BOOLEAN NEGATION AND NON-CONSERVATIVITY II
THE VARIABLE SHARING PROPERTY

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ABSTRACT: Many relevant logics are conservatively extended by Boolean negation. Not all, however. This paper shows an acute form of non-conservativeness, namely that the Boolean free fragment of the Boolean extension of a relevant logic need not always satisfy the variable sharing property. In fact, it is shown that such an extension can in fact yield classical logic. For a vast range of relevant logic, however, it is shown that the variable sharing property, restricted to the Boolean-free fragment, still holds for the Boolean extended logic.

KEYWORDS: *Boolean negation, non-conservative extension, relevant logics, variable sharing*

1. INTRODUCTION

One of the hallmarks of relevant logics is the *variable sharing property*, that if $A \to B$ is a logical theorem, then A and B share a propositional variable. The reason the property is treasured is that it is thought to satisfy the requirement that for an entailment statement to be true, the statement need to have related relata; related by some kind of relevance. One way of making this more precise is to demand that there needs to be some kind of commonality of meaning between the antecedent and consequent of a true entailment statement. After noting that in the propositional case "commonality of meaning is carried by identity of propositional variables" ([2, p. 144]), Belnap then suggested the variable sharing property as a way to cash out the relevance criterion and showed that the logic **E** satisfies the property.

One of the objections to both classical and modal logics and a motivating factor for investigating relevance as a logical concept in the first place, was that these logics validate the implicational paradoxes expressed by $A \wedge \neg A \rightsquigarrow B$ and $A \rightsquigarrow B \vee \neg B$, where $\rightsquigarrow$ is either the material conditional

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$\supset$, or the strict conditional $\prec$.[1] These two axioms express that negation is Boolean over the relation expressed by the implication conditional. The standard relevant negation has therefore ever since been the weaker *De Morgan* negation $\sim$. However, Robert Meyer and Richard Routley started in the early seventies to investigate so-called *classical relevant logics*—relevant logics with a Boolean negation added as an additional primitive negation. The addition is got by simply adding the two Boolean axioms

(B1)	$A \wedge \neg A \rightarrow B$	Boolean explosion axiom
(B2)	$A \rightarrow B \vee \neg B$	Boolean excl. middle axiom,

where $\rightarrow$ is the relevant conditional. Of course, the variable sharing property does not hold unrestrictedly for such logics, but the addition was in many cases found to be conservative, and so it follows that at least the Boolean-free fragment of these logics still satisfies it. In fact, even for those logics, logics such as **E** and **NR**, for which the extension turned out to be non-conservative, it is often quite trivial to show that the Boolean extension is not so non-conservative as to rob the Boolean free fragment of its variable sharing property. But then again, not always. The purpose of this short paper is simply to show that the Boolean-free-fragment-restricted variable sharing property holds for a vast range of Boolean extended logics, but that this is not always the case. As we shall see, there are cases where **L** is a relevant logic, but where the Boolean extension of **L** is in fact classical logic. The example which is presented in this paper is $\mathbf{T}_3$—**T** augmented by the $\mathbf{RM}_3$-distinctive axiom $A \vee (A \rightarrow B)$.

The plan for the paper is as follows: Sect. 2 provides the axioms and rules of some common relevant logics and some extensions thereof before Sect. 3 shows forth Belnap's proof of the variable sharing property and Méndez et al.'s variant of it. Sect. 4 shows the main result that this property is not always passed on to the Boolean extension of a relevant logic. Sect. 5 then summarizes.

This is the second of in all three essays on Boolean negation and non-conservativeness pertaining to relevant logics. The first essay, [11], dealt with modal relevant logics, whereas the third essay, [12], deals with the question whether relevant logics with the truth-constant known as the Ackermann constant can be conservatively extended by Boolean negation. Together the three essays paint a picture of relevant logics being quite often non-conservatively extended by Boolean negation. It should therefore be noted that many relevant logics in fact are conservatively extended by Boolean negation. Neither of the three papers make any effort to survey such proofs, however. The interested reader should consult [4], [5], [9] and [13].

T	A1–A10, A12–A13, R1–R2	**E**	**T** +A14, +A15
R	**T** (or **E**) +A11	**LM**	**L** +A16
$\mathbf{L}_3$	**L** +A17	$\mathcal{C}$**L**	**L** +B1–B2

TABLE 1. Definitions of **T**, **E** and **R** and some extensions thereof.

2. DEFINITION OF LOGICS

(A1)	$A \to A$	identity
(A2)	$A \to A \vee B$ and $B \to A \vee B$	$\vee$-introduction
(A3)	$A \wedge B \to A$ and $A \wedge B \to B$	$\wedge$-elimination
(A4)	$\sim\sim A \to A$	double negation elimination
(A5)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	distribution
(A6)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	strong lattice $\wedge$
(A7)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	strong lattice $\vee$
(A8)	$(A \to \sim B) \to (B \to \sim A)$	contraposition axiom
(A9)	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
(A10)	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
(A11)	$A \to ((A \to B) \to B)$	assertion axiom
(A12)	$(A \to \sim A) \to \sim A$	reductio
(A13)	$(A \to (A \to B)) \to (A \to B)$	contraction axiom
(A14)	$((A \to A) \to B) \to B$	1. **E**-distinctive axiom
(A15)	$\Box A \wedge \Box B \to \Box(A \wedge B)$	2. **E**-distinctive axiom
		$(\Box C =_{df} (C \to C) \to C)$
(A16)	$A \to (A \to A)$	Mingle
(A17)	$A \vee (A \to B)$	$\mathbf{RM}_3$-distinctive axiom
(B1)	$A \wedge \neg A \to B$	Boolean explosion axiom
(B2)	$A \to B \vee \neg B$	Boolean excl. middle axiom
(R1)	$A, B \vdash A \wedge B$	adjunction
(R2)	$A, A \to B \vdash B$	modus ponens

3. THE VARIABLE SHARING PROPERTY FOR MODAL AND NON-MODAL RELEVANT LOGICS

Belnap proved the variable sharing property to hold of **E** by showing forth the eight-element algebra displayed in Fig. 1.[2] The set $\mathcal{T}$ is the set of designated elements; conjunction and disjunction are interpreted as infimum and supremum, and $\to$, $\sim$, and $\Box$ are interpreted according to the

[1]Intuitionistic logic is similarly charged of validating implicational paradoxes. Although $A \to B \vee \neg B$ does not hold in intuitionistic logic, $A \wedge \neg A \to B$ does.

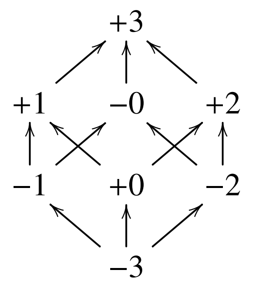
[2]All models depicted in this paper have been found with the help of `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney ([14]). I have made heavy use of both it as well as William McCune's theorem prover/model generator package *Prover9/Mace4* ([7]) in arriving at the results reported in this essay.

displayed matrices. The following theorem and proof thereof is that given by Belnap in [2]:

Theorem 1 (Belnap's vsp-theorem)**.** **E** *has the variable sharing property.*

Proof. Assume that A and B share no propositional variable. Assign to every propositional variable in A the value $+1$, and $+2$ to every variable in B. It is easy to check that both $\{-1, +1\}$ and $\{-2, +2\}$ are closed under the functions which interprets $\sim, \Box, \wedge, \vee, \rightarrow$, and so A will be assigned either -1 or $+1$ and B either -2 or $+2$. It is then easy to check that $A \rightarrow B$ will be assigned -3. Since the model is a model for all the axioms and rules of **E** (left for the reader), it follows that $A \rightarrow B$ is not a theorem. Thus if $A \rightarrow B$ *is* a logical theorem, then A and B share a propositional variable. $\square$

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\rightarrow$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$
$\sim$	$+3$	$+2$	$+1$	$+0$	-0	-1	-2	-3
$\Box$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$

FIGURE 1. Belnap's model of relevance

E's $\Box$ is a defined operator—$\Box A =_{df} (A \rightarrow A) \rightarrow A$—which expresses a **S4** modality. The stronger logic **R**, however, is not a modal logic in any meaningful sense. It was later noted that Belnap's model is also a model of the stronger logic **R**, and since $\Box$ is interpreted as the identity operator in the model, it also validate Meyer's **NR** as well as stronger modal logics such as $\mathbf{R5}^{\Box}$ got by adding the following modal axioms and rule, where now $\Box$ is taken as primitive, to **R**:[3]

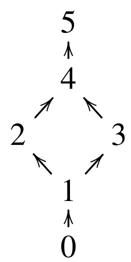
(**NEC**)	$\varnothing \vdash A \Rightarrow \varnothing \vdash \Box A$	
(**K**)	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	
($^{\Box}/_{\wedge}$)	$\Box A \wedge \Box B \rightarrow \Box(A \wedge B)$	
($^{\Box}/_{\vee}$)	$\Box(A \vee B) \rightarrow \Box A \vee \Diamond B$	$\Diamond A =_{df} \sim\Box\sim A$
(**T**)	$\Box A \rightarrow A$	
(**4**)	$\Box A \rightarrow \Box\Box A$	
(**5**)	$\Box A \rightarrow \Box\Diamond A$	

[3] **NR** is simply **R5** without the **5**- and $^{\Box}/_{\vee}$-axioms.

This, then, shows that a large class of logics satisfies the variable sharing property. Early on it was thought that also **RM**—**R** augmented by the *mingle* axiom $A \to (A \to A)$—would also turn out to satisfy the variable sharing property; Meyer then reported that "[s]urprisingly, R-mingle doesn't—Dunn and I found the counterexample CNCppCqq" ([8]).[4] However, **RM** and its three-valued extension $\mathbf{RM}_3$ was found to satisfy the weaker property that if $A \to B$ is a logical theorem, then either A and B share a propositional variable, or both $\sim A$ and B are theorems. Anderson and Belnap thought this to be an insufficient property for ensuring relevance and so claimed that "relevance and mingle are incompatible" ([1, p. 98]). Whereas it is true that neither **RM** nor **EM** satisfies the variable sharing property, Méndez et al. showed in [10] that **TM** does satisfies the variable sharing property.[5]

The proof of Méndez et al. is basically like Belnap's but uses a different six-element algebraic structure—displayed in Fig. 2—instead of Belnap's eight-element structure. Méndez et al. do not consider modal extensions of **TM**, but by simply interpreting $\Box$ as the identity operator, and let $\Diamond =_{df} \sim\Box\sim$, their model can easily be checked be a model for $\mathbf{TM5}^{\Box}$ as well. They do note that their model not only validates **TM**, but also the **E**-axiom $((A \to A) \to B) \to B$ (A14). As can easily be checked, their model in fact also validates the $\mathbf{RM}_3$-distinctive axiom $A \vee (A \to B)$. It follows, then, that $\mathbf{TM5}^{\Box}_{3}[A14]$ has the variable sharing property.

$\mathcal{T} = \{1, 2, 3, 4, 5\}$



$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	4	4	4	4	4
2	0	0	2	0	4	5
3	0	0	0	3	4	5
4	0	0	0	0	4	5
5	0	0	0	0	0	5
$\sim$	5	4	2	3	1	0

FIGURE 2. Méndez et al.'s 6-element model for variable sharing

4. THE VARIABLE SHARING PROPERTY FOR CLASSICAL RELEVANT LOGICS

Robert Meyer and Richard Routley showed in [9] that C**R**—**R** extended by the Boolean axioms B1 and B2—is a conservative extension. Thus since **R** has the variable sharing property, so does C**R**, provided, of course, that

[4] *CNCppCqq* is Polish for $\sim(p \to p) \to (q \to q)$.

[5] **EM** is sometimes taken to be the logic **E** augmented by $(A \to B) \to ((A \to B) \to (A \to B))$. It is still an open question whether **E** extended by this $\to$-restricted mingle axiom satisfies the variable sharing property or not.

one restricts to the $\neg$-free fragment. Since not all relevant logics are conservatively extended by Boolean negation—**E** being a noteworthy example[6]—an easier proof of this $\neg$-free restricted variable sharing property would be preferable. A moments notice suffices for realizing that Belnap's model can simply be augmented with the following matrix for $\neg$:

	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
$\neg$	$+3$	$+1$	$+2$	$+0$	-0	-2	-1	-3

We therefore get the following corollary:

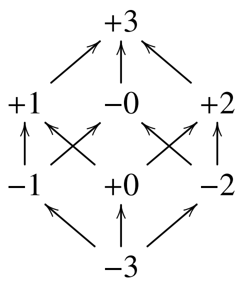
Corollary 1. *All sublogics of* C**R5**$^\Box$ *have the* $\neg$*-restricted variable sharing property.*

TM is not a sublogic of C**R5**$^\Box$ and it is easy to see that Méndez et al.'s model can't be extended to a model for $\neg$ so as to validate the two Boolean axioms.[7] There is, however, a different model, quite similar to Belnap's, which can be used to show the same result as for C**R5**$^\Box$:

Theorem 2. C**TM5**$^\Box[A14]$ *has the* $\neg$*-restricted variable sharing property.*

Proof. The model in Fig. 3 is a model for C**TM5**$^\Box[A14]$. Inspecting the model it is evident that $\{-1, +1\}$ and $\{-2, +2\}$ are also in this case closed under all connectives except $\neg$, and so the result follows by the same type of proof as that of Belnap's vsp-theorem. ∎

$\mathcal{T} = \{+0, +1, +2, +3\}$



$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
-2	-3	$+2$	-3	$+3$	-3	-3	$+2$	$+3$
-1	-3	-3	$+1$	$+3$	-3	$+1$	-3	$+3$
-0	-3	-3	-3	$+3$	-3	-3	-3	$+3$
$+0$	-0	-0	-0	-0	$+3$	$+3$	$+3$	$+3$
$+1$	-3	-3	-1	-0	-3	$+1$	-3	$+3$
$+2$	-3	-2	-3	-0	-3	-3	$+2$	$+3$
$+3$	-3	-3	-3	-0	-3	-3	-3	$+3$
$\sim$	$+3$	$+2$	$+1$	$+0$	-0	-1	-2	-3
$\neg$	$+3$	$+1$	$+2$	$+0$	-0	-2	-1	-3
$\Box$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$

FIGURE 3. A 8-element model for C**TM5**$^\Box[A14]$

Note that the **RM**$_3$-distinctive axiom $A \vee (A \to B)$ (A17) is not a theorem of C**TM5**$^\Box[A14]$. As the proof of following theorem show, that axiom together with the Boolean axioms is too potent a mix:

[6]Mares proved this in [6]. See [11] for an easy proof which applies to a range of logics.

[7]If the first axiom $A \wedge \neg A \to B$ is to be satisfied, one needs to set $\neg x = 0$ for all non-zero elements in the algebra. But then $[\![5 \to 1 \vee \neg 1]\!] = 0$, and so the second Boolean axiom, $A \to B \vee \neg B$, will not be satisfied.

Theorem 3. *There are logics* $\mathbf{L}$ *which have the variable sharing property but are such that* $C\mathbf{L}$ *do not have the* $\neg$*-restricted variable sharing property.*

Proof. We saw earlier that even $\mathbf{TM}_3[A14]$ has the variable sharing property. However, $C\mathbf{T}_3$ does not have the $\neg$-restricted variable sharing property as the following derivation of the weakening rule $A \vdash B \to A$ shows:[8]

(1)	A	assumption
(2)	$\neg A \vee (\neg A \to A)$	A17
(3)	$A \wedge (\neg A \vee (\neg A \to A)$	1, 2, R1
(4)	$(A \wedge \neg A) \vee (A \wedge (\neg A \to A))$	3, A5
(5)	$A \wedge \neg A \to (\neg A \to A)$	B1
(6)	$A \wedge (\neg A \to A) \to (\neg A \to A)$	A3
(7)	$(A \wedge \neg A) \vee (A \wedge (\neg A \to A)) \to (\neg A \to A)$	6, A7
(8)	$\neg A \to A$	4, 7, R2
(9)	$A \vee \neg A \to A$	8, fiddling
(10)	$B \to A \vee \neg A$	B2
(11)	$B \to A$	9, 10, transitivity of $\to$

□

Thus even though $\mathbf{T}_3$ is a relevant logic, $C\mathbf{T}_3$ is arguably not seeing as it does not have the $\neg$-restricted variable sharing property.[9] That this is so is underscored by the following corollary:

Corollary 2. *There are relevant logics* $\mathbf{L}$ *for which the Boolean extension is identical to classical logic.*

Proof. The proof is to the effect that weakening *axiom*, $A \to (B \to A)$, is a theorem of $C\mathbf{T}_3$. The meta-rule of reasoning by cases, $\dfrac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}$, is provable for any axiomatic extension of $\mathbf{T}$.[10] Since A17 is an axiom, we get as one of its instance $A \vee (A \to (B \to A))$. Using Thm. 3 twice we get that $A \vdash A \to (B \to A)$. And since $A \to (B \to A) \vdash A \to (B \to A)$, reasoning by cases yields that $\vdash A \to (B \to A)$. Since even $\mathbf{T}$ augmented by the weakening axiom is identical to classical logic, it follows that also $C\mathbf{T}_3$ is. □

[8] Note that one does not need the full power of $C\mathbf{T}_3$ for the proof to go through; besides the Boolean and $\mathbf{RM}_3$-distinctive axiom it only requires a minimum of logical resources; the weak relevant logic $\mathbf{BB}$—got by dropping axioms A12 and A13 from $\mathbf{T}$ and weakening axioms A6–A10 to rule form—would suffice. The weakening rule destroys any hope of satisfying the variable sharing property for all logics with logical theorems—and therefore also for $\mathbf{BB}$—since if $\varnothing \vdash A$, the weakening rule yields $\varnothing \vdash B \to A$ for every B.

[9] That $\mathbf{T}_3$ has the variable sharing property follows also from the fact that it is a sublogic of the logic of the Chrystal lattice axiomatized by adding both the $\mathbf{RM}_3$-distinctive axiom as well as the axiom $\sim A \wedge B \to (\sim A \to A) \vee (A \to B)$ to $\mathbf{R}$. See [3, §§9.7–9.8] for this axiomatization and proof of the variable sharing property.

[10] See [11, thm. 2] for a proof.

5. Summary

This paper shows an acute form of non-conservativeness, namely that the Boolean free fragment of the Boolean extension of a relevant logic need not always satisfy the variable sharing property. We saw that this was the case with the logic $\mathbf{T}_3$—$\mathbf{T}$ augmented by the $\mathbf{RM}_3$-distinctive axiom $A \vee (A \to B)$—which itself satisfies the variable sharing property, but that the weakening rule $A \vdash B \to A$ is a derivable rule of its Boolean extension. This, then, was shown to be sufficient for deriving all of classical logic. For a vast range of Boolean extended relevant logics, however—even **S5**-modal extensions of both $\mathbf{TM}[A14]$ and $\mathbf{R}$—the variable sharing property, restricted to the Boolean-free fragment, was shown to hold.

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