Department
of
APPLIED MATHEMATICS

The effects of trapped and untrapped particles on an electrostatic wave packet

by

Magne S. Espedal

Report No. 46 December 1973

UNIVERSITY OF BERGEN
Bergen, Norway

ISSN 0084-778x

The effects of trapped and untrapped particles on an electrostatic wave packet

by

Magne S. Espedal

Report No. 46 December 1973

Abstract.

The propagation of an electrostativ wavepacket in a collissionless plasma is studied. We get a change in amplitude caused by interaction between the packet and particles propagating with velocities near to the group velocity. Also, we get modulation of the plasma in the front of the plasma caused by trapping effects.

Introduction.

In this paper we shall study the interaction between particles and an electrostatic wave packet. The evolution of a large amplitude wave packet, has been studied earlier by numerical simulation (J. Denavit and R.N. Sudan 1972), but a more complete theory has not been given. As in nonlinear optics and water-wave theory, we shall try to find a waveequation. In collisionless plasmas, the nonlinearity often comes from the trapping of particles in the potential troughs of the waves. Therefore we have to find a procedure which takes care of this effect.

I. The wave equation.

The equations to govern the onedimensional motion of collisionless plasmas are the Vlasov - Poisson equations:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f_j'(x,v,t) + \frac{e_j}{m_j} E'(x,t) \frac{\partial}{\partial v} f_j'(x,v,t) = 0 \qquad (1.1)$$

$$\frac{\partial}{\partial x} E'(x,t) = 4\pi \sum_j e_j \int f_j'(x,v,t)\, dv \qquad (1.2)$$

The suffix j denotes the species of plasma particles, representing $e_j = -e$, $m_j = m$ for electrons, and $e_j = e$, $m_j = M$ for the ions.

We shall solve eqs. (1.1 - 2) as an initialvalue problem, where:

$$f_j'(x,v,0) = f_0(v) + f_j(x,v,0)\, e^{i\chi(x,0)} \tag{1.3}$$

$$E'(x,0) = E(x,0)\, e^{i\chi(x,0)} \tag{1.4}$$

are given consistently.

In order to solve eqs. (1.1 - 4), we assume that:

$$f_j'(x,v,t) = f_0(v) + f_j(x,v,t)\, e^{i\chi(x,t)} \tag{1.5}$$

$$E'(x,t) = E(x,t)\, e^{i\chi(x,t)} \tag{1.6}$$

Further we define:

$$\frac{\partial}{\partial t}\chi(x,t) = -\omega(x,t) \tag{1.7}$$

$$\frac{\partial}{\partial x}\chi(x,t) = k(x,t) \tag{1.8}$$

Now, eqs. (1.1 - 8) gives:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f_j(x,v,t) + \frac{e_j}{m_j} E(x,t)\, e^{i\chi(x,t)} \frac{\partial}{\partial v} f_j(x,v,t) = i(\omega - kv)\, f_j(x,v,t) - \frac{e_j}{m_j} E(x,t) \frac{\partial f_0}{\partial v} \tag{1.9}$$

$$\left(\frac{\partial}{\partial x} + ik\right) E(x,t) = 4\pi \sum_j e_j \int f_j(x,v,t)\, dv \qquad (1.10)$$

Integrating eq. (1.9) along the characteristics we get:

$$\frac{dt}{d\tau} = 1$$

$$\frac{dx_j}{d\tau} = v_j \qquad (1.11)$$

$$\frac{dv_j}{d\tau} = \frac{e_j}{m_j} E e^{i\chi}$$

$$\frac{df_j}{d\tau} = i(\omega - kv_j) f_j - \frac{e_j}{m_j} E \frac{\partial f_0}{\partial v} \qquad (1.12)$$

We shall solve eq. (1.11) with the following conditions:

$$X_j(\tau = t) = x$$

$$V_j(\tau = t) = v \qquad (1.13)$$

which gives:

$$f_j(\tau) = f_j(x,v,t)\, e^{i\int_t^\tau (\omega - kv_j)\, d\tau} + $$

$$- e^{i\int_t^\tau (\omega - kv_j)\, d\tau} \frac{e_j}{m_j} \int_t^\tau \left(E(\tau)\, e^{-i\int_t^\tau (\omega - kv_j)\, ds} \frac{\partial f_0}{\partial v_j} \right) d\tau \qquad (1.14)$$

Now, eqs. (1.10) and (1.14) combine to.

$$\frac{\partial E}{\partial x} + ikE = 4\pi \sum_j e_j \left(\int f_j(\tau=0)\, e^{i\int_0^t (\omega - kv_j)\, d\tau} dv \right) +$$

$$- \sum_j \frac{4\pi e_j^2}{m_j} \left(\int dv \int_0^t E(\tau) \frac{\partial f_0}{\partial v_j} e^{-i\int_t^\tau (\omega - kv_j)\, ds} d\tau \right) \tag{1.15}$$

We assume that $f_j(x, v, t)$ exists in the sense that the right hand side of eq. (1.15) is finite.

Integrating the last integral in eq. (1.15) by parts, we get:

$$\frac{\partial E}{\partial x} + ikE = 4\pi \sum_j e_j \int f_j(\tau=0)\, e^{i\int_0^t (\omega - kv_j)\, d\tau} dv +$$

$$- \sum_j \frac{4\pi e_j^2}{m_j} \int dv \Bigg[i \left(\frac{E}{\omega - kv_j} \frac{\partial f_0}{\partial v_j} e^{-i\int_t^\tau (\omega - kv_j)\, ds} \right) \Bigg|_0^t +$$

$$+ \left(\frac{1}{\omega - kv_j} \frac{d}{d\tau} \left(\frac{E}{\omega - kv_j} \frac{\partial f_0}{\partial v_j} \right) e^{-i\int_t^\tau (\omega - kv_j)\, ds} \right) \Bigg|_0^t - \tag{1.16}$$

$$- i \left(\frac{1}{\omega - kv_j} \frac{d}{d\tau} \left(\frac{1}{\omega - kv_j} \frac{d}{d\tau} \left(\frac{E}{\omega - kv_j} \frac{\partial f_0}{\partial v_j} \right) \right) e^{-i\int_t^\tau (\omega - kv_j)\, ds} \right) \Bigg|_0^t$$

$$+ i \int_0^t d\tau \left\{ \frac{d}{d\tau} \left(\frac{1}{\omega - kv_j} \frac{d}{d\tau} \left(\frac{1}{\omega - kv_j} \frac{d}{d\tau} \left(\frac{E}{\omega - kv_j} \frac{\partial f_0}{\partial v_j} \right) \right) \right) e^{-i\int_t^\tau (\omega - kv_j)\, ds} \right\} \Bigg]$$

Eq. (1.16) is rather complicated, so we want to write it in a more attractive form.

We define:

$$\mathcal{E}(\omega, K) = K + \sum_j \frac{4\pi e_j^2}{m_j} \int \frac{\frac{\partial f_0}{\partial v}}{\omega - Kv} \, dv \tag{1.17}$$

$$v_g(x,t) = - \frac{\partial \mathcal{E}}{\partial K} \Big/ \frac{\partial \mathcal{E}}{\partial \omega} \tag{1.18}$$

$$v_D(x,t) = \left(\frac{\partial \mathcal{E}}{\partial \omega} \right)^{-1} \tag{1.19}$$

$$\Gamma(\omega, K) = \mathcal{E}(\omega, K) - K \tag{1.20}$$

$$\begin{aligned}
\Omega(x,t) = {} & \frac{1}{2}\left(\frac{\partial}{\partial t}\frac{\partial \Gamma}{\partial \omega} - \frac{\partial}{\partial x}\frac{\partial \Gamma}{\partial K} \right) + \\
& + \sum_j \frac{4\pi e_j^2}{m_j} \Big[i \int \frac{\frac{\partial f_0}{\partial v}}{\omega - Kv} \frac{d}{dt}\left(\frac{1}{\omega - Kv} \frac{d}{dt}\left(\frac{1}{\omega - Kv} \right)\right) dv + \\
& - \frac{e_j}{m_j} e^{i\phi(x,t)} \int \Big[\frac{1}{\omega - Kv} \frac{d}{dv}\left(\frac{\frac{\partial f_0}{\partial v}}{\omega - Kv} \frac{d}{dt}\left(\frac{E}{\omega - Kv} \right)\right) + \\
& \qquad \frac{1}{\omega - Kv} \frac{d}{dt}\left(\frac{E}{\omega - Kv} \frac{d}{dv}\left(\frac{\frac{\partial f_0}{\partial v}}{\omega - Kv} \right)\right)\Big] dv + \\
& - i\left(\frac{e_j}{m_j}\right)^2 (E)^2 e^{i2\phi(x,t)} \int \frac{dv}{\omega - Kv} \frac{d}{dv}\left(\frac{1}{\omega - Kv} \frac{d}{dv}\left(\frac{\frac{\partial f_0}{\partial v}}{\omega - Kv} \right)\right) \Big]
\end{aligned} \tag{1.21}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$; and

$$\chi(\tau = t) = \varphi(x,t) + \frac{\pi}{2} \tag{1.22}$$

Using eqs. (1.17 - 22), eq. (1.16) reduces to:

$$\frac{\partial E}{\partial t} + v_g \frac{\partial E}{\partial x} - i v_D\, \varepsilon(\omega,k) E - v_D \Omega E +$$

$$-i v_D \frac{1}{2}\left(\frac{\partial}{\partial t}\left(\frac{\partial^2 \Gamma}{\partial \omega^2}\frac{\partial}{\partial t}E - \frac{\partial^2 \Gamma}{\partial \omega \partial k}\frac{\partial}{\partial x}E\right) + \frac{\partial}{\partial x}\left(-\frac{\partial^2 \Gamma}{\partial \omega \partial k}\frac{\partial}{\partial t}E + \frac{\partial^2 \Gamma}{\partial k^2}\frac{\partial}{\partial x}E\right)\right) = \tag{1.23}$$

$$v_D \sum_{i=1}^{4} I_i(x,t) + v_D I_5(E,x,t)$$

where:

$$I_1 = -4\pi \sum_j \int f_j(\tau = 0)\, e^{i\int_0^t (\omega - k v_j)\,ds}\, dv \tag{1.24}$$

$$I_2 = -i \sum_j \frac{4\pi e_j^2}{m_j} \int \left\{ E(\tau)\, \frac{\frac{\partial f_0}{\partial v_j}}{\omega - k v_j} \right\}_{\tau=0} e^{i\int_0^t (\omega - k v_j)\,ds}\, dv \tag{1.25}$$

$$I_3 = -\sum_j \frac{4\pi e_j^2}{m_j} \int \left\{ \frac{1}{\omega - k v_j} \frac{d}{d\tau}\left(\frac{E \frac{\partial f_0}{\partial v_j}}{\omega - k v_j} \right) \right\}_{\tau=0} e^{i\int_0^t (\omega - k v_j)\,ds}\, dv \tag{1.26}$$

$$I_4 = i \sum_j \frac{4\pi e_j^2}{m_j} \int \left\{ \frac{1}{\omega - k v_j} \frac{d}{d\tau}\left(\frac{1}{\omega - k v_j} \frac{d}{d\tau}\left(\frac{E \frac{\partial f_0}{\partial v_j}}{\omega - k v_j} \right)\right) \right\}_{\tau=0} e^{i\int_0^t (\omega - k v_j)\,ds}\, dv \tag{1.27}$$

$$I_5 = -i \sum_j \frac{4\pi e_j^2}{m_j} \int dv \int_0^t d\tau \left\{ \frac{d}{d\tau}\left(\frac{1}{\omega - k v_j} \frac{d}{d\tau}\left(\frac{1}{\omega - k v_j} \frac{d}{d\tau}\left(\frac{E \frac{\partial f_0}{\partial v_j}}{\omega - k v_j} \right)\right)\right) e^{-i\int_t^{\tau} (\omega - k v_j)\,ds} \right\} \tag{1.28}$$

The left hand side of eq. (1.23) is a nonlinear Schroedinger type of wave equation. But the I_5 term on the right hand side contains $E(x,t)$, so we still have a integrodifferential equation to solve.

II. The lowest order solution.

In order to solve eq. (1.23), we shall introduce the characteristic time and space scales connected to the problem.

The frequency and wavenumber of the waves making up the packet are given by $\omega(x,t)$ and $k(x,t)$, and define the fast time- and space-scales.

If ℓ is the characteristic length for the variation of the amplitude of the wave packet, we may define:

$$\varepsilon = \frac{1}{k\ell}$$

and we shall assume that

$$\varepsilon \ll 1 \tag{2.1}$$

Therefore we may define the slow space- and timescales by:

$$\begin{aligned} x_1 &= \varepsilon x \\ t_1 &= \varepsilon t \end{aligned} \tag{2.2}$$

Now, our basic assumtions are that the amplitude of the wave packet, the frequency and the wavenumber vary only on the slow time- and space-scales.

$$E(x,t) = E(x_1,t_1)$$
$$\omega(x,t) = \omega(x_1,t_1) \qquad (2.3)$$
$$k(x,t) = k(x_1,t_1)$$

There are two other characteristic timescales which enter into the problem:

$$\tau_p = \frac{2\ell}{|V_g - \frac{\omega}{k}|} \qquad (2.4)$$

$$\tau_{tr}^j = \frac{2\pi}{\omega_{Bj}} = \left(|\frac{e_j}{m_j} E k|\right)^{-\frac{1}{2}}$$

τ_p is the typical time which a particle with the velocity $\frac{\omega}{k}$ uses to get through the wave packet. We may note that if we have a very long wave packet, or a finite amplitude wave, ℓ should be taken as the damping or growth scale of the amplitude.

τ_{tr}^j is the oscillation time for the trapped particles, and it depends on the particle mass m_j:

$$\tau_{tr}^i = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \tau_{tr}^e \qquad (2.5)$$

This means that we have to distinguish between electron-waves and ion-waves.

We assume that

$$\tau_{tr}^e > \tau_p \qquad (2.6)$$

for electron-waves, and

$$\tau_{tr}^{i} > \tau_{p} \tag{2.7}$$

for ion-waves.

Eq. (2,5-6) means that the trapped electrons make less than one oscillation in the potential well, while the ions feel no trapping effects.

Eqs. (2.5) and (2.7) means that the ions make less than one oscillation in the potential well, in which the electrons may oscillate several times. However, in many cases the electrons behave as an ideal fluid and the electron trapping effects may be neglected.

The phase velocities, $\frac{\omega}{k}$, of the waves making up the wave packet, are given by:

$$V_m \leq \frac{\omega}{k} \leq V_M \tag{2.8}$$

and trapping effects will be important in the same range of the velocity-space.

Using eq. (2.3), we notice that $\Omega(x, t, x_1, t_1)$ is the only term on the left hand side of eq. (1.23) which depends on the fast time and space scales.

In order to eliminate this dependence, we integrate over the fast variables in the following way:

$$\bar{A}(x_1, t_1) = \frac{1}{2\pi} \int_0^{2\pi} A(\varphi(x, t, x_1, t_1), x_1, t_1)\, d\varphi \tag{2.9}$$

where φ is defined by eq. (1.22).

Eqs. (1.7-8), (1.11) and (1.22) gives

$$\chi(\tau,t) = \frac{\pi}{2} + \varphi(x,t) - \int_t^\tau (\omega - \kappa v_j)\, ds \qquad (2.10)$$

Writing eqs. (1.26-28) in a more explicit form, we have:

$$I_3(x_1,t_1,\varphi) = -\sum_j \frac{4\pi e_j^2}{m_j}\Big[\varepsilon\int dv \Big\{ \frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j} \frac{d}{d\varepsilon\tau}\Big(\frac{E}{\omega-\kappa v_j}\Big)\Big\}_{\tau=0} e^{i\int_0^t(\omega-\kappa v_j)ds} +$$

$$i e^{i\varphi}\int dv \Big\{\frac{E}{\omega-\kappa v_j}\frac{d}{dv_j}\Big(\frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j}\Big)\frac{e_j}{m_j}E\Big\}_{\tau=0} e^{2i\int_0^t(\omega-\kappa v_j)ds}\Big] \qquad (2.11)$$

$$I_4(x_1,t_1,\varphi) = i\sum_j \frac{4\pi e_j^2}{m_j}\Big[\varepsilon^2\int dv \Big\{\frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j}\frac{d}{d\varepsilon\tau}\Big(\frac{1}{\omega-\kappa v_j}\frac{d}{d\varepsilon\tau}\Big(\frac{E}{\omega-\kappa v_j}\Big)\Big)\Big\}_{\tau=0} e^{i\int_0^t(\omega-\kappa v_j)ds} +$$

$$+\varepsilon i e^{i\varphi}\Big(\int dv \Big\{\frac{1}{\omega-\kappa v_j}\frac{d}{dv_j}\Big(\frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j}\frac{d}{d\varepsilon\tau}\Big(\frac{E}{\omega-\kappa v_j}\Big)\Big)\frac{e_j}{m_j}E\Big\}_{\tau=0} e^{i2\int_0^t(\omega-\kappa v_j)ds} +$$

$$+\int dv\Big\{\frac{1}{\omega-\kappa v_j}\frac{d}{d\varepsilon\tau}\Big(\frac{E}{\omega-\kappa v_j}\frac{d}{dv_j}\Big(\frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j}\Big)\frac{e_j}{m_j}E\Big\}_{\tau=0} e^{i2\int_0^t(\omega-\kappa v_j)ds}\Big) + \qquad (2.12)$$

$$-e^{i2\varphi}\int dv\Big\{\frac{E}{\omega-\kappa v_j}\frac{d}{dv_j}\Big(\frac{1}{\omega-\kappa v_j}\frac{d}{dv_j}\Big(\frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j}\Big)\Big)\Big(\frac{e_j}{m_j}E\Big)^2\Big\}_{\tau=0} e^{i3\int_0^t(\omega-\kappa v_j)ds} +$$

$$+ e^{i\varphi}\int dv\Big\{\frac{E}{\omega-\kappa v_j}\frac{d}{dv_j}\Big(\frac{\frac{\partial f_0}{\partial v_j}}{\omega-\kappa v_j}\Big)\frac{e_j}{m_j}E\Big\}_{\tau=0} e^{i2\int_0^t(\omega-\kappa v_j)ds}\Big]$$

$$I_5(x_1,t_1,\varphi,E) = -i\sum_j \frac{4\pi e_j^2}{m_j}\int dv\Big[\varepsilon^3\int_0^t\Big\{\frac{d}{d\varepsilon\tau}\Big(\frac{\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{E}{\omega-kv_j}\Big)\Big)\Big)\cdot$$

$$e^{-i\int_t^\tau(\omega-kv_j)ds}\Big\}d\tau +$$

$$+i\varepsilon^2 e^{i\varphi}\int_0^t d\tau\, e^{-i2\int_t^\tau(\omega-kv_j)ds}\Big\{\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{E}{\omega-kv_j}\Big)\Big)\Big)\frac{e_j}{m_j}E +$$

$$+\frac{d}{d\varepsilon\tau}\Big(\frac{\frac{e_j}{m_j}E}{\omega-kv_j}\Big(\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)+\frac{d}{d\varepsilon\tau}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\Big)\Big)\Big\} +$$

$$-\varepsilon\Big(e^{i2\varphi}\int_0^t d\tau\, e^{-i3\int_t^\tau(\omega-kv_j)ds}\Big\{\frac{d}{d\varepsilon\tau}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\Big)\Big(\frac{e_j}{m_j}E\Big)^2\Big) +$$

$$+\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\Big[\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)+\frac{d}{d\varepsilon\tau}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\Big]\Big)\Big(\frac{e_j}{m_j}E\Big)^2\Big) + \qquad (2.13)$$

$$-e^{i\varphi}\int_0^t d\tau\, e^{-i2\int_t^\tau(\omega-kv_j)ds}\Big\{\frac{d}{d\varepsilon\tau}\Big(\frac{2}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\frac{e_j}{m_j}E +$$

$$+\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{d\varepsilon\tau}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\frac{e_j}{m_j}E\Big\}\Big) +$$

$$-i e^{i3\varphi}\int_0^t d\tau\, e^{-i4\int_t^\tau(\omega-kv_j)ds}\Big\{\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\Big)\Big(\frac{e_j}{m_j}E\Big)^3\Big\} +$$

$$+2i e^{i2\varphi}\int_0^t d\tau\, e^{-i3\int_t^\tau(\omega-kv_j)ds}\Big\{\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)\Big(\frac{e_j}{m_j}E\Big)^2\Big\} +$$

$$+e^{i\varphi}\int_0^t d\tau\, e^{-i2\int_t^\tau(\omega-kv_j)ds}\Big\{\frac{d}{dv_j}\Big(\frac{1}{\omega-kv_j}\,\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big)+i\frac{d}{dv_j}\Big(\frac{E\frac{\partial f_0}{\partial v_j}}{\omega-kv_j}\Big)\Big\}\Big(\frac{e_j}{m_j}E\Big)\Big]$$

Assuming that

$$\frac{\omega_{ir}^{j}}{\omega} = O(\varepsilon) \tag{2.14}$$

all the coefficients of $e^{in\varphi}$, n = 1. 2. 3., are slowly varying functions compared to $e^{in\varphi}$. One should note that the coefficients depend on φ through $v_j(\tau)$ which is periodic in φ. Therefore, taking the mean value of eq. (1.23) and eqs. (2.11-13), we get to $O(\varepsilon^3)$:

$$\varepsilon\left(\frac{\partial E}{\partial t_1} + v_g\frac{\partial E}{\partial x_1}\right) - i v_D \mathcal{E}(\omega,k) E - v_D \bar{\Omega} E + \tag{2.15}$$

$$-\frac{1}{2} i v_D \varepsilon^2\left(\frac{\partial}{\partial t_1}\left(\frac{\partial^2\Gamma}{\partial\omega^2}\frac{\partial}{\partial t_1}E - \frac{\partial^2\Gamma}{\partial\omega\partial k}\frac{\partial}{\partial x_1}E\right) + \frac{\partial}{\partial x_1}\left(-\frac{\partial^2\Gamma}{\partial\omega\partial k}\frac{\partial}{\partial t_1}E + \frac{\partial^2\Gamma}{\partial k^2}\frac{\partial}{\partial x_1}E\right)\right) =$$

$$\sum_{i=1}^{4} \bar{I}_i$$

where

$$\bar{\Omega} = \varepsilon \frac{1}{2}\left(\frac{\partial}{\partial t_1}\frac{\partial\Gamma}{\partial\omega} - \frac{\partial}{\partial x_1}\frac{\partial\Gamma}{\partial k}\right) + \varepsilon^2 i \sum_j \frac{4\pi e_j^2}{m_j}\left[\int \frac{\frac{\partial f_0}{\partial v}}{\omega - kv}\frac{d}{dt_1}\left(\frac{1}{\omega-kv}\frac{d}{dt_1}\left(\frac{1}{\omega-kv}\right)\right)dv \tag{2.16}$$

$$= \varepsilon\Omega_1 + \varepsilon^2\Omega_2$$

$$\bar{I}_1 = I_1(\varphi=0) \tag{2.17}$$

$$\bar{I}_2 = I_2(\varphi=0) \tag{2.18}$$

$$\bar{I}_3 = -\varepsilon \sum_j \frac{4\pi e_j^2}{m_j} \int \left[\left\{ \frac{1}{\omega - kv_j} \frac{d}{d\varepsilon\tau} \left(\frac{\frac{\partial f_0}{\partial v_j} E}{\omega - kv_j} \right) \right\}_{\tau=0} e^{i\int_0^t (\omega - kv_j)\,ds} dv \right]_{\varphi=0} \tag{2.19}$$

$$\bar{I}_4 = i\varepsilon^2 \sum_j \frac{4\pi e_j^2}{m_j} \int \left[\left\{ \frac{1}{\omega - kv_j} \frac{d}{d\varepsilon\tau} \left(\frac{1}{\omega - kv_j} \frac{d}{d\varepsilon\tau} \left(\frac{\frac{\partial f_0}{\partial v_0} E}{\omega - kv_j} \right) \right) \right\}_{\tau=0} e^{i\int_0^t (\omega - kv_j)\,ds} dv \right]_{\varphi=0} \tag{2.20}$$

In eqs.(2.16) and (2.20), we have neglected the self action term (Dysthe 1974), which gives an amplitude dependent frequency shift.

In order to solve eq.(2.8), we shall make the following assumptions:

$$R(\omega, k) = (T_{\omega k})^2 - T_{\omega\omega} T_{kk} > 0 \tag{2.21}$$

$$-\frac{T_{\omega k}}{T_{\omega\omega}} - \frac{\sqrt{R}}{T_{\omega\omega}} \le v_g \le -\frac{T_{\omega k}}{T_{\omega\omega}} + \frac{\sqrt{R}}{T_{\omega\omega}} \tag{2.21a}$$

With the condition (2.21), eq. (2.15) is a hyperbolic type of equation.

Eq. (2.21a) means that the subcharacteristics (J.D. Cole 1968) given by:

$$\frac{dt_1}{ds} = 1$$

$$\frac{dx_1}{ds} = v_g(s) \tag{2.22}$$

are timelike, and the initial value problem may be solved.

Furthermore, we divide the (x_1, t_1) space into two parts according to:

$$\frac{\partial}{\partial t_1} + v_g \frac{\partial}{\partial x_1} \gtrsim \frac{\omega}{\varepsilon} \tag{2.23}$$

$$\frac{\partial}{\partial t_1} + v_g \frac{\partial}{\partial x_1} < \frac{\omega}{\varepsilon} \tag{2.24}$$

The solution in the region (2.12), we shall call the outer solution, and the other one the inner solution.

III. The outer solution.

In this region it is natural to search for a solution in the form

$$E(x_1, t_1) = \sum_{n=0}^{\infty} \varepsilon^n E_n^{out}(x_1, t_1) \tag{3.1}$$

where

$$\begin{aligned} E_0^{out}(x_1, 0) &= E_0(x_1) \\ E_i^{out}(x_1, 0) &= 0 \qquad ; \qquad i = 1, 2, 3 \ldots \end{aligned} \tag{3.2}$$

Eqs. (2.15) and (3.1-2) give:

$$\left(\frac{\partial}{\partial t_1} + v_g \frac{\partial}{\partial x_1}\right) E_i^{out} - \frac{1}{\varepsilon} i v_D \mathcal{E}(\omega, k) E_i^{out} - v_D \Omega_1 E_i^{out} = \varepsilon^{-i-1} L_i(x_1, t_1) \tag{3.3}$$

where

$$L_0(x_1, t_1) = v_D \sum_{i=1}^{3} \overline{I}_i(x_1, t_1) \tag{3.4}$$

$$L_1(x_1,t_1) = V_D \bar{I}_4(x_1,t_1) + \varepsilon^2 V_D \Omega_2 E_0^{out} +$$

$$+ \tfrac{1}{2} i \varepsilon^2 V_D \Big(\frac{\partial}{\partial t_1}\big(T_{\omega\omega} \frac{\partial}{\partial t_1} - T_{\omega\kappa} \frac{\partial}{\partial x_1}\big) + \frac{\partial}{\partial x_1}\big(-T_{\omega\kappa} \frac{\partial}{\partial t_1} + T_{\kappa\kappa} \frac{\partial}{\partial x_1}\big)\Big) E_0^{out} \tag{3.5}$$

$$L_i(x_1,t_1) = \tfrac{1}{2} i \varepsilon V_D \Big(\frac{\partial}{\partial t_1}\big(T_{\omega\omega} \frac{\partial}{\partial t_1} - T_{\omega\kappa} \frac{\partial}{\partial x_1}\big) + \frac{\partial}{\partial x_1}\big(-T_{\omega\kappa} \frac{\partial}{\partial t_1} + T_{\kappa\kappa} \frac{\partial}{\partial x_1}\big)\Big) E_{i-1}^{out} +$$

$$+ \varepsilon^2 V_D \Omega_2 E_{i-1}^{out} + V_D \bar{I}_5 (E_{i-2}^{out}, x_1, t_1) \tag{3.6}$$

$i = 2, 3, 4 \ldots \ldots$

The solution of eq. (3.3) is given by:

$$E_i^{out}(x_1,t_1) = \exp\Big(i \frac{1}{\varepsilon} \int_0^{t_1} V_D(\mathcal{E}(\omega,\kappa) + \varepsilon \Omega_1)\, ds\Big) E_i^{out}(x_1(s=0), 0) +$$

$$+ \varepsilon^{-i-1} \int_0^{t_1} ds \Big(\exp\big(-i \frac{1}{\varepsilon} \int_{t_1}^{s} V_D(\mathcal{E}(\omega,\kappa) + \varepsilon \Omega_1)\, dw\big) L_i(x_1(s), s)\Big) \tag{3.7}$$

$i = 0, 1, 2, \ldots \ldots$

$$\frac{dt_1}{ds} = 1$$

$$\frac{dx_1}{ds} = V_g(s) \tag{3.8}$$

$$x_1(t_1) = x_1$$

Because $E_i^{out}(x_1, t_1)$ is a slowly varying quantity, $V_D \mathcal{E}(\omega, K)$ must be zero to lowest order:

$$\mathcal{E}(\omega_0, K_0) = 0 \tag{3.9}$$

Expanding $V_D \mathcal{E}(\omega, K)$ in a Taylor-series around $K = K_0$, $\omega = \omega_0$, we get:

$$V_D \mathcal{E}(\omega, K) = \Delta\omega - V_g\Big|_{\substack{K=K_0\\ \omega=\omega_0}} \Delta K + \frac{1}{2}\frac{dV_g}{dK}\Big|_{\substack{K=K_0\\ \omega=\omega_0}} (\Delta K)^2 + O\left((\Delta K)^3, (\Delta\omega)^3\right) \tag{3.10}$$

where $\Delta\omega = \omega - \omega_0$, $\Delta K = K - K_0$

We may note that $\Delta\omega - V_g \Delta K = O\left((\Delta K)^2, (\Delta\omega)^2\right)$
Because $V_D \mathcal{E}(\omega, K) = O(\varepsilon\omega)$, $V_D \Omega_1 = O(\varepsilon\omega)$, which means that we may meglect this term to lowest order.

This gives:

$$E_0^{out}(x_1, t_1) = Re\Big[\exp\Big(i\frac{1}{\varepsilon}\int_0^{t_1}\Big(\big(\Delta\omega - V_g(K_0, \omega_0)\Delta K\big) + \frac{dV_g}{dK}\Big|_{\substack{K=K_0\\ \omega=\omega_0}}(\Delta K)^2\Big)ds\Big) E_0(x_1(s=0), 0) +$$

$$+ \int_0^{t_1} ds \Big(\exp -i\frac{1}{\varepsilon}\int_{t_1}^{s}\Big(\big(\Delta\omega - V_g(K_0, \omega_0)\Delta K\big) + \frac{dV_g}{dK}\Big|_{\substack{K=K_0\\ \omega=\omega_0}}(\Delta K)^2\Big)ds\Big) L_0(x_1(s), s) \tag{3.11}$$

To get the explicit expressions of eqs. (3.7) and (3.11) we have to solve eqs. (1.11) and (3.8), which will be done in section V.

IV. The inner solution.

In the inner region, we define:

$$\frac{\partial}{\partial t_1} = -V_g \frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial t_2} + \varepsilon^2 \frac{\partial}{\partial t_3} + \cdots$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial x_2} + \varepsilon^2 \frac{\partial}{\partial x_3} + \cdots \qquad (4.1)$$

where $t_i = \varepsilon^i t$, $x_i = \varepsilon^i x$; $\quad i = 1, 2, 3, \cdots$

Eq. (4.1) is consistent with the assumption (2.24). Furthermore, we have from Eqs. (1.7-8):

$$\frac{\partial \omega}{\partial x_1} + \frac{\partial \kappa}{\partial t_1} = 0 \text{ , or} \qquad (4.2)$$

$$\frac{\partial \omega}{\partial x_1} - V_g \frac{\partial \kappa}{\partial x_1} = -\left(\varepsilon \frac{\partial}{\partial x_2} + \varepsilon^2 \frac{\partial}{\partial x_3} + \cdots\right)\omega - \left(\varepsilon \frac{\partial}{\partial t_2} + \varepsilon^2 \frac{\partial}{\partial t_3} + \cdots\right)$$

We define:

$$E^{in}(x_1, x_2, \cdots, t_1, t_2, \cdots) = \sum_{m=0}^{\infty} \varepsilon^m E_m^{in} \qquad (4.3)$$

$$E_0^{in}(x_1, x_2, \cdots, 0, 0, \cdots) = E_0(x_1)$$

$$\qquad (4.4)$$

$$E_i^{in}(x_1, x_2, \cdots, 0, 0, \cdots) = 0 \quad ; \quad i = 1, 2, 3, \cdots$$

Now, eqs. (2.15), (4.1-3) give to $O(\varepsilon^3)$:

$$\left(\frac{\partial}{\partial t_2} + V_g \frac{\partial}{\partial x_2}\right) E_0^{in} - \frac{1}{\varepsilon^2}\left(i V_D \mathcal{E}(\omega,\kappa) + V_D \Omega\right) E_0^{in} - \frac{1}{2} i \left(\frac{dV_g}{d\kappa} \frac{\partial^2}{\partial x_1^2} E_0^{in} + \right.$$

$$\left. - \frac{1}{T_\omega}\left(\frac{\partial}{\partial x_1}\left(T_\omega \frac{\partial}{\partial \kappa} V_g\right) + V_g \frac{\partial}{\partial x_1}\left(T_\omega \frac{\partial}{\partial \omega} V_g\right)\right)\frac{\partial}{\partial x_1} E_0^{in} = \varepsilon^{-2} V_D \sum_{i=1}^{4} \overline{I}_i (x_1, x_2, \ldots, t_1, t_2 \ldots)\right. \quad (4.5)$$

Using eqs. (1.18), (4.1-2), we have:

$$\left(\frac{\partial}{\partial x_1} + \varepsilon \frac{\partial}{\partial x_2} + \cdots\right)\mathcal{E}(\omega,\kappa) = -\frac{\partial \mathcal{E}(\omega,\kappa)}{\partial \omega}\left(\varepsilon\left(\frac{\partial \kappa}{\partial t_2} + V_g \frac{\partial \kappa}{\partial x_2}\right) + \varepsilon^2\left(\frac{\partial \kappa}{\partial t_3} + V_g \frac{\partial \kappa}{\partial x_2}\right) + \cdots\right) \quad (4.6)$$

which gives: $$\frac{\partial \mathcal{E}(\omega,\kappa)}{\partial x_1} = 0 \quad (4.7)$$

$$V_g = V_g(x_2, x_3, \ldots, t_1, t_2, \ldots) \quad (4.8)$$

As in the outer region, $\mathcal{E}(\omega,\kappa) = O(\varepsilon\omega)$. Otherwise $E(x,t)$ should have a variation on the fast time scale.

Now, eq. (4.5) reduces to:

$$\frac{d}{du} E_0^{in} - \left(i\varepsilon^{-2} V_D \mathcal{E}(\omega,\kappa) + V_D \Omega_2\right) E_0^{in} - \frac{1}{2} i \frac{dV_g}{d\kappa} \frac{\partial^2}{\partial x_1^2} E_0^{in} = \varepsilon^{-2} V_D \sum_{i=0}^{4} \overline{I}_i \quad (4.9)$$

$$\frac{dt_2}{du} = 1$$

$$\frac{dx_2}{du} = V_g(u) \quad ; \quad x_2(u = t_2) = x_2 \quad (4.10)$$

We define:

$$\tilde{E}_0^{in} = E_0^{in} \exp\left(-\varepsilon^{-2}\int_{t_2}^{u}(i V_D \mathcal{E}(\omega,\kappa) + \varepsilon^2 V_D \Omega_2)\,du\right) \tag{4.11}$$

noting that $E_0^{in}(u = t_1) = E_0^{in}(x,t)$. Eqs. (4.9), (4.11) give:

$$\frac{d}{du}\tilde{E}_0^{in} - \frac{1}{2} i \frac{dv_g}{d\kappa}\frac{\partial^2}{\partial x_1^2}\tilde{E}_0^{in} = \varepsilon^{-2} V_D \exp\left(-\varepsilon^{-2}\int_{t_2}^{u}(i V_D \mathcal{E}(\omega,\kappa) + \varepsilon^2 V_D \Omega_2)\,du\right)\sum_{i=0}^{3}\bar{\bar{I}}_i$$

$$= M(x_1, t_1, \ldots) \tag{4.12}$$

Using eqs. (4.7-8), we may fouriertransform eq. (4.12)

$$\tilde{E}_\xi = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{i\xi x_1}\tilde{E}_0^{in}(x_1)\,dx_1 \tag{4.13}$$

$$\frac{d}{du}\tilde{E}_\xi + \frac{1}{2} i \frac{dv_g}{d\kappa}\xi^2 \tilde{E}_\xi = M_\xi(u) \tag{4.14}$$

$$\tilde{E}_0^{in}(x_1 t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left(\tilde{E}_\xi(u=0)\, e^{-\frac{1}{2} i \int_0^{t_2}\frac{dv_g}{d\kappa}\xi^2 du}\right) e^{-i\xi x_1}\, d\xi \; +$$

$$+ (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left(\int_0^{t_2} e^{\frac{1}{2} i \int_{t_2}^{u}\frac{dv_g}{d\kappa}\xi^2 du} M_\xi(u)\,du\right) e^{-i\xi x_1}\, d\xi \tag{4.15}$$

Noting that $e^{i\alpha\xi^2}$ is the fouriertransform of $f(x) = (1+i)\alpha^{-\frac{1}{2}} e^{-i\frac{x^2}{4\alpha}}$, so, using the convolution theorem for fouriertransforms, we have that:

$$E_0^{in}(x,t) = R_e\Big[\frac{1+i}{\sqrt{2\pi}}\Big(\Big(-\frac{1}{2}\int_0^{t_2}\frac{dv_g}{dk}du\Big)^{-\frac{1}{2}}\int_{-\infty}^{\infty}d\eta\Big(\tilde{E}_0^{in}(\eta, u=0)\cdot e^{-i\frac{1}{2}(x-\eta)^2\left(-\int_0^{t_2}\frac{dv_g}{dk}du\right)^{-1}}\Big) + \int_0^{t_2}ds\Big(\Big(\int_{t_2}^{s}\frac{dv_g}{dk}du\Big)^{-\frac{1}{2}}\int_{-\infty}e^{-i\frac{1}{2}(x-\eta)^2\left(\int_{t_2}^{s}\frac{dv_g}{dk}du\right)^{-1}}M(\eta,s)d\eta\Big)\Big)\Big] \tag{4.16}$$

Eq. (4.16) describes a diffusion in X_1 -space, during the evolution.

Introducing an intermediate scale t_η,

$$t_1 = \eta^{-1}t_\eta \quad ; \quad t_2 = \eta\, t_\eta$$

$$t_1 \to \infty \qquad t_2 \to 0$$
$$\eta \to 0 \qquad \eta \to 0$$
$$t_\eta = \text{const} \qquad t_\eta = \text{const}$$

(J.D. Cole 1968).

Choosing $\eta = \varepsilon^{\frac{1}{2}}$, it is easy to show that:

$$E_0^{in} \xrightarrow[\substack{\eta\to 0\\ t_\eta = \text{const}}]{} E_0^{out}$$

As in the outer solution we have to solve eqs. (1.11), (4.10) to get an explicit expression in eq. (4.16).

V. The uniform solution.

To solve eqs. (1.11), (1.13), there are two different effects which have to be taken into account.

Particles with velocities:

$$\frac{\omega}{k} - S_k < V_j < \frac{\omega}{k} + S_k \tag{5.1}$$

$$V_m < \frac{\omega}{k} < V_M \tag{2.8}$$

$$S_k = \left(\frac{e_j}{m_j} E \frac{1}{k}\right)^{\frac{1}{2}} \tag{5.2}$$

are trapped by the waves making up the wave packet, while particles with velocities outside this region, are untrapped.

Furthermore, the wave packet will behave similar to an electrostatic pulse. This means that particles with velocities:

$$V_g - S_g < V_j < V_g \tag{5.3}$$

where $$S_g = \left(\frac{e_j}{m_j} E \frac{V_g}{\omega - kV_g}\right)^{\frac{1}{2}} \tag{5.4}$$

are accellerated by the packet, while particles with velocities

$$V_g < V_j < V_g + S_g \tag{5.5}$$

are retarded (M.S. Espedal 1971).

Particles outside the regions (5.1), (5.3) and (5.5) are passing the wave packet, and get no final change in velocity. They may get a change in phase if the packet is unsymmetrical.

Because of eqs. (2.6-7), the regions (5.1) and (5.3), (5.5) are seperated.

Using eqs. (1.11), (1.13) and (2.10), we get:

$$\frac{dV_j}{d\tau} = \frac{e_j}{m_j} E(\varepsilon\tau) \sin\left(\chi(\tau) + \varphi(x,t)\right) \tag{5.6}$$

$$\chi(\tau) = -\int_t^{\tau} \left(\omega(\varepsilon\tau) - k(\varepsilon\tau) V_j(\tau)\right) d\tau \tag{5.7}$$

Eqs. (5.6-7), we shall solve approximately, dividing the velocity space into the following regions:

$$V < V_g - S_g \tag{5.8}$$

$$V_g - S_g \leq V \leq V_g + S_g \tag{5.9}$$

$$V_g + S_g < V < V_m - S_K \tag{5.10}$$

$$V_m - S_K \leq V \leq V_M + S_K \tag{5.11}$$

$$V_M + S_K < V \tag{5.12}$$

Further, we shall approximate $E(\varepsilon\tau)$ to lowest order by:

$$E_0(\varepsilon\tau) = E_0(\varepsilon X(\tau), 0) \tag{5.13}$$

In the regions (5.8), (5.10) and (5.12), the particles are passing the packet. Therefore we take $V_j(\tau) \approx V$ to lowest order, which gives:

$$V_{0j}^1(\tau) = V + \frac{e_j}{m_j}\int_t^\tau E_0(\varepsilon(x+V(\tau-t)), 0) \sin(\chi_0^1(\tau) + \phi(x,t))\, d\tau \tag{5.14}$$

where $\chi_0^1(\tau) = -\int_t^\tau (\omega(\varepsilon\tau) - \kappa(\varepsilon\tau)V)\, d\tau$

Now, in the region (5.9), we define:

$$V_j(\tau) = V_g(\varepsilon\tau) + S_j(\tau) \tag{5.13}$$

$$\chi_0^2(\tau) = -\int_x^{X(\tau)} (\omega - \kappa V_g)\frac{1}{V_g}\, dX(\tau) \tag{5.14}$$

$$X'(\tau) = \int_t^\tau S(\tau)\, d\tau = X(\tau) - x - \int_t^\tau V_g\, d\tau \tag{5.15}$$

Eqs. (5.6-7) and (5.13-14) give:

$$\frac{d}{d\tau} S_j = \frac{e_j}{m_j} E_0(\varepsilon X(\tau), 0) \sin(\chi_0^2(X(\tau)) + \varphi(x,t)) - \varepsilon \frac{dv_g}{d\varepsilon\tau} \tag{5.16}$$

The lowest order solution of eq. (5.16) is:

$$S_{j0}(\tau) = \pm\left((V - V_g)^2 + \frac{e_j}{m_j}\int_0^{X'(\tau)} E_0(\varepsilon X(\tau), 0) \sin(\chi_0^2 + \varphi)\, dX'(\tau)\right)^{\frac{1}{2}}$$

$$V_{j0}^2(\tau) = V_g(\varepsilon\tau) + S_{j0}(\tau) \tag{5.17}$$

We should note that particles which have velocities $V = V_g \pm \alpha$, $|\alpha| \leq S_g$, before the interaction with the packet, get a velocity $V = V_g \mp \alpha$ after the interaction.

Similarly, in the region of trapped particles, (5.10), we define:

$$V_j(\tau) = \frac{\omega}{\kappa}(\varepsilon\tau) + u(\tau) \tag{5.18}$$

$$X_j''(\tau) = X(\tau) - x - \int_t^{\tau} \frac{\omega}{\kappa}\, d\tau = \int_t^{\tau} u(\tau)\, d\tau \tag{5.19}$$

$$\chi_0^3(\tau) = K(\varepsilon\tau) X_j''(\tau) \tag{5.20}$$

So, eq. (5.6) reduces to:

$$\frac{d}{d\tau} u_j(\tau) = \frac{e_j}{m_j} E_0(\varepsilon X(\tau), 0) \sin(\chi_0^3 + \varphi) - \varepsilon \frac{d}{d\varepsilon\tau}\left(\frac{\omega}{\kappa}\right) \tag{5.21}$$

which gives to lowest order:

$$U_{j0}(\tau) = \pm\left(\left(V - \frac{\omega}{k}\right)^2 + \frac{e_j}{m_j}\int_0^{x''(\tau)} E_0(\varepsilon x(\tau),0)\sin(x_0^3+\varphi)\,dx_j''(\tau)\right)^{\frac{1}{2}}$$

$$V_{j0}^3(\tau) = \frac{\omega}{k}(\varepsilon\tau) + U_{j0}(\tau) \tag{5.22}$$

If $\tau = p$ is the time when the interaction between the particle and the packet start, and τ_{int} the interaction time, we may write eqs. (5.14), (5.17) and (5.23):

$$V_{j0}(\tau) = \begin{cases} V & \tau < p \\ V_{j0}^{i}(\tau) & p \leq \tau \leq p + \tau_{int} \\ V_{j0}^{i}(\tau = p + \tau_{int}) & p + \tau_{int} < \tau \end{cases} \tag{5.23}$$

$i = 1, 2, 3$

To solve eqs. (3.8) and (4.10) to lowest order, we have to approximate V_g. This may be done, using the fact that

$$\mathcal{E}(\omega, k) = O(\varepsilon K) \tag{5.24}$$

Therefore eqs. (3.11), (4.16) and (5.23) may be used as a first step in a successive approximation procedure.

VI. The electron plasma wave packet.

In this section, we shall study the podulation of a electron plasma wace packet propagating through a collisionless electron plasma in a background of fixed ions.

We assume that:

$$f_0(v) = n(2\pi v_e^2)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\left(\frac{v}{v_e}\right)^2\right) \tag{6.1}$$

$$E_0(x,0) = E_0 \exp\left(-\tfrac{1}{2}\left(\tfrac{x_1}{\ell}\right)^2\right) \tag{6.2}$$

$$f_j(\tau = 0) = 0$$

$$V_e^2 = \left(\frac{\varkappa T_e}{m}\right)^{\frac{1}{2}}\ ;\ \omega_{pe} = \left(\frac{4\pi n e^2}{m}\right)^{\frac{1}{2}}\ ;\ K_D = \frac{\omega_{pe}^2}{V_e^2} \tag{6.3}$$

Further, we assume that:

$$\left(\frac{K}{K_D}\right)^2 = O(\varepsilon) \tag{6.4}$$

Using eq. (5.24), we get:

$$\omega^2(K) = \omega_{pe}^2\left(1 + 3\left(\frac{K}{K_D}\right)^2 + 6\left(\frac{K}{K_D}\right)^3\right) + O(\varepsilon^3 \omega_{pe}^2) \tag{6.5}$$

Eq. (6.5) gives:

$$V_g = \frac{3K}{\omega} V_e^2\left(1 + 4\left(\frac{K}{K_D}\right)^2\right) + O\left(\varepsilon^2\ \frac{V_e^2 K}{\omega}\right) \tag{6.6}$$

Also, we assume that:

$$K = K(x_3, t_3) \tag{6.7}$$

To calculate $\overline{I}_i$, i = 1, 2, 3, 4, we have to estimate the interaction time, T_{int} . In the regions (5.8), (5.10), (5.12), T_{int} is the passing time:

$$\tau_{int} \approx \frac{2\ell}{|V - V_g|} \tag{6.8}$$

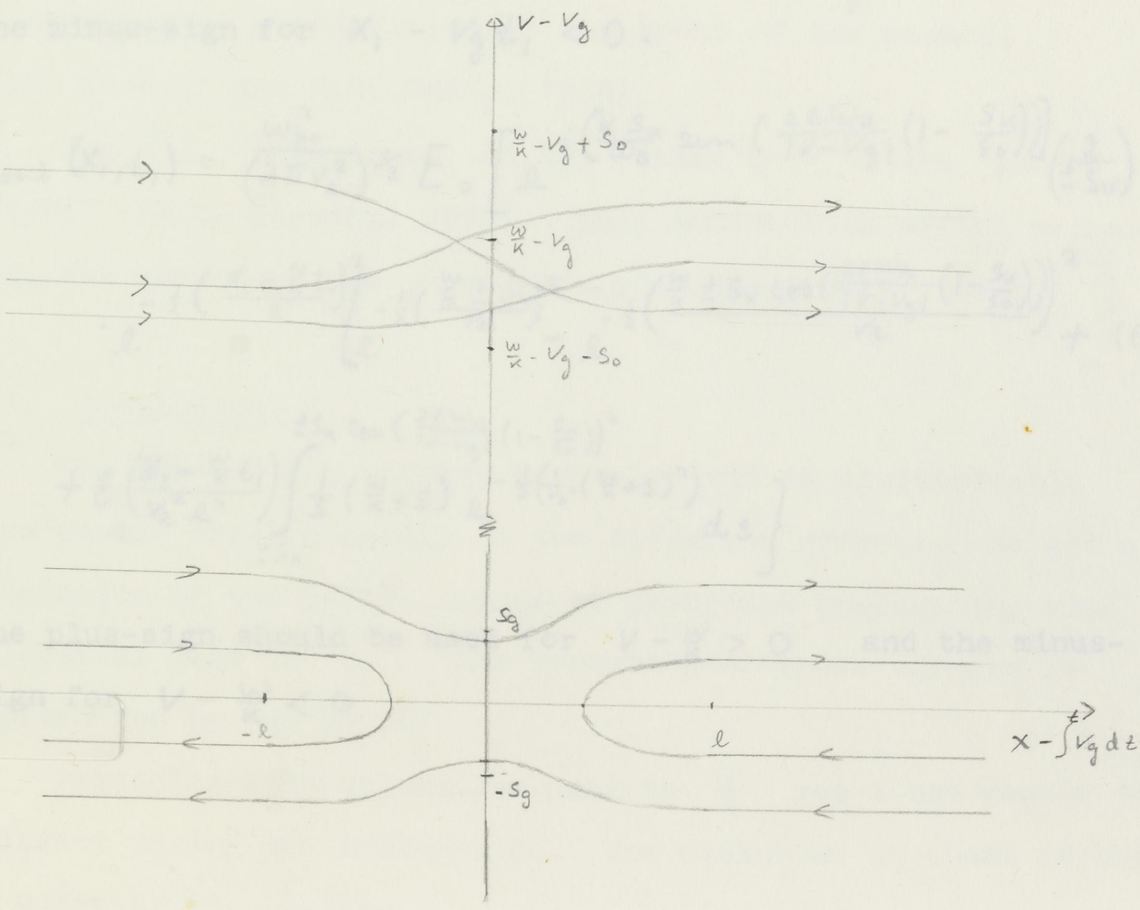
In the regions (5.9), τ_{int} is twice the time needed to accelerate a particle from $V = V_g$ to $V = V_g + S$. $|S| < S_g$. This is approximately:

$$\tau_{int} \approx 2\left(\frac{\ell}{S_g}\right)\left(\left|\frac{E_0(x_1)}{E_0}\right|\right)^{\frac{1}{2}} \tag{6.9}$$

The interaction time in the region (5.10), is also the passing time, which is approximately:

$$\tau_{int} \approx \frac{2\ell}{|V - V_g|}\left(1 - \frac{\omega - kV}{k S_0}\right)$$
$$S_0 = \left(\frac{\ell}{m} E_0 \frac{1}{k}\right)^{\frac{1}{2}} \tag{6.10}$$

A rough figure of the phase-plane is:



With these assumptions, the main contribution from the integrals (2.18-20), is:

$$\overline{I}_{in}(x_1,t_1,t_2) = \frac{\omega_{pe}^2}{(2\pi v_e^2)^{\frac{1}{2}}} E_0 \Big[e^{i(\omega-kv_g)^{3/2} \ell \left(\frac{e}{m} E_0 v_g\right)^{-\frac{1}{2}}} e^{-\frac{1}{2}\left(\frac{x_1 - v_g t_1}{\ell}\right)^2} \cdot$$

$$\cdot e^{-\frac{1}{2}\left(\frac{v_g \pm s_g}{v_e}\right)^2} \left(1 - e^{\pm \frac{2 v_g s_g}{v_e^2}}\right) \frac{1}{\omega - k v_g} \Big] \Big(-i +$$

$$- \varepsilon \frac{v_g}{\omega - k v_g} \left(\frac{x_1 - v_g t_1}{\ell^2} \right) + \qquad (6.11)$$

$$+ i \varepsilon^2 \frac{v_g^2}{(\omega - k v_g)^2 \ell^2} \left(1 - \frac{(x_1 - v_g t_1)^2}{\ell^2} \right) \Big) + O(\varepsilon^3)$$

where the plus-sign should be used for $x_1 - v_g t_1 > 0$ and the minus-sign for $x_1 - v_g t_1 < 0$.

$$\overline{I}_{out}(x_1,t_1) = \frac{\omega_{pe}^2}{(2\pi v_e^2)^{\frac{1}{2}}} E_0 \Big[e^{i\left(\frac{k s_k}{\omega_B} \sin\left(\frac{2\ell\omega_B}{|v - v_g|}\left(1 - \frac{s_k}{s_0}\right)\right)\right)} \left(\frac{2}{\pm s_0}\right) \cdot$$

$$\cdot e^{-\frac{1}{2}\left(\frac{x_1 - \frac{\omega}{k} t_1}{\ell}\right)^2} \Big\{ e^{-\frac{1}{2}\left(\frac{\frac{\omega}{k} \pm s_k}{v_e}\right)^2} - e^{-\frac{1}{2}\left(\frac{\frac{\omega}{k} \pm s_k \cos\left(\frac{2\ell\omega_B}{|v - v_g|}\left(1 - \frac{s_k}{s_0}\right)\right)}{v_e}\right)^2} + \qquad (6.12)$$

$$+ \varepsilon \left(\frac{x_1 - \frac{\omega}{k} t_1}{v_e^2 \ell^2} \right) \int_{\pm s_k}^{\pm s_k \cos\left(\frac{2\ell\omega_B}{|v - v_g|}\left(1 - \frac{s_k}{s_0}\right)\right)} \frac{1}{s}\left(\frac{\omega}{k} + s\right) e^{-\frac{1}{2}\left(\frac{1}{v_e^2}\left(\frac{\omega}{k} + s\right)^2\right)} ds \Big\}$$

The plus-sign should be used for $v - \frac{\omega}{k} > 0$ and the minus-sign for $v - \frac{\omega}{k} < 0$

Eqs. (3.11), (4.16), (6.12-13) give:

$$E_0(x_1,t_1,t_2) = E_0(x_1+v_g t_1, 0) + Re\int_0^{t_1} \bar{I}_{out}(x_1+v_g s, s)\, ds \; +$$

$$+ Re\left(\frac{1+i}{\sqrt{2\pi}}\int_0^{t_2} ds \left(\int_{t_2}^{s}\frac{dV_g}{d\kappa}du\right)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-i(x-\eta)^2\frac{1}{2}\left(\int_{t_2}^{s}\frac{dV_g}{d\kappa}du\right)^{-1}} \bar{I}_{in}(\eta+v_g t_1, s)\, d\eta\right) \qquad (6.13)$$

Eq. (6.13) give that the effect of trapped particles, $\bar{I}_{out}$, propagates as a free streaming effect. Because $\frac{\omega}{k} - V_g > 0$, it propagates faster than the packet, and should be observed as a modulation in the front of the packet. (J.N. Denavit and R.N. Sudan, 1972).

The $\bar{I}_{in}$ term in eq. (6.13) takes care of the "pulse effect" (M.S. Espedal, 1971). This interaction effect propagates with the velocity V_g, and modulates the packet itself.

VII. Conclusion.

The interaction between particles and an electrostatic wave packet results mainly in two different effects. We get a modulation of the packet caused by particles propagating with velocities near to V_g. The evolution of these effects is represented by eq. (4.16).

Particles with velocities near to $\frac{\omega}{k}$ get a net change in velocity during the interaction. The evolution of these effects is given by eq. (3.11).

We may note that, taking into account wave-wave interaction effects, the average equation is no longer linear. In models where these effects appear, we may get similar equation as those obtained by Y.H.Ichikawa and T.Taniuti.

References.

J.Denavit and R.N.Sudan. Phys. Rev. Letters. Vol.28 (1972) pp. 404-407.

K.B.Dysthe. To appear in J. of Plasma Physics (1974).

M.S.Espedal. J. Plasma Physics. Vol.5 (1971) pp. 343-355.

Y.H.Ichikawa and T.Taniuti. J. of Physical Society of Japan. Vol.34 (1973) pp. 513-521.