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Brill–Noether general K3 surfaces with the maximal number of elliptic pencils of minimal degree

Michael Hoff[1] · **Andreas Leopold Knutsen**[2]

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Abstract
We explicitly construct Brill–Noether general $K3$ surfaces of genus 4, 6 and 8 having the maximal number of elliptic pencils of degrees 3, 4 and 5, respectively, and study their moduli spaces and moduli maps to the moduli space of curves. As an application we prove the existence of Brill–Noether general $K3$ surfaces of genus 4 and 6 without stable Lazarsfeld–Mukai bundles of minimal c_2.

Keywords K3 surfaces · Unirationality · Moduli map · Lazarsfeld–Mukai bundle

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1 Introduction

It is well-known that a general curve of genus $g \leq 9$ or $g = 11$ can be realized as a linear section of a primitively polarized $K3$ surface, cf. [26,28]. Since for even g a general curve C carries a finite number of pencils of minimal degree $\frac{g}{2}+1$, it is natural to ask whether one can simultaneously extend C and all or some of these pencils to some $K3$ surfaces for $g = 4, 6, 8$. This question is connected to the existence of non-stable Lazarsfeld–Mukai bundles. Indeed, the Lazarsfeld–Mukai bundle associated to a pencil on a smooth curve on the $K3$ surface induced by an elliptic pencil on the surface is necessarily not stable, cf. Lemma 5.1.

Using vector bundle methods, Mukai [29] showed that the projective model of any *Brill–Noether general* $K3$ surface (S, L) is obtained as sections of homogeneous varieties for $g \in \{6, \ldots, 10, 12\}$. By definition, cf. [29, Def. 3.8], a polarized $K3$ surface (S, L) of genus g is Brill–Noether general if $h^0(M)h^0(N) < g+1 = h^0(L)$ for any non-trivial decomposition $L \sim M + N$. In these low genera this is equivalent to all the smooth curves in the linear system $|L|$ being Brill–Noether general, due to techniques in [13,22] (see [14, Lemma 1.7]).

✉ Michael Hoff
hahn@math.uni-sb.de

Andreas Leopold Knutsen
andreas.knutsen@math.uib.no

1 Universität des Saarlandes, Campus E2 4, 66123 Saarbrücken, Germany

2 Department of Mathematics, University of Bergen, 5020 Bergen, Norway

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Using Mukai's results, we will study projective models of Brill–Noether general $K3$ surfaces of genus $g \in \{4, 6, 8\}$ containing the maximal possible number of elliptic pencils of degree $\frac{g}{2}+1$.

The goal of our paper is threefold:

(1) We provide explicit constructions/equations of $K3$ surfaces with special geometric features.
(2) We describe their moduli spaces as lattice polarized $K3$ surfaces and the corresponding moduli map to the moduli space of curves of genus g.
(3) We study the slope-stability of Lazarsfeld–Mukai bundles of hyperplane sections on such $K3$ surfaces.

Our main results are the following.

- Section 3: We prove that a general curve C of genus 4 is a linear section of a smooth $K3$ surface S such that its two g^1_3s (which are well-known to be auto-residual) are induced by two elliptic pencils $|E_1|$ and $|E_2|$ on S satisfying $C \sim E_1 + E_2$, cf. Proposition 3.4. Furthermore, the moduli space parametrizing such $K3$ surfaces is unirational (and 18-dimensional), cf. Proposition 3.2. We believe that these results should be known, but could not find any reference.
- Section 4: A general curve C of genus 6 carries precisely five pencils $|A_1|, \dots, |A_5|$ of minimal degree 4 which satisfy $2K_C \sim A_1 + \cdots + A_5$ (see [4, p. 209ff]). We prove that C is a linear section of a smooth $K3$ surface S such that its five g^1_4s are induced by five elliptic pencils $|E_1|, \dots, |E_5|$ on S satisfying $2C \sim E_1 + \cdots + E_5$, cf Theorem 4.3(a). We prove that the moduli space parametrizing such pairs (S, C) is unirational, cf. Theorem 4.3(b). The moduli space of the underlying $K3$ surfaces was already studied in [5] where it was shown to be birational to the moduli space $\mathcal{M}_6$ of curves of genus 6 (and therefore, rational, cf. [34]). Our approach shows that this moduli space is exactly the locus of Brill–Noether general $K3$ surfaces that cannot be realized as quadratic sections of a smooth quintic Del Pezzo threefold (but as quadratic sections of a cone over a smooth quintic Del Pezzo surface), cf. Remark 4.4(b).
- Section 6: A general curve C of genus 8 carries precisely 14 pencils of degree 5. An easy lattice computation shows that at most 9 can be extended to a $K3$ surface containing C. We prove that this bound is reached in codimension 3 in the moduli space $\mathcal{M}_8$, and for a general curve only six out of its 14 pencils can be extended to elliptic pencils on a $K3$ surface, cf. Corollary 6.11. We prove that the moduli spaces of such $K3$ surfaces containing i elliptic pencils are unirational for $1 \le i \le 6$ and $i = 9$, cf. Theorems 6.7 and 6.8.
- Section 5: The $K3$ surfaces constructed in Sect. 3 (respectively 4) provide examples of $K3$ surfaces without stable (resp. semistable) Lazarsfeld–Mukai bundles with $c_2 = 3$ (resp. 4), cf. Corollary 5.2 (resp. 5.3). This shows in particular the sharpness of a result of Lelli-Chiesa [23, Thm. 4.3], cf. Remark 5.4.

Notation and conventions

We work over $\mathbb{C}$. We will denote V_n an n-dimensional vector space and $G(k, V_n)$ (respectively $G(V_n, k)$) the Grassmannian of k-dimensional sub- (resp. quotient-) spaces of V_n. The projective space of one-dimensional sub- (resp. quotient-) spaces is denoted $\mathbb{P}_*(V_n)$ (resp. $\mathbb{P}^*(V_n)$).

2 Lattice polarized $K3$ surfaces and their moduli spaces

Let $\mathfrak{h}$ be a lattice. The moduli space $\mathcal{F}^{\mathfrak{h}}$ of $\mathfrak{h}$-polarized $K3$ surfaces parametrizes pairs (S, φ) (up to isomorphism) consisting of a $K3$ surface S and a primitive lattice embedding $\varphi : \mathfrak{h} \to \mathrm{Pic}(S)$ such that $\varphi(\mathfrak{h})$ contains an ample class. It is a quasi-projective irreducible $(20 - \mathrm{rk}(\mathfrak{h}))$-dimensional variety by [11].

If $(S, \varphi) \in \mathcal{F}^{\mathfrak{h}}$ is an $\mathfrak{h}$-polarized $K3$ surface and $L \in \mathfrak{h} \cong \varphi(\mathfrak{h})$ is a distinguished class with $L^2 = 2g - 2 \geq 2$, one may consider the open subset

$$\mathcal{F}_g^{\mathfrak{h}} = \left\{ (S, \varphi) \;\middle|\; (S, \varphi) \in \mathcal{F}^{\mathfrak{h}} \text{ and } L \text{ ample} \right\}$$

of the moduli space $\mathcal{F}^{\mathfrak{h}}$, which may also be considered as a subset of the moduli space $\mathcal{F}_g$ of polarized $K3$ surfaces of genus g. Furthermore, let $\mathcal{P}_g^{\mathfrak{h}}$ denote the moduli space of triples (S, φ, C) where $C \in |L|$ is a smooth irreducible curve in the distinguished linear system. Then we have moduli maps

$$m_g : \mathcal{P}_g^{\mathfrak{h}} \to \mathcal{M}_g.$$

Since in our cases of study it will be clear what the distinguished class L will be, we will often skip the index g in $\mathcal{F}_g^{\mathfrak{h}}$ and $\mathcal{P}_g^{\mathfrak{h}}$.

3 $K3$ surfaces of genus 4

We will show the unirationality of the moduli space $\mathcal{F}^{\mathfrak{U}(3)}$ of lattice polarized $K3$ surfaces where $\mathfrak{U}$ is the hyperbolic lattice of rank 2. We believe that this result should be well-known, but we could not find any reference.

The following example is well-known, but we include it for the sake of the reader and it serves as an introduction for our next results and constructions.

Example 3.1 (The moduli space of $K3$ surfaces of genus 4) A smooth polarized $K3$ surface $S \subset \mathbb{P}^4$ of genus 4 is the complete intersection of a quadric Q and a cubic hypersurface Y in $\mathbb{P}^4$. The quadric $Q = V(q)$ and the cubic $Y = V(y)$ are given by polynomials $q \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ and $y \in H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$ of degrees 2 and 3, respectively.

The moduli space $\mathcal{F}_4$ of $K3$ surfaces of genus 4 is described as follows. The quadric has to be of rank at least 4 since otherwise S will be singular. Let $V \subset H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ be the open subset consisting of quadratic equations of rank ≥ 4. For a chosen equation q we need to pick a cubic y such that y is no multiple of q, and the intersection of Q and Y should be smooth. Let V_q be the five-codimensional quotient of $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$ parametrizing non-multiples of q. The desired cubic equations are parametrized by an open subset $W_q \subset V_q$. Let W be the iterated Grassmannian

$$W \xrightarrow{G(1, W_q)} \mathbb{P}_*(V) \cong \mathbb{P}^{14}$$

whose fibers are Grassmannians of one-dimensional subspaces of W_q. Then $\mathcal{F}_4$ is birational to W modulo the automorphism group of $\mathbb{P}^4$ and therefore $\mathcal{F}_4$ is unirational. Note further that a dimension count yields

$$\dim V + \dim W_q - \dim PGL(5) = \left(\binom{6}{2} - 1\right) + \left(\binom{7}{3} - 1 - 5\right) - (5^2 - 1) = 19,$$

as expected.

3.1 K3 surfaces of genus 4 with an elliptic pencil of degree 3

With notation as in the previous example let $S \subset \mathbb{P}^4$ be a smooth $K3$ surface of genus 4 with polarization $L = \mathcal{O}_S(1)$. Assume that there exists a class $E \in \mathrm{Pic}(S)$ such that $E^2 = 0$ and $E.L = 3$. By Riemann–Roch, $h^0(S, E) = 2$ and E' is a smooth elliptic normal curve for general $E' \in |E|$. Hence we get a pencil of elliptic normal curves. The pencil induces a rational normal scroll

$$X = \bigcup_{E' \in |E|} \overline{E'} \subset \mathbb{P}^4$$

of dimension 3 and degree 2 where $\overline{E'} = \mathbb{P}^2$ is the linear span of E'. Thus the scroll X is the unique quadric hypersurface containing S. Furthermore, the scroll X is singular in a point (since any two different projective planes in $\mathbb{P}^4$ intersect and X cannot be singular along a line), that is, X is a rank 4 quadric.

We remark that the residual class $L - E$ is a second elliptic pencil of degree 3 on S and the maximal number of such pencils is two since $S \subset \mathbb{P}^4$ is generated by a unique quadric. We get a $K3$ surface whose Picard lattice contains the intersection matrix with respect to the ordered basis $\{L, E\}$ (respectively $\{L - E, E\}$)

$$\begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix} \left(\text{resp. } \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = \mathfrak{U}(3) \right)$$

where $\mathfrak{U}$ is the hyperbolic lattice of rank 2 and L is the sum of the two basis elements of square 0. In general $\mathrm{Pic}(S) \cong \mathfrak{U}(3)$ (such $K3$ surfaces exist by [24, Thm. 2.9(i)] or [31]), in which case L is the unique element (up to sign) of square 6, hence genus 4, which is easily seen to be very ample by the classical results of Saint-Donat [32]. Furthermore, such a $K3$ surface (S, L) is Brill–Noether general.

Recall from the introduction that $\mathcal{F}^{\mathfrak{U}(3)}$ is the moduli space of $\mathfrak{U}(3)$-polarized $K3$ surfaces.

Proposition 3.2 *The moduli space $\mathcal{F}^{\mathfrak{U}(3)}$ is unirational.*

Proof By what we said, a general element in $\mathcal{F}^{\mathfrak{U}(3)}$ comes equipped with a unique embedding into $\mathbb{P}^4$ (up to the action of the projective linear group), as a complete intersection of a cubic and a rank 4 quadric, singular in a point. The converse holds true: if a smooth surface $S \subset \mathbb{P}^4$ is a complete intersection of a rank 4 quadric hypersurface Q and a cubic hypersurface, then the two rulings on Q cut out two residual elliptic pencils of degree 3 on S.

We describe a birational model of the moduli space $\mathcal{F}^{\mathfrak{U}(3)}$ by modifying the construction in Example 3.1, keeping the notation therein.

Let $V' \subset H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ be the subset of quadratic equations of rank 4. Since a rank 4 quadric is a cone over a smooth quadric in $\mathbb{P}^3$, the space V' is isomorphic to an open subset of a $\mathbb{P}^4$-bundle over $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ and is therefore unirational. Pick $q \in V'$. Then the moduli space $\mathcal{F}^{\mathfrak{U}(3)}$ is birational to the iterated Grassmannian

$$W' \xrightarrow{G(1, W_q)} V'$$

modulo automorphisms and is therefore unirational, too. (Since $\dim V' = \binom{5}{2} - 1 + 4 = 13$, a dimension count yields that $\mathcal{F}^{\mathfrak{U}(3)}$ is a codimension one subspace of $\mathcal{F}_4$, as expected.) $\square$

Remark 3.3 Let $\mathfrak{U}$ be the hyperbolic lattice of rank 2. Even if the example above should be classically known, we only found in the literature unirationality results of $\mathcal{F}^{\mathfrak{U}(n)}$ for $n = 1$ and

2 (cf. [9]). Elliptic surfaces are parametrized by $\mathcal{F}^{\mathfrak{U}}$ and double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a curve of bidegree (4, 4) are parametrized by $\mathcal{F}^{\mathfrak{U}(2)}$.

Recall from the introduction that $\mathcal{P}^{\mathfrak{U}(3)}$ is the moduli space of triples (S, φ, C) where $(S, \varphi) \in \mathcal{F}^{\mathfrak{U}(3)}$ and $C \in |L|$ is a smooth curve of genus 4 in the distinguished linear system. Also recall that a general curve of genus 4 has exactly two distinct g^1_3s, which are autoresidual.

Proposition 3.4 *The moduli map $\mathcal{P}^{\mathfrak{U}(3)} \to \mathcal{M}_4$ is dominant. In particular, a general curve C of genus 4 is a linear section of a smooth $K3$ surface S such that its two g^1_3s are induced by two elliptic pencils $|E_1|$ and $|E_2|$ on S satisfying $C \sim E_1 + E_2$.*

Proof We consider a general curve $C \subset \mathbb{P}^3$ of genus 4, canonically embedded into $\mathbb{P}^3$, which is a complete intersection of a smooth quadric Q' and a cubic Y' (the quadric Q' is smooth since the two g^1_3s are distinct). We will construct a $K3$ surface $S \in \mathcal{F}^{\mathfrak{U}(3)}$ with the curve C as a linear section. Therefore, we choose a $\mathbb{P}^4$ containing the ambient space $\mathbb{P}^3$ of the curve. Let $Q \subset \mathbb{P}^4$ be a cone over the quadric $Q' \subset \mathbb{P}^3$, that is, a rank 4 quadric whose hyperplane section with the given $\mathbb{P}^3$ is Q'. Let $Y \subset \mathbb{P}^4$ be any cubic hypersurface such that $Y \cap \mathbb{P}^3 = Y'$. The surface $S \subset \mathbb{P}^4$ can be chosen as the complete intersection of Q and Y. Then, the pair (S, C) is an element of $\mathcal{P}^{\mathfrak{U}(3)}$ by construction, and the dominance of the moduli map follows. The last statement is immediate. □

Remark 3.5 Similarly in [21] it is shown that the moduli space of $K3$ surfaces admitting a special automorphism of order 3 is birational to the moduli space of curves of genus 4 (see also [6] for its generalization).

4 K3 surfaces of genus 6

Inspired by the seminal work of Mukai [27], we will construct a Brill–Noether general $K3$ surface S of genus 6 where every complete pencil of degree 4 on a hyperplane section of S is induced by an elliptic pencil on S. Furthermore, we show that the moduli space of such lattice polarized $K3$ surfaces is unirational.

We briefly recall Mukai's construction. Let (S, L) be a Brill–Noether general $K3$ surface of genus 6. There exists a unique stable (rigid) vector bundle $\mathcal{E}$ of rank 2 on S with $c_1(\mathcal{E}) = L$, $h^0(S, \mathcal{E}) = 5$ and $h^i(S, \mathcal{E}) = 0$ for $i = 1, 2$ [16, Prop. 5.2.7]. This bundle induces an embedding of S into the Grassmannian $G(V_5, 2)$, where $V_5 = H^0(S, \mathcal{E})$, by sending $s \in S$ to the fiber $\mathcal{E}_s = \mathcal{E} \otimes \mathcal{O}_s$. As described in [27], a Brill–Noether general $K3$ surface S is the intersection of a linear section of codimension 3 (or 4) and a quadratic section of either the Plücker embedding $G(V_5, 2) \subset \mathbb{P}^9$ or of its cone $\widehat{G(V_5, 2)} \subset \mathbb{P}^{10}$, respectively.

In order to get an elliptic pencil of degree 4 on a $K3$ surface, we need special sections of the following form. If the linear section of codimension 3 cuts a sub-Grassmannian of type $G(4, 2)$ in a quadric surface, we get an elliptic normal curve of degree 4 on S as the intersection of this quadric surface with the quadric section. A pencil of Grassmannians of type $G(4, 2)$ induces a pencil of elliptic curves on S and can be controlled in the dual space in the following way.

Lemma 4.1 *A hyperplane corresponds to a point in the dual Grassmannian $G(2, V_5) \subset \mathbb{P}^{9\vee}$ if and only if it cuts out a Schubert subvariety. Moreover, the Schubert variety is a one-dimensional union of Grassmannians of type $G(4, 2)$ contained in $G(V_5, 2)$.*

We will prove the same statement for the Grassmannian $G(V_6, 2)$ in the next section (cf. Sect. 6.1.1) and leave this proof to the readers. Note that two Grassmannians of type $G(4, 2)$ in $G(V_5, 2)$ intersect in a 2-plane. Hence, two elliptic curves of distinct pencils of degree 4 with respect to L intersect in two points. This can also be seen in the following way: if E_1 and E_2 are such elliptic curves, then $E_1.E_2 \geq 2$ (as each $|E_i|$ is a pencil); moreover, since $(L - E_1)^2 = 2$, one also has $4 - E_1.E_2 = E_2.(L - E_1) \geq 2$, whence $E_1.E_2 \leq 2$. Also inspired by the previous example of $K3$ surfaces of genus 4, we will construct a $K3$ surface with Picard lattice of the following form:

$$\begin{pmatrix} 10 & 4 & 4 & \dots & 4 \\ 4 & 0 & 2 & \dots & 2 \\ 4 & 2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 \\ 4 & 2 & \dots & 2 & 0 \end{pmatrix}$$

An easy computation shows that the rank can be at most five (otherwise the matrix has at least two non-negative eigenvalues). Let $\mathfrak{M}$ be the lattice given by the following intersection matrix

$$\mathfrak{M} = \begin{pmatrix} 10 & 4 & 4 & 4 & 4 \\ 4 & 0 & 2 & 2 & 2 \\ 4 & 2 & 0 & 2 & 2 \\ 4 & 2 & 2 & 0 & 2 \\ 4 & 2 & 2 & 2 & 0 \end{pmatrix}.$$

We denote S a $K3$ surface with the above Picard lattice $\mathfrak{M}$ of rank 5 (which exists by [24, Thm. 2.9(i)] or [31]) and let L be the basis element of square 10. Let E_i, $i = 1, \dots, 4$, be the generators of square zero. Note that $E_5 := 2L - E_1 - E_2 - E_3 - E_4$ is also an element of square zero and degree 4 with respect to L.

The lattice $\mathfrak{M}$ is also generated by elements $s_0, s_1, \dots, s_4$ where $s_0 = E_1 + \dots + E_4 - L$ and $s_i = s_0 - E_i$, $i = 1, \dots, 4$, with intersection matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

(This is the lattice considered in [5].) We may assume that s_0 is big and nef by standard arguments (see [7, VIII, Prop. 3.10]). Note that $L = 3s_0 - \sum_{i=1}^4 s_i$, $E_i = s_0 - s_i$ for $i = 1, \dots, 4$ and $E_5 = 6s_0 - 3\sum_{i=1}^4 s_i$.

Lemma 4.2 *(a) The class L is ample.*
(b) The $K3$ surface (S, L) is Brill–Noether general.
(c) The classes $E_1, \dots, E_5$ define elliptic pencils and are the only classes in $\mathrm{Pic}(S)$ *of square* 0 *and degree* 4 *with respect to L.*

Proof Let $\Delta = \sum_{i=0}^4 a_i s_i$ be an arbitrary class. Then $\Delta^2 = 2a_0 - 2\sum_{i=1}^4 a_i$, thus $L.\Delta = 8a_0 - \Delta^2$. If Δ is effective, then $a_0 = \frac{1}{2}s_0.\Delta \geq 0$ since s_0 is nef. It follows that $L.\Delta \geq 2$ for any (-2)-curve Δ, and we conclude (a). It also immediately follows that there exists no nontrivial effective class Δ such that either $\Delta^2 = 0$ and $\Delta.L \leq 3$ or $\Delta^2 = 2$ and $\Delta.L = 5$.

This implies (b) by either a direct computation using the definition of Brill–Noether generality or invoking, e.g., [17, Prop. 10.5] and [32], or [14, Lemma 1.7].

To prove that $|E_i|$ is an elliptic pencil, it suffices to show that E_i is nef by [32]. If E_i for some $i \in \{1, \dots, 5\}$ is not nef, there exists a (-2)-curve Γ with $\Gamma.E_i \le 0$. Let $k := -\Gamma.E_i \ge 1$. Then $(E_i - k\Gamma)^2 = 0$ and $E_i - k\Gamma$ is effective and nontrivial with $(E_i - k\Gamma).L \le 4 - k \le 3$ by ampleness of L, a contradiction to the Brill–Noether generality. Finally, if F is another effective class with $F^2 = 0$, then $F.E_i \ge 2$ for all i, since F moves in (at least) a pencil. Thus $F.L = \frac{1}{2}F.(E_1 + \cdots + E_5) \ge 5$. □

We will show that the general curve lies on a six-dimensional family of such $K3$ surfaces of Picard rank 5. We will use the cone over the Grassmannian $G(V_5, 2)$ in $\mathbb{P}^{10}$.

4.1 K3 sections of a cone of the Grassmannian $G(V_5, 2)$

Let $\mathfrak{M}$ be the rank 5 lattice above. Let $\mathcal{F}^{\mathfrak{M}}$ be the moduli space of $\mathfrak{M}$-polarized $K3$ surfaces and $\mathcal{P}^{\mathfrak{M}}$ be as in the introduction. Recall that $\dim \mathcal{F}^{\mathfrak{M}} = 15$ and $\dim \mathcal{P}^{\mathfrak{M}} = 21$. Also recall that a general genus 6 curve carries precisely five elliptic pencils $|A_1|, \dots, |A_5|$ of degree four, which satisfy $2K_C \sim A_1 + \cdots + A_5$.

By [5] the moduli space $\mathcal{F}^{\mathfrak{M}}$ is birational to $\mathcal{M}_6$, which is well-known to be rational by [34]. More precisely, Artebani and Kondō show that $\mathcal{F}^{\mathfrak{M}}$ is the locus of $K3$ surfaces admitting a double cover to a quintic Del Pezzo surface branched along a curve of genus 6. In particular, this shows that the moduli map $\psi : \mathcal{P}^{\mathfrak{M}} \to \mathcal{M}_6$ is dominant since we get a section. However, the pairs (S, L) admit automorphisms fixing L, therefore $\mathcal{P}^{\mathfrak{M}}$ is not birational to a $\mathbb{P}^6$-bundle over $\mathcal{F}^{\mathfrak{M}}$ and one cannot conclude its unirationality from the rationality of $\mathcal{F}^{\mathfrak{M}}$. We will show by our construction that $\mathcal{P}^{\mathfrak{M}}$ is unirational and that $\mathcal{F}^{\mathfrak{M}}$ is the space of polarized $K3$ surfaces of genus 6 such that all the five g^1_4s of their smooth curve sections are induced by elliptic pencils on the surfaces.

Theorem 4.3 *(a) The moduli map $\psi : \mathcal{P}^{\mathfrak{M}} \to \mathcal{M}_6$ is dominant. Furthermore, a general curve C of genus 6 is a linear section of a smooth $K3$ surface S such that its five g^1_4s are induced by five elliptic pencils $|E_1|, \dots, |E_5|$ on S satisfying $2C \sim E_1 + \cdots + E_5$.*
(b) $\mathcal{P}^{\mathfrak{M}}$ is unirational.

Proof (a) We will describe a $K3$ surface containing the general curve in $\mathcal{M}_6$ as well as the geometry describing the elliptic pencils on the $K3$ surface. This is based on Mukai's result [27, §6].

Let $C \in \mathcal{M}_6$ be a general curve of genus 6 which is given as follows. We fix a Plücker embedding of the Grassmannian $G(V_5, 2) \subset \mathbb{P}^9$. Then there exists a projective 5-space $P \subset \mathbb{P}^9$ as well as a quadric hypersurface $Q \subset P$ such that $C = P \cap Q \cap G(V_5, 2)$.

Let $P^\vee = \mathbb{P}^3 \subset \mathbb{P}^{9\vee}$ be the dual space. As C is assumed to be general, $W^1_4(C)$ is finite-dimensional, more precisely $W^1_4(C)$ consists of five smooth points, and is isomorphic to $P^\vee \cap G(2, V_5) \subset \mathbb{P}^{9\vee}$, that is, the intersection of $P^\vee$ and the dual Grassmannian $G(2, V_5) = G(V_5, 2)^\vee \subset \mathbb{P}^{9\vee}$. By Lemma 4.1 each point of $P^\vee \cap G(2, V_5)$ corresponds to a pencil of Grassmannians of type $G(4, 2)$ in $\mathbb{P}^9$. This pencil induces a cubic scroll in $\mathbb{P}^9$ whose restriction to C cuts out the corresponding point of $W^1_4(C)$.

Now let $\widehat{G(V_5, 2)} \subset \mathbb{P}^{10}$ be the cone over the Grassmannian $G(V_5, 2)$ with vertex point v. We denote $\widehat{G(2, V_5)} \subset \mathbb{P}^{10\vee}$ the cone over the dual Grassmannian with vertex w such that $\widehat{G(2, V_5)} = \widehat{G(V_5, 2)}^\vee$. We consider the given projective 5-space P as a subspace of $\mathbb{P}^{10}$.

Let $P_v = \overline{P+v}$ be the span of P and the vertex v. Let $Q' \subset P_v$ be a quadric hypersurface such that $Q' \cap P = Q$. We get a $K3$ surface $S = \widehat{G(V_5, 2)} \cap P_v \cap Q'$, which we can assume to be smooth for general Q'. Then the dual space of this P_v is exactly the above $P^\vee$. As above the five intersection points $P^\vee \cap \widehat{G(V_5, 2)} = P^\vee \cap G(V_5, 2)$ correspond to five pencils of Grassmannians in $\mathbb{P}^{10}$ whose restriction to S are the five elliptic pencils of degree 4 on S. We get the desired $K3$ surface with the right Picard lattice.

(b) Recall that any canonical model of a general curve of genus 6 can be realized as a quadratic section of a fixed quintic Del Pezzo surface $Y \subset \mathbb{P}^5$ (see [34]).

We fix a $\mathbb{P}^6 \supset \mathbb{P}^5$ and a point $v \in \mathbb{P}^6$. Let $\widehat{Y}$ be the cone over Y with vertex v. For a general curve $C \in \mathcal{M}_6$ we consider the linear system $\mathfrak{L}_C$ of quadratic sections of $\widehat{Y}$ containing C. We have $\dim \mathfrak{L}_C = h^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) - h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)) - 1 = 6$. We define the incidence correspondence

$$I = \{(C, S) \mid C \subset S\} \subset |\mathcal{O}_Y(2)| \times |\mathcal{O}_{\widehat{Y}}(2)| = \mathbb{P}^{15} \times \mathbb{P}^{22}$$

together with the projection $\pi : I \to |\mathcal{O}_Y(2)|$, whose fibers are given by $\mathfrak{L}_C$. It follows that π has the structure of a $\mathbb{P}^6$-bundle, whence $\dim(I) = 15 + 6 = 21$.

By the proof of part (a) the general member of $\mathfrak{L}_C$ is a smooth $K3$ surface in $\mathcal{F}^{\mathfrak{M}}$ (note that $P = \mathbb{P}^5$, $P_v = \mathbb{P}^6$, $Y = P \cap G(V_5, 2)$ and $\widehat{Y} = \widehat{G(V_5, 2)} \cap P_v$ in the notation of that proof). Hence, we get a natural rational moduli map $\varphi : I \dashrightarrow \mathcal{P}^{\mathfrak{M}}$. Since I is unirational, the corollary will follow if we prove that φ is dominant, equivalently, generically finite, since $\mathcal{P}^{\mathfrak{M}}$ is irreducible of the same dimension as I.

Assume therefore that φ has positive-dimensional fibers. Since the rational moduli map $|\mathcal{O}_Y(2)| \dashrightarrow \mathcal{M}_6$ is finite, the fibers of φ lie in fibers of π. Hence, the $K3$ surfaces in $\mathfrak{L}_C$ do not have maximal variation in moduli. Note that $\mathfrak{L}_C$ contains the quadratic sections of the form $Y \cup Y'$ where $Y' \in \mathbb{P}H^0(\widehat{Y}, \mathcal{O}_{\widehat{Y}}(1))$ which form a hypersurface in $\mathfrak{L}_C$. Hence a general one-dimensional family in $\mathfrak{L}_C$ is non-isotrivial, a contradiction. □

Remark 4.4 (a) The proof of Corollary 4.3 shows that our construction dominates the moduli space $\mathcal{F}^{\mathfrak{M}}$, that is, the general $K3$ surface in $\mathcal{F}^{\mathfrak{M}}$ is a quadratic section of a cone over a quintic Del Pezzo surface in $\mathbb{P}^5$.

(b) By [27], all Brill–Noether general $K3$ surfaces of genus 6 can be realized as a quadratic section of either a smooth quintic Del Pezzo threefold in $\mathbb{P}^6$ or a cone over a quintic Del Pezzo surface. Item (a) shows that $\mathcal{F}^{\mathfrak{M}}$ is precisely the locus of $K3$ surfaces that cannot be realized in a smooth Del Pezzo threefold.

5 Lazarsfeld–Mukai bundles and their stability

For $K3$ surfaces constructed in Sects. 3 and 4 we will show that these are $K3$ surfaces without any stable rank 2 Lazarsfeld–Mukai bundle with determinant L and $c_2 = 3$ or 4, respectively. This shows in particular that the result of Lelli-Chiesa [23, Thm. 4.3] about stability of rank 2 vector bundles on $K3$ surfaces is optimal.

We recall the definition and basic properties of Lazarsfeld-Mukai bundles, which will also be needed in Sect. 6. Let S be a $K3$ surface and let $C \subset S$ be a smooth curve of genus g with a globally generated line bundle A of degree d with $h^0(C, A) = r + 1$. The *Lazarsfeld-Mukai bundle* $\mathcal{E}_{C,A}$ is defined via an elementary transformation on S:

$$0 \longrightarrow \mathcal{E}_{C,A}^\vee \longrightarrow H^0(C, A) \otimes \mathcal{O}_S \longrightarrow A \longrightarrow 0, \tag{5.1}$$

where A is considered as a coherent sheaf on S supported on C. Hence, it is a bundle of rank $r+1$ satisfying $c_1(\mathcal{E}_{C,A}) = [C]$, $c_2(\mathcal{E}_{C,A}) = \deg A = d$ and $H^i(S, \mathcal{E}_{C,A}) = 0$ for $i = 1, 2$. The bundles have been introduced by Lazarsfeld [22] and Mukai [30]. Dualizing the above sequence, we get

$$0 \longrightarrow H^0(C, A)^* \otimes \mathcal{O}_S \longrightarrow \mathcal{E}_{C,A} \longrightarrow \omega_C \otimes A^* \longrightarrow 0,$$

and in particular a distinguished $(r+1)$-dimensional subspace $H^0(C, A)^* \subset H^0(\mathcal{E}_{C,A})$. Equivalently, by [2, Prop. 1.3], a rank $(r+1)$-bundle $\mathcal{E}$ on S is a Lazarsfeld-Mukai bundle if and only if $h^1(S, \mathcal{E}) = h^2(S, \mathcal{E}) = 0$ and there exists an $(r+1)$-dimensional subspace $V \subset H^0(S, \mathcal{E})$ such that the degeneracy locus of the evaluation morphism $V \otimes \mathcal{O}_S \to \mathcal{E}$ is a smooth curve.

Lemma 5.1 *If $A \in W^1_d(C)$ with $d \leq g-1$ is induced by an elliptic pencil $|E|$ on the $K3$ surface S, then $\mathcal{E}_{C,A}$ is not L-stable, where $L = \mathcal{O}_S(C)$. Furthermore, the bundle $\mathcal{E}_{C,A}$ is L-unstable, if $d < g-1$.*

Proof This is essentially already contained in [1, Proof of Thm. 1.1]. Using the snake lemma, we get the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & E \otimes L^* \arrow[d] & \\
0 \arrow[r] & E^* \arrow[r] \arrow[d] & H^0(S, E) \otimes \mathcal{O}_S \arrow[r] \arrow[d, "\cong"] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}^\vee_{C,A} \arrow[r] \arrow[d] & H^0(C, A) \otimes \mathcal{O}_S \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & E \otimes L^* \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Dualizing the left column, we see that $L \otimes E^*$ is a subbundle of $\mathcal{E}_{C,A}$. Computing slopes, we get $\mu(L \otimes E^*) = 2g - 2 - d \geq g - 1 = \mu(\mathcal{E}_{C,A})$. □

Corollary 5.2 *Let $(S, L) \in \mathcal{F}_4^{\mathfrak{U}(3)}$ be a Brill–Noether general polarized $K3$ surface as in Sect. 3.1. Then S contains only L-strictly semistable Lazarsfeld–Mukai bundles $\mathcal{E}_{C,A}$ of rank 2 and $\det(\mathcal{E}_{C,A}) = L$, $c_2(\mathcal{E}_{C,A}) = 3$ for $C \in |L|$ smooth.*

Proof Note that $W^1_3(C)$ consists of exactly two residual pencils of divisors which extend to two elliptic pencils on S. We can apply Lemma 5.1, and the corollary follows. □

Corollary 5.3 *Let $(S, L) \in \mathcal{F}_6^{\mathfrak{M}}$ be a Brill–Noether general polarized $K3$ surface as in Sect. 4. Then S contains only L-unstable Lazarsfeld–Mukai bundles $\mathcal{E}_{C,A}$ of rank 2 and $\det(\mathcal{E}_{C,A}) = L$, $c_2(\mathcal{E}_{C,A}) = 4$ for $C \in |L|$ smooth.*

Proof Since C is Brill–Noether general, every pencil in $W^1_4(C)$ is induced by an elliptic pencil on the $K3$ surface S. The result follows from Lemma 5.1. $\square$

Remark 5.4 Part (i) of [23, Thm. 4.3] implies that on any Brill–Noether general $K3$ surface (S, L) of genus g there are L-stable Lazarsfeld–Mukai bundles of determinant L and c_2 equal to d as soon as $\rho(g, 1, d) > 0$. (Indeed, sections of Brill–Noether general $K3$ surfaces have maximal gonality as a consequence of the definition and have Clifford dimension 1 by ampleness of L, cf. [18, Thm. 1.2] or [10, Prop. 3.3]). The above corollaries show that this does not always hold for $\rho(g, 1, d) = 0$ (at least when $g = 4$ or 6).

6 K3 surfaces of genus 8

In this section we construct $K3$ surfaces of genus 8 with the maximal number of elliptic pencils of degree 5. We recall Mukai's construction from [27,29] and fix our notation.

Let (S, L) be a Brill–Noether general polarized $K3$ surface of genus 8. Then there exists a unique globally generated stable vector bundle $\mathcal{E}$ of rank 2 with determinant L and Euler characteristic 6 (this can be constructed as the Lazarsfeld–Mukai bundle associated to a g^1_5 on any smooth $C \in |L|$ not induced by an elliptic pencil on S by [2, Prop. 1.3]). It is known that $V_6 = H^0(S, \mathcal{E})$ is six-dimensional. Every fiber $\mathcal{E}_s$ of $\mathcal{E}$ for $s \in S$ is a 2-dimensional quotient space of V_6, which induces a morphism $\phi_{\mathcal{E}} : S \to G(V_6, 2)$, $s \mapsto \mathcal{E}_s$. The Grassmannian $G(V_6, 2)$ is naturally embedded into $\mathbb{P}^*(\bigwedge^2 V_6) = \mathbb{P}^{14}$ via the Plücker embedding. The second exterior product induces a surjective map on global sections

$$\lambda : \bigwedge^2 H^0(S, \mathcal{E}) \to H^0(S, \bigwedge^2 \mathcal{E}),$$

and we get the following commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\phi_{\mathcal{E}}"] \arrow[d, "\phi_{\bigwedge^2 \mathcal{E}}"'] & & G(V_6, 2) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}^8 = \mathbb{P}^*(H^0(S, \bigwedge^2 \mathcal{E})) \arrow[rr, "\mathbb{P}^*(\lambda)"] & & \mathbb{P}^{14} \end{tikzcd}$$

where $\mathbb{P}^*(\lambda)$ is the linear embedding induced by λ. Since $\bigwedge^2 \mathcal{E} = c_1(\mathcal{E}) = L$, the map $\phi_{\bigwedge^2 \mathcal{E}}$ is given by the linear system $|L|$. The above diagram is cartesian, that is, $S = \mathbb{P}^8 \cap G(V_6, 2)$.

Hyperplane sections of $G(V_6, 2)$ are parametrized by $\mathbb{P}_*(\bigwedge^2 V_6)$. The dual of $\mathbb{P}^8$ is a five-dimensional projective space $\mathbb{P}^5 = \mathbb{P}_*(\ker \lambda) \subset \mathbb{P}_*(\bigwedge^2 V_6)$.

Let $C \in |L|$ be a smooth curve. The Brill–Noether generality of (S, L) is equivalent to C not containing a g^2_7 (arguing as in [13,22] or see [14, Lemma 1.7]). Let $\mathcal{E}_C$ be the restriction of $\mathcal{E}$ to C, which is stable by [27, §3] and $H^0(S, \mathcal{E}) \cong H^0(C, \mathcal{E}_C)$. As above we get a surjective morphism $\lambda_C : \bigwedge^2 H^0(C, \mathcal{E}_C) \to H^0(C, \omega_C)$ and a commutative cartesian diagram

$$\begin{tikzcd} C \arrow[rr] \arrow[d] & & G(V_6, 2) \arrow[d, "\text{Plücker}"] \\ \mathbb{P}^7 = \mathbb{P}(H^0(C, \omega_C)^*) \arrow[rr, "\mathbb{P}^*(\lambda_C)"] & & \mathbb{P}^{14} \end{tikzcd}$$

since $\mathbb{P}_*(\lambda_C) \cap G(2, V_6) \cong W^1_5(C)$ is finite (see [27, Thm. C]). Note that $\mathbb{P}_*(\lambda_C)$ is a six-dimensional space containing $\mathbb{P}_*(\lambda)$.

For our purpose we state Mukai's result in the following form.

Lemma 6.1 *(Mukai) A linear intersection of $G(V_6, 2)$ and $\mathbb{P}^8$ is a surface (in particular a Brill–Noether general $K3$ surface if smooth) if and only if the dual projective space $\mathbb{P}^5$ intersects the Grassmannian $G(2, V_6)$ in the following way: for every $\mathbb{P}^6 \supset \mathbb{P}^5$ the intersection with $G(2, V_6) \subset \mathbb{P}_*(\bigwedge^2 V_6)$ is finite.*

Proof The ”only if“ part follows from the above. Conversely, the second condition is equivalent to any hyperplane section of the given linear section being a curve. □

6.1 Linear sections of $G(V_6, 2)$ and elliptic pencils

We are interested in $K3$ surfaces $S \subset \mathbb{P}^8$ with an elliptic pencil of minimal degree 5. We describe a way of constructing such $K3$ surfaces.

We use the notation above. Let V_6 be a 6-dimensional complex vector space, and let V_5 be a 5-dimensional subspace of V_6. We consider $G(V_5, 2) \subset G(V_6, 2) \subset \mathbb{P}^*(\bigwedge^2 V_6)$. By a dimension count, a general 8-dimensional linear subspace of $\mathbb{P}^{14}$ intersects $G(V_5, 2)$ in 5 points. Assume instead that our $\mathbb{P}^8$ intersects $G(V_6, 2)$ transversally and $\mathbb{P}^8 \cap G(V_5, 2)$ is a smooth curve, which is then an irreducible elliptic normal curve of degree 5. Then we get a $K3$ surface S with an elliptic pencil.

6.1.1 Dual Grassmannian and Schubert varieties

Even more is true. As Mukai already notices in [27, end of p.3], a hyperplane corresponds to a point in the dual Grassmannian $G(2, V_6) \subset \mathbb{P}_*(\bigwedge^2 V_6)$ if and only if it cuts out a Schubert subvariety. We will explain this fact in detail.

Let $U \in G(2, V_6)$ be a point in the Grassmannian, that is, $U \subset V_6$ be a 2-dimensional subspace of V_6. Hence, $U^\perp = V_6/U$ is a 4-dimensional quotient of V_6. By the perfect pairing $\bigwedge^2 V_6 \otimes \bigwedge^4 V_6 \to \mathbb{C}$ we may interpret $U^\perp$ as a linear function on $\bigwedge^2 V_6$, denoted by H_U. We compute the hyperplane section $H_U \cap G(V_6, 2)$. By definition $H_U : \ker(\bigwedge^2 V_6 \xrightarrow{\wedge^4 U^\perp} \bigwedge^6 V_6 = \mathbb{C})$. Thus,

$$\begin{aligned} H_U \cap G(V_6, 2) &= \{U' \in G(V_6, 2) \mid \bigwedge^2 U' \wedge \bigwedge^4 U^\perp = 0\} \\ &= \{U' \in G(V_6, 2) | \dim(U' \cap U^\perp) \geq 1\} =: \Sigma_1(U^\perp) \end{aligned}$$

is a Schubert variety. Note that $\dim(U' \cup U^\perp) \leq 5$ for $U' \in H_U \cap G(V_6, 2)$, and it is easy to check that

$$\Sigma_1(U^\perp) = \bigcup_{v \in W} G(U^\perp \cup v, 2),$$

where $W \oplus U^\perp = V_6$. Note that everything is compatible with projectivization. Finally, we see that $\mathbb{P}^*(H_U) \cap G(V_6, 2) \subset \mathbb{P}^{14}$ is a pencil of Grassmannians of type $G(5, 2)$. The converse direction can be shown similarly.

We conclude that every intersection point of $\mathbb{P}_*(\ker \lambda) \cap G(2, V_6)$ gives a pencil of elliptic curves on S. In order to get $K3$ surfaces with many elliptic pencils of degree 5, we have to construct a transversal linear section $\mathbb{P}^8$ such that its dual $\mathbb{P}_*(\ker \lambda)$ intersects the Grassmannian $G(2, V_6)$ in as many points as possible.

6.1.2 Extension of elliptic curves to the Grassmannian $G(V_6, 2)$

Let (S, L) be a Brill–Noether general polarized $K3$ surface of genus 8 with an elliptic pencil $|E|$ satisfying $L.E = 5$. As S can be embedded (as a linear section) into the Grassmannian $G(V_6, 2)$, we will show that every elliptic curve $E' \in |E|$ is a linear section of a sub-Grassmannian of type $G(5, 2)$ of $G(V_6, 2)$.

We need some lemmas. We note that $(L - E)^2 = 4$ and $(L - E).L = 9$, whence $h^0(L - E) \geq 4$ by Serre duality and Riemann–Roch.

Lemma 6.2 *The complete linear system $|L-E|$ is base point free and maps S birationally onto a quartic surface in $\mathbb{P}^3$ having at most isolated A_1-singularities coming from contractions of smooth rational curves Γ satisfying $\Gamma.L = \Gamma.E = 1$.*

Proof Assume there exists an effective divisor Δ such that $\Delta^2 = -2$ and $\Delta.(L - E) \leq 0$. In particular, $\Delta.E \geq \Delta.L > 0$. Then $(L - E - \Delta)^2 \geq 2$, whence $h^0(L - E - \Delta) \geq 3$. As (S, L) is assumed to be Brill–Noether general, we must have $h^0(E + \Delta) = h^0(E) = 2$, thus $\Delta.E = 1$, and consequently $\Delta.L = 1$ and $\Delta.(L - E) = 0$. It follows that $L - E$ is nef. It also follows, once we have proved that $|L - E|$ defines a birational morphism, that any connected curve contracted by this morphism is an irreducible rational curve of degree one with respect to L and E, proving that the image surface has at most isolated rational A_1-singularities.

To prove that $|L - E|$ defines a birational morphism, it suffices by the well-known results of Saint-Donat [32] to prove that there is no irreducible curve D on S satisfying $D^2 = 0$ and $D.(L - E) = 1$ or 2. If such a D exists, then it is easily seen to satisfy $D.L \geq 5$ by Brill–Noether generality. Hence, $D.E \geq 3$, so that $(D + E)^2 \geq 6$. It follows that $h^0(D + E) \geq 5$. Since $(L - E - D)^2 \geq 0$ and $(L - E - D).D \geq 1$, we have $h^0(L - E - D) \geq 2$ by Riemann–Roch and Serre duality, contradicting Brill–Noether generality. □

Let $C \in |L|$ be a smooth curve and let $\mathcal{E} = \mathcal{E}_{C,A}$ be the Lazarsfeld–Mukai bundle associated to C and a pencil $|A|$ of degree 5 on C. Note that the bundle $\mathcal{E}_{C,A}$ is the unique L-stable bundle on S with determinant L and Euler characteristic 6. We write $A_E = E \otimes \mathcal{O}_C$ and note that $A \ncong A_E$ by Lemma 5.1.

Lemma 6.3 *Let (S, L), E and $\mathcal{E} = \mathcal{E}_{C,A}$ be as above. Then $h^0(\mathcal{E}(-E)) = 1$ and $h^1(\mathcal{E}(-E)) = h^2(\mathcal{E}(-E)) = 0$. In particular, $H^0(\mathcal{E}|_E)$ is a five-dimensional quotient of $H^0(S, \mathcal{E})$.*

Proof Since we know that $h^0(\mathcal{E}) = 6$, the last assertion immediately follows from the claimed cohomology of $\mathcal{E}(-E)$ by the obvious restriction sequence.

We will compute the cohomology of $\mathcal{E}(-E)$ using Serre duality and the sequence

$$0 \longrightarrow \mathcal{E}^\vee(E) \longrightarrow H^0(C, A) \otimes \mathcal{O}_S(E) \longrightarrow A \otimes A_E \longrightarrow 0, \tag{6.1}$$

which is (5.1) tensored by $\mathcal{O}_S(E)$.

Since $\mathcal{E}^\vee(E)$ is semi-stable of degree -4, one has $h^0(\mathcal{E}^\vee(E)) = 0$. Moreover, $h^0(\mathcal{O}_S(E)) = 2$ and $h^1(\mathcal{O}_S(E)) = h^2(\mathcal{O}_S(E)) = 0$, as E is an irreducible elliptic curve. Hence, the desired cohomology of $\mathcal{E}(-E)$ will follow once we prove that

$$h^0(C, A \otimes A_E) = 4 \text{ and } h^1(C, A \otimes A_E) = 1. \tag{6.2}$$

To prove the latter, note that $h^0(C, A \otimes A_E) = \chi(C, A \otimes A_E) + h^1(C, A \otimes A_E) = 3 + h^1(C, A \otimes A_E)$ by Riemann–Roch. Since $A \ncong A_E$, we have $h^0(C, A \otimes A_E) \geq 4$; moreover, equality must hold, as otherwise $h^0(C, \omega_C \otimes (A \otimes A_E)^{-1}) = h^1(C, A \otimes A_E) \geq 2$ and $\deg(\omega_C \otimes (A \otimes A_E)^{-1}) = 4$, hence C would contain a g^1_4, a contradiction to Brill–Noether generality. This proves (6.2). $\square$

By abuse of notation, let E be an elliptic curve of the pencil $|E|$ on S. Since $H^0(\mathcal{E}|_E)$ is a 5-dimensional quotient space of $V_6 = H^0(S, \mathcal{E})$, each fiber $\mathcal{E}_s$ for $s \in E$ is a 2-dimensional quotient of $H^0(\mathcal{E}|_E)$ and hence of V_6. The image $\phi_{\mathcal{E}}(E)$ of the elliptic curve is contained in $G(H^0(\mathcal{E}|_E), 2)$. Since λ is surjective and E is projectively normal, we have the following commutative diagram

$$\begin{tikzcd}
\bigwedge^2 H^0(S, \mathcal{E}) \arrow[r, two heads, "\lambda"] \arrow[d] & H^0(S, \bigwedge^2 \mathcal{E}) \cong H^0(S, L) \arrow[d] \\
\bigwedge^2 H^0(E, \mathcal{E}|_E) \arrow[r] & H^0(E, \bigwedge^2 \mathcal{E}|_E) \cong H^0(E, L|_E).
\end{tikzcd}$$

So, we obtain the commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\phi_{\mathcal{E}|_E}"] \arrow[d, "\phi_{\bigwedge^2 \mathcal{E}|_E}"'] & G(H^0(\mathcal{E}|_E), 2) \arrow[r, hook] \arrow[d, "\text{Plücker}"] & G(V_6, 2) \arrow[d] \\
\mathbb{P}^4 = \mathbb{P}^*(H^0(E, L|_E)) \arrow[r, "\alpha"'] & \mathbb{P}^*(\bigwedge^2 H^0(E, \mathcal{E}|_E)) \arrow[r, hook] & \mathbb{P}^*(\bigwedge^2 H^0(S, \mathcal{E}))
\end{tikzcd}$$

where α is an embedding. The diagram is also cartesian. Indeed, let $\mathbb{P}^4 = \overline{E}$ be the linear span, then

$$E \subset \mathbb{P}^4 \cap G(H^0(\mathcal{E}|_E), 2) \subset \mathbb{P}^4 \cap G(V_6, 2) = \mathbb{P}^4 \cap \mathbb{P}^8 \cap G(V_6, 2) = S \cap \mathbb{P}^4.$$

But $E = S \cap \mathbb{P}^4$ since $|E|$ and $|L - E|$ are base point free (c.f. Lemma 6.2). Hence, it follows that $E = \mathbb{P}^4 \cap G(H^0(\mathcal{E}|_E), 2)$. By Section 6.1.1, the elliptic pencil $|E|$ on S is cut out by the Schubert cycle $\Sigma_1(V_4)$ on $G(V_6, 2)$ for some four-dimensional quotient V_4. Recall further that there is a one-to-one correspondence between such Schubert cycles and points on the dual Grassmannian $G(2, V_6)$.

The following corollary follows immediately from our discussion.

Corollary 6.4 *Let (S, L) be a Brill–Noether general polarized $K3$ surface of genus 8. Let $\mathbb{P}^5_{(S)} \subset \mathbb{P}_*(\bigwedge^2 H^0(S, \mathcal{E}))$ be the dual space of $\mathbb{P}^8 = \mathbb{P}^* H^0(S, L) \subset \mathbb{P}^*(\bigwedge^2 H^0(S, \mathcal{E}))$. There is a one-to-one correspondence between elliptic pencils $|E|$ on S satisfying $L.E = 5$ and points of $G(2, V_6) \cap \mathbb{P}^5_{(S)}$.*

6.1.3 Maximal number of distinct elliptic pencils

Let (S, L) be a Brill–Noether general $K3$ surface of genus 8, and let E_1, E_2 be two classes with $E_1^2 = E_2^2 = 0$ and $E_1.L = E_2.L = 5$. Then $E_1.E_2 = 2$. Indeed, the Hodge Index Theorem on $E_1 + E_2$ and L yields $E_1.E_2 \leq 3$. Equality implies $(E_1 + E_2)^2 = 6$ and $(L - E_1 - E_2)^2 = 0$, whence $h^0(S, E_1 + E_2) \geq 5$ and $h^0(S, L - E_1 - E_2) \geq 2$, a contradiction to Brill–Noether generality.

On can also see this fact geometrically using the notation of the previous section. Let V_5, V_5' be two distinct 5-dimensional quotients of V_6. The intersection of the Grassmannians $G(V_5, 2)$ and $G(V_5', 2)$ is the Grassmannian $G(V_5 \cap V_5', 2)$. The Grassmannian $G(V_5 \cap V_5', 2)$

is a 4-dimensional quadric. Hence, if $\mathbb{P}^8$ is a general linear subspace such that its intersection with $G(V_5, 2)$ and $G(V_5', 2)$ are elliptic curves, then these elliptic curves intersect in two points, namely $\mathbb{P}^8 \cap G(V_5 \cap V_5', 2)$.

If all our above assumptions are satisfied, we get a $K3$ surface with Picard lattice containing the following lattice

$$\begin{pmatrix} 14 & 5 & 5 & \dots & 5 \\ 5 & 0 & 2 & \dots & 2 \\ 5 & 2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 \\ 5 & 2 & \dots & 2 & 0 \end{pmatrix}.$$

An easy computation shows that the maximal possible rank is 10 (otherwise the matrix has at least two positive eigenvalues). Let $\mathfrak{G}_9$ be such a lattice of maximal possible rank which is given by the following intersection matrix

$$\mathfrak{G}_9 = \underbrace{\begin{pmatrix} 14 & 5 & 5 & \dots & 5 \\ 5 & 0 & 2 & \dots & 2 \\ 5 & 2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 \\ 5 & 2 & \dots & 2 & 0 \end{pmatrix}}_{\text{10 columns}}.$$

We denote S a $K3$ surface with the above Picard lattice $\mathfrak{G}_9$ of rank 10 (which again exists by [24, Thm. 2.9(i)] or [31]), and let L be the basis element of square 14, which can be taken to be big and nef by standard arguments (see [7, VIII, Prop. 3.10]). Let E_i, $i = 1, \dots, 9$, be the generators of square zero.

Lemma 6.5 *(a) The class L is ample.*
(b) The $K3$ surface (S, L) is Brill–Noether general.
(c) The classes $E_1, \dots, E_9$ define elliptic pencils.

This can probably be proved arguing as in the proof of Lemma 4.2, but the computations are much more tedious. Instead we will give a constructive proof in the next subsection.

6.2 A unirational construction of *K*3 surfaces with nine distinct elliptic pencils

Recall that any projective equivalence of two $K3$ surfaces that are linear sections of the Grassmannian $G(V_6, 2)$ is induced by an automorphism of V_6 (see [28, Theorem 0.2]).

By Corollary 6.4, any Brill–Noether general polarized $K3$ surface S of genus 8 with exactly nine elliptic pencils of degree five induces and is induced by a unique five-dimensional space $\mathbb{P}^5_{(S)}$ intersecting $G(2, V_6) \subset \mathbb{P}^{14}$ in exactly nine points. We reformulate this fact in the following proposition. To state it we denote $\mathcal{H}_{9,5}(G(2, V_6))$ the space of 9-secant 5-planes of the Grassmannian $G(2, V_6) \subset \mathbb{P}^{14}$ intersecting the latter in exactly nine points and $\widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$ this space modulo the automorphisms of V_6.

Proposition 6.6 *The moduli space of Brill–Noether general polarized $K3$ surfaces of genus 8 with exactly nine elliptic pencils of degree 5 is birational to $\widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$, and both spaces are non-empty.*

Proof By Corollary 6.4, we only need to prove the non-emptiness of $\mathcal{H}_{9,5}(G(2, V_6))$. A general intersection of $G(2, V_6)$ and a $\mathbb{P}^7$ is a smooth curve C of genus 8 and the general curve of genus 8 is obtained in this way (cf. [27]). Furthermore, a 9-secant 5-plane of $G(2, V_6)$ contained in this $\mathbb{P}^7$ is also a 9-secant of C, which is a divisor in a g^3_9 by the geometric Riemann–Roch. Note that the g^3_9 is automatically base point free as otherwise the curve would not be Brill–Noether general and thus could not be a linear section of the $G(2, V_6)$ by [27]. Hence a general divisor in the g^3_9 induces an element of $\mathcal{H}_{9,5}(G(2, V_6))$.

We have reduced the problem to constructing a curve of genus 8 as a linear section of $G(2, V_6)$ carrying a g^3_9, or equivalently, taking residuals, a g^1_5. Such a curve can be realized as follows: We get a divisor D of degree 5 in a g^1_5 on a curve C of genus 8 if we fix a $G(2, V_5)$ (where V_5 is a 5-dimensional subspace of V_6) and choose a $\mathbb{P}^7$ such that $C = \mathbb{P}^7 \cap G(2, V_6)$ and $D = \mathbb{P}^7 \cap G(2, V_5)$ induces the $g^1_5 = |D|$. In an ancillary file, cf. [15], we have implemented this construction in *Macaulay2* (see [12]) as well as the construction of the corresponding $K3$ surface. □

The Picard lattice of the $K3$ surfaces in the moduli space in Proposition 6.6 contains the lattice $\mathfrak{G}_9$ and the generator of square 14 is (very) ample and the generators of square 0 are nef. Let $\mathcal{F}^{\mathfrak{G}_9}$ be the moduli space of $\mathfrak{G}_9$-lattice polarized $K3$ surfaces. By standard deformation arguments (see [20, Thm. 14]) the very general element in $\mathcal{F}^{\mathfrak{G}_9}$ has Picard lattice equal to $\mathfrak{G}_9$, is Brill–Noether general with ample generator of square 14 and the generators of square 0 define elliptic pencils.

Proof of Lemma 6.5 The last discussion proves the lemma for the very general element in $\mathcal{F}^{\mathfrak{G}_9}$ having Picard lattice equal to $\mathfrak{G}_9$. Since the properties (a)-(c) of the lemma only depend on the lattice, this finishes the proof. □

We also have the following

Theorem 6.7 *The moduli space $\mathcal{F}^{\mathfrak{G}_9}$ of $\mathfrak{G}_9$-lattice polarized $K3$ surfaces is unirational.*

Proof The above discussion shows that $\mathcal{F}^{\mathfrak{G}_9}$ is birational to $\widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$. In particular, $\widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$ is irreducible.

Consider the following incidence variety

$$\{(V^9_5, \mathbb{P}^7) \in \mathcal{H}_{9,5}(G(2, V_6)) \times G(8, \Lambda^2 V_6) \mid V^9_5 \subset \mathbb{P}^7,\\ C = \mathbb{P}^7 \cap G(2, V_6) \text{ a smooth curve}\}$$

and denote I its quotient with the automorphisms of V_6 acting diagonally. Then I admits a natural first projection map $\pi_1 : I \to \widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$ and a second projection to the moduli space of curves of genus 8. As for $K3$ surfaces, any projective equivalence of two curves of genus 8 that are linear sections of the Grassmannian $G(2, V_6)$ is induced by an automorphism of V_6.

The proof of Proposition 6.6 shows that I is non-empty and is therefore birational to a $\mathbb{P}^3$-bundle over the universal Brill–Noether variety $\mathcal{W}^3_{8,9}$ by the universal Abel–Jacobi map. Hence, I is unirational and irreducible since $\mathcal{W}^3_{8,9} \cong \mathcal{W}^1_{8,5}$ is unirational (and irreducible) by [3]. Since π_1 is dominant (because $\widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$ is irreducible), $\widetilde{\mathcal{H}}_{9,5}(G(2, V_6))$ is unirational. The theorem follows. □

One may also consider, for $i \in \{0, \ldots, 8\}$, the moduli spaces $\mathcal{F}^{\mathfrak{G}_i}$ of $\mathfrak{G}_i$-lattice polarized $K3$ surfaces, where $\mathfrak{G}_i$ is the rank $i+1$ lattice

$$\mathfrak{G}_i = \underbrace{\begin{pmatrix} 14 & 5 & 5 & \ldots & 5 \\ 5 & 0 & 2 & \ldots & 2 \\ 5 & 2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 2 \\ 5 & 2 & \ldots & 2 & 0 \end{pmatrix}}_{i+1 \text{ columns}}$$

Then $\dim \mathcal{F}^{\mathfrak{G}_i} = 19 - i$ and $\mathcal{F}^{\mathfrak{G}_{i+1}} \subset \mathcal{F}^{\mathfrak{G}_i}$ for each $i \in \{0, \ldots, 8\}$. Note that $\mathcal{F}^{\mathfrak{G}_0} = \mathcal{F}_8$.

Theorem 6.8 *The moduli spaces $\mathcal{F}^{\mathfrak{G}_i}$ of $\mathfrak{G}_i$-lattice polarized $K3$ surfaces are unirational for $i \leq 6$.*

Proof The case $i = 0$ is proved in [28]. By Corollary 6.4 and Lemma 6.5, the general $K3$ surface in $\mathcal{F}^{\mathfrak{G}_i}$ corresponds uniquely to a five-dimensional projective space intersecting the Grassmannian $G(2, V_6) \subset \mathbb{P}^{14}$ in exactly i points modulo automorphisms of V_6. Such i-secant 5-planes are unirationally parametrized by the product of the i-th symmetric product of $G(2, V_6)$ and $(6 - i)$-th symmetric product of $\mathbb{P}^{14}$. □

We remark that the unirationality of $\mathcal{F}^{\mathfrak{G}_1}$ can also be shown using quartic surfaces in $\mathbb{P}^3$ containing an elliptic quintic curve. The question of (uni)rationality of $\mathcal{F}^{\mathfrak{G}_7}$ and $\mathcal{F}^{\mathfrak{G}_8}$ is open.

6.3 The moduli map

Let $\mathcal{F}_8$ denote the 19-dimensional moduli space of polarized $K3$ surface of genus 8 and $\mathcal{P}_8$ the moduli space of triples (S, L, C) where $(S, L) \in \mathcal{F}_8$ and $C \in |L|$ is a smooth irreducible curve. Let $m_8 : \mathcal{P}_8 \longrightarrow \mathcal{M}_8$ be the moduli map.

Proposition 6.9 *Let $(S, L) \in \mathcal{F}_8$ be a Brill–Noether general $K3$ surface such that S contains an elliptic pencil $|E|$ satisfying $E.L = 5$. Then the fiber of m_8 is smooth and 6-dimensional at any point represented by a smooth curve C in $|L|$.*

Proof By comparing dimensions, the fibers of m_8 are at least 6-dimensional. (It is known that m_8 is dominant, and therefore its general fibers are precisely 6-dimensional, but we will not use this.) By [33, §3.4.4] or [8], the kernel of the differential of m_8 at a point (S, L, C) is isomorphic to $H^1(\mathcal{T}_S(-L))$. To prove the proposition, it therefore suffices by Serre duality to prove that $h^1(\Omega_S(L)) \leq 6$.

Let $\varphi : S \to \mathbb{P}^3$ be the morphism defined by $|L - E|$ and S_0 be its image, which is a quartic surface. By Lemma 6.2 its possible singularities are images of contracted disjoint rational curves Γ_i on S, $i = 1, \ldots, k$. By [25, Thm. 2.1] we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\Gamma_1 + \cdots + \Gamma_k} \longrightarrow \varphi^* \Omega_{S_0} \longrightarrow \Omega_S \longrightarrow \mathcal{O}_{\Gamma_1 + \cdots + \Gamma_k} \longrightarrow 0. \tag{6.3}$$

Twisting by $\mathcal{O}_S(L)$, taking cohomology and using the fact that $\Gamma_i \cdot L = 1$ by Lemma 6.2, we obtain

$$h^1(\Omega_S(L)) \leq h^1(\varphi^* \Omega_{S_0}(L)). \tag{6.4}$$

Pulling back the conormal bundle sequence

$$\mathcal{O}_{S_0}(-4) \cong \mathcal{I}_{S_0/\mathbb{P}^3}/\mathcal{I}^2_{S_0/\mathbb{P}^3} \longrightarrow \Omega_{\mathbb{P}^3}|_{S_0} \longrightarrow \Omega_{S_0} \longrightarrow 0$$

and twisting by $\mathcal{O}_S(L)$, we obtain

$$\mathcal{O}_S(-3L+4E) \longrightarrow \varphi^*\Omega_{\mathbb{P}^3}|_{S_0}(L) \longrightarrow \varphi^*\Omega_{S_0}(L) \longrightarrow 0.$$

The left hand map is injective, as $\mathcal{O}_S(-3L+4E)$ is locally free. Thus,

$$h^1(\varphi^*\Omega_{S_0}(L)) \leq h^1(\varphi^*\Omega_{\mathbb{P}^3}|_{S_0}(L)) + h^0(3L-4E), \tag{6.5}$$

using Serre duality. Pulling back the dual of the Euler sequence,

$$0 \longrightarrow \Omega_{\mathbb{P}^3}|_{S_0} \longrightarrow H^0(\mathcal{O}_{S_0}(1)) \otimes \mathcal{O}_{S_0}(-1) \longrightarrow \mathcal{O}_{S_0} \longrightarrow 0$$

and twisting by $\mathcal{O}_S(L)$, we obtain

$$0 \longrightarrow \varphi^*\Omega_{\mathbb{P}^3}|_{S_0}(L) \longrightarrow H^0(L-E) \otimes \mathcal{O}_S(E) \longrightarrow \mathcal{O}_S(L) \longrightarrow 0.$$

Hence, since $h^1(E) = 0$ as E is irreducible, we obtain

$$h^1(\varphi^*\Omega_{\mathbb{P}^3}|_{S_0}(L)) \leq \operatorname{cork} \mu, \tag{6.6}$$

where μ is the multiplication map of sections

$$\mu : H^0(L-E) \otimes H^0(E) \longrightarrow H^0(L).$$

Combining (6.4), (6.5) and (6.6), we see that we obtain the desired inequality $h^1(\Omega_S(L)) \leq 6$ if we prove that

$$h^0(3L-4E) = 5 \tag{6.7}$$

and

$$\operatorname{cork} \mu = 1. \tag{6.8}$$

To prove (6.8), note that the evaluation map $H^0(E) \otimes \mathcal{O}_S \to \mathcal{O}_S(E)$ is surjective as $|E|$ is base point free and has kernel $\mathcal{O}_S(-E)$. Twisting by $\mathcal{O}_S(L-E)$, we obtain

$$0 \longrightarrow \mathcal{O}_S(L-2E) \longrightarrow H^0(E) \otimes \mathcal{O}_S(L-E) \longrightarrow \mathcal{O}_S(L) \longrightarrow 0$$

Taking cohomology and using the fact that $h^1(L-E) = 0$ as $L-E$ is big and nef by Lemma 6.2, we obtain that $\operatorname{cork} \mu = h^1(L-2E)$.

We have $(L-2E).L = 4$, hence $h^2(L-2E) = h^0(2E-L) = 0$, as L is ample. Similarly, $h^0(L-2E) = 0$, since $(L-2E).(L-E) = -1$ and $L-E$ is nef. Since $(L-2E)^2 = -6$, Riemann–Roch yields $h^1(L-2E) = 1$, and (6.8) is proved.

To prove (6.7), note that $(3L-4E)^2 = 6$ and $h^2(3L-4E) = h^0(4E-3L) = 0$, as $(4E-3L).E < 0$ and E is nef. Hence, (6.7) is equivalent to $h^1(3L-4E) = 0$.

To get a contradiction, assume that $h^1(3L-4E) > 0$. Then, by [19], there exists an effective divisor Δ such that $\Delta^2 = -2$ and $k := -\Delta.(3L-4E) \geq 2$. Since $\Delta.L > 0$, as L is ample, we must have

$$\Delta.E \geq 2. \tag{6.9}$$

One computes $(3L - 4E - k\Delta)^2 = 6$ and $(3L - 4E - k\Delta).(L - E) = 7 - k\Delta.(L - E)$. By the Hodge index theorem,

$$24 = (3L - 4E - k\Delta)^2 \cdot (L - E)^2 \leq [7 - k\Delta.(L - E)]^2 ,$$

and the only possibilities are

(I) $\Delta.(L - E) = 0$; or
(II) $\Delta.(L - E) = 1$ and $k = 2$.

In case (I) we find $(L - E - \Delta)^2 = 2$ and $(L - E - \Delta).(L - E) = 4$, whence $h^0(L - E - \Delta) \geq 3$ by Riemann–Roch and Serre duality. By (6.9) we have $(E + \Delta)^2 \geq 2$, thus also $h^0(E + \Delta) \geq 3$ by Riemann–Roch. But then $h^0(L - E - \Delta)h^0(E + \Delta) \geq 9 = 8 + 1$, contradicting Brill–Noether generality.

In case (II) we have $\Delta.L = \Delta.E + 1$ and $-2 = \Delta.(3L - 4E)$, which together yield $\Delta.E = 5$ and $\Delta.L = 6$. Therefore, $(L - E - \Delta)^2 = 0$ and $(L - E - \Delta).L = 3$, it follows $h^0(L - E - \Delta) \geq 2$ by Riemann–Roch and Serre duality. Moreover, $(E + \Delta)^2 = 8$, whence $h^0(E + \Delta) \geq 6$ by Riemann–Roch. Similarly to the previous case, we obtain a contradiction to Brill–Noether generality.

This shows that (6.7) holds and finishes the proof of the proposition. $\square$

For $i \in \{0, \ldots, 9\}$, let $\mathfrak{G}_i$ and $\mathcal{F}^{\mathfrak{G}_i}$ be as in the previous subsection and let $\mathcal{P}^{\mathfrak{G}_i}$ be the moduli space of triples as in Sect. 2. Note that $\mathcal{P}^{\mathfrak{G}_i}$ is birational to the open part of the tautological $\mathbb{P}^8$-bundle over $\mathcal{F}^{\mathfrak{G}_i}$ consisting of pairs (S, C) with $[S] \in \mathcal{F}^{\mathfrak{G}_i}$ and $[C]$ representing a smooth curve in $|L|$, where L is the generator class of square 14 in $\mathfrak{G}_i$. We have $\mathcal{P}^{\mathfrak{G}_{i+1}} \subset \mathcal{P}^{\mathfrak{G}_i}$ for each $i \in \{0, \ldots, 8\}$.

Let $m_8^{\mathfrak{G}_i} : \mathcal{P}^{\mathfrak{G}_i} \to \mathcal{M}_8$ be the moduli map.

Proposition 6.10 *For each $i \in \{0, \ldots, 9\}$, a general fiber of $m_8^{\mathfrak{G}_i}$ has dimension* $\max\{0, 6 - i\}$.

Proof By Proposition 6.9, the fiber of $m_8^{\mathfrak{G}_0}$ is smooth and 6-dimensional at any point $(S, C) \in \mathcal{P}^{\mathfrak{G}_9}$. Fix such an (S, C).

We will show that there exists a chain of irreducible components $F_i \subset (m_8^{\mathfrak{G}_i})^{-1}([C])$ of the fiber of $m_8^{\mathfrak{G}_i}$ for $i \in \{0, \ldots, 5\}$, respectively, containing $(S, C) \in \mathcal{P}^{\mathfrak{G}_9}$ such that

$$(S, C) \in F_5 \subsetneq F_4 \subsetneq \cdots \subsetneq F_1 \subsetneq F_0.$$

Consequently, there exist $K3$ surfaces $S_i \in \mathcal{F}^{\mathfrak{G}_i} \backslash \mathcal{F}^{\mathfrak{G}_{i+1}}$ for $i \in \{0, \ldots, 5\}$ containing C. Since $\dim F_0 = 6$ by Proposition 6.9, the dimension of F_i is $6 - i$ for $i \in \{0, \ldots, 5\}$ and the proposition will follow.

By construction, S (resp. C) is the intersection of $G(V_6, 2)$ with a $\mathbb{P}^8$ (respectively a $\mathbb{P}^7$) in $\mathbb{P}^{14}$. The dual $\mathbb{P}^5$ of the $\mathbb{P}^8$, which we henceforth call $\mathbb{P}^5_{(S)}$, intersects the dual $G(2, V_6)$ in 9 points, call them $x_1, \ldots, x_9$, and the dual $\mathbb{P}^6$ of the $\mathbb{P}^7$, which we henceforth call $\mathbb{P}^6_{(C)}$, contains $\mathbb{P}^5_{(S)}$.

By construction, the nine points $x_1, \ldots, x_9$ span $\mathbb{P}^5_{(S)}$. Thus, we may find inside $\mathbb{P}^6_{(C)}$ a set of six additional hyperplanes $\mathbb{P}^5_{(i)}$, $i \in \{0, \ldots, 5\}$ containing precisely i of the points $x_1, \ldots, x_9$; in particular $\mathbb{P}^5_{(i)}$ intersects $G(2, V_6)$ in precisely i points.

Denote by $\mathbb{P}^8_{(i)}$ the dual $\mathbb{P}^8$ of $\mathbb{P}^5_{(i)}$. Then $\mathbb{P}^8_{(i)} \cap G(V_6, 2)$ is a $K3$ surface S_i containing C and precisely i elliptic pencils of degree 5 (and mutually intersecting in 2 points) by Corollary 6.4. As the nine elliptic pencils together with C generate $\mathfrak{G}_9 \subset \mathrm{Pic}(S)$, we also have that C

and the i elliptic pencils generate $\mathfrak{G}_i \subset \mathrm{Pic}(S_i)$, whence $S_i \in \mathcal{F}^{\mathfrak{G}_i} \backslash \mathcal{F}^{\mathfrak{G}_{i+1}}$. Each pair (S_i, C) therefore lies in $F_i \backslash F_{i+1}$. This concludes the proof. □

Corollary 6.11 *For each* $i \in \{0, \ldots, 9\}$*, the codimension of the image of the moduli map* $m_8^{\mathfrak{G}_i}$ *is* $\max\{0, i-6\}$*. In particular, a general curve of genus* 8 *is a linear section of a* $K3$ *surface such that precisely six out of its* 14 g^1_5*s are induced by elliptic pencils on the* $K3$ *surface. Moreover, there is a codimension k family of curves lying on a* $K3$ *surface such that precisely* $6+k$ *of its* g^1_5*s are induced by elliptic pencils on the* $K3$ *surface for* $k \in \{1, 2, 3\}$.

Remark 6.12 One can ask similar questions for $K3$ surfaces of higher even genus. For instance, how many elliptic pencils of minimal degree exist on a Brill–Noether general $K3$ surface? But the methods in this article cannot be applied to $K3$ surfaces of higher genus. Indeed, let C be a Brill–Noether general curve of even genus $g \geq 10$. Note on the one hand that the curve C does not lie on a $K3$ surface and on the other hand that the (finite) number of pencils of minimal degree on C is bigger that 19 (the maximal rank of the Picard lattice of a smooth $K3$ surface). Furthermore, a characterization of Brill–Noether general $K3$ surfaces is only known for $g \leq 10$ and 12.

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