfrak{h}^0$.

Now we define a decomposition of the roots. Let $\Sigma = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{h}) | i^*\alpha \neq 0\}$ be the set of all roots which do not vanish everywhere on $\mathfrak{a}$. This can be decomposed into the set of complex roots $\Phi_{\mathbb{C}}$ and the set of real roots $\Phi_{\mathbb{R}}$, i.e. $\Sigma = \Phi_{\mathbb{C}} \cup \Phi_{\mathbb{R}}$, which restricts to the root system $i^*\Sigma = \Phi(\mathfrak{g}^0, \mathfrak{a})$. Furthermore, let $\Delta_0 = \Delta(\mathfrak{g}, \mathfrak{h}) \cap \Phi_{i\mathbb{R}}$ be the set of simple imaginary roots and let $\Delta_1 = \Delta(\mathfrak{g}, \mathfrak{h}) \cap \Sigma$ be the set of simple complex or real roots.

Now there exists a Chevalley basis $\mathcal{C} = \{x_\alpha, h_i | \alpha \in \Phi(\mathfrak{g}, \mathfrak{h})\}$ of $(\mathfrak{g}, \mathfrak{h})$ such that

(i) $\tau(x_\alpha) = x_{\alpha^\tau} = x_{-\alpha}$ for each $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$,

(ii) $\sigma(x_\alpha) = \pm x_{\alpha^\sigma}$ for each $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$,
$\sigma(x_\alpha) = x_{\alpha^\sigma}$ for each $\alpha \in \Phi_{i\mathbb{R}} \cup \Delta_1$.

Now we use this basis to construct our desired basis. Set $X_\alpha := x_\alpha + \sigma(x_\alpha)$, $Y_\alpha := i(x_\alpha - \sigma(x_\alpha))$, $H^1_\alpha := h_\alpha + h_{\alpha^\sigma}$, $H^0_\alpha := i(h_\alpha - h_{\alpha^\sigma})$ and $Z_\alpha := X_\alpha + Y_\alpha$ for $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. We note that X_α and H^1_α are twice the real part and Y_α and H^0_α are twice the negative imaginary part of x_α and h_α in the complex vector space $\mathfrak{g}$ with respect to the real structure σ. We define the set $\Phi^{+*}_{\mathbb{C}} \subset \Phi^+_{\mathbb{C}}$ such that for all $\alpha \in \Phi^+_{\mathbb{C}}$ the following is true:

$$\{\alpha, \alpha^\sigma\} \not\subset \Phi^{+*}_{\mathbb{C}} \quad \text{and} \quad \Phi^{+*}_{\mathbb{C}} \cap \{\alpha, \alpha^\sigma\} \neq \emptyset.$$

Analogously we define the set $\Delta_1^* \subset \Delta_1$ such that for all $\alpha \in \Delta_1$ the following is true:

$$\{\alpha, \omega(\alpha)\} \not\subset \Delta_1^* \quad \text{and} \quad \Delta_1^* \cap \{\alpha, \omega(\alpha)\} \neq \emptyset,$$

where $\omega\colon \Delta_1 \to \Delta_1$ is a unique involutive permutation with unique nonnegative integers $n_{\alpha\beta}$ with $\alpha \in \Delta_1$ and $\beta \in \Delta_0$ such that for each α:

(i) $\alpha^\theta = -\omega(\alpha) - \sum_{\beta \in \Delta_0} n_{\beta\alpha}\beta$,

(ii) $n_{\beta\omega(\alpha)} = n_{\beta\alpha}$,

(iii) ω extends to a Dynkin diagram automorphism $\omega\colon \Delta(\mathfrak{g}, \mathfrak{h}) \to \Delta(\mathfrak{g}, \mathfrak{h})$.

We defined these two sets to obtain the following basis $\mathfrak{B}$, which is the union of:

$$\begin{aligned} \mathfrak{B}_{\mathbb{R}} &:= \{Z_\alpha | \alpha \in \Phi_{\mathbb{R}}\}, \quad \mathfrak{B}_{i\mathbb{R}} := \{X_\alpha, Y_\alpha | \alpha \in \Phi^+_{i\mathbb{R}}\}, \quad \mathfrak{B}_{\mathbb{C}} := \{X_\alpha, Y_\alpha | \alpha \in \Phi^*_{i\mathbb{C}}\}, \\ \mathcal{H}^1 &:= \{H^1_\alpha | \alpha \in \Delta_1 \setminus \Delta_1^*\}, \quad \mathcal{H}^0 := \{H^0_\alpha | \alpha \in \Delta_0 \cup \Delta_1^*\}. \end{aligned}$$

Then the basis $2\mathfrak{B}$ has integer structure constants by [58]. Let us denote this special basis by $\mathcal{C}_{\mathcal{L}}$.

Definition 5.5.1. *Let $\mathfrak{g}$ be a Lie algebra such that with respect to a basis $\mathcal{B}_{\mathfrak{g}}$ the Lie algebra $\mathfrak{g}$ has rational structure constants. Then the set* $\operatorname{span}_{\mathbb{Q}}\{\mathcal{B}_{\mathfrak{g}}\}$ *is called the rational structure of the Lie algebra $\mathfrak{g}$. A subspace U of $\mathfrak{g}$ is called rational subspace with respect to the rational structure* $\operatorname{span}_{\mathbb{Q}}\{\mathcal{B}_{\mathfrak{g}}\}$ *if there is a basis B_U such that* $B_U \subset \operatorname{span}_{\mathbb{Q}}\{\mathcal{B}_{\mathfrak{g}}\}$.

Proposition 5.5.2. *If W is a Lie triple system of $\mathfrak{so}(p,q)$, and $\mathcal{L} = W \oplus [W\,, W] = [\mathcal{L}\,, \mathcal{L}]$ is semisimple, then W is a rational subspace of $\mathcal{L}$ with respect to the rational structure* $\operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$.

Proof. We choose the basis $\mathcal{C}_{\mathcal{L}}$. Then it is obvious that there is a basis of W and $[W\,, W]$ contained in $\operatorname{span}_{\mathbb{Z}}\{\mathcal{C}_{\mathcal{L}}\} \subset \operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$. □

Now let us assume that $\mathcal{L}_1 = W \cap [W\,, W] \neq \{0\}$. We need to show that W has a basis in the rational structure $\operatorname{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}}\}$.

Note that $\mathcal{L}_1$ is an ideal of $\mathcal{L} = [W\,, W] + W$ since $\mathcal{L}_1$ is ad_W and $\operatorname{ad}_{[W\,,W]}$ invariant, see proof of item 3 of Proposition 5.4.11. Let $\mathcal{L}_2$ be the orthogonal complement of $\mathcal{L}_1$ with respect to any ad-invariant inner product $(\cdot\,,\cdot)_{\mathcal{L}}$ on $\mathcal{L}$. Then $\mathcal{L}_2$ is also an ideal of $\mathcal{L}$. Indeed

$$([X\,, \mathcal{L}_2]\,, \mathcal{L}_1)_{\mathcal{L}} = -(\mathcal{L}_2\,, [X\,, \mathcal{L}_1])_{\mathcal{L}} = 0,$$

as $-(\mathcal{L}_2, \mathcal{L}_1)_{\mathcal{L}} = 0$. Thus we have two ideals $\mathcal{L}_1$, $\mathcal{L}_2$ of $\mathcal{L}$, such that $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$. This implies that they are also orthogonal with respect to the Killing form $B_{\mathcal{L}}$ and therefore $B_{\mathcal{L}}$ is non-degenerate on both $\mathcal{L}_1$ and $\mathcal{L}_2$. (If it would be degenerate on one of it, then it would be degenerate on the other one too and then it would be degenerate on $\mathcal{L}$, which would be a contradiction.) Moreover, the restrictions of $B_{\mathcal{L}}$ on ideals $\mathcal{L}_1$ and $\mathcal{L}_2$ define the Killing forms $B_{\mathcal{L}_1}$ and $B_{\mathcal{L}_2}$ of them.

Proposition 5.5.3. *The Lie triple system W of $\mathfrak{so}(p,q)$ is a rational subspace of $\mathcal{L} = W + [W\,, W] = [\mathcal{L}\,, \mathcal{L}]$ with respect to the rational structure* $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_1}\} \oplus \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}$.

Proof. First we observe that the semisimple Lie algebra $\mathcal{L}_2$ admits a decomposition $\mathcal{L}_2 = W_2 \oplus [W_2\,, W_2]$, where W_2 is a Lie triple system of $\mathfrak{so}(p,q)$, see proof of item 3 of Proposition 5.4.11. Moreover there is the basis $\mathcal{C}_{\mathcal{L}_2}$ such that W_2 is a rational subspace of $\mathcal{L}_2$ with respect to $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}$ by Proposition 5.5.2.

As a semisimple Lie algebra $\mathcal{L}_1$ admits the basis $\mathcal{C}_{\mathcal{L}_1}$. Then the basis $\mathcal{C} = \mathcal{C}_{\mathcal{L}_1} \cup \mathcal{C}_{\mathcal{L}_2}$ is the (Chevalley) basis of the Lie algebra $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$. We define the rational structure of $\mathcal{L}$ by

$$\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}\} = \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_1}\} \oplus \mathrm{span}_{\mathbb{Q}}\{\mathcal{C}_{\mathcal{L}_2}\}.$$

Now $W = \mathcal{L}_1 \oplus W_2$ is a rational subspace of $\mathcal{L}$ with respect to $\mathrm{span}_{\mathbb{Q}}\{\mathcal{C}\}$. $\square$

Chapter 6

The sub-Riemannian geodesic equation in the octonionic H-type group

In the present chapter we study sub-Riemannian geodesics in the octonionic H-type group G_7^1, which is a nilpotent group of step two and, as a manifold, diffeomorphic to $\mathbb{R}^{15}$.

The Lie group structure of G_7^1, obtained via the Cayley-Dickson construction of real division algebras, induces a natural Riemannian metric and a bracket-generating distribution $\mathcal{H}$ of rank eight and step two on G_7^1. Restricting the metric to $\mathcal{H}$ we obtain a sub-Riemannian structure on G_7^1.

The class of curves we are interested in are horizontal with respect to $\mathcal{H}$ and, most importantly, critical points of the natural sub-Riemannian length functional. We present a characterization of these critical points via a differential equation, similar to the geodesic equation in Riemannian geometry, which states that for critical points of the length functional the intrinsic acceleration $\nabla_{\dot{\gamma}}\dot{\gamma}$ is a linear combination with constant coefficients of some special rotations of the velocity $\dot{\gamma}$.

This chapter is the result of a productive cooperation between Godoy Molina and me. The main results are accepted and will be published in Springer Proceedings in Mathematics & Statistics [10]. Finally, we would like to mention that Prof. Fabrice Baudoin informed us that our main theorem is contained as a special case in the non-published manuscript [14].

6.1 Introduction

The H(eisenberg)-type algebras were introduced by A. Kaplan in his foundational work [59]. Their Lie algebra structure is intimately related to the existence of a Clifford algebra representation over a certain inner product space. To make this claim more precise, recall that a composition of two positive definite real quadratic forms φ and λ on two vector spaces H and U, respectively, is a bilinear map $\mu\colon H \times U \to H$ such that for any

$h \in H$, $u \in U$

$$\varphi(h)\lambda(u) = \varphi(\mu(h,u)).$$

One can always assume there exists a vector $u_0 \in U$ such that $\mu(h,u_0) = h$ for all $h \in H$. Setting V as the orthogonal complement of $\mathbb{R}u_0$ in U, one can introduce a Lie bracket $[\cdot\,,\cdot]\colon H \times H \to V$ that induces a Lie algebra structure of step 2 on $H \oplus V$. The Clifford algebra representation mentioned before refers to the fact that

$$\mu(\mu(h,v),v) = -\lambda(v)h,$$

i.e., the existence of μ induces an H-representation of the Clifford algebra $\mathrm{C}\ell(V,-\lambda)$.

Among the plethora of H-type algebras, one can distinguish the class of those satisfying the so-called J^2-condition, which is Clifford-algebraic in its very nature. This family of algebras was introduced in [39], and has been the subject of intense study by analysts for the past twenty years. A major result, obtained in the previous reference, is the fact that the nilpotent, connected and simply connected groups corresponding to H-type algebras can be singled out as those appearing in Iwasawa decompositions of real rank one simple Lie groups, and thus, there are but a few classes of H-type algebras satisfying the J^2-condition. These families of H-type algebras are the trivial Euclidean spaces $\mathbb{R}^n$, the Heisenberg Lie algebras $\mathfrak{g}_1^{2n+1}$, the quaternionic H-type algebras $\mathfrak{g}_3^{4n+3}$ and the octonionic H-type algebra $\mathfrak{g}_7^1$. Note that, although there are nontrivial H-type algebras with centers of arbitrary dimension [59, Corollary 1], those that satisfy the J^2-condition are either abelian or have centers of dimension 1, 3 and 7.

There is a natural connection between H-type algebras and sub-Riemannian geometry, which we proceed to explain. Recall that a sub-Riemannian manifold is a triplet $(M,\mathcal{H},\langle\cdot\,,\cdot\rangle)$, where $\mathcal{H} \hookrightarrow TM$ is a distribution, i.e., a subbundle of the tangent bundle of M, and $\langle\cdot\,,\cdot\rangle$ is a fiber inner product defined on $\mathcal{H}$ called the sub-Riemannian metric. For most applications, it is assumed that the distribution $\mathcal{H}$ is bracket-generating, that is,

$$\operatorname{Lie}\mathcal{H} = \text{ Lie algebra generated by sections of } \mathcal{H} = \Gamma(TM),$$

where $\Gamma(TM)$ denotes the space of vector fields on M. The step of $\mathcal{H}$ is, by convention, the minimal length of brackets needed to generate all the vector fields on M plus one. Associated to an H-type algebra $\mathfrak{g} = H \oplus V$ there is a unique (up to isomorphism) connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$. By left-translating the subspace H of $\mathfrak{g}$, we obtain a bracket-generating distribution $\mathcal{H} \hookrightarrow TG$ of step 2. The quadratic form φ induces a sub-Riemannian metric on $\mathcal{H}$.

From now on, we focus our attention on the sub-Riemannian octonionic H-type group, that is, the sub-Riemannian structure defined on the connected and simply connected Lie group G_7^1 with Lie algebra $\mathfrak{g}_7^1$. The main purpose of this note is to give a variational description of the critical points of the length functional

$$L(\gamma) = \int \sqrt{\varphi(\dot\gamma(t))}\,dt$$

defined for horizontal curves in G_7^1, that is, piecewise smooth curves γ whose velocity vector satisfies the constraint $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$, whenever $\dot\gamma$ is defined. We will refer to these critical points as sub-Riemannian geodesics. An alternative description of these curves, from a Hamiltonian point of view, has been obtained in [28]. Let us stress the fact that we use their model of the group G_7^1, which is obtained from the Cayley-Dickson construction of division algebras, instead of the Clifford algebraic model defined in [39].

This chapter is organized as follows. In Section 6.2, we recall briefly the definition and main properties of the octonionic H-type group and its natural sub-Riemannian structure, following [28]. In Section 6.3, we prove the main result of this chapter, following the lines of [50, 81]. The two major difficulties to overcome when dealing with G_7^1 are the fact that as a manifold it is 15-dimensional and that underlying its structure we are using the octonions, the only normed division algebra which is non-associative. Finally we conclude with two appendices, where we collect all the formulas that are too large to be displayed in an aesthetically pleasing way within the main line of argumentation.

6.2 The Octonionic H-type group G_7^1

In this section, we give a short introduction to the octonionic H-type algebra $\mathfrak{g}_7^1$ and the sub-Riemannian geometry of its (unique connected and simply connected) Lie group G_7^1, both concretely realized in $\mathbb{R}^{15}$. For a deeper study, and some interesting facts about its horizontal curves, we recommend [28].

Let us start by giving a description of $\mathfrak{g}_7^1$ through vector fields defined on $\mathbb{R}^{15} = \mathbb{R}^8 \oplus \mathbb{R}^7$, with coordinates $x_1, \ldots, x_8, z_1, \ldots, z_7$. Consider the 8×8 matrices $\mathcal{J}_1, \ldots, \mathcal{J}_7$ with real coefficients given in Appendix 1. The horizontal space $H = \operatorname{span}\{X_1, \ldots, X_8\}$ corresponds to the distribution generated by the vector fields

$$X_l(x,z) = \partial_{x_l} + \frac{1}{2}\sum_{m=1}^{7} (x\mathcal{J}_m)_l \partial_{z_m}, \quad l \in \{1, \ldots, 8\},$$

where $x = (x_1, \ldots, x_8)$ and $(x\mathcal{J}_m)_l$ denotes the l-th coordinate of the row vector $x\mathcal{J}_m$. Explicitly, these vector fields are given by

$$\begin{aligned}
X_1(x,z) &= \partial x_1 + \frac{1}{2}(-x_2\partial_{z_1} - x_3\partial_{z_2} - x_4\partial_{z_3} - x_5\partial_{z_4} - x_6\partial_{z_5} - x_7\partial_{z_6} - x_8\partial_{z_7}),\\
X_2(x,z) &= \partial x_2 + \frac{1}{2}(x_1\partial_{z_1} + x_4\partial_{z_2} - x_3\partial_{z_3} + x_6\partial_{z_4} - x_5\partial_{z_5} - x_8\partial_{z_6} + x_7\partial_{z_7}),\\
X_3(x,z) &= \partial x_3 + \frac{1}{2}(-x_4\partial_{z_1} + x_1\partial_{z_2} + x_2\partial_{z_3} + x_7\partial_{z_4} + x_8\partial_{z_5} - x_5\partial_{z_6} - x_6\partial_{z_7}),\\
X_4(x,z) &= \partial x_4 + \frac{1}{2}(x_3\partial_{z_1} - x_2\partial_{z_2} + x_1\partial_{z_3} + x_8\partial_{z_4} - x_7\partial_{z_5} + x_6\partial_{z_6} - x_5\partial_{z_7}),\\
X_5(x,z) &= \partial x_5 + \frac{1}{2}(-x_6\partial_{z_1} - x_7\partial_{z_2} - x_8\partial_{z_3} + x_1\partial_{z_4} + x_2\partial_{z_5} + x_3\partial_{z_6} + x_4\partial_{z_7}),\\
X_6(x,z) &= \partial x_6 + \frac{1}{2}(x_5\partial_{z_1} - x_8\partial_{z_2} + x_7\partial_{z_3} - x_2\partial_{z_4} + x_1\partial_{z_5} - x_4\partial_{z_6} + x_3\partial_{z_7}),\\
X_7(x,z) &= \partial x_7 + \frac{1}{2}(x_8\partial_{z_1} + x_5\partial_{z_2} - x_6\partial_{z_3} - x_3\partial_{z_4} + x_4\partial_{z_5} + x_1\partial_{z_6} - x_2\partial_{z_7}),\\
X_8(x,z) &= \partial x_8 + \frac{1}{2}(-x_7\partial_{z_1} + x_6\partial_{z_2} + x_5\partial_{z_3} - x_4\partial_{z_4} - x_3\partial_{z_5} + x_2\partial_{z_6} + x_1\partial_{z_7}).
\end{aligned}$$

Table 6.1: Nontrivial Lie bracket relations in $\mathfrak{g}_7^1$.

$[row\,, col.]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
X_2	$-Z_1$	0	Z_3	$-Z_2$	Z_5	$-Z_4$	$-Z_7$	Z_6
X_3	$-Z_2$	$-Z_3$	0	Z_1	Z_6	Z_7	$-Z_4$	$-Z_5$
X_4	$-Z_3$	Z_2	$-Z_1$	0	Z_7	$-Z_6$	Z_5	$-Z_4$
X_5	$-Z_4$	$-Z_5$	$-Z_6$	$-Z_7$	0	Z_1	Z_2	Z_3
X_6	$-Z_5$	Z_4	$-Z_7$	Z_6	$-Z_1$	0	$-Z_3$	Z_2
X_7	$-Z_6$	Z_7	Z_4	$-Z_5$	$-Z_2$	Z_3	0	$-Z_1$
X_8	$-Z_7$	$-Z_6$	Z_5	Z_4	$-Z_3$	$-Z_2$	Z_1	0

The vertical distribution V, i.e., the center of the Lie algebra $\mathfrak{g}_7^1$, is defined by

$$V = \operatorname{span}\{Z_1, \ldots, Z_7\},$$

where $Z_i(x, z) = \partial_{z_i}$. The Lie algebra $\mathfrak{g}_7^1$ is the algebra spanned by the vector fields $X_1, \ldots, X_8, Z_1, \ldots, Z_7$ with the usual commutator of vector fields in $\mathbb{R}^{15}$, see Table 6.1.

The Lie group G_7^1 is the nilpotent Lie group structure on $\mathbb{R}^{15}$ of step 2 induced by the Lie algebra $\mathfrak{g}_7^1$ via the Baker-Campbell-Hausdorff formula. An explicit expression for the product rule can be found in [28, Equation (3.7)].

We define an inner product $\langle \cdot\,, \cdot \rangle$ on $\mathfrak{g}_7^1$ such that the vector fields $X_1, \ldots, X_8$, $Z_1, \ldots, Z_7$ form an orthonormal frame. The left-invariant distribution

$$\mathcal{H} := \operatorname{span}\{X_1, \ldots, X_8\},$$

and the restriction of $\langle \cdot\,, \cdot \rangle$ to $\mathcal{H}$ give us the sub-Riemannian structure on G_7^1 we want to study further. The group G_7^1 with the structure introduced before is called the octonionic H-type group, since the map

$$\operatorname{ad}_X \colon \ker(\operatorname{ad}_X)^\perp \subset H \to V,$$

is a surjective isometry for any $X \in H$ of norm one, see [59]. From this definition, it follows immediately that the distribution $\mathcal{H}$ is strongly bracket generating and, thus, all length-minimizing curves are normal, i.e., they all solve a natural Hamiltonian equation, see [71, Chapter 1]. Explicit solutions to this equation in the case of the octonionic H-group can be found in [28]. The method employed to find these solutions in [28] uses explicitly the coordinates of $\mathbb{R}^{15}$, instead our approach is entirely coordinate-free.

With all these ingredients at hand, we can compute explicitly the Levi-Civita connection of the metric $\langle \cdot\,, \cdot \rangle$. To do this, we employ the well-known Koszul formula

$$\begin{aligned}\langle Z\,, \nabla_Y X \rangle \;=\; & \frac{1}{2}(X\langle Y\,, Z\rangle + Y\langle Z\,, X\rangle - Z\langle X\,, Y\rangle \\ & -\langle [X\,, Z]\,, Y\rangle - \langle [Y\,, Z]\,, X\rangle - \langle [X\,, Y]\,, Z\rangle),\end{aligned}$$

and we immediately notice that the following equations

$$\langle X_b \,, \nabla_{X_a} Z_r \rangle = -\frac{1}{2} \langle [X_a \,, X_b] \,, Z_r \rangle, \quad \langle Z_s \,, \nabla_{X_a} Z_r \rangle = 0,$$

hold, for all $a, b \in \{1, \dots, 8\}$, $r, s \in \{1, \dots, 7\}$. We conclude that $\nabla_{X_a} Z_r$ has trivial vertical part, and thus

$$\nabla_{X_a} Z_r = -\frac{1}{2} \sum_{b=1}^{8} \langle [X_a \,, X_b] \,, Z_r \rangle X_b.$$

From this and the information in Table 6.1, we can deduce the expressions found in Appendix 2. From these, it is natural to define the operators $J_r \colon \mathcal{H} \to \mathcal{H}$, $r \in \{1, \dots, 7\}$, by

$$J_r(X) := 2\nabla_X Z_r, \quad r \in \{1, \dots, 7\}.$$

These are almost complex structures on $\mathcal{H}$, i.e., $J_r^2 = -Id|_{\mathcal{H}}$, with the property that

$$\langle J_r(X) \,, Y \rangle + \langle X \,, J_r(Y) \rangle = 0, \tag{6.1}$$

for every $r \in \{1, \dots, 7\}$ and all $X, Y \in \mathcal{H}$. Furthermore, we note that this equation implies that $\langle X \,, J_r(X) \rangle = 0$, for all $X \in \mathcal{H}$.

6.3 Geodesic equation on G_7^1

In this section we follow the arguments in [50, 80, 81] to find an intrinsic differential equation for the sub-Riemannian geodesics of G_7^1 with respect to the sub-Riemannian structure introduced in Section 6.2. An earlier attempt to this problem can be found in [82], where the author obtained a differential equation for geodesics in CR sub-Riemannian 3-manifolds using the Tanaka-Webster connection. We conclude with some examples and interpretations.

6.3.1 Main result

Recall that a piecewise smooth curve $\gamma \colon [a, b] \to G_7^1$ is called horizontal if $\dot{\gamma}(s) \in \mathcal{H}_{\gamma(s)}$, whenever $\dot{\gamma}$ is defined. A variation of a curve $\gamma : [a, b] \to G_7^1$ is a C^2-map $\tilde{\gamma} : [a, b] \times I \to G_7^1$, where I is an open interval containing 0 and $\tilde{\gamma}(s, 0) = \gamma(s)$. As customary, we will denote $\tilde{\gamma}(s, \varepsilon) = \gamma_\varepsilon(s)$. If γ is horizontal, we say that $\tilde{\gamma}$ is an admissible variation if all curves $\gamma_\varepsilon : [a, b] \to G_7^1$ are horizontal, $\gamma_\varepsilon(a) = \gamma(a)$ and $\gamma_\varepsilon(b) = \gamma(b)$. As an abuse of notation, we call γ_ε an admissible variation of γ.

Given a vector $v \in \mathfrak{g}_7^1$, we write v_H for its orthogonal projection to the horizontal space H. We will use the same notation for the horizontal components of vector fields, vector fields along curves, etc.

Lemma 6.3.1. *Let $\gamma\colon [a,b] \to G^1_7$ be a horizontal curve parameterized by arc length, and let W be any C^1 vector field along γ such that $W(\gamma(a)) = W(\gamma(b)) = 0$ satisfying*

$$0 = \dot\gamma\langle W\,, Z_r\rangle - 2\langle W_H\,, J_r(\dot\gamma)\rangle, \quad r \in \{1, \dots, 7\}. \tag{6.2}$$

Then there exists an admissible variation γ_ε of γ such that $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\gamma_\varepsilon(s) = W$.

Proof. Note that there exists a vector field $\tilde W$ along γ, orthogonal to $\dot\gamma$, such that we can write $W = f\dot\gamma + \tilde W$ for some smooth function f satisfying $f(a) = f(b) = 0$. From the choice of $\tilde W$, the definition of the almost complex structures J_r, the arc length parameterization and horizontality of γ we can immediately see that

$$\begin{aligned}
\langle W\,, \dot\gamma\rangle &= f\langle\dot\gamma\,, \dot\gamma\rangle + \langle\tilde W\,, \dot\gamma\rangle = f,\\
\langle W\,, J_r(\dot\gamma)\rangle &= f\langle\dot\gamma\,, J_r(\dot\gamma)\rangle + \langle\tilde W\,, J_r(\dot\gamma)\rangle = \langle\tilde W\,, J_r(\dot\gamma)\rangle,\\
\langle W\,, Z_r\rangle &= f\langle\dot\gamma\,, Z_r\rangle + \langle\tilde W\,, Z_r\rangle = \langle\tilde W\,, Z_r\rangle,
\end{aligned}$$

for all $r \in \{1, \dots, 7\}$.

It is easy to see that if there exists a (not necessarily admissible) variation $\gamma(s,\varepsilon)$ for which $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\gamma(s,\varepsilon) = \tilde W$, then there exists $\gamma_1(s,\varepsilon)$ satisfying $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}\gamma_1(s,\varepsilon) = W$. This implies that, without loss of generality, we can and will assume that $W \perp \dot\gamma$.

We have to distinguish the cases in which the vector field W is horizontal or not. Let us first examine the case when W is horizontal on some non-empty interval $I_0 \subset [a,b]$. By definition, we have that $W = W_H$ for all $s \in I_0$, and since we are assuming that W satisfies condition (6.2), we have the equalities

$$\langle W_H\,, J_r(\dot\gamma)\rangle = \langle W\,, J_r(\dot\gamma)\rangle = \frac{1}{2}\,\dot\gamma\langle W_H\,, Z_r\rangle = 0,$$

for all $r \in \{1, \dots, 7\}$. This implies that $W_H \in \operatorname{span}\{\dot\gamma\}$, and since W_H is also orthogonal to $\dot\gamma$, we can conclude that $W_H = 0$.

The non-horizontal case requires more care. If exp is the exponential map associated to the (Riemannian) metric $\langle\cdot\,,\cdot\rangle$ on G^1_7, we can define the mapping

$$F(s,\varepsilon) = \exp_{\gamma(s)}(\varepsilon W(s))$$

for sufficiently small $\varepsilon > 0$ and $s \in [a,b]$. Let us assume there exists $s_0 \in [a,b]$ such that $W(s_0) \notin \mathcal{H}_{\gamma(s_0)}$. We note that $F(s,\varepsilon)$ defines locally a surface, which is transverse to the horizontal space $\mathcal{H}_{\gamma(s_0)}$, as it contains curves in non-horizontal directions by definition. Furthermore, it is foliated by horizontal curves. These two facts together imply that there exists a function $g(s,\varepsilon)$ of class C^2 such that we can define a family of horizontal curves

$$\gamma_\varepsilon(s) = \exp_{\gamma(s)}(g(s,\varepsilon)W(s)).$$

If we choose g such that $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}g(s_0,\varepsilon) = 1$, it follows that γ_ε is an admissible variation of γ with associated vector field W. $\square$

Simple computations show that the converse of Lemma 6.3.1 also holds. For completeness, we include it here. Given an admissible variation γ_ε of a horizontal curve γ with variational vector field W, then

$$0 = \dot\gamma\langle W\,, Z_r\rangle - 2\langle W_H\,, J_r(\dot\gamma)\rangle, \quad r \in \{1, \ldots, 7\}.$$

Since $\langle \dot\gamma_\epsilon\,, Z_r\rangle = 0$, for all $r \in \{1, \ldots, 7\}$, it follows trivially that $\frac{d}{d\epsilon}\big|_{\epsilon=0} \langle \dot\gamma_\epsilon\,, Z_r\rangle = 0$. From this equality, the fact that $\nabla_{Z_l} Z_r = 0$ for all $r, l \in \{1, \ldots, 7\}$, and equation (6.1), we deduce that

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \langle \dot\gamma_\varepsilon\,, Z_r\rangle = \langle \nabla_W \dot\gamma\,, Z_r\rangle + \langle \dot\gamma\,, \nabla_W Z_r\rangle \\ &= \langle \nabla_{\dot\gamma} W\,, Z_r\rangle + \langle \dot\gamma\,, \nabla_{W_H} Z_r\rangle \\ &= \dot\gamma\langle W\,, Z_r\rangle - \langle W\,, \nabla_{\dot\gamma} Z_r\rangle + \langle \dot\gamma\,, J_r(W_H)\rangle \\ &= \dot\gamma\langle W\,, Z_r\rangle - \langle W_H\,, J_r(\dot\gamma)\rangle - \langle J_r(\dot\gamma)\,, W_H\rangle \\ &= \dot\gamma\langle W\,, Z_r\rangle - 2\langle W_H\,, J_r(\dot\gamma)\rangle. \end{aligned}$$

Now we have all tools to prove the main theorem.

Theorem 6.3.2. *Let $\gamma\colon [a, b] \to G_7^1$ be a horizontal curve of class C^2, parametrized by arc length. Then γ is a critical point of the length functional (with respect to admissible variations) if, and only if, there exist constants $\lambda_1, \ldots, \lambda_7 \in \mathbb{R}$ such that γ satisfies the second order differential equation*

$$\nabla_{\dot\gamma}\dot\gamma - 2\sum_{r=1}^{7} \lambda_r J_r(\dot\gamma) = 0. \tag{6.3}$$

Proof. Let us first assume that $\gamma\colon [a, b] \to G_7^1$ is a horizontal curve, parametrized by arc length, satisfying equation (6.3) for some constants $\lambda_1, \ldots, \lambda_7 \in \mathbb{R}$. We consider a C^1-smooth vector field W, vanishing at the endpoints of γ and satisfying

$$\dot\gamma\langle W\,, Z_r\rangle = 2\langle W_H\,, J_r(\dot\gamma)\rangle, \tag{6.4}$$

for all $r \in \{1, \ldots, 7\}$. It is well-known, see [32], that the length functional L satisfies

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\gamma_\varepsilon) = -\int_a^b \langle \nabla_{\dot\gamma}\dot\gamma\,, W\rangle,$$

therefore, to prove that γ is a critical point of L with respect to admissible variations, we need to show that $\int_a^b \langle \nabla_{\dot\gamma}\dot\gamma\,, W\rangle = 0$. Decompose $W = W_H + W_V$ in its horizontal and vertical parts, where $W_V = \sum_{r=1}^{7} g_r Z_r$ for some smooth functions $g_1, \ldots, g_7$ satisfying

$g_r(a) = g_r(b) = 0$. Then

$$\begin{aligned}
\int_a^b \langle \nabla_{\dot\gamma}\dot\gamma\,, W\rangle &\overset{(6.3)}{=} 2\sum_{r=1}^7 \lambda_r \int_a^b \langle J_r(\dot\gamma)\,, W\rangle \overset{J_r(\dot\gamma)\in\mathcal{H}}{=} 2\sum_{r=1}^7 \lambda_r \int_a^b \langle J_r(\dot\gamma)\,, W_H\rangle \\
&\overset{(6.4)}{=} \sum_{r=1}^7 \lambda_r \int_a^b \dot\gamma\langle W\,, Z_r\rangle \overset{Z_r\in V}{=} \sum_{r=1}^7 \lambda_r \int_a^b \dot\gamma\langle W_V\,, Z_r\rangle \\
&= \sum_{r=1}^7 \lambda_r \int_a^b \dot\gamma \left\langle \sum_{l=1}^7 g_l Z_l\,, Z_r \right\rangle = \sum_{r=1}^7 \lambda_r \int_a^b \dot\gamma(g_r) \\
&= \sum_{r=1}^7 \lambda_r \int_a^b \frac{d}{dt}(g_r(t)) \overset{g_r(a)=g_r(b)=0}{=} 0.
\end{aligned}$$

For the converse, let γ be a critical point of the length functional, which is horizontal and parametrized by arc length. This implies that

$$0 = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} L(\gamma_\epsilon) = -\int_a^b \langle \nabla_{\dot\gamma}\dot\gamma\,, W\rangle,$$

where W is the vector field of the variation γ_ϵ.

We know that the condition $\|\dot\gamma\|^2 = \langle\dot\gamma\,, \dot\gamma\rangle = 1$ implies

$$\langle \nabla_{\dot\gamma}\dot\gamma\,, \dot\gamma\rangle = \frac{1}{2}\frac{d}{dt}\langle\dot\gamma\,, \dot\gamma\rangle = \frac{1}{2}\frac{d}{dt}1 = 0.$$

Furthermore, since γ is horizontal, then $\langle\dot\gamma\,, Z_r\rangle = 0$ for all $r \in \{1, \ldots, 7\}$, and thus

$$\begin{aligned}
0 &\overset{\langle\dot\gamma\,,Z_r\rangle=0}{=} \dot\gamma\langle\dot\gamma\,, Z_r\rangle = \langle\nabla_{\dot\gamma}\dot\gamma\,, Z_r\rangle + \langle\dot\gamma\,, \nabla_{\dot\gamma}Z_r\rangle = \langle\nabla_{\dot\gamma}\dot\gamma\,, Z_r\rangle + \langle\dot\gamma\,, J_r(\dot\gamma)\rangle \\
&\overset{\langle X\,,J_r(X)\rangle=0}{=} \langle\nabla_{\dot\gamma}\dot\gamma\,, Z_r\rangle,
\end{aligned}$$

for all $r \in \{1, \ldots, 7\}$. In summary, we have shown that $\nabla_{\dot\gamma}\dot\gamma \perp \dot\gamma$ and $\nabla_{\dot\gamma}\dot\gamma \perp Z_r$ for all $r \in \{1, \ldots, 7\}$. Therefore the vector field $\nabla_{\dot\gamma}\dot\gamma$ has to be contained in the seven dimensional subspace $\operatorname{span}\{J_1(\dot\gamma), \ldots, J_7(\dot\gamma)\}$, that is

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{r=1}^7 g_r J_r(\dot\gamma).$$

It remains to show that the functions g_r are in fact constant. We fix $f_r\colon [a, b] \to \mathbb{R}$ for $r \in \{1, \ldots, 7\}$ such that $f_r(a) = f_r(b) = 0$ and $\int_a^b f_r = 0$. Furthermore, we consider a vector field $\tilde W$ such that its horizontal part satisfies $\tilde W_H = \sum_{r=1}^7 f_r J_r(\dot\gamma)$ and satisfies $\langle\tilde W\,, Z_r\rangle(s) = 2\int_a^s f_r(t)dt$.

The last condition for the vertical part of $\tilde W$ yields

$$\dot\gamma\langle\tilde W\,, Z_r\rangle = \frac{d}{ds}\left(2\int_a^s f_r(t)dt\right) = 2f_r(s),$$

for all $r \in \{1, \dots, 7\}$. The horizontal condition and the orthonormality of the family $\{J_1(\dot\gamma), \dots, J_7(\dot\gamma)\}$, see Appendix 2, imply

$$\langle \tilde W_H \,, J_r(\dot\gamma)\rangle = \left\langle \sum_{l=1}^{7} f_l J_l(\dot\gamma) \,, J_r(\dot\gamma) \right\rangle = f_r(s),$$

for all $r \in \{1, \dots, 7\}$. These two equations together imply the condition (6.2) of Lemma 6.3.1, which reads

$$\dot\gamma \langle \tilde W \,, Z_r\rangle = 2\langle \tilde W_H \,, J_r(\dot\gamma)\rangle,$$

for all $r \in \{1, \dots, 7\}$. Using Lemma 6.3.1 we conclude that $\tilde W$ is a vector field for an admissible variation of γ. We obtain

$$0 = \int_a^b \langle \nabla_{\dot\gamma}\dot\gamma \,, \tilde W\rangle = \sum_{r=1}^{7} \int_a^b f_r \langle \nabla_{\dot\gamma}\dot\gamma \,, J_r(\dot\gamma)\rangle,$$

which is valid for any seven functions with mean zero, which implies that $\langle \nabla_{\dot\gamma}\dot\gamma \,, J_r(\dot\gamma)\rangle$ is constant for all $r \in \{1, \dots, 7\}$. We obtain equation (6.3) for suitable constants $\lambda_1, \dots, \lambda_7 \in \mathbb{R}$. □

6.3.2 Interpretations and examples

Similar equations to the one in our main theorem can be found in the literature in different guises, and with various geometric and physical interpretations.

As mentioned in [55], when studying the case of the natural CR sub-Riemannian structure on the three dimensional sphere S^3, the admissible C^2 critical points of the length functional satisfy the equation

$$\nabla_{\dot\gamma}\dot\gamma + 2\lambda J(\dot\gamma) = 0, \tag{6.5}$$

where J is the almost complex structure on the horizontal distribution of S^3 induced by the CR structure. In that case, the constant λ corresponds to a curvature in the following sense: if γ solves the equation (6.5) with parameter λ, then the projection of γ to S^2 via the Hopf fibration produces a piece of a geodesic circle with constant geodesic curvature λ (see [55, Lemma 3.2]).

In the case of Theorem 6.3.2, after a rather tedious computation, we can show that the curves in G_7^1 starting from the origin and satisfying equation (6.3) with $\lambda_1 = \cdots = \lambda_7 = 0$ are straight lines in $\mathbb{R}^{15}$ contained in the 8-plane $z_1 = \cdots = z_7 = 0$. This fact indicates that we can again interpret the constants as curvatures. In a sense, the values of $\lambda_1, \dots, \lambda_7$ measure how far are the curves solving (6.3) from being a Riemannian geodesic. We are currently working on making this claim precise and applying it to all the similar cases known to us.

Finally, it is of worth mentioning this equation has a very similar structure to the so-called Wong's equation, see [71, Chapter 12], which corresponds to a nonabelian version of Lorentz equations for the dynamics of a particle. In that case, the parameter λ corresponds to the charge of the particle which satisfies an additional restriction in the form of an evolution equation. It would be of interest to study the precise relation between Wong's equation and the general formulation of critical points of length in sub-Riemannian manifolds with transverse symmetries, see [14].

6.4 Appendix

We present the matrices $\mathcal{J}_1, \dots, \mathcal{J}_7$ used in Section 6.2 to define the vector fields $X_1, \dots, X_8$.

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{J}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here we present the components of the Levi-Civita connection for the Riemannian metric on G_7^1 defined in Section 6.2.

$$
\begin{array}{llll}
\nabla_{X_1}Z_1 = -\frac{1}{2}X_2, & \nabla_{X_2}Z_1 = \frac{1}{2}X_1, & \nabla_{X_3}Z_1 = -\frac{1}{2}X_4, & \nabla_{X_4}Z_1 = \frac{1}{2}X_3, \\
\nabla_{X_5}Z_1 = -\frac{1}{2}X_6, & \nabla_{X_6}Z_1 = \frac{1}{2}X_5, & \nabla_{X_7}Z_1 = \frac{1}{2}X_8, & \nabla_{X_8}Z_1 = -\frac{1}{2}X_7, \\
\nabla_{X_1}Z_2 = -\frac{1}{2}X_3, & \nabla_{X_2}Z_2 = \frac{1}{2}X_4, & \nabla_{X_3}Z_2 = \frac{1}{2}X_1, & \nabla_{X_4}Z_2 = -\frac{1}{2}X_2, \\
\nabla_{X_5}Z_2 = -\frac{1}{2}X_7, & \nabla_{X_6}Z_2 = -\frac{1}{2}X_8, & \nabla_{X_7}Z_2 = \frac{1}{2}X_5, & \nabla_{X_8}Z_2 = \tfrac{1}{2}X_6, \\
\nabla_{X_1}Z_3 = -\frac{1}{2}X_4, & \nabla_{X_2}Z_3 = -\frac{1}{2}X_3, & \nabla_{X_3}Z_3 = \frac{1}{2}X_2, & \nabla_{X_4}Z_3 = \tfrac{1}{2}X_1, \\
\nabla_{X_5}Z_3 = -\frac{1}{2}X_8, & \nabla_{X_6}Z_3 = \frac{1}{2}X_7, & \nabla_{X_7}Z_3 = -\frac{1}{2}X_6, & \nabla_{X_8}Z_3 = \frac{1}{2}X_5, \\
\nabla_{X_1}Z_4 = -\frac{1}{2}X_5, & \nabla_{X_2}Z_4 = \frac{1}{2}X_6, & \nabla_{X_3}Z_4 = \frac{1}{2}X_7, & \nabla_{X_4}Z_4 = \frac{1}{2}X_8, \\
\nabla_{X_5}Z_4 = \frac{1}{2}X_1, & \nabla_{X_6}Z_4 = -\frac{1}{2}X_2, & \nabla_{X_7}Z_4 = -\frac{1}{2}X_3, & \nabla_{X_8}Z_4 = -\frac{1}{2}X_4, \\
\nabla_{X_1}Z_5 = -\frac{1}{2}X_6, & \nabla_{X_2}Z_5 = -\frac{1}{2}X_5, & \nabla_{X_3}Z_5 = \frac{1}{2}X_8, & \nabla_{X_4}Z_5 = -\frac{1}{2}X_7, \\
\nabla_{X_5}Z_5 = \frac{1}{2}X_2, & \nabla_{X_6}Z_5 = \frac{1}{2}X_1, & \nabla_{X_7}Z_5 = \frac{1}{2}X_4, & \nabla_{X_8}Z_5 = -\frac{1}{2}X_3, \\
\nabla_{X_1}Z_6 = -\frac{1}{2}X_7, & \nabla_{X_2}Z_6 = -\frac{1}{2}X_8, & \nabla_{X_3}Z_6 = -\frac{1}{2}X_5, & \nabla_{X_4}Z_6 = \frac{1}{2}X_6, \\
\nabla_{X_5}Z_6 = \frac{1}{2}X_3, & \nabla_{X_6}Z_6 = -\frac{1}{2}X_4, & \nabla_{X_7}Z_6 = \frac{1}{2}X_1, & \nabla_{X_8}Z_6 = \frac{1}{2}X_2, \\
\nabla_{X_1}Z_7 = -\frac{1}{2}X_8, & \nabla_{X_2}Z_7 = \frac{1}{2}X_7, & \nabla_{X_3}Z_7 = -\frac{1}{2}X_6, & \nabla_{X_4}Z_7 = -\frac{1}{2}X_5, \\
\nabla_{X_5}Z_7 = \frac{1}{2}X_4, & \nabla_{X_6}Z_7 = \frac{1}{2}X_3, & \nabla_{X_7}Z_7 = -\frac{1}{2}X_2, & \nabla_{X_8}Z_7 = \frac{1}{2}X_1.
\end{array}
$$

Part III

Sub-Riemannian Cut-locus

Chapter 7

The Sub-Riemannian cut locus of H-type groups

In the present chapter we present a proof of the fact that the sub-Riemannian cut locus of a wide class of nilpotent groups of step two, called H-type groups, starting from the origin corresponds to the center of the group. We obtain this result by completely describing the sub-Riemannian geodesics in the group, and using these to obtain three disjoint sets of points in the group determined by the number of geodesics joining them to the origin.

7.1 Introduction

The H(eisenberg)-type algebras, which are one of the most important examples of nilpotent Lie algebras of step 2, were introduced by A. Kaplan in his foundational work [59]. Their Lie algebra structure is intimately related to the existence of certain Clifford algebra representations, which we will introduce carefully later on. These Lie algebras have a deep connection to sub-Riemannian geometry, which we will proceed to explain.

It is well-known that for a nilpotent algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ there is a unique (up to isomorphism) connected and simply connected Lie group N with Lie algebra $\mathfrak{n}$. Applying this idea to an H-type algebra, and by left-translating the subspace $\mathfrak{v}$ of $\mathfrak{n}$, we obtain a bracket-generating distribution $\mathcal{H} \hookrightarrow TN$ of step 2. Any inner product defined on $\mathfrak{v}$ induces a sub-Riemannian metric on $\mathcal{H}$. Explicit equations for the sub-Riemannian geodesics in H-type groups can be found in [51].

A fundamental tool in the analysis of sub-Riemannian manifolds is the so-called cut locus. Recall that the (sub-Riemannian) cut locus of $(Q, \mathcal{H}, \langle \cdot\,, \cdot \rangle)$, with respect to a point $p \in Q$ is defined as the set of points $q \in Q$ such that there is more than one minimizing sub-Riemannian geodesic connecting p to q, for details see Definition 1.2.8. One of the most relevant applications where knowledge of this set plays an important part, is to describe the short-time asymptotic behavior of the heat kernel associated to a naturally defined sub-elliptic operator, see [12]. The aim of this chapter is to give a complete characterization of the sub-Riemannian cut locus for the H-type groups.

This chapter is organized as follows. In Section 7.2, we briefly recall some necessary definitions and give a precise form of the sub-Riemannian geodesics on an H-type group. In Section 7.3, we prove the main result of this chapter, which follows from studying carefully three different sets of points in the group. In order to understand these sets completely, we need to obtain results similar to those in [28], but valid in the full generality of H-type groups.

7.2 Sub-Riemannian geodesics on H-type groups

7.2.1 Sub-Riemannian H-type groups

Let us first recall the construction of the H-type algebras $\mathfrak{n}_{r,0}$. In what follows, all inner products are positive definite and $Cl_{r,0}$ is the Clifford algebra generated by the vector space $\mathfrak{z}_{r,0} = \mathbb{R}^{r,0}$ with inner product $\langle \cdot , \cdot \rangle_{\mathfrak{z}_{r,0}}$.

Consider a $Cl_{r,0}$-module $\mathfrak{v}_{r,0}$, where $J\colon \mathfrak{z}_{r,0} \to \mathrm{End}(\mathfrak{v}_{r,0})$ is the corresponding Clifford algebra representation. If $\langle \cdot , \cdot \rangle_{\mathfrak{v}_{r,0}}$ is an inner product on $\mathfrak{v}_{r,0}$, we can define a Lie bracket by

$$\langle J_Z v \, , w \rangle_{\mathfrak{v}_{r,0}} = \langle Z \, , [v \, , w] \rangle_{\mathfrak{z}_{r,0}}, \quad \text{for all } v, w \in \mathfrak{v}_{r,0} \, .$$

We set all brackets with elements in $\mathfrak{z}_{r,0}$ to be zero. This induces a Lie algebra structure of step 2 on $\mathfrak{n}_{r,0} = \mathfrak{v}_{r,0} \oplus \mathfrak{z}_{r,0}$. We define the inner product $\langle \cdot , \cdot \rangle := \langle \cdot , \cdot \rangle_{\mathfrak{v}_{r,0}} + \langle \cdot , \cdot \rangle_{\mathfrak{z}_{r,0}}$ on $\mathfrak{n}_{r,0}$. Then

$$J_{Z'} J_Z + J_Z J_{Z'} = -2 \langle Z \, , Z' \rangle_{\mathfrak{z}_{r,0}} \mathrm{Id}_{\mathfrak{v}_{r,0}} \ \text{ for all } Z \in \mathfrak{z}_{r,0} \, .$$

As it is usual in the literature, we call $\mathfrak{v}_{r,0}$ the horizontal space and the center $\mathfrak{z}_{r,0}$ the vertical space.

Let $\{v_1, \dots , v_m\}$ and $\{Z_1, \dots , Z_n\}$ be orthonormal bases of $\mathfrak{v}_{r,0}$ and $\mathfrak{z}_{r,0}$ respectively. The structure constants C_{ij}^k and the coefficients B_{ij}^k of the representation J are defined by

$$[v_i \, , v_j] = \sum_{k=1}^{n} C_{ij}^k Z_k \qquad \text{and} \qquad J_{Z_k} v_i = \sum_{j=1}^{m} B_{ij}^k v_j .$$

It is easy to see that $B_{ij}^k = C_{ij}^k$, and we will use this fact freely throughout many of the forthcoming computations.

It follows that the structure matrices $\{C^1, \dots , C^n\} \subset \mathfrak{so}(m)$ defined by $C^k = (C_{ij}^k)_{ij}$ for all $k = 1, \dots , n$ satisfy the relations

$$C^k C^p = -C^p C^k, \qquad \text{for } k \neq p,$$

and $(C^k)^2 = -\mathrm{Id}_{\mathfrak{v}_{r,0}}$. Furthermore, we note that $C^k C^p \in \mathfrak{so}(m)$ for $k \neq p$, since

$$(C^k C^p)^T = (C^p)^T (C^k)^T = (-C^p)(-C^k) = C^p C^k = -C^k C^p,$$

where $(C^k)^T$ is the transposed matrix of C^k.

The H-type group $N_{r,0}$ is the unique (up to isomorphism) connected and simply connected Lie group with Lie algebra $\mathfrak{n}_{r,0}$. The subspace $\mathfrak{v}_{r,0}$ defines a bracket generating distribution of step 2 over $N_{r,0}$ by left-translation, and the translations of the inner product $\langle \cdot , \cdot \rangle_{\mathfrak{v}_{r,0}}$ makes $N_{r,0}$ into a sub-Riemannian manifold.

Recall that for a simply connected nilpotent Lie group, the exponential map is a diffeomorphism, see for example [40]. Therefore we can identify $N_{r,0}$ with $\mathfrak{v}_{r,0} \oplus \mathfrak{z}_{r,0}$. We will use this identification through this chapter. Under this identification, we denote by $V_{r,0}$ and $Z_{r,0}$ the image of $\mathfrak{v}_{r,0}$ and $\mathfrak{z}_{r,0}$ respectively.

7.2.2 Sub-Riemannian geodesics on $N_{r,0}$

From now on, we write a horizontal sub-Riemannian geodesic $c\colon [0,1] \to N_{r,0}$ by $c(t) = (x(t), z(t))$, where $x(t)$ is in $V_{r,0}$ and $z(t) = (z^1(t), \dots, z^n(t))$ is in $Z_{r,0}$. From [51, Theorem 2], we have the following formulas:

$$x(t) = \frac{\sin(t|\theta|)}{|\theta|}\dot{x}(0) + \frac{(1-\cos(t|\theta|))}{|\theta|^2}\Omega\dot{x}(0), \tag{7.1}$$

$$\begin{aligned}\dot{z}^k(t) = \frac{1}{2}\dot{x}(0)^T\Bigg[&\frac{C^k}{|\theta|}\cos(t|\theta|)\sin(t|\theta|) + \frac{C^k\Omega}{|\theta|^2}\cos(t|\theta|)(1-\cos(t|\theta|)) \\ &+ \frac{\Omega^T C^k}{|\theta|^2}\sin^2(t|\theta|) + \frac{\Omega^T C^k\Omega}{|\theta|^3}\sin(t|\theta|)(1-\cos(t|\theta|))\Bigg]\dot{x}(0),\end{aligned} \tag{7.2}$$

where $\theta = (\theta_1, \dots, \theta_n) \neq (0, \dots, 0)$ is a vector of parameters coming from the Hamiltonian formulation, $\Omega = \sum_{k=1}^n C^k\theta_k$, $\theta = (\theta_1, \dots, \theta_n) \neq (0, \dots, 0)$ and $|\theta| = \left(\sum_{k=1}^n \theta_k^2\right)^{1/2}$.

For $\theta = (0, \dots, 0)$ the sub-Riemannian geodesics are straight lines, i.e., the geodesic starting at the point $(0,0)$ and reaching the point $(x,0)$ at time $t=1$ is given by $c(t) = (tx, 0)$.

Proposition 7.2.1. *The vertical part $z(t)$ of a horizontal geodesic $c(t)$ for the H-type group $N_{r,0}$, corresponding to $\theta = (\theta_1, \dots, \theta_n) \neq (0, \dots, 0)$, is given by*

$$z(t) = \frac{|\dot{x}(0)|^2}{2|\theta|^2}\left(t - \frac{\sin(t|\theta|)}{|\theta|}\right)\theta. \tag{7.3}$$

Proof. First we note that given a skew-symmetric matrix $A \in \mathfrak{so}(m)$, i.e. $A^T = -A$, then $v^TAv = 0$ for any vector $v \in \mathbb{R}^m$ as

$$\mathbb{R} \ni v^TAv = (v^TAv)^T = v^TA^Tv = -v^TAv.$$

Now we calculate the matrices $C^k\Omega$, Ω^TC^k and $\Omega^TC^k\Omega$.

$$\begin{aligned} C^k\Omega &= C^k\sum_{l=1}^n C^l\theta_l = \sum_{l=1}^n C^kC^l\theta_l = -\theta_k \operatorname{Id}_{\mathfrak{v}_r} + \sum_{l\neq k}^n C^kC^l\theta_l, \\ \Omega^TC^k &= -\Omega C^k = -\sum_{l=1}^n C^lC^k\theta_l = \theta_k\operatorname{Id}_{\mathfrak{v}_r} - \sum_{l\neq k}^n C^lC^k\theta_l \\ &= \theta_k\operatorname{Id}_{\mathfrak{v}_r} + \sum_{l\neq k}^n C^kC^l\theta_l \end{aligned}$$

$$\begin{aligned}
\Omega^T C^k \Omega &= \left(\theta_k \operatorname{Id}_{\mathfrak{v}_r} + \sum_{l \neq k}^{n} C^k C^l \theta_l\right) \Omega = \theta_k \Omega + \sum_{l \neq k}^{n} \sum_{p=1}^{n} C^k C^l C^p \theta_l \theta_p \\
&= \theta_k \Omega + \sum_{l \neq k}^{n} \sum_{p \neq k}^{n} C^k C^l C^p \theta_l \theta_p + \sum_{l \neq k}^{n} C^k C^l C^k \theta_l \theta_p \\
&= \theta_k \Omega + \sum_{l \neq k}^{n} \sum_{p \neq k}^{n} C^k C^l C^p \theta_l \theta_p + \sum_{l \neq k}^{n} C^l \theta_l \theta_p \\
&= \theta_k \Omega + \sum_{l \neq k}^{n} -C^k \theta_l^2 + \sum_{l \neq k}^{n} C^l \theta_l \theta_p,
\end{aligned}$$

where the last equation is obtained as $C^l C^p = -C^p C^l$ for $p \neq l$. As Ω, C^p, $C^p C^l$ are skew-symmetric for any $p \neq l$, we obtain that

$$\begin{aligned}
\dot{x}(0)^T C^k \dot{x}(0) &= 0, \\
\dot{x}(0)^T C^k \Omega \dot{x}(0) &= -\theta_k |\dot{x}(0)|^2 + \sum_{l \neq k}^{n} \dot{x}(0)^T C^k C^l \dot{x}(0) \theta_l = -\theta_k |\dot{x}(0)|^2, \\
\dot{x}(0)^T \Omega^T C^k \dot{x}(0) &= \theta_k |\dot{x}(0)|^2 + \sum_{l \neq k}^{n} \dot{x}(0)^T C^k C^l \dot{x}(0) \theta_l = \theta_k |\dot{x}(0)|^2, \\
\dot{x}(0)^T \Omega^T C^k \Omega \dot{x}(0) &= \theta_k \dot{x}(0)^T \Omega \dot{x}(0) + \sum_{l \neq k}^{n} -\dot{x}(0)^T C^k \dot{x}(0) \theta_l^2 \\
&\quad + \sum_{l \neq k}^{n} \dot{x}(0)^T C^l \dot{x}(0) \theta_l \theta_p = 0.
\end{aligned}$$

It follows that for any $k = 1, \dots, n$

$$\begin{aligned}
\dot{z}^k(t) &= \theta_k |\dot{x}(0)|^2 \frac{-\cos(t|\theta|)(1 - \cos(t|\theta|)) + \sin^2(t|\theta|)}{2|\theta|^2} \\
&= \theta_k |\dot{x}(0)|^2 \frac{1 - \cos(t|\theta|)}{2|\theta|^2}, \\
z^k(t) &= \frac{\theta_k |\dot{x}(0)|^2}{2|\theta|^2} \left(t - \frac{\sin(t|\theta|)}{|\theta|}\right).
\end{aligned}$$ □

This simplification will allow us to determine concretely the points in any H-type group where minimizing sub-Riemannian geodesics starting from the origin stop being unique.

7.3 Sub-Riemannian cut locus of H-type groups

In this section, we give a precise description of the sub-Riemannian cut locus of curves starting from the identity $(0, 0)$ in the H-type groups introduced previously. More precisely, we want to prove the following

Theorem 7.3.1. *The cut locus $K_{(0,0)}$ of the H-type group $N_{r,0}$ is given by the points of the form $(0,z)$.*

This result will be achieved in three steps: points of the form $(0,z)$ are in $K_{(0,0)}$; points of the form (x,z), with $x \neq 0$ and $z \neq 0$, are not in $K_{(0,0)}$; and neither are the points of the form $(x,0)$.

7.3.1 The vertical space is contained in the cut locus

As a first step, we show that the points $(0,z) \in N_{r,0}$ are in the cut locus. The geodesics connecting the origin $(0,0)$ and $(0,z)$ and their length are given by the following Theorem which generalizes [28, Theorem 6.3] to arbitrary H-type groups.

Theorem 7.3.2. *For each natural number $k \in \mathbb{N}$, there exists a sub-Riemannian geodesic $c_k(t) = (x_k(t), z_k(t))$ in $N_{r,0}$ joining the origin with the point $(0,z)$. These curves have lengths $l_1, l_2, \dots$, where $l_k^2 = 4k\pi|z|$, $k \in \mathbb{N}$, and their equations are*

$$x_k(t) = 4\frac{\sin^2(k\pi t)}{|\dot{x}(0)|^2}\mathcal{Z}\dot{x}(0) + \frac{\sin(2k\pi t)}{2k\pi}\dot{x}(0), \quad k \in \mathbb{N},$$

where $\mathcal{Z} = \sum_{r=1}^n z^r C^r$ and

$$z_k(t) = \left(t - \frac{\sin(2\pi k t)}{2\pi k}\right) z, \quad k \in \mathbb{N}.$$

Proof. We follow a similar scheme as in [28]. Evaluating equation (7.1) at $t = 1$, and after some simple computations, we see that

$$0 = |x(1)|^2 = \frac{4|\dot{x}(0)|^2}{|\theta|^2}\sin^2\left(\frac{|\theta|}{2}\right).$$

Since we can assume that $|\dot{x}(0)| \neq 0$, it follows that $|\theta| = 2k\pi$, for $k \in \mathbb{N}$. This, in turn, implies that

$$z = z(1) = \frac{|\dot{x}(0)|^2}{8k^2\pi^2}\theta,$$

thus $|\dot{x}(0)|^2 = 4k\pi|z|$. We immediately obtain the geodesic equations.

The length of the geodesics follows easily, since

$$l_k^2 = l(c_k)^2 = \left(\int_0^1 \sqrt{|\dot{x}(t)|}dt\right)^2 = |\dot{x}(0)|^2 = 4k\pi|z|. \qquad \square$$

Suppose $c(t)$ is the minimizing geodesic between the origin and $(0,z)$ with length $4\pi|z|$ and with initial vector $\dot{x}(0) \neq 0$. We define the geodesic $\tilde{c}(t) = (\tilde{x}(t), \tilde{z}(t))$ with initial vector $-\dot{x}(0)$ by

$$\begin{aligned}
\tilde{x}(t) &= -4\frac{\sin^2(\pi m t)}{|\dot{x}(0)|^2}\mathcal{Z}\dot{x}(0) - \frac{\sin(2\pi t)}{2\pi}\dot{x}(0),\\
\tilde{z}(t) &= \left(t - \frac{\sin(2\pi t)}{2\pi}\right) z.
\end{aligned}$$

This geodesic is minimizing between the origin and $(0,z)$ as it has length $4\pi|z|$ and clearly $c \neq \tilde{c}$. It follows that $(0,z)$ is an element of the cut locus of the origin.

7.3.2 If $x \neq 0$ and $z \neq 0$, then (x, z) is not in the cut locus

Theorem 7.3.1 will follow after proving that points not contained in the vertical space are not elements in the cut locus. We start the analysis by proving the following extension of [28, Theorem 6.5].

Theorem 7.3.3. *Given a point $(x,z) \in N_{r,0}$ with $x \neq 0$, $z \neq 0$, there are finitely many sub-Riemannian geodesics joining the origin $(0,0)$ with (x,z). Let $|\theta|_1, \dots, |\theta|_N$ be solutions of the equation*

$$\frac{4|z|}{|x|^2} = \mu(|\theta|/2),$$

where $\mu(\alpha) = \frac{\alpha}{\sin^2(\alpha)} - \cot(\alpha)$. Then the equation of the geodesic $c_k(t) = (x_k(t), z_k(t))$, $t \in [0,1]$, corresponding to $|\theta|_k$, is

$$\begin{aligned} x_k(t) &= \sin\left(\frac{t|\theta|_k}{2}\right)\cos\left(\frac{t|\theta|_k}{2}\right)\left(\frac{8\sin^2\left(\frac{|\theta|_k}{2}\right)\left(\tan\left(\frac{t|\theta|_k}{2}\right)\cot\left(\frac{|\theta|_k}{2}\right)-1\right)}{|x|^2(|\theta|_k - \sin(|\theta|_k))}\mathcal{Z}x \right. \\ &\quad \left. + \left(\tan\left(\frac{t|\theta|_k}{2}\right) + \cot\left(\frac{|\theta|_k}{2}\right)\right)x\right), \\ z_k(t) &= \frac{t|\theta|_k - \sin(t|\theta|_k)}{|\theta|_k - \sin(|\theta|_k)}\, z \end{aligned}$$

with $\mathcal{Z} = \sum_{r=1}^n z^r C^r$ and $k = 1, 2, \dots, N$. The lengths of these geodesics are $l_k^2 = \nu(|\theta|_k)(|x|^2 + 4|z|)$, where

$$\nu(\alpha) = \frac{\alpha^2}{2(1 + \alpha - \cos(\alpha) - \sin(\alpha))}.$$

Proof. Putting $s = 1$ into equations (7.1) and (7.3), we see that

$$|x|^2 = |x(1)|^2 = \frac{4\sin^2(|\theta|/2)}{|\theta|^2}\,|\dot{x}(0)|^2, \tag{7.4}$$

$$z = z(1) = \frac{|x|^2}{8\sin^2(|\theta|/2)}\left(1 - \frac{\sin(|\theta|)}{|\theta|}\right)\theta = \frac{|x|^2\mu(|\theta|/2)}{4|\theta|}\,\theta. \tag{7.5}$$

It follows that $|z| = \frac{1}{4}|x|^2\mu(|\theta|/2)$. Let $|\theta|_1, \dots, |\theta|_N$ be the solutions of this equation. We fix a solution $|\theta|_k$ and obtain by the use of equation (7.4) in (7.5) that

$$\theta = \frac{2|\theta|_k^3\, z}{|\dot{x}(0)|^2(|\theta|_k - \sin(|\theta|_k))}, \tag{7.6}$$

and therefore we have that

$$z_k(t) = \frac{t|\theta|_k - \sin(t|\theta|_k)}{|\theta|_k - \sin(|\theta|_k)}\, z.$$

To find the expression for $x_k(t)$, let us first observe that

$$\left(\frac{|\theta|_k}{2}\cot\left(\frac{|\theta|_k}{2}\right)\mathrm{Id}_{\mathfrak{v}_r} - \frac{\Omega}{2}\right)\left(\frac{\sin(|\theta|_k)}{|\theta|_k}\mathrm{Id}_{\mathfrak{v}_r} + \frac{1-\cos(|\theta|_k)}{|\theta|_k^2}\,\Omega\right) = \mathrm{Id}_{\mathfrak{v}_r},$$

which follows from simple trigonometric identities and the fact that choosing the corresponding covector θ, we have that $\Omega^2 = -|\theta|_k^2 \operatorname{Id}_{\mathfrak{v}_r}$. Since equation (7.1) can be written as

$$x = \left(\frac{\sin(|\theta|_k)}{|\theta|_k} \operatorname{Id}_{\mathfrak{v}_r} + \frac{(1-\cos(|\theta|_k))}{|\theta|_k^2} \Omega \right) \dot{x}(0),$$

then $\dot{x}(0) = \left(\frac{|\theta|_k}{2} \cot\left(\frac{|\theta|_k}{2}\right) \operatorname{Id}_{\mathfrak{v}_r} - \frac{\Omega}{2} \right) x$, and therefore

$$\begin{aligned} x_k(t) &= \left(\frac{\sin(t|\theta|_k)}{|\theta|_k} \operatorname{Id}_{\mathfrak{v}_r} + \frac{(1-\cos(t|\theta|_k))}{|\theta|_k^2} \Omega \right) \dot{x}(0) \\ &= \left(\frac{\sin(t|\theta|_k)}{|\theta|_k} \operatorname{Id}_{\mathfrak{v}_r} + \frac{(1-\cos(t|\theta|_k))}{|\theta|_k^2} \Omega \right) \left(\frac{|\theta|_k}{2} \cot\left(\frac{|\theta|_k}{2}\right) \operatorname{Id}_{\mathfrak{v}_r} - \frac{\Omega}{2} \right) x. \end{aligned}$$

The equation for $x_k(t)$ in the statement follows from a simple computation, using the formula above and equation (7.6).

We calculate the length of our obtained geodesics. For a fixed solution $|\theta|_k$ we obtain

$$\begin{aligned} \dot{x}_k(t) &= \frac{1}{2}\Big(\Big(-\cos(t|\theta|_k) + \cot\left(\frac{|\theta|_k}{2}\right) \sin(t|\theta|_k)\Big)\Omega x \\ &+ |\theta|_k \Big(\cos(t|\theta|_k) \cot\left(\frac{|\theta|_k}{2}\right) + \sin(t|\theta|_k)\Big) x\Big), \end{aligned}$$

such that

$$\begin{aligned} \langle \dot{x}_k(t)\,,\dot{x}_k(t)\rangle &= \langle \Omega x\,,\Omega x\rangle \frac{1}{4}\Big(-\cos(t|\theta|_k) + \cot\left(\frac{|\theta|_k}{2}\right)\sin(t|\theta|_k)\Big)^2 \\ &+ \langle x\,,x\rangle \frac{1}{4}|\theta|_k^2 \Big(\cos(t|\theta|_k)\cot\left(\frac{|\theta|_k}{2}\right) + \sin(t|\theta|_k)\Big)^2 \\ &= \langle x\,,x\rangle \frac{1}{4}|\theta|_k^2 \Big(\Big(-\cos(t|\theta|_k) + \cot\left(\frac{|\theta|_k}{2}\right)\sin(t|\theta|_k)\Big)^2 \\ &+ \Big(\cos(t|\theta|_k)\cot\left(\frac{|\theta|_k}{2}\right) + \sin(t|\theta|_k)\Big)^2\Big) \\ &= \langle x\,,x\rangle \frac{|\theta|_k^2}{2-2\cos(|\theta|_k)}. \end{aligned}$$

Hence $l_k^2 = \nu(|\theta|_k)(|x|^2 + 4|z|)$, as $4|z| = |x|^2 \mu(|\theta|_k/2)$. $\square$

Given a point (x,z) with $x \neq 0$, $z \neq 0$. Then there exists N solutions $|\theta|_1, \dots, |\theta|_N$ of the equation

$$\frac{4|z|}{|x|^2} = \mu\left(\frac{|\theta|}{2}\right).$$

If $N = 1$, then there does not exist a second minimizing geodesic. Hence (x,z) is not in the cut locus.

For $N > 1$, we have to examine the solutions $|\theta|_k$ in detail. Without loss of generality, we assume that $|\theta|_k < |\theta|_{k+1}$. We know that μ is an increasing diffeomorphism on the interval

$(-2\pi\,,2\pi)$ onto $\mathbb{R}$ such that $|\theta|_1 < 2\pi$ and $|\theta|_k > 2\pi$ for all $1 < k \leq N$, see Figure 7.1. In Figure 7.2, we see that $\nu(2\pi) = \pi$ and that $\nu(x) < \nu(y)$ for all $x \in [0,\pi)$, $y \in (\pi,\infty)$. Hence

$$\nu(|\theta|_1) < \nu(|\theta|_k), \qquad \text{for } 1 < k \leq N.$$

This implies that the geodesics $c_k(t) = (x_k(t), z_k(t))$ cannot be minimizing for $1 < k \leq N$. This implies that the only minimizing geodesic between the origin and (x,z) with $x \neq 0$, $z \neq 0$ is given by $c_1(t) = (x_1(t), z_1(t))$, hence (x,z) is not in the cut locus.

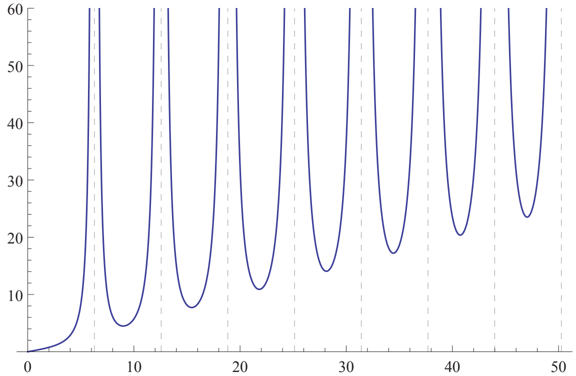

Figure 7.1: $\mu\left(\frac{\alpha}{2}\right) = \frac{\left(\frac{\alpha}{2}\right)^2}{\sin^2\left(\frac{\alpha}{2}\right)} - \cot\left(\frac{\alpha}{2}\right)$ on the interval $[0\,,16\pi]$ with vertical lines at the points $2n\pi$, $n \in \mathbb{N}$.

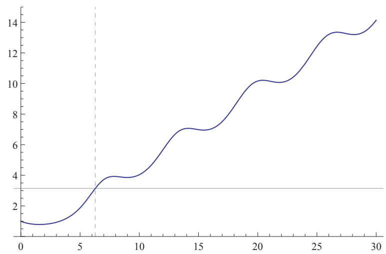

Figure 7.2: $\frac{\alpha^2}{2(1+\alpha-\cos(\alpha)-\sin(\alpha))}$ on the interval $[0\,,30]$ with vertical line at the point 2π and horizontal line at the point π.

7.3.3 Points of the form $(x,0)$ are not in the cut locus

To conclude the proof of Theorem 7.3.1, we prove the following result.

Theorem 7.3.4. *A sub-Riemannian geodesic $c(t)$ in $N_{r,0}$ is horizontal with constant z-coordinate $z_0 \in Z_{r,0}$ if and only if $c(t) = (at, z_0)$ for some vector $a \in V_{r,0}$ such that $|a| \neq 0$.*

Proof. Since $z(t) = z_0$ is constant, then $0 = \dot{z}(t)$. If we assume $|\theta| \neq 0$, then we can apply Proposition 7.2.1, to see that

$$0 = \dot{z}(t) = \frac{|\dot{x}(0)|^2(1 - \cos(t|\theta|))}{2|\theta|^2}\theta.$$

Since $|\dot{x}(0)|^2 \neq 0$ and $(1 - \cos(t|\theta|)) \neq 0$ for all $t \in [0, 1]$, we obtain a contradiction. It follows that θ must vanish, and thus

$$x(t) = t\,\dot{x}(0),$$

from the characterization of geodesics in Subsection 7.2.2. Setting $a = \dot{x}(0)$, the result is proved. □

It remains to show that there is no geodesic connecting the origin $(0, 0)$ with $(x, 0)$ with non-constant vertical component $z(t)$. Let assume that there exist such a geodesic which reaches $(x, 0)$ at time $t_0 = 1$, then the non-constant part $z(t)$ is given by

$$z(t) = \frac{|\dot{x}(0)|^2}{4|\theta|^2}\left(t - \frac{\sin(2t|\theta|)}{2|\theta|}\right)\theta.$$

It follows that $1 = \frac{\sin(2|\theta|)}{2|\theta|}$ if and only if $|\theta| = 0$, see Figure 7.3, which implies that $z(t)$ is constant. This is a contradiction to our assumption, hence there does not exist a geodesic connecting the origin $(0, 0)$ with $(x, 0)$ with non-constant vertical component $z(t)$. Hence the geodesic given in Theorem 7.3.4 is the unique geodesic connecting $(0, 0)$ with $(x, 0)$.

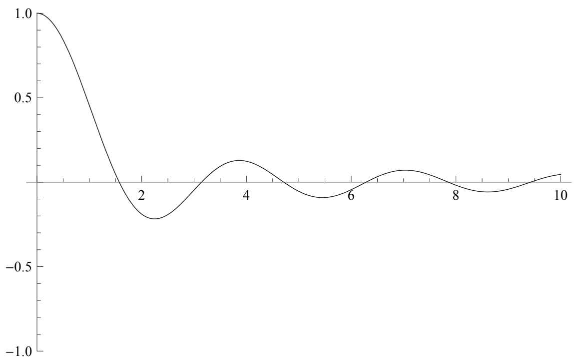

Figure 7.3: $\frac{\sin(2\alpha)}{2\alpha}$ on the interval $[0\,, 10]$.

Chapter 8

Sub-Riemannian geometry of Stiefel manifolds

In this chapter we consider the Stiefel manifold $V_{n,k}$ in the real and complex case as a principal $U(k)$-bundle over the Grassmann manifold and study the cut locus from the unit element. We give the complete description of this cut locus on $V_{n,1}$ and present a sufficient condition on the general case. At the end, we study the complement to the cut locus of $V_{2k,k}$ and extend our results to the real case.

The main results of this chapter are summarized and published in the paper [9].

8.1 Introduction

One of the main objects of interest in sub-Riemannian geometry are normal and abnormal geodesics which are two different but not mutually disjoint families. Contrary to the Riemannian geometry, the exponential map is not a local diffeomorphism. Nevertheless, the singularities of the exponential map, as in Riemannian geometry, are closely related to the cut locus and failure of the optimality for geodesics. The cut locus in sub-Riemannian geometry is an object which is of great interest, but rather poorly studied. There exist very few results concerning the global and local structure of it and most of them are restricted to low dimensional manifolds. The work [74] studies the one dimensional Heisenberg group, and the results easily can be extended to higher dimensions. A full description of the global structure of the cut locus for the groups $SU(2)$, $SO(3)$, $SL(2)$, and lens spaces is given in [24]. For the groups $SO(3)$, $SL(2)$, and lens spaces the cut locus is a stratified set, whereas in $SU(2)$ it is a maximal circle S^1 without one point. The reader will find similar structures to those that have been obtained in the present chapter. The global structure of the exponential map and the cut locus of the identity on the group $SE(2)$ is completely presented in [83]. The nature of normal and abnormal geodesics and complexity of the cut locus structure in sub-Riemannian geometry on the example of the Martinet manifold is pointed out in the work [2]. More interesting results can be found also in [13, 72, 73].

In the present chapter we consider the Stiefel manifold $V_{n,k}$ as a principal $U(k)$-bundle with the Grassmann manifold as a base space. We completely describe the cut locus from the unit element for the case $V_{n,1}$. Technical difficulties do not allow to extend these results to the general case $V_{n,k}$. Nevertheless, we present a partial description of the cut locus, which is to our knowledge an almost unique example for manifolds of higher dimensions.

The structure of the chapter is the following. In Section 8.2, we define the Stiefel and Grassmann manifolds embedded in $U(n)$, their metrics of constant bi-invariant type and their normal geodesics based on the general Theorem 1.2.18 that can be found in [71]. In Section 8.3, we describe the cut locus for the equivalence class of the unit element on the principal $U(1)$-bundle structure on the Stiefel manifold $V_{n,1}$. Since the considered manifold is homogeneous it gives the structure of the cut locus for any point. Section 8.4 is dedicated to the cut locus for the general case of the Stiefel manifold $V_{n,k}$. In Section 8.5, we briefly review some particular cases of the Stiefel manifold embedded in $SO(n)$.

8.2 Stiefel and Grassmann manifolds embedded in $U(n)$

We use the following notation in the present section. Let $\mathbb{C}^n$ denote the n-dimensional complex vector space and $\mathbb{C}^{m\times n}$ the set of $(m \times n)$-matrices with complex entries. We want to apply Theorem 1.2.18 to the submersion $\pi\colon V_{n,k}(\mathbb{C}^n) \to G_{n,k}(\mathbb{C}^n)$, where $V_{n,k}(\mathbb{C}^n) = V_{n.k}$ is the Stiefel manifold and $G_{n,k}(\mathbb{C}^n) = G_{n,k}$ is the Grassmann manifold for $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$.

We start from the description of a general construction. Given a group G with an invariant inner product on its Lie algebra $\mathfrak{g}$ and two subgroups $H, K \subset G$, we form the quotient spaces G/H and $G/(H \times K)$. The submersion $G/H \to G/(H \times K)$ is a principal K-bundle, with Riemannian metrics on G/H and $G/(H\times K)$ induced from the bi-invariant Riemannian metric on G generated by an invariant inner product. The Riemannian metrics are induced by the projections $G \to G/H$ and $G \to G/(H \times K)$. Both manifolds in the submersion $G/H \to G/(H \times K)$ are homogeneous manifolds, where the group G acts transitively. The induced Riemannian metric on G/H is also bi-invariant under the action of the group G. The geodesics on G/H are the projections from G of one-parameter subgroups $\exp(t\xi)$ with ξ orthogonal to the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of H. We introduce the specific subgroups of $U(n)$:

$$U_n^{up}(k) := \left\{ \begin{pmatrix} U_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \,\middle|\, U_k \in U(k) \right\} \subset U(n) \quad \text{and}$$

$$U_n^{l}(k) := \left\{ \begin{pmatrix} I_{n-k} & 0 \\ 0 & U_k \end{pmatrix} \,\middle|\, U_k \in U(k) \right\} \subset U(n).$$

Note that we use the notation $U_n^{up}(k)$ and $U_n^l(k)$ with the lower subscript n in the current section to emphasize that the elements of these subgroups are written as $(n \times n)$-matrices and the upper-script indicates that the subgroups $U(k)$ are given by matrices in upper left or lower right angle in the $(n \times n)$ matrices. The subgroups $U_n^{up}(k)$ and $U_n^l(k)$ are different, but isomorphic. Set $G = U(n)$, $H = U_n^l(n-k)$, $K = U_n^{up}(k)$. Then the quotient $G/H = U(n)/U_n^l(n-k)$ is isomorphic to the Stiefel manifold $V_{n,k}$ and $G/(H \times K) = U(n)/(U_n^l(n-k) \times U_n^{up}(k))$ is isomorphic to the Grassmann manifold $G_{n,k}$.

8.2.1 Unitary group and bi-invariant metric

Before giving a detailed definition of the Stiefel and Grassmann manifolds, we remind that the unitary group $U(n)$ is a matrix Lie group, whose elements X satisfy the condition

$$U(n) = \{X \in \mathbb{C}^{n\times n} |\ \bar{X}^T X = X\bar{X}^T = I_n\}.$$

Here I_n is the unit $(n \times n)$-matrix and $\bar{X}^T$ is the complex conjugate and transposed of the matrix X. The Lie algebra $\mathfrak{u}(n)$ consists of all skew-Hermitian matrices:

$$\mathfrak{u}(n) = \{\mathcal{X} \in \mathbb{C}^{n\times n} |\ \mathcal{X} = -\bar{\mathcal{X}}^T\}.$$

We remind that a matrix $X \in U(n)$ is of full rank, its columns and rows are orthonormal with respect to the standard Hermitian product in $\mathbb{C}^n$ and that the main diagonal of the skew-Hermitian matrices are purely imaginary. Moreover, the Hermitian product in $\mathbb{C}^n$ is invariant under the action of $U(n)$, that particularly means that the orthogonality is preserved under this action. The Lie algebra $\mathfrak{u}(n)$ can be endowed with the inner product $(\mathcal{X}, \mathcal{Y})_{\mathfrak{u}(n)} := -\frac{1}{n}\operatorname{tr}(\mathcal{X}\mathcal{Y})$, $\mathcal{X}, \mathcal{Y} \in \mathfrak{u}(n)$. Considering $U(n)$ as a real analytic manifold, we denote its points by q and the metric at this point by $\langle\cdot\,,\cdot\rangle_{U(n)}(q)$ or, if it is clear from the context, simply by g_q. Then a left-invariant metric on $U(n)$ with respect to the group action of $U(n)$ is given by

$$\begin{array}{rrlcl} \langle\cdot\,,\cdot\rangle_{U(n)}(q): & T_qU(n) \times T_qU(n) \cong & q\mathfrak{u}(n) \times q\mathfrak{u}(n) & \to & \mathbb{R} \\ & & (q\mathcal{X}\ ,\ q\mathcal{Y}) & \mapsto & -\frac{1}{n}\operatorname{tr}(\mathcal{X}\mathcal{Y}) \end{array}$$

$q \in U(n)$, $\mathcal{X}, \mathcal{Y} \in \mathfrak{u}(n)$. We claim, that this metric is actually bi-invariant, which follows from the observation that can be found, for instance, in [47] and [68]. We present some details.

Definition 8.2.1. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G endowed with an inner product $(\cdot\,,\cdot)_{\mathfrak{g}}$. An inner product $(\cdot\,,\cdot)_{\mathfrak{g}}$ is called invariant if it is invariant under the adjoint action of G, i. e. $(q^{-1}\eta q, q^{-1}\xi q)_{\mathfrak{g}} = (\eta, \xi)_{\mathfrak{g}}$ for all $\eta, \xi \in \mathfrak{g}$ and $q \in G$.*

Then it is well known, see for instance [62], that an invariant inner product $(\cdot\,,\cdot)_{\mathfrak{g}}$ on a Lie algebra $\mathfrak{g}$ determines a bi-invariant metric $\langle\cdot\,,\cdot\rangle_G$ on the group G via

$$\langle\eta\,, \xi\rangle_G(q) := (q^{-1}\eta, q^{-1}\xi)_{\mathfrak{g}} = (\eta q^{-1}, \xi q^{-1})_{\mathfrak{g}}$$

for all $\eta, \xi \in T_qG$.

We only need to check that the inner product $(\mathcal{X}, \mathcal{Y})_{\mathfrak{u}(n)} = -\frac{1}{n}\operatorname{tr}(\mathcal{X}\mathcal{Y})$ on $\mathfrak{u}(n)$ is invariant. Indeed,

$$\begin{aligned} (q^{-1}\mathcal{X}q, q^{-1}\mathcal{Y}q)_{\mathfrak{u}(n)} &= -n^{-1}\operatorname{tr}(q^{-1}\mathcal{X}qq^{-1}\mathcal{Y}q) = -n^{-1}\operatorname{tr}(q^{-1}\mathcal{X}\mathcal{Y}q) \\ &= -n^{-1}\operatorname{tr}(\mathcal{Y}qq^{-1}\mathcal{X}) = -n^{-1}\operatorname{tr}(\mathcal{X}\mathcal{Y}) = (\mathcal{X}, \mathcal{Y})_{\mathfrak{u}(n)} \end{aligned}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{u}(n)$ and $q \in U(n)$.

Remark 8.2.2. *The left and right action of any subgroup $U_n^{up}(k)$, $U_n^l(k)$, $1 \le k \le n$ on the group $U(n)$ and on the Lie algebra $\mathfrak{u}(n)$ are defined as a matrix multiplication from the left or from the right. The inner product $(\cdot\,,\cdot)_{\mathfrak{g}} = -\frac{1}{n}\operatorname{tr}(\cdot\,,\cdot)$ on the Lie algebra $\mathfrak{u}(n)$ is invariant under the adjoint action of $U_n^{up}(k)$ or $U_n^l(k)$ and therefore the metric $\langle\cdot\,,\cdot\rangle_{U(n)}$, defined by left or right translations by the action of $U_n^{up}(k)$ or $U_n^l(k)$, is bi-invariant under this action.*

8.2.2 Stiefel manifold and metric of constant bi-invariant type

The Stiefel manifold $V_{n,k}$ is the set of all k-tuples $(q_1, \dots, q_k)$ of vectors $q_i \in \mathbb{C}^n$, $i \in \{1, \dots, k\}$, which are orthonormal with respect to the standard Hermitian metric. This is a compact real analytic manifold which can be equivalently defined as

$$V_{n,k} := \{X \in \mathbb{C}^{n\times k} | \ \ \bar{X}^T X = I_k\}.$$

The condition $\bar{X}^T X = I_k$ is equivalent to the orthonormality of columns in X. These k orthonormal columns can be considered as arbitrary k columns in a matrix $X \in U(n)$. This allows us to realize the Stiefel manifold as a quotient set of $U(n)$ by the subgroup $U_n^l(n-k)$. To do this we introduce the equivalence relation $\backsim_1$ on $U(n)$ by

$$q \backsim_1 p \quad \Longleftrightarrow \quad q = p\begin{pmatrix} I_k & 0 \\ 0 & U_{n-k}\end{pmatrix}, \qquad q, p \in U(n), \quad U_{n-k} \in U(n-k).$$

This yields to the equivalence class for $q \in U(n)$

$$[q]^{\backsim_1} = \left\{q\begin{pmatrix} I_k & 0 \\ 0 & U_{n-k}\end{pmatrix} \Big| U_{n-k} \in U(n-k)\right\} \in U(n)/U_n^l(n-k), \quad q \in U(n).$$

The quotient $U(n)/U_n^l(n-k)$ is a real analytic manifold with the quotient topology and we denote by π_1 the natural projection from $U(n)$ to the quotient $U(n)/U_n^l(n-k)$. We identify the equivalence class $[q]^{\backsim_1}$ with a point in the Stiefel manifold and write $[q]_{V_{n,k}} \in V_{n,k}$ instead of $[q]^{\backsim_1}$ to emphasize that the point belongs to the Stiefel manifold. The real dimension of $V_{n,k}$ is $2nk - k^2$.

The tangent space to the Stiefel manifold is the quotient of the tangent space to $U(n)$ by the tangent space of the equivalence classes. To obtain it we differentiate the curves $c(t) \in [q]_{V_{n,k}}$ at $t = 0$ for a fixed $q \in U(n)$ and receive the space $\mathcal{R} = \left\{q\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{C}\end{pmatrix} \mid \mathcal{C} \in \mathfrak{u}(n-k)\right\}$. Intuitively, movements in the direction $\mathcal{R}$ make no change in the quotient space. It follows that the tangent space $T_{[q]_{V_{n,k}}} V_{n,k}$ to the Stiefel manifold at $[q]_{V_{n,k}} \in V_{n,k}$ is given by the quotient of the tangent space $T_q U(n)$, which is isomorphic to $q\mathfrak{u}(n)$, by $\mathcal{R}$:

$$T_{[q]_{V_{n,k}}} V_{n,k} = \left\{[q]_{V_{n,k}}\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}_2}^T \\ \mathcal{X}_2 & 0\end{pmatrix} \Bigg| \ \ \mathcal{X}_1 \in \mathfrak{u}(k), \mathcal{X}_2 \in \mathbb{C}^{(n-k)\times k}\right\}.$$

Similar results can be found in [44, 67].

Now we define a metric $\langle \cdot\,, \cdot\rangle_{V_{n,k}}$ on $V_{n,k}$ by

$$\left\langle [q]_{V_{n,k}}\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}_2}^T \\ \mathcal{X}_2 & 0\end{pmatrix}, [q]_{V_{n,k}}\begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}_2}^T \\ \mathcal{Y}_2 & 0\end{pmatrix}\right\rangle_{V_{n,k}} \left([q]_{V_{n,k}}\right)$$
$$:= \left\langle q\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}_2}^T \\ \mathcal{X}_2 & 0\end{pmatrix}, q\begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}_2}^T \\ \mathcal{Y}_2 & 0\end{pmatrix}\right\rangle_{U(n)} (q) = \left(\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}_2}^T \\ \mathcal{X}_2 & 0\end{pmatrix}, \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}_2}^T \\ \mathcal{Y}_2 & 0\end{pmatrix}\right)_{\mathfrak{u}(n)},$$

where $q \in [q]_{V_{n,k}}$ is any representative of the equivalence class $[q]_{V_{n,k}}$. It is clear from this definition that the metric $\langle \cdot\,, \cdot\rangle_{V_{n,k}}$ is independent of the choice of a representative.

Since $U_k[q]_{V_{n,k}} = [U_k q]_{V_{n,k}}$ and $[q]_{V_{n,k}} U_k = [qU_k]_{V_{n,k}}$, $U_k \in U_n^{up}(k)$, it follows directly from the definition of the metric on $T_{[q]_{V_{n,k}}} V_{n,k}$ and the bi-invariance of the metric $\langle \cdot\,,\cdot\rangle_{U(n)}$ with respect to $U_n^{up}(k)$ that

$$\begin{aligned}
&\left\langle [U_k q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [U_k q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}} \\
= \;& \left(\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right)_{\mathfrak{u}(n)} \\
= \;& \left\langle [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle [qU_k]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [qU_k]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}} \\
= \;& \left(\begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right)_{\mathfrak{u}(n)} \\
= \;& \left\langle [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\bar{\mathcal{X}}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\bar{\mathcal{Y}}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}},
\end{aligned}$$

where U_k is any element in $U_n^{up}(k) \subset U(n)$. So the metric of $\langle \cdot\,,\cdot\rangle_{V_{n,k}}$ is invariant under the action of $U_n^{up}(k)$.

Now we show that the metric $\langle \cdot\,,\cdot\rangle_{V_{n,k}}$ on $V_{n,k}$ is of constant bi-invariant type with respect to the right group action of $U_n^{up}(k)$, i. e. satisfies Definition 1.2.16. To prove it we recall that the infinitesimal generator $\sigma_{[q]_{V_{n,k}}} : \mathfrak{u}_n^{up}(k) \to T_{[q]_{V_{n,k}}} V_{n,k}$ is given by $\sigma_{[q]_{V_{n,k}}}(\xi) = [q]_{V_{n,k}}\xi$, where $\mathfrak{u}_n^{up}(k)$ is the Lie algebra of $U_n^{up}(k)$. It follows that

$$\mathbb{I}_{[q]_{V_{n,k}}}(\xi,\eta) = \langle [q]_{V_{n,k}}\xi\,, [q]_{V_{n,k}}\eta\rangle_{V_{n,k}} = -n^{-1}\operatorname{tr}(\xi\eta), \qquad \text{where} \quad [q]_{V_{n,k}} \in V_{n,k}.$$

This implies that $\mathbb{I}_{[q]_{V_{n,k}}}(\xi,\eta)$ is independent of $[q]_{V_{n,k}}$.

8.2.3 Grassmann manifold

The Grassmann manifold $G_{n,k}$ is defined as a collection of all k-dimensional subspaces Λ of $\mathbb{C}^n$. Equivalently, an element Λ of $G_{n,k}$ can be written as an $(n \times k)$-matrix with columns $e_1, \ldots, e_k$, such that $\operatorname{span}(e_1, \ldots, e_k) = \Lambda$. We are interested in the representation of $G_{n,k}$ as a quotient of $U(n)$ by some subgroup. As in the previous case of the Stiefel manifold, we quotient $U(n)$ by $U_n^l(n-k)$, but moreover, since the definition of $G_{n,k}$ does not depend on the choice of an orthonormal basis $e_1, \ldots, e_k$ for Λ, but only on its span, we also quotient $U(n)$ by the group $U_n^{up}(k)$ that leaves $\operatorname{span}\{e_1, \ldots, e_k\}$ invariant. Therefore, we define the equivalence relation $\backsim_2$ in $U(n)$ by

$$m_1 \backsim_2 m_2 \quad \Longleftrightarrow \quad m_1 = m_2 \begin{pmatrix} U_k & 0 \\ 0 & U_{n-k} \end{pmatrix}, \qquad m_1, m_2 \in U(n),$$

where $U_k \in U(k)$, $U_{n-k} \in U(n-k)$. This leads to the equivalence class

$$[m]^{\smallfrown_2} = \left\{ m \begin{pmatrix} U_k & 0 \\ 0 & U_{n-k} \end{pmatrix} \;\Big|\; U_k \in U(k), U_{n-k} \in U(n-k) \right\} \subset U(n), \quad m \in U(n),$$

which is isomorphic to $U(k) \times U(n-k) \cong U_n^{up}(k) \times U_n^l(n-k)$. We identify $G_{n,k}$ with the quotient space $U(n)/(U_n^{up}(k) \times U_n^l(n-k))$ and use the notation $[m]_{G_{n,k}}$ for $[m]^{\smallfrown_2}$ in the present Section 8.2.

Furthermore, we obtain that the tangent space to the equivalence class $[m]_{G_{n,k}}$ is

$$\left\{ m \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & \mathcal{X}_4 \end{pmatrix} \;\Big|\; \mathcal{X}_1 \in \mathfrak{u}(k),\ \mathcal{X}_4 \in \mathfrak{u}(n-k) \right\}, \quad m \in U(n),$$

and it implies that the tangent space of $G_{n,k}$ at the point $[m]_{G_{n,k}}$ is given by

$$T_{[m]_{G_{n,k}}} G_{n,k} = \left\{ [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \;\Big|\; \mathcal{X}_2 \in \mathbb{C}^{k \times (n-k)} \right\}.$$

It has real dimension $2k(n-k)$ that defines the real dimension of $G_{n,k}$, see also [44, 67].

We define a metric $\langle \cdot\,, \cdot \rangle_{G_{n,k}}$ on $G_{n,k}$ by

$$\begin{aligned}
&\left\langle [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix}, [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{Y}_2 \\ -\bar{\mathcal{Y}}_2{}^T & 0 \end{pmatrix} \right\rangle_{G_{n,k}} \left([m]_{G_{n,k}}\right) \\
:= &\left\langle m \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix}, m \begin{pmatrix} 0 & \mathcal{Y}_2 \\ -\bar{\mathcal{Y}}_2{}^T & 0 \end{pmatrix} \right\rangle_{U(n)} (m) \\
= &\left(\begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathcal{Y}_2 \\ -\bar{\mathcal{Y}}_2{}^T & 0 \end{pmatrix} \right)_{\mathfrak{u}(n)},
\end{aligned}$$

where $m \in U(n)$ is any representative of $[m]_{G_{n,k}}$.

8.2.4 Submersion $\pi\colon V_{n,k} \to G_{n,k}$ and sub-Riemannian geodesics.

Starting from now, we consider the matrices q and m as elements in $U(n)$ and define the submersion

$$\begin{aligned} \pi\colon V_{n,k} &\to G_{n,k}, \\ [q]_{V_{n,k}} &\mapsto [m]_{G_{n,k}}. \end{aligned}$$

The projection π sends the equivalence class $[q]_{V_{n,k}}$ to the equivalence class $[m]_{G_{n,k}}$, where $m \in U(n)$ can be any matrix from the set

$$\left\{ q \begin{pmatrix} U_k & 0 \\ 0 & U_{n-k} \end{pmatrix} \;\Big|\; U_k \in U(k), U_{n-k} \in U(n-k) \right\}.$$

Note that the latter set consists of all unitary matrices whose first k columns from the left span the same space as the first left k columns of q. This implies that a fibre over a point

$[m]_{G_{n,k}} \in G_{n,k}$ is given by

$$\begin{aligned}\pi^{-1}([m]_{G_{n,k}}) &= \left\{ \left[m \begin{pmatrix} U_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \right]_{V_{n,k}} \Big| \ U_k \in U(k) \right\} \\ &= \left\{ [m]_{V_{n,k}} \begin{pmatrix} U_k & 0 \\ 0 & I_{n-k} \end{pmatrix} \Big| \ U_k \in U(k) \right\}, \quad m \in U(n),\end{aligned}$$

which is homeomorphic to $U_n^{up}(k) \cong U(k)$.

The submersion π is also a principal $U_n^{up}(k)$-bundle, where the right group action is defined by the multiplication from the right by an element from $U_n^{up}(k)$. It remains to show that the right action of $U_n^{up}(k)$ is continuous, preserves the fibre and acts freely and transitively on the fibre.

The multiplication of $[q]_{V_{n,k}} \in V_{n,k}$ from the right by an element $U_k^0 \in U(k)$ is given by

$$q \begin{pmatrix} I_k & 0 \\ 0 & U_{n-k} \end{pmatrix} \begin{pmatrix} U_k^0 & 0 \\ 0 & I_{n-k} \end{pmatrix} = q \begin{pmatrix} U_k^0 & 0 \\ 0 & U_{n-k} \end{pmatrix}, \quad q \in U(n),$$

where U_{n-k} is an arbitrary element of $U(n-k)$ and U_k^0 is a fixed element of $U(k)$. It follows that the right multiplication is well defined and continuous. It can also be seen, that it preserves the fibre $\pi^{-1}(\pi([q]_{V_{n,k}}))$. By definition of the fibre it is clear that $[q]_{V_{n,k}} U_n^{up}(k) = \pi^{-1}(\pi([q]_{V_{n,k}}))$ and this implies the transitivity of the $U_n^{up}(k)$ action.

To show that $U_n^{up}(k)$ acts freely, we assume that $\tilde{U_1} = \begin{pmatrix} U_1 & 0 \\ 0 & I_{n-k} \end{pmatrix} \in U_n^{up}(k)$, $\tilde{U_2} = \begin{pmatrix} U_2 & 0 \\ 0 & I_{n-k} \end{pmatrix} \in U_n^{up}(k)$ and $[q]_{V_{n,k}} \tilde{U_1} = [q]_{V_{n,k}} \tilde{U_2}$ with $[q]_{V_{n,k}} = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$, $q_1 \in \mathbb{C}^{k \times k}$, $q_2 \in \mathbb{C}^{k \times (n-k)}$, $q_3 \in \mathbb{C}^{(n-k) \times k}$ and $q_4 \in \mathbb{C}^{(n-k) \times (n-k)}$. Then we get the equations

$$\begin{aligned} q_1 U_1 = q_1 U_2 \quad &\Longleftrightarrow \quad q_1 = q_1 U_2 U_1^{-1} = q_1 U_1 U_2^{-1}, \\ q_3 U_1 = q_3 U_2 \quad &\Longleftrightarrow \quad q_3 = q_3 U_2 U_1^{-1} = q_3 U_1 U_2^{-1}, \end{aligned}$$

which leads to $U_1 = U_2$ and so $\tilde{U_1} = \tilde{U_2}$. Thus, we have shown that $\pi\colon V_{n,k} \to G_{n,k}$ is a principal $U_n^{up}(k)$-bundle.

The differential of π defines the vertical and horizontal spaces. The differential $d_{[q]_{V_{n,k}}} \pi$ at $[q]_{V_{n,k}}$ acts as

$$[q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2^T & 0 \end{pmatrix} \mapsto [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2^T & 0 \end{pmatrix},$$

where m is defined as above for π. So, the kernel of $d_{[q]_{V_{n,k}}} \pi$ or the vertical space $\mathcal{V}_{[q]_{V_{n,k}}}$ is given by

$$\mathcal{V}_{[q]_{V_{n,k}}} = \left\{ [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & 0 \end{pmatrix} \Big| \ \mathcal{X}_1 \in \mathfrak{u}(k) \right\}, \qquad q \in U(n).$$

We choose the horizontal space of $V_{n,k}$ by setting

$$\mathcal{H}_{[q]_{V_{n,k}}} = \left\{ [q]_{V_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2^T & 0 \end{pmatrix} \Big| \ \mathcal{X}_2 \in \mathbb{C}^{k \times (n-k)} \right\}, \qquad q \in U(n). \tag{8.1}$$

It is clear that $d\pi\colon TV_{n,k} \to TG_{n,k}$ is a linear isometry if we restrict it to the horizontal space, $\mathcal{H}_{[q]_{V_{n,k}}} \to T_{\pi([q]_{V_{n,k}})}G_{n,k}$ for each $[q]_{V_{n,k}} \in V_{n,k}$, therefore π is a Riemannian submersion.

The $\mathfrak{u}_n^{up}(k)$-valued connection one-form $A_{[q]_{V_{n,k}}}\colon T_{[q]_{V_{n,k}}}V_{n,k} \to \mathfrak{u}_n^{up}(k)$ is given by

$$A_{[q]_{V_{n,k}}}\left([q]_{V_{n,k}}\begin{pmatrix}\mathcal{X}_1 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2{}^T & 0\end{pmatrix}\right) := \begin{pmatrix}\mathcal{X}_1 & 0\\ 0 & 0\end{pmatrix} \in \mathfrak{u}_n^{up}(k), \qquad \mathcal{X}_2 \in \mathbb{C}^{k\times(n-k)}.$$

Now we can write precisely the normal sub-Riemannian geodesic on $V_{n,k}$ starting from a point $[q]_{V_{n,k}}$ with initial velocity $v \in T_{[q]_{V_{n,k}}}V_{n,k}$ by using Theorem 1.2.18. It is given by

$$\begin{aligned}\gamma_v(t) &= \exp_R(tv)\exp_{U_n(k)}(-tA(v))\\ &= \pi_1\left(q\exp_{U(n)}\left(t\begin{pmatrix}\mathcal{X}_1 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2{}^T & 0\end{pmatrix}\right)\right)\exp_{U_n(k)}\left(-t\begin{pmatrix}\mathcal{X}_1 & 0\\ 0 & 0\end{pmatrix}\right),\end{aligned} \tag{8.2}$$

where $q \in U(n)$, $v = [q]_{V_{n,k}}\begin{pmatrix}\mathcal{X}_1 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2{}^T & 0\end{pmatrix} \in T_{[q]_{V_{n,k}}}V_{n,k}$ with $\begin{pmatrix}\mathcal{X}_1 & \mathcal{X}_2\\ -\bar{\mathcal{X}}_2{}^T & 0\end{pmatrix} \in \mathfrak{u}(n)$.

We simplify notation and write $q \in V_{n,k}$, $m \in G_{n,k}$, $U(k)$ for $U_n^{up}(k)$, $U(n-k)$ for $U_n^{l}(n-k)$, and g for a Riemannian metric of constant bi-invariant type.

8.2.5 The group $SO(n)$, Stiefel and Grassmann manifolds

We recall that the special orthogonal group $SO(n)$ is the set of matrices

$$SO(n) := \{X \in \mathbb{R}^{n\times n} |\ X^TX = XX^T = I_n\ ,\ \det(X) = 1\}.$$

This is a compact Lie group with the Lie algebra $\mathfrak{so}(n)$ given by

$$\mathfrak{so}(n) := \{\mathcal{X} \in \mathbb{R}^{n\times n} |\ \mathcal{X} = -\mathcal{X}^T\}.$$

Every entry on the diagonal of $\mathcal{X} \in \mathfrak{so}(n)$ is zero and the real dimension of the manifold is $\frac{1}{2}n(n-1)$.

We define a bi-invariant Riemannian metric on $SO(n)$ by

$$\begin{aligned}\langle\cdot\,,\cdot\rangle\colon q\mathfrak{so}(n)\times q\mathfrak{so}(n) &\to \mathbb{R}\\ \langle q\mathcal{X}\,,q\mathcal{Y}\rangle &:= -\operatorname{tr}(\mathcal{X}\mathcal{Y})\end{aligned}$$

with $\mathcal{X},\mathcal{Y} \in \mathfrak{so}(n)$.

The Stiefel manifold $V_{n,k}$ for $k < n$ is the set of all k-tuples $(q_1,\dots,q_k)$ of vectors $q_i \in \mathbb{R}^n$, $i \in \{1,\dots,k\}$, which are orthonormal with respect to the standard Euclidean metric. This compact manifold can be equivalently defined as

$$V_{n,k} := \{X \in \mathbb{R}^{n\times k} |\ \ X^TX = I_k\}.$$

Another way to define the Stiefel manifold $V_{n,k}$ is to introduce the equivalence relation $\backsim_1$ in $SO(n)$ by

$$q \backsim_1 p \quad\Longleftrightarrow\quad q = p\begin{pmatrix}I_k & 0\\ 0 & S_{n-k}\end{pmatrix}, \qquad q,p \in SO(n), \quad S_{n-k} \in SO(n-k),$$

such that the equivalence class $[q]^{\backsim_1}$ of a point $q \in SO(n)$ is given by

$$[q]^{\backsim_1} = \left\{ q \begin{pmatrix} I_k & 0 \\ 0 & S_{n-k} \end{pmatrix} \middle| S_{n-k} \in SO(n-k) \right\} \in SO(n)/SO_n(n-k).$$

The Stiefel manifold $V_{n,k}$ can be identified with $SO(n)/SO_n(n-k)$. We use the notation $[q]_{V_{n,k}}$ for $[q]^{\backsim_1}$ in the present subsection.

The tangent space at a point $[q]_{V_{n,k}} \in V_{n,k}$ is given by

$$T_{[q]_{V_{n,k}}} V_{n,k} = \left\{ [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\mathcal{X}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix} \middle| \; \mathcal{X}_1 \in \mathfrak{so}(k), \mathcal{X}_2 \in \mathbb{R}^{(n-k)\times k} \right\}.$$

The induced metric $\langle \cdot\,,\cdot \rangle_{V_{n,k}}$ on $V_{n,k}$ is given by

$$\begin{aligned} &\left\langle [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & -\mathcal{X}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, [q]_{V_{n,k}} \begin{pmatrix} \mathcal{Y}_1 & -\mathcal{Y}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{V_{n,k}} \left([q]_{V_{n,k}}\right) \\ &:= \left\langle q \begin{pmatrix} \mathcal{X}_1 & -\mathcal{X}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix}, q \begin{pmatrix} \mathcal{Y}_1 & -\mathcal{Y}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right\rangle_{SO(n)} (q) \\ &= -\operatorname{tr}\left(\begin{pmatrix} \mathcal{X}_1 & -\mathcal{X}_2^T \\ \mathcal{X}_2 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{Y}_1 & -\mathcal{Y}_2^T \\ \mathcal{Y}_2 & 0 \end{pmatrix} \right), \end{aligned}$$

where $q \in [q]_{V_{n,k}}$ is any representative of the equivalence class $[q]_{V_{n,k}}$.

The Grassmann manifold $G_{n,k}$ is defined as a collection of all k-dimensional subspaces Λ of $\mathbb{R}^n$. Equivalently, an element Λ of $G_{n,k}$ can be written as an $(n \times k)$-matrix with columns $w_1, \dots, w_k \in \mathbb{R}^n$, such that $\operatorname{span}\{w_1, \dots, w_k\} = \Lambda$, or, it can be defined as a quotient space in $SO(n)$ with respect to the following equivalence relation

$$m_1 \backsim_2 m_2 \quad \Longleftrightarrow \quad m_1 = m_2 \begin{pmatrix} S_k & 0 \\ 0 & S_{n-k} \end{pmatrix}, \qquad m_1, m_2 \in SO(n),$$

where $S_k \in O(k)$, $S_{n-k} \in O(n-k)$, such that $\det(S_k) = \det(S_{n-k}) \in \{-1, 1\}$. This leads to the equivalence classes

$$[m]^{\backsim_2} = \left\{ m \begin{pmatrix} S_k & 0 \\ 0 & S_{n-k} \end{pmatrix} \middle| \; S_k \in O(k), S_{n-k} \in O(n-k)\ , \det(S_k) = \det(S_{n-k}) \right\},$$

$m \in SO(n)$, which is isomorphic to $O(k) \times SO(n-k) \cong O_n(k) \times SO_n(n-k)$, as

$$m \begin{pmatrix} S_k & 0 \\ 0 & S_{n-k} \end{pmatrix} \mapsto \left(S_k, S_{n-k} \begin{pmatrix} \det(S_k) & 0 \\ 0 & \mathrm{Id}_{n-1} \end{pmatrix} \right) \in O(k) \times SO(n-k).$$

We identify $G_{n,k}$ with the quotient space $SO(n)/(O_n(k) \times SO_n(n-k))$ and use the notation $[m]_{G_{n,k}}$ for $[m]^{\backsim_2}$ in the current subsection and again in Section 8.5.

The tangent space of $G_{n,k}$ at the point $[m]_{G_{n,k}}$ is given by

$$T_{[m]_{G_{n,k}}} G_{n,k} = \left\{ [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\mathcal{X}_2^T & 0 \end{pmatrix} \middle| \; \mathcal{X}_2 \in \mathbb{R}^{k \times (n-k)} \right\}.$$

It has real dimension $k(n-k)$ that gives the real dimension of $G_{n,k}$.

The induced metric $\langle \cdot\,,\cdot\rangle_{G_{n,k}}$ on $G_{n,k}$ is given by

$$\begin{aligned}
&\left\langle [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\mathcal{X}_2^T & 0 \end{pmatrix}, [m]_{G_{n,k}} \begin{pmatrix} 0 & \mathcal{Y}_2 \\ -\mathcal{Y}_2^T & 0 \end{pmatrix} \right\rangle_{G_{n,k}} \left([m]_{G_{n,k}}\right) \\
&:= \left\langle m \begin{pmatrix} 0 & \mathcal{X}_2 \\ -\mathcal{X}_2^T & 0 \end{pmatrix}, m \begin{pmatrix} 0 & \mathcal{Y}_2 \\ -\mathcal{Y}_2^T & 0 \end{pmatrix} \right\rangle_{SO(n)} (m) \\
&= -\operatorname{tr}\left(\begin{pmatrix} 0 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathcal{Y}_2 \\ -\bar{\mathcal{Y}}_2{}^T & 0 \end{pmatrix} \right),
\end{aligned}$$

where $m \in SO(n)$ is any representative of $[m]_{G_{n,k}}$.

A normal sub-Riemannian geodesic on $V_{n,k}$ starting from $[q]_{V_{n,k}}$ is given by the formula similar to (8.2) presented in Subsection 8.2.4.

$$\begin{aligned}
\gamma(t) &= \exp_{V_{n,k}}(tv)\exp_{O_n(k)}(-tA(v)) \\
&= \pi_1\left[q \exp_{SO(n)}\left(t \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \right) \right] \exp_{O_n(k)}\left(-t \begin{pmatrix} \mathcal{X}_1 & 0 \\ 0 & 0 \end{pmatrix} \right),
\end{aligned} \tag{8.3}$$

where $q \in SO(n)$, $v = [q]_{V_{n,k}} \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \in T_{[q]_{V_{n,k}}} V_{n,k}$ with $\begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ -\bar{\mathcal{X}}_2{}^T & 0 \end{pmatrix} \in \mathfrak{so}(n)$, $\pi_1\colon SO(n) \to SO(n)/SO_n(n-k)$ is the natural projection from $SO(n)$ to the quotient space, and $A\colon TV_{n,k} \to \mathfrak{so}_n(k)$ is the $\mathfrak{so}_n(k)$-valued connection one form.

8.3 The cut-locus of $V_{n,1}$

In this section we study the cut locus of the complex Stiefel manifold $V_{n,1}$ considered as a sub-Riemannian manifold by making use of the normal sub-Riemannian geodesics (8.2). We recall that the definition of the cut locus is given by 1.2.8.

As a motivation for studying this problem we mention that $V_{n,1}$ is also an example of a contact manifold, which was studied, for instance, in [15, 31, 50, 71]. To present the contact structure, we note that the submersion $U(1) \to V_{n,1} \to G_{n,1}$ can be written as $U(1) \to S^{2n-1} \to \mathbb{C}P^{n-1}$. In [50], it is shown that for submersion $S^{2n-1} \to \mathbb{C}P^{n-1}$ the vertical vector space is spanned by

$$V(q) = -y_0\partial_{x_0} + x_0\partial_{y_0} - \ldots - y_{n-1}\partial_{x_{n-1}} + x_{n-1}\partial_{y_{n-1}}, \quad q \in S^{2n-1}.$$

The horizontal distribution D on S^{2n-1} is defined as the orthogonal complement to $\operatorname{span}\{V\}$ in TS^{2n-1} with respect to the Euclidean metric in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. At the point $(1, 0, \ldots, 0) \in S^{2n-1}$ the vertical vector $V = (i, 0, \ldots, 0)$ coincides with the generator $\xi = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$ of the Lie algebra $\mathfrak{u}_n(1)$ and the horizontal distribution $D = V^\perp$ coincides with the horizontal distribution $\mathcal{H} = \left\{ \begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix} \;\middle|\; B \in \mathbb{C}^{1\times(n-1)} \right\}$, which is orthogonal to ξ with respect to the trace metric. Since trace metric and Euclidean metric, vertical and horizontal distributions are invariant under the action of $U(n)$ we conclude that the sub-Riemannian geometries are essentially the same. It is shown in [50], that the distribution D coincides with the holomorphic tangent space HS^{2n-1} of the sphere S^{2n-1} considered as an embedded CR-manifold and that it also coincides with the contact distribution given by $\ker(\omega)$ with respect to the contact form

$$\omega = -y_0 dx_0 + x_0 dy_0 - \ldots - y_{n-1} dx_{n-1} + x_{n-1} dy_{n-1}.$$

Thus the contact and CR structures can be transferred to the Stiefel manifold $V_{n,1}$.

We note that we write Id for the equivalence class $[I_n]_{V_{n,k}} \in V_{n,k}$. The main theorem is stated as follows.

Theorem 8.3.1. *The cut locus K_{Id} on $V_{n,1}$ is given by*

$$L_{n,1} := \left\{ \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,1}} \;\middle|\; C \in U(1),\ D \in U(n-1) \right\} \setminus \{\mathrm{Id}\}.$$

Proof. We only need to show the inclusion $K_{\mathrm{Id}} \subset L_{n,1}$ since the converse inclusion $L_{n,1} \subset K_{\mathrm{Id}}$ will be proved in Theorem 8.4.4 for the more general case $V_{n,k}$.

First of all we claim that in the case of $V_{n,1}$ there are no abnormal minimizing geodesics because the distribution is strongly bracket generating. Remind that a smooth distribution $\mathcal{H}$ on a manifold is strongly bracket generating if for any non-zero section $\mathcal{Z}$ of $\mathcal{H}$, the tangent bundle of the manifold is generated by $\mathcal{H}$ and $[\mathcal{Z}\,, \mathcal{H}]$. We actually mentioned at the beginning of the section that $V_{n,1}$ can be considered as a contact manifold and therefore it is strongly bracket generating, see for instance [71].

Thus all the possible minimizers are normal and they are given by Theorem 1.2.18. We calculate the precise form of the geodesic γ_v, paying special attention to the components γ_v^1 and γ_v^3, where v is the initial vector of the Riemannian geodesic in formula (1.9), see also Remark 1.2.19. The forthcoming calculations are well defined since the sub-Riemannian Stiefel manifold is analytic. Let $v = \begin{pmatrix} ix & B \\ -\bar{B}^T & 0 \end{pmatrix}$, where $x \in \mathbb{R}$ and $B \in \mathbb{C}^{1\times(n-1)}$. Recall that $\exp(tv) = \sum_{n=0}^{\infty} \frac{t^n}{n!} v^n$. First we will calculate the two upper parts of the n-th power of v, $v^n := v(n) = \begin{pmatrix} v_1(n) & v_2(n) \\ v_3(n) & v_4(n) \end{pmatrix}$, namely $v_1(n)$ and $v_2(n)$. From the recursion formula $v^n = v^{n-1} v$ it follows that

$$v_1(n) = v_1(n-1)ix - v_2(n-1)\bar{B}^T = v_1(n-1)ix - v_1(n-2)B\bar{B}^T,$$

as $v_2(n) = v_1(n-1)B$. Furthermore, as $v^n = vv^{n-1}$ we deduce $v_3(n) = -\bar{B}^T v_1(n-1)$. Having the initial values $v_1(0) = 1$, $v_1(1) = ix$, and $v_3(0) = 0$ we obtain that

$$\begin{aligned} v_1(n) &= \frac{2^{-n-1}}{i\sqrt{x^2 + 4B\bar{B}^T}} \Big(ix\big((i\sqrt{x^2 + 4B\bar{B}^T} + ix)^n - (ix - i\sqrt{x^2 + 4B\bar{B}^T})^n\big) \\ &+ i\sqrt{x^2 + 4B\bar{B}^T}\big((ix - i\sqrt{x^2 + 4B\bar{B}^T})^n + (i\sqrt{x^2 + 4B\bar{B}^T} + ix)^n\big)\Big), \end{aligned}$$

which implies for $\exp(tv) := \begin{pmatrix} \exp(tv)_1 & \exp(tv)_2 \\ \exp(tv)_3 & \exp(tv)_4 \end{pmatrix}$ that

$$\begin{aligned} \exp(tv)_1 &= \sum_{n=0}^{\infty} \frac{t^n}{n!} v_1(n) = \frac{1}{2i\sqrt{x^2+4B\bar{B}^T}} \left(e^{-\frac{it}{2}(\sqrt{x^2+4B\bar{B}^T}-x} \right) \\ &\times \left(i\sqrt{x^2+4B\bar{B}^T}\big(e^{it\sqrt{x^2+4B\bar{B}^T}}+1\big) + ix\big(e^{it\sqrt{x^2+4B\bar{B}^T}}-1\big)\right) \\ &= \frac{1}{2\sqrt{x^2+4B\bar{B}^T}} \left(e^{-\frac{it}{2}(\sqrt{x^2+4B\bar{B}^T}-x} \right) \\ &\times \left(\sqrt{x^2+4B\bar{B}^T}\big(e^{it\sqrt{x^2+4B\bar{B}^T}}+1\big) + x\big(e^{it\sqrt{x^2+4B\bar{B}^T}}-1\big)\right). \end{aligned}$$

The first component $\gamma_v^1(t)$ of the normal geodesic $\gamma_v(t) = \begin{pmatrix} \gamma_v^1(t) & \gamma_v^2(t) \\ \gamma_v^3(t) & \gamma_v^4(t) \end{pmatrix}$ is written as

$$\begin{aligned} \gamma_v^1(t) &= \exp_{U(n)}(tv)_1 \exp_{U(1)}(-tix) = \frac{1}{2\sqrt{x^2+4B\bar{B}^T}} e^{-\frac{it}{2}(\sqrt{x^2+4B\bar{B}^T}+x)} \\ &\times \left(\sqrt{x^2+4B\bar{B}^T}(e^{it\sqrt{x^2+4B\bar{B}^T}}+1\big) + x(e^{it\sqrt{x^2+4B\bar{B}^T}}-1\big) \right). \end{aligned} \tag{8.4}$$

The second important component of the geodesic γ_v is

$$\begin{aligned} \exp(tv)_3 &= \sum_{n=0}^{\infty} \frac{t^n}{n!} v_3(n) = \sum_{n=1}^{\infty} \frac{t^n}{n!} v_3(n) \\ &= \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!} v_3(n+1) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!}\big(-\bar{B}^T v^1(n)\big) \\ &= -\bar{B}^T \frac{1}{i\sqrt{x^2+4B\bar{B}^T}} e^{-\frac{ti}{2}(\sqrt{x^2+4B\bar{B}^T}-x)} \left(e^{ti\sqrt{x^2+4B\bar{B}^T}} - 1 \right), \end{aligned}$$

$$\begin{aligned} \gamma_v^3(t) &= \exp_{U(n)}(tv)_3 \exp_{U(1)}(-tix) \\ &= -\bar{B}^T \frac{1}{i\sqrt{x^2+4B\bar{B}^T}} e^{-\frac{ti}{2}(\sqrt{x^2+4B\bar{B}^T}+x)} \left(e^{ti\sqrt{x^2+4B\bar{B}^T}} - 1 \right). \end{aligned} \tag{8.5}$$

It follows that $\gamma_v^3(t) = 0$ first at the time $t_0 = \frac{2\pi}{\sqrt{x^2+4B\bar{B}^T}}$. That implies that the geodesic $\gamma_v(t)$ reaches the set $L_{n,1}$ first at the time t_0. Since $L_{n,1} \subset K_{\mathrm{Id}}$, $\gamma_v(t)$ reaches the cut locus at the time t_0, it follows that the geodesic $\gamma_v(t)$ loses its optimality at the latest t_0.

Having exact formulas for the coordinates of the geodesics we proceed to the core of the proof. Suppose $q \in V_{n,1} \setminus L_{n,1}$ but $q \in K_{\mathrm{Id}}$, and there exist two different minimizing normal geodesics γ_{v_1} and γ_{v_2} with $\gamma_{v_1}(0) = \gamma_{v_2}(0) = \mathrm{Id}$, $\gamma_{v_1}(T^*) = \gamma_{v_2}(T^*) = q$ and $v_1 = \begin{pmatrix} ix_1 & B \\ -\bar{B}^T & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} ix_2 & E \\ -\bar{E}^T & 0 \end{pmatrix}$ and $x_j \in \mathbb{R}$, $j = 1, 2$ and $B, E \in \mathbb{C}^{1\times(n-1)}$.

Claim. Under the above assumptions, we claim that $B\bar{B}^T = E\bar{E}^T$. Since both geodesics are minimizing, they have equal length at time T^*. Then Proposition 8.4.3 implies

$$T^*\sqrt{2n^{-1}\,\mathrm{tr}(B\bar{B}^T)} = l(\gamma_{v_1}, T^*) = l(\gamma_{v_2}, T^*) = T^*\sqrt{2n^{-1}\,\mathrm{tr}(E\bar{E}^T)}.$$

It proves the claim since B and E are complex vectors and $B\bar{B}^T = \operatorname{tr}(B\bar{B}^T) = \operatorname{tr}(E\bar{E}^T) = E\bar{E}^T$.

The consideration of the following two cases will finish the proof.

Case 1. Suppose $x_1 = x_2$ and $B\bar{B}^T = E\bar{E}^T$. Since $q \notin L_{n,1}$, we know that $\gamma^3_{v_1}(T^*) = \gamma^3_{v_2}(T^*) \neq 0$. Then $\gamma^3_{v_1}(T^*) = \gamma^3_{v_2}(T^*) \quad \Longleftrightarrow$

$$\begin{aligned} & - \quad \bar{B}^T \frac{1}{i\sqrt{x_1^2 + 4B\bar{B}^T}} e^{-\frac{iT^*}{2}(\sqrt{x_1^2+4B\bar{B}^T}+x_1)}(e^{iT^*\sqrt{x_1^2+4B\bar{B}^T}} - 1) \\ = & - \quad \bar{E}^T \frac{1}{i\sqrt{x_2^2 + 4E\bar{E}^T}} e^{-\frac{iT^*}{2}(\sqrt{x_2^2+4E\bar{E}^T}+x_2)}(e^{iT^*\sqrt{x_2^2+4E\bar{E}^T}} - 1). \end{aligned}$$

Hence $\bar{B}^T = \bar{E}^T$ and so $B = E$, which leads to the equality $v_1 = v_2$. Thus $\gamma_{v_1}(t) = \gamma_{v_2}(t)$ for all t according to formulas (8.4) and (8.5) of geodesics. This contradicts to the assumption that the geodesics are different.

Case 2. Let now $x_1 \neq x_2$ and $B\bar{B}^T = E\bar{E}^T$. Since $q \notin L_{n,1}$, we know that $\gamma^3_{v_1}(T^*) = \gamma^3_{v_2}(T^*) \neq 0$. The assumption $q \in K_{\mathrm{Id}}$ implies $\gamma^3_{v_1}(T^*) = \gamma^3_{v_2}(T^*)$, which yields $\|\gamma^3_{v_1}(T^*)\| = \|\gamma^3_{v_2}(T^*)\| \neq 0$. Thus

$$\frac{\|B\|}{\sqrt{x_1^2 + 4B\bar{B}^T}} \left| e^{T^*\sqrt{x_1^2+4B\bar{B}^T}} - 1 \right| = \frac{\|E\|}{\sqrt{x_2^2 + 4E\bar{E}^T}} \left| e^{T^*\sqrt{x_2^2+4E\bar{E}^T}} - 1 \right|,$$

and

$$\frac{\sin(\frac{T^*}{2}\sqrt{x_1^2 + 4B\bar{B}^T})}{\frac{T^*}{2}\sqrt{x_1^2 + 4B\bar{B}^T}} = \frac{\sin(\frac{T^*}{2}\sqrt{x_2^2 + 4B\bar{B}^T})}{\frac{T^*}{2}\sqrt{x_2^2 + 4B\bar{B}^T}}. \tag{8.6}$$

Note that $0 < T^* \leq \min\left\{\frac{2\pi}{\sqrt{x_1^2+4B\bar{B}^T}}, \frac{2\pi}{\sqrt{x_2^2+4B\bar{B}^T}}\right\}$ by assumption $q \in K_{\mathrm{Id}}$ and therefore $\sin(\frac{T^*}{2}\sqrt{x_j^2 + 4B\bar{B}^T}) > 0$ for $j = 1, 2$. Since the function $\frac{\sin x}{x}$ is injective on the interval $(0, \pi]$, we obtain $x_1 = x_2$ or $x_1 = -x_2$. In the first case we already get a contradiction. In the case of the assumption $x_1 = -x_2$ we turn our attention to the first component of the geodesics. Then the equality

$$\gamma^1_{v_1}(T^*) = \gamma^1_{v_2}(T^*), \tag{8.7}$$

implies

$$\frac{\tan(\frac{T^*}{2}\sqrt{x_1^2 + 4B\bar{B}^T})}{\sqrt{x_1^2 + 4B\bar{B}^T}} = \frac{\tan(\frac{T^*x_1}{2})}{x_1}. \tag{8.8}$$

Since $0 < \frac{T^*x_1}{2} < \frac{T^*}{2}\sqrt{x_1^2 + 4B\bar{B}^T} < \pi$ equality (8.8) is not true, which is equivalent to say that equality (8.7) is not true.

Figure 8.1 illustrates that $\lambda_1 < \lambda_2$ implies $\frac{\tan \lambda_1}{\lambda_1} \neq \frac{\tan \lambda_2}{\lambda_2}$. Similar arguments can be found in [24, p. 1871]. $\square$

8.4 The cut locus of $V_{n,k}$

In the present section we show that some of the properties of the cut locus of $V_{n,1}$ are preserved in the case $V_{n,k}$. In general, we are not able to describe the total cut locus, since the

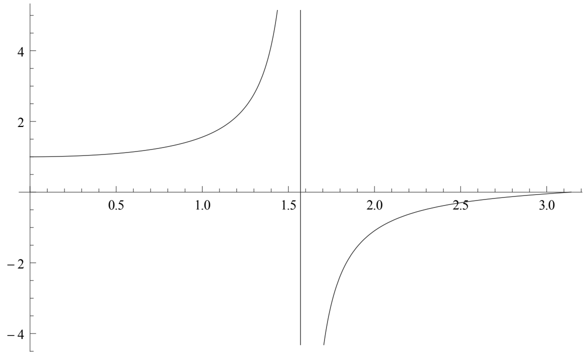

Figure 8.1: $\frac{\tan(x)}{x}$ on the interval $[0\,,\pi]$.

exact formulas of the geodesics are very complicated. Additionally, we have the problem that the distribution is, in general, not strongly bracket generating, which follows from Proposition 1.2.22. The conditions of Proposition 1.2.22 are obviously not always fulfilled for $V_{n,k}$, where $m = 2nk - k^2$ and $l = 2nk - 2k^2$ and therefore the distribution on an arbitrary $V_{n,k}$ is not necessarily strongly bracket generating. But it is always bracket generating of step 2, as stated in the following proposition.

8.4.1 Partial description of the cut locus of $V_{n,k}$

Proposition 8.4.1. *The distribution $\mathcal{H}$ on $V_{n,k}$ is bracket generating of step 2.*

Proof. First we note that the commutator $[\mathcal{H}\,,\mathcal{H}]$ is given by

$$\left[\begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}, \begin{pmatrix} 0 & C \\ -\bar{C}^T & 0 \end{pmatrix}\right] \;=\; \begin{pmatrix} -B\bar{C}^T + C\bar{B}^T & 0 \\ 0 & -\bar{C}^T B + \bar{B}^T C \end{pmatrix}.$$

It is sufficient to show that for every upper triangular $(k \times k)$-matrix D_{lm}, $m > l$ with an entry $d_{lm} \neq 0$ on the intersection of l-th row and m-th column and all other entries vanish we can find $B, C \in \mathbb{C}^{k\times(n-k)}$ such that $D_{lm} = -B\bar{C}^T$. For instance, if we choose

$$B = \big(b_{\alpha\beta}\big) \quad \text{by} \quad b_{\alpha\beta} = \begin{cases} d_{lm} & \text{for } \alpha = l, \beta = \min\{m, n-k\}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and}$$

$$-C^T = (c_{\alpha\beta}) \quad \text{by} \quad c_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \min\{m, n-k\}, \beta = m, \\ 0 & \text{otherwise,} \end{cases}$$

then we deduce that $D_{lm} = -B\bar{C}^T$.

We also need to construct diagonal $(k \times k)$-matrices D_j with $i \in \mathbb{C}$ on the intersection of j-th row and j-th column and all other entries vanish and show that there are $B, C \in \mathbb{C}^{k\times(n-k)}$ such that $D_j = -B\bar{C}^T$. In this case we choose

$$B = (b_{\alpha\beta}) \quad \text{by} \quad b_{\alpha\beta} = \begin{cases} i & \text{for } \alpha = j, \beta = \min\{j, n-k\}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and}$$

$$-\bar{C}^T = (c_{\alpha\beta}) \quad \text{by} \quad c_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \min\{j, n-k\}, \beta = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain that $D_j = -B\bar{C}^T$. It implies that $\mathcal{H}$ is bracket generating of step 2. □

Before we proceed further we note recent results about the existence of normal and abnormal geodesics on sub-Riemannian manifolds with a bracket generating distribution of step 2.

Proposition 8.4.2. *[69, Theorem 4] On a sub-Riemannian manifold $(Q, \mathcal{H}, g_{\mathcal{H}})$ with bracket generating distribution $\mathcal{H}$ of step 2, any length minimizing curve is C^∞-smooth, or in other words there are no strictly abnormal minimizing geodesics in this case.*

Thus, if a minimizing geodesic is abnormal on the sub-Riemannian Stiefel manifold, then its projection to the manifold coincides with the projection of some normal geodesic by Proposition 8.4.2, and we can use the precise formula (1.9) for all minimizing geodesics.

Proposition 8.4.3. *Suppose $\gamma_v(t)$ is a sub-Riemannian geodesic, which connects the identity* Id *with a point $q \in V_{n,k}$, $q \neq \mathrm{Id}$, at the time $T > 0$, and $v = \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}$. The length of γ_v is given by $l(\gamma_v, T) = T\sqrt{\frac{2}{n}\operatorname{tr}(B\bar{B}^T)}$.*

Proof. First we calculate the velocity vector $\dot{\gamma}_v(t)$ at $\gamma_v(t)$. The velocity vector will have the form $\dot{\gamma}_v(t) = \gamma_v(t) w_{\mathcal{H}}(t)$, where $w_{\mathcal{H}}(t) \in \mathfrak{u}(n)$ for each t and $w_{\mathcal{H}}(t)$ has to be of the form $\begin{pmatrix} 0 & \mathcal{X}(t) \\ -\bar{\mathcal{X}(t)}^T & 0 \end{pmatrix}$. We omit the subscript $U(n)$ or $U(k)$ from $\exp_{(\cdot)}$, since it is clear which one we use from the context. By the chain rule we get that

$$\begin{aligned} \dot{\gamma}_v(t) &= d_{p(t)}\pi_1\Bigg[\left(\exp\left\{t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right\}\right)\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\left(\exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) \\ &+ \left(\exp\left\{t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right\}\right)\left(\exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\right)\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\Bigg], \end{aligned}$$

where $p(t) := \exp\left(t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right)\exp\left(t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right)$. We note that

$$\begin{aligned} &\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} A\exp(-tA) & B \\ -\bar{B}^T\exp(-tA) & 0 \end{pmatrix} \\ &= \exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} \exp(tA)A\exp(-tA) & \exp(tA)B \\ -\bar{B}^T\exp(-tA) & 0 \end{pmatrix} \\ &= \exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\begin{pmatrix} A & \exp(tA)B \\ -\bar{B}^T\exp(-tA) & 0 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \dot{\gamma}_v(t) &= d_{p(t)}\pi_1\Bigg[\exp\left\{t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right\}\exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\} \\ &\times \left(\begin{pmatrix} A & \exp(tA)B \\ -\bar{B}^T\exp(-tA) & 0 \end{pmatrix} + \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right)\Bigg] \\ &= \gamma_v(t)\begin{pmatrix} 0 & \exp(tA)B \\ -\bar{B}^T\exp(-tA) & 0 \end{pmatrix} \end{aligned}$$

and

$$w_{\mathcal{H}} = \begin{pmatrix} 0 & \exp(tA)B \\ -\bar{B}^T \exp(-tA) & 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned} g(\dot{\gamma}_v(t), \dot{\gamma}_v(t)) &= -n^{-1} \operatorname{tr}(w_{\mathcal{H}}^2) = -n^{-1} \operatorname{tr}\left(\begin{pmatrix} -\exp(tA)B\bar{B}^T \exp(-tA) & 0 \\ 0 & -\bar{B}^T B \end{pmatrix}\right) \\ &= -n^{-1}\Big(-\operatorname{tr}\left(\exp(tA)B\bar{B}^T \exp(-tA)\right) - \operatorname{tr}(\bar{B}^T B)\Big) = 2n^{-1} \operatorname{tr}(B\bar{B}^T). \end{aligned}$$

In the last equation we used $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ and $\operatorname{tr}(-X) = -\operatorname{tr}(X)$.

We conclude that the length of γ_v does not depend on A, but depends on the final time T and the trace of the matrix $B\bar{B}^T$. □

Theorem 8.4.4. *The set*

$$L_{n,k} = \left\{ \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,k}} \;\Big|\; C \in U(k),\ D \in U(n-k)\} \setminus \mathrm{Id} \right\}$$

belongs to the cut locus K_{Id} on $V_{n,k}$.

Proof. Suppose the point $[g]_{V_{n,k}} = \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,k}} \in L_{n,k}$. Then there exists a minimizing geodesic γ_v of the form (1.9) with $v = \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix} \in \mathfrak{u}(n)$ connecting Id with $[g]_{V_{n,k}} = \gamma_v(T)$ at some time T by Propositions 1.2.7 and 8.4.2 . We write

$$\gamma_v(t) \;=\; \pi_1\left(\exp\left\{t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right\}\exp\left\{t\begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) = \left[\begin{pmatrix} \gamma_v^1(t) & \gamma_v^2(t) \\ \gamma_v^3(t) & \gamma_v^4(t) \end{pmatrix}\right]_{V_{n,k}}$$

and see how γ_v^j, $j = 1, 2, 3, 4$, depend on A and B. We calculate $\exp\left(t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right) = \begin{pmatrix} v_1(t) & v_2(t) \\ v_3(t) & v_4(t) \end{pmatrix}$. Using the notation $\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}^n := \begin{pmatrix} v_1(n) & v_2(n) \\ v_3(n) & v_4(n) \end{pmatrix}$, we receive that $v_1(n) = v_1(n-1)A - v_1(n-2)B\bar{B}^T$, $n \geq 2$, for initial values $v_1(0) = \mathrm{Id}$ and $v_1(1) = A$. This implies that v_1 as a function of t depends on A and $B\bar{B}^T$. Furthermore, we obtain the formulas $v_2(n) = v_1(n-1)B$, $v_3(n) = -\bar{B}^T v_1(n-1)$ and $v_4(n) = -\bar{B}^T v_1(n-2)B$.

Now we claim that the geodesic γ_{v^*} with $v^* := \begin{pmatrix} A & -B \\ \bar{B}^T & 0 \end{pmatrix}$ is also minimizing from Id to $[g]_{V_{n,k}}$ with $\gamma_{v^*}(T) = [g]_{V_{n,k}}$. Indeed, since $(-B)(-\bar{B}^T) = B\bar{B}^T$ and $(-\bar{B}^T)(-B) = \bar{B}^T B$ the length of both geodesics coincides. It remains to show that $\gamma_{v^*}(T) = [g]_{V_{n,k}}$. Observe, that the value $v_1(t)$ depends on A, $B\bar{B}^T$ and t, and therefore $\gamma_{v^*}^1(T) = \gamma_v^1(T)$. Finally $\gamma_v^2(T) = \gamma_v^3(T) = 0$ implies $\gamma_{v^*}^2(T) = -\gamma_v^2(T) = 0 = \gamma_v^2(T)$ and $\gamma_{v^*}^3(T) = -\gamma_v^3(T) = 0 = \gamma_v^3(T)$. We conclude that $\gamma_{v^*}(T) = \gamma_v(T)$. Furthermore, it follows from $\gamma_{v^*}^3(t) = -\gamma_v^3(t) \neq 0$ for $t \in (0, T)$, that $\gamma_{v^*}(t) \neq \gamma_v(t)$ for $t \in (0, t)$, i.e. $\gamma_{v^*} \neq \gamma_v$. We conclude that $L_{n,k} \subset K_{\mathrm{Id}}$. □

Corollary 8.4.5. *There are infinitely many minimizing geodesics connecting* Id *with any point $q \in L_{n,k}$.*

Proof. The geodesic γ_{v^*} in the proof of Theorem 8.4.4 can be replaced by $\gamma_{\hat{v}}$ with

$$\hat{v} = \begin{pmatrix} A & -BU \\ (\overline{BU})^T & 0 \end{pmatrix}$$

for all $U \in U(n-k)$. This is also a minimizing geodesic from Id to $[g]_{V_{n,k}}$, with $\gamma_{\hat{v}}(T) = [g]_{V_{n,k}}$, as the length just depends on the final time T and $B\bar{B}^T$. □

8.4.2 Uniqueness results for minimizing geodesics on $V_{2k,k}$

Since the description of the cut locus for general Stiefel manifolds is very complicated we focus on the Stiefel manifolds $V_{n,k}$ with $n = 2k$ and present some additional information in this case. The main result of this section is stated in Theorem 8.4.8.

Lemma 8.4.6. *The points* $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}} \in G_{2k,k}$ *are reached by Riemannian geodesics starting from* $[I_n]_{G_{2k,k}}$ *only if the initial velocity vector* v *has the form* $v = \begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}$, $B \in U(k)$. *If we assume that* $\operatorname{tr}(B\bar{B}^T) = 1$, *then the condition* $B \in U(k)$ *is changed to* $\sqrt{k}B \in U(k)$.

Proof. Geodesics of the Grassmann manifold $G_{2k,k}$ are given by

$$\gamma_v(t) = \left[\exp\left(t\begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}\right)\right]_{G_{2k,k}} = \begin{pmatrix} \gamma_v^1(t) & \gamma_v^2(t) \\ \gamma_v^3(t) & \gamma_v^4(t) \end{pmatrix}, \tag{8.9}$$

where

$$\gamma_v^1(t) = \cos(t\sqrt{B\bar{B}^T}), \qquad \gamma_v^3(t) = -\bar{B}^T\sin(t\sqrt{B\bar{B}^T})(\sqrt{B\bar{B}^T})^{-1}.$$

We are looking for all geodesics for which there exists $T_0 > 0$, such that $\gamma_v^1(T_0) = 0$ and $\gamma_v^3(T_0) = C$. As $C \in U(k)$ and particularly is invertible it follows from the form of $\gamma_v^3(T_0)$ that B is invertible. Therefore, the matrix $B\bar{B}^T$ is positive definite and diagonalizable: $B\bar{B}^T = PDP^{-1}$, where $D = \operatorname{diag}(d_1, \dots, d_k)$ is a diagonal matrix with $d_i > 0$ for $i \in \{1, \dots, k\}$. This implies that

$$\cos(t\sqrt{B\bar{B}^T}) = P\cos(t\sqrt{D})P^{-1}$$

and so $\gamma_v^1(T_0) = \cos(T_0\sqrt{B\bar{B}^T}) = 0$ if and only if $\cos(T_0\sqrt{d_1}) = \dots = \cos(T_0\sqrt{d_k}) = 0$.

If $B \in U(k)$, then, using the normalization $\operatorname{tr}(B\bar{B}^T) = 1$, we get $\sqrt{k}B \in U(k)$. Thus $B\bar{B}^T = \frac{1}{k}\operatorname{Id}_k = \operatorname{diag}(\frac{1}{k}, \dots, \frac{1}{k})$, and $T_0 := \min\{t > 0 | \cos(t\sqrt{B\bar{B}^T}) = 0\} = \frac{\pi\sqrt{k}}{2}$.

Now we claim that no other minimizing geodesics exist except for those with initial velocity defined by matrices from $U(k)$. Let B be an arbitrary invertible matrix, not necessarily from $U(k)$. If we again assume the normalization $\operatorname{tr}(B\bar{B}^T) = 1$, then we obtain that there exist at least two eigenvalues $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ of $B\bar{B}^T$ with $0 < \frac{1}{\lambda_1} < \frac{1}{k} < \frac{1}{\lambda_2}$. It follows that if $\cos(T_0\sqrt{B\bar{B}^T}) = 0$, then $\cos(\frac{T_0}{\sqrt{\lambda_1}}) = 0$. We conclude that $T_0 \geq \frac{\pi\sqrt{\lambda_1}}{2} > \frac{\pi\sqrt{k}}{2}$. Thus the geodesic with initial velocity defined by the matrix B and reaching the point $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ at time T_0 is not minimizing. □

Corollary 8.4.7. *Let* $p = \left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}} \in V_{2k,k}$ *with* $C, D \in U(k)$ *and* $v = \begin{pmatrix} 0 & B \\ -\bar{B}^T & 0 \end{pmatrix}$ *with* $\sqrt{k}B \in U(k)$, $\operatorname{tr}(B\bar{B}^T) = 1$. *Then sub-Riemannian geodesics* $\gamma_v(t)$ *in* $V_{2k,k}$ *reaching the points* p *at time* $T_0 = \frac{\pi\sqrt{k}}{2}$ *are minimizing. Furthermore, if* $B_1 \neq B_2$*, then* $\gamma^3_{v_1}(T_0) \neq \gamma^3_{v_2}(T_0)$.

Proof. First we note that geodesics in $G_{2k,k}$ defined by v satisfying the assumption of Lemma 8.4.6 are minimizing geodesics from $[\mathrm{Id}]_{G_{2k,k}}$ to $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ by Lemma 8.4.6. The time of reaching the points $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ is $T_0 = \frac{\pi\sqrt{k}}{2}$. Furthermore,

$$\gamma^3_v(T_0) = -\bar{B}^T \operatorname{diag}\left(\sin(\frac{T_0}{\sqrt{k}}), \dots, \sin(\frac{T_0}{\sqrt{k}})\right)\sqrt{k} = -\sqrt{k}\bar{B}^T \in U(k). \tag{8.10}$$

The unique horizontal lift of (8.9) is a minimizing geodesic between fibers passing through $[\mathrm{Id}]_{V_{2k,k}}$ and p and moreover they are geodesics since they are horizontal lifts of geodesics. Fix a point p_0 at the fiber passing through $[\mathrm{Id}]_{V_{2k,k}}$. Then the unique horizontal lift $\gamma_v(t)_{V_{2k,k}} = [\exp(tv)]_{V_{2k,k}}$ of (8.9) starting from p_0 always reaches different points at the fiber

$$\pi^{-1}\left(\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}\right)$$

at the time T_0 since $\gamma^3_v(T_0)$ depends on $\bar{B}^T$ but not on $B\bar{B}^T$ as shows (8.10). □

Theorem 8.4.8. *For any point* $s = \left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ *with* $C, D \in U(k)$ *there is a unique minimizing geodesic connecting* Id *with* s.

Proof. Let us assume that a point $s = \left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ belongs to the cut locus from $[\mathrm{Id}]_{V_{2k,k}}$. Let

$$\gamma_{v^*}(t) = \left[\exp\left(t\begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}\right)\right]_{V_{2k,k}} \exp\left(-t\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right)$$

be a minimizing normal geodesic from $[\mathrm{Id}]_{V_{2k,k}}$ to s such that $\gamma(T_0) = s$. Here $v^* = \begin{pmatrix} A & B \\ -\bar{B}^T & 0 \end{pmatrix}$ with $A \neq 0$. Then its projection $\tilde{\gamma}$ to $G_{2k,k}$ is a minimizing geodesic from $[\mathrm{Id}]_{G_{2k,k}}$ to $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$. This implies that $\tilde{\gamma}$ has to coincide with a geodesic in $G_{2k,k}$ having form (8.9) for some B_1 satisfying $\sqrt{k}B_1 \in U(k)$. It is also clear that $\gamma_{v^*}(t)$ is a horizontal lift of $\tilde{\gamma}$ starting at the point $[\mathrm{Id}]_{V_{2k,k}}$. On the other hand the horizontal lift of a geodesic having form (8.9) is equal to $\left[\exp\left(t\begin{pmatrix} 0 & B_1 \\ -\bar{B}_1^T & 0 \end{pmatrix}\right)\right]_{V_{2k,k}}$ which is different from $\gamma_{v^*}(t)$. This is a contradiction to the fact that horizontal lift starting from the same point is unique. We conclude that the points $s = \left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ can not belong to the cut locus and there is a unique minimizing geodesic connecting $[\mathrm{Id}]_{V_{2k,k}}$ with s. □

8.5 Stiefel and Grassmann manifold as embedded into $SO(n)$

In this section we assume that the Stiefel and Grassmann manifolds are embedded into $SO(n)$. We use similar notation for the Stiefel and the Grassmann manifolds as in the previous sections.

We would like to emphasize that we do not input different sub-Riemannian structures on the same Stiefel manifold in this section. We consider different Stiefel manifolds. One of them arise from $U(n)$ factorized by a subgroup of $U(n)$ and others from $SO(n)$, factorized by a subgroup of $SO(n)$. For example all manifolds $V_{n,1}$, related to $U(n)$ group possess the CR structure, but $V_{n,1}$ related to $SO(n)$ does not possess any sub-Riemannian structure.

8.5.1 The cut locus of $V_{n,1}$, $n \in \mathbb{N}$

In this case $\dim(V_{n,1}) = \dim(G_{n,1}) = n-1$ and all sub-Riemannian geodesics are Riemannian ones. For the reason of completeness we present the cut locus in this case, because it is strongly related to the cut locus of $V_{n,1}$ embedded in $U(n)$.

Two parts $\gamma_v^1(t), \gamma_v^3(t)$ of the geodesic $\gamma_v(t) = \left[\begin{pmatrix} \gamma^1(t) & \gamma^2(t) \\ \gamma^3(t) & \gamma^4(t) \end{pmatrix}\right]_{V_{n,1}}$ for an initial velocity $v = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$ are given by

$$
\begin{aligned}
\gamma_v^1(t) &= \frac{1}{4\sqrt{BB^T}} e^{-it\sqrt{BB^T}} 2\sqrt{BB^T}(e^{2it\sqrt{BB^T}}+1) \\
&= \frac{1}{2} e^{-it\sqrt{BB^T}}(e^{2it\sqrt{BB^T}}+1) = \cos(t\sqrt{BB^T}), \\
\gamma_v^3(t) &= -B^T \frac{1}{i2\sqrt{BB^T}} e^{-it\sqrt{BB^T}}(e^{2it\sqrt{BB^T}}-1) \\
&= \frac{-B^T}{\sqrt{BB^T}} \sin(t\sqrt{BB^T}).
\end{aligned}
$$

These formulas are a particular case of formulas (8.4) and (8.5) for the choice of the initial velocity $v = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} \in \left\{ \begin{pmatrix} xi & E \\ -\bar{E}^T & 0 \end{pmatrix} \,\middle|\, E \in \mathbb{C}^{1\times(n-1)}, x \in \mathbb{R} \right\}$. Thus we can use arguments of Theorem 8.3.1 and state that the cut locus of the Stiefel manifold $V_{n,1}$ embedded in $SO(n)$ consists of exactly one point:

$$
\begin{aligned}
&\left\{ \left[\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,k}} \,\middle|\, C \in O(1),\ D \in O(n-1) : \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \in SO(n) \right\} \setminus \left\{ [Id]_{V_{n,k}} \right\} = \\
&\left\{ \left[\begin{pmatrix} \pm 1 & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,k}} \,\middle|\, D \in O(n-1) : \begin{pmatrix} \pm 1 & 0 \\ 0 & D \end{pmatrix} \in SO(n) \right\} \setminus \left\{ [Id]_{V_{n,k}} \right\} = \\
&\left\{ \left[\begin{pmatrix} -1 & 0 \\ 0 & D \end{pmatrix}\right]_{V_{n,k}} \,\middle|\, D \in O(n-1) : \begin{pmatrix} -1 & 0 \\ 0 & D \end{pmatrix} \in SO(n) \right\}.
\end{aligned}
$$

8.5.2 Partial description of the cut locus of $V_{2k,k}$

Inspired by the example of $V_{4,2}$ embedded in $SO(4)$, which can be found in the Appendix, we will exclude the points $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ from the cut locus.

Proposition 8.5.1. *All the points of the form* $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ *with* $C, D \in O(k)$ *are not in the normal cut locus of* $V_{2k,k}$.

For reasons of simplicity for the proof of this proposition, we will prove the following lemma.

Lemma 8.5.2. *The point* $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ *in* $Gr_{2k,k}$ *is reached only by geodesics starting from* $\left[\begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}\right]_{G_{2k,k}}$ *with initial value* $v = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$ *with* $B \in O(k)$. *If we assume* $\mathrm{tr}(BB^T) = 1$, *then this condition will change to* $\sqrt{k}B \in O(k)$.

Proof. Geodesics of $Gr_{2k,k}$ are given by

$$\gamma_v(t) = \left[\exp\left(t\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}\right)\right]_{Gr_{2k,k}},$$

where

$$\begin{aligned} \gamma_v^1(t) &= \cos(t\sqrt{BB^T}), \\ \gamma_v^3(t) &= -B^T \sin(t\sqrt{BB^T})(\sqrt{BB^T})^{-1}. \end{aligned}$$

We are looking for all geodesics for which there exists a $T > 0$, s.t. $\gamma_v^3(T) = C$. As $C \in O(k)$ and so invertible it follows that $\gamma_v^3(T)$ have to be invertible, which directly implies that B have to be invertible. This implies that the matrix BB^T is positive definite. This together with the fact that BB^T is symmetric implies that BB^T is diagonalizable with only positive eigenvalues, i.e. $BB^T = PDP^{-1}$ where $D = \mathrm{diag}(d_1, \dots, d_k)$ is a diagonal matrix with $d_i > 0$ for $i \in \{1, \dots, k\}$. This implies that

$$\cos(t\sqrt{BB^T}) = P\cos(t\sqrt{D})P^{-1}$$

and so $\cos(T_0\sqrt{BB^T}) = 0$ if and only if $\cos(T_0\sqrt{D}) = \mathrm{diag}(\cos(T_0\sqrt{d_1}), \dots, \cos(T_0\sqrt{d_k})) = 0$ if and only if $\cos(T_0\sqrt{d_1}) = \ldots = \cos(T_0\sqrt{d_k}) = 0$.
Let assume without loss of generality $\mathrm{tr}(BB^T) = 1$ and $\sqrt{k}B \in O(k)$. This implies that BB^T is a diagonal matrix of the form $\mathrm{diag}(\frac{1}{k}, \dots, \frac{1}{k})$. Which implies that $\cos(t\sqrt{BB^T}) = \mathrm{diag}(\cos(\frac{t}{\sqrt{k}}), \dots, \cos(\frac{t}{\sqrt{k}}))$. This implies that $\min\{t > 0 | \cos(t\sqrt{BB^T}) = 0\}$ is $T_0 = \frac{\pi\sqrt{k}}{2}$. Furthermore, for v defined by $\sqrt{k}B \in O(k)$ it follows that

$$\gamma_v^3(t) = -B^T \mathrm{diag}(\sin(\frac{t}{\sqrt{k}}), \dots, \sin(\frac{t}{\sqrt{k}}))\sqrt{k}Id_k,$$

which leads to $\gamma_v^3(T_0) = -\sqrt{k}B^T \in O(k)$. This implies that the unique horizontal lift $\gamma_v(t)_{V_{2k,k}} = [\exp(tv)]_{V_{2k,k}}$ of $\gamma_v(t)$ always reaches different points at T_0.

Now we want to show that no other minimizing geodesic exist. Let's have a look at all invertible B with $\sqrt{k}B \notin O(k)$ with $\operatorname{tr}(BB^T) = 1$. It follows that there exist at least two eigenvalues $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ of BB^T with $\lambda_1, \lambda_2 > 0$ with $\frac{1}{\lambda_1} < \frac{1}{k} < \frac{1}{\lambda_2}$. It follows that if $\cos(T\sqrt{BB^T}) = 0$, then $\cos(\frac{T}{\sqrt{\lambda_1}}) = 0$ which implies that $T_0 \geq \frac{\pi\sqrt{\lambda_1}}{2} > \frac{\pi\sqrt{k}}{2}$. This implies that B can not define a minimizing geodesic for the point $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$. □

Corollary 8.5.3. *The geodesics $\gamma_v(t)$ with $v = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$, $\sqrt{k}B \in O(k)$ and $\operatorname{tr}(BB^T) = 1$ in $V_{2k,k}$ are reaching the points $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ with $C, D \in O(k)$ at the length minimizing time $T_0 = \frac{\pi\sqrt{k}}{2}$. Furthermore, if $B_1 \neq B_2$, then $\gamma^3_{v_1}(T_0) \neq \gamma^3_{v_2}(T_0)$.*

Now we know that all geodesics in $Gr_{2k,k}$ defined by B with $\sqrt{k}B \in O(k)$ and $\operatorname{tr}(BB^T) = 1$ are minimizing geodesics for the point $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ starting from $\left[\begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}\right]_{G_{2k,k}}$. They reach the point $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$ at $T_0 = \frac{\pi\sqrt{k}}{2}$. We also know that the horizontal lifts of geodesics starting at the point $\left[\begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}\right]_{V_{2k,k}}$ are minimizing between the fibers of $\left[\begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}\right]_{V_{2k,k}}$ and $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$. As we can reach every point in the fiber $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ at time T_0 by the horizontal lifts, we know that all the horizontal lifts are geodesics for different points in the fiber $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$. Furthermore, we know that our horizontal lifts are minimizing between the points as they are minimizing between the fibers. Now we are able to proof the proposition.

Proof. Let's assume there exists a matrix $\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}$ with $A \neq 0$ which defines a minimizing normal horizontal geodesic $\gamma^*(t) = \left[\exp\left(t\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\right)\right]_{V_{2k,k}} \exp\left(-t\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right)$ from $\left[\begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}\right]_{V_{2k,k}}$ to $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ at time T_0. Then we know the length of it's projection in $Gr_{2k,k}$ is the same, s.t. it's projection is a minimizing geodesic for $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{G_{2k,k}}$. This implies that it have to coincide with a geodesic in $Gr_{2k,k}$ which is defined by B_1 with $\sqrt{k}B_1 \in O(k)$. It is clear that $\gamma^*(t)$ is a horizontal lift for this geodesic at the point $\left[\begin{pmatrix} Id_k & 0 \\ 0 & Id_k \end{pmatrix}\right]_{V_{2k,k}}$. But we already know a horizontal lift of this geodesic, which is $\left[\exp\left(t\begin{pmatrix} 0 & B_1 \\ -B_1^T & 0 \end{pmatrix}\right)\right]_{V_{2k,k}}$ which is clearly different of $\gamma^*(t)$. This is a contradiction to the fact that a horizontal lift is unique. We conclude that the points of the form $\left[\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}\right]_{V_{2k,k}}$ can not be in the cut locus.

□

8.6 Appendix

In this Appendix we collect several particular examples of the cut locus in the complex and in the real case.

8.6.1 The cut locus of $V_{2,1}$ embedded in $U(2)$ and its equivalence to $SU(2)$

In this subsection we show that the results obtained in Section 8.3 recover the results obtained in [24]. In particular, for the three dimensional manifold $V_{2,1}$ we get the following simple formulas. The tangent spaces at Id are given by

$$T_{\mathrm{Id}}V_{2,1} = \left\{ \mathrm{Id}\begin{pmatrix} ix & b \\ -\bar{b} & 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R},\ b \in \mathbb{C} \right\}, \quad T_{\mathrm{Id}}G_{2,1} = \left\{ \mathrm{Id}\begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \;\middle|\; b \in \mathbb{C} \right\}.$$

We obtain the following corollary from Theorem 8.3.1.

Corollary 8.6.1. *The circle given by*

$$L_{2,1} := \left\{ \left[\begin{pmatrix} e^{ci} & 0 \\ 0 & e^{di} \end{pmatrix} \right]_{V_{2,1}} \;\middle|\; c, d \in \mathbb{R} \right\} \setminus \{\mathrm{Id}\}$$

is the cut locus K_{Id} of $V_{2,1}$.

An element q of $V_{2,1}$ is an equivalence class which can be written as

$$[q]_{V_{2,1}} = \left\{ \begin{pmatrix} \alpha & \exp(\lambda i)\bar{\beta} \\ \beta & -\exp(\lambda i)\bar{\alpha} \end{pmatrix} \;\middle|\; \lambda \in (0, 2\pi) \right\}.$$

Since $\begin{pmatrix} \alpha & \exp(\lambda i)\bar{\beta} \\ \beta & -\exp(\lambda i)\bar{\alpha} \end{pmatrix}$ is a unitary matrix, the norm $\|\alpha\|^2 + \|\beta\|^2$ of the vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is equal one. Thus, points $q \in V_{2,1}$ can be parametrized by the vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Recall the definition of the group $SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \;\middle|\; \|\alpha\|^2 + \|\beta\|^2 = 1 \right\}$. So, it is clear that every element of $SU(2)$ can be represented by the vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. It follows that the both manifolds are diffeomorphic under the mapping $f\colon V_{2,1} \to SU(2)$, $[g]_{V_{2,1}} \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. The metric in both cases is left invariant, arising from an inner product on the Lie algebras making the basis of the Lie algebras orthogonal. The horizontal distribution is orthogonal to the vertical one.

The set $L_{2,1}$ as a subset of $V_{2,1}$ depends only on $c \in (0, 2\pi)$, since the part depending on d is quotient out. This implies that the cut locus of $SU(2)$, given by the circle $\{e^{ci}\}$ without the point $1 \in SU(2)$ [24], has a bijective relation under the map f to the cut locus of $V_{2,1}$, given in Corollary 8.6.1.

8.6.2 The cut locus of $V_{3,2}$ embedded in $SO(3)$

Since $V_{3,2} \cong SO(3)/SO(1)$ and $SO(1)$ is a normal subgroup of $SO(3)$, one can identify the sub-Riemannian structure of $V_{3,2}$ with the sub-Riemannian structure on the group $SO(3)$, that was studied in [24]. In particular all equivalences classes contain exactly one matrix

$$\left[\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]_{V_{n,k}} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & S_1 \end{pmatrix} \Big| S_1 \in S(1) \right\}$$
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \right\},$$

such that we can identify $V_{3,2}$ with $SO(3)$. Furthermore, the induced horizontal and vertical space coincide with the horizontal and vertical space of the $k \oplus p$ problem on $SO(3)$ stated in [24].

8.6.3 Partial description of the cut locus of $V_{4,2}$ embedded in $SO(4)$

The normal geodesic $\gamma(t) = (\gamma_{ij})_{i,j \in \{1,\dots,4\}}$ in this case is again given by the formula

$$\gamma(t) = [\exp(tv)]_{V_{4,2}} \exp_{SO(2)}(-tA),$$

where $v := \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \in \mathfrak{so}(4)$, $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in \mathfrak{so}(2)$, $B = \begin{pmatrix} b & c \\ d & e \end{pmatrix} \in \mathbb{R}^{2\times 2}$, $a, b, c, d, e \in \mathbb{R}$. We note that the dimension of $V_{4,2}$ is 5, the dimension of the horizontal distribution is 4 and of the vertical distribution is 1. We define $x := a^2 + b^2 + c^2 + d^2 + e^2 = \frac{\mathrm{tr}(vv^T)}{2}$, $r := \sqrt{x^2 - 4(cd - be)^2} \geq 0$, which is well defined as

$$x^2 - 4(cd - be)^2 = (a^2 + c^2 + b^2 - (d^2 + e^2))^2 + 4(bd + ce)^2 + 4a^2(d^2 + e^2) \geq 0.$$

We further note that $x \geq r \geq 0$, s.t. it follows that $x - r \geq 0$. The definition of x and r will be of interest for the calculations of the eigenvalues and eigenvectors of v. The characteristic polynomial of v is given by

$$p_v(\lambda) = \lambda^4 + x\lambda^2 + \det(B)^2.$$

It follows that the eigenvalues are given by

$$\lambda_1 = -i\sqrt{\frac{x+r}{2}}, \qquad \lambda_2 = i\sqrt{\frac{x+r}{2}}, \qquad \lambda_3 = -i\sqrt{\frac{x-r}{2}}, \qquad \lambda_4 = i\sqrt{\frac{x-r}{2}}.$$

As v is a skew-symmetric matrix, hence normal, which implies that it is diagonizable by its eigenvalues in the diagonal matrix, we can calculate the exponential by the help of decomposition:

$$\gamma_v(t) = O\begin{pmatrix} \exp(-ti\sqrt{\frac{x+r}{2}}) & 0 & 0 & 0 \\ 0 & \exp(ti\sqrt{\frac{x+r}{2}}) & 0 & 0 \\ 0 & 0 & \exp(-ti\sqrt{\frac{x-r}{2}}) & 0 \\ 0 & 0 & 0 & \exp(ti\sqrt{\frac{x-r}{2}}) \end{pmatrix} O^T \times$$
$$\times \begin{pmatrix} \cos(at) & -\sin(at) & 0 & 0 \\ \sin(at) & \cos(at) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where O is an orthogonal matrix.
Without loss of generality we assume that $\mathrm{tr}(BB^T) = b^2 + c^2 + d^2 + e^2 = 1$ and remind the notation

$$BB^T = \begin{pmatrix} b^2 + c^2 & bd + ce \\ bd + ce & d^2 + e^2 \end{pmatrix}.$$

Then the long calculations lead to the formulas for geodesics. For $x^2 - 4(cd - be)^2 =: r^2 \neq 0$ the geodesic is given by

$$\begin{aligned}\gamma_{11}(t) &= \frac{1}{2r}[\cos(at)[\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)(x + r - 2(d^2 + e^2)) \\ &- \cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)(x - r - 2(d^2 + e^2))] + 2\sin(at)[(bd + ce)(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right) \\ &- \cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)) + a(\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right) \\ &- \frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]],\end{aligned}$$

$$\begin{aligned}\gamma_{12}(t) &= \frac{1}{2r}[\sin(at)[\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)(x - r - 2(d^2 + e^2)) \\ &- \cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)(x + r - 2(d^2 + e^2))] + 2\cos(at)[(bd + ce)(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right) \\ &- \cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)) + a(\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right) \\ &- \frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]],\end{aligned}$$

$$\begin{aligned}
\gamma_{21}(t) &= \frac{1}{2r}[\sin(at)[-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)(x-r-2(b^2+c^2))\\
&+ \cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)(x+r-2(b^2+c^2))]+2\cos(at)[(bd+ce)(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\\
&- \cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))+a(-\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\\
&+ \frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]],
\end{aligned}$$

$$\begin{aligned}
\gamma_{22}(t) &= \frac{1}{2r}[\cos(at)[-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)(x-r-2(b^2+c^2))\\
&+ \cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)(x+r-2(b^2+c^2))]-2\sin(at)[(bd+ce)(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\\
&- \cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))+a(-\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\\
&+ \frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]].
\end{aligned}$$

We note that $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$ depend on t, BB^T and a.

$$\begin{aligned}
\gamma_{31}(t) &= \frac{1}{\sqrt{2}r}[\cos(at)[-\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{b(x+r)+2e(cd-be)}{\sqrt{x+r}}\\
&+ \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{b(x-r)+2e(cd-be)}{\sqrt{x-r}}+\sqrt{2}ad(\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right))]\\
&+ \sin(at)[-\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{d(x+r)+2c(be-cd)}{\sqrt{x+r}}\\
&+ \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{d(x-r)+2c(be-cd)}{\sqrt{x-r}}+\sqrt{2}ab(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]],
\end{aligned}$$

$$\begin{aligned}
\gamma_{32}(t) &= \frac{1}{\sqrt{2}r}[\cos(at)[-\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{d(x+r)+2c(be-cd)}{\sqrt{x+r}}\\
&+ \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{d(x-r)+2c(be-cd)}{\sqrt{x-r}}+\sqrt{2}ab(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]\\
&+ \sin(at)[\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{b(x+r)+2e(cd-be)}{\sqrt{x+r}}\\
&- \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{b(x-r)+2e(cd-be)}{\sqrt{x-r}}+\sqrt{2}ad(-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)+\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right))]].
\end{aligned}$$

We note that γ_{31}, γ_{32} depend on t, BB^T, b, d and a, as

$$b(x+r)+2e(cd-be) = b(a^2+b^2+c^2-(d^2+e^2))+2d(ce+bd).$$

$$\begin{aligned}
\gamma_{41}(t) &= \frac{1}{\sqrt{2}r}[\cos(at)[-\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{c(x+r)+2d(be-cd)}{\sqrt{x+r}}\\
&+ \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{c(x-r)+2d(be-cd)}{\sqrt{x-r}}+\sqrt{2}ae(\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right))]\\
&+ \sin(at)[-\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{e(x+r)+2b(cd-be)}{\sqrt{x+r}}\\
&+ \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{e(x-r)+2b(cd-be)}{\sqrt{x-r}}+\sqrt{2}ac(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]],
\end{aligned}$$

$$\begin{aligned}
\gamma_{42}(t) &= \frac{1}{\sqrt{2}r}[\cos(at)[-\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{e(x+r)+2b(cd-be)}{\sqrt{x+r}}\\
&+ \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{e(x-r)+2b(cd-be)}{\sqrt{x-r}}+\sqrt{2}ac(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]\\
&+ \sin(at)[\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)\frac{c(x+r)+2d(be-cd)}{\sqrt{x+r}}\\
&- \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)\frac{c(x-r)+2d(be-cd)}{\sqrt{x-r}}+\sqrt{2}ae(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]].
\end{aligned}$$

We note that γ_{41}, γ_{42} depend on t, BB^T, c, e and a.
Furthermore, we note that r just depends on BB^T and a as

$$x^2-4(cd-be)^2 = (a^2+c^2+b^2-(d^2+e^2))^2+4(bd+ce)^2+4a^2(d^2+e^2).$$

Now to the case that $x^2-4(cd-be)^2=r^2=0$. We first note that in this case a have to be equal to 0. Furthermore, we get, together with the assumption that $b^2+c^2+d^2+e^2=1$, just the following four cases for $-\frac{1}{\sqrt{2}} \le b \le \frac{1}{\sqrt{2}}$:

$$(I) \qquad e=b \qquad c=\pm\sqrt{\frac{1-2b^2}{2}}=-d,$$

$$(II) \qquad e=-b \qquad c=\pm\sqrt{\frac{1-2b^2}{2}}=d.$$

From this it follows the normal geodesic for $e=b$ and $c=\pm\sqrt{\frac{1-2b^2}{2}}=-d$:

$$\begin{aligned}
\gamma_I(t) &= \begin{pmatrix}
\cos\left(\frac{t}{\sqrt{2}}\right) & 0 & \sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) & \pm\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right)\\
0 & \cos\left(\frac{t}{\sqrt{2}}\right) & \mp\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) & \sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right)\\
-\sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) & \pm\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) & \cos\left(\frac{t}{\sqrt{2}}\right) & 0\\
\mp\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) & -\sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) & 0 & \cos\left(\frac{t}{\sqrt{2}}\right)
\end{pmatrix}\\
&= \begin{pmatrix}
\cos\left(\frac{t}{\sqrt{2}}\right)Id_2 & \sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix}\sqrt{2}b & \pm\sqrt{1-2b^2}\\ \mp\sqrt{1-2b^2} & \sqrt{2}b\end{pmatrix}\\
\sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix}-\sqrt{2}b & \pm\sqrt{1-2b^2}\\ \mp\sqrt{1-2b^2} & -\sqrt{2}b\end{pmatrix} & \cos\left(\frac{t}{\sqrt{2}}\right)Id_2
\end{pmatrix}.
\end{aligned}$$

Furthermore, it follows the normal geodesic for $e=-b$ and $c=\pm\sqrt{\frac{1-2b^2}{2}}=d$:

$$\gamma_{II}(t) = \begin{pmatrix} \cos\left(\frac{t}{\sqrt{2}}\right) & 0 & \sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) & \pm\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) \\ 0 & \cos\left(\frac{t}{\sqrt{2}}\right) & \pm\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) & -\sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) \\ -\sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) & \mp\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) & \cos\left(\frac{t}{\sqrt{2}}\right) & 0 \\ \mp\sqrt{1-2b^2}\sin\left(\frac{t}{\sqrt{2}}\right) & \sqrt{2}b\sin\left(\frac{t}{\sqrt{2}}\right) & 0 & \cos\left(\frac{t}{\sqrt{2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{t}{\sqrt{2}}\right) Id_2 & \sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix} \sqrt{2}b & \pm\sqrt{1-2b^2} \\ \mp\sqrt{1-2b^2} & -\sqrt{2}b \end{pmatrix} \\ \sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix} -\sqrt{2}b & \mp\sqrt{1-2b^2} \\ \mp\sqrt{1-2b^2} & \sqrt{2}b \end{pmatrix} & \cos\left(\frac{t}{\sqrt{2}}\right) Id_2 \end{pmatrix}.$$

It is clear, that these two geodesics just can reach the same points if

$$\sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix} -\sqrt{2}b & \pm\sqrt{1-2b^2} \\ \mp\sqrt{1-2b^2} & -\sqrt{2}b \end{pmatrix} = \sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix} -\sqrt{2}b & \mp\sqrt{1-2b^2} \\ \mp\sqrt{1-2b^2} & \sqrt{2}b \end{pmatrix} = 0.$$

It follows that we just have to check if points which are reached by this two geodesics can be reached at the same time of geodesics with $r \neq 0$.

Proposition 8.6.2. *The points of the form* $\left[\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}\right]_{V_{4,2}}$ *where* $X, Y \in O(2)$*, s.t.* $\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} \in SO(4)$ *are not in the normal cut locus.*

Proof. First we are interested in the geodesics starting from the identity and reaching the points of the form $\left[\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}\right]_{V_{4,2}}$ where $X, Y \in O(2)$. We will call such kind of points matrices with "upper zeros". We already noticed that the geodesics with the restriction that $r = 0$ are equal if and only if

$$\sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix} -\sqrt{2}b & \pm\sqrt{1-2b^2} \\ \mp\sqrt{1-2b^2} & -\sqrt{2}b \end{pmatrix} = \sin\left(\frac{t}{\sqrt{2}}\right)\begin{pmatrix} -\sqrt{2}b & \mp\sqrt{1-2b^2} \\ \mp\sqrt{1-2b^2} & \sqrt{2}b \end{pmatrix} = 0.$$

It follows as $\gamma_v^3(t) \neq 0$ that we just have to check if there exists a minimizing geodesic with $r \neq 0$ and one with $r = 0$.
For geodesics with $r = 0$, we reach a point of the form $\left[\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}\right]_{V_{4,2}}$ for $T_0 = \frac{\pi}{\sqrt{2}}$. If we are now able to show that for $r \neq 0$, T_0, the minimal time where the geodesic reaches $\left[\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}\right]_{V_{4,2}}$, is strictly bigger than $\frac{\pi}{\sqrt{2}}$, then we are done.
First we note that the geodesic $\gamma(t) = \exp\left(t\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\right)\exp\left(-t\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right)$ has "upper zeros" if and only if the matrix $\exp\left(t\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\right)$ has "upper zeros", as the exponential of A is invertible, i.e. $\gamma_{11}(T_0) = \gamma_{12}(T_0) = \gamma_{21}(T_0) = \gamma_{22}(T_0) = 0$. This implies the following

system of equations:

$$\frac{\cos\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)}{2r}(x+r-2(d^2+e^2))-\frac{\cos\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right)}{2r}(x-r-2(d^2+e^2))=0, \tag{8.11}$$

$$\begin{aligned}&\frac{1}{2r}((bd+ce)(\cos\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right))+\\&a(\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right))])=0,\end{aligned} \tag{8.12}$$

$$-\frac{\cos\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right)}{2r}(x-r-2(b^2+c^2))+\frac{\cos\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)}{2r}(x+r-2(b^2+c^2))=0, \tag{8.13}$$

$$\begin{aligned}&\frac{1}{2r}((bd+ce)(\cos\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right))+\\&a(-\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)+\frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right))])=0.\end{aligned} \tag{8.14}$$

As equation (8.12) = (8.14) and (8.11) = (8.13) are equal, it follows that

$$\begin{aligned}a(-\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)+\frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right))]&=0,\\(b^2+c^2-(d^2+e^2))(\cos\left(t\sqrt{\frac{x+r}{2}}\right)-\cos\left(t\sqrt{\frac{x-r}{2}}\right))&=0,\\(bd+ce)(\cos\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)-\cos\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)&=0.\end{aligned}$$

If we assume that $\cos\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)=\cos\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right)$ and put it into equation (8.11) we will get that $\cos\left(T_0\frac{\sqrt{x+r}}{\sqrt{2}}\right)=\cos\left(T_0\frac{\sqrt{x-r}}{\sqrt{2}}\right)=0$ and as $\sqrt{x-r}<1$ it follows that $T_0>\frac{\pi}{\sqrt{2}}$, which tells us that it can not be a minimizing geodesic for our points (as the minimizing time for the minimizing geodesics was $\frac{\pi}{\sqrt{2}}$).
So we can assume that $bd+ce=0$ and $b^2+c^2=d^2+e^2\Leftrightarrow b^2+c^2=\frac{1}{2}$, which gives us

$$\begin{aligned}e=b\qquad c=\pm\sqrt{\frac{1-2b^2}{2}}=-d,\\e=-b\qquad c=\pm\sqrt{\frac{1-2b^2}{2}}=d.\end{aligned}$$

for $-\frac{1}{\sqrt{2}}\leq b\leq\frac{1}{\sqrt{2}}$.

So now we just have to look at the two remaining cases with $a=0$ or

$$-\frac{\sqrt{x+r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right)+\frac{\sqrt{x-r}}{\sqrt{2}}\sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right)=0.$$

If $a = 0$, then also $r = 0$. This would be a contradiction, as we assumed $r \neq 0$, s.t. there is just the case left with $a \neq 0$ and $-\frac{\sqrt{x+r}}{\sqrt{2}} \sin\left(t\frac{\sqrt{x+r}}{\sqrt{2}}\right) + \frac{\sqrt{x-r}}{\sqrt{2}} \sin\left(t\frac{\sqrt{x-r}}{\sqrt{2}}\right) = 0$. It is enough to check the conditions for one case as for our restrictions on b, c, d, e the matrix product BB^T is always equal, s.t. the first upper matrix is the same. So without limit of generality we observe the case $b = e$ and $c = -d$ and get the following conditions:

$$\begin{aligned}
& (a^2 + \sqrt{a^2(2+a^2+2b^2)}) \cos\left(t\frac{\sqrt{1+a^2+b^2+\sqrt{a^2(2+a^2+2b^2)}}}{\sqrt{2}}\right) \\
+ \quad & (-a^2 + \sqrt{a^2(2+a^2+2b^2)}) \cos\left(t\frac{\sqrt{1+a^2+b^2-\sqrt{a^2(2+a^2+2b^2)}}}{\sqrt{2}}\right) = 0, \\
- \quad & \sqrt{1+a^2+b^2+\sqrt{a^2(2+a^2+2b^2)}} \sin\left(t\frac{\sqrt{1+a^2+b^2+\sqrt{a^2(2+a^2+2b^2)}}}{\sqrt{2}}\right) \\
+ \quad & \sqrt{1+a^2+b^2-\sqrt{a^2(2+a^2+2b^2)}} \sin\left(t\frac{1+a^2+b^2-\sqrt{a^2(2+a^2+2b^2)}}{\sqrt{2}}\right) = 0.
\end{aligned}$$

This can be generally formulated as

$$\begin{aligned}
\alpha_1 \sin(x) - \beta_1 \sin(y) = 0 \quad &\Leftrightarrow \quad \sin(x) = \frac{\beta_1}{\alpha_1} \sin(y), \\
\alpha_2 \cos(x) + \beta_2 \cos(y) = 0 \quad &\Leftrightarrow \quad \cos(x) = -\frac{\beta_2}{\alpha_2} \cos(y),
\end{aligned}$$

for fixed $x, y \in \mathbb{R}$ and fixed $0 < \beta_i < \alpha_i$, $i \in \{1, 2\}$. But this is not possible as

$$\begin{aligned}
1 \quad &= \quad \cos^2(x) + \sin^2(x) = \frac{\beta_2^2}{\alpha_2^2} \cos^2(y) + \frac{\beta_1^2}{\alpha_1^2} \sin^2(y) \\
&< \quad \sin^2(y) + \cos^2(y) = 1
\end{aligned}$$

for $0 < \frac{\beta_i}{\alpha_i} < 1$, $i \in \{1, 2\}$.

This implies that all points of the form $\left[\begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix}\right]_{V_{4,2}}$ can not be in the cut locus. $\square$

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