A note on the multiphase balance equations

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Abstract

The formulation and use of the momentum equations for multiphase systems are not always consistent in the literature. The objective of this note is to clarify an issue, which long has dogged the literature on the multiphase flow equations. The standard book by Gidaspow, which historically has formed the basis of at least two of the major CFD software packages in common use today, describes three formulations of the momentum equations for fluid (continuous phase)-solids (dispersed phase) flows that are often not clearly distinguished, as this note will demonstrate.

This note focuses on Models "A" and "B", which are more widely used than model "C". Their form and significance are discussed, as are the most common flaws in their interpretation and use.

The equations are listed in their differential form, the form in which they are normally used, although in their nature, and when derived, the equations are not differential.

1 Continuity equations

The continuity equations are the same for models A and B. They are 1-D equations derived by considering the flow in a tube of cylindrical cross-section.

Fluid:

$$\frac{d}{dx}\left[\epsilon\rho_g v_g\right] = 0 \tag{1}$$

Solid:

$$\frac{d}{dx}\left[(1-\epsilon)\rho_s v_s\right] = 0 \tag{2}$$

Since the book of Gidaspow [1] discusses gas fluidization, the phase indices are "g" for gas (the continuous or carrier phase) and "s" for solid (the dispersed phase(s)). ϵ is the "voidage fraction" or the volume fraction of the continuous phase; x is the coordinate in the direction of the axis of the containing tube. The rest of the symbols have their usual meaning.

2 Momentum equations

2.1 Equations Common to the two models

The equation for the mixture momentum is the same for models A and B:

$$\frac{d\left[(1-\epsilon)\rho_s v_s^2\right]}{dx} + \frac{d\left[\epsilon\rho_g v_g^2\right]}{dx} = -\frac{dP}{dx} - \frac{d\sigma}{dx} - g\rho_s(1-\epsilon) - g\rho_g\epsilon - \frac{4(\tau_{wg}+\tau_{ws})}{D_t}. \tag{3}$$

The accumulation terms can be rewritten using the equation of continuity, for example:

$$\frac{d\left[\epsilon\rho_g v_g^2\right]}{dx} = \epsilon\rho_g v_g\frac{dv_g}{dx} + v_g\frac{d\left[\epsilon\rho_g v_g\right]}{dx} = \epsilon\rho_g v_g\frac{dv_g}{dx},$$

since the second term in the middle part of this equation is zero by the continuity equation.

2.2 Equations that differ between models A and B

2.2.1 Model A

Fluid momentum, modelling the upflow of a fluid through a bed of particles in a cylindrical tube of diameter D_t:

$$\frac{d\left[\epsilon\rho_g v_g^2\right]}{dx} = -\epsilon\frac{dP}{dx} - g\rho_g\epsilon - \frac{4\tau_{wg}}{D_t} - \beta_A(v_g - v_s). \tag{4}$$

Solid momentum:

$$\frac{d\left[(1-\epsilon)\rho_s v_s^2\right]}{dx} = -(1-\epsilon)\frac{dP}{dx} - \frac{d\sigma}{dx} - g\rho_s(1-\epsilon) - \frac{4\tau_{ws}}{D_t} - \beta_A(v_s - v_g). \tag{5}$$

These equations are symmetrical for the two phases if the solids stress, σ, arising from particle-particle collisions, is neglected. The symbols have their usual meaning.

2.2.2 Model B

Fluid momentum:

$$\frac{d\left[\epsilon\rho_g v_g^2\right]}{dx} = -\frac{dP}{dx} - g\rho_g - \frac{4\tau_{wg}}{D_t} - \beta_B(v_g - v_s). \tag{6}$$

Solid momentum:

$$\frac{d\left[(1-\epsilon)\rho_s v_s^2\right]}{dx} = -\frac{d\sigma}{dx} - g(\rho_s - \rho_g)(1-\epsilon) - \frac{4\tau_{ws}}{D_t} - \beta_B(v_s - v_g). \tag{7}$$

2.3 Difference between the models

A key point here is that in model A both the dynamic pressure and the hydrostatic pressure are considered to act through both the phases, while in Model B the pressure is considered to act only through the continuous-phase fluid.

Model A thus describes the situation where both phases are continuous and interpenetrating, such that neither plays the role of a dispersed phase and the shared pressure *acts through both phases.* In the derivation of each phase equation the pressure on the in- and outflow faces of the control volume is applied *only to the area filled with that particular phase.* The symmetry of the phase equations in model A means that they are suitable also for systems involving phase inversion, which may be one reason that this model is preferred in CFD simulation software packages. The main issue with the equations of this model is that analyses by the method of characteristics [2] show them to be only conditionally well-posed as discussed by Lyczkowski et al. [3] and Pannala et al. [4].

Model B is directly formulated for the situation where one of the phases is dispersed as particles or droplets in the other, and the pressure does not, therefore, act through the dispersed phase but only through the continuous one. In this model the pressure is applied *over the entire cross-sectional area* in the derivation of the continuous-phase equation. In model B an explicit buoyancy term is included in the solid phase equation and the phase equations are not symmetrical.

In both models the fluid and solid equations can be obtained by subtracting the solid and fluid equations, respectively, from the mixture equation.

2.4 Drag

Note that:

$$\beta_B = \beta_C = \frac{\beta_A}{\epsilon}. \tag{8}$$

β_B and β_C are the actual drag coefficients[1] and can be found directly from the Ergun equation or from the definition of C_D with correction for the presence of other particles, while to find β_A these expressions must be multiplied by ϵ (Equations (2.11) and (2.12) in the book of Gidaspow [1] showing that $\beta_A(v_g - v_s)$ is drag multiplied by ϵ as stated in his Equation (2.17)).

2.5 Buoyancy

To recover, also in Model A, that in fluidized beds the dissipative drag balances the force of gravity minus the classical, Archimedean, buoyancy force Gidaspow subtracts the solid equation divided by $(1-\epsilon)$ from the fluid equation divided by ϵ and neglects acceleration, solids and fluid wall friction and solids pressure to obtain:

$$\begin{aligned} g(\rho_s - \rho_g)(1-\epsilon) &= \frac{\beta_A(v_g - v_s)}{\epsilon} \\ \text{net gravity} &= \text{drag}. \end{aligned} \tag{9}$$

where "net gravity" can be seen to include the effect of Archimedian buoyancy.

[1] drag force per unit volume of suspension per unit of velocity difference between the phases

In Model B it follows directly from the solids momentum equation that:

$$g(\rho_s - \rho_g)(1-\epsilon) = \beta_B(v_g - v_s). \tag{10}$$

2.6 Relation to Crowe et al. [5]

An example, among many, of the misinterpretations of the above we choose the standard, and otherwise very useful—especially the first edition—textbook of Crowe et al. [5], because it is the most used and quoted reference representing this type of interpretation.

In the following we concentrate on the continuous-phase, i.e. fluid, momentum equation.

The final form of the continuous-phase momentum equation derived in Crowe et al. [5] is consistent with Model A above, it is:

$$\frac{\partial(\rho_c\alpha_c u)}{\partial t} + \frac{1}{A}\frac{\partial(\alpha_c\rho_c u^2 A)}{\partial x} = s_{\mathrm{mass}}v - \alpha_c\frac{\partial p}{\partial x} + \beta_V(v-u) - \frac{1}{R_h}\tau_w + \alpha_c\rho_c g \tag{11}$$

Tranlating this to the notation used in this paper, which is largely consistent with that used by Gidaspow, gives:

$$\frac{\partial(\rho_c\epsilon v_g)}{\partial t} + \frac{1}{A}\frac{\partial(\epsilon\rho_c v_g^2 A)}{\partial x} = s_{\mathrm{mass}}v_s - \epsilon\frac{\partial p}{\partial x} + \beta_A(v_s - v_g) - \frac{1}{R_h}\tau_w + \epsilon\rho_c g \tag{12}$$

where Crowe et al. account for the momentum transport associated with a mass exchange, s_{mass}, between the phases and for a tube of varying cross-sectional area, A, which Gidaspow does not. R_h is the hydraulic radius of the conduit and in the drag term the velocities are reversed in the equation giving by Crowe et al. compared to Equation (4).

Two aspects of the derivation of this equation are problematic, their effects cancel out to leave the correct form of the final equation.

1. Crowe et al. apply the pressure in the fluid over the *entire* cross-section, as Gidaspow does in Model B, using arguments that would be correct for model B but

not for model A (see equations (6.38) and (6.41) in [5] and the arguments leading up to them, noting that $\Delta A = 0$ in [1]) and

2. they include the reaction on the continuous phase of a force assumed to be acting on the particles proportional to the macroscopic pressure drop (their equation (6.44)). This is often referred to as a "pressure gradient force".

while the operation in item 1 is inconsistent with the derivation in ref. [1], the operation in item 2 brings Crowe et al. back to the proper form of the model A equation. However, as pointed out here, and also shown by the extended Bernoulli equation and by the analysis of Happel and Brenner [6] (their equations (3-6.6) and (3-6.7))[2], the pressure drop in excess of the hydrostatic pressure gradient is due to dissipative drag, and is therefore properly and completely accounted for in the term for the dissipative drag acting on the particles. Including an extra force, $V_p \frac{\Delta P}{\Delta L}$, which is conservative, acting on the particles proportional to the macroscopic pressure drop, is therefore, although bringing them back to the correct form of Gidaspow's Model A equation, not consistent with the extended Bernoulli equation. Several publications in addition to [1] and [7], many of which are related to fluidized beds (e.g. refs. [8–10]), have pointed out that such a force is not physically correct: all the fluid-particle interaction, aside from Archimedean buoyancy, is due to dissipative drag (except when the entire system is accelerating, e.g. as in a pipeline bend) consistent with equations (9) and (10) above.

Maxey and Riley [11] point out that if a force due to the pressure gradient in the undisturbed flow around the particle is included, then also a dissipative force due to the undisturbed flow should be included, such that the force acting on a neutral-density particle due to the undisturbed flow is equal to the force that would act on a fluid particle in that position.

As discussed above, if Crowe et al. had omitted the force assumed to act between the particle and fluid phases proportional to the part of the macroscopic pressure drop

[2]The analysis of Happel and Brenner also shows that the force proportional to the macroscopic pressure drop acting on larger objects in a bed of small particles is associated with dissipation

generated by the flow, and also had applied the pressure forces on in-and outflow faces of their fluid control volume only to the continuous phase rather than the whole cross-section, their resulting equations would also have been consistent with Gidaspow's model A.

The argument made in this note is a theoretical one, but it can be interesting to see whether the presence or absence of a pressure gradient force will make a detectable difference in model predictions. In fluidized beds the issue translates to whether the buoyancy force on the fluidized particles is calculated using the suspension density (assuming a pressure gradient force) or the fluid density (without it) [9, 12]. Foscolo and Gibilaro proposed an innovative model to determine at which voidage, ϵ_{mb}, a Geldart group A powder would transition from particulate to aggregative (bubbling) fluidization when increasing the fluidization velocity, an important design variable for fluidized beds. They assumed the presence of a "pressure gradient force", and others, namely Jean and Fan [13] and Mazzei et al. [12] reformulated the model to calculate the buoyancy using the fluid density so that on basis of these papers it is possible to assess the effect in practice (Mazzei et al. also reformulated the constitutive equations involved in the model in some other respects). Figure 1 shows a parity plot, plotted on basis of experimental results and predictions given in tabular form in [12]. The figure clearly shows that there is a correlation between experiment and the predictions of the model of Foscolo and Gibilaro. It also shows that the corrections proposed by Jean and Fan and Mazzei et al., eliminating the "pressure gradient force", do make a significant difference to the model predictions, over most of the range improving the agreement with experiment. The powders used by Mazzei et al. are mostly relatively fine FCC powders, which have a low envelope density.

[Figure 1 about here.]

3 Conclusions

It could be argued that since the resulting continuous phase momentum equations are the same, the inconsistencies pointed out here are only of academic interest, having practical ramifications only in some specific contexts. However, it is always damaging to research if incorrect arguments are being promulgated in the literature. Some CFD software packages offer an option for a "pressure gradient force" acting on the dispersed phase to be turned off or on [15, 16] and in some cases the "on" option has even been made the default.

And also there is the "academic interest": to make sense, teaching should be consistent with the fundamental result that, in an inertial system, any force, except for Archimedean buoyancy, acting between the dispersed and continuous phases is dissipative drag, no additional conservative force acts.

4 Acknowledgements

The author would like to thank professor Boris Balakin for useful discussions.

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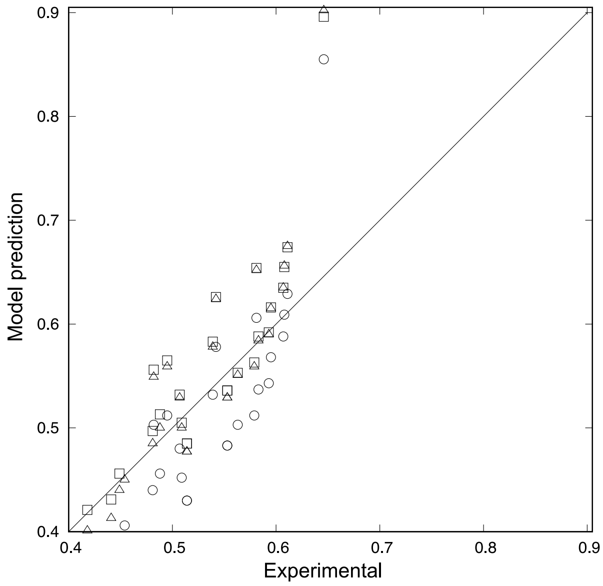

Figure 1: Parity plot between experimental and model [14] predicted values for the voidage fraction at the transition from particulate to bubbling fluidization for Geldart group A powders according to the tabulated data of Mazzei et al. [12]. Circles: Foscolo and Gibilaro; triangles: Mazzei et al.; squares: Jean and Fan.