Department
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APPLIED MATHEMATICS

Perturbation about neutral solutions occurring in shear flows in stratified, incompressible and inviscid fluids.

by

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Summary.

The unstable solution $\varphi(y,k^2,c)$ contiguous to the neutral one $\varphi_s(y,k_s^2,c_s)$ which may occur in shear flows in stratified, incompressible and inviscid fluids, can be expressed as

$$\varphi = \varphi_s + \sum_{l=1}^{\infty} \varphi_l (c-c_s)^l \text{ , where } k^2 - k_s^2 = \sum_{l=1}^{\infty} k_l (c-c_s)^l \text{ .}$$

Here k is the wave-number and c the wave-velocity corresponding to the unstable solution, and k_s and c_s the wave-number and wave-velocity of the neutral solution. Expressions for φ_l and k_l are given.

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$$c = c_s + \sum_{i=1}^{\infty} a_i (k-k_s)^i, \quad \text{where} \quad k^2 - k_s^2 = \sum_{i=1}^{\infty} b_i (c-c_s)^i .$$

Here k is the wave-number and c the wave-velocity corresponding to the unstable solution, and k_s and c_s the wave-number and wave-velocity of the neutral solution. Expressions for a_1 and b_1 are given.

I. Introduction.

In this paper we are concerned with the unstable solutions contiguous to the neutral ones which may occur in shear flows in stratified, incompressible and inviscid fluids. In this connection we have to find solutions of the equation:

$$(1.1) \qquad \varphi'' + \left\{\frac{\beta g}{(U-c)^2} - \frac{U''}{U-c} - k^2\right\}\varphi = 0 .$$

Here $U(y)$ denotes the unperturbed flow velocity, and $\beta(y) = -\rho'(y)/\rho(y)$, where $\rho(y)$ is the unperturbed density field. The prime denotes differentiation with respect to y . $U(y)$ and $\rho(y)$ vary in the y-direction perpendicular to the flow direction, which is taken to be the x-direction. This basic state is perturbed, and the perturbation stream function is $\Psi(x,y,t) = \mathrm{Re}\{\varphi(y)\, e^{ik(x-ct)}\}$, where k is the wave-number (real), and $c = c_r + ic_i$ is the wave-velocity (which may be complex, $c_i \neq 0$). $\mathrm{Re}\{\ldots\}$ means the real part of the quantity within the brackets. Eq.(1.1) is the equation for the amplitude function $\varphi(y)$. The fluid is supposed to be confined between two rigid horizontal planes at $y = y_1$ and $y = y_2$. The boundary conditions to be satisfied are therefore:

$$(1.2) \qquad \varphi = 0 \quad \text{at} \quad y = y_1,\ y_2 .$$

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$$(1.2) \qquad \varphi = 0 \quad \text{at} \quad y = y_1, y_2 .$$

In this paper we assume that $U(y)$ and $\beta(y)$ are analytic functions of y on the interval $I = \{y | y_1 \leq y \leq y_2\}$ of the real axis. Then $U(y)$ and $\beta(y)$ are analytic in some region in the complex plane close to this interval. Further it is assumed that $U'(y) \neq 0$ on I. Let us take $U' > 0$ on I. The case $U' < 0$ can be treated in an analogous way.

We consider the case when the fluid is statically stable. In this case there may exist singular neutral solutions, i.e. solutions which are located on the stability boundary in a wave number - Richardson number - plane, see for instance [1]. The singular neutral solution φ_s with the wave velocity c_s and the wave number k_s satisfies eq.(1.1) with $c = c_s$ and $k = k_s$ and the boundary conditions eq.(1.2). Since $U' \neq 0$ for $y \in [y_1, y_2]$, φ_s must be of the form, see [1]:

(1.3) $$\varphi_s = (U-c_s)^{\frac{1}{2}+\mu} Y_s , \quad \text{where} \quad \mu \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

$\mu = (\frac{1}{4} - R_s)^{\frac{1}{2}}$, where $R_s = g\beta/(U')^2$ is the Richardson number at the critical layer defined by $U(y_s) - c_s = 0$. Y_s is analytic on I since U and β are assumed to be analytic there. We define $\arg(U-c_s)$ in the following manner: $\arg(U-c_s) = 0$ when $U - c_s > 0$, and $\arg(U-c_s) = -\pi$ when $U - c_s < 0$. If $\arg(U-c_s)$ is defined in this way, we have shown in [2] that φ_s coincides almost everywhere with the viscous solution within the limit of zero viscosity. Also with this definition of $\arg(U-c_s)$ φ_s will

be the limit when $c_i \to 0^+$ of the unstable solution. Let L be the contour shown in fig.1. φ_s given by the eq.(1.3) is analytic along L if ρ is made small enough. (Note that $U' > 0$ for $y \in [y_1, y_2]$).

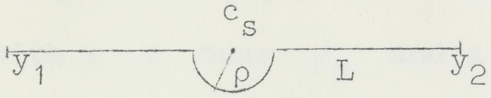

Fig.1

We also observe that an unstable solution is analytic on I, and will therefore also be analytic along L if ρ is made small enough. We also see that $-\pi \leqq \arg(U-c) \leqq 0$ when $c_i \geqq 0$ both when $y \in I$ and $y \in L$.

II. Perturbation about the neutral solution.

We assume that there exists a neutral solution φ_s as defined in eq.(1.3). As mentioned this solution is analytic along L if ρ is made snall enough. Let $|c - c_s| \leqq \rho_1$, where $\rho_1 < \rho$. Further let us define $\arg(U-c)$ in the following manner: $-\pi - \epsilon_2 < \arg(U-c) < \epsilon_1$, where $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Our choice of ϵ_1 and ϵ_2 will depend on ρ_1. We see that if $c_i \geqq 0$, $-\pi \leqq \arg(U-c) \leqq 0$; and if $c_i < 0$, $-\pi - \epsilon_2 < \arg(U-c) < \epsilon_1$. A solution φ of eq.(1.1) is an analytic function of $y \in L$, $c \in \left\{ c \,\middle|\, |c - c_s| \leqq \rho_1 \right\}$ and k^2, with $\arg(U-c)$ defined as above. This solution can therefore be expanded in a series:

(2.1) $$\varphi = \varphi_o + \left(\frac{\partial \varphi}{\partial c}\right)_s (c - c_s) + \left(\frac{\partial \varphi}{\partial k^2}\right)_s (k^2 - k_s^2) + \ldots ,$$

where $(\ldots)_s$ means that the quantity within the brackets is calculated at $c = c_s$ and $k^2 = k_s^2$.
Assume that there exists a solution φ of eq.(1.1), with a c near c_s and a k^2 near k_s^2, that satisfies the boundary conditions eq.(1.2). By introducing eq.(2.1) into eq.(1.1) and eq.(1.2) we get within the limit when $c \to c_s$ and $k^2 \to k_s^2$:

$$(2.2) \quad \varphi_o'' + \left\{\frac{\beta g}{(U - c_s)^2} - \frac{U''}{U-c_s} - k_s^2\right\} \varphi_o = 0 \text{ , and } \varphi_o = 0$$

$$\text{for } \; y = y_1, y_2.$$

Wee see that φ_o must satisfy the same equations as φ_s, and we get that $\varphi_o = A_o\varphi_s$, where A_o is a constant.
By using the equations which govern φ and φ_s , we get:

$$(2.3) \quad \int_L \left[\left(\frac{\beta g}{(U-c)^2} - \frac{U''}{U-c} - k^2\right) - \left(\frac{\beta g}{(U-c_s)^2} - \frac{U''}{U-c_s} - k_s^2\right)\right] \varphi\varphi_s \, dy = 0.$$

Let us denote by E the expression $\dfrac{\beta g}{(U-c)^2} - \dfrac{U''}{U-c} - k^2$, and by E_s the same expression with c_s and k_s^2 instead of c and k^2. By introducing eq.(2.1) into (2.3) we get:

$$
(2.4)\quad
\begin{cases}
(k^2-k_s^2) \int_L \left[\frac{\partial}{\partial k^2}\{(E-E_s)\varphi\}\right]_s \varphi_s \, dy \; + \\
\qquad + (c-c_s) \int_L \left[\frac{\partial}{\partial c}\{(E-E_s)\varphi\}\right]_s \varphi_s \, dy + \ldots \\
\frac{1}{l!} \int_L \left[\left\{(c-c_s)\frac{\partial}{\partial c} + (k^2-k_s^2)\frac{\partial}{\partial k^2}\right\}^l \{(E-E_s)\varphi\}\right]_s \varphi_s dy + \ldots = 0,
\end{cases}
$$

where $[\ldots]_s$ means that the expression within the brackets is calculated at $c = c_s$ and $k^2 = k_s^2$. The integrals in eq.(2.4) will exist because φ is an analytic function on L.

We assume that the coefficient of $(k^2-k_s^2)$ in eq.(2.4), i.e. $\int_L \left[\frac{\partial}{\partial k^2}\{(E-E_s)\varphi\}\right]_s \varphi_s dy = -\int_L \varphi_s^2 dy$, is not equal to zero. Then if eq.(2.4) is to be satisfied within the limit when $c \to c_s$, we must have that:

$$
(2.5)\qquad k^2-k_s^2 = k_1(c-c_s) + k_2(c-c_s)^2 + \ldots k_l(c-c_s)^l + \ldots ,
$$

where k_l, $l = 1,2, \ldots$ are constants.

When U is an odd and β is an even function of y , $y_1 = -y_2$, and φ_s is a singular neutral solution with wave velocity $c_s = 0$, the coefficient of $(k^2-k_s^2)$ in eq.(2.4) is equal to $(e^{-i2\pi\mu} - 1)\int_0^{y_2} U^{1+2\mu} Y_s^2 dy$, where it has been assumed that $U' > 0$. This expression is not equal to zero when $|\mu| \in (0,\frac{1}{2}]$. In section III we have considered an example of this type.

When $\mu = 0$, i.e. $R_s = \frac{1}{4}$, the coefficient of $(k^2-k_s^2)$

is zero. However, to find the unstable solution close to the neutral one on this point of the stability boundary, we should expand φ and $(c-c_s)$ in a series in $(R-R_s)$ keeping $k^2 = k_s^2$ fixed, rather than expanding φ and $(k^2-k_s^2)$ in a series in $(c-c_s)$ keeping $R = R_s$ fixed, as is done in this paper. In [1] we have found the formula for $\left(\frac{\partial c}{\partial R}\right)_{k_s}$,i.e. we have found the first term in the series for $(c-c_s)$ in powers of $(R-R_s)$, keeping $k^2 = k_s^2$ fixed.

The coefficient of $(c-c_s)$ in eq.(2.4) may be zero, (see the example in section III), and that is the reason why we have expanded $(k^2-k_s^2)$ in a series of $(c-c_s)$. If the coefficient of $(c-c_s)$ is not equal to zero, $k_1 \neq 0$, and $(k^2-k_s^2)$ behaves as $k_1(c-c_s)$ for c close to c_s. If this coefficient is equal to zero, $k_1 = 0$, and $(k^2-k_s^2)$ behaves as $k_2(c-c_s)^2$ for c close to c_s. This shows that if we had expanded $(c-c_s)$ in a series in $(k^2-k_s^2)$, we would have had to treat these two cases separately.

We have assumed that there exists a solution φ for a c near c_s and for a k^2 near k_s^2, which satisfies the boundary conditions eq.(1.2), i.e.:

(2.6) $$\varphi(y_1,k^2,c) = 0 \quad \text{and} \quad \varphi(y_2,k^2,c) = 0 .$$

The functions in the eqs.(2.6) are analytic functions of $c \in \left\{c \middle| |c-c_s| \leq \rho_1\right\}$ and k^2 , so that all the derivatives with respect to c and k^2 exist . Derivating the eqs. (2.6) with respect to c , yields:

$$(2.7) \qquad \frac{\partial\varphi}{\partial c} + \frac{\partial\varphi}{\partial k^2}\frac{dk^2}{dc} = 0,$$

where the value of y is either y_1 or y_2.

$\left(\frac{\partial\varphi}{\partial K^2}\right)_s \neq 0$ because of the assumption above that the coefficient of $(k^2-k_s^{\,2})$ in eq.(2.4) is not equal to zero. It then follows from eq.(2.7) that $\frac{dk^2}{dc}$ exists in some region around c_s. Therefore $(k^2-k_s^{\,2})$ is analytic in this region $|c-c_s| < \rho_2$, and can be expanded in a power series, which is valid for $|c-c_s| < \rho_2$. Consequently the series given by eq.(2.5) will converge whithin this region.

Taking into account eq.(2.5), we find that eq.(2.4) is satisfied if:

$$(2.8) \qquad \begin{cases} \int_L \left[\frac{d}{dc}\left\{(E-E_s)\varphi\right\}\right]_s \varphi_s \, dy = 0 \\ \vdots \\ \int_L \left[\frac{d^l}{dc^l}\left\{(E-E_s)\varphi\right\}\right]_s \varphi_s \, dy = 0 \\ \vdots \end{cases}$$

where $\frac{d^l}{dc^l} = \left(\frac{\partial}{\partial c} + \frac{dk^2}{dc}\frac{\partial}{\partial k^2}\right)^l$, and the derivatives of k^2 with respect to c at $c=c_s$ are given by eq.(2.5).
By introducing eq.(2.5) into eq.(2.1), the solution φ can be written as:

$$(2.9) \quad \varphi = \varphi_o + \varphi_1(c-c_s) + \dots \frac{1}{l!}\varphi_l(c-c_s)^l + \dots ,$$

$$\text{where} \quad \varphi_l = \left(\frac{d^l\varphi}{dc^l}\right)_s$$

From the above it follows that this series is valid in some region $|c-c_s| < \rho_3$ for all $y \in L$.

From the first of the eqs.(2.8) we find that we have to know φ_o in order to find k_1. But $\varphi_o = A_o\varphi_s$, where φ_s is known. We find that k_1 is independent of the value of A_o. A_o must of course not be equal to zero. In the expression for k_1, we may therefore put $A_o=1$, which is done. To find k_l we have to know $\varphi_o, \dots ,\varphi_{l-1}$ and $k_1, \dots ,k_{l-1}$. We observe that k_l does not depend on φ_l, which follows from the fact that

$$\left[\frac{d^l}{dc^l}\left\{(E-E_s)\varphi\right\}\right]_s = \left[(E-E_s)\frac{d^l}{dc^l}\varphi\right]_s + l\left[\frac{d^{l-1}}{dc^{l-1}}\varphi\frac{d}{dc}(E-E_s)\right]_s + \dots$$

$$+\left[\varphi\frac{d^l}{dc^l}(E-E_s)\right]_s$$, where the first term on the right hand side of this expression is equal to zero.
The equation for φ_l is obtained by differentiating eq(1.1) l times with respect to c. We write:

$$
(2.10)\quad \begin{cases} \varphi_l'' + \left\{\dfrac{\beta g}{(U-c_s)^2} - \dfrac{U''}{U-c_s} - k_s^2\right\}\varphi_l = \\ \qquad\qquad\qquad\qquad - \left[\dfrac{d^l}{dc^l}\left\{(E-E_s)\varphi\right\}\right]_s , \\ \varphi_l = 0 \quad \text{for} \quad y = y_1, y_2 . \end{cases}
$$

We observe that the expression on the right hand side of eq.(2.10) is known if $\varphi_o, \ldots, \varphi_{l-1}$ and $k_1, \ldots, k_l$ are known.

The homogeneous equation corresponding to eq.(2.10) has the two linearly independent solutions φ_s and θ_s. $\theta_s \neq 0$ for $y = y_1, y_2$, since $\varphi_s = 0$ for $y = y_1, y_2$. The general solution of eq.(2.10) is easily found by the method of variation of parameters:

$$
\varphi_l = A_l\varphi_s + B_l\theta_s + \varphi_s \int_{y_1}^{y} \frac{J_l\theta_s}{W}\, dt + \theta_s \int_{y}^{y_2} \frac{J_l\varphi_s}{W}\, dt ,
$$

where $J_l = -\left[\frac{d^l}{dc^l}\left\{(E-E_s)\varphi\right\}\right]_s$. The integration is along the contour L. A_l and B_l are constants, and $W = \varphi_s'\theta_s - \varphi_s\theta_s'$ is the Wronskian which is a constant in this case because φ' does not appear in eq.(2.10). φ_l satisfies the boundary conditions if $B_l = 0$, because $\int_{y_1}^{y_2} J_l\varphi_s\, dt = -\int_{y_1}^{y_2} \left[\frac{d^l}{dc^l}\left\{(E-E_s)\varphi\right\}\right]_s \varphi_s dt = 0$, where the integration is along L. This follows from (2.8). φ_l which satisfies the boundary conditions, is therefore :

$$(2.11) \qquad \varphi_l = A_l\varphi_s + \varphi_s \int_{y_1}^{y} \frac{J_l\theta_s}{W}\,dt + \theta_s \int_{y}^{y_2} \frac{J_l\varphi_s}{W}\,dt \ ,$$

where the integration is along L .

We see that if $\varphi_o, \ldots ,\varphi_{l-1}$ and $k_1, \ldots ,k_l$ are known, φ_l can be determined except for the constant A_l .

It has been mentioned previously that k_1 is independent of the value of A_o , except that A_o shall not be equal to zero. From the second of the equations in (2.8) it is easily found that k_2 is independent of both A_o ($A_o \neq 0$) and A_1 . We may therefore put $A_o = 1$ and $A_1 = 0$ when calculating k_1 , and this is done. Generally k_l must be independent of A_o ($A_o \neq 0$), $A_1, \ldots, A_{l-1}$. This is equivalent to saying that the value of $(k^2 - k_s^2)$ for a given c close to c_s is independent of the choice of the constants A_o ($A_o \neq 0$), A_l $l = 1,2,\ldots$. Let us show this. Let ψ_1 be the solution given by eq.(2.9) when the constants are chosen to be $A_o = C_o \neq 0$, $A_l = C_l$ $l = 1,2,\ldots$. Let ψ_2 be the solution when the constants are $A_o = D_o \neq 0$, $A_l = D_l$ $l = 1,2,\ldots$. The wave number for a given c close to c_s which corresponds to ψ_1 and ψ_2 is κ_1 and κ_2 respectively. ψ_1 and ψ_2 satisfy the equation:

$$\varphi'' + E_s\varphi = - (E - E_s)\varphi \ ,$$

and the boundary conditions eq.(1.2). Note that in the expression for E we have to put $k^2 = \kappa_1^2$ when $\varphi = \psi_1$, and $k^2 = \kappa_2^2$ when $\varphi = \psi_2$. By using the equation for ψ_1 , the equation for ψ_2 and the boundary conditions, it is easily obtained that $(\kappa_1^2 - \kappa_2^2)\int_L \psi_1\psi_2\,dy = 0$. We have assumed previously that $\int_L \varphi_s^2\,dy \neq 0$, from which it follows that $\int_L \psi_1\psi_2\,dy \neq 0$ in some region close to c_s . But then it follows that $\kappa_1^2 = \kappa_2^2$ in that

region, which means that the series for $(k^2 - k_s^2)$ is independent of the choice of the constants, except that $A_o \neq 0$.

ψ_1 and ψ_2 satisfy the same differential equation and the boundary conditions eq.(1.2). The Wronskian $\psi_1\psi_2' - \psi_1'\psi_2$ is zero, and ψ_1 and ψ_2 are therefore linearly dependent, i.e. $\psi_1 = A(c)\psi_2$, where $A(c)$ is a function of c . This can also be shown directly by using the expressions for ψ_1 and ψ_2 , and then $A(c)$ is also found. This means that the solutions, eq.(2.9), which are obtained by different choices of the constants, are linearly dependent solutions.

Above we have shown that if there exists a solution φ of eq.(1.1) which satisfies the boundary conditions eq.(1.2), and which tends to φ_s given by eq.(1.3) when $c \to c_s$ and $k^2 \to k_s^2$, it must be given by the eq.(2.9) with $\varphi_o = \varphi_s$, φ_1 given by eq.(2.11) and $(k^2 - k_s^2)$ by eq.(2.5) .

Now, if there exists a singular neutral solution φ_s , there will always exist a solution φ close to φ_s with a c close to c_s and a k^2 close to k_s^2 which satisfies the eq.(1.1) and the boundary conditions eq.(1.2). This solution φ tends to φ_s and k^2 tends to k_s^2 when $c \to c_s$. This follows from the fact that the solutions of eq.(1.1) are analytic functions of $c \in \{c \mid |c-c_s| < \rho\}$ and of k^2 especially for $k^2 \in \{k^2 \mid |k^2 - k_s^2| < \gamma\}$ for all $y \in L$. From the analysis above it follows that φ is given by the eq.(2.9) with $\varphi_o = \varphi_s$, φ_l ($l = 1,2,\ldots$) given by the eq.(2.11) and $(k^2 - k_s^2)$ given by eq.(2.5) . We see that both φ_1 and k_1 are given when φ_s and θ_s are known, so that φ and $(k^2 - k_s^2)$ can be found.

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Above we have shown that if there exists a solution φ of eq.(1.1) which satisfies the boundary conditions eq.(1.2), and which tends to φ_s given by eq.(1.3) when $c \to c_s$ and $\kappa^2 \to \kappa_s^2$, it must be given by the eq.(2.9) with $\varphi_o = \varphi_s$; φ_1 given by eq.(2.11) and $(\kappa^2 - \kappa_s^2)$ by eq.(2.5).

Now, if there exists a singular neutral solution φ_s, there will always exist a solution φ close to φ_s with a c close to c_s and a κ^2 close to κ_s^2 which satisfies the eq.(1.1) and the boundary conditions eq.(1.2). This solution φ tends to φ_s and κ^2 tends to κ_s^2 when $c \to c_s$. This follows from the fact that the solutions of eq.(1.1) are analytic functions of $c \in \{\delta \mid |c - c_s| < \rho\}$ and of κ^2 especially for

$\kappa^2 \in \{\kappa^2 \mid |\kappa^2 - \kappa_s^2| < \gamma\}$ for all $y \in I$. From the analysis above it follows that φ is given by the eq.(2.9) with $\varphi_o = \varphi_s$, φ_i ($i = 1, 2, \dots$) given by the eq.(2.11) and $(\kappa^2 - \kappa_s^2)$ given by eq.(2.5). We see that both φ_i and κ_i are given when φ_s and c_s are known, so that φ and $(\kappa^2 - \kappa_s^2)$ can be found.

It is important to be aware of the following. φ given by eq.(2.9) is valid in some region $|c - c_s| < \rho_3$ for all $y \in L$. However, it is only the solution with $c_i > 0$ for real values of $(k^2 - k_s^2)$ which is relevant to the stability problem of shear flows in stratified, incompressible and inviscid fluids. This unstable solution with $\varphi_o = \varphi_s$ tends, when $c \to c_s$, $c_i \to 0^+$, to the singular neutral solution φ_s defined in eq.(1.3), where $\arg(U - c_s) = 0$ when $(U - c_s) > 0$, and $\arg(U - c_s) = -\pi$ when $(U - c_s) < 0$. The solution with $c_i < 0$ for real values of $(k^2 - k_s^2)$ which is obtained from eq.(2.9), has <u>no relevance</u> to our stability problem. This solution with $\varphi_o = \varphi_s$ will also tend, when $c \to c_s$, $c_i \to 0^-$, to φ_s given by eq.(1.3), with the definition of $\arg(U - c_s)$ given above. The stable solution $(c_i < 0)$ which has relevance to our stability problem, is the one which is obtained by taking the complex conjugate of the unstable solution, and this stable solution will tend, when $c \to c_s$, $c_i \to 0^-$, to $(U - c_s)^{\frac{1}{2}+\mu}Y_s$ where $\arg(U - c_s) = 0$ when $(U - c_s) > 0$, and $\arg(U - c_s) = \pi$ when $(U - c_s) < 0$.

From eq.(2.5) we find for what real values of k^2 close to k_s^2 there is instability. It is for those real values which make $c_i > 0$.

Note that φ_s and θ_s in general have singularities at $c = c_s$. Then also $\varphi_o, \ldots, \varphi_1, \ldots$ have singularities

at $c = c_s$, Let L_r be a contour of the same kind as L , but with the radius r of the small semicircle instead of ρ . $\varphi_0 \dots \varphi_1 \dots$ will be analytic on L_r for every r such that $0 < r \leq \rho$, and for the integrals in eq. (2.8) we therefore have :

$$\int_L (\dots)dy = \lim_{r \to 0} \int_{L_r} (\dots)dy$$

We may use this when evaluating the constants k_1 , ... k_l ... .

III An example.

In this section we will use k_1 and k_2 , and let us therefore write out the explicit expressions for k_1 and k_2 . From (2.8) we get :

$$(3.1) \qquad k_1 = \frac{\lim\limits_{\rho \to 0} \int_L \left\{ \frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} \right\} \varphi_s^2 dy}{\lim\limits_{\rho \to 0} \int_L \varphi_s^2 dy} ,$$

which is the inverse of the expression for $\left(\frac{\partial c}{\partial k^2} \right)_{R_s}$ obtained in [1] .

$$(3.2) \quad k_2 = \frac{\lim\limits_{\rho\to 0} \int\limits_L \left\{ \frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1 \right\} \varphi_1 \varphi_s dy}{\lim\limits_{\rho\to 0} \int\limits_L \varphi_s^2 \, dy} +$$

$$+ \frac{\lim\limits_{\rho\to 0} \int\limits_L \left\{ \frac{3\beta g}{(U-c_s)^4} - \frac{U''}{(U-c_s)^3} \right\} \varphi_s^2 \, dy}{\lim\limits_{\rho\to 0} \int\limits_L \varphi_s^2 \, dy} ,$$

where

$$(3.3) \quad \varphi_1 = - \frac{\varphi_s}{W} \int\limits_{y_1}^{y} \left\{ \frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1 \right\} \varphi_s \theta_s dy -$$

$$- \frac{\theta_s}{W} \int\limits_{y}^{y_2} \left\{ \frac{2\beta g}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2} - k_1 \right\} \varphi_s^2 \, dy \quad ,$$

where the integration is along L .

Let us consider the case :

$$(3.4) \quad U = y \ , \ \beta g = Qy^2 + R_o \ , \text{ where } \ Q \geqq 0 \ \text{ and } \ R_o \geqq 0 \ .$$

The horizontal rigid planes are at $y_1 = -1$ and $y_2 = 1$. This case has been studied in [1], where

we have found that there may exist singular neutral solutions with $c_s = 0$ when $R_o \leq \frac{1}{4}$. These singular neutral solutions are:

$$(3.5) \quad \varphi_s = y^{\frac{1}{2}} J_\mu(\lambda_{j,\mu} y) \quad j = 1,2,\ldots,n \text{ , where}$$

$|\mu| = (\frac{1}{4} - R_o)^{\frac{1}{2}}$ and $|\mu| \in [0,\frac{1}{2}]$. $\lambda_{j,\mu}$ is the j^{th} zero of the Besselfunction J_μ . The wave number corresponding to the j^{th} singular neutral solution is :

$$(3.6) \quad k^2_{j,\mu} = Q - \lambda^2_{j,\mu} .$$

The number of solutions is given by the number n which satisfies $Q - \lambda^2_{n,\mu} \geq 0$, but $Q - \lambda^2_{n+1,\mu} < 0$.

When $\mu = -\frac{1}{2}$, φ_s is proportional to $\cos \lambda_{j,-\frac{1}{2}} = \cos \left(\frac{2j-1}{2}\pi\right) y$, and when $\mu = \frac{1}{2}$, φ_s is proportional to $\sin \lambda_{j,\frac{1}{2}} y = \sin (j\pi) y$.

In the following we will discuss the case when $Q = 15$. In this case $\pi^2 < Q < (\frac{3}{2}\pi)^2$, which means that the number n in the eq.(3.5) is equal to 1 , and that $|\mu| \in [0,\frac{1}{2}]$.

Let us consider the cases $|\mu| \in (0,\frac{1}{2})$, $\mu = -\frac{1}{2}$, $\mu = \frac{1}{2}$.

1) When $|\mu| \in (0,\frac{1}{2})$, the singular neutral solution is

$$(3,7)\quad \begin{cases} \varphi_s = y^{\frac{1}{2}} J_\mu(\lambda_{1,\mu} y) \text{ , and the corresponding wave} \\ \text{number is:} \\ k^2_{1,\mu} = Q - \lambda^2_{1,\mu} \quad (Q = 15). \end{cases}$$

The function θ_s defined in section II is :

$$(3.8)\quad \theta_s = y^{\frac{1}{2}} J_{-\mu}(\lambda_{1,\mu} y) .$$

By introducing eq.(3.4), eq.(3.7) and $c_s = 0$ into eq.(3.1) we get:

$$(3.9)\quad k_1 = -\, i \,\text{cotan}\, \pi\mu \;\frac{\text{Pf.} \int_0^1 (Qy^2 + R_o) y^{-2} J^2_\mu dy}{\int_0^1 y J^2_\mu \, dy} ,$$

where Pf. in front of the integral sign means the finite part.

We see from eq.(3.9) that k_1 is purely imaginary. In [1] we have shown that the integrals in eq.(3.9) are positive when $|\mu| \in (0,\frac{1}{2})$, so that k_1 changes sign with cotan $\pi\mu$. Taking into account the expression for k_1 , we get from eq.(2.5) that there is instability $(c_i > 0)$ for $k > k_{1,\mu}$ when $\mu \in (0,\frac{1}{2})$, and for $k < k_{1,\mu}$ when $\mu \in (-\frac{1}{2},0), (k \geq 0)$. Both φ_s and θ_s are known, and by using the formulae in section II we can calculate the unstable solution for a given k in the vicinity of $k_{1,\mu}$. From eq.(2.5) we can find c which corresponds to a given k near $k_{1,\mu}$.

2) When $\mu = -\frac{1}{2}$, the singular neutral solution is:

(3.10) $\varphi_s = \cos \frac{\pi}{2} y$, and $k^2_{1,-\frac{1}{2}} = Q - (\frac{\pi}{2})^2$ $(Q = 15)$.

$\theta_s = \sin \frac{\pi}{2} y$.

By introducing eq.(3.10) together with eq.(3.4) and $c_s = 0$, into eq.(3.1), we get: $k_1 = 2i\pi Q$, which together with the eq.(2.5) yields instability for $k < k_{1,-\frac{1}{2}}$. Again the unstable solution for a given k near $k_{1,-\frac{1}{2}}$ can be calculated by the formulae in section II .

3) When $\mu = \frac{1}{2}$, the singular neutral solution is :

(3.11) $\varphi_s = \sin \pi y$, and $k^2_{1,\frac{1}{2}} = Q - \pi^2$ $(Q = 15)$,

$\theta_s = \cos \pi y$.

We find that $k_1 = 0$ in this case. From eq.(3.2) we get:

$$(3.12) \quad k_2 = \frac{\lim\limits_{\rho\to 0}\left[\int_L \frac{2Q}{y}\varphi_1\varphi_s dy + \int_L \frac{3Q}{y^2}\varphi_s^2 dy\right]}{\lim\limits_{\rho\to 0}\int_L \varphi_s^2 dy} ,$$

where φ_s is given by eq.(3.11) and φ_1 by eq.(3.3), i.e.:

$$(3.13)\quad \varphi_1 = -\frac{Q}{\pi}\sin\pi y\int_{-1}^{y}\frac{\sin 2\pi t}{t}dt - \frac{Q}{\pi}\cos\pi y\int_{y}^{1}\frac{1-\cos 2\pi t}{t}dt \ .$$

Introducing eq.(3.11) and eq.(3.13) into eq.(3.12), we get:

$$(3.14)\quad k_2 = 6Q\pi\int_0^1\frac{\sin 2\pi t}{t}dt - \frac{2Q^2}{\pi}\int_0^1\frac{1-\cos 2\pi y}{y}dy\int_0^y\frac{\sin 2\pi t}{t}dt +$$

$$+ \frac{2Q^2}{\pi}\int_0^1 \frac{\sin 2\pi y}{y}dy\int_0^y\frac{1-\cos 2\pi t}{t}dt -$$

$$- \frac{2Q^2}{\pi}\int_0^1\frac{\sin 2\pi y}{y}dy\int_0^1\frac{1-\cos 2\pi t}{t}dt ,$$

where we have used that

$$\int_0^1\frac{(\sin\pi t)^2}{t^2}dt = \pi\int_0^1\frac{\sin 2\pi t}{t}dt \ .$$

In the case $Q = 15$, $\varphi_s = \sin\pi y$ is the only neutral solution with $c_s = 0$ when $\mu = \frac{1}{2}$. In the general case when the value of Q is such that $\sin n\pi y$ is a neutral solution, we also find that $k_1 = 0$, and that k_2 is given by :

$$(3.15)\quad k_2 = 6Qn\pi \int_0^1 \frac{\sin 2n\pi t}{t}dt - \frac{2Q^2}{n\pi}\int_0^1 \frac{1-\cos 2n\pi y}{y}dy \int_0^y \frac{\sin 2n\pi t}{t}dt +$$

$$+ \frac{2Q^2}{n\pi}\int_0^1 \frac{\sin 2n\pi y}{y}dy \int_0^y \frac{1-\cos 2n\pi t}{t}\, dt -$$

$$- \frac{2Q^2}{n\pi}\int_0^1 \frac{\sin 2n\pi y}{y}dy \int_0^1 \frac{1-\cos 2n\pi t}{t}dt \;.$$

Using the result from Appendix I, we find that this expression for k_2 is equivalent to the one found in [3] by a less general method.

When $Q = 15$ we have shown in Appendix II that k_2 given by eq.(3.14) is negative. From eq.(2.5) we find that $(k^2 - k^2_{1,\frac{1}{2}}) = k_2 c^2 + \cdots$, and we see that there is instability for $k > k_{1,\frac{1}{2}}$. Again the unstable solution for a given k near $k_{1,\frac{1}{2}}$ can be found since φ_s and θ_s are known.

The case $\mu = 0$ remains. In this case $\varphi_s = y^{\frac{1}{2}} J_o(\lambda_{1,o} y)$ and $k^2_{1,o} = Q - \lambda^2_{1,o}$ $(Q = 15)$. We have shown in [1] that $\left(\frac{\partial c}{\partial k^2}\right)_{R_o}$, which is equal to k_1^{-1}, is equal to zero in this case. We have also shown that $\left(\frac{\partial c_i}{\partial R}\right)_{k_{1,o}} < 0$, so that there is instability

for $R < \frac{1}{4}$. The curve in the $k - R_o$ - plane on which $c_s = 0$, has a maximum $R_o = \frac{1}{4}$ at $k = k_{1,o}$, see [1] , and therefore we should expand φ and c in a series of powers of $(R - \frac{1}{4})$ keeping $k = k_{1,o}$ fixed in order to find an unstable solution close to this point on the curve. We would not find any unstable solution by expanding φ and $(k^2 - k^2_{1,o})$ in a series of powers of c , keeping $R = \frac{1}{4}$ fixed, which is the method used in this paper.

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Appendix I.

We will show that:

$$\text{(A1.1)}\quad \begin{cases} \frac{2}{n\pi}\left[\int_0^1 \frac{1-\cos 2n\pi y}{y}\,dy \int_0^y \frac{\sin 2n\pi t}{t}\,dt \right. \\ \qquad \left. - \int_0^1 \frac{\sin 2n\pi y}{y}\,dy \int_0^y \frac{1-\cos 2n\pi t}{t}\,dt\right] \\ = \int_0^1 \cos 2n\pi t \log^2\left(\frac{t}{1-t}\right)dt \,. \end{cases}$$

Proof.

It is easily found that:

$$\text{(A1.2)}\quad \begin{cases} \int_0^1 \frac{1-\cos 2n\pi y}{y}\,dy \int_0^y \frac{\sin 2n\pi t}{t}\,dt \\ \qquad - \int_0^1 \frac{\sin 2n\pi y}{y}\,dy \int_0^y \frac{1-\cos 2n\pi t}{t}\,dt \\ = 2n\pi\left[\int_0^1 \cos 2n\pi y \log^2 y\,dy - \int_0^1 \log^2 y\,dy\right] \\ + (2n\pi)^2\left[\int_0^1 \sin 2n\pi y \log y\,dy \int_0^y \cos 2n\pi t \log t\,dt \right. \\ \qquad \left. - \int_0^1 \cos 2n\pi y \log y\,dy \int_0^y \sin 2n\pi t \log t\,dt\right] . \end{cases}$$

Put:

$$\text{(A1.3)}\quad \begin{cases} I(\alpha) = \int_0^1 \sin\alpha y \log y \, dy \int_0^y \cos\alpha t \log t \, dt \\ \qquad - \int_0^1 \cos\alpha y \log y \, dy \int_0^y \sin\alpha t \log t \, dt \, . \end{cases}$$

The expression on the right hand side of eq.(A1.2) is then equal to:

$$2n\pi \left[\int_0^1 \cos 2n\pi y \log^2 y \, dy - \int_0^1 \log^2 y \, dy \right] + (2n\pi)^2 I(2n\pi) \, .$$

From eq.(A1.3) it follows that:

$$\text{(A1.4)}\qquad I(0) = 0 \, .$$

We differentiate eq.(A1.3) with respect to α, and find that:

$$\text{(A1.5)}\quad \begin{cases} \frac{dI}{d\alpha} + \frac{2}{\alpha} I = \frac{1}{\alpha^2} \int_0^1 \cos\alpha y \log y \, dy - \frac{2}{\alpha^2} \int_0^1 \log y \, dy \\ \qquad + \frac{1}{\alpha^2} \int_0^1 \cos\alpha(1-y) \log y \, dy . \end{cases}$$

The solution of this equation which satisfies the condition eq.(A1.4), is:

$$\text{(A1.6)}\qquad I(\alpha) = \frac{1}{\alpha^2} \left[-2\alpha \int_0^1 \log y \, dy + \int_0^1 \frac{\sin\alpha y}{y} \log y \, dy + \int_0^1 \frac{\sin\alpha(1-y)}{1-y} \log y \, dy \right] .$$

Now:

$$\int_0^1 \frac{\sin 2n\pi(1-y)}{1-y}\log y \, dy = \int_0^1 \frac{\sin 2n\pi(1-y)}{y}\log(1-y)dy -$$

$$- 2n\pi \int_0^1 \cos 2n\pi(1-y)\log y \log(1-y)dy =$$

$$= -\int_0^1 \frac{\sin 2n\pi(1-y)}{1-y}\log y \, dy - 2n\pi \int_0^1 \cos 2n\pi y \log y \log(1-y)dy,$$

from which it follows that:

$$\text{(A1.7)} \quad \int_0^1 \frac{\sin 2n\pi(1-y)}{1-y}\log y \, dy = - n\pi \int_0^1 \cos 2n\pi y \log y \log(1-y)dy.$$

Further:

$$\text{(A1.8)} \quad \int_0^1 \frac{\sin \alpha y}{y} \log y \, dy = - \frac{\alpha}{2} \int_0^1 \cos \alpha y \log^2 y \, dy ,$$

$$\text{and} \quad \int_0^1 \log y \, dy = - 1.$$

Taking into account eq.(A1.7) and eq.(A1.8), we find that:

$$\text{(A1.9)} \begin{cases} I(2n\pi) = \dfrac{1}{(2n\pi)^2}\Big[4n\pi - n\pi \displaystyle\int_0^1 \cos 2n\pi y \log^2 y \, dy - \\ \qquad - n\pi \displaystyle\int_0^1 \cos 2n\pi y \log y \log(1-y)dy\Big] . \end{cases}$$

Now:

$$(A1.10)\quad \begin{cases} \int_0^1 \cos 2n\pi y \, \log^2 y \, dy = \int_0^1 \cos 2n\pi y \, \log^2(1-y)\,dy, \\ \qquad\qquad\qquad\qquad \text{and} \quad \int_0^1 \log^2 y \, dy = 2 . \end{cases}$$

Introducing eq.(A1.9) into eq.(A1.2) and using eq.(A1.10), we get:

$$\int_0^1 \frac{1-\cos 2n\pi y}{y}\,dy \int_0^y \frac{\sin 2n\pi t}{t}\,dt - \int_0^1 \frac{\sin 2n\pi y}{y}\,dy \int_0^y \frac{1-\cos 2n\pi t}{t}\,dt =$$

$$\frac{1}{2}\,n\pi \int_0^1 \cos 2n\pi y \left[\log^2 y + \log^2(1-y) - 2\log y \,\log(1-y)\right] dy ,$$

which is equivalent to (A1.1).

Appendix II.

By using the result from Appendix I, k_2 given by eq.(3.14) can be written as:

$$(A2.1)\qquad k_2 = 6Q\pi \,\mathrm{Si}(2\pi) - Q^2 \int_0^1 \cos 2\pi t \,\log^2\left(\frac{t}{1-t}\right)dt -$$

$$- \frac{2Q^2}{\pi}\,\mathrm{Si}(2\pi)\,\mathrm{Cin}(2\pi),$$

where $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$, $\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,dt$.

It will be shown that k_2 given by eq.(A2.1) is negative. $Q = 15$. $\mathrm{Cin}(\pi x)$ is tabulated in [4], and we find that $\mathrm{Cin}(2\pi) = 2,44$ approximately. From this it follows that $6Q\pi\, \mathrm{Si}(2\pi) < \frac{2Q^2}{\pi}\, \mathrm{Si}(2\pi)\, \mathrm{Cin}(2\pi)$. Further:

$$\int_0^1 \cos 2\pi t \log^2\left(\frac{t}{1-t}\right) dt = 2\left[\int_0^{\frac{1}{4}} \cos 2\pi t \left\{\log^2\left(\frac{t}{1-t}\right) - \log^2\left(\frac{\frac{1}{2} - t}{\frac{1}{2} + t}\right)\right\} dt\right] > 0,$$

since $\log^2\left(\frac{t}{1-t}\right) \geq \log^2\left(\frac{\frac{1}{2} - t}{\frac{1}{2} + t}\right)$ when $t \in (0, \frac{1}{4}]$.

From the above it follows that $k_2 < 0$.

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