Brauer groups of bielliptic surfaces and classification of irregular surfaces in positive characteristic

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Thesis for the degree of Philosophiae Doctor (PhD)
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Abstract

In this work we tackle three problems about surfaces.

In Part I (Chapter 2) we study the Brauer groups of bielliptic surfaces in characteristic zero. More precisely, given a bielliptic surface X, we give explicit generators for the torsion of the second cohomology group $\mathrm{H}^2(X,\mathbb{Z})$ of each type of bielliptic surface, and we determine the injectivity (and possibly the triviality) of the Brauer maps arising from canonical covers and bielliptic covers. This part is based on [FTVB19].

In Part II (Chapter 3 and Chapter 4) we deal with two problems of characterisation of surfaces in positive characteristic.

In Chapter 3 we show that a smooth projective surface over an algebraically closed field of characteristic at least five is birational to an abelian surface if and only if $P_1(S) = P_4(S) = 1$ and $h^1(S, \mathcal{O}_S) = 2$ ([Fe19]). Also, we discuss the fact that K3 surfaces are characterised by $P_1(S) = P_2(S) = 1$ and $h^1(S, \mathcal{O}_S) = 0$.

In Chapter 4 we study surfaces of general type with $p_g(S) = h^1(S, \mathcal{O}_S) = 3$ in positive characteristic. We compare our results to those of [HP02] and [Pi02] in characteristic zero.

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Contents

Introduction

In this thesis we study three problems that have as a common factor the feature of being all problems about surfaces. Surfaces and curves are among the most studied objects in algebraic geometry, and they have been in the spotlight since the early days of the discipline in the 19th century.

The first results in what was to become the theory of surfaces where studies by Cremona and Clebsch about rational surfaces. Subsequently, Clebsch and Noether started to generalise to surfaces concepts that had been used for curves. For example, in 1868 Clebsch extended to surfaces the concept of genus (conceptualising what later on would be called the *geometric genus* p_g), while Noether in 1886 introduced *linear systems of curves* on surfaces as a generalisation of the already-in-use idea of linear series on curves. Furthermore, Noether arrived at the definitions of an invariant p_a, called the *arithmetic genus* of the surface, and also of the *irregularity* q of the surface, at the time defined as $q := p_g - p_a$.

Both the geometric genus and the arithmetic genus were shown to be birational invariants; birational transformations were introduced and then were studied by Cremona from 1863 onwards (see [BC95, §1.1, §1.2]). The study of surfaces continued, in particular with the Italian school, and one of the principal topics in the theory of surfaces became their birational classification; this topic was taken up by Castelnuovo after 1981, about twenty years after the seminal work of Clebsch and Noether. Castelnuovo also directed Enriques towards these issues. Plurigenera and minimal models made their appearance during this period (see [BC95, §1.5]). Algebraic geometers, in particular Severi and De Franchis, were also very interested in the study of *irregular surfaces*, i.e. surfaces with $q > 0$. Noether in 1875 had famously erroneously conjectured that all surfaces but the ruled ones should have $q = 0$; as the knowledge about algebraic surfaces deepened, mathematicians started to realise how rich the geometry of irregular surfaces is.

In Part I - Chapter 2 - of this thesis we deal with a modern problem about a classical type of surface (Brauer groups and Brauer maps of bielliptic surfaces) in characteristic zero, while in Part II we address a very classical kind of question (the characterisation of surfaces via numerical birational invariants), but in positive characteristic. We deal with abelian surfaces in Chapter 3, and with surfaces of general type with $p_g = q = 3$ in Chapter 4. To prove our results, we will use classical results from the theory of surfaces, and we will need to employ several results from the theory developed around abelian varieties. In particular, we will often have to study fibrations on surfaces and elliptic surfaces.

In the 1900s, Bagnera and De Franchis published their famous classification result for bielliptic surfaces over the complex numbers. They showed that bielliptic surfaces can be divided into seven types, and they explained how each type is constructed as a quotient of the product of two complex elliptic curves A, B by the action of a finite group G.

Bielliptic surfaces, together with the other surfaces of Kodaira dimension zero, are among the most studied and well-understood algebraic surfaces; in this thesis we work with their Brauer groups. The Brauer group of an elliptic surface is trivial for three of the types identified in the Bagnera-De Franchis classification, thus we focus on the other four types (type one, two, three and five). First, for such a surface S, we find explicit generators for the torsion of $\mathrm{H}^2(X, \mathbb{Z})$, which is non-canonically isomorphic to the Brauer group $\mathrm{Br}(S)$ of S, in terms of the reduction of the multiple fibres arising from the fibration to $\mathbb{P}^1$ given in the Bagnera-De Franchis classification. More specifically, we prove that $\mathrm{H}^2(X, \mathbb{Z})_{\mathrm{tor}}$ is generated by the differences of such divisors (Proposition 2.2.1).

To any bielliptic surface S we can associate an étale cyclic cover $X \to S$ induced by ω_S, where X is an abelian surface, called the *canonical cover* of S. This cover is a (possibly non-trivial) intermediate quotient of the quotient morphism $A \times B \to S$. We study the induced homomorphism $\mathrm{Br}(S) \to \mathrm{Br}(X)$ and determine when it is trivial and when it is injective.

The analogous problem for the canonical cover of Enriques surfaces (which is a K3 surface) was studied by Beauville in [Bea09]. The Brauer group of an Enriques surface is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so that the Brauer map of the canonical cover is non-trivial if and only if it is injective. Beauville showed that, in the moduli space of Enriques surfaces, the surfaces with trivial Brauer map of the canonical cover belong to a countable union of hypersurfaces. In our investigation, one pivotal tool is the description given in [Bea09] of the kernel of the Brauer map induced by an étale cyclic cover.

For bielliptic surfaces we show that the answer to the problem depends on the geometry of the surface. More precisely,

- if the curves A and B are non-isogenous, then the Brauer map induced by the canonical cover is always injective (Theorem 2.5.4);
- if the curves A and B are isogenous, then
    - ∗ if S is of type one or two we distingush between the case in which B has complex multiplication and the case in which it does not have it; for both cases we show that the failure to be injective is equivalent to the triviality of at least one of some line bundles on B and points on A that depend on the construction of S (Theorem 2.5.9 and Theorem 2.5.22). For type-two bielliptic surfaces triviality and injectivity of this Brauer map are the same (since the Brauer group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$), while for type-one bielliptic surfaces they are not, thus we specify a necessary and sufficient condition for triviality in Theorem 2.5.11;

* if S is of type three or five injectivity (and therefore triviality, since the Brauer groups are cyclical of prime order) is equivalent to the non-triviality of a line bundle on B coming from the construction of the surface (Theorem 2.5.15 and Theorem 2.5.20)

As seen in the work of Nuer ([Nue]), bielliptic surfaces, in the case when the group G is non-cyclic or cyclic but not of prime order, also have an étale cyclic cover $\tilde{S} \to S$, where $\tilde{S}$ is another bielliptic surface. We study the Brauer map induced by this cover in the cases in which $\mathrm{Br}(S)$ is non-trivial, i.e. for type-two and type-three bielliptic surfaces. We show that (Corollary 2.4.2 and Theorem 2.4.3):

Theorem. *Consider the étale cyclic cover $\tilde{S} \to S$ introduced above. Then:*

* *if S is of type two, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is trivial;*
* *if S is of type three, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is injective.*

This result, interesting on its own, is also used as an intermediate step to settle the problem we have already discussed of injectivity of the Brauer map induced by the canonical cover.

In Part II of this thesis we shift our focus to a very classical problem, namely the birational classification of surfaces in terms of invariants. We tackle two problems that have been solved in characteristic zero, and we study them in positive characteristic. As we have already mentioned, the problem of finding which sets of birational invariants, and in particular numerical birational invariants, correspond to which surfaces, has been studied since the 19th century. One of the most famous results among these is Castelnuovo's Rationality Criterion, which says that complex rational surfaces are characterised by $q = P_2 = 0$. Surfaces of Kodaira dimension zero are of special interest, all the more so taking into account that each of their minimal models belongs to one of only four families. Their invariants were detemined thanks to the work of Castelnuovo and Enriques in particular.

In modern terminology, Enriques in [En1905] proved the following result about complex abelian surfaces:

Theorem (Enriques)**.** *Let S be a smooth complex projective surface. If*

$$P_1(S) = P_4(S) = 1, \qquad h^1(S, \mathcal{O}_S) = 2,$$

then S is birational to an abelian surface.

Unlike the common greater generality of today, Enriques was working with zero loci of polynomials in $\mathbb{P}^3$. Also, in his result of 1905 he does not mention the term *abelian surfaces*; instead he writes of *hyperelliptic surfaces* (not what today is also known as bielliptic surfaces), that at the time meant surfaces representable through functions which are four times periodical in two parameters and that can be referred to the variety of the couples of points of a curve of genus two (the Jacobian of a curve of genus two). At the time, *abelian varieties* was a term meaning, very loosely speaking, all the projective algebraic varieties whoose field of rational functions was a certain field of abelian functions (and Lefschetz in 1919 proved that there is a distinguished one that is a quotient of some copies of $\mathbb{C}$ by a lattice; see [Kl05, §1.]). Our modern definition of abelian variety is the one introduced by Weil in 1948.

Abelian varieties, in the modern sense, occupy a special place in algebraic geometry because of their many interesting properties and the way they often appear in connection to other varieties, for example as Albanese variety. Many efforts have been made to find a birational characterisation of abelian varieties; among them:

* in a paper of 1981 Kawamata showed that, if $\kappa(X) = 0$, then $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space. In particular abelian varieties are characterized by $\kappa(X) = 0$ and irregularity equal to $\dim X$;

* Kóllar first showed in a paper of 1986 that if $P_1(X) = P_4(X) = 1$ and X has maximal Albanese dimension, then X is birational to an abelian variety. He subsequently improved his result by taking as hypotheses $P_3(X) = 1$ and irregularity equal to $\dim X$. He also conjectured that the same should hold taking $P_2(X)$ instead of $P_3(X)$;

* in an article of 1997 Ein and Lazarsfeld developed some Generic Vanishing techniques that enabled them to reprove the result of Kóllar's paper of 1986.

In [CH01], Chen and Hacon used a result of the paper by Ein and Lazarsfeld to show that a complex abelian variety X is birationally characterised by $P_1(X) = P_2(X) = 1$ and $h^1(X, \mathcal{O}_X) = \dim X$. Thus, they improved the result of Enriques. The Generic Vanishing techniques used to prove this result are known to fail in positive characteristic, and furthermore it is no longer true in general that

$$h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X) = \frac{1}{2} b_1(X) = \dim \mathrm{Alb}(X),$$

so that one has to be careful about which invariants to fix. Hacon, Patakfalvi and Zhang refined a previous results of [HP16] and showed that

Theorem ([HPZ17]). *Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$. Then X is birational to an abelian variety if and only if $\kappa(X) = 0$ and $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is generically finite.*

This characterisation is not numerical; however, they also gave a numerical birational characterisation of ordinary abelian varieties by fixing the Kodaira Stable dimension and the dimension of the Albanese variety. Even so, neither the Kodaira dimension, nor the Kodaira Stable dimension are effective invariants (as they do not predict which plurigenera will be equal to one), and fixing either of them is a much stronger requirement than fixing some of the lower plurigenera (as the Kodaira and the Kodaira Stable dimension bound all the plurigenera). The lower the fixed plurigenera are, the stronger the result is.

In Chapter 3 we prove a version of the theorem of Enriques about the characterisation of abelian surfaces for surfaces defined over an algebraically closed field of characteristic at least five (Theorem 3.A), thus providing a stronger characterisation, albeit just in dimension two:

Theorem. *Let S be a smooth projective surface over an algebraically closed field of characteristic $p > 3$. If*

$$P_1(S) = P_4(S) = 1, \qquad h^1(S, \mathcal{O}_S) = 2,$$

then S is birational to an abelian surface.

We use more classical methods, in particular we study elliptic surfaces, and we obtain a numerical birational characterisation valid for all abelian surfaces, not only ordinary ones. We also show that our way of working can be applied to tackle the characterisation problem for K3 surfaces.

In Chapter 4 we deal with a second problem of birational classification, but this time we are interested in a specific type of irregular surfaces of general type: those with geometric genus and irregularity equal to three.

The surfaces S we consider belong to the class of surfaces of general type with $\chi(S) = 1$, which is a class of special interest: over any algebraically closed field the Euler characteristic of a surface of general type is positive, and therefore $\chi(S) = 1$ is a limit case. This fact is a very classical result over the complex numbers, and it is a consequence of the Castelnuovo-De Franchis Theorem, saying that two linearly independent 1-forms with wedge product zero on a surface S are the pullback of two 1-forms on a curve C such that there is a fibration $S \to C$ (see [Bea96, Proposition X.9]). The positivity of $\chi(S)$ is however a recent result in characteristic p. It was first Sheperd-Barron in the 1990s who showed that $\chi(S) > 0$ if p is at least seven ([SB91, Theorem 8]). Afterwards, Gu proved that $\chi(S) > 0$ if the characteristic is not two ([Gu16]). Finally, Gu, Sun and Zhou ([GSZ19]) showed that $\chi(S) > 0$ is true for every p.

As for the case of abelian surfaces, where it was interesting to consider the problem of classification for abelian varieties of any dimension, here it is worthwhile to consider what happens for smooth projective varieties X of general type with $\chi(X) = 1$. In 2005, using techniques of Generic Vanishing, Hacon and Pardini showed that

Theorem ([HP05])**.** *Let X be a smooth projective variety over the complex numbers. Assume that X has maximal Albanese dimension. If $\chi(X) = 1$, then $q(X) \leq 2\dim(X)$, and, if equality holds, then X is birational to a product of curves of genus two.*

A decade later, Jiang, Lahoz and Tirabassi proved a classification result for the value of $q(X)$ that is next in line:

Theorem ([JLT14])**.** *Let X be a smooth projective variety over the complex numbers. Assume that X has maximal Albanese dimension. If $\chi(X) = 1$ and $q(X) = 2\dim(X) - 1$, then X is birational to one of the following varieties:*

- *a product of smooth curves of genus two with the two-dimensional symmetric product of a curve of genus three;*
- *a quotient $(C_1 \times Z)/ < \tau >$, where C_1 is a bielliptic curve of genus two, $Z \to C_1 \times ... \times C_{n-1}$ is an étale double cover of a product of smooth projective curves of genus two, and τ is an involution acting diagonally on C_1 and Z via the involutions corresponding respectively to the double covers.*

At the moment there are no classification results for smaller values of $q(X)$.

These recent results for arbitrary dimension are generalisatons of what happens for complex surfaces. Beauville had shown in the 1980s that a smooth complex surface of general type with $p_g = q = 4$ is birational to a product of curves of genus two ([Be82]). In 2002 it had been proved independently in [HP02] and [Pi02] the classification result for surfaces with $p_g = q = 3$; we transcribe and explain more in detail this latter result in Chapter 4. It should be noticed that, as we explain at the beginning of Chapter 4, for smooth complex surfaces with $\chi = 1$ we have $p_g = q \leq 4$ thanks to inequalities from the theory of surfaces (there is no need to put maximal Albanese dimension as hypothesis).

We do not have a complete list of possibilities for what a birational model of a complex surface of general type with $p_g = q \leq 2$ might be, but there seem to appear more cases as the values of p_g and q get smaller. Several classical examples of surfaces of general type are to be found among these.

Almost nothing is known about the classification of varieties of general type with Euler characteristic equal to one over algebraically closed fields k of characteristic p. Also, in this setting it is no longer true in general that p_g and $h^1(S, \mathcal{O}_S)$ are smaller than four (that is true, however, if one assumes liftability to $W_2(k)$ and $p \neq 2$). Wang proved, with some technical hypotheses, a result corresponding to Beauville's classification for surfaces with $p_g = q = 4$:

Theorem ([Wa17]). *Let S be a smooth projective surface of general type defined over an algebraically closed field k with $\mathrm{char}(k) \geq 11$. Let $\chi(S) = 1$. Assume that S is of maximal Albanese dimension, that it lifts to $W_2(k)$, its Picard variety has no supersingular factors, the Albanese morphism is separable, and $\dim \mathrm{Alb}(S) = 4$. Then S is birational to the product of two smooth curves of genus two.*

In his proof, Wang used results from the Generic Vanishing theory that hold in positive characteristic, and the cost of doing so is the additional hypotheses.

In Chapter 4 we study smooth surfaces of general type S with $p_g(S) = h^1(S, \mathcal{O}_S) = 3$ in positive characteristic. Ideally, we would want to arrive to a statement analogous to that of the aforementioned theorem of [JLT14]. We work mainly with classical methods, and by adding some hypotheses (S of maximal Albanese dimension, $\dim \mathrm{Alb}(S) = 3$ and separable Albanese morphism) we compute (Theorem 4.A) some numerical birational invariants of a resolution of singularities of the image of the Albanese morphism:

Theorem. *Let S be a smooth minimal surface of general type over an algebraically closed field; assume $\dim \mathrm{alb}_S(S) = 2$, the Albanese morphism separable and $\mathrm{Pic}^0(S)$ reduced. Assume $p_g(S) = h^1(S, \mathcal{O}_S) = 3$. Consider a resolution of singularities Y of $\mathrm{alb}_S(S)$. Then*

1. *if* $\mathrm{alb}_S(S)$ *is ample, then* Y *is a surface of general type with* $p_g(Y) = h^1(Y, \mathcal{O}_Y) = 3$ *and its Picard variety is reduced;*

2. *if* $\mathrm{alb}_S(S)$ *is not ample, then* $\kappa(Y) = 1$*, and* Y *has a structure of elliptic surface. Moreover,* $\dim \mathrm{Alb}(Y) = 3$*,* $\chi(Y) = 0$ *and* Y *has one of the following sets of invariants:*

$h^0(Y, \omega_Y)$	$h^1(Y, \omega_Y)$	$\mathrm{Pic}^0(Y)$
2	3	reduced
3	4	non-reduced

Table 1: Possible invariants of Y*.*

The first case is the one that ideally should correspond to when the surface is birational to the symmetric product of a curve of genus three (without the product of curves of genus two for reasons of dimension); as we will explain in greater detail in Chapter 4, in characteristic zero that would follow from the fact that S would be birational to $\mathrm{alb}_S(S)$ and that the latter is a theta divisor. Here we could not prove the birationality in full generality, nor that $\mathrm{alb}_S(S)$ is a theta divisor, but, as written in the statement above, we show that S and the resolution of singularities of $\mathrm{alb}_S(S)$ have those same birational invariants and are both surfaces of general type.

The point of the theorem with non-ample $\mathrm{alb}_S(S)$ should correspond to the second case of the aforementioned theorem of [JLT14], which for surfaces means that S should be birational to the quotient of a product of a curve of genus two and a curve of genus three by $\mathbb{Z}/2\mathbb{Z}$ (see Theorem 4.0.1). The first line in Table 1 above corresponds to the invariants of such a quotient. The second line of Table 1 would be written off in characteristic zero since the Picard variety can be non-reduced only in positive characteristic; thus either these invariants correspond to a surface which does not appear in the complex case, or they should be somehow eliminated.

Moreover, in the case in which the image of the Albanese morphism is not an ample divisor, we build a pencil on S from which one would hope to prove that S is birational to the quotient of a product of curves. Nevertheless, we could not rule out the possibility of having all singular fibres in this pencil, and therefore also here some characteristic p phenomena might appear.

Subsequently, in Chapter 4 we also add some more hypotheses and, by using techniques of Generic Vanishing, improve our previous result (Theorem 4.B):

Theorem. *Let* S *be a smooth minimal surface of general type over an algebraically closed field* k*; assume* S *of maximal Albanese dimension, the Albanese morphism separable and* $\mathrm{Pic}^0(S)$ *reduced. Assume* $p_g(S) = h^1(S, \mathcal{O}_S) = 3$*. Furthermore, we assume that* $\mathrm{alb}_S(S)$ *is an ample divisor and that it is normal, that* S *lifts to* $W_2(k)$*, and that* $\mathrm{Pic}^0(S)$ *has no supersingular factors. Then the Albanese morphism is birational onto its image.*

We shall also discuss more thoroughly how our results relate to the classification theorem of [HP02] and [Pi02] in characteristic zero and what remains to be proved in positive characteristic.

Notation

Here we introduce some notation that we will use throughout this work. When needed, additional notation will be explained in the relevant chapters.

We will always work over an algebraically closed field.

Let X be a smooth projective algebraic variety defined over an algebraically closed field k. We write ω_X for the canonical bundle of X and K_X for a canonical divisor in ω_X.

We will use the notation $\kappa(X)$ for the Kodaira dimension of X, and for $i \in \mathbb{Z}$ we define $h^i(X,\cdot) := \dim \mathrm{H}^i(X,\cdot)$. Also, we use the notation $\chi(X) := \sum_i (-1)^i h^i(X,\mathcal{O}_X)$ for the Euler characteristic of X.
The plurigenera of X are $P_n(X) := \dim \mathrm{H}^0(X,\omega_X^{\otimes n})$ for n a positive integer.

If D and E are two linearly equivalent divisors on X we write $D \sim E$; in addition, $\mathcal{O}_X(D)$ will denote the line bundle associated to the divisor D.

If X is a proper scheme of dimension n in characteristic zero, we define the Betti numbers of X as $b_i(X) := \dim_{\mathbb{C}} \mathrm{H}^i(X,\mathbb{C})$, where $\mathrm{H}^i(X,\mathbb{C})$ is the i-th singular cohomology group of X. Over a field of any characteristic, we define the Betti numbers as $b_i(X;l) := \dim_{\mathbb{Q}_l} \mathrm{H}^i_{\text{ét}}(X,\mathbb{Q}_l)$, via the l-adic cohomology of X, for a prime integer l different from p. These numbers do not depend on l, and they coincide with the b_is in characteristic zero. Therefore we will use the notation $b_i(X)$ in any characteristic without risk of confusion.
The Euler-Poincaré characteristic of X (or Euler characteristic of X) is defined as $e(X) := \sum_{i=0}^{2n} (-1)^i b_i(X)$. For a smooth surface X we have $e(X) = c_2(X)$, and for a smooth curve X we have $e(X) = 2 - 2g(X)$.

Chapter 1

Background Material

In this chapter we review several results and concepts we will use throughout this work, in particular about surfaces defined over algebraically closed fields.

1.1 A Miscellanea of Results about Surfaces

We begin by recalling, for the sake of completeness, some well-known classical formulas whose validity is not restricted to the characteristic zero setting. These results can be found for example in [Li12, 3.] and [Bea96, I.15]. Let S be a smooth projective surface defined over an algebraically closed field and consider any $\mathcal{O}_S(D) \in \mathrm{Pic}(S)$; then we have the formula given by the Riemann-Roch Theorem:

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{D^2 - D \cdot K_S}{2}. \tag{1.1}$$

Moreover, if S is minimal, also the equality given by the Noether's Formula holds:

$$\chi(\mathcal{O}_S) = \frac{K_S^2 + c_2(S)}{12}. \tag{1.2}$$

Since the Riemann-Roch Theorem holds also in positive characteristic, we recover the genus formula in this setting as well:

$$h^1(C, \mathcal{O}_C) = 1 + \frac{C^2 + K_S \cdot C}{2}, \tag{1.3}$$

where C is an irreducible curve and S is no longer assumed minimal. We will make use of the following result (see for example [Ba01, Corollary 2.4]):

Theorem 1.1.1 (Hodge Index Theorem)**.** *Let S be a smooth projective surface over an algebraically closed field. Assume D is a divisor on S such that $D^2 > 0$. Then, for any divisor F on S such that $D \cdot F = 0$, it must be that $F^2 \leq 0$, with equality holding if and only if F is numerically trivial.*

The Néron-Severi group of a variety X, $\mathrm{NS}(X)$, is defined as the quotient of the group $\mathrm{Pic}(X)$ by the subgroup $\mathrm{Pic}^0(X)$. Its rank (which is finite) is called the Picard number of X and written $\rho(X)$. A theorem by Igusa gives it an upper bound in the case of surfaces.

Theorem 1.1.2 ([Ig60]). *Let S be a smooth projective surface. Then*

$$\rho(S) \leq b_2(S). \tag{1.4}$$

1.2 Some Results about Higher Direct Images

This section is about some tools we will need to use when dealing with higher direct images of sheaves.

To begin with, we will need this basic property of higher direct images (see [Ha77, III, Proposition 8.1] and [Hu06, Theorem 3.22]):

Proposition 1.2.1. *Let $f : X \to Y$ be a morphism of noetherian schemes, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then, for each integer $i \geq 0$, the higher direct image $R^i f_* \mathscr{F}$ is the sheaf associated to the presheaf*

$$V \mapsto \mathrm{H}^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)})$$

on Y. In particular, the sheaves $R^i f_* \mathscr{F}$ are trivial for $i > \dim X$.

Also, recall that in the situation of Proposition 1.2.1 we get the *Leray spectral sequence* (see for example [Hu06, (3.3)]):

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(Y, R^q f_* \mathscr{F}) \qquad \Longrightarrow \qquad E^{p+q} = \mathrm{H}^{p+q}(X, \mathscr{F}). \tag{1.5}$$

Later in this work we will avail the following statement of Cohomology and Base Change (see [Mu14, 5., Corollary 3]):

Theorem 1.2.2 (Cohomology and Base Change). *Let $f : X \to Y$ be a proper morphism of noetherian schemes, Y connected. Let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y. For $y \in Y$, let $X_y := X \times_Y \operatorname{Spec} k(y)$ (as a scheme over $k(y)$) be the fibre of f over y. Also, we define $\mathscr{F}_y := \mathscr{F} \otimes_{\mathcal{O}_Y} k(y)$.*
Let q be an integer such that $\mathrm{H}^q(X_y, \mathscr{F}_y) = 0$ for all $y \in Y$. Then there is an isomorphism

$$R^{q-1} f_*(\mathscr{F}) \otimes_{\mathcal{O}_y} k(y) \longrightarrow \mathrm{H}^{q-1}(X_y, \mathscr{F}_y) \tag{1.6}$$

for all $y \in Y$.

In the case of the canonical sheaf, we may have more information about its higher direct images. The Grauert-Riemenschneider Vanishing Theorem (see for example [La04, Theorem 4.3.9]) is a well-known result for varieties over defined over the complex numbers. It also holds for smooth surfaces defined over algebraically closed fields in any characteristic. We recall it in the form found in [Wa17, Theorem 2.3]:

Theorem 1.2.3 (Grauert-Riemenschneider Vanishing Theorem for Surfaces). *Let $f : S \to W$ be a projective generically finite morphism from a smooth surface S to a normal, quasi-projective surface W. Then $R^1 f_*(\omega_S \otimes \alpha) = 0$ for any $\alpha \in \operatorname{Pic}^0(S)$.*

1.3 Fibrations on Surfaces

We will often have to deal with surjective morphisms $f : S \to B$ from a smooth projective surface S to a smooth projective curve B. Assume, up to Stein factorisation, that the generic fibre F is connected, i.e. that $f : S \to B$ is a fibration. As it is customary in the literature (see for example [CCM98, 0.]), we will sometimes refer to a surjective rational map $f : S \to B$ as a *pencil of genus* $g(B)$ or a *pencil of curves of genus* $g(F)$. An *irrational pencil* will be understood to be a pencil with $g(B) \geq 1$, while a *rational pencil* will be a pencil with $g(B) = 0$.

By Generic Smoothness (see for example [Ha77, Corollary 10.7]), in characteristic zero the generic fibre of such a morphism is smooth. On the other hand, in positive characteristic the generic fibre could be a singular curve. A theorem by Tate states that if the generic fibre is singular then the only type of singularity it can have is cusps (unibranch singularities), and that is to say that the generic fibre turns out to be homeomorphic to its normalisation (see for example [Li12, 4.]).

A result of Beauville ([Be82, Corollaire] and [Be82, Remarque]) relates some invariants in the case in which the generic fibre is smooth:

Theorem 1.3.1 ([Be82]). *Let S be a smooth minimal surface, B a smooth curve, $f : S \to B$ a surjective morphism whose generic fibre F is a smooth connected curve of genus $g(F) \geq 2$. Then*

1. $K_S^2 \geq 8(g(B) - 1)(g(F) - 1)$;
2. $c_2(S) \geq 4(g(B) - 1)(g(F) - 1)$;
3. $\chi(S) \geq (g(B) - 1)(g(F) - 1)$.

Moreover, equality in 1. implies that the fibration has constant moduli; equality in 2. implies that the fibraton is smooth; equality in 3. implies that the fibration is smooth and with constant moduli.

More in general, over any algebraically closed field we have a formula relating the Euler-Poincaré characteristic of the fibred surface to those of the base curve and the fibres (see [Do72, Theorem 1.1], [CD89, Proposition 5.1.6]; see also [IS12, Remark 7.2] for comments on two misprints in [Do72, Theorem 1.1]).

Theorem 1.3.2. *Let S be a smooth projective surface, and B a smooth projective curve. Let $f : S \to B$ be a surjective morphism with geometrically connected generic fibre. Then*

$$c_2(S) = e(S_\eta)e(B) + \sum_{b \in \overline{B}} (e(S_b) - e(S_\eta) + \delta_b) \tag{1.7}$$

where S_η is the geometric generic fibre, $\overline{B}$ is the set of closed points of B, and the Serre's measure of wild ramification δ_b is ≥ 0.

1.4 A Miscellanea of Results about Abelian Varieties

We recall this important result about the fibres of morphisms that have as domain abelian varieties (see [Mu14, p. 84]):

Proposition 1.4.1. *Consider a morphism of varieties $f : A \to X$, with A an abelian variety. Define, for all $a \in A$, F_a to be the connected component of $f^{-1}f(a)$ that contains a. Then there exists a closed connected subgroup F of A such that $F_a = t_a F$ for all $a \in A$.*

In particular, fibres of morphisms from abelian varieties are disjoint unions of translates of abelian subvarieties.

We recall an important property of morphisms to abelian varieties from [Mi08, Theorem 3.2].

Theorem 1.4.2. *A rational map from a nonsingular variety W to an abelian variety A is defined on the whole of W.*

We will need the following result, for which a proof valid in an characteristic can be found for example in [AB15, Lemma 12]:

Lemma 1.4.3. *Let A be an abelian variety defined over an algebraically closed field, and let D be a prime divisor on A. If D is not ample, then there exist a surjective morphism of abelian varieties $f : A \to X$ and an ample divisor B on X such that $f^{-1}B = D$ as schemes.*

1.5 The Albanese Variety and the Picard Variety

We give a brief characteristic-free summary of some results concerning the Albanese variety. Refernces for the next definitions and facts are for example [Ba01, 5.] and [La59, II., §3.].

Definition 1.5.1. Let X be a smooth projective variety and A an abelian variety. Let $f : X \to A$ be a morphism. Then we say that the couple (X, f) *generates* A if there exists an integer n such that the morphism

$$\begin{aligned} F : X^{\times n} &\longrightarrow A \\ (x_i)_{1 \le i \le n} &\mapsto \sum_{i=1}^{n} f(x_i) \end{aligned}$$

is generically surjective. This is equivalent to asking that, up to translation on A, the smallest abelian subvariety of A containing $f(X)$ is A itself.

Definition 1.5.2. Let X be a smooth projective variety. An *Albanese variety* for X is a couple (A, f), with A abelian variety and $f : X \to A$ morphism, such that (X, f) generates A and, for any morphism $g : X \to B$ to an abelian variety B, there exist (up to translation on B) a morphism of abelian varieties $\varphi : A \to B$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] \arrow[r, phantom, shift right=4ex, "\circlearrowright"] & A \arrow[d, "\varphi"'] \\ & B \end{tikzcd}$$

commutes.

Such a couple (A, f) exists for any X (see for example [Se58–59, Théorème 5]), and A is unique up to isomorphism and f up to composition with a translation. In what follows we will therefore, by abuse of language, refer to *the* Albanese variety of X and write $(\mathrm{Alb}(X), \mathrm{alb}_X)$ for any such couple; also, we will often forget the morphism.

We will adopt the following common terminology:

Definition 1.5.3. Let X be a smooth projective variety. Then X is *mAd* (maximal Albanese dimension) if $\dim X = \dim \mathrm{alb}_X(X)$, or, equivalently, if the Albanese morphism is generically finite onto its image.

In reviewing the definitions and properties in the next paragraphs we follow closely [Ba01, 5.], [Li12, 2.]. See also [Kl05] for further information and an historical introduction.

At the beginning of the 1960s, Grothendieck associated to any ringed space X a functor (the Picard functor) that classifies invertible sheaves on X. We will consider a smooth projective variety X defined over an algebraically closed field k. In this situation the Picard functor turns out to be representable, and the corresponding fine moduli scheme M (the *Picard scheme* of X) is such that its k-rational points are in a natural bijective correspondence with the elements of $\mathrm{Pic}(X)$, the Picard group of X. By abuse of notation we will write $\mathrm{Pic}(X)$ for both the Picard group of X and the fine moduli scheme M, and the context will clarify what we are referring to.
The scheme $\mathrm{Pic}(X)$ is a disjoint union of an infinite family of proper k-schemes. By abuse of notation, we will write $\mathrm{Pic}^0(X)$ for the connected component containing the point corresponding to $\mathcal{O}_X$, as in fact its k-rational points are in a natural bijective correspondence with the elements of $\mathrm{Pic}^0(X)$ (the group of invertible $\mathcal{O}_X$-modules modulo algebraic equivalence).

Furthermore, $\mathrm{Pic}^0(X)$ is a group scheme and, as such, either it is an abelian variety or it is non-reduced. A theorem by Cartier says that group schemes in characteristic zero are always reduced, and therefore in that case one always has an abelian variety.

For the sake of convenience, we group together in the next theorem some known results that we will often use.

Theorem 1.5.4. *Let X be a smooth projective variety defined over an algebraically closed field k. Then*

1. *by arguments of deformation theory (see for example [Kl05, Theorem 5.11] for a proof),*
$$\mathrm{T}_{\mathcal{O}_X}\mathrm{Pic}(S) \simeq \mathrm{H}^1(X,\mathcal{O}_X), \tag{1.8}$$
where $\mathrm{T}_{\mathcal{O}_X}\mathrm{Pic}(S)$ *is the Zariski tangent space to* $\mathrm{Pic}(S)$ *at* $\mathcal{O}_X$*;*

2. *the reduction of* $\mathrm{Pic}^0(S)$ *is the dual abelian variety of* $\mathrm{Alb}(X)$*;*

3. *the following equalities hold:*
$$\frac{b_1(X)}{2} = \dim \mathrm{Alb}(X) = \dim \mathrm{Pic}(X). \tag{1.9}$$

In light of Theorem 1.5.4, we always have that

$$\Delta := 2h^1(S,\mathcal{O}_X) - b_1(X) \geq 0, \tag{1.10}$$

and equality holds if and only if $\mathrm{Pic}^0(X)$ is smooth, i.e. if and only if $\mathrm{Pic}^0(X)$ is reduced. We will review later what is known about the reducedness of $\mathrm{Pic}^0(X)$ in the case of surfaces.

We recall an important property of all the sheaves of in $\mathrm{Pic}^0(X)$ (see [Mu14, 8. (vii)]).

Proposition 1.5.5. *Let* X *be a smooth projective variety defined over an algebraically closed field* k*. Then, for any* $L \in \mathrm{Pic}^0(X)$ *such that* L *is not* $\mathcal{O}_X$ *and for any* $i \in \mathbb{N}$*, we have*
$$\mathrm{H}^i(X,L) = 0.$$

1.6 The Enriques-Kodaira Classification

A very classical and still popular topic in algebraic geometry is the study and classification of surfaces. The study of complex algebraic surfaces in its budding phase received many contributions in particular by Max Noether and the Italian school of algebraic geometry, notably by Guido Castelnuovo and especially Federigo Enriques. The classification of algebraic surfaces consists essentially in their subdivision first according to their Kodaira dimension, and then possibly into subclasses. The classification of complex algebraic surfaces was fully-fledged at the moment of the publication of Enriques' celebrated work [En1949]. Kodaira in the sixties extended the classification to non-algebraic surfaces, and so the classification is often referred to as *the Enriques-Kodaira classification*, even in a context where only algebraic surfaces are being considered.

Enrico Bombieri and David Mumford extended Enriques' classification to positive characteristic in three articles: [Mu69], [BM77] and [BM76].

In this work we will need to exploit the knowledge of how the canonical divisor

behaves in terms of intersection numbers according to the Kodaira dimension of a surface.
For this reason, we recall the very basics of the Enriques' classification, as found in [BM77]. Therefore, assume S to be a smooth projective (algebraic) minimal surface over an algebraically closed field of any characteristic. Then

* if $\kappa(S) = -\infty$, there exists a curve C on S such that $K_S \cdot C < 0$;
* if $\kappa(S) = 0$, for any curve C on S one has that $K_S \cdot C = 0$;
* if $\kappa(S) = 1$, for any curve C on S one has that $K_S \cdot C \geq 0$. Moreover, $K_S^2 = 0$ and for any ample divisor H on S one has that $K_S \cdot H > 0$;
* if $\kappa(S) = 2$, for any curve C on S one has again that $K_S \cdot C \geq 0$ and for any ample divisor H on S one has that $K_S \cdot H > 0$. But this time $K_S^2 > 0$.

Much effort has been devoted to studying which invariants correspond to which surfaces. One of the main results, due to Castelnuovo and Enriques in characteristic zero and to Catanese and Li in general is the P_{12}-Theorem ([CL19]).

Theorem 1.6.1 (P_{12}-Theorem)**.** *Let S be a smooth projective surface defined over an algebraically closed field. Assume S minimal. Then*

* $\kappa(S) = -\infty$ *if and only if* $P_{12}(S) = 0$*;*
* $\kappa(S) = 0$ *if and only if* $P_{12}(S) = 1$*;*
* $\kappa(S) = 1$ *if and only if* $P_{12}(S) \geq 2$ *and* $K_S^2 = 0$*;*
* $\kappa(S) = 2$ *if and only if* $P_{12}(S) \geq 2$ *and* $K_S^2 > 0$*.*

A classical result concerning the numerical invariants of surfaces over the complex numbers is the following (which we report from [Bea96, Theorem X.4]):

Theorem 1.6.2 (Castelnuovo)**.** *Let S be a smooth projective surface defined over the complex numbers with $\kappa(S) \neq -\infty$. Assume S minimal. Then $c_2(S) \geq 0$ and $\chi(S) \geq 0$. If S is of general type, then $\chi(S) > 0$.*

The proof of the result above depends on the Castelnuovo-De Franchis Theorem, which holds only in characteristic zero. Also, the inequality $\chi(S) > 0$ for surfaces of general type is improved by the Bogomolov-Miyaoka-Yau inequality for which several counterexamples have been built in characteristic p, starting from those found by Szpiro in 1979.
Moreover, in characteristic p it is no longer true that the topological Euler characteristic is non-negative for all surfaces that are of non-negative Kodaira dimension (for example, Liedtke in [Li08a] found minimal surfaces of general type in characteristic 2 having $c_2 = -2$). In [SB91] Shepherd-Barron studies some properties of surfaces of general type according to the behaviour of c_2; also, he shows that $\chi(S) > 0$ for surfaces of general type if the characteristic p is at least 11. After that article, the question of what happens for smaller p remained open for long. Now we know the following:

Theorem 1.6.3. *Let S be a smooth projective surface defined over an algebraically closed field of characteristic p. Assume $\kappa(S) \neq -\infty$. Then $\chi(S) \geq 0$, and if S is of general type one has that $\chi(S) > 0$.*

If $\kappa(S) = 0$, then $\chi(S) \geq 0$ by the classification of surfaces (see Table 1.1). The fact that $\chi(S) \geq 0$ when $\kappa(S) = 1$ is observed for example in [KU85, (1.5)]. Gu settled the case $\kappa(S) = 2$ when $p \neq 2$ ([Gu16, Theorem 1.3]), and the case $\kappa(S) = 2$ in characteristic 2 has been recently solved ([GSZ19, Theorem 2.]).

Next, we recall a theorem that is due in characteristic zero to Kodaira ([Ko68, Theorem 5]) and to Ekedahl in positive characteristic ([Ek88, Main Theorem]):

Theorem 1.6.4. *Let S be a minimal smooth projective surface of general type over an algebraically closed field k. Then, for any integer $m > 0$,*

$$\mathrm{H}^1(S, \omega_S^{\otimes -m}) = 0, \tag{1.11}$$

unless possibly when $m = 1$, $\mathrm{char}(k) = 2$, $\chi(S) = 1$ and S is birational to an inseparable double cover of a K3-surface or to a rational surface. In any case, $h^1(S, \omega_S^{\otimes -m}) \leq 1$.

So, for the sake of simplicity, assume $\mathrm{char}(k) \neq 2$ and take an integer $n \geq 2$. Then, for a minimal surface S of general type it is true that

$$\chi(\omega_S^{\otimes n}) = P_n(S), \tag{1.12}$$

since $h^1(S, \omega_S^{\otimes n}) = h^1\left(S, \omega_S^{\otimes (1-n)}\right) = 0$ by Theorem 1.6.4 and clearly $h^2(S, \omega_S^{\otimes n}) = h^0\left(S, \omega_S^{\otimes (1-n)}\right) = 0$. Then the Riemann-Roch Theorem implies that

$$P_n(S) = \chi(S) + \frac{n(n-1)}{2} K_S^2. \tag{1.13}$$

The above equation proves, thanks to Theorem 1.6.3, the following classical result when $\mathrm{char}(k) \neq 2$.

Corollary 1.6.5. *Let S be a smooth projective surface of general type over an algebraically closed field k. Then $P_2(S) \geq 2$.*

However, Ekedahl proved Corollary 1.6.5 in any characteristic ([Ek88, Corollary 1.8]) even before knowing that for a surface of general type the Euler characteristic is always strictly positive.

About surfaces with negative Kodaira dimension, before we move on to surfaces of Kodaira dimension zero and one, we will need the following result by Nagata (see for example [Li12, Theorem 3.5]) which holds in any characteristic.

Theorem 1.6.6. *Consider a smooth projective minimal surface S with $\kappa(S) = -\infty$.*

* *if $h^1(S, \mathcal{O}_S) \geq 1$, then $\mathrm{alb}_S(S)$ is a smooth curve and $\mathrm{alb}_S : S \to \mathrm{alb}_S(S)$ is isomorphic to $\mathbb{P}(\mathcal{E}) \to \mathrm{alb}_S(S)$ for some rank two vector bundle $\mathcal{E}$ on $\mathrm{alb}_S(S)$;*

* *if $h^1(S, \mathcal{O}_S) = 0$, then S is isomorphic to either $\mathbb{P}^2$ or to a Hirzebruch surface $\mathbb{F}_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1$ with $d \neq 1$.*

1.7 Surfaces of Kodaira Dimension Zero

Let S be a smooth projective surface over an algebraically closed field k. If $\kappa(S) = 0$, then the minimal model of S belongs to one of a finite number of well-known families.
Assume S minimal and $\kappa(S) = 0$. Then $K_S^2 = 0$. Following [BM77], Noether's formula $12\chi(S) = K_S^2 + c_2(S)$ becomes

$$10 + 12p_g(S) = 8h^1(S,\mathcal{O}_S) + 2\overbrace{(2h^1(S,\mathcal{O}_S) - b_1(S))}^{\Delta} + b_2(S),$$

where Δ had been introduced and discussed in (1.10). Since $\kappa(S) = 0$, then $p_g(S)$ can be only either 0 or 1, and one can see that there are only seven possible sets of invariants that satisfy the above equation. In [BM77] the authors show that actually one of those sets of invariants does not correspond to any existing surface. Each of the other six remaining sets of invariants corresponds to exactly one type of surface, up to taking the minimal model. The possible sets of solutions are listed in Table 1.1.

	$b_2(S)$	$b_1(S)$	$c_2(S)$	$\chi(S)$	$h^1(S,\mathcal{O}_S)$	$p_g(S)$	Δ
K3 surfaces	22	0	24	2	0	1	0
Enriques surfaces	10	0	12	1	0	0	0
					1	1	2
Abelian surfaces	6	4	0	0	2	1	0
Bielliptic surfaces	2	2	0	0	1	0	0
					2	1	2

Table 1.1: Table of Invariants for Surfaces with $\kappa(S) = 0$.

1.8 Elliptic Surfaces

An *elliptic surface* (resp. a *quasi-elliptic surface*) is a fibration $f : S \to B$ from a smooth projective surface S to a smooth projective curve B satisfying $f_*\mathcal{O}_S \simeq \mathcal{O}_B$ and such that the generic fibre is a smooth curve of genus one (resp. a rational curve with a cusp). Quasi-elliptic fibrations exist only in characteristic 2 and 3. Observe that we follow the definitions of for example [BM77] and [Li12] in that we do not require the existence of sections in the definition of elliptic fibration.

Quasi-elliptic surfaces teem with characteristic p features. For example, as seen in [Li12, Theorem 8.3], they are always uniruled (i.e. there exist a smooth curve C and a dominant rational map $\mathbb{P}^1 \times C \dashrightarrow S$), but while in characteristic zero all uniruled surfaces have negative Kodaira dimension, quasi-elliptic surfaces exist in higher Kodaira dimension.

By adjunction formula, if F is the generic fibre of the fibration of an elliptic surface, then $\mathcal{O}_F \simeq \omega_F \simeq (\omega_S + \mathcal{O}_S(F))|_F = (\omega_S)|_F$, and we see that S cannot be a surface with $\kappa(S) = 2$, but it can have any other Kodaira dimension. Moreover, it is well-known that all surfaces of Kodaira dimension one are elliptic or quasi-elliptic. We summarise the main results to this effect in the next theorem (see [KU85, Lemma 5.1, Theorem 5.2] and [Li12, Theorem 5.3]).

Theorem 1.8.1. *Let S be a minimal algebraic surface with $\kappa(S) = 1$. Then the Stein factorisation of the Iitaka fibration is a morphism which gives S a structure of relatively minimal*[1] *elliptic or quasi-elliptic fibration.*
In particular, the elliptic fibration on a surface with $\kappa(S) = 1$ is unique, and the complete linear system $|mK_S|$ for $m \geq 14$ gives to S this unique structure.

Consider an elliptic (or quasi-elliptic) surface $f : S \to B$. Let $b_1, \ldots b_r \in B$ be the finitely many points at which the fibre $f^{-1}(b_\alpha)$ is multiple, that is to say:

$$f^{-1}(b_\alpha) = m_\alpha P_\alpha \tag{1.14}$$

with $m_\alpha \geq 2$ and P_α indecomposable of canonical type[2]. As in [BM77, Proposition 4] and [KU85, 1. Preliminaries] we define, for each α,

$$\nu_\alpha = \mathrm{order}(\mathcal{O}_{P_\alpha} \otimes \mathscr{I}_{P_\alpha}^{-1}), \tag{1.15}$$

where $\mathscr{I}_{P_\alpha}$ is the ideal sheaf of $\mathscr{I}_{P_\alpha}$. In characteristic zero $m_\alpha = \nu_\alpha$, while in characteristic p there exist, for each $\alpha = 1, \ldots r$, integers $\gamma_\alpha \in \mathbb{N}$ such that

$$m_\alpha = p^{\gamma_\alpha} \nu_\alpha, \tag{1.16}$$

as recalled in [KU85, (1.6)]. Also let

$$R^1 f_* \mathcal{O}_S \simeq L \oplus T \tag{1.17}$$

be the decomposition of $R^1 f_* \mathcal{O}_S$ into an invertible sheaf L and a torsion sheaf T; the latter is always zero in characteristic zero. It is a fact (see [BM77, Proposition3]) that the support of T, as a set, is contained in the set of the points of B whose inverse image is multiple fibre.

Definition 1.8.2. The fibres arising from points in the support of T are called *wild fibres*.

The following theorem holds ([BM77, Theorem 2]):

Theorem 1.8.3 (Canonical Bundle Formula). *Let $f : S \to B$ be a relatively minimal elliptic or quasi-elliptic fibration and let $R^1 f_* \mathcal{O}_S \simeq L \oplus T$. Then*

$$\omega_S = f^*(L^{-1} \otimes \omega_B) \otimes \mathcal{O}\big(\textstyle\sum a_\alpha P_\alpha\big), \tag{1.18}$$

where

[1] A *relatively minimal fibration* is a fibration such that none of the fibres contains an exceptional curve of the first kind (a smooth rational curve with self-intersection -1).
[2] A curve $C = \sum n_i C_i$ is *of canonical type* if for all i one has $(K_S \cdot C_i) = (C \cdot C_i) = 0$.

a. *$m_\alpha P_\alpha$ are the multiple fibres;*

b. *$0 \le a_\alpha < m_\alpha$;*

c. *$a_\alpha = m_\alpha - 1$ if $m_\alpha P_\alpha$ is not wild;*

d. *$\deg(L^{-1} \otimes \omega_B) = 2g(B) - 2 + \chi(\mathcal{O}_S) + \mathrm{length}(T)$.*

In some situations we actually know more about the relation between the m_α's and the a_α's appearing in the theorem. We recall [KU85, Lemma 2.4]:

Lemma 1.8.4. *Consider a relatively minimal elliptic fibration $f : S \to B$. With the notation already introduced in this section,*

1. *if $h^0(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}) = 2$, then either $a_\alpha + 1 = m_\alpha$ or $a_\alpha + \nu_\alpha + 1 = m_\alpha$;*
2. *if $h^0(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}) = 3$, then $a_\alpha + 1 = m_\alpha$, $a_\alpha + \nu_\alpha + 1 = m_\alpha$, $a_\alpha + 2\nu_\alpha + 1 = m_\alpha$ or $a_\alpha + (p+1)\nu_\alpha + 1 = m_\alpha$.*

Kodaira and Néron classified the possible non-smooth fibres of an elliptic fibration. The list does not depend on the characteristic, and after reduction these fibres must be one of the following (see [Li12, 4.] and [Sil94, IV. Theorem 8.2]):

I_0 a non-singular curve of genus one;

I_1 a rational curve with a node;

I_n for $n \ge 2$, n non-singular rational curves arranged in the shape of an n-gon;

II a rational curve with a cusp;

III two non-singular rational curves intersecting tangentially at a single point;

IV three non-singular rational curves intersecting at a single point;

I_0^* a non-singular rational curve of multiplicity two with four non-singular rational curves of multiplicity one attached;

I_n^* a chain of $n+1$ non-singular rational curves of multiplicity two with two non-singular rational curves of multiplicity one attached at either end;

IV^* seven non-singular rational curves arranged in a way that can be described by the Dynkin diagram $\tilde{E}_6$;

III^* eight non-singular rational curves arranged in a way that can be described by the Dynkin diagram $\tilde{E}_7$;

II^* nine non-singular rational curves arranged in a way that can be described by the Dynkin diagram $\tilde{E}_8$.

Also, in the case of an elliptic fibration on a surface S we can rewrite the formula of Theorem 1.3.2. Clearly, the Euler-Poincaré characteristic of the generic fibre is zero, as it is zero the Euler-Poincaré characteristic of any multiple fibre of type I_0. Computing also the Euler-Poincaré characteristic of the singular fibres, we get (see [Li12, 4.])

$$c_2(S) = \sum_{P_\alpha \in D} \nu(\Delta_{P_\alpha}) \tag{1.19}$$

where D is the set of singular fibres of the elliptic fibration, and, if P_α has m irreducible components,

$$\nu(\Delta_{P_\alpha}) = \begin{cases} 0 & \text{if } P_\alpha \text{ is of type } I_0, \\ m & \text{if } P_\alpha \text{ is of type } I_1 \text{ or } I_n, \\ m+1+\delta & \text{otherwise;} \end{cases} \tag{1.20}$$

here δ is non-negative, and in particular it is zero if the characteristic is neither 2 nor 3.

Finally, we will have to consider the Albanese variety for some elliptic surfaces, and in those situations we will need [KU85, Lemma 3.4] (adding information which is found in its proof) together with [KU85, Lemma 3.5]:

Lemma 1.8.5. *Let $f: S \to B$ be a relatively minimal elliptic surface. Then either $\mathrm{Alb}(S) \simeq J(B)$ and $\mathrm{alb}_S(S)$ is a curve, or $\dim \mathrm{Alb}(S) = \dim J(B) + 1$.*
In particular, if $B \simeq \mathbb{P}^1$ and $\chi(S) = 0$, then $\dim \mathrm{Alb}(S) = 1$.

Moreover, the condition $\mathrm{Alb}(S) \simeq J(B)$ is equivalent to the existence of a point $\hat{b} \in B$ such that $\mathrm{alb}_S(f^{-1}(\hat{b}))$ is a point, i.e. the fibre $f^{-1}(\hat{b})$ is contracted by the Albanese morphism.

1.9 Non-reducedness of the Picard Scheme for Surfaces

In this section we report some results and observations found in [Li09] concerning the non-reducedness of the Picard scheme for surfaces.
If S is a smooth surface in positive characteristic, then the Kodaira dimension is an indicator of the possible non-reducedness of $\mathrm{Pic}^0(S)$: surfaces with $\kappa(S) = -\infty, 0$ are the least unruly, and surfaces of general type admit some level of control if the characteristic is large enough.

Theorem 1.9.1 ([Li09]). *Let S be a smooth surface over an algebraically closed field of characteristic p. Then*

* *if $\kappa(S) = -\infty$, then $\mathrm{Pic}^0(S)$ is reduced;*
* *if $\kappa(S) = 0$, then $\mathrm{Pic}^0(S)$ is reduced unless $p = 2,3$ and the minimal model of S is one of the surfaces with $\Delta \neq 0$ in Table 1.1;*
* *if $\kappa(S) = 2$, then for any $m \in \mathbb{N}$ there exists $\bar{p}_m \in \mathbb{N}$ such that (assuming S minimal) if $K_S^2 = m$ and $p \geq \bar{p}_m$ then $\mathrm{Pic}^0(S)$ is reduced.*

The case $\kappa(S) = 1$ is the most unruly. Indeed, in any characteristic p it is possible to find a surface S with $\kappa(S) = 1$ admitting an iso-trivial elliptic fibration and such that Δ is arbitrarily large, thus in particular $\mathrm{Pic}^0(S)$ is non-reduced (see [Li09, Proposition 2.3]).
Even more strikingly, in [Li09, Theorem 2.2], it is shown that, starting from any relatively minimal elliptic fibration $S \to B$ which is not generically constant, one can find an elliptic fibration $\tilde{S} \to B$ from a surface $\tilde{S}$ with $\kappa(\tilde{S}) = 1$ such that S and $\tilde{S}$ have the same Euler characteristic, the same Betti numbers, $\mathrm{Pic}^0(\tilde{S})$ is non-reduced and $\tilde{S}$ has a Δ which is arbitrarily large.

1.10 Normalisation and Stein Factorisation

As in this work we will need to normalise varieties, we recall here the universal property of normalisation (see for example [GW10, Proposition 12.44]):

Proposition 1.10.1 (Universal Property of the Normalisation)**.** *Let $\nu : N \to X$ be a morphism of integral schemes with N normal. Then $\nu : N \to X$ is the normalisation of X if and only if for every integral normal scheme Y and every dominant morphism $f : Y \to X$ there exists a unique morphism $\bar{f} : Y \to N$ such that the following diagram*

$$\begin{tikzcd} & N \arrow[d, "\nu"] \\ Y \arrow[ur, "\exists! \bar{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & X \end{tikzcd}$$

commutes.

We will often be interested in the connectedness of the fibres of morphisms. Recall that for a projective morphism $f : X \to Y$ of noetherian schemes the condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ implies that all the fibres of f are connected (see [Ha77, III. Corollary 11.3]). If X and Y are defined over an algebraically closed field of characteristic zero, then the converse is also true; on the other hand, in positive characteristic it is possible to have a morphism with connected fibres that does not satisfy $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ because of issues related to the separability of the morphism ([De01, §1.12]).

The way to overcome a situation in which the fibres are not connected or the condition $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ does not hold is to consider the *Stein factorisation* of the given morphism. We recall this result using as sources [Ha77, III. Corollary 11.5] and [De01, §1.13].

Theorem 1.10.2 (Stein Factorisation)**.** *Given a projective morphism $f : X \longrightarrow Y$ of noetherian schemes, we can always obtain a diagram*

$$\begin{tikzcd} & \mathbf{Spec} f_*\mathcal{O}_X \arrow[dr, "g"] & \\ X \arrow[ur, "f'"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where $f'_*\mathcal{O}_X \simeq \mathcal{O}_{\mathbf{Spec} f_*\mathcal{O}_X}$, $\dim \mathbf{Spec} f_*\mathcal{O}_X = \dim Y$, *and* g *is a finite map.*

In characteristic zero, if g is bijective then it is an isomorphism. However, in characteristic p it could happen that g is bijective but not an isomophism. In this case, g would be an inseparable morphism and the Stein factorisation essentially brings about changes only at the level of the function fields.

Part I

Brauer Groups of Quotient Varieties

Chapter 2

Brauer Groups of Bielliptic Surfaces

This chapter is based on [FTVB19], a joint work with S. Tirabassi and M. Vodrup including an appendix by S. Tirabassi and J. Bergström.

Given a smooth complex projective variety Z its (cohomological) Brauer group is defined as $\mathrm{Br}(Z) := \mathrm{H}^2_{\text{ét}}(Z, \mathcal{O}_Z^*)_{\text{tor}}$. A morphism of projective varieties $f : Z \to Y$ induces, via pullbacks, a homomorphism $f_{\mathrm{Br}} : \mathrm{Br}(Y) \to \mathrm{Br}(Z)$, which we call the *Brauer map induced by f*. In [Bea09] Beauville studies this map in the case of a complex Enriques surface S and that of its K3 canonical cover $\pi : X \to S$. More precisely the author of [Bea09] identifies the locus in the moduli space of Enriques surfaces in which π_{Br} is not injective (and therefore trivial). Here we carry out a similar investigation in the case of bielliptic surfaces.

A bielliptic surface is constructed by taking the quotient of a product of ellipic curves $A \times B$ by the action of a finite group G. They were classified in 7 different types by Bagnera-De Franchis, as illustrated in Table 2.1. Since the canonical

Type	G	Order of ω_S in $\mathrm{Pic}(S)$	$\mathrm{H}^2(X, \mathbb{Z})_{\text{tor}}$
1	$\mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2	$\mathbb{Z}/2\mathbb{Z}$
3	$\mathbb{Z}/4\mathbb{Z}$	4	$\mathbb{Z}/2\mathbb{Z}$
4	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4	0
5	$\mathbb{Z}/3\mathbb{Z}$	3	$\mathbb{Z}/3\mathbb{Z}$
6	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	3	0
7	$\mathbb{Z}/6\mathbb{Z}$	6	0

Table 2.1: Types of bielliptic surfaces and torsion of their second cohomology.

bundle of a bielliptic surface S is a torsion element in $\mathrm{Pic}(S)$, it can be used to define an étale cyclic cover $\pi : X \to S$, where X is an abelian variety isogenous to $A \times B$. We then obtain a homomorphism between the respective Brauer groups: $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$. A very natural question is the following:

Question. When is π_{Br} injective? When is it trivial?

As for Enriques surfaces, using the long exact exponential sequence and Poincaré duality, we have a non-canonical isomorphism

$$\mathrm{Br}(S) \simeq \mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tor}},$$

so from the fourth column of Table 2.1, we easily see that this map is trivial when S is of type 4, 6 or 7. Thus we will limit ourselves to surfaces of type 1, 2, 3 and 5. We will find that the behaviour of the Brauer map depends heavily on the geometry of the bielliptic surface S.

Our first step in this investigation is to focus on bielliptic surfaces of type 2 and 3. By a costruction of Nuer ([Nue]) they admit a degree 2 étale cover $\tilde{\pi} : \tilde{S} \to S$, with $\tilde{S}$ a bielliptic surface of type 1 (see Examples 2.1.4 and 2.1.6 below for more details). We investigate the properties of the induced Brauer map $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ finding how this behaves differently in the two cases:

Theorem 2.A. *(a) If S is of type 2, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is trivial.*

(b) If S is of type 3, then $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is injective.

The main tool behind our argument is a result of Beauville (see Section 2.1 for more details) which states that the kernel of the Brauer map of a cyclic étale cover $X \to X/\sigma$ is naturally isomorphic to the kernel of the norm map $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(X/\sigma)$ quotiented by $\mathrm{Im}(1-\sigma^*)$. We prove that a line bundle on $\tilde{S}$ is in the kernel of the norm map only if it is numerically trivial. Then we reach our conclusion by carefully computing the norm map of numerically trivial line bundles. The different behaviour of the two type of surfaces is motivated by the different "values" taken by the norm map on torsion elements of $\mathrm{H}^2(\tilde{S},\mathbb{Z})$: in the type 2 case they are sent to topologically trivial line bundles, while this is not true in the type 3 case.

Apart from being of interest on its own, Theorem 2.A, or more precisely some parts of its proof, will be useful in order to study the Brauer map to the canonical cover for bielliptic surfaces of type 2.

We then turn our attention to the main focus of this work, and study the norm map to the canonical cover of a bielliptic surface. We give necessary and sufficient conditions for it to be injective, trivial, and, in the case of type 1 surfaces, neither trivial nor injective. This is done in Theorems 2.5.4, 2.5.9, 2.5.15, 2.5.20, and 2.5.22. Unfortunately the statements are particularly involved and it is not possible reproduce them here without a lengthy explanation of the notation used. Some examples of our results are the following.

Theorem 2.B. *Given a bielliptic surface S, let $\pi : X \to S$ be its canonical cover. If the two elliptic curves A and B are not isogenous, then the pullback map*

$$\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$$

is injective.

The proof of this statement uses the same ideas of the proof of Theorem 2.A. In fact we can leverage the fact that X and S have the same Picard number (as it happened in the case of a bielliptic cover) to show that line bundles in the kernel of the norm map are topologically trivial. The result is then obtained by showing that line bundles in $\mathrm{Pic}^0(X)$ which are also in the kernel of the norm map are always in $\mathrm{Im}(1-\sigma^*)$. As a corollary of both Theorem 2.A and 2.B we find an example of isogeny between two abelian varieties $\varphi : X \to Y$ such that the corresponding group homomorphism φ_{Br} is not injective.

When the two curves A and B are isogenous, we see the first examples of bielliptic surfaces with a non-injective Brauer map to the canonical cover.

This chapter is organized as follows. Section 2.1 contains all the background and preliminary results. More precisely we outline some classical facts on the geometry of bielliptic surfaces, and present the construction, due to Nuer, of the bielliptic covers of surfaces of type 2 and 3. We also expound the work of Beauville [Bea09] which allows us to study the kernel of the Brauer map in terms of the norm homomorphism of the cover. We describe the Neron-Severi group of a product of elliptic curves. Before doing that, in §2.1.6, which was an appendix in the original article and is a joint work of S.Tirabassi and J. Bergström, a structure theorem for the homomorphism ring of two elliptic curves is given in the case of j-invariant 0 or 1728. This will give, in turn, a really useful description of the Picard group of the product of such curves. We conclude Section 2.1 by recalling some results about abelian varieties and providing details about some computations we will use many times.

In Section 2.2 we provide explicit generators for $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}}$, when S is a bielliptic surface of type 1, 2, 3 or 5. Section 2.3 we recall some facts about pullbacks and, using the results of Section 2.2, we make computations that shall be needed in the sections afterwards. We prove Theorem 2.A in Section 2.4, while we completely describe the norm map to the canonical cover in Section 2.5.

Notation. In this chapter we are working over the field of the complex numbers $\mathbb{C}$. If X is a complex abelian variety over $\mathbb{C}$, and $n \in \mathbb{Z}$, then we will write $X[n]$ for the subscheme of n-torsion points of X, while $n_X : X \to X$ will stand for the "multiplication by n isogeny". Given $x \in X$ a point, then we will write t_x for the translation by x. In addition, if $\dim X = 1$ – that is, X is an elliptic curve – then P_x will be the line bundle $\mathcal{O}_X(x - p_0) \simeq t^*_{-x}\mathcal{O}_X(p_0) \otimes \mathcal{O}_X(-p_0)$ in $\mathrm{Pic}^0(X)$, where $p_0 \in X$ is the identity element.

For any smooth complex projective variety Y we will write 1_Y for the identity homomorphism (or simply 1 if there is no chance of confusion).

2.1 Background and Preliminary Results

2.1.1 Bielliptic Surfaces

As seen in Table 1.1, a complex *bielliptic* (or *hyperelliptic*) surface S is a minimal smooth projective surface over the field of complex numbers with Kodaira dimension $\kappa(S) = 0$, irregularity $q(S) = 1$, and geometric genus $p_g(S) = 0$. By the work of Bagnera-De Franchis (see for example [Ba01, 10.24-10.27]), the canonical bundle ω_S has order either 2, 3, 4 or 6 in $\mathrm{Pic}(S)$, and S occurs as a finite étale quotient of a product $A \times B$ of elliptic curves by a finite group G acting on A by translations and on B in such a way that $B/G \simeq \mathbb{P}^1$. More precisely we have the following classification result (see [BDF10], [BM77, p. 37]).

Theorem 2.1.1 (Bagnera-De Franchis). *Any bielliptic surface is of the form $S = A \times B/G$, where A and B are elliptic curves and G is a finite group of translations of A acting on B by automorphisms. They are divided into seven types according to what G is as shown in Table 2.1.*

There are natural maps $a_S\colon S \to A/G$ and $g\colon S \to B/G \simeq \mathbb{P}^1$ which are both elliptic fibrations. The morphism a_S is smooth, and coincides with the Albanese morphism of S. On the other hand, g admits multiple fibres, corresponding to the branch points of the quotient $B \to B/G$, with multiplicity equal to that of the associated branch point. The smooth fibres of a_S and g are isomorphic to B and A, respectively. We will write a and b for the classes of these fibres in $\mathrm{Num}(S)$, $\mathrm{H}^2(S,\mathbb{Z})$ and $\mathrm{H}^2(S,\mathbb{Q})$.

It is well known (see for example [Ser90a, p. 529]) that a and b span $\mathrm{H}^2(S,\mathbb{Q})$ and satisfy $a^2 = b^2 = 0$, $ab = |G|$. Furthermore, we have the following description of the second cohomology of S:

Proposition 2.1.2. *The decomposition of $\mathrm{H}^2(S,\mathbb{Z})$ is described according to the type of S and the multiplicities $(m_1,\ldots,m_s)$ of the singular fibres of $g\colon S \to \mathbb{P}^1$ as follows:*

Type	$(m_1,\ldots,m_s)$	$\mathrm{H}^2(S,\mathbb{Z})$	$\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}}$
1	(2,2,2,2)	$\mathbb{Z}[\frac{1}{2}a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
2	(2,2,2,2)	$\mathbb{Z}[\frac{1}{2}a] \oplus \mathbb{Z}[\frac{1}{2}b] \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	(2,4,4)	$\mathbb{Z}[\frac{1}{4}a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
4	(2,4,4)	$\mathbb{Z}[\frac{1}{4}a] \oplus \mathbb{Z}[\frac{1}{2}b]$	0
5	(3,3,3)	$\mathbb{Z}[\frac{1}{3}a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
6	(3,3,3)	$\mathbb{Z}[\frac{1}{3}a] \oplus \mathbb{Z}[\frac{1}{3}b]$	0
7	(2,3,6)	$\mathbb{Z}[\frac{1}{6}a] \oplus \mathbb{Z}[b]$	0

Proof. See [Ser90a, Tables 2 and 3]. The computation of the torsion of $\mathrm{H}^2(S,\mathbb{Z})$ can be found also in [Iit70, Ser91, Suw70, Ume75]. □

Since $\mathrm{H}^2(S,\mathcal{O}_S) = 0$, the first Chern class map $c_1\colon \mathrm{Pic}(S) \to \mathrm{H}^2(S,\mathbb{Z})$ is surjective, so the Néron-Severi group $NS(S) \simeq \mathrm{H}^2(S,\mathbb{Z})$. Modulo torsion we then get

$$\mathrm{Num}(S) = \mathbb{Z}[a_0] \oplus \mathbb{Z}[b_0]$$

where $a_0 = \frac{1}{\mathrm{ord}\,(\omega_S)} a$ and $b_0 = \frac{\mathrm{ord}\,(\omega_S)}{|G|} b$.

2.1.2 Canonical Covers

Let S be a bielliptic surface and let n be the order of its canonical bundle. Then, by a classical construction (see for example [BM98, Section 2]), ω_S induces an étale cyclic cover $\pi_S \colon X \to S$, called the *canonical cover of S*. From now on, when no confusion can arise, we will omit the subscript S and write simply $\pi : X \to S$.

If we let $\lambda_S := |G| / \operatorname{ord}(\omega_S)$, we have that $G \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/\lambda_S\mathbb{Z}$, and X is the abelian surface sitting as an intermediate quotient

$$\begin{tikzcd} A \times B \arrow[rr] \arrow[dr] & & S \simeq A \times B/G \\ & X \simeq A \times B/H \arrow[ur, "\pi"'] & \end{tikzcd}$$

where $H \simeq \mathbb{Z}/\lambda_S\mathbb{Z}$. The abelian surface X thus comes with homomorphisms of abelian varieties $p_A : X \to A/H$ and $p_B : X \to B/H$ with kernels isomorphic to B and A, respectively. Denoting by a_X and b_X the classes of the fibres A and B in $\operatorname{Num}(X)$, we have $a_X \cdot b_X = \lambda_S$ and the embedding $\pi^* \colon \operatorname{Num}(S) \hookrightarrow \operatorname{Num}(X)$ satisfies

$$\pi^* a_0 = a_X, \qquad \pi^* b_0 = \frac{n}{\lambda_S} b_X. \tag{2.1}$$

There is a fixed-point-free action of the group $\mathbb{Z}/n\mathbb{Z}$ on the abelian variety X such that the quotient is exactly S. We will call σ an automorphism of X that generates $\mathbb{Z}/n\mathbb{Z}$. In what follows it will be useful to have an explicit description of σ when S is of type 1, 2, 3, or 5.

Suppose first that S is of type 1, 3, or 5; so G is cyclic, H is trivial, and $X \simeq A \times B$. If S is of type 3, then the j-invariant of B is 1728, and B admits an automorphism $\omega : B \to B$ of order 4. If S is of type 5, B has j-invariant 0 and admits an automorphism ρ of order 3 (see for example[BM77, p. 37], [Ba01, List 10.27] or [BHPvdV15, p. 199]). With this notation we have that the automorphism σ of $A \times B$ inducing the covering π is given by

$$\sigma(x,y) = \begin{cases} (x+\tau, -y), & \text{if } S \text{ is of type 1,} \\ (x+\epsilon, \omega(y)), & \text{if } S \text{ is of type 3,} \\ (x+\eta, \rho(y)), & \text{if } S \text{ is of type 5,} \end{cases} \tag{2.2}$$

where τ, ϵ, and η are points of A of order 2, 4, and 3 respectively. We remark that different choices for the automorphism ρ and ω – there are two possible choices in each case – will lead to isomorphic bielliptic surfaces.

If S is otherwise of type 2, then there are points $\theta_1 \in A$ and $\theta_2 \in B$, both of order two, such that X is the quotient of $A \times B$ by the involution $(x,y) \mapsto (x+\theta_1, y+\theta_2)$. If we let $[x,y]$ denote the image of (x,y) through the quotient map, we have that

$$\sigma[x,y] = [x+\tau, -y], \tag{2.3}$$

where $\tau \in A$ is a point of order 2, $\tau \neq \theta_1$.

2.1.3 Covers of Bielliptic Surfaces by Other Bielliptic Surfaces

When G is not a cyclic group, or when G is cyclic, but the order of G is not a prime number, then the bielliptic surface S admits a cyclic cover $\tilde{\pi} : \tilde{S} \to S$, where $\tilde{S}$ is another bielliptic surface. This construction, as well as the statements of Lemmas 2.1.3 and 2.1.5, are present in the unpublished work of Nuer [Nue]. Since we were not able to find another source, in this section we give the details of the construction and provide proofs for the aforementioned Lemmas for the reader's convenience. The main point that we will need in Section 2.4 is the description of the pullback map $\mathrm{Num}(S) \to \mathrm{Num}(\tilde{S})$.

We begin with the case in which the order of the canonical bundle is not prime.

Lemma 2.1.3. *Let S be a bielliptic surface such that $\mathrm{ord}(\omega_S)$ is not a prime number and take d a proper divisor of n. Then there is a bielliptic surface $\tilde{S}$ sitting as an intermediate étale cover between S and X,*

$$\begin{tikzcd} X \arrow[r, "\pi_{\tilde{S}}"] \arrow[rr, bend right, "\pi_S"'] & \tilde{S} \arrow[r, "\tilde{\pi}"] & S \end{tikzcd}$$

such that $\mathrm{ord}\left(\omega_{\tilde{S}}\right) = \frac{\mathrm{ord}\left(\omega_S\right)}{d}$ and

$$\tilde{\pi}^* a_0 = \tilde{a_0}, \qquad \tilde{\pi}^* b_0 = d\tilde{b_0},$$

where $\tilde{a_0}, \tilde{b_0}$ are the natural generators of $\mathrm{Num}\left(\tilde{S}\right)$.

Proof. Let $\mathrm{ord}(\omega_S) = k \cdot d$ and $\tilde{\pi} \colon \tilde{S} \to S$ be the cyclic covering of order d associated to ω_S^k. Here $\omega_{\tilde{S}}^k = \tilde{\pi}^* \omega_S^k \simeq \mathcal{O}_{\tilde{S}}$, and by looking at the table for bielliptic surfaces we see that $k = 2$ or 3, hence $6K_{\tilde{S}} \sim 0$ and $\kappa(\tilde{S}) = 0$. Since $\omega_{\tilde{S}}$ is not trivial, $\tilde{S}$ is either an Enriques or a bielliptic surface. It cannot be Enriques, because taking the canonical cover of $\tilde{S}$ we get the canonical cover X of S by composition.

Alternatively, one can see this by letting g be a generator of $G/H \simeq \mathbb{Z}/n\mathbb{Z}$, and setting $\tilde{S} = X/\langle g^d \rangle$. □

Example 2.1.4. Suppose that S is a bielliptic surface of type 3. Then the canonical bundle has order 4. In addition the canonical cover X of S is a product of elliptic curves, that is $X \simeq A \times B$. Using the notation of (2.2), we obtain $\tilde{S}$ from $A \times B$ by taking the quotient with respect to the involution $(x, y) \mapsto (x + 2\epsilon, -y)$. Thus we have that $\tilde{S}$ is a bielliptic surface of type 1. The map $\tilde{\pi} : \tilde{S} \to S$ is an étale double cover with associated involution $\tilde{\sigma}$. Thus, given $s \in \tilde{S}$, we can see it as an equivalence class $[x, y]$ of a point $(x, y) \in A \times B$. Then we have an explicit expression for $\tilde{\sigma}$:

$$\tilde{\sigma}(s) = [x + \epsilon, \omega(y)]. \tag{2.4}$$

On the other hand, when the group G is not cyclic we have the following lemma.

Lemma 2.1.5. *Let S be a bielliptic surface with $\lambda_S > 1$, i.e., with G not cyclic. Then there is a bielliptic surface $\tilde{S}$ sitting as an intermediate étale cover between S and $A \times B$*

$$\begin{tikzcd} A \times B \arrow[r, "\pi_{\tilde{S}}"] \arrow[rr, bend right, "\pi_S"'] & \tilde{S} \arrow[r, "\tilde{\pi}"] & S \end{tikzcd}$$

such that $\lambda_{\tilde{S}} = 1$, $\mathrm{ord}(\omega_{\tilde{S}}) = \mathrm{ord}(\omega_S)$ and

$$\tilde{\pi}^* a_0 = \lambda_S \tilde{a_0},\ \tilde{\pi}^* b_0 = \tilde{b_0},$$

where $\tilde{a_0}, \tilde{b_0}$ are the natural generators of $\mathrm{Num}(\tilde{S})$.

Proof. By the assumption $\lambda_S > 1$, S is of type 2, 4 or 6. For these types, we recall, for example from [Bea96, List VI.20] or [GH94, pp. 585–590], that the action of the generators of G on B can be described respectively as

$$x \mapsto -x,\ x \mapsto x + \epsilon \text{ with } 2\epsilon = 0,$$

$$x \mapsto ix,\ x \mapsto x + \frac{1+i}{2},$$

$$x \mapsto e^{\frac{2\pi i}{3}} x,\ x \mapsto x + \frac{1 - e^{\frac{2\pi i}{3}}}{3}.$$

Viewing G via its action on B as above, we can take $\tilde{G}$ to be the subgroup G generated by -1, i and $e^{\frac{2\pi i}{3}}$, respectively. Then $\tilde{S} := A \times B/\tilde{G}$ is again a bielliptic surface, and more precisely (by Table 2.1) of type 1, 3, or 5; and the map $\pi_S \colon A \times B \to S$ factors as required.

□

Example 2.1.6. Take S to be a bielliptic surface of type 2. Then the group G is isomorphic to the product $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then we obtain $\tilde{S}$ from $A \times B$ by taking the quotient with respect to $(x,y) \mapsto (x + \tau, -y)$, where we are using the notation of (2.3). Thus, as in 2.1.4, $\tilde{S}$ is a bielliptic surface of type 1 and each $s \in \tilde{S}$ can be written as an equivalence class $[x,y]$ of a point $(x,y) \in A \times B$. If we write again $\tilde{\sigma}$ for the involution induced by the cover $\tilde{\pi} : \tilde{S} \to S$, we have the following:

$$\tilde{\sigma}(s) = [x + \theta_1, y + \theta_2]. \tag{2.5}$$

2.1.4 Norm Homomorphisms

Let $\pi : X \to Y$ be a finite locally free morphism of projective varieties of degree n. To it we can associate a group homomorphism $\mathrm{Nm}_\pi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ called the *norm homomorphism associated to π*. This is constructed in the following manner. First, one lets $\mathscr{B} := \pi_* \mathcal{O}_X$, and defines a morphism of sheaves of multiplicative monoids $N : \mathscr{B} \to \mathcal{O}_Y$: given s a section of $\mathscr{B}$ on an open set U, let m_s be the endomorphism of $\mathscr{B}(U)$ induced by the multiplication by s; we set $N(s) := \det(m_s) \in \mathcal{O}_Y(U)$ (see [Gro61, § 6.4, and §6.5] or [Sta19, Lemma 0BD2]). The restriction of N to invertible sections induces a morphism of sheaves of groups $N : \mathscr{B}^* \to \mathcal{O}_Y^*$. Now, given L an invertible sheaf on X, $\pi_* L$ is an invertible $\mathscr{B}$-module and, as such is represented by a cocycle $\{u_{ij}, U_i\}$ for an open cover $\{U_i\}$ of Y. Observe that $u_{ij} \in \mathscr{B}^*(U_{ij})$. The fact that N is multiplicative ensures that also the $v_{ij} := N(u_{ij})$ satisfies the cocycle condition and so uniquely identifies a line bundle $\mathrm{Nm}_\pi(L)$ on Y. The map $L \mapsto \mathrm{Nm}_\pi(L)$ is a group homomorphism by [Gro61, (6.5.2.1)]. In addition [Gro61, (6.5.2.4)] ensures that

$$\mathrm{Nm}_\pi(\pi^* M) \simeq M^{\otimes n}, \tag{2.6}$$

and we also have the following important property:

Proposition 2.1.7. *Given two finite locally free morphism $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$, then*

$$\mathrm{Nm}_{\pi_2 \circ \pi_1} = \mathrm{Nm}_{\pi_2} \circ \mathrm{Nm}_{\pi_1}.$$

Proof. See [Gro67, Lemma 21.5.7.2]. □

Suppose now that $\pi : X \to Y$ is an étale cyclic cover of degree n. Then there is a fixed-point-free automorphism $\sigma : X \to X$ of order n such that $Y \simeq X/\sigma$. In addition we can write $\mathscr{B} \simeq \bigoplus_{h=0}^{n-1} M^{\otimes h}$ with M a line bundle of order n in $\mathrm{Pic}(Y)$. In this particular setting the norm homomorphism satisfies some additional useful properties. First, as Nm_π behaves well with base change ([Gro61, Proposition 6.5.8]), it is not difficult to see that

$$\mathrm{Nm}_\pi \circ (1_X - \sigma^*) = 0. \tag{2.7}$$

Indeed, we consider the following cartesian diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "1_Y"] & Y; \end{tikzcd}$$

then, applying the proposition we mentioned, we get that $1_Y^* \mathrm{Nm}_\pi(L) \simeq \mathrm{Nm}_\pi(\sigma^* L)$ and we conclude because the norm is a group homomorphism.

In addition, as discussed by Beauville in [Bea09], we have that

$$\pi^* \mathrm{Nm}_\pi(L) \simeq \bigotimes_{h=0}^{n} (\sigma^h)^* L. \tag{2.8}$$

In fact, by the definition of pushforward of divisors ([Gro67, Definition 21.5.5]), if $L \simeq \mathcal{O}_X(\sum a_i \cdot D_i)$ with prime divisors on X, then $\mathrm{Nm}_\pi(L) \simeq \mathcal{O}_Y(\sum a_i \cdot \pi_* D_i)$. Therefore (2.8) follows from the fact that for a prime divisor D we have that $\pi^* \pi_* D \sim \sum_{h=0}^{n-1} (\sigma^h)^* D$.

Remark 2.1.8 (Pic^0-trick). In what follows it will be important to provide elements in the kernel of the norm homomorphism. We will often use the following trick. Let $\pi : X \to Y$ be an étale morphism of degree n and suppose that there is a line bundle L on X such that $\mathrm{Nm}_\pi(L) \in \mathrm{Pic}^0(Y)$. Then there is an element $\alpha \in \mathrm{Pic}^0(X)$

such that $\mathrm{Nm}_\pi(L \otimes \alpha)$ is trivial. In fact, as abelian varieties are divisible groups, it is possible to find $\beta \in \mathrm{Pic}^0(Y)$ such that $\beta^{\otimes n} \simeq \mathrm{Nm}_\pi(L)^{-1}$. Then, by (2.6) we get

$$\mathrm{Nm}_\pi(L \otimes \pi^*\beta) \simeq \mathrm{Nm}_\pi(L) \otimes \beta^{\otimes n} \simeq \mathcal{O}_Y.$$

We conclude this section by saying that, from now on, if there is no chance of confusion, we will omit the subscript when denoting the norm. That is, we will write Nm instead of Nm_π.

2.1.5 Brauer Groups and Brauer Maps

For a scheme X, the *cohomological Brauer group* $\mathrm{Br}'(X)$ is defined as the étale cohomology group $\mathrm{H}^2_{\text{ét}}(X,\mathcal{O}_X^*)$. For complex varieties, this is isomorphic to the torsion of $\mathrm{H}^2(X,\mathcal{O}_X^*)$ in the analytic topology. In addition, when X is quasi-compact and separated, by a theorem of Gabber (see, for example, [dJ] for more details) the cohomological Brauer group of X is canonically isomorphic to the *Brauer group* $\mathrm{Br}(X)$ of Morita-equivalence classes of Azumaya algebras on X. As far as our work here is concerned, we will deal only with smooth complex projective varieties, in which case these three groups will be isomorphic and we will use the notation $\mathrm{Br}(X)$ without risk of confusion. Furthermore, we will write simply of the *Brauer group of* X, without any additional adjective.

If S is a bielliptic surface, the exponential sequence yields that $\mathrm{H}^3(S,\mathbb{Z}) \simeq \mathrm{H}^2(S,\mathcal{O}_S^*)$, so that the Brauer group of S is isomorphic to the torsion of $\mathrm{H}^3(S,\mathbb{Z})$. By the universal coefficients theorem, the torsion of $\mathrm{H}^3(S,\mathbb{Z})$ is (non-canonically) isomorphic to the torsion of $\mathrm{H}_2(S,\mathbb{Z})$, and this in turn is isomorphic to the torsion of $\mathrm{H}^2(S,\mathbb{Z})$ by Poincaré duality; this implies that the Brauer group of S can be described in terms of Proposition 2.1.2.
Crucial to our purposes will be the following result of Beauville which describes the kernel of the Brauer map π_{Br} when π is a cyclic étale cover.

Proposition 2.1.9 ([Bea09, Prop. 4.1]). *Let $\pi\colon X \to S$ be an étale cyclic covering of smooth projective varieties. Let σ be a generator of the Galois group of π, $\mathrm{Nm}\colon \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ be the norm map and $\pi_{\mathrm{Br}}\colon \mathrm{Br}(S) \to \mathrm{Br}(X)$ be the pullback. Then we have a canonical isomorphism*

$$\mathrm{Ker}(\pi_{\mathrm{Br}}) \simeq \mathrm{Ker}\,\mathrm{Nm}/(1-\sigma^*)\,\mathrm{Pic}(X).$$

2.1.6 The Homomorphism Lattice of Two Elliptic Curves

This section (the appendix in the original paper) contains the results of Jonas Bergström and Sofia Tirabassi about the formulation and proof of a structure theorem for the $\mathbb{Z}$-module $\mathrm{Hom}(B,A)$, where A and B are two complex elliptic curves with $j(B) = 0, 1728$. This result is used in 2.1.7 in order to make a clever choice of generators for $\mathrm{Num}(A \times B)$ which in turn allows an accurate description of the action of the automorphism σ^* on the Neron-Severi group of the product $A \times B$.

If B is an elliptic curve with j-invariant either 0 or 1728, then B admits an automorphism λ_B of order 3 or 4 respectively. The main result of this section is that the group $\mathrm{Hom}(B,A)$ can be completely described in terms of λ_B and an isogeny $\psi : B \to A$. More precisely, we have the following statement:

Theorem 2.1.10. *Let A and B be two isogenous complex elliptic curves and assume that $j(B)$ is either 0 or 1728. Then there exist an isogeny $\psi : B \to A$ such that*

$$\mathrm{Hom}(B,A) =< \psi, \psi \circ \lambda_B > .$$

This section is organized into three subsections. In the first one we outline some classical results about imaginary quadratic fields and their orders. The second focuses on complex elliptic curves with complex multiplication. Theorem 2.1.10 is proven the third subsection. The key idea of our argument is to describe $\mathrm{Hom}(B,A)$ as a fractional ideal of $\mathrm{End}(B)$ homothetic to $\mathrm{End}(B)$. This is done by observing that the class number of $\mathrm{End}(B)$ is 1.

Preliminaries on Orders in Imaginary Quadratic Fields

An *imaginary quadratic field* is a subfield $K \subseteq \mathbb{C}$ of the form $\mathbb{Q}(\sqrt{-d})$, with d a positive, square-free integer. The *discriminant of* K is the integer d_k defined as

$$d_K = \begin{cases} -d, & \text{if } d \equiv 1 \mod 4, \\ -4d, & \text{otherwise.} \end{cases}$$

The *ring of integers of* K, $\mathcal{O}_K$, is the largest subring of K which is a finitely generated abelian group. Then we have that $\mathcal{O}_K = \mathbb{Z}[\delta]$, where

$$\delta = \begin{cases} \frac{1+\sqrt{-d}}{2}, & \text{if } d \equiv 3 \mod 4, \\ \sqrt{-d}, & \text{otherwise.} \end{cases} \tag{2.9}$$

An *order* in an imaginary quadratic field K is a subring $\mathcal{O}$ of $\mathcal{O}_K$ which properly contains $\mathbb{Z}$. It turns out that $\mathcal{O} \simeq \mathbb{Z} + \mathbb{Z} \cdot (n\delta)$ for some positive integer n.

Given an order $\mathcal{O}$ in an imaginary quadratic field K, a *fractional ideal* of $\mathcal{O}$ is a non-zero finitely generated sub $\mathcal{O}$-module of K. For every M fractional ideal of $\mathcal{O}$ there is an $\alpha \in K^*$ and an ideal $\mathfrak{a}$ of $\mathcal{O}$ such that $M = \alpha \cdot \mathfrak{a}$. We will need the following notions.

Definition 2.1.11. (i) Two fractional $\mathcal{O}$-ideals M and M' are *homothetic* if there is $\alpha \in K^*$ such that $M = \alpha M'$.

(ii) A fractional $\mathcal{O}$-ideal M is called *proper* if

$$\mathcal{O} = \{\alpha \in K \mid \alpha M \subseteq M\}.$$

(iii) A fractional $\mathcal{O}$-ideal is *invertible* if there is a fractional ideal M' such that $M \cdot M' \simeq \mathcal{O}$. The class of invertible $\mathcal{O}$-ideals is denoted by $I(\mathcal{O})$.

(iv) A fractional $\mathcal{O}$-ideal M is *principal* if it is of the form $\alpha \cdot \mathcal{O}$ for some $\alpha \in K^*$. Therefore, principal ideals are precisely the fractional ideals homothetic to $\mathcal{O}$. The class of principal $\mathcal{O}$-ideals is denoted by $P(\mathcal{O})$.

Principal ideals are clearly invertible. Not all fractional ideals are invertible, but proper fractional ideals are ([Cox11, Proposition 7.4]). In particular we have that, if $\mathcal{O} \simeq \mathcal{O}_K$, then all fractional ideals are invertible (see also [Cox11, Proposition 5.7]). The quotient

$$\mathfrak{Cl}(\mathcal{O}) := I(\mathcal{O})/P(\mathcal{O})$$

describes the homothety classes of invertible $\mathcal{O}$-ideals. It is a group with the product and it is called the *ideal class group of* $\mathcal{O}$. Its order is called the *class number of* $\mathcal{O}$. When $\mathcal{O} \simeq \mathcal{O}_K$, then the class number of $\mathcal{O}$ is exactly the class number of the field K, which is a function of the discriminant of K (see [Cox11, Theorem 5.30(ii)]).

Example 2.1.12. If K is either $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, then all the fractional ideals of $\mathcal{O}_K$ are homothetic to $\mathcal{O}_K$. In fact the class number of the field K in this cases is 1, as it was computed by Gauss in his book *Disquisitiones arithmeticae*, [Gau66].

Elliptic Curves with Complex Multiplication

The importance of orders in the study of the geometry of elliptic curves is that they describe the endomorphism ring of a complex elliptic curve:

Theorem 2.1.13. *Let A be an elliptic curve over $\mathbb{C}$, then $\mathrm{End}(A)$ is either isomorphic to $\mathbb{Z}$ or to an order in an imaginary quadratic field.*

Proof. See [Was08, Theorem 10.2]. □

We say that a (complex) elliptic curve has *complex multiplication* if its endomorphism ring is larger than $\mathbb{Z}$. Observe that in this case $\mathrm{End}(A) \otimes \mathbb{Q}$ is a quadratic field K and $\mathrm{End}(A)$ is an order in K.

Given a complex elliptic curve A there is a canonical way to identify its endomorphism ring with a subring of $\mathbb{C}$. More generally let A and B two elliptic curves, then there are two lattices Λ_A and Λ_B in $\mathbb{C}$ such that $A \simeq \mathbb{C}/\Lambda_A$ and $B \simeq \mathbb{C}/\Lambda_B$. Given a complex number ζ such that $\zeta \cdot \Lambda_B \subseteq \Lambda_A$, the map $\Phi_\zeta : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto \zeta \cdot z$ descends to an (algebraic) homomorphism $\varphi_\zeta : B \to A$. It is possible to show (see [Sil09, VI.5.3(d)]) that any morphism of elliptic curves preserving the origin is obtained in this way, and in particular we get an isomorphism of abelian groups

$$\mathrm{Hom}(B, A) \simeq \{\zeta \in \mathbb{C} \mid \zeta \cdot \Lambda_B \subseteq \Lambda_A\} \subseteq \mathbb{C}. \tag{2.10}$$

By setting $B = A$ we get a ring isomorphism

$$\mathrm{End}(A) \simeq \mathcal{O} := \{\zeta \in \mathbb{C} \mid \zeta \cdot \Lambda_A \subseteq \Lambda_A\} \subseteq \mathbb{C}.$$

The isomorphism $\zeta \mapsto \varphi_\zeta$ is characterized as the unique isomorphism $f : \mathcal{O} \to \mathrm{End}(A)$ such that, for any $\zeta \in \mathcal{O}$ and for every invariant form ω on A we have that $f(\zeta)^*\omega = \zeta \cdot \omega$ ([Sil94, II.1.1]). We will make use of the following notation.

Notation 2.1.14. For an elliptic curve with complex multiplication A such that $\mathrm{End}(A) \simeq \mathbb{Z} + \mathbb{Z} \cdot n\delta$, we will write λ_A for the isogeny $\varphi_{n\delta} : A \to A$ and we will say that A *has complex multiplication by* λ_A.

It is clear that, as a $\mathbb{Z}$-module, $\mathrm{End}(A) =< 1_A, \lambda_A >$.

Example 2.1.15. (a) Suppose that B is an elliptic curve with j-invariant 0. Then we can write $B \simeq \mathbb{C}/\Lambda_B$, with $\Lambda_B =< 1, e^{\frac{2\pi i}{3}} >$. Then $\mathrm{End}(B) \otimes \mathbb{Q} \simeq \mathbb{Q}(\sqrt{-3})$ and $\mathrm{End}(B) \simeq \mathcal{O}_K = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$. We have that λ_B is induced by the multiplication by $\frac{-1+\sqrt{-3}}{2}$ and is an automorphism of B satisfying $\lambda_B^2 + \lambda_B + 1_B = 0$. This is exactly the automorphism which in (2.2) was called ρ and used to construct bielliptic surfaces of type 5.

(b) Suppose now that the j-invariant of B is 1728. Then we can take $\Lambda_B =< 1, i >$ and we have that $\mathrm{End}(B) \otimes \mathbb{Q} \simeq \mathbb{Q}(i)$. The endomorphism ring of B is isomorphic to $\mathbb{Z}[i]$ and the multiplication by i induces an automorphism λ_B such that $\lambda_B^2 + 1_B = 0$. This is the automorphism ω of B used to construct bielliptic surfaces of type 3 in (2.2).

Proof of the Main Result

We are now ready to provide a proof for Theorem 2.1.10. Our key point will be the following:

Claim: the $\mathbb{Z}$-module $\mathrm{Hom}(B, A)$ is isomorphic to a fractional ideal of $\mathcal{O}_K$, where K is $\mathbb{Q}(i)$ if $j(B) = 0$ and $\mathbb{Q}(\sqrt{-3})$ if $j(B) = 1728$.

Before proceeding to show that this Claim is true, let us see how it implies the statement. By Example 2.1.12 all fractional $\mathcal{O}_K$-ideals are homothetic to $\mathcal{O}_K$. Therefore, for any fractional ideal M there exists an $\alpha \in K^*$ such that

$$M \simeq \alpha \cdot \mathcal{O}_K = \alpha \cdot < 1, \delta >,$$

where δ is like in (2.9), which is, $\delta = i$ if K is $\mathbb{Q}(i)$ and $\delta = \frac{1+\sqrt{-3}}{2}$ if K is $\mathbb{Q}(\sqrt{-3})$. Then the fact that

$$M =< \alpha, \alpha \cdot \delta >$$

implies that $\mathrm{Hom}(B, A) =< \varphi_\alpha, \varphi_\alpha \circ \lambda_B >$, and the statement is true.

Proof of the Claim. Let $\Lambda_A =< 1, \tau >$ a lattice in $\mathbb{C}$ such that $A \simeq \mathbb{C}/\Lambda_A$, and write $K \subseteq \mathbb{C}$ for the quadratic field $\mathrm{End}(B) \otimes \mathbb{Q} = \mathrm{End}(A) \otimes \mathbb{Q}$. Then the ring $\mathrm{End}(B)$ is exactly the ring of integers $\mathcal{O}_K$. Observe that this is isomorphic to a lattice in $\mathbb{C}$, and that $B \simeq \mathbb{C}/\mathcal{O}_K$ (See Example 2.1.15).

By (2.10) we can identify $M := \mathrm{Hom}(B, A)$ with a subgroup of $\mathbb{C}$. Composition on the right with endomorphisms of B gives to M a structure of $\mathcal{O}_K$-module.

We also have that $\mathfrak{a} := \mathrm{Hom}(A, B)$ is isomorphic to a subgroup of $\mathbb{C}$. Using the fact that $\mathrm{Hom}(A, B) \simeq \{\zeta \in \mathbb{C} \mid \zeta \cdot \Lambda_A \subseteq \mathcal{O}_K\}$, it follows that for every $\alpha \in \mathfrak{a}$ we have that $\alpha = \alpha \cdot 1 \in \mathcal{O}_K$. Therefore $\mathfrak{a}$ is indeed a subgroup of $\mathcal{O}_K$, and hence it is a subgroup of K.

Let $\alpha \neq 0$ denote an element of $\mathfrak{a}$, then clearly $\alpha \cdot M \subseteq \mathcal{O}_K$. Since $\mathfrak{a}$ is a subgroup of K, the inverse of α belongs to K, and we deduce that $M \subseteq K$. It follows that M is a fractional ideal of $\mathcal{O}_K$, and the Claim is proven. □

Remark 2.1.16. (a) It is clear from the proof that the role of A and B can be exchanged, so we have proven a structure theorem for $\mathrm{Hom}(A,B)$ when one of the two curves has j invariant 0 or 1728.

(b) More generally we can see that the same argument works whenever the class number of $\mathcal{O} := \mathrm{End}(B)$ is 1. The tricky part is to show that also in this case we have that $B \simeq \mathbb{C}/\mathcal{O}$, but this is a consequence of [Cox11, Corollary 10.20] which yields that, since $\mathcal{O}$ has class number 1, that there is just one elliptic curve up to isomorphism with endomorphism ring $\mathcal{O}$.

In particular it will work also when $\mathrm{End}(B)$ is the ring of integers of $\mathbb{Q}(\sqrt{-d})$ with $d = 2, 7, 11, 19, 43, 67,$ and 163.

2.1.7 The Neron-Severi Lattice of a Product of Elliptic Curves

In this section we want to describe $\mathrm{Num}(A \times B)$ when A and B are two elliptic curves. Many of these topics might be well-known to experts, but we were not able to find a rigorous reference about them, thus we wrote this for the reader's convenience. In the first part of this section we will follow closely the exposition in [HLT20].

Let A be an elliptic curve over $\mathbb{C}$ with identity element p_0, then there is a lattice Λ such that $A \simeq \mathbb{C}/\Lambda$. Identify A with its dual and consider $\mathscr{P}_A$ the *normalized Poincaré bundle* on $A \times A$:

$$\mathscr{P}_A \;\simeq\; \mathcal{O}_{A\times A}(\Delta_A) \otimes \mathrm{pr}_1^*\mathcal{O}_A(-p_0) \otimes \mathrm{pr}_2^*\mathcal{O}_A(-p_0)$$

where $\Delta_A \subset A \times A$ is the diagonal divisor and pr_1, pr_2 are the projections of $A \times A$ onto the first and second factor respectively. Observe that if x is a point in A, then the topologically trivial line bundle P_x is simply $\mathscr{P}_{A|A\times\{x\}} \simeq \mathscr{P}_{A|\{x\}\times A}$.

Given another elliptic curve B, line bundles L_A and L_B on A and B respectively, and a morphism $\varphi : B \to A$, we define a line bundle on the product $A \times B$

$$L(L_A, L_B, \varphi) := (1_A \times \varphi)^* \mathscr{P}_A \otimes \mathrm{pr}_A^* L_A \otimes \mathrm{pr}_B^* L_B \tag{2.11}$$

where pr_A and pr_B are the projections onto A and B respectively. As a direct consequence of the See-Saw Principle it is possible to see that, if M_A and M_B are two other line bundles on A and B, and $\psi : B \to A$ is another homomorphism, then

$$L(L_A \otimes M_A, L_B \otimes M_B, \varphi + \psi) \simeq L(L_A, L_B, \varphi) \otimes L(M_A, M_B, \psi).$$

In addition, the universal property of the dual abelian variety ensures that every line bundle $L \in \mathrm{Pic}(A \times B)$ is of the form $L(L_A, L_B, \varphi)$ for some invertible sheaves L_A and L_B and a morphism φ. Therefore we have an isomorphism

$$\mathrm{Pic}(A \times B) \simeq \mathrm{Pic}(A) \times \mathrm{Pic}(B) \times \mathrm{Hom}(B, A).$$

If we quotient by numerically trivial line bundles, we find that

$$\mathrm{H}^2(A \times B, \mathbb{Z}) \simeq \mathrm{Num}(A \times B) \simeq \mathbb{Z} \cdot [B] \times \mathbb{Z} \cdot [A] \times \mathrm{Hom}(B, A), \tag{2.12}$$

where $[A]$ and $[B]$ are the classes of the fibres of the two projections. Let us write $l(\deg(L_A), \deg(L_B), \varphi)$ for the first Chern class of $L(L_A, L_B, \varphi)$. Then every class in $\mathrm{Num}(A \times B)$ can be written as $l(m, n, \varphi)$ for some integers n and m and an isogeny φ. In what follows we will often refer to line bundles (or numerical classes) in $\mathrm{Hom}(B, A)$ as elements of the *Hom-part* of $\mathrm{Pic}(A \times B)$ (or of $\mathrm{Num}(A \times B)$). For our purposes it will be really important to pick explicit generators for $\mathrm{Num}(A \times B)$ to see how the automorphism σ acts on $\mathrm{H}^2(A \times B, \mathbb{Z})$. In order to do that, we need to investigate the $\mathbb{Z}$-module structure on $\mathrm{Hom}(B, A)$.

So suppose that there is a non-trivial isogeny $\varphi : B \to A$. Then we know that $\mathrm{Hom}(B, A)$ has rank 1 if A does not have complex multiplication, and 2 otherwise (more details about elliptic curves with complex multiplication can be found in §2.1.6).

Suppose we are in the first case, so that there exists an isogeny $\psi : B \to A$ such that $l(0, 0, \psi)$ generates the Hom-part of $\mathrm{H}^2(A \times B, \mathbb{Z})$. We will call such isogeny a *generating isogeny for* $\mathrm{Num}(A \times B)$. Observe that, since $l(0, 0, \psi)$ is necessarily a primitive class, ψ cannot factor through any "multiplication by n" map. That is, we cannot write $\psi = n \cdot \psi'$ for any n. In particular, for any integer n we have that $\mathrm{Ker}\,\psi$ does not contain $B[n]$ as a subscheme.
Indeed, given an elliptic curve E and two nonconstant separable isogenies on it, if one of the two kernels is contained in the other, then one isogeny factorises through the other (see e.g. [Sil09, III.4 Corollary 4.11]); moreover every finite subgroup of an elliptic curve is the kernel of a unique isogeny (see e.g. [Sil09, III.4 Proposition 4.12]). So, if $B[n]$ were contained in $\mathrm{Ker}\,\psi$, by the results we just mentioned we would have a commuting diagram

$$\begin{tikzcd} B \arrow[r, "n"] \arrow[dr, "\psi"'] & B/B[n] \arrow[d, dashed, "\psi'"] \\ & A \end{tikzcd}$$

and this is absurd.

Suppose now that A has complex multiplication, and again fix a non-trivial isogeny $\varphi : B \to A$. Then also B has complex multiplication, and $\mathrm{Hom}(B, A)$ is a rank 2 free $\mathbb{Z}$-module. We pick generators ψ_1 and ψ_2, and we have that for any line bundle L on $A \times B$ there are two integers h and k such that

$$L \simeq L(M_A, M_B, h \cdot \psi_1 + k \cdot \psi_2), \tag{2.13}$$

where M_A and M_B are element of $\mathrm{Pic}(A)$ and $\mathrm{Pic}(B)$ respectively. In addition we can write

$$\mathrm{H}^2(A \times B, \mathbb{Z}) = \langle l(1,0,0), l(0,1,0), l(0,0,\psi_1), l(0,0,\psi_2) \rangle\,. \tag{2.14}$$

In the particular cases in which the j-invariant of B is either 0 or 1728, then Theorem 2.1.10 in §2.1.6 yields a more accurate description: if we write $\lambda_B : B \to B$ for the automorphism ρ or ω (see §2.1.2), we have that there exists an isogeny $\psi : B \to A$ such that in (2.13) and (2.14) we can take $\psi_1 = \psi$ and $\psi_2 = \psi \circ \lambda_B$. So we have that

$$\mathrm{H}^2(A \times B, \mathbb{Z}) = \langle l(1,0,0), l(0,1,0), l(0,0,\psi), l(0,0,\psi \circ \lambda_B) \rangle . \tag{2.15}$$

In this case we say that ψ is again a *generating isogeny* for $\mathrm{H}^2(A \times B, \mathbb{Z})$. Observe again that the isogenies ψ_i, as well as ψ, cannot factor through the multiplication by an integer or they could not generate the whole $\mathrm{Hom}(B, A)$.

2.1.8 The See-Saw Principle and Complex Abelian Varieties

We recall from [Mi08, Corollary 5.18] the following result which was first introduced by André Weil, the See-Saw Principle.

Theorem 2.1.17 (See-Saw Principle)**.** *Let W and Y be varieties over an algebraically closed field k, assume W complete. Let $\mathcal{S}$ and $\mathcal{M}$ be invertible sheaves on the product variety $W \times Y$. Assume that*

- *for all closed points $y \in Y$ we have $\mathcal{S}_y \simeq \mathcal{M}_y$;*
- *for at least one closed point $\bar{w} \in W$ we have $\mathcal{S}_{\bar{w}} \simeq \mathcal{M}_{\bar{w}}$.*

Then there is an isomorphism $\mathcal{S} \simeq \mathcal{M}$.

Remark 2.1.18. For any complex torus T of dimension g, the $\mathrm{H}^n(T, \mathbb{Z})$, i.e. the singular cohomology groups with values in $\mathbb{Z}$, are free abelian groups of rank $\binom{2g}{n}$ for any integer $n \geq 1$ (see for example [BL04, Corollary 1.3.3]).

In particular, for any complex abelian variety X we have that $\mathrm{H}^2(X, \mathbb{Z})_{\mathrm{tor}} \simeq 0$, and therefore algebraic equivalence and numerical equivalence coincide.

We also recall a characterization of the Picard variety in the case of abelian varieties. See [Mu14, Corollary 4, §6] and [Mu14, Definition, p.70].

Lemma 2.1.19. *Let X be an abelian variety over an algebraically closed field. Then*

$$\mathrm{Pic}^0(X) \simeq \{L \in \mathrm{Pic}(X) | \forall \gamma \in X \; t_\gamma^* L \simeq L\}. \tag{2.16}$$

We will need the following results (see [Mu14, §8.]):

Lemma 2.1.20. *Let X be an abelian variety over an algebraically closed field and consider any $L \in \mathrm{Pic}^0(X)$. Then*

i. for all schemes S and morphisms $f, g : S \longrightarrow X$ we have

$$(f+g)^* L \simeq f^* L \otimes g^* L;$$

ii. for any $n \in \mathbb{Z}$ we have
$$n_X^* L \simeq L^{\otimes n}.$$

We will now state and proof some lemmas that we will use in this work. The first lemma is a very easy consequence of the See-Saw Principle.

Lemma 2.1.21. *Let A be an elliptic curve, then*

$$(1_A \times 0_{\hat{A}})^* \mathscr{P}_A \simeq \mathcal{O}_{A \times \hat{A}}. \tag{2.17}$$

Proof. We avail ourselves of the See-Saw Principle, and since any restriction of $\mathcal{O}_{A\times\hat{A}}$ is trivial, we prove the statement by showing that $(1_A \times 0_{\hat{A}})^* \mathscr{P}_A$ restricts trivially to both $A \times 0$ and $\{a\} \times \hat{A}$ for any $a \in A$.
By the commutative diagram

$$\begin{tikzcd} A \times 0 \arrow[r, hook] \arrow[dr, hook] & A \times \hat{A} \arrow[d, "1_A \times 0_{\hat{A}}"] \\ & A \times \hat{A}, \end{tikzcd}$$

we get that
$$(1_A \times 0_{\hat{A}})^* \mathscr{P}_A|_{A\times 0} \simeq \mathscr{P}_A|_{A\times 0} \simeq \mathcal{O}_A.$$
Now, for any $a \in A$, we have the diagram

$$\begin{tikzcd} \{a\} \times \hat{A} \arrow[r, hook] \arrow[dr, "\psi"'] & A \times \hat{A} \arrow[d, "1_A \times 0_{\hat{A}}"] \\ & A \times \hat{A}, \end{tikzcd}$$

where ψ is the constant map sending any point to $(a, 0)$. Therefore it follows that

$$(1_A \times 0_{\hat{A}})^* \mathscr{P}_A|_{\{a\}\times\hat{A}} \simeq \psi^* \mathscr{P}_A \simeq \mathcal{O}_{\hat{A}},$$

and the lemma is proven. □

Now let A, B be elliptic curves, let $\sigma : A \times B \longrightarrow A \times B$ be defined as $\sigma = t_\xi \times \zeta$, where ξ is a point on A and ζ an automorphism of B. Before proceeding with the statement and proof of Lemma 2.1.23, we begin with a preliminary observation which depends on the See-Saw Principle:

Lemma 2.1.22. *With the notation introduced above,*

$$(t_\xi \times 1_{\hat{A}})^* \mathscr{P}_A \simeq \mathscr{P}_A \otimes p_{\hat{A}}^* P_\xi. \tag{2.18}$$

Proof. By the See-Saw Principle, we prove the statement by showing that the two line bundles are isomorphic when restricted to $0 \times \hat{A}$ and $A \times \{\hat{a}\}$ for any $\hat{a} \in \hat{A}$.
We begin by restricting to $0 \times \hat{A}$. Consider the diagram

$$\begin{tikzcd}
0 \times \hat{A} \arrow[r, hook] \arrow[d, "\xi \times 1_{\hat{A}}"'] & A \times \hat{A} \arrow[d, "t_\xi \times 1_{\hat{A}}"] \\
\{\xi\} \times \hat{A} \arrow[r, hook] & A \times \hat{A},
\end{tikzcd}$$

from which it follows that

$$(t_\xi \times 1_{\hat{A}})^* \mathscr{P}_A|_{0\times\hat{A}} \simeq (\xi \times 1_{\hat{A}})^*(\mathscr{P}_A|_{\{\xi\}\times\hat{A}}) \simeq (\xi \times 1_{\hat{A}})^* P_\xi \simeq P_\xi.$$

Also, by

$$\begin{tikzcd}
0 \times \hat{A} \arrow[r, hook] \arrow[dr, "(0,\hat{a}) \mapsto \hat{a}"'] & A \times \hat{A} \arrow[d, "p_{\hat{A}}"] \\
& \hat{A},
\end{tikzcd}$$

it follows that

$$\begin{aligned}(\mathscr{P}_A \otimes p_{\hat{A}}^* P_\xi)|_{0\times\hat{A}} &\simeq \mathscr{P}_A|_{0\times\hat{A}} \otimes (p_{\hat{A}}^* P_\xi)|_{0\times\hat{A}} \\ &\simeq \mathcal{O}_{\hat{A}} \otimes P_\xi \simeq P_\xi.\end{aligned}$$

By the diagram

$$\begin{tikzcd}
A \times \{\hat{a}\} \arrow[r, hook] \arrow[d, "t_\xi \times 1_{\{\hat{a}\}}"'] & A \times \hat{A} \arrow[d, "t_\xi \times 1_{\hat{A}}"] \\
A \times \{\hat{a}\} \arrow[r, hook] & A \times \hat{A},
\end{tikzcd}$$

we get

$$\begin{aligned}(t_\xi \times 1_{\hat{A}})^* \mathscr{P}_A|_{A\times\{\hat{a}\}} &\simeq (t_\xi \times 1_{\{\hat{a}\}})^*(\mathscr{P}_A|_{A\times\{\hat{a}\}}) \\ &\simeq t_\xi^* P_{\hat{a}} \simeq P_{\hat{a}},\end{aligned}$$

where the last step follows from Lemma 2.1.19.
Finally, since the composition of $p_{\hat{A}}$ with the inclusion of $A \times \{\hat{a}\}$ in $A \times \hat{A}$ is the constant map $\hat{a}$, we get

$$(\mathscr{P}_A \otimes p_{\hat{A}}^* P_\xi)|_{A\times\{\hat{a}\}} \simeq P_{\hat{a}} \otimes \mathcal{O}_A \simeq P_{\hat{a}},$$

and the statement follows. □

So we can show

Lemma 2.1.23. *Let* $\varphi : B \to A$ *be a morphism, consider* $L_A \in \mathrm{Pic}(A)$ *and* $L_B \in \mathrm{Pic}(B)$*. Then*

$$\begin{aligned}\sigma^*\big((1_A \times \varphi)^* \mathscr{P}_A \otimes p_A^* L_A \otimes p_B^* L_B\big) &\simeq \\ (1_A \times \varphi \circ \zeta)^* \mathscr{P}_A \otimes p_A^* t_\xi^* L_A &\otimes p_B^*(\zeta^* L_B \otimes (\varphi \circ \zeta)^* P_\xi).\end{aligned} \tag{2.19}$$

In particular, if $\alpha \in \mathrm{Pic}^0(A)$ *and* $\beta \in \mathrm{Pic}^0(B)$*, then*

$$\begin{aligned}\sigma^*\big((1_A \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta\big) \simeq \\ (1_A \times \varphi \circ \zeta)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^*(\zeta^* \beta \otimes (\varphi \circ \zeta)^* P_\xi).\end{aligned} \tag{2.20}$$

Proof. We prove the statement by applying σ^* separately to $p_A^* L_A$, $p_B^* L_B$ and $(1_A \times \varphi)^* \mathscr{P}_A$.
First, observe that $p_A \circ \sigma = t_\xi \circ p_A$ and $p_B \circ \sigma = \zeta \circ p_B$. From the latter it follows directly that

$$\sigma^* p_B^* L_B \simeq p_B^* \zeta^* L_B.$$

Similarly, $p_A \circ \sigma = t_\xi \circ p_A$ implies that

$$\sigma^* p_A^* L_A \simeq p_A^* t_\xi^* L_A.$$

Moreover, if $\alpha \in \mathrm{Pic}^0(A)$, Lemma 2.1.19 implies that we can further simplify to

$$\sigma^* p_A^* \alpha \simeq p_A^* t_\xi^* \alpha \simeq p_A^* \alpha.$$

Finally, by Lemma 2.1.22,

$$\begin{aligned}\sigma^*(1_A \times \varphi)^* \mathscr{P}_A &\simeq (t_\xi \times \zeta)^* (1_A \times \varphi)^* \mathscr{P}_A \\ &\simeq (1_A \times \zeta)^* (1_A \times \varphi)^* (t_\xi \times 1_{\hat{A}})^* \mathscr{P}_A \\ &\simeq (1_A \times \zeta)^* (1_A \times \varphi)^* \big(\mathscr{P}_A \otimes p_{\hat{A}}^* P_\xi\big) \\ &\simeq (1_A \times \varphi \circ \zeta)^* \mathscr{P}_A \otimes (1_A \times \varphi \circ \zeta)^* p_{\hat{A}}^* P_\xi \\ &\simeq (1_A \times \varphi \circ \zeta)^* \mathscr{P}_A \otimes p_B^* (\varphi \circ \zeta)^* P_\xi,\end{aligned}$$

where the last step follows from the fact that $p_{\hat{A}} \circ (1_A \times \varphi \circ \zeta) = \varphi \circ \zeta \circ p_B$ and that we identify $\hat{A}$ with A. □

Finally, we observe the following:

Remark 2.1.24. Let $\varphi : B \to A$ be a morphism, let $L_A \in \mathrm{Pic}(A)$ and $L_B \in \mathrm{Pic}(B)$. Then $\sigma^*\big((1_A \times \varphi)^* \mathscr{P}_A \otimes p_A^* L_A \otimes p_B^* L_B\big)$ and $(1_A \times \varphi \circ \zeta)^* \mathscr{P}_A \otimes p_A^* L_A \otimes p_B^* L_B$ are in the same class of algebraic equivalence.

Proof. Indeed, by Lemma 2.1.23 we know that

$$\begin{aligned}\sigma^*\big((1_A \times \varphi)^* \mathscr{P}_A \otimes p_A^* L_A \otimes p_B^* L_B\big) \simeq \\ (1_A \times \varphi \circ \zeta)^* \mathscr{P}_A \otimes p_A^* t_\xi^* L_A \otimes p_B^*(\zeta^* L_B \otimes (\varphi \circ \zeta)^* P_\xi).\end{aligned}$$

Then $p_A^* L_A$ (resp. $p_B^* L_B$) is in the same class of algebraic equivalence of $p_A^* t_\xi^* L_A$ (resp. $p_B^* \zeta^* L_B$) since t_ξ (resp. ζ) is an automorphism, and therefore it preserves the degree of the divisor through pullback. Moreover, $(\varphi \circ \zeta)^* P_\xi$ can be discarded as it is algebraically trivial. □

2.2 Generators for the Second Cohomology

In this section we give explicit generators for the torsion of $\mathrm{H}^2(S,\mathbb{Z})$ in terms of the reduced multiple fibres of the elliptic fibration $g\colon S \to \mathbb{P}^1$. More precisely, we will prove the following statement:

Proposition 2.2.1. *Let $S = A \times B/G$ be a bielliptic surface. Let D_i be the reduced multiple fibres of $g\colon S \to \mathbb{P}^1$ with the same multiplicity. Then the torsion of $\mathrm{H}^2(S,\mathbb{Z})$ is generated by the classes of the differences $D_i - D_j$ for $i \neq j$.*

The reader who is familiar with the work of Serrano might find similarities between the above statement and Serrano's description of the torsion of $\mathrm{H}^2(X,\mathbb{Z})$ when there is an elliptic fibration $\varphi : X \to C$ with multiple fibres (cfr. [Ser90b, Corollary 1.5 and Proposition 1.6]). However in [Ser90b] it is used the additional assumption that $h^1(X,\mathcal{O}_X) = h^1(C,\mathcal{O}_C)$. This clearly does not hold in our context.

Before proving Proposition 2.2.1 we need two preliminary lemmas.

Lemma 2.2.2. *Let $g\colon S \to \mathbb{P}^1$ be an elliptic pencil with connected fibres. Let D_1 and D_2 be two reduced multiple fibres. Let m_1 and m_2 be the corresponding multiplicities. Then, for all non-negative integers n,*

$$D_1 \nsim nD_2. \tag{2.21}$$

Proof. The statement is obvious for $n = 0$, so we have to prove the statement for $n > 0$. By contradiction, assume $D_1 \sim nD_2$. Let F be the generic fibre of g. Then

$$\begin{aligned} h^0(S,\mathcal{O}_S(F)) &= h^0(\mathbb{P}^1, g_*\mathcal{O}_S(F)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes g_*\mathcal{O}_S) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2, \end{aligned}$$

where the second equality follows by projection formula. Since $h^0(S,\mathcal{O}_S(D_1)) \leq h^0(S,\mathcal{O}_S(m_1D_1)) = h^0(S,\mathcal{O}_S(F))$, it follows that $h^0(S,\mathcal{O}_S(D_1)) \leq 2$.
The absurd hypothesis is used here: if $D_1 \sim nD_2$, then, since the supports of D_1 and D_2 are disjoint, $\mathrm{H}^0(S,\mathcal{O}_S(D_1))$ has at least two independent sections, and therefore the dimension of $\mathrm{H}^0(S,\mathcal{O}_S(D_1))$ is 2. Thus, since $D_1^2 = 0$ implies that there are no basepoints (see for example [Bea96, II.5]), the map is actually a morphism $\varphi_{|D_1|} : S \to \mathbb{P}^1$. Note that both D_1 and nD_2 are fibres of this morphism.

Let now C be the generic fibre of φ (which is irreducible by semicontinuity). Since $C \cdot D_1 = 0$, one gets $C \cdot F = 0$ for any fibre F of g. This implies that g and $\varphi_{|D_1|}$ have the same generic fibre. So we can write $C = F$. But then

$$D_1 \sim F \sim m_1 D_1,$$

which in turn implies that $\mathcal{O}_S(D_1)^{\otimes(m_1-1)} \simeq \mathcal{O}_S$, which is a contradiction. □

In order to prove the next lemma, we need to look closely into the geometry of the structure of bielliptic surfaces. In particular, recall that every class $G \cdot b$ of B/G is the image of (at least one) point $b \in B$ having as isotropy group a subgroup H of G. Recall also that the multiple fibres of g are the inverse image of the branch points of $\Theta : B \longrightarrow B/G$, and that the multiplicity of the fibre over $G \cdot b$ is exactly $|H|$. We have the diagram

$$\begin{tikzcd} A \times B \arrow[r, "\pi"] \arrow[d, "p_B"'] & S \arrow[d, "g"] \\ B \arrow[r, "\Theta"] & B/G. \end{tikzcd}$$

For effective divisors we can think of the pullback in terms of inverse image of subschemes (see for example [GW10, Corollary 11.49]). Clearly, $p_B^{-1}(b) = A \times \{b\}$ for any $b \in B$. Let $\bar{b}_i$ be the branch point of B/G corresponding to the multiple fibre $m_i D_i$. Then

$$\begin{aligned} m_i \pi^{-1} D_i = \pi^{-1}(m_i D_i) &= \pi^{-1} g^{-1}(\bar{b}_i) \\ &= p_B^{-1} \Theta^{-1}(\bar{b}_i) = m_i \sum_{\substack{b \in B \\ \Theta(b) = \bar{b}_i}} A \times \{b\}. \end{aligned}$$

In particular we obtain that

$$\pi^* \mathcal{O}_S(D_i) \simeq \mathcal{O}_S\left(\sum_{\substack{b \in B \\ \Theta(b) = \bar{b}_i}} A \times \{b\} \right) \simeq p_B^* \mathcal{O}_B\left(\sum_{\substack{b \in B \\ \Theta(b) = \bar{b}_i}} b \right). \tag{2.22}$$

In the next example we spell out what happens in each of the cases of bielliptic surfaces with which we are going to work.

Example 2.2.3.

- **Type 1.** In this case $G \simeq \mathbb{Z}/2\mathbb{Z}$, and G is generated by the automorphism given by taking the inverse. Clearly, the fixed points of B are the four elements of $B[2]$. Also, the class in B/G of any of these four points contains only the point itself, while every other class contains two points of B. We have four ramification points and four branch points. Since the isotropy group of any of the fixed points $b \in B[2]$ is the whole group, which has order two, the multiple fibre over each branch point $\Theta(b)$ has multiplicity 2.

- **Type 2.** Here $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. One of the two factors of G is generated by an order-two translation, therefore it has no fixed points. The other factor, which we call N, is generated by the automorphism given by taking the inverse. Therefore we can repeat the reasoning used for type 1 bielliptic surfaces and we find four multiple fibres each of multiplicity two, four branch points and four ramification points.

- **Type 3.** In this case $G \simeq \mathbb{Z}/4\mathbb{Z}$, and G is generated by multiplication by i, which we call ω; this generating automorphism has two fixed points, and we

call them b_1 and b_2.
Indeed, recall that $B \simeq \mathbb{C}/\Lambda$, with $\Lambda =< 1, i >$. Then, for $a, b \in \mathbb{R}$, we have that $\omega(a + bi + \Lambda) = -b + ai + \Lambda$. Therefore, $a + bi + \Lambda$ is a fixed point of ω if and only if there exist two integers m, n such that $a + b = n$ and $a - b = m$. All the lines parametrised by these equations meet only in the two classes $0 + \Lambda$ and $\frac{1}{2} + \frac{1}{2}i + \Lambda$. Therefore ω has exactly two fixed points, the two we just found. Clearly, the two other points of order two of B, $\frac{1}{2} + \Lambda$ and $\frac{1}{2}i + \Lambda$, are exchanged by ω.
The subgroup $H = \mathbb{Z}/2\mathbb{Z}$ of G is again generated by taking the inverse and it has four fixed points, two more than the whole G, we call these two new points e_1 and e_2. The class in B/G of each b_i contains only the point itself as an element, and the multiple fibre over either of these two classes has multiplicity four. The other two ramification points, e_1 and e_2, end up in the same class in B/G, as they are swapped by multiplication by i. The fibre over this class of B/G has multiplicity two. We have therefore three branch points and four ramification points.

- **Type 5.** In this case $G \simeq \mathbb{Z}/3\mathbb{Z}$, and G is generated by multiplication by $\frac{-1+\sqrt{3}i}{2}$, which we call ρ. This automorphism has three fixed points: $0 + \Lambda$, $\frac{\sqrt{3}}{3}i + \Lambda$ and $\frac{1}{2} + \frac{\sqrt{3}}{6}i + \Lambda$. Therefore the branch points on B/G corresponding to these three points are classes containing only the point itself, and the corresponding multiple fibres have multiplicity 3. Any point of B/G which is not a branch point is the class containing three points of B.

Lemma 2.2.4. *Let $S = A \times B/G$ be a bielliptic surface with its fibrations $a_S \colon S \to A/G$ and $g \colon S \to \mathbb{P}^1$. Let D_1 and D_2 be two reduced multiple fibres of g. Then the restriction of $\mathcal{O}_S(D_1 - D_2)$ to the generic fibre of a_S is trivial.*

Proof. Let $F = g^{-1}(\bar{p})$ be a smooth fibre of g. Here $\bar{p}$ is the orbit $G \cdot y$ of a point $y \in B$ not fixed under any element of G that is not the identity. We will choose an embedding of A into S via an isomorphism $\varphi \colon A \to F$ such that we get a commutative diagram

$$\begin{tikzcd}
A \arrow[r, hook, "j"] \arrow[d, "\varphi"'] & A \times B \arrow[r, "p_B"] \arrow[d, "\pi"'] & B \arrow[d, "\Theta"'] \\
F \arrow[r, hook, "i"] \arrow[dr, "\psi"'] & S \arrow[r, "g"] \arrow[d, "a_S"] & B/G \\
& A/G &
\end{tikzcd}$$

where i is just the natural inclusion of the fibre F into S and π is the quotient map. To this end we let $\varphi \colon A \to F$ be the isomorphism $a \mapsto G \cdot (a, y)$ and j be the embedding $a \mapsto (a, y)$.

By the discussion preceeding the statement of the lemma,

$$\begin{aligned}\varphi^* i^* \mathcal{O}_S(D_1 - D_2) &\simeq j^* \pi^* \mathcal{O}_S(D_1 - D_2) \\ &\simeq j^* p_B^* \mathcal{O}_B\Bigg(\sum_{\substack{b\in B \\ \Theta(b)=\bar{b}_1}} b - \sum_{\substack{b\in B \\ \Theta(b)=\bar{b}_2}} b \Bigg).\end{aligned}$$

As $p_B \circ j$ is the constant map, we have that it is clearly trivial. Hence $\varphi^* i^* \mathcal{O}_S(D_1 - D_2)$ is trivial, and since φ is an isomorphism we deduce the statement. $\square$

Hereinafter we identify F and A via the isomorphism φ defined in the proof above. This way, we get the following commutative triangle:

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "\psi"'] & & S \arrow[dl, "a_S"] \\ & A/G. & \end{tikzcd} \tag{2.23}$$

Note that ψ is an isogeny of degree $|G|$. In particular, we have that the dual isogeny $\psi^* : \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(A)$ has degree $|G|$, too (see, for example [BL04, Proposition 2.4.3]).

More precisely, we recall [Mi08, Theorem 9.1]:

Theorem 2.2.5. *Let $\varrho : W \longrightarrow Z$ be an isogeny of two abelian varieties W and Z, and let K be the kernel of ϱ. Then the dual isogeny $\varrho^* : \mathrm{Pic}^0(Z) \longrightarrow \mathrm{Pic}^0(W)$ has kernel $\hat{K}$, the Cartier dual of K.*

This implies that, in the cases at hand, we have $\mathrm{Ker}(\psi^*) \simeq \hat{G}$.

Having fixed the setting, we are now ready to begin the proof of Proposition 2.2.1. To start with, we observe that by the Canonical Bundle Formula (Theorem 1.8.3) applied to $g : S \to \mathbb{P}^1$ we can write

$$\omega_S \simeq g^* \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_S\big(\sum_k (m_k - 1) D_k\big),$$

where the D_ks are all the multiple fibres of g, and m_k is the multiplicity of D_k. We can rewrite $g^* \mathcal{O}_{\mathbb{P}^1}(-2)$ by choosing, for any i and j, two points b_i and b_j on $\mathbb{P}^1$ giving rise to the fibres $m_i D_i$ and $m_j D_j$. So we obtain

$$K_S \sim -D_i - D_j + \sum_{k \neq i,j} (m_k - 1) D_k. \tag{2.24}$$

Since ω_S is a non-trivial element in $\mathrm{Pic}^0(S)$, we conclude that the classes of $D_i + D_j$ and $\sum_{k\neq i,j}(m_k - 1) D_k$ coincide in $\mathrm{H}^2(S,\mathbb{Z})$. Moreover, we observe that K_S restricts trivially to A, so ω_S yields a non-trivial element in $\mathrm{Ker}(\psi^*)$. Note that if D_i and D_j have the same multiplicity m, the difference $D_i - D_j$ induces a (possibly trivial) torsion element in $\mathrm{H}^2(S,\mathbb{Z})$ of order m. We prove Proposition

2.2.1 by showing that a sufficient number of these differences is non-trivial so to generate the torsion of $\mathrm{H}^2(S,\mathbb{Z})$. We proceed by a case-by-case analysis, studying separately bielliptic surfaces of type 1, 2, 3, and 5. The key point in the argument is the observation that, if $[D_i - D_j]$ is trivial, then the line bundle $\mathcal{O}_S(D_i - D_j)$ belongs to $\mathrm{Pic}^0(S)$. In addition, using Lemma 2.2.4 and the commutativity of diagram (2.23), we would have that $\psi^*\mathcal{O}_S(D_i - D_j) \simeq \mathcal{O}_S$, in particular $\mathcal{O}_S(D_i - D_j) \in \mathrm{Ker}(\psi^*)$, while Lemma 2.2.2 ensures that $\mathcal{O}_S(D_i - D_j)$ cannot be $\mathcal{O}_S$. A closer study of the structure of $\mathrm{Ker}(\psi^*) \simeq \hat{G}$ will bring us to the desired conclusion.

2.2.1 Type 1 Bielliptic Surfaces

In this case we have that $\mathrm{Ker}(\psi^*)$ is the reduced group scheme $\mathbb{Z}/2\mathbb{Z}$ and the fibration $g : S \to \mathbb{P}^1$ has four multiple fibres all of multiplicity 2. Hence, up to reordering the indices, (2.24) yields

$$K_S \sim D_i - D_j + D_k - D_l. \tag{2.25}$$

In particular, as the canonical divisor is algebraically equivalent to zero, for distinct indices i, j, k and l we have that $D_j - D_i$ is algebraically equivalent to $D_k - D_l$. Thus we get three classes in $\mathrm{H}^2(S,\mathbb{Z})$:

$$\begin{aligned}
[D_1 - D_2] &= \{D_1 - D_2,\, D_3 - D_4\},\\
[D_1 - D_3] &= \{D_1 - D_3,\, D_2 - D_4\},\\
[D_1 - D_4] &= \{D_1 - D_4,\, D_2 - D_3\},
\end{aligned} \tag{2.26}$$

which a priori are neither distinct nor non-trivial. Since $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}}$ is isomorphic to the Klein four-group, we need to show that they are indeed different classes and that they are not zero. Note that, if two classes were equal, since they are both 2-torsion and the third class is clearly equal to the sum of the first two, then the remaining class would be trivial. Thus, in order to prove that the three classes listed above are distinct and are all non-trivial, it will be enough to show that for any two distinct indices i and j the divisor $D_i - D_j$ is not algebraically equivalent to zero.
Suppose otherwise that for some indices we had $\mathcal{O}_S(D_i - D_j) \in \mathrm{Pic}^0(S)$, then (2.25) implies that also $\mathcal{O}_S(D_k - D_l)$ is in $\mathrm{Pic}^0(S)$. By the above discussion both $\mathcal{O}_S(D_i - D_j)$ and $\mathcal{O}_S(D_i - D_j)$ are non-trivial elements of $\mathrm{Ker}(\psi^*)$, which has only one non-trivial element, ω_S. Then we write

$$\omega_S \simeq \mathcal{O}_S(D_i - D_j) \otimes \mathcal{O}_S(D_k - D_l) \simeq \omega_S^{\otimes 2} \simeq \mathcal{O}_S,$$

getting a contradiction, and thus we can conclude.

2.2.2 Type 2 Bielliptic Surfaces

Here $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ker}(\psi^*) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Like in the type 1 case, there are four multiple fibres, each of multiplicity 2. As above we get the

three classes induced by $D_1 - D_2$, $D_1 - D_3$ and $D_1 - D_4$. We want to show that they cannot be all trivial.
By contradiction, suppose that two of these classes, say, $[D_1 - D_2]$ and $[D_1 - D_3]$, are trivial in $\mathrm{H}^2(S,\mathbb{Z})$. For $i = 2,3$ set $L_i := \mathcal{O}_S(D_1 - D_i)$ and $M_i := \mathcal{O}_S(D_i - D_4)$; then the L_i's and the M_i's determine non-trivial elements of $\mathrm{Ker}(\psi^*)$, which has only three nonzero elements. We deduce that some of these must be the same line bundle. The only combination that does not contradict Lemma 2.2.2 is having $L_i \simeq M_j$ for some $i \neq j$. But then we would have

$$\omega_S \simeq L_i \otimes M_j \simeq L_i^{\otimes 2} \simeq \mathcal{O}_S,$$

which is false, and we have reached a contradiction.
Hence, at most one of the three classes can be trivial, and one must indeed be trivial: since $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}} \simeq \mathbb{Z}/2\mathbb{Z}$, any two non-trivial classes must coincide, implying that the third one, being their sum, is trivial.

2.2.3 Type 3 Bielliptic Surfaces

Here $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}} \simeq \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ker}(\psi^*) \simeq \mathbb{Z}/4\mathbb{Z}$, but now we have two fibres of multiplicity 4 and one of multiplicity 2. Let E be the reduced multiple fibre of multiplicity 2 and let D_1, D_2 be the reduced multiple fibres of multiplicity 4. By the Canonical Bundle Formula, we get

$$K_S \sim E - D_1 - D_2.$$

Since K_S is algebraically trivial, the relation above implies that in $\mathrm{H}^2(S,\mathbb{Z})$ we have the following equalities:

$$[E - 2D_1] = [D_2 - D_1] \quad \text{and} \quad [E - 2D_2] = [D_1 - D_2],$$

which in particular imply, by looking at the right-hand sides, that if one of those two classes is trivial the other must be trivial too.
We need to show that they are non-trivial. Suppose by contradiction that they are both zero in $\mathrm{H}^2(S,\mathbb{Z})$; then, as before, we have that $\mathcal{O}_S(E - 2D_1)$ and $\mathcal{O}_S(E - 2D_2)$ are non-trivial elements of $\mathrm{Ker}(\psi^*)$. Since both these line bundles have order two in $\mathrm{Pic}(S)$ and $\mathrm{Ker}(\psi^*)$ has only one element of order two, we deduce that

$$\mathcal{O}_S(E - 2D_1) \simeq \mathcal{O}_S(E - 2D_2).$$

But then

$$\begin{aligned}\omega_S^{\otimes 2} &\simeq \mathcal{O}_S(E - D_1 - D_2)^{\otimes 2} \simeq \mathcal{O}_S(E - 2D_1) \otimes \mathcal{O}_S(E - 2D_2) \\ &\simeq \mathcal{O}_S(E - 2D_1)^{\otimes 2} \simeq \mathcal{O}_S,\end{aligned}$$

which is impossible because ω_S has order 4. Therefore $E - 2D_1$ and $E - 2D_2$ induce the same non-trivial torsion element of $\mathrm{H}^2(S,\mathbb{Z})$.

2.2.4 Type 5 Bielliptic Surfaces

Here $\mathrm{H}^2(S,\mathbb{Z})_{\mathrm{tor}} \simeq \mathbb{Z}/3\mathbb{Z}$, $\mathrm{Ker}(\psi^*) \simeq \mathbb{Z}/3\mathbb{Z}$ and there are three multiple fibres, each of multiplicity 3. By the Canonical Bundle Formula, we get

$$K_S \sim -D_i - D_j + 2D_k = (D_k - D_i) + (D_k - D_j).$$

Again, K_S is algebraically equivalent to zero, so we get that $[D_k - D_i] = [D_j - D_k]$ in $\mathrm{H}^2(S,\mathbb{Z})$. Running through the indices we get the two classes

$$[D_1 - D_2] = \{D_1 - D_2,\ D_3 - D_1,\ D_2 - D_3\},$$
$$[D_1 - D_3] = \{D_1 - D_3,\ D_3 - D_2,\ D_2 - D_1\}.$$

We need to show that they are distinct and both non-trivial. Observe that, if they were the same class, then both classes would be trivial; therefore it is enough to show that they are not the zero class. Again, suppose by contradiction that $[D_k - D_i] = 0$ in $\mathrm{H}^2(S,\mathbb{Z})$, then we can write

$$\omega_S \simeq \mathcal{O}_S(D_1 - D_2) \otimes \mathcal{O}_S(D_1 - D_3),$$

with $\mathcal{O}_S(D_1 - D_2)$ and $\mathcal{O}_S(D_1 - D_3)$ non-trivial elements in $\mathrm{Ker}(\psi^*)$. Neither $\mathcal{O}_S(D_1 - D_2)$ nor $\mathcal{O}_S(D_1 - D_3)$ can be isomorphic to the canonical bundle ω_S, or we would have $\mathcal{O}_S(D_k - D_i) \simeq \mathcal{O}_S$, contradicting Lemma 2.2.2. As $\mathrm{Ker}\,\psi^*$ has only two non-trivial elements, we necessarily have

$$\mathcal{O}_S(D_1 - D_2) \simeq \mathcal{O}_S(D_1 - D_3),$$

and so $\mathcal{O}_S(D_2 - D_3) \simeq \mathcal{O}_S$, which contradicts again Lemma 2.2.2, thus we can conclude.

2.3 Triviality of Pullbacks

Given a morphism $f : Y \to X$ and a line bundle $\mathcal{O}_X(L) \in \mathrm{Pic}(X)$, we will need to deduce triviality properties of $\mathcal{O}_X(L)$ from triviality properties of $f^*\mathcal{O}_X(L)$, and viceversa.
To begin with, we recall a basic result from [Kl66, Chapter I, §4, Corollary 1] which follows from projection formula:

Lemma 2.3.1. *Let X be a complete algebraic scheme and let $f : Y \to X$ be a morphism; consider $\mathcal{O}_X(L) \in \mathrm{Pic}(X)$. Then*

1. *if $\mathcal{O}_X(L)$ is numerically trivial, then $f^*\mathcal{O}_X(L)$ is also numerically trivial;*
2. *if f is surjective, then $f^*\mathcal{O}_X(L)$ numerically trivial implies that $\mathcal{O}_X(L)$ is numerically trivial.*

More in particular, we observe that

Remark 2.3.2. Let $\pi : A \times B \to S$ be the canonical cover of a bielliptic surface S of type 1,3 or 5. Take $L \in \mathrm{Pic}(S)$. Then π^*L is algebraically trivial if and only if L is numerically trivial; moreover, if $\pi^*L \simeq \mathcal{O}_{A\times B}$, then $L \in \mathrm{Pic}^0(S)$.

To prove this remark we observe the following:

Remark 2.3.3. With the notation of § 2.2, observe that, if $m_i = m_j$, then

$$\pi^*\mathcal{O}_S(D_i - D_j) \simeq p_B^*\mathcal{O}_B\left(\sum_{\substack{b\in B\\ \Theta(b)=\bar{b}_i}} b - \sum_{\substack{b\in B\\ \Theta(b)=\bar{b}_j}} b \right) \in \mathrm{Pic}^0(A\times B). \tag{2.27}$$

Indeed, we see directly that the pullback is algebraically trivial as the pullback of a numerically trivial divisor is numerically trivial, and all numerically trivial divisors on abelian varieties are also algebraically trivial.
Moreover, thanks to Preposition 2.2.1, all the generators of $\mathrm{H}^2(S,\mathbb{Z})_{\text{tor}}$ can be written this way.

Proof of Remark 2.3.2. Any $L \in \mathrm{Pic}(S)$ can be written as

$$L \simeq a_0^{\otimes n} \otimes b_0^{\otimes m} \otimes \tau^{\otimes s} \otimes \nu^{\otimes t} \otimes \alpha,$$

where n,m,s,t are integers, $\alpha \in \mathrm{Pic}^0(S)$ and τ,ν are the generators of $\mathrm{H}^2(S,\mathbb{Z})_{\text{tor}}$ (meaning that ν is non-trivial only in the type 1 case). Then

$$\pi^*L \simeq a_X^{\otimes n} \otimes b_X^{\otimes m} \otimes \pi^*(\tau^{\otimes s} \otimes \nu^{\otimes t}) \otimes \pi^*\alpha.$$

From Lemma 2.3.1 and the fact that numerical and algebraic equivalence coincide on abelian varieties it is obvious that L is numerically trivial if and only if π^*L is algebraically trivial.

Assume $\pi^*L \simeq \mathcal{O}_{A\times B}$. This means that $L \simeq \tau^{\otimes s} \otimes \nu^{\otimes t} \otimes \alpha$ with

$$\pi^*(\tau^{\otimes s} \otimes \nu^{\otimes t}) \otimes \pi^*\alpha \simeq \mathcal{O}_{A\times B}.$$

We have the commutative diagram

$$\begin{tikzcd} A\times B \arrow[r, "\pi"] \arrow[d, "p_A"'] & S \arrow[d, "a_S"] \\ A \arrow[r, "\Psi"] & A/G, \end{tikzcd}$$

by which we see that, since a_S is the Albanese map of S, $\pi^*\alpha \simeq p_A^*\Psi^*\alpha$, where $\Psi^*\alpha \in \mathrm{Pic}^0(A)$. As we saw, $\pi^*(\tau^{\otimes s} \otimes \nu^{\otimes t}) \simeq p_B^*\beta$, for some $\beta \in \mathrm{Pic}^0(B)$. Therefore we have that

$$\mathcal{O}_{A\times B} \simeq \pi^*(\tau^{\otimes s} \otimes \nu^{\otimes t}) \otimes \pi^*\alpha \simeq p_A^*\Psi^*\alpha \otimes p_B^*\beta,$$

which implies that β and $\Psi^*\alpha$ must be both trivial. In particular, $\pi^*(\tau^{\otimes s} \otimes \nu^{\otimes t}) \simeq \mathcal{O}_{A\times B}$, which is true if and only if $s = t = 0$. This shows that $L \in \mathrm{Pic}^0(S)$. □

2.4 The Brauer Map to Another Bielliptic Surface

Let S be a bielliptic surface of type 2 or 3. Then by Examples 2.1.4 and 2.1.6 there is a 2:1 cyclic cover $\tilde{\pi}: \tilde{S} \to S$, where $\tilde{S}$ is a bielliptic surface of type 1. As in section 2.1.3, we will write $\tilde{\sigma}$ for the involution induced by $\tilde{\pi}$. In this section we study the Brauer map $\tilde{\pi}_{\mathrm{Br}}: \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$. Surprisingly, we reach two antipodal conclusions depending on the type of the bielliptic surface in question.

Recall that, as $\tilde{S}$ is a bielliptic surface of type 1, the elliptic fibration $q_B: \tilde{S} \to \mathbb{P}^1$ has four multiple fibres $D_1, \ldots, D_4$ of multiplicity 2, corresponding to the four 2-torsion points of B. We will write τ_{ij} for the line bundle $\mathcal{O}_{\tilde{S}}(D_i - D_j)$.

2.4.1 Bielliptic Surfaces of Type 2

Suppose that S is of type 2, and note that the involution $\tilde{\sigma}$ acts on the set of the D_i's by exchanging them pairwise. Up to relabeling, we can assume that $\tilde{\sigma}^* D_1 \sim D_2$ and $\tilde{\sigma}^* D_3 \sim D_4$. By (2.8), we therefore have that

$$\tilde{\pi}^*(\mathrm{Nm}(\tau_{13})) \simeq \tau_{13} \otimes \tilde{\sigma}^* \tau_{13} \simeq \tau_{13} \otimes \tau_{24} \simeq \omega_{\tilde{S}}, \tag{2.28}$$

where the last equality is a consequence of (2.25). Thus, if we call γ the generator of $\mathrm{Ker}\, \tilde{\pi}^*$, we get that

$$\mathrm{Nm}(\tau_{13}) \in \{\omega_S, \omega_S \otimes \gamma\} \subset \mathrm{Pic}^0(S).$$

Indeed, we need only to show that $\mathrm{Nm}(\tau_{13}) \in \mathrm{Pic}^0(S)$. We can write

$$\mathrm{Nm}(\tau_{13}) \simeq a_0^{\otimes n} \otimes b_0^{\otimes m} \otimes \tau^{\otimes s} \otimes \alpha,$$

where n, m, s are integers, $\alpha \in \mathrm{Pic}^0(S)$ and τ is the generator of $\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{tor}}$. Now, by Lemma 2.1.5, we have

$$\tilde{\pi}^* \mathrm{Nm}(\tau_{13}) \simeq (2\tilde{a}_0)^{\otimes n} \otimes \tilde{b}_0^{\otimes m} \otimes \tilde{\pi}^* \tau^{\otimes s} \otimes \tilde{\pi}^* \alpha.$$

Since $\tilde{\pi}^* \mathrm{Nm}(\tau_{13}) \simeq \omega_{\tilde{S}} \in \mathrm{Pic}^0(\tilde{S})$, it must be $n = m = 0$. Thus it remains to show that $s = 0$, and in order to prove that we want to compute $\tilde{\pi}^* \tau$. Observe that of the six line bundles τ_{12}, τ_{13}, τ_{14}, τ_{34}, τ_{24} and τ_{23} it is only τ_{12} and τ_{34} that are fixed under the action of $\tilde{\sigma}^*$, therefore they must be pull-backs of line bundles from S. In particular, reasoning as above, $\tau_{12} \simeq \tilde{\pi}^* \tau \otimes \tilde{\pi}^* \beta$, for some $\beta \in \mathrm{Pic}^0(S)$, and we see that $\tilde{\pi}^* \tau$ cannot be algebraically trivial because τ_{12} is not algebracally trivial, and this implies that $s = 0$.

Then we can use the Pic^0-trick (Remark 2.1.8) and we find a $\beta \in \mathrm{Pic}^0(S)$ such that $\mathrm{Nm}(\tilde{\pi}^* \beta \otimes \tau_{13})$ is trivial.

Lemma 2.4.1. *In the above notation, the line bundle $\tilde{\pi}^* \beta \otimes \tau_{13}$ does not belong to the image of $1 - \tilde{\sigma}^*$.*

Before we proceed with the proof, let us notice how, as an easy corollary, we get

Corollary 2.4.2. *If S is of type 2, then the induced map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ is trivial.*

Proof of Lemma 2.4.1. We will show that the class of τ_{13} in $\mathrm{H}^2(\tilde{S},\mathbb{Z})$ is not in the image of $1-\tilde{\sigma}^*$. Write $[\tau_{ij}]$ for the algebraic equivalence class of the line bundle τ_{ij}. Then, by Proposition 2.1.2 and (2.26), for every L in $\mathrm{Pic}(\tilde{S})$ there are integers n, m, and h, and k such that

$$c_1(L) = \frac{n}{2}\cdot a + m\cdot b + h\cdot[\tau_{13}] + k\cdot[\tau_{14}].$$

Then $\tilde{\sigma}^*\tau_{13} \simeq \tau_{24}$ and $\tilde{\sigma}^*\tau_{14} \simeq \tau_{23}$; since $[\tau_{13}] = [\tau_{24}]$ and $[\tau_{14}] = [\tau_{23}]$ we have $\tilde{\sigma}^*[\tau_{13}] = [\tau_{13}]$ and $\tilde{\sigma}^*[\tau_{14}] = [\tau_{14}]$, therefore

$$(1-\tilde{\sigma}^*)c_1(L) = 0.$$

We deduce that $\mathrm{Im}(1-\tilde{\sigma}^*) = 0$. But, on the other hand, we have that $c_1(\tilde{\pi}^*\beta\otimes\tau_{13}) = [\tau_{13}]$ is not trivial, thus $\tilde{\pi}^*\beta\otimes\tau_{13}$ cannot possibly lie in the image of $(1-\tilde{\sigma}^*)$, and the lemma is proven. □

2.4.2 Bielliptic Surfaces of Type 3

In this section we will prove the following statement:

Theorem 2.4.3. *If S is a bielliptic surface of type 3, then the Brauer map $\tilde{\pi}_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(\tilde{S})$ induced by the cover $\tilde{\pi} : \tilde{S} \to S$, where $\tilde{S}$ is bielliptic of type 1, is injective.*

We will use Proposition 2.1.9 and we will show that $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1-\sigma^*)$ is trivial. There are two main key steps:

1. we first study the norm map when applied to numerically trivial line bundles;
2. then we prove that all the line bundles L in $\mathrm{Ker}(\mathrm{Nm})$ are numerically trivial.

Norm of Numerically Trivial Line Bundles

We will use the notation of Example 2.1.4. Observe that we have the diagram

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\pi}"] \arrow[d, "a_{\tilde{S}}"'] & S \arrow[d, "a_S"] \\
A/G \arrow[r, "\varphi"'] & A/H,
\end{tikzcd} \tag{2.29}$$

where $G \simeq \mathbb{Z}/2\mathbb{Z}$, and H is $\mathbb{Z}/4\mathbb{Z}$.

Remark 2.4.4. Note that the bottom arrow, φ, is an isogeny of degree 2. As the vertical arrows are the Albanese maps of $\tilde{S}$ and S respectively, we have that $\tilde{\pi}^* : \mathrm{Pic}^0(S) \to \mathrm{Pic}^0(\tilde{S})$ coincides with the isogeny dual to φ. In particular it is surjective.

Our first step in the study of the norm homomorphism for numerically trivial line bundles is to see how the norm behaves when applied to the generators of the torsion of $\mathrm{H}^2(\tilde{S}, \mathbb{Z})$. In order to do that, we need to see how $\tilde{\sigma}$ behaves on the reduction of the multilple fibres. We have already seen how the automorphism ω acts on the points of B in Example 2.2.3. Here we arrive to the same conclusion in a different fashion.

First we observe that ω acts on $B[2] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with at least one fixed point, the one corresponding to the identity element of B. Since ω has order 4, it cannot act transitively on the remaining three points on $B[2]$. Thus the action has at least two fixed points.

We deduce that $\tilde{\sigma}$ acts on the set of the reduced multiple fibres by leaving fixed at least two of them, let us say D_1 and D_2. If the action were trivial, then we would have that all the line bundles τ_{ij} would be invariant under the action of $\tilde{\sigma}$ and as a consequence they would be pullbacks of line bundles coming from S. We would deduce that all the torsion classes of $\mathrm{H}^2(\tilde{S}, \mathbb{Z})$ are pullbacks of classes from $\mathrm{H}^2(S, \mathbb{Z})$, which is impossible. Thus we know that $\tilde{\sigma}$ exchanges D_3 and D_4.

Now we can prove the following Lemma.

Lemma 2.4.5. *Let n and m be two integers. Then the norm of the line bundle $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$ is zero if and only if n and m have the same parity. In addition we have that $\mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m})$ is not in $\mathrm{Pic}^0(S)$ if n and m are not congruent modulo 2.*

Proof. Observe first of all that, thanks to the above discussion, the line bundle $\tau_{34} \simeq \tau_{13} \otimes \tau_{14}$ is invariant with respect to the action of $\tilde{\sigma}$. In particular we can write $\tau_{34} \simeq \tilde{\pi}^* \tau$ where τ is a line bundle on S whose algebraic equivalence class is the only non-trivial class in $\mathrm{H}^2(S, \mathbb{Z})_{\mathrm{tor}}$.

Now, if n and m are both even, then $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$ is the trivial line bundle, and there is nothing to prove. Otherwise, if n and m are odd, then

$$\mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}) \simeq \mathrm{Nm}(\tau_{34}) \simeq \tau^{\otimes 2} \simeq \mathcal{O}_S.$$

Conversely suppose that n and m are not congruent modulo 2. Up to exchanging n and m we can assume that m is even, while n is odd. Then $\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m} \simeq \tau_{13}$. Again by (2.8) we get

$$\tilde{\pi}^* \mathrm{Nm}(\tau_{13}) \simeq \tau_{13} \otimes \tilde{\sigma}^* \tau_{13} \simeq \tau_{34} \simeq \tilde{\pi}^* \tau.$$

We deduce that $\mathrm{Nm}(\tau_{13})$ is either equal to τ or to $\tau \otimes \omega_S^{\otimes 2}$. In any case it is not algebraically equivalent to zero and so the statement is proven. □

Remark 2.4.6. (a) Observe that τ_{34} is in the image of $1 - \tilde{\sigma}^*$; indeed we have that $\tau_{34} \simeq \mathcal{O}_{\tilde{S}}(D_3) \otimes \tilde{\sigma}^* \mathcal{O}_{\tilde{S}}(-D_3)$.

(b) We will see in what follows that the different behaviour of the norm map applied to torsion classes is what determines the contrast between the type 2 and the type 3 bielliptic surfaces. In particular, the fact that the norm map of a torsion class is not necessarily algebraically trivial is what does not allow us to use Remark 2.1.8 in order to provide a non-trivial class in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1-\tilde{\sigma}^*)$.

Now we turn our attention to the elements of $\mathrm{Pic}^0(\tilde{S})$ whose norm is trivial. We will show that they never determine nonzero classes in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1-\tilde{\sigma}^*)$.

Lemma 2.4.7. *Let $\mathrm{Nm} : \mathrm{Pic}(\tilde{S}) \to \mathrm{Pic}(S)$ be the norm homomorphism. Take $L \in \mathrm{Pic}^0(\tilde{S})$ such that $\mathrm{Nm}(L) = \mathcal{O}_S$. Then the class of L in the kernel of the Brauer map is trivial.*

Proof. We have to show that such an L is in the image of the morphism $1-\tilde{\sigma}^*$. By Remark 2.4.4, we can write $L \simeq \tilde{\pi}^* M$ with $M \in \mathrm{Pic}^0(S)$. Then our assumption warrants that

$$\mathcal{O}_S \simeq \mathrm{Nm}(L) \simeq M^{\otimes 2}.$$

We deduce that M is a 2-torsion point in $\mathrm{Pic}^0(S)$. We know that $\mathrm{Pic}^0(S)[2]$ is a group scheme isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Let γ be the element $\omega_S^{\otimes 2} \in \mathrm{Pic}^0(S)[2]$; then we can find $\beta \in \mathrm{Pic}^0(S)[2]$, β non-trivial, such that

$$\mathrm{Pic}^0(S)[2] = \{\mathcal{O}_S, \gamma, \beta, \gamma \otimes \beta\}.$$

In particular, as $\tilde{\pi}^*\gamma \simeq \mathcal{O}_{\tilde{S}}$,

$$\mathrm{Ker}(\mathrm{Nm}) \cap \mathrm{Pic}^0(\tilde{S}) = \{\mathcal{O}_{\tilde{S}}, \tilde{\pi}^*\beta\}. \tag{2.30}$$

Our goal now is to produce a line bundle $\alpha \in \mathrm{Pic}^0(\tilde{S}) \cap \mathrm{Im}(1-\tilde{\sigma}^*)$, $\alpha \not\simeq \mathcal{O}_{\tilde{S}}$. Thus we would have that $\mathrm{Pic}^0(\tilde{S}) \cap \mathrm{Im}(1-\tilde{\sigma}^*)$ is a non-trivial subgroup of $\mathrm{Ker}(\mathrm{Nm}) \cap \mathrm{Pic}^0(\tilde{S})$. From (2.30) we would deduce that

$$\mathrm{Ker}(\mathrm{Nm}) \cap \mathrm{Pic}^0(\tilde{S}) = \mathrm{Pic}^0(\tilde{S}) \cap \mathrm{Im}(1-\tilde{\sigma}^*),$$

and so the statement.

To the aforementioned goal, let $\overline{\epsilon} \in A' := A/G$ be the image of the point $\epsilon \in A$ defining the involution $\tilde{\sigma}$ (see (2.4)). Also, let p_0 be the identity element of A'; observe that by the construction of bielliptic surfaces $\overline{\epsilon} \neq p_0$. Consider the following line bundle on $\tilde{S}$:

$$\alpha := a_{\tilde{S}}^*(\mathcal{O}_{A'}(p_0) \otimes t_{\overline{\epsilon}}^* \mathcal{O}_{A'}(-p_0)).$$

Clearly α is a non-trivial element in $\mathrm{Pic}^0(\tilde{S})$. In addition, by (2.4) we see that

$$\alpha \simeq a_{\tilde{S}}^*(\mathcal{O}_{A'}(p_0)) \otimes \tilde{\sigma}^* a_{\tilde{S}}^*(\mathcal{O}_{A'}(-p_0)),$$

and therefore it is in the image of $1-\tilde{\sigma}^*$. Thus we can conclude. □

Injectivity of the Brauer Map

We are now ready to prove Theorem 2.4.3. We will do so by showing the following statement.

Proposition 2.4.8. *If $L \in \mathrm{Ker}(\mathrm{Nm})$, then L is numerically trivial.*

Before proceeding with the proof, let us show how this implies Theorem 2.4.3. Let L be a line bundle in the kernel of the norm map. Then Proposition 2.4.8 yields that
$$L \simeq \alpha \otimes \tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$$
for some positive integers n and m, and for some $\alpha \in \mathrm{Pic}^0(\tilde{S})$. Write again $\alpha \simeq \tilde{\pi}^*\beta$, and since
$$\mathcal{O}_S \simeq \mathrm{Nm}(L) \simeq \beta^{\otimes 2} \otimes \mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}),$$
if n and m do not have the same parity, then Lemma 2.4.5 leads us to a contradiction, as $\mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m})$ should not be algebraically trivial in this case. We deduce that n and m must have the same parity.

Now, by the first part of Lemma 2.4.5, we see that $\mathrm{Nm}(L) \simeq \mathrm{Nm}(\alpha)$; so we deduce that $\alpha \in \mathrm{Ker}(\mathrm{Nm})$. In particular, Lemma 2.4.7 implies that $\alpha \in \mathrm{Im}(1-\tilde{\sigma}^*)$, and so the class of L in $\mathrm{Ker}(\mathrm{Nm})/\mathrm{Im}(1-\tilde{\sigma}^*)$ is the same as the class of τ_{34}. But Remark 2.4.6(a) tells us that the latter is trivial and so Theorem 2.4.3 is proven.

Proof of Proposition 2.4.8. We have L in the kernel of the norm map. Lemmas 2.1.3 and 2.1.5 imply that $\tilde{\pi}^*\mathrm{Num}(S)$ is a sublattice of index 2 of $\mathrm{Num}(\tilde{S})$. In particular $L^{\otimes 2}$ is numerically equivalent to the pullback of a line bundle from S. Thus we can write
$$L^{\otimes 2} \simeq \tilde{\pi}^*M \otimes \alpha \otimes \tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}$$
for some positive integers n and m, and for some $\alpha \in \mathrm{Pic}^0(\tilde{S})$. Again, by Remark 2.4.4 we can write $\alpha \simeq \tilde{\pi}^*\beta$ for some $\beta \in \mathrm{Pic}^0(S)$, and so, up to substituting M with $M \otimes \beta$ we have that
$$L^{\otimes 2} \simeq \tilde{\pi}^*M \otimes \tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m}.$$
If we show that M is numerically trivial we can conclude. Observe that
$$\begin{aligned}\mathcal{O}_S &\simeq \mathrm{Nm}(L) \otimes \mathrm{Nm}(L) \simeq \mathrm{Nm}(L^{\otimes 2})\\ &\simeq M^{\otimes 2} \otimes \mathrm{Nm}(\tau_{13}^{\otimes n} \otimes \tau_{14}^{\otimes m})\\ &\simeq M^{\otimes 2} \otimes \tau^{\otimes (n+m)},\end{aligned}$$
where the last equality is a consequence of Lemma 2.4.5. As τ is numerically trivial we conclude that the same is true for M. □

2.5 Brauer Map to the Canonical Cover

In this section we study the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$ when S is a bielliptic surface and X is its canonical cover. Then there is an n to 1 étale cyclic cover $\pi : X \to S$, where n is the order of the canonical bundle ω_S. Thus, as in the previous section, we can use Beauville's work [Bea09] to study the kernel of the map π_{Br} via the norm homomorphism $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$. As in the other cases the Brauer group is trivial, we can assume that S is of type 1, 2, 3 or 5. Recall that, independently from the case at hand, there are two elliptic curves A and B such that X is isogenous to $A \times B$. In what follows we will see that the geometry of the Brauer maps depends to a large extent on the geometry of $A \times B$, and in particular on whether there are isogenies between A and B or not. Throughout this section we will use the notation introduced in section 2.1.2.

2.5.1 The Norm of Numerically Trivial Line Bundles

Our first step will be proving the following proposition, which will allow us to study the norm map from a strictly numerical point of view.

Proposition 2.5.1. *Let* $L \in \mathrm{Pic}^0(X) \cap \mathrm{Ker}(\mathrm{Nm})$. *Then* L *is in* $\mathrm{Im}(1-\sigma^*)$.

Before going any further we need to describe more precisely our setting and introduce some notation.

Observe first that, if we let as in 2.1.2 $p_A : X \to A/H$ and $p_B : X \to B/H$ be the two elliptic fibrations of the abelian variety X, then $\mathrm{Pic}^0(X)$ is generated by $p_A^* \mathrm{Pic}^0(A/H)$ and $p_B^* \mathrm{Pic}^0(B/H)$, thus we can write any $L \in \mathrm{Pic}^0(X)$ as $p_A^*\alpha \otimes p_B^*\beta$, where $\alpha \in \mathrm{Pic}^0(A/H)$, $\beta \in \mathrm{Pic}^0(B/H)$. In this notation we have the following:

Lemma 2.5.2. *For every* $\beta \in \mathrm{Pic}^0(B/H)$ *we have that* $p_B^*\beta$ *is in the image of* $1-\sigma^*$. *In particular, these line bundles are in the kernel of the norm homomorphism.*

Proof. Suppose first that G is cyclic; in this setting the group H is trivial and $X \simeq A \times B$. We proceed by a case-by-case analysis. Recall from (2.2) what σ is in each of the three cases.

Type 1 case. Since abelian varieties are divisible groups, there exists $\gamma \in \mathrm{Pic}^0(B)$ such that $\gamma^{\otimes 2} \simeq \beta$. Observe that from diagram

$$\begin{tikzcd} A \times B \arrow[r, "\sigma"] \arrow[d, "p_B"'] & A \times B \arrow[d, "p_B"] \\ B \arrow[r, "-1_B"] & B \end{tikzcd}$$

and Lemma 2.1.20 we get that

$$(\sigma^* p_B^*\gamma)^{-1} \simeq (p_B^*(-1_B)^*\gamma)^{-1} \simeq (p_B^*\gamma^{-1})^{-1} \simeq p_B^*\gamma.$$

Then, the computation we just made and again Lemma 2.1.20 imply that

$$\begin{aligned} (1-\sigma^*)p_B^*\gamma &\simeq p_B^*\gamma \otimes (\sigma^* p_B^*\gamma)^{-1} \\ &\simeq p_B^* 1_B^*\gamma \otimes p_B^* 1_B^*\gamma \\ &\simeq p_B^*(1_B^* + 1_B^*)\gamma \simeq p_B^* 2_B^*\gamma \simeq p_B^*\beta, \end{aligned}$$

and the statement is proven in this case.

Type 3 case. In this case the j-invariant of B is 1728 and there is an automorphism ω of B of order 4. Consider the map $1-\omega : B \to B$. Since this is not trivial it is an isogeny, and in particular $(1-\omega)^* : \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(B)$ is surjective. Take $\gamma \in \mathrm{Pic}^0(B)$ such that $(1-\omega)^*\gamma \simeq \beta$, then by (2.2) we have

$$(1-\sigma^*)p_B^*\gamma \simeq p_B^*(1-\omega)^*\gamma \simeq p_B^*\beta,$$

and the statement is proven in this case.

Type 5 case. This case is similar to the previous one, with the only difference that instead of ω we use the automorphism ρ. Again, we note that $(1-\rho): B \to B$ is non-trivial, and therefore an isogeny. In particular, the dual map $(1-\rho)^*: \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(B)$ is surjective; it follows that we can find a γ such that $(1-\rho)^*\gamma \simeq \beta$. Again (2.2) yields:

$$(1-\sigma^*)p_B^*\gamma \simeq p_B^*(1-\rho)^*\gamma \simeq p_B^*\beta,$$

and the statement is proven.

Type 2 case. In order to conclude we need to analize the case of bielliptic surfaces of type 2. Under this assumption the group H is not trivial; indeed it is cyclic of order 2. Let $B' := B/H$ and observe that, by virtue of (2.3), we have the following diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "p_B"'] & X \arrow[d, "p_B"] \\ B' \arrow[r, "-1_{B'}"] & B'. \end{tikzcd}$$

So, as in the type 1 case, take $\gamma \in \mathrm{Pic}^0(B')$ such that $2_{B'}^*\gamma \simeq \beta$. Then, by the same computation, we have that $(1-\sigma^*)p_B^*\gamma \simeq p_B^*\beta$ and the proof is concluded. □

We will need the following easy remark.

Remark 2.5.3. With the notation already introduced, consider the diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "p_A"'] & S \arrow[d, "a_S"] \\ A/H \arrow[r, "\varphi"'] & A/G. \end{tikzcd}$$

Then for any $\alpha \in \mathrm{Pic}^0(A/H)$ one can find $M \in \mathrm{Pic}^0(S)$ such that $p_A^*\alpha \simeq \pi^*M$.

Proof. As usual, we identify elliptic curves with their dual curves. Since φ is an isogeny, it has its dual isogeny $\varphi^*: A/G \longrightarrow A/H$ and we can compose

$$n \circ p_A \simeq \varphi^* \circ \varphi \circ p_A \simeq \varphi^* \circ a_S \circ \pi.$$

Since abelian varieties are divisible groups we can find $\gamma \in \mathrm{Pic}^0(A/H)$ such that $\alpha \simeq \gamma^{\otimes n} \simeq n^*\gamma$ (where we use Lemma 2.1.20). Define $M := a_S^*\varphi(\gamma)$. Then clearly M satisfies the requirement of the statement. □

And now we can proceed with the proof of the proposition.

Proof of Proposition 2.5.1. Now take $L = p_A^*\alpha \otimes p_B^*\beta \in \mathrm{Pic}^0(X)$ such that $\mathrm{Nm}(L) \simeq \mathcal{O}_S$. Lemma 2.5.2 implies that also $p_A^*\alpha$ is in the kernel of the norm homomorphism. In addition, we have that the class of L in the quotient $\mathrm{Ker}\,\mathrm{Nm}/(1-\sigma^*)\,\mathrm{Pic}(X)$ is just the class of $p_A^*\alpha$. We have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "p_A"'] & S \arrow[d, "a_S"] \\ A/H \arrow[r, "\varphi"'] & A/G, \end{tikzcd}$$

where, as mentioned above, the bottom arrow is an isogeny of degree n. In particular, by Remark 2.5.3, we can write $p_A^*\alpha \simeq \pi^* M$ with $M \in \mathrm{Pic}^0(S)$. It follows that

$$\mathcal{O}_S \simeq \mathrm{Nm}(p_A^*\alpha) \simeq M^{\otimes n},$$

thus we have that

$$p_A^*\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Ker}(\mathrm{Nm}) = \pi^*\left(\mathrm{Pic}^0(S)[n]\right).$$

It easy to see that the right-hand-side above is a group isomorphic to the cyclic group of order n. Indeed, $\mathrm{Ker}\,\pi^* \simeq \mathbb{Z}/n\mathbb{Z}$, therefore $\mathrm{Pic}^0(S)[n] \simeq < g > \times \mathrm{Ker}\,\pi^*$ for some element g of order n. Since $< g >$ and $\mathrm{Ker}\,\pi^*$ intersect trivially, it is clear that $\pi^*(< g >) \simeq \mathbb{Z}/n\mathbb{Z}$, and therefore $\pi^*\left(\mathrm{Pic}^0(S)[n]\right) \simeq \mathbb{Z}/n\mathbb{Z}$.
Since $\mathrm{Im}(1-\sigma^*)$ is a subgroup of the kernel of the norm, if we provided an element of order n in $p_A^*\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Im}(1-\sigma^*)$ we would conclude that

$$p_A^*\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Im}(1-\sigma^*) = p_A^*\left(\mathrm{Pic}^0(A/H)\right) \cap \mathrm{Ker}(\mathrm{Nm})$$

and consequently the statement of Proposition 2.5.1. Let p_0 be the identity element of A/H; using the notation of (2.2) and (2.3) we set

$$\gamma := \begin{cases} \mathcal{O}_A(p_0) \otimes t_\tau^*(\mathcal{O}_A(-p_0)), & \text{if } S \text{ is of type 1,} \\ \mathcal{O}_{A/H}(p_0) \otimes t_{\tau'}^*(\mathcal{O}_{A/H}(-p_0)), & \text{if } S \text{ is of type 2,} \\ \mathcal{O}_A(p_0) \otimes t_\epsilon^*(\mathcal{O}_A(-p_0)), & \text{if } S \text{ is of type 3,} \\ \mathcal{O}_A(p_0) \otimes t_\eta^*(\mathcal{O}_A(-p_0)), & \text{if } S \text{ is of type 5;} \end{cases}$$

where τ' is the image of τ under the isogeny $A \to A/H$. Then γ is a non-trivial element of $\mathrm{Pic}^0(A/H)$ with the desired property. In addition, by (2.2) and (2.3), we have that $p_A^*\gamma \simeq (1-\sigma^*)p_A^*\mathcal{O}_A(p_0)$, and so we can conclude. □

We are ready to start our investigation of the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$. We first put ourselves in the special situation in which there are no non-trivial morphisms between A and B.

2.5.2 The Brauer Map When the Two Elliptic Curves Are Not Isogenous

If there are no isogenies between A and B, then the lattice $\mathrm{Num}(X)$ has rank 2 and it is generated by the classes of the two fibres, a_X and b_X. In addition, $\pi^*\mathrm{Num}(S)$ is a sublattice of $\mathrm{Num}(X)$ of index n. So, let L be in the kernel of the norm map. We have that $L^{\otimes n}$ is numerically equivalent to the pullback of a line bundle L' from S. More precisely we can write

$$L^{\otimes n} \simeq \pi^* L' \otimes p_A^*\alpha \otimes p_B^*\beta,$$

with $\alpha \in \mathrm{Pic}^0(A/H)$ and $\beta \in \mathrm{Pic}^0(B/H)$. Thanks to Remark 2.5.3 we can rewrite $p_A^*\alpha$ as the pullback of a line-bundle from S, so that in the end we get

$$L^{\otimes n} \simeq \pi^* M \otimes p_B^*\beta,$$

for some line bundle M on S. Lemma 2.5.2 ensures that π^*M is in the kernel of the norm map. In particular, M is an n-torsion element in $\mathrm{Pic}(S)$. We deduce that it is numerically trivial, and so L was numerically trivial to begin with. Now we apply Proposition 2.5.1 and deduce the following statement.

Theorem 2.5.4. *If $S := A \times B/G$ is a bielliptic surface such that the elliptic curves A and B are not isogenous, then the Brauer map to the canonical cover $\pi_{Br} : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is injective.*

Before moving on to the next case, observe that if S is a bielliptic surface of type 2, then we have the following diagram:

$$\begin{tikzcd}
A \times B \arrow[r, "\pi_{\tilde{S}}"] \arrow[d, "\phi"'] & \tilde{S} \arrow[d, "\tilde{\pi}"] \\
X \arrow[r, "\pi_S"'] & S.
\end{tikzcd}$$

If A and B are not isogenous Theorem 2.5.4 above implies that the Brauer map induced by π_S is injective. On the other hand, since we proved that the Brauer map induced by $\tilde{\pi}$ is trivial, the Brauer map induced by $\pi_S \circ \phi$ must be trivial. Thus the Brauer map induced by ϕ cannot be injective and we have

Corollary 2.5.5. *If $\varphi : X \to Y$ is an isogeny of abelian varieties, the map $\varphi_{\mathrm{Br}} : \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is not necessarily injective.*

2.5.3 The Brauer Map When the Two Elliptic Curves Are Isogenous

Suppose now that A and B are isogenous. Our first step will be to use the description the Picard group and of the Neron-Severi group of $A \times B$ that we outlined in 2.1.7 in order to find the image of $1 - \sigma^*$ and potential elements in the kernel of the norm homomorphism when S is a cyclic bielliptic surface. We begin with the following lemma.

Lemma 2.5.6. *Suppose that G is a cyclic group, so that $X \simeq A \times B$. If $L \in \mathrm{Pic}(A \times B)$ is in the kernel of the norm map, then $c_1(L) = l(0,0,\varphi)$ for some isogeny $\varphi : B \to A$.*

Proof. By the result of 2.1.7 we have that $c_1(L) = l(m,n,\varphi)$ for two integers n and m and an isogeny φ.

Suppose that S is of type 1 and L is in the kernel of the norm map. Since σ has order 2, by (2.8) we get that $L \otimes \sigma^*L$ is trivial: $\mathcal{O}_{A\times B} \simeq \pi^*\mathrm{Nm}(L) \simeq L \otimes \sigma^*L$. In particular, $c_1(L \otimes \sigma^* L)$ is zero. But then we get the following:

$$\begin{aligned} 0 &= c_1(L \otimes \sigma^* L) \\ &= c_1(L) + \sigma^* c_1(L) \\ &= l(m,n,\varphi) + l(m,n,-\varphi) = l(2m,2n,0), \end{aligned}$$

where we used the fact that $\sigma^* c_1(L) = l(m,n,-\varphi)$ by Remark 2.1.24. We conclude that $n = m = 0$.

Analogously, if S is of type 5, σ has order 3 and, by Example 2.1.15(a), B has j-invariant 0 and the automorphism ρ satisfies $1_B + \rho + \rho^2 = 0$. By (2.8) and Remark 2.1.24 we have that

$$\begin{aligned} 0 &= c_1(L) + \sigma^* c_1(L) + (\sigma^2)^* c_1(L) \\ &= l(m,n,\varphi) + l(m,n,\varphi \circ \rho) + l(m,n,\varphi \circ \rho^2) \\ &= l(3m,3n,\varphi \circ (1_B + \rho + \rho^2)) \\ &= l(3m,3n,0). \end{aligned}$$

Finally, if S is of type 3, σ has order 4 and, by Example 2.1.15(b), B has j-invariant 1728 and the automorphism ω satisfies $1_B + \omega^2 = 0$. By (2.8) and Remark 2.1.24 we get

$$\begin{aligned} 0 &= c_1(L) + \sigma^* c_1(L) + (\sigma^2)^* c_1(L) + (\sigma^3)^* c_1(L) \\ &= l(m,n,\varphi) + l(m,n,\varphi \circ \omega) + l(m,n,\varphi \circ \omega^2) + l(m,n,\varphi \circ \omega^3) \\ &= l(m,n,\varphi \circ (1_B + \omega - 1_B - \omega)) \\ &= l(4m,4n,0), \end{aligned}$$

so the statement is proven. □

We turn now our attention to the Brauer map in general and we study it by performing a case-by-case analysis on the different types of bielliptic surfaces. We will describe in complete detail what happens for type 1 and type 2 bielliptic surfaces, while for the remaining cases we will just point out the small differences in the argument.

Bielliptic Surfaces of Type 1

In this section we study the Brauer map to the canonical cover of bielliptic surfaces of type 1. If B does not have complex multiplication, we fix, once and for all, $\psi : B \to A$ a generating isogeny. Otherwise we fix $\psi_i : B \to A$, for $i = 1,2$, two generators of $\mathrm{Hom}(B,A)$. Our first step is to describe $(1 - \sigma^*)\,\mathrm{Pic}(A \times B)$.

Lemma 2.5.7. *Let S be a bielliptic surface of type 1 and consider $L \in (1 - \sigma^*)\,\mathrm{Pic}(A \times B)$, then there exist three integers n, h and k, and a line bundle $\beta \in \mathrm{Pic}^0(B)$ such that*

$$L \simeq \begin{cases} L(P_\tau^{\otimes n}, \beta, 2h \cdot \psi), & \text{if } B \text{ does not have complex multiplication;} \\ L(P_\tau^{\otimes n}, \beta, 2h \cdot \psi_1 + 2k \cdot \psi_2), & \text{if } B \text{ has complex multiplication.} \end{cases}$$

Proof. We do the complex multiplication case, the other is similar. Let $M \in \mathrm{Pic}(A \times B)$, then by the results of 2.1.7 we have that $M \simeq L(M_A, M_B, h \cdot \psi_1 + k \cdot \psi_2)$. We can write $M_A \simeq \mathcal{O}_A(n \cdot p_0) \otimes \alpha$ and $M_B \simeq \mathcal{O}_B(m \cdot q_0) \otimes \gamma$ for p_0 and q_0 the identity elements of A and B respectively, some integers n and m and some algebraically trivial line bundles α and γ. With this notation, we apply Lemma 2.1.23 and we find that

$$\begin{aligned}\sigma^* M \simeq L(t_\tau^* \mathcal{O}_A(n \cdot p_0) \otimes \alpha,&\\ \mathcal{O}_B(m \cdot q_0) \otimes \gamma^{-1} \otimes (-h \cdot \psi_1 - k \cdot \psi_2)^* P_\tau,&\\ - h \cdot \psi_1 - k \cdot \psi_2),&\end{aligned}$$

where $t_\tau^* \alpha \simeq \alpha$ by Lemma 2.1.19, $(-1_B)^* \gamma \simeq \gamma^{-1}$ by Lemma 2.1.20, and $(-1_B)^* \mathcal{O}_B(m \cdot q_0) \simeq \mathcal{O}_B(m \cdot q_0)$ because q_0 was chosen as the identity of B. Define $\beta := \gamma^{\otimes 2} \otimes (h \cdot \psi_1 + k \cdot \psi_2)^* P_\tau$ and observe that since γ ranges over all of $\mathrm{Pic}^0(B)$ also β ranges over the whole $\mathrm{Pic}^0(B)$. Having computed $\sigma^* M$, we immediately get that

$$(1 - \sigma^*) M \simeq L\left(P_\tau^{\otimes n}, \beta, 2h \cdot \psi_1 + 2k \cdot \psi_2\right),$$

and the lemma is proven. □

Remark 2.5.8. In the case without complex multiplication, for any integer h the line bundle $L(0,0,2h \cdot \psi)$ belongs to $\mathrm{Im}(1 - \sigma^*)$; analogously, in the case with complex multiplication, for any two integers h and k, $L(0,0,2h \cdot \psi_1 + 2k \cdot \psi_2)$ belongs to $\mathrm{Im}(1 - \sigma^*)$.

Proof. Indeed, in the case with complex multiplication, take $\gamma \in \mathrm{Pic}^0(B)$ such that $\gamma^2 = (-h \cdot \psi_1 - k \cdot \psi_2)^* P_\tau$. Then, as seen in the preceding proof, by Lemma 2.1.23

$$\sigma^* L(0, \gamma, h \cdot \psi_1 + k \cdot \psi_2) \simeq L(0, \gamma^{-1} \otimes (-h \cdot \psi_1 - k \cdot \psi_2)^* P_\tau, -h \cdot \psi_1 - k \cdot \psi_2).$$

And therefore it is immediate that

$$L(0,0,2h \cdot \psi_1 + 2k \cdot \psi_2) \simeq L(0, \gamma, h \cdot \psi_1 + k \cdot \psi_2) \otimes \sigma^* L(0, \gamma, h \cdot \psi_1 + k \cdot \psi_2)^{-1}.$$

The case without complex multiplication is essentially treated in the same way, with the obvious changes. □

We are now ready to prove one of the main statements of this section:

Theorem 2.5.9. *Suppose that S is a bielliptic surface of type 1 whose canonical cover is $A \times B$ with A and B isogenous elliptic curves. Then the Brauer map to the canonical cover of S is not injective if, and only if, one of the following mutually exclusive conditions is satisfied:*

1. *the elliptic curve B (and so A) does not have complex multiplication and $\psi^* P_\tau$ is trivial;*
2. *the elliptic curve B (and so A) has complex multiplication and we have that at least one of the following line bundles is trivial:*

$$L_1 := \psi_1^* P_\tau, \quad L_2 := \psi_2^* P_\tau, \quad L_3 := (\psi_1 + \psi_2)^* P_\tau. \tag{2.31}$$

Proof. We deal with the complex multiplication case since it is slightly more involved. The argument for the other case is very similar.

Before explaining the details of our reasoning we would like to give a quick outline of the proof for the convenience of the reader. The key observation is that the assumption on the line bundles (2.31) is equivalent to the norm of one of the invertible sheaves

$$(1 \times \psi_1)^* \mathscr{P}_A, \ (1 \times \psi_2)^* \mathscr{P}_A, \ (1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A$$

being topologically trivial. Therefore, if the assumption in the statement of the theorem is verified, we can use the Pic^0-trick (Remark 2.1.8) to provide an element in the kernel of the norm map. Such an element will give by construction a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$. Conversely, if none of the line bundles is trivial, then an element in the kernel of the norm map will be forced to be numerically equivalent to $(1 \times 2 \cdot \varphi)^* \mathscr{P}_A$ for some isogeny $\varphi \in \mathrm{Hom}(B, A)$. Then we will apply Lemma 2.5.7 and see that such a line bundle lies in $\mathrm{Im}(1-\sigma^*)$, so no element of $\mathrm{Pic}(A \times B)$ yields a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$.

Now, for the complete argument, by (2.8), Lemma 2.1.23, Lemma 2.1.20(i.), Lemma 2.1.21 and recalling that τ has order 2, we get that, for every α in $\mathrm{Pic}^0(A)$ and every isogeny $\varphi : B \to A$,

$$\begin{aligned}
\pi^* \mathrm{Nm}(&(1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha) \\
&\simeq (1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes \sigma^*((1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha) \\
&\simeq (1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes (1 \times \varphi \circ (-1_B))^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \varphi^* P_\tau \qquad (2.32) \\
&\simeq (1 \times 0)^* \mathscr{P}_A \otimes p_A^* \alpha^{\otimes 2} \otimes p_B^* \varphi^* P_\tau \\
&\simeq p_A^* \alpha^{\otimes 2} \otimes p_B^* \varphi^* P_\tau.
\end{aligned}$$

Suppose first that one of the three line bundles in (2.31) is trivial. To demonstrate our reasoning we can assume that $\psi_1^* P_\tau$ is trivial, the argument is identical in the other cases. Then by (2.32), since the $p_A^* \alpha^{\otimes 2}$ comes from $\pi^* \mathrm{Nm}(p_A^* \alpha)$, we have that $\mathrm{Nm}((1 \times \psi_1)^* \mathscr{P}_A)$ is in the kernel of π^*, so in particular it is in $\mathrm{Pic}^0(S)$. We can therefore apply the Pic^0-trick (Remark 2.1.8) and find $\gamma \in \mathrm{Pic}^0(S)$ such that the norm of $(1 \times \psi_1)^* \mathscr{P}_A \otimes \pi^* \gamma$ is trivial. But by Lemma 2.5.7 we have that $(1 \times \psi_1)^* \mathscr{P}_A \otimes \pi^* \gamma$ is not in the image of $1 - \sigma^*$ and so it defines a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$. Thus one direction of the statement is proven.

Conversely, suppose that there is a line bundle L on X which identifies a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$. By Lemmas 2.5.6 and 2.5.7 we can write

$$L \simeq (1 \times h \cdot \psi_1 + k \cdot \psi_2)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta,$$

for two integers h and k, and two topologically trivial line bundles α and β. Note that h and k cannot be both even, for otherwise Lemma 2.5.7 and Remark 2.5.8 yield that $[L] = [p_A^* \alpha \otimes p_B^* \beta] \in \mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$ which, by Proposition 2.5.1, implies that $[L] = 0$. Thus we can assume that one between h and k is odd. Then by Lemma 2.5.2 and Lemma 2.5.7 we have that

$$L \simeq (1 \times \psi_i)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes M, \quad \text{or} \quad L \simeq (1 \times \psi_1 + \psi_2)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes M,$$

with M in $\mathrm{Im}(1 - \sigma^*)$. In particular, one of the following line bundles

$$(1 \times \psi_1)^* \mathscr{P}_A, \quad (1 \times \psi_2)^* \mathscr{P}_A, \quad (1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A$$

has trivial norm. We deduce by (2.32) that one of the line bundles in (2.31) is trivial and the statement is proven. □

Example 2.5.10. (a) Suppose that $A \simeq B$. If A does not have complex multiplication, then we can take $\psi = \pm 1_A$. In particular we have that $\psi^* P_\tau$ is never trivial and the Brauer map is injective.

(b) Suppose again that $A \simeq B$ and that the j-invariant of A is 1728. Then $\mathrm{End}(A) \simeq \mathbb{Z}[i]$ and the multiplication by i induces an automorphism ω of A of order 4, and we can take 1_A and ω as generators of $\mathrm{End}(A)$. Suppose that P_τ is a fixed point[2] of the dual automorphism ω^*. Then $(1_A + \omega)^* P_\tau$ is zero and the Brauer map is not injective.

In order to complete our description of the Brauer map for type 1 bielliptic surfaces we need to give necessary and sufficient conditions for it to be trivial. To this end, we want to provide two distinct non-zero classes in $\mathrm{Ker}\,\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$. We can assume that the Brauer map is already non-injective, and therefore the triviality of the line bundles as by Theorem 2.5.9 holds in either case.

Suppose first that B does not have complex multiplication. Consider an L in the kernel of the Brauer map that yields a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$. Then, as before, we have that

$$L \simeq (1 \times h \cdot \psi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta.$$

Observe first that, since L is not trivial in the kernel of the Brauer map, $h \neq 0$ by Proposition 2.5.1. Again by Lemma 2.5.7 and Remark 2.5.8, we can assume that h is odd, and the same result, together with Lemma 2.5.2, also yields that in $\mathrm{Ker}\,\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$ the class of L and that of $(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \alpha$ are the same. Consider any non-trivial class in $\mathrm{Ker}\,\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$ having as a representative $(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \delta$ for $\delta \in \mathrm{Pic}^0(A)$. Then

$$(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes (1 \times \psi)^* \mathscr{P}_A^{-1} \otimes p_A^* \delta^{-1} \simeq p_A^* (\alpha \otimes \delta^{-1}).$$

Clearly the line bundle above is in the kernel of the norm homomorphism, but being also algebraically trivial it lies in the image of $(1 - \sigma^*)$ by Proposition 2.5.1. We deduce that, in $\mathrm{Ker}\,\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$,

$$[L] = [(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \gamma] = [(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \delta],$$

[2]for example we can identify A with its dual and ω^* with ω and take $\tau = (\frac{1}{2}, \frac{1}{2}) + \Lambda$, where $\Lambda = < 1, i >$ and $A \simeq \mathbb{C}/\Lambda$.

for every $\delta \in \mathrm{Pic}^0(A)$ such that $(1 \times \psi)^* \mathscr{P}_A \otimes p_A^* \delta$ is in the kernel of the Brauer map. In particular, there is only one non-trivial element in $\operatorname{Ker} \pi_{\mathrm{Br}}$.

Thus we can assume that B has complex multiplication and that, as before, we have fixed ψ_1 and ψ_2 a system of generators for $\mathrm{Hom}(A,B)$.

Suppose first that only one among the line bundles in (2.31) is trivial, for example L_1, and as usual take L in the kernel of the norm map. Let

$$M_1 := (1 \times \psi_1)^* \mathscr{P}_A, \quad M_2 := (1 \times \psi_2)^* \mathscr{P}_A, \quad M_3 := (1 \times (\psi_1 + \psi_2))^* \mathscr{P}_A.$$

As before, we can write $L \simeq M_i \otimes p_A^* \alpha \otimes M$ with M in the image of $(1 - \sigma^*)$. We deduce by (2.32) that $i = 1$ and that the class of L in $\operatorname{Ker}\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$ is equal to the class of $M_1 \otimes p_A^* \gamma$ for every $\gamma \in \mathrm{Pic}^0(A)$ such that $\mathrm{Nm}(M_1 \otimes p_A^* \gamma)$ is trivial. Thus, there is just one non-zero class and the Brauer map is again non-trivial.

Finally, suppose that two (and so all) line bundles in (2.31) are trivial. We have by (2.32) that the norms of both both M_1 and M_2 are algebraically trivial. By the Pic^0-trick (Remark 2.1.8) and Lemma 2.5.2 we get that $M_1 \otimes p_A^* \alpha$ and $M_2 \otimes p_A^* \delta$ are in the kernel of the norm homomorphism for some $\alpha, \delta \in \mathrm{Pic}^0(A)$. Now,

$$M_1 \otimes p_A^* \alpha \otimes (M_2 \otimes p_A^* \delta)^{-1} \simeq (1 \times (\psi_1 - \psi_2))^* \mathscr{P}_A \otimes p_A^* (\alpha \otimes \delta^{-1}),$$

which by Lemma 2.5.7 is not in the image of $(1 - \sigma^*)$. Therefore we deduce that those line bundles determine two different classes in $\operatorname{Ker}\mathrm{Nm} / \mathrm{Im}(1 - \sigma^*)$, and hence the Brauer map is trivial. We have thus proven the following statement:

Theorem 2.5.11. *The Brauer map to the canonical cover of a type 1 bielliptic surface is trivial if, and only if, the elliptic curves A and B are isogenous, B has complex multiplication, and all the line bundles in* (2.31) *are trivial.*

Example 2.5.12. If $A \simeq B$ then the Brauer map is never trivial. Suppose otherwise that there are ψ_1 and ψ_2 generators of $\mathrm{End}(A)$ such that both $\psi_1^* P_\tau$ and $\psi_2^* P_\tau$ are zero. Then we can write $1_A = h \cdot \psi_1 + k \cdot \psi_2$ and we would get that $P_\tau \simeq 1_A^* P_\tau$ is trivial, reaching an obvious contradiction.

Bielliptic Surfaces of Type 3

Let S be a bielliptic surface of type 3. Then the canonical cover of S is isomorphic to $A \times B$ with $j(B) = 1728$ and multiplication by i induces an automorphism ω of B of order 4, ω. By the discussion in 2.1.7, it is possible to find a generating isogeny ψ such that

$$\mathrm{Num}(X) = \langle l(1,0,0),\, l(0,1,0),\, l(0,0,\psi),\, l(0,0,\psi \circ \omega) \rangle,$$

We fix such a ψ once and for all and prove the following Lemma, which yields a precise description of $(1 - \sigma^*)\,\mathrm{Pic}(X)$.

Lemma 2.5.13. *Let $\varphi : B \to A$ be an isogeny. Consider the two integers h and k such that $\varphi = h \cdot \psi + k \cdot \psi \circ \omega$. Then we have that $(1 \times \varphi)^* \mathscr{P}_A \in \mathrm{Im}(1 - \sigma^*)$ if and only if $h + k$ is even.*

Proof. Let $T : \mathrm{Hom}(B,A) \to \mathrm{Hom}(B,A)$ be the linear operator obtained by precomposing with $(1_B - \omega)$. Then, an isogeny φ as in the statement is in the image of T if, and only if, $h+k$ is even. Indeed, if $a \cdot \psi + b \cdot \psi \circ \omega$ is an isogeny in $\mathrm{Hom}(B,A)$, then, using the fact that $\omega^2 = -1_B$, we get that

$$\begin{aligned} T(a \cdot \psi + b \cdot \psi \circ \omega) &= (a \cdot \psi + b \cdot \psi \circ \omega) \circ (1_B - \omega) \\ &= a \cdot \psi + b \cdot \psi \circ \omega - a \cdot \psi \circ \omega - b \cdot \psi \circ \omega^2 \\ &= (a+b) \cdot \psi + (b-a) \cdot \psi \circ \omega, \end{aligned}$$

and $(a+b)+(b-a) = 2b$ is even. Viceversa, if we have a φ such that $h+k$ is even, then the system

$$\begin{cases} a+b = h \\ b-a = k \end{cases}$$

has integer solutions

$$\begin{cases} a = \frac{h-k}{2} \\ b = \frac{k+h}{2} \end{cases},$$

and therefore φ is in the image of T.

Suppose now that $\varphi = h \cdot \psi + k \cdot \psi \circ \omega$ with $h+k$ an even number. By the above argument, we can find an isogeny $\gamma : B \to A$ such that $\varphi = \gamma \circ (1_B - \omega)$. Then, using Lemma 2.1.23, we have

$$\begin{aligned} (1-\sigma^*)(1 \times \gamma)^* \mathscr{P}_A &\simeq (1 \times \gamma \circ (1_B - \omega))^* \mathscr{P}_A \otimes p_B^* \omega^* \gamma^* P_\epsilon^{-1} \\ &\simeq (1 \times \varphi)^* \mathscr{P}_A \otimes p_B^* \omega^* \gamma^* P_\epsilon^{-1}. \end{aligned}$$

By Lemma 2.5.2, elements of the form $p_B^*\beta$ with $\beta \in \mathrm{Pic}^0(B)$ are in the image of $(1-\sigma^*)$, so we conclude that $(1 \times \varphi)^* \mathscr{P}_A \in \mathrm{Im}(1-\sigma^*)$.

We also observe that if $h+k$ is not even, then $(1 \times \varphi)^* \mathscr{P}_A \notin \mathrm{Im}(1-\sigma^*)$. □

Remark 2.5.14. Observe that, if we identify $\mathrm{Hom}(B,A)$ with the corresponding subgroup of $\mathrm{Num}(A \times B)$, Lemma 2.5.13 implies easily that the quotient $\mathrm{Hom}(B,A)/\mathrm{Im}(1-\sigma^*)$ is cyclic generated by the coset $(1_A \times \psi)^* \mathscr{P}_A + \mathrm{Im}(1-\sigma^*)$.

Now we are ready to study the kernel of the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$. Our result is the following:

Theorem 2.5.15. *Let S be a bielliptic surface of type 3 with canonical cover $A \times B$ such that A and B are isogenous. Then the Brauer map to the canonical cover is identically zero if, and only if, $(1_B + \omega)^* \psi^* P_{2\epsilon}$ is trivial.*

Proof. For any isogeny $\varphi : B \to A$, $\alpha \in \mathrm{Pic}^0(A)$ and $\beta \in \mathrm{Pic}^0(B)$, using that the norm of $p_B^*\beta$ is trivial by Lemma 2.5.2, we have that

$$\begin{aligned} \pi^* \mathrm{Nm}((1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes p_B^* \beta) \simeq\ & (1 \times \varphi)^* \mathscr{P}_A \otimes p_A^* \alpha \otimes \\ & (1 \times \varphi \circ \omega)^* \mathscr{P}_A \otimes p_B^* \omega^* \varphi^* P_\epsilon \otimes p_A^* \alpha \otimes \\ & (1 \times -\varphi)^* \mathscr{P}_A \otimes p_B^* (-1_B)^* \varphi^* P_{2\epsilon} \otimes p_A^* \alpha \otimes \\ & (1 \times -\varphi \circ \omega)^* \mathscr{P}_A \otimes p_B^* (-\omega)^* \varphi^* P_{3\epsilon} \otimes p_A^* \alpha \otimes \\ \simeq\ & p_A^* \alpha^{\otimes 4} \otimes p_B^* (1_B + \omega)^* \varphi^* P_{2\epsilon}. \end{aligned} \tag{2.33}$$

Suppose that $(1_B+\omega)^*\psi^*P_{2\epsilon}\simeq\mathcal{O}_B$. Since $P_{2\epsilon}$ is a two-torsion point, this is equivalent to asking for $(1_B-\omega)^*\psi^*P_{2\epsilon}$ to be also trivial. Therefore (2.33) implies that the norms of $(1\times\psi)^*\mathscr{P}_A$ and of $(1\times\psi\circ\omega)^*\mathscr{P}_A$ lie in $\mathrm{Pic}^0(S)$. Then, using the Pic^0-trick (Remark 2.1.8) and Lemma 2.5.13, we can find a non-zero class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$, and the Brauer map is trivial.

Conversely, let L be a line bundle yielding a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$. Then, as we did in the case of type 1 surfaces, we can write

$$L\simeq(1\times h\cdot\psi+k\cdot\psi\circ\omega)^*\mathscr{P}_A\otimes p_A^*\alpha\otimes p_B^*\beta,$$

with α and β in $\mathrm{Pic}^0(A)$ and $\mathrm{Pic}^0(B)$ respectively. Lemma 2.5.13 implies that the integer $h+k$ is odd or we would have that $p_A^*\alpha$ is in the kernel of the norm map, and consequently, by Proposition 2.5.1, $L\in\mathrm{Im}(1-\sigma^*)$. Thus we can write

$$L\simeq M\otimes M',$$

where M' is in the image of $1-\sigma^*$, and M is numerically equivalent to $(1\times\psi)^*\mathscr{P}_A$ (this is a consequence of Lemma 2.5.13 and Remark 2.5.14). We deduce that M is in the kernel of the norm map. But then (2.33) implies that $(1+\omega)^*\psi^*P_{2\epsilon}$ is trivial, proving the statement. □

Example 2.5.16. Suppose that $A\simeq B$, so we can take $\psi=1_A$. If $P_{2\epsilon}$ is a fixed point of ω, then we have that $\mathscr{P}_A$ yields a non-trivial element in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$. Conversely, if $P_{2\epsilon}$ is not a fixed point of ω, we have that the Brauer map is injective.

Bielliptic Surfaces of Type 5

Let S be a bielliptic surface of type 5. We will treat this case similarly to that of bielliptic surfaces of type 3. In the type-5 case, the canonical cover is isomorphic to an abelian surface $A\times B$ with $j(B)=0$. As already seen, B admits an automorphism ρ of order 3 such that $\rho^2+\rho+1=0$. Again, thanks to Theorem 2.5.4, we need to study only the case in which A and B are isogenous. Also in this case, by the results of 2.1.7, there is a generating isogeny $\psi: B\to A$ such that

$$\mathrm{Num}(X)=\langle l(1,0,0),\, l(0,1,0),\, l(0,0,\psi),\, l(0,0,\psi\circ\rho)\rangle\,.$$

With this notation, we prove a statement analogous to Lemma 2.5.13:

Lemma 2.5.17. *Let $\varphi: B\to A$ be an isogeny. Then there are two integers h and k such that $\varphi=h\cdot\psi+k\cdot\psi\circ\rho$. If $h+k$ is not divisible by 3, then $(1\times\varphi)^*\mathscr{P}_A\notin\mathrm{Im}(1-\sigma^*)$. Conversely, if 3 divides $h+k$, then $(1\times\varphi)^*\mathscr{P}_A\otimes p_B^*\beta\in\mathrm{Im}(1-\sigma^*)$ for every $\beta\in\mathrm{Pic}^0(B)$.*

Proof. The argument is completely analogous to the one used in the proof of Lemma 2.5.13 after observing that, if $T:\mathrm{Hom}(B,A)\to\mathrm{Hom}(B,A)$ is the operator defined by pre-composing with $1_B-\rho$, then the image of T are exactly the homomorphisms $h\cdot\psi+k\cdot\psi\circ\rho$ such that 3 divides $k+h$.

Indeed, if $a \cdot \psi + b \cdot \psi \circ \rho$ is an isogeny in $\mathrm{Hom}(B, A)$, then, using the fact that $-\rho^2 = \rho + 1_B$, we get that

$$\begin{aligned} T(a \cdot \psi + b \cdot \psi \circ \rho) &= (a \cdot \psi + b \cdot \psi \circ \omega) \circ (1_B - \rho) \\ &= a \cdot \psi + b \cdot \psi \circ \rho - a \cdot \psi \circ \rho - b \cdot \psi \circ \rho^2 \\ &= (a+b) \cdot \psi + (2b - a) \cdot \psi \circ \rho, \end{aligned}$$

and $(a+b) + (2b-a) = 3b$ is divisible by 3. Viceversa, if we have a φ such that $h + k$ is even, then the system

$$\begin{cases} a + b = h \\ 2b - a = k \end{cases}$$

has integer solutions

$$\begin{cases} a = \frac{2h-k}{2} \\ b = \frac{k+h}{3} \end{cases},$$

and therefore φ is in the image of T.

Similarly to the type-3 case, take $\varphi = h \cdot \psi + k \cdot \psi \circ \omega$ with $h + k$ divisible by 3. By what we proved about T, there exists an isogeny $\gamma : B \to A$ such that $\varphi = \gamma \circ (1_B - \omega)$. Again, by Lemma 2.1.23 we have

$$\begin{aligned} (1 - \sigma^*)(1 \times \gamma)^* \mathscr{P}_A &\simeq (1 \times \gamma \circ (1_B - \rho))^* \mathscr{P}_A \otimes p_B^* \rho^* \gamma^* P_\eta^{-1} \\ &\simeq (1 \times \varphi)^* \mathscr{P}_A \otimes p_B^* \rho^* \gamma^* P_\eta^{-1}. \end{aligned}$$

We know that $p_B^* \beta$ with $\beta \in \mathrm{Pic}^0(B)$ is in the image of $(1 - \sigma^*)$ thanks to Lemma 2.5.2, and therefore $(1 \times \varphi)^* \mathscr{P}_A \in \mathrm{Im}(1 - \sigma^*)$. □

Remark 2.5.18. This Lemma implies easily that the quotient of the Hom-part of $\mathrm{Num}(A \times B)$ by the action of $1 - \sigma^*$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ with elements $(1_A \times \psi)^* \mathscr{P}_A + \mathrm{Im}(1 - \sigma^*)$ and $(1_A \times (\psi + \psi \circ \rho))^* \mathscr{P}_A + \mathrm{Im}(1 - \sigma^*) = (1_A \times 2 \cdot \psi)^* \mathscr{P}_A + \mathrm{Im}(1 - \sigma^*)$.

We will also need the following statement:

Lemma 2.5.19. *Let B be an elliptic curve with j-invariant 0 and let β be an element of $\mathrm{Pic}^0(B)$. Consider the following line bundles:*

$$P_1 := (2 \cdot \rho + 1_B)^* \beta, \quad P_\rho := (2 \cdot \rho + 1_B)^* \rho^* \beta,$$
$$P_{1+\rho} := (2 \cdot \rho + 1_B)^* (1_B + \rho)^* \beta.$$

If one of them is trivial, then all of them are trivial.

Proof. Observe first that $(2 \cdot \rho + 1_B)^* \rho^* \beta \simeq \rho^* (2 \cdot \rho + 1_B)^* \beta$. Since ρ is an automorphism, the triviality of P_ρ is equivalent to the triviality of P_1. In addition, as $P_{1+\rho} \simeq P_1 \otimes P_\rho$, we have that, if P_1 and P_ρ are both trivial, then also $P_{1+\rho}$ is trivial.

It remains to show that, if $P_{1+\rho} \simeq \mathcal{O}_B$, then also P_1 and P_ρ are trivial. We note that $P_{1+\rho} \simeq \mathcal{O}_B$ if, and only if, $\rho^* P_{1+\rho} \simeq \mathcal{O}_B$. On the other side we have

$$\begin{aligned} \rho^* P_{1+\rho} &\simeq \rho^*(2\cdot\rho + 1_B)^*(1_B + \rho)^*\beta \\ &\simeq \rho^*(\rho - 1_B)^*\beta \simeq (-2\cdot\rho - 1_B)^*\beta \simeq P_1^{-1}. \end{aligned}$$

We conclude that the triviality of $P_{1+\rho}$ is equivalent to the triviality of P_1 as required by the statement. □

Now we are ready to prove the main result of this section:

Theorem 2.5.20. *Let S be a bielliptic surface of type 5 such that the two elliptic curves A and B are isogenous. Let ψ be a generating isogeny. Then we have that the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(A \times B)$ is trivial if, and only if, the line bundle $(2\cdot\rho + 1_B)^*\psi^* P_\eta$ is trivial.*

Proof. The argument is really similar to the one for type 3 bielliptic surfaces. We first note that, for any isogeny $\varphi : B \to A$, and every α and β in $\mathrm{Pic}^0(A)$ and $\mathrm{Pic}^0(B)$ respectively, we have that

$$\begin{aligned} \pi^* \mathrm{Nm}((1\times\varphi)^*\mathscr{P}_A \otimes p_A^*\alpha \otimes p_B^*\beta) &\simeq (1\times\varphi)^*\mathscr{P}_A \otimes p_A^*\alpha \otimes p_B^*\beta \\ &\quad \otimes (1\times\varphi\circ\rho)^*\mathscr{P}_A \otimes p_A^*\alpha \otimes p_B^*\rho^*\beta \otimes p_B^*(\varphi\circ\rho)^*P_\eta \\ &\quad \otimes (1\times\varphi\circ\rho^2)^*\mathscr{P}_A \otimes p_A^*\alpha \otimes p_B^*\rho^{2*}\beta \otimes p_B^*(\varphi\circ\rho^2)^*P_{2\eta} \\ &\simeq p_A^*\alpha^{\otimes 3} \otimes p_B^*(\rho - \rho^2)^*\varphi^* P_\eta \\ &\simeq p_A^*\alpha^{\otimes 3} \otimes p_B^*(2\cdot\rho + 1_B)^*\varphi^* P_\eta. \end{aligned} \tag{2.34}$$

Suppose first that $(2\cdot\rho + 1_B)^*\psi^* P_\eta$ is trivial. Then (2.34) ensures that the norm of $M_1 := (1\times\psi)^*\mathscr{P}_A$ is topologically trivial. By Lemma 2.5.17 we know that no line bundle numerically equivalent to M_1 is in the image of $1 - \sigma^*$. Thus we use Remark 2.1.8 to provide an element in $\mathrm{Ker}\,\mathrm{Nm}$ inducing a non-trivial class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$.

Conversely, assume that L is a line bundle in the kernel of the norm map whose class in $\mathrm{Ker}\,\mathrm{Nm}\,/\,\mathrm{Im}(1-\sigma^*)$ is not trivial. As before, we can write

$$L \simeq (1 \times h\cdot\psi + k\cdot\psi\circ\rho)^*\mathscr{P}_A \otimes p_A^*\alpha \otimes p_B^*\beta.$$

We apply Lemma 2.5.17 and write $L \simeq M \otimes M'$ with $M' \in \mathrm{Im}(1-\sigma^*)$ and M a line bundle numerically equivalent to either

$$M_1 := (1_A \times \psi)^*\mathscr{P}_A \quad \text{or} \quad M_{1+\rho} := (1_A \times (\psi + \psi\circ\rho))^*\mathscr{P}_A. \tag{2.35}$$

Clearly M is in the kernel of the norm map, which implies, by (2.34), that at least one between

$$P_1 := (2\cdot\rho + 1_B)^*\psi^* P_\eta \quad \text{and} \quad P_{1+\rho} := (2\cdot\rho + 1_B)^*(1_B + \rho)^*\psi^* P_\eta$$

is trivial. We conclude by applying Lemma 2.5.19 and deducing that $P_1 \simeq \mathcal{O}_B$. □

Example 2.5.21. Suppose that $A \simeq B$. Note that the isogeny $\varphi := (2 \cdot \rho + 1_B) : B \to B$ has degree 3, and its kernel is contained in $B[3]$, which has order 9. Indeed, we see that

$$
\begin{aligned}
(2 \cdot \rho + 1_B) \circ (\rho - 1_B) \circ \rho &= (2 \cdot \rho + 1_B) \circ (-\rho - 1_B - \rho) \\
&= -(2 \cdot \rho + 1_B)^2 \\
&= -(4(-\rho - 1_B) + 4\rho + 1_B) = 3_B;
\end{aligned}
$$

since ρ is an automorphism with three fixed points, $(\rho - 1_B)$ is an isogeny of degree 3. The isogeny 3_B has degree 9, thus $(2 \cdot \rho + 1_B)$ must have degree 3.

If η is in the kernel of φ, then the bielliptic surface obtained by the action of $\sigma(x,y) = (x + \eta, \rho(y))$ has trivial Brauer map. Otherwise the Brauer map is injective.

Bielliptic Surfaces of Type 2

We kept for last the bielliptic surfaces of type 2 since in this case we need an *ad hoc* argument.
Therefore, we now consider an S bielliptic surface of type 2 and we take X to be its canonical cover. Then $X \simeq A \times B / < t_{(\theta_1,\theta_2)} >$ for two elliptic curves A, B and θ_1, θ_2 points of order 2 in A and B respectively.
We also fix generators for $\mathrm{Hom}(B,A)$: if B does not have complex multiplication, then $\mathrm{Hom}(B,A) =< \psi >$ with $\psi : B \to A$ an isogeny; otherwise there are two isogenies $\psi_1, \psi_2 : B \to A$ such that $\mathrm{Hom}(B,A) =< \psi_1, \psi_2 >$.

Our goal is to prove the following statement:

Theorem 2.5.22. *In the above notation, the Brauer map $\pi_{\mathrm{Br}} : \mathrm{Br}(S) \to \mathrm{Br}(X)$ is not injective if, and only if, one of the following conditions is satisfied:*

1. *the elliptic curve B does not have complex multiplication and either $\psi(\theta_2)$ is not the identity element of A or $\psi^* P_{\theta_1}$ is not trivial;*
2. *the elliptic curve B has complex multiplication and not all of the following elements are the identity element in the elliptic curve to which they belong:*

$$\left\{\psi_1(\theta_2),\, \psi_2(\theta_2),\, \psi_1^* P_{\theta_1}, \psi_2^* P_{\theta_1},\, (\psi_1 + \psi_2)(\theta_2),\, (\psi_1 + \psi_2)^*(P_{\theta_1})\right\}.$$

Before proceeding with the proof we need to set up some notation. Recall that we have the diagram

$$
\begin{tikzcd}
A \times B \arrow[r, "\pi_{\tilde{S}}"] \arrow[d, "\phi"'] & \tilde{S} \arrow[d, "\tilde{\pi}"] \\
X \arrow[r, "\pi_S"'] & S,
\end{tikzcd}
$$

where $\tilde{S}$ is a bielliptic surface of type 1. We have that $S \simeq X / < \sigma >$, $\tilde{S} \simeq A \times B / < \tilde{\sigma} >$ and $X \simeq A \times B / < \Sigma >$, where Σ denotes the translation

$t_{(\theta_1,\theta_2)}$.

We are going to deal only with the case in which B has complex multiplication. The proof in the other case will be identical, provided that one drops one of the two generators of $\mathrm{Hom}(B,A)$. We first observe the following fact:

Lemma 2.5.23. *In the notation above, suppose that B has complex multiplication and let L_i be the line bundle $(1\times\psi_i)^*\mathscr{P}_A$, for $i=1,2$. Then the conditions of Theorem 2.5.22 are satisfied if, and only if, for every $\gamma\in\mathrm{Pic}^0(A\times B)$ one of the following line bundles is not Σ-invariant:*

$$L_1\otimes\gamma,\qquad L_2\otimes\gamma,\qquad L_1\otimes L_2\otimes\gamma. \tag{2.36}$$

Proof. Using the See-Saw Principle, we show that

$$\begin{aligned}\Sigma^*[(1\times\psi_i)^*\mathscr{P}_A\otimes\gamma]&\simeq(1\times\psi_i)^*\mathscr{P}_A\otimes\gamma\otimes p_A^*P_{\psi_i(\theta_2)}\otimes p_B^*\psi_i^*P_{\theta_1},\\ \Sigma^*[(1\times(\psi_1+\psi_2))^*\mathscr{P}_A\otimes\gamma]&\simeq(1\times(\psi_1+\psi_2))^*\mathscr{P}_A\otimes\gamma\\ &\qquad\otimes p_A^*P_{(\psi_1+\psi_2)(\theta_2)}\otimes p_B^*(\psi_1+\psi_2)^*P_{\theta_1};\end{aligned}$$

from which the statement of the lemma follows directly. We prove the first one, as the difference between the two is just formal. We use repeatedly Lemma 2.1.19 and Lemma 2.1.22.

$$\begin{aligned}(t_{\theta_1}\times t_{\theta_2})^*(1\times\psi_i)^*\mathscr{P}_A&\simeq(1\times\psi_i)^*(1\times t_{\psi_i(\theta_2)})^*(t_{\theta_1}\times1)^*\mathscr{P}_A\\ &\simeq(1\times\psi_i)^*(1\times t_{\psi_i(\theta_2)})^*\mathscr{P}_A\otimes(1\times\psi_i)^*(1\times t_{\psi_i(\theta_2)})^*p_2^*P_{\theta_1}\\ &\simeq(1\times\psi_i)^*\mathscr{P}_A\otimes(1\times\psi_i)^*p_A^*P_{\psi_i(\theta_2)}\otimes p_B^*\psi_i^*t_{\psi_i(\theta_2)}^*P_{\theta_1}\\ &\simeq(1\times\psi_i)^*\mathscr{P}_A\otimes p_A^*P_{\psi_i(\theta_2)}\otimes p_B^*\psi_i^*P_{\theta_1},\end{aligned}$$

where to compute $(1\times\psi_i)^*(1\times t_{\psi_i(\theta_2)})^*p_2^*P_{\theta_1}$ (p_2 being the projection onto the second factor) we used the diagram

$$\begin{array}{ccccc}A\times B & \xrightarrow{1\times\psi_i} & A\times A & \xrightarrow{1\times t_{\psi_i(\theta_2)}} & A\times A\\ \Big\downarrow{\scriptstyle p_B} & & & & \Big\downarrow{\scriptstyle p_2}\\ B & \xrightarrow{\psi_i} & A & \xrightarrow{t_{\psi_i(\theta_2)}} & A.\end{array}$$

Clearly $\Sigma^*\gamma\simeq\gamma$ by Lemma 2.1.19, and so we proved the isomorphism we set out to prove. □

Proof of the Sufficiency of the Conditions of Theorem 2.5.22. Suppose that the conditions of the statement are satisfied. Then, by Lemma 2.5.23, one of the line bundles in (2.36) is not Σ-invariant.

Suppose first that $L_1\otimes\gamma$ is not Σ-invariant for any topologically trivial γ. Thus we have that $l(0,0,\psi_1)$ is not in $\phi^*\mathrm{Num}(X)$. We deduce that

$$2\cdot\psi_1\notin(1-\tilde{\sigma}^*)\phi^*\mathrm{Num}(X); \tag{2.37}$$

if that were not the case, using

$$\begin{aligned}\tilde{\sigma}^*(1_A \times (h\cdot\psi_1 + k\cdot\psi_2))^*\mathscr{P}_A &= (t_\tau \times (-1_B))^*(1_A \times (h\cdot\psi_1 + k\cdot\psi_2))^*\mathscr{P}_A \\ &\simeq (1_A \times (-h\cdot\psi_1 - k\cdot\psi_2))^*\mathscr{P}_A \otimes p_B^*(-h\cdot\psi_1 - k\cdot\psi_2)^* P_\tau,\end{aligned}$$

we would have

$$\begin{aligned}2\cdot\psi_1 &= (1-\tilde{\sigma}^*)\phi^*\varphi \\ &= (1-\tilde{\sigma}^*)(h\cdot\psi_1 + k\cdot\psi_2) \\ &= 2h\cdot\psi_1 + 2k\cdot\psi_2,\end{aligned}$$

therefore $h=1$, $k=0$ and $\phi^*\varphi = \psi_1$, contradicting our previous conclusion.

Now consider the line bundle $L := \mathrm{Nm}_\phi((1\times\psi_1)^*\mathscr{P}_A)$. We want to show that there is $\beta \in \mathrm{Pic}^0(X)$ such that $\mathrm{Nm}_{\pi_S}(L\otimes\beta)$ is trivial. We use the functoriality of the norm map (Proposition 2.1.7) and we obtain that

$$\mathrm{Nm}_{\pi_S}(L) \simeq \mathrm{Nm}_{\tilde{\pi}} \circ \mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi_1)^*\mathscr{P}_A).$$

Observe that by (2.32) we have that $\pi_{\tilde{S}}^*\mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi_1)^*\mathscr{P}_A)$ is numerically trivial. Therefore we have that $\mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi_1)^*\mathscr{P}_A)$ is itself numerically trivial. This implies that

$$\mathrm{Nm}_{\tilde{\pi}} \circ \mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi)^*\mathscr{P}_A) \in \mathrm{Pic}^0(S);$$

indeed, if we are in the case in which $\mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi_1)^*\mathscr{P}_A) := \alpha \in \mathrm{Pic}^0(\tilde{S})$, then we can write $\alpha \simeq \tilde{\pi}^*\gamma$ and we get that

$$\mathrm{Nm}_{\pi_S}(L) \simeq \mathrm{Nm}_{\tilde{\pi}} \circ \mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi_1)^*\mathscr{P}_A) \simeq \gamma^{\otimes 2}.$$

Otherwise, if $\mathrm{Nm}_{\pi_{\tilde{S}}}((1\times\psi_1)^*\mathscr{P}_A) := T$ is a numerically trivial but not an algebraically trivial line bundle, then, as in (2.28), we have that $\mathrm{Nm}_{\tilde{\pi}}(T)$ is topologically trivial. Thus, as before, we can find β such that $\mathrm{Nm}_{\pi_S}(L\otimes\beta) \simeq \mathcal{O}_S$ via the Pic^0-trick (Remark 2.1.8).

In order to determine the non-injectivity of the Brauer map we have to ensure that $L\otimes\beta$ is not in $\mathrm{Im}(1-\sigma^*)$. Suppose that this were not the case, and consider the commutative diagram

$$\begin{tikzcd} A\times B \arrow[r, "\tilde{\sigma}"] \arrow[d, "\phi"'] & A\times B \arrow[d, "\phi"] \\ X \arrow[r, "\sigma"'] & X. \end{tikzcd}$$

Then $c_1(\phi^*L) \in (1-\tilde{\sigma}^*)\phi^*\mathrm{Num}(X)$. However, (2.8) and the computation in the proof of Lemma 2.5.23 ensure that $c_1(\phi^*L) = l(0,0,2\cdot\psi_1)$, since

$$\begin{aligned}\phi^*L \simeq \phi^*\mathrm{Nm}_\phi((1\times\psi_1)^*\mathscr{P}_A) &\simeq (1\times\psi_1)^*\mathscr{P}_A \otimes \Sigma^*(1\times\psi_1)^*\mathscr{P}_A \\ &\simeq (1\times 2\cdot\psi_1)^*\mathscr{P}_A \otimes p_A^* P_{\psi_1(\theta_2)} \otimes p_B^*\psi_1^* P_{\theta_1};\end{aligned}$$

thus we would have that $l(0,0,2\cdot\psi_1)\in(1-\tilde{\sigma}^*)\phi^*\operatorname{Num}(X)$, contradicting (2.37).

If $L_2\otimes\gamma$ is not Σ-invariant for any $\gamma\in\operatorname{Pic}^0(A\times B)$, then we proceed as before by exchanging the role of ψ_1 and ψ_2. Thus, it remains only to see what happens if $L_1\otimes L_2\otimes\gamma$ is not Σ-invariant for any γ. In this case we have that $l(0,0,\psi_1+\psi_2)\notin\phi^*\operatorname{Num}(A\times B)$, and so $l(0,0,\psi_1)$ and $l(0,0,\psi_2)$ cannot be both in the image of ϕ^*. Without loss of generality we can assume the first. Then we still have (2.37) and we can repeat the above argument.

□

In order to complete the proof of Theorem 2.5.22 we need to show that, if the three line bundles $(1\times\psi_1)^*\mathscr{P}_A$, $(1\times\psi_2)^*\mathscr{P}_A$, and $(1\times(\psi_1+\psi_2))^*\mathscr{P}_A$ are Σ-invariant, then the Brauer map to X is injective. Observe that, under this assumption, we can write

$$(1\times\psi_1)^*\mathscr{P}_A\simeq\phi^*L_1,\quad(1\times\psi_2)^*\mathscr{P}_A\simeq\phi^*L_2,\quad(1\times(\psi_1+\psi_2))^*\mathscr{P}_A\simeq\phi^*L_3$$

for some line bundles L_1, L_2, and L_3 in $\operatorname{Pic}(X)$. Then, for $\alpha\in\operatorname{Pic}^0(X)$ write $\phi^*\alpha\simeq p_A^*\alpha_1\otimes p_B^*\alpha_2$; and for $i=1,2$ we have

$$\begin{aligned}\phi^*(\pi_S^*\operatorname{Nm}_{\pi_S}(L_i\otimes\alpha))&\simeq\phi^*(L_i\otimes\alpha\otimes\sigma^*(L_i\otimes\alpha))\\&\simeq\phi^*L_i\otimes\phi^*\alpha\otimes(t_\tau\times-1)^*(\phi^*L_i\otimes p_A^*\alpha_1\otimes p_B^*\alpha_2)\\&\simeq\phi^*L_i\otimes\phi^*\alpha\otimes\phi^*L_i^{-1}\otimes p_A^*\alpha_1\otimes p_B^*\alpha_2^{-1}\otimes p_B^*(\psi_i^*P_\tau)^{-1}\\&\simeq p_A^*\alpha_1^{\otimes 2}\otimes p_B^*(\psi_i^*P_\tau),\end{aligned}$$

$$\begin{aligned}\phi^*(\pi_S^*\operatorname{Nm}_{\pi_S}(L_1\otimes L_2\otimes\alpha))&\simeq\phi^*(L_1\otimes L_2\otimes\alpha\otimes\sigma^*(L_1\otimes L_2\otimes\alpha))\\&\simeq p_A^*\alpha_1^{\otimes 2}\otimes p_B^*(\psi_1^*P_\tau\otimes\psi_2^*P_\tau).\end{aligned}$$

Observe that neither the ψ_i's nor $\psi_1+\psi_2$ can factor through the multiplication by 2 isogeny, or we would have that ψ_1 and ψ_2 cannot generate $\operatorname{Hom}(B,A)$. In particular, since by hypothesis P_{θ_1} is in the kernel of all of those homomorphisms, neither the $\psi_i^*P_\tau$'s nor $(\psi_1+\psi_2)^*P_\tau$ can be trivial. We deduce that

$$\begin{aligned}\phi^*(\pi_S^*\operatorname{Nm}_{\pi_S}(L_i\otimes\alpha))&\not\simeq\mathcal{O}_{A\times B},\\\phi^*(\pi_S^*\operatorname{Nm}_{\pi_S}(L_1\otimes L_2\otimes\alpha))&\not\simeq\mathcal{O}_{A\times B}.\end{aligned}$$

From this discussion, we obtained the following lemma:

Lemma 2.5.24. *In the above notation, if the conditions of Theorem 2.5.22 are not satisfied, then any line bundle numerically equivalent to L_i or $L_1\otimes L_2$ is not in the kernel of the norm map $\operatorname{Nm}_{\pi_S}$.*

Before going further we need an intermediate step:

Lemma 2.5.25. *For any integer n, $L_i^{\otimes 2n}$ and $(L_1\otimes L_2)^{\otimes 2n}$ are in $\operatorname{Im}(1-\sigma^*)$.*

Proof. Obviously it is enough to show that $L_i^{\otimes 2}$ is in the image of $(1-\sigma^*)$. To this end, we pullback $L_i \otimes \sigma^* L_i$ to $A \times B$ and apply (2.32). We see that

$$\phi^*(L_i \otimes \sigma^*(L_i)) \in p_B^* \operatorname{Pic}^0(B) \subseteq \operatorname{Pic}^0(A \times B),$$

and we deduce that $\gamma := L_i \otimes \sigma^*(L_i)$ is a line bundle in $p_B^* \operatorname{Pic}^0(B/H)$. By Lemma 2.5.2 we know that $\gamma \in \operatorname{Im}(1-\sigma^*)$. Thus we can write

$$L_i^{\otimes 2} \simeq \gamma \otimes \sigma^* L_i^{-1} \otimes L_i.$$

□

Conclusion of the Proof of Theorem 2.5.22. Let M be a line bundle such that $\operatorname{Nm}_{\pi_S}(M) \simeq \mathcal{O}_S$, we will show that M is in the image of $(1-\sigma^*)$. Using (2.8), we know that $M \otimes \sigma^* M \simeq \mathcal{O}_X$. By pulling back via ϕ, we get that $\phi^* M \otimes \tilde{\sigma}^* \phi^* M$ is again trivial. As in the proof of Lemma 2.5.6 we see that, if $c_1(\phi^* M) = l(m,n,\varphi)$, then

$$\begin{aligned} 0 &= c_1(\phi^* M \otimes \tilde{\sigma}^* \phi^* M) \\ &= l(m,n,\varphi) + l(m,n,-\varphi) = l(2m,2n,0)), \end{aligned}$$

implying $m = n = 0$; it follows that $c_1(\phi^* M) = l(0,0,h \cdot \psi_1 + k \cdot \psi_2)$ for two integers h and k. Then we can write

$$\phi^* M \simeq (1 \times h \cdot \psi_1)^* \mathscr{P}_A \otimes (1 \times k \cdot \psi_2)^* \mathscr{P}_A \otimes \gamma \simeq \phi^*(L_1^{\otimes h} \otimes L_2^{\otimes k}) \otimes \gamma,$$

for some γ in $\operatorname{Pic}^0(A \times B)$. Therefore $\phi^*(M \otimes L_1^{\otimes -h} \otimes L_2^{\otimes -k}) \simeq \gamma$, and we deduce that $M \simeq L_1^{\otimes h} \otimes L_2^{\otimes k} \otimes \alpha$ for some $\alpha \in \operatorname{Pic}^0(X)$. If h and k are both even, then by Lemma 2.5.25 we know that $\alpha \in \operatorname{Ker} \operatorname{Nm}_{\pi_S}$, and the class of M in $\operatorname{Ker} \operatorname{Nm}_{\pi_S} / \operatorname{Im}(1-\sigma^*)$ is exactly $[\alpha]$. We apply Proposition 2.5.1 and deduce that $[M] = 0$.

We will now show that neither h nor k can be odd. Suppose otherwise that h and k are not both even. For instance, assume that h is odd and k is even, the proof in the two other cases is very similar. Under this hypothesis, Lemma 2.5.25 ensures that $L_1 \otimes \alpha$ is in the kernel of the norm map. But this contradicts Lemma 2.5.24, and our proof is complete. □

Example 2.5.26. (a) Suppose that $A \simeq B$, then the isogenies ψ_1 and ψ_2 are indeed isomorphisms and thus the Brauer map can never be injective.

(b) Let B be an elliptic curve without complex multiplication and consider θ_2 a point of order 2 in B. Let A be the elliptic curve $B/ < \theta_2 >$ and $\psi : B \to A$ the quotient map. The dual map ψ^* has degree 2. Let $\theta_1 \in A$ be the point such that $\psi^* P_{\theta_1}$ is trivial and let τ be another order-two element of A. These data identify a bielliptic surface of type 2 whose Brauer map to the canonical cover is injective.

Part II

Classification of Irregular Surfaces in Positive Characteristic

Chapter 3

Characterisation of Abelian Surfaces

This chapter is based on [Fe19]. To that material we added in particular a section (§3.5) about the case of K3 surfaces.

An often-tackled problem in algebraic geometry is that of characterizing projective varieties in terms of their birational invariants. In this frame, a classical result of Enriques states that a smooth complex surface S with plurigenera $P_1(S) = P_4(S) = 1$ and irregularity $h^1(S, \mathcal{O}_S) = \dim S$ is birationally equivalent to an abelian surface ([En1905]).

In the literature there are now several theorems birationally characterizing complex abelian varieties in terms of certain plurigenera and some other hypotheses. Among these results, Chen and Hacon proved in [CH01] that a smooth complex projective variety X with $P_1(X) = P_2(X) = 1$ and $h^1(X, \mathcal{O}_X) = \dim X$ is birational to an abelian variety.
What these works have in common is that they rely on the theory of Generic-Vanishing, and in particular on statements that are known to fail in positive characteristic ([HK15], [Fi18]). Therefore it is still an open question whether and what kind of generalizations of Enriques' theorem hold when dealing with varieties defined over fields of positive characteristic.

When in characteristic zero, given a variety X, the dimension of $\mathrm{H}^1(X, \mathcal{O}_X)$ equals the dimension of $\mathrm{Alb}(X)$, the Albanese variety of X. In this situation, one can proceed to prove that the Albanese morphism gives a birational morphism between X and $\mathrm{Alb}(X)$, provided that $h^1(X, \mathcal{O}_X) = \dim X$. In positive characteristic $h^1(X, \mathcal{O}_X)$ does not necessarily equal $\dim \mathrm{Alb}(X)$; thus if for a surface S one fixes $h^1(S, \mathcal{O}_S) = 2$, then the Albanese variety of S is not necessarily a surface.

The result proved here is

Theorem 3.A. *Let S be a smooth projective surface over an algebraically closed field of characteristic $p > 3$. If*

$$P_1(S) = P_4(S) = 1, \qquad h^1(S, \mathcal{O}_S) = 2, \tag{3.1}$$

then S is birational to an abelian surface.

The conditions $P_1(S) = P_4(S) = 1$ imply that $P_2(S) = 1$. The statement would be stronger if one could fix $P_1(S) = P_2(S) = 1$ and make no requirement about $P_4(S)$. By the work done here, if one asks only for $P_1(S) = P_2(S) = 1$ there would be only a few specific cases of surfaces that might cause the conclusion of the theorem to fail.

In this chapter, we first see that we need to study elliptic fibrations and we find relations about numerical invariants on them (§3.1). We then proceed with the proof of Theorem 3.A in §3.2. In §3.3 we compare our hypotheses to those of the corresponding problem in characteristic zero, while in §3.4 we comment on the cases of characteristic 2 and 3. Finally, we observe that our computations can be used to solve an analogous problem for K3 surfaces (§3.5).

3.1 Initial Reductions.

We prove Theorem 3.A by considering the Kodaira dimension of an S satisfying the hypotheses. We will see that the invariants that we have fixed lead to a contradiction in all cases but when $\kappa(S) = 0$ and S is birational to an abelian surface.

Indeed, clearly, having a non-zero plurigenus implies that $\kappa(S) \neq -\infty$.

As seen in Corollary 1.6.5, $P_2(S) \geq 2$ for any surface S of general type, and therefore an S as in the statement of the theorem cannot have Kodaira dimension 2.

Assume $\kappa(S) = 0$ as we want. As seen in Theorem 1.9.1, if the characteristic is neither 2 nor 3 the Pic^0 of a surface is reduced and $\Delta := 2h^1(S, \mathcal{O}_S) - b_1$ is zero.
Therefore, looking at Table 1.1, it is immediate that, since $h^1(S, \mathcal{O}_S) = 2$, S must be birational to an abelian surface.

Requiring only $P_1(S) = 1$ would have ruled out the cases of Enriques and hyperelliptic surfaces, but not K3 surfaces. From this discussion,

Lemma 3.1.1. *Let S satisfy the hypotheses in Theorem 3.A. Then either $\kappa(S) = 1$ or S is birational to an abelian surface.*

Therefore we are left with ruling out the case $\kappa(S) = 1$.

Since we are interested in classifying S birationally, we can assume S to be a minimal surface. As by Theorem 1.8.1, for a minimal surface S with $\kappa(S) = 1$ in characteristic neither 2 nor 3, the Stein factorisation of the Iitaka fibration yields a relatively minimal elliptic fibration

$$f : S \longrightarrow B \tag{3.2}$$

onto a non-singular curve B. We will study this hypothetical fibration in order to find a contradiction and show that it cannot exist.

3.1.1 Relations among Numbers of the Elliptic Fibration

From our previous discussion, to prove that an S such as in Theorem 3.A is an abelian surface, we have to prove that there cannot exist an elliptic fibration $f : S \to B$. We begin by showing that if such a fibration existed the genus of the base curve would be bounded since we have fixed $h^1(S, \mathcal{O}_S)$.

Lemma 3.1.2. *Let $f : S \to B$ be a quasi-elliptic surface or an elliptic surface. Then*

$$g(B) \leq \dim \mathrm{H}^1(S, \mathcal{O}_S) \tag{3.3}$$

Proof. Indeed, we consider the Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(B, R^q f_* \mathcal{O}_S) \qquad \Longrightarrow \qquad E^{p+q} = \mathrm{H}^{p+q}(S, \mathcal{O}_S) \tag{3.4}$$

By Proposition 1.2.1 the sheaves $R^q f_* \mathcal{O}_S$ are trivial except possibly when $0 \leq q \leq 2$, and so the page two of the above spectral sequence has zeroes except for a rectangle consisting of two objects in the horizontal direction and three in the vertical one. It follows that at page two there are no differentials between two nonzero vector spaces, and thus the sequence degenerates here and therefore $\mathrm{H}^1(S, \mathcal{O}_S)$ can be split as:

$$\begin{aligned} \mathrm{H}^1(S, \mathcal{O}_S) &= \mathrm{H}^0(B, R^1 f_* \mathcal{O}_S) \oplus \mathrm{H}^1(B, f_* \mathcal{O}_S) \\ &= \mathrm{H}^0(B, L) \oplus \mathrm{H}^0(B, T) \oplus \mathrm{H}^1(B, \mathcal{O}_B) \end{aligned} \tag{3.5}$$

and $\mathrm{H}^1(B, \mathcal{O}_B)$ has dimension $g(B)$ since B is smooth. The conclusion follows. □

The following lemma provides some useful relations among numbers linked to objects on the base curve B. From now on in this chapter, let L and T be sheaves such that

$$R^1 f_* \mathcal{O}_S \simeq L \oplus T, \tag{3.6}$$

as in the standard notation introduced in (1.17).

Lemma 3.1.3. *Let $f : S \to B$ be a minimal elliptic surface with $h^1(S, \mathcal{O}_S) = 2$ and $P_1(S) = 1$. Then*

(i) $h^0(B, L) - \deg L + g(B) = 2;$

(ii) $\deg L = -h^0(B, T) \leq 0;$

(iii) $h^1(B, L) = 1;$

(iv) $g(B) \leq 2.$

Proof. Given those invariants, $\chi(S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - 2 + 1 = 0$. Therefore, from Theorem 1.8.3 it follows that

$$\deg L = -\mathrm{length}(T) = -h^0(B, T); \tag{3.7}$$

and this proves (ii).

From Lemma 3.1.2, one immediately has (iv).

From (3.5) and (ii) one gets (i). By (i) together with the Riemann-Roch Theorem for curves

$$h^0(B,L) - \deg L + g(B) = h^1(B,L) + 1, \tag{3.8}$$

one gets (iii). □

3.1.2 An Inequality for the Plurigenera

When, with the notation of Theorem 1.8.3, $\omega_S = \mathcal{O}\Big(\sum a_\alpha P_\alpha\Big)$, we get a lower bound for the plurigernera depending on the number of the fibres that give rise to an a_α as large as possible (i.e. either multiple fibres that are not wild, or *strange fibres*: wild fibres with $a_\alpha = m_\alpha - 1$, following the terminology of [KU85]).

Lemma 3.1.4. *Let $f : S \to B$ be a minimal elliptic surface. Assume that, with the notation of Theorem 1.8.3, $\omega_S = \mathcal{O}\Big(\sum a_\alpha P_\alpha\Big)$.*

Let I be the set of the indices α such that $a_\alpha = m_\alpha - 1$. If $n, t \in \mathbb{N}_{>0}$ are such that $t \le \frac{n}{2}$, then

$$P_n(S) \ge |I| \cdot t + 1 - g(B), \tag{3.9}$$

where $|I|$ is the cardinality of I.

Proof. Assume that

$$h^0(S, \omega_S^{\otimes n}) \ge h^0\left(S, \mathcal{O}\left(\sum_{\alpha \in I} t m_\alpha P_\alpha\right)\right). \tag{3.10}$$

Then

$$\begin{aligned}
P_n(S) = h^0(S, \omega_S^{\otimes n}) &\ge h^0\left(S, \mathcal{O}\left(\sum_{\alpha \in I} t m_\alpha P_\alpha\right)\right) \\
&= h^0\left(S, f^*\left(\mathcal{O}\left(\sum_{\alpha \in I} t b_\alpha\right)\right)\right) \\
&= h^0\left(B, \mathcal{O}\left(\sum_{\alpha \in I} t b_\alpha\right)\right) && \text{(connected fibres)} \\
&= h^1\left(B, \mathcal{O}\left(\sum_{\alpha \in I} t b_\alpha\right)\right) + \sum_{\alpha \in I} t + 1 - g(B)
\end{aligned}$$

by Riemann-Roch for curves, thence the statement of the lemma.

It remains to verify (3.10). That is true if

$$nK_S \ge \sum_{\alpha \in I} n(m_\alpha - 1) P_\alpha \ge \sum_{\alpha \in I} t m_\alpha P_\alpha,$$

and in turn that holds if, for every α,

$$n(m_\alpha - 1) \geq t m_\alpha,$$

which, being $n > t$, is equivalent to

$$m_\alpha \geq 1 + \frac{t}{n-t}.$$

Since $m_\alpha \geq 2$, this latter inequality is satisfied if

$$\frac{t}{n-t} \leq 1,$$

that is, again by $n > t$, when $t \leq \frac{n}{2}$. $\square$

3.2 Proof of Theorem 3.A

From what said up to now, in order to prove Theorem 3.A we only have to show that a minimal surface S with $P_1(S) = P_4(S) = 1$ and $h^1(S, \mathcal{O}_S) = 2$ and Kodaira dimension one cannot have an elliptic fibration $f:\ S \to B$ onto a curve of genus $g(B) \leq 2$.

As a first reduction, $g(B)$ cannot be 2 because of $P_2(S) = 1$. Indeed, given an elliptic fibration $f:\ S \to B$ with $P_2(S) = 1$,

$$\begin{aligned}
1 = h^0(S, \omega_S^{\otimes 2}) &= h^0(S, f^*(L^{-1} \otimes \omega_B)^2 \otimes \text{eff.divisor}) && \text{(Theorem 1.8.3)}\\
&\geq h^0(S, f^*(L^{-2} \otimes \omega_B^2))\\
&= h^0(B, L^{-2} \otimes \omega_B^2)\\
&\geq h^0(B, L^{-2} \otimes \omega_B^2) - h^1(B, L^{-2} \otimes \omega_B^2)\\
&= 1 - g(B) + \deg(L^{-2} \otimes \omega_B^2) && \text{(Riemann} - \text{Roch)}\\
&= 1 - g(B) + 4g(B) - 4 - 2\deg L\\
&= 3g(B) - 3 - 2\deg L\\
&= 3 \cdot 2 - 3 - 2 \cdot 0\\
&= 3
\end{aligned}$$

where the second to last equality holds if $g(B) = 2$ because of (i) of Lemma 3.1.3: it must be $h^0(B, L) - \deg L = 0$, and by (ii) of Lemma 3.1.3 both $h^0(B, L)$ and $-\deg L$ are non-negative, therefore $\deg L = 0$.

The following sections deal with the remaining cases $g(B) = 0$ and $g(B) = 1$. In both these cases the relations of Lemma 3.1.3 together with the formulas in Theorem 1.8.3 allow to write ω_S as sheaf associated to a particular effective divisor coming from the multiple fibres of the elliptic fibration.

3.2.1 Genus of the Base Curve Equals One

Assume to have an elliptic fibration $f : S \to B$, where S has the birational invariants fixed as in Theorem 3.A. The purpose of this section is to show that the genus of B cannot be one. If $g(B) = 1$, by (i) of Lemma 3.1.3, one gets that either $h^0(B,L) = 0$ and $\deg L = -1$ or $h^0(B,L) = 1$ and $\deg L = 0$.

The first case can be written off because of $P_2(S) = 1$ in a similar fashion to what has been done to rule out $g(B) = 2$. Indeed, again by Theorem 1.8.3, projection formula and the Riemann-Roch Theorem for curves:

$$1 = h^0(S, \omega_S^{\otimes 2}) \geq 3g(B) - 3 - 2\deg L = 3 \cdot 1 - 3 - 2 \cdot (-1) = 2. \tag{3.11}$$

So, assume $h^0(B,L) = 1$ and $\deg L = 0$. Recall from [Ha77, IV, Lemma 1.2] that

Lemma 3.2.1. *Let C be a smooth projective curve over an algebraically closed field k. Let D be a divisor on C such that $h^0(C, \mathcal{O}_C(D)) \neq 0$.*
Then $\deg D \geq 0$, and if equality holds we get $\mathcal{O}_C(D) \simeq \mathcal{O}_C$.

It follows that L must be $\mathcal{O}_B$. Since $h^0(B,T) = -\deg L = 0$ and T is torsion, it follows that $R^1 f_* \mathcal{O}_S = \mathcal{O}_B$ and that there are no wild fibres.
By Theorem 1.8.3,

$$\omega_S = \mathcal{O}\Big(\sum (m_\alpha - 1) P_\alpha\Big). \tag{3.12}$$

If there were at least two multiple fibres, Lemma 3.1.4 would imply that $P_2(S) \geq 2$. If there were no multiple fibres the canonical bundle of S would be trivial, so $\kappa(S) = 0$.

Remark 3.2.2. Without asking $P_4(S) = 1$, the only case left open for $g(B) = 1$ is when the fibration has exactly one multiple fibre which is not wild. Then Lemma 3.1.4 yields $P_n(S) \geq 2$ for $n \in \mathbb{N}_{\geq 4}$.

3.2.2 Genus of the Base Curve Equals Zero

From what previously done, a surface S which satisfies the conditions of Theorem 3.A, is either birational to an abelian surface or it has $\kappa(S) = 1$ and there is an elliptic fibration $f : S \to B$ onto a smooth curve B with $g(B) = 0$. Therefore, Therem 3.A is proven on condition of proving the following:

Proposition 3.2.3. *Let S be a minimal surface over an algebraically closed field of characteristic $p > 5$ with $P_1(S) = P_2(S) = 1$, $h^1(S, \mathcal{O}_S) = 2$ and Kodaira dimension 1. Then the elliptic fibration of S cannot be onto a curve of genus 0.*

Having fixed $h^1(S, \mathcal{O}_S)$, $P_1(S)$ and characteristic $p > 3$, by (i) of Lemma 3.1.3 there would be three possible cases:

1. $h^0(B,L) = 2$, $\deg L = 0$. By Lemma 3.2.1, $L \simeq \mathcal{O}_B$, but then $h^0(B,L) = 1$, contradiction.

2. $h^0(B,L) = 1$, $\deg L = -1$. Impossible, an effective divisor has positive degree.

3. $h^0(B,L) = 0$, $\deg L = -2$. Since the genus is 0, $L \simeq \mathcal{O}(-2) \simeq \omega_B$. Therefore, by Theorem 1.8.3,

$$\omega_S = \mathcal{O}\Big(\sum a_\alpha P_\alpha\Big). \tag{3.13}$$

So, if such a surface as in the proposition existed, we would be in the last case.

If no fibre appeared in the expression of the canonical bundle, then the latter would be trivial, and the Kodaira dimension would not be 1. If there was at least one multiple fibre not wild, then, by Lemma 3.1.4, $P_2(S) \geq 2$, contradicting $P_2(S) = 1$.
By (ii) of Lemma 3.1.3 and $\deg L = -2$, there can be at most two wild fibres. So it remains to exclude the two cases when there is exactly one multiple fibre which is wild and when there are exactly two multiple fibres, both wild. These two cases are addressed by the next two lemmas. Both of them are based on the following fact, used many times in [KU85]:

Lemma 3.2.4. *Let $f : S \to B$ be an elliptic surface. Then, with the notation already introduced,*

$$h^0(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}) = 1 + h^0(B, T \otimes_{\mathcal{O}_B} k(b_\alpha)) \tag{3.14}$$

Proof. As mentioned in the proof of [KU85, Lemma 1.2], we have

$$h^0(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}) = h^1(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}).$$

Indeed, by

$$0 \longrightarrow \mathcal{O}_S(-m_\alpha P_\alpha) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{m_\alpha P_\alpha} \longrightarrow 0 \tag{3.15}$$

one gets $\chi(\mathcal{O}_S(-m_\alpha P_\alpha)) + \chi(\mathcal{O}_{m_\alpha P_\alpha}) = \chi(\mathcal{O}_S)$, and the claim $\chi(\mathcal{O}_{m_\alpha P_\alpha}) = 0$ follows from the Riemann-Roch theorem for surfaces and the fact that the fibres of f are curves of canonical type:

$$\chi(\mathcal{O}_S(-m_\alpha P_\alpha)) = \chi(\mathcal{O}_S) + \frac{-m_\alpha P_\alpha \cdot (-m_\alpha P_\alpha - K_S)}{2} = \chi(\mathcal{O}_S). \tag{3.16}$$

Then the statement of the Lemma holds because of the equalities

$$\begin{aligned} h^0(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}) &= h^1(m_\alpha P_\alpha, \mathcal{O}_{m_\alpha P_\alpha}) = h^0(B, R^1 f_* \mathcal{O}_S \otimes k(b_\alpha)) \\ &= h^0(B, L \otimes k(b_\alpha)) + h^0(B, T \otimes k(b_\alpha)) = 1 + h^0(B, T \otimes k(b_\alpha)), \end{aligned}$$

where the second equality holds by Theorem 1.2.2. □

Lemma 3.2.5. *Let S be a minimal surface over an algebraically closed field of characteristic $p > 3$ with $P_1(S) = P_2(S) = 1$, $h^1(S, \mathcal{O}_S) = 2$ and Kodaira dimension 1. Assume that the elliptic fibration of S is onto a curve B of genus 0. Then it is not possible to have exactly 2 multiple fibres, both wild.*

Proof. Let m_1P_1 and m_2P_2 be the two wild fibres. By Lemma 3.2.4, $h^0(m_iP_i,\mathcal{O}_{m_iP_i}) = 2$. So one can apply Lemma 1.8.4 and see that

$$a_i = \begin{cases} m_i - 1 \\ \text{or} \\ m_i - \nu_i - 1, \end{cases} \tag{3.17}$$

where ν_i are positive integers defined in (1.15). If for at least one of the two fibres the first equality held, with $P_2(S)$ one would get a contradiction applying Lemma 3.1.4. So, one can assume that for both fibres

$$a_i = p^{\delta_i}\nu_i - \nu_i - 1, \tag{3.18}$$

where $m_i = p^{\delta_i}\nu_i$ as seen in (1.16), and $\delta_i \geq 1$ because $a_i \geq 0$. It is worth noticing that at least one of the two a_i must be strictly positive, otherwise by (3.13) ω_S would be trivial, impossible because of $\kappa(S) = 1$.

The goal here is to reach a contradiction by showing that this would lead to $P_2(S) \geq 2$.

First, for an i such that $a_i \neq 0$,

$$2a_i = 2(p^{\delta_i}\nu_i - \nu_i - 1) \geq p^{\delta_i}\nu_i = m_i.$$

Indeed, that inequality is equivalent to:

$$2 \geq \frac{p^{\delta_i}\nu_i}{p^{\delta_i}\nu_i - \nu_i - 1} = \frac{p^{\delta_i}\nu_i - \nu_i - 1 + \nu_i + 1}{p^{\delta_i}\nu_i - \nu_i - 1} = 1 + \frac{\nu_i + 1}{p^{\delta_i}\nu_i - \nu_i - 1},$$

which is true if $\frac{\nu_i+1}{p^{\delta_i}\nu_i-\nu_i-1} \leq 1$, that is: $p^{\delta_i}\nu_i - 2\nu_i - 2 \geq 0$. Since $\nu_i,\delta_i \geq 1$, that equality is satisfied if $p^{\delta_i} - 4 \geq 0$ holds. Since the characteristic is neither 2 nor 3, that inequality is satisfied.

Having proved above that $2a_i \geq m_i$, for a fixed i we have that

$$2K_S \geq 2a_iP_i \geq m_iP_i \tag{3.19}$$

and so

$$h^0(S,\omega_S^{\otimes 2}) \geq h^0(S,\mathcal{O}(m_iP_i)) \tag{3.20}$$

then, similarly to the proof of Lemma 3.1.4,

$$\begin{aligned} P_2(S) = h^0(S,\omega_S^{\otimes 2}) &\geq h^0(S,\mathcal{O}(m_iP_i)) \\ &= h^0(S,f^*(\mathcal{O}(b_i))) \\ &= h^0(B,\mathcal{O}(b_i)) \\ &= h^1(B,\mathcal{O}(b_i)) + 1 + 1 - g(B) \geq 2 \end{aligned}$$

which contradicts $P_2(S) = 1$. □

Finally, to conclude the proof of Proposition 3.2.3 we have to exclude the case of an elliptic fibration with exactly one multiple fibre which is wild, and Lemma 3.2.6 deals precisely with that case.

Lemma 3.2.6. *Let S be a minimal surface over an algebraically closed field of characteristic $p > 5$ with $P_1(S) = P_2(S) = 1$, $h^1(S, \mathcal{O}_S) = 2$ and Kodaira dimension 1. Assume that the elliptic fibration of S is onto a curve B of genus 0. Then it is not possible to have exactly 1 multiple fibre which is wild.*

To prove Lemma 3.2.6 we will need [KU85, Corollary 4.2], which we recall here:

Lemma 3.2.7. *Let $f : S \to \mathbb{P}^1$ be an elliptic surface with $\chi(S) = 0$ and only one multiple fibre. We define m and ν coherently with our usual notation of (1.14) and (1.15) by dropping the indices.*
Then the unique multiple fibre is a a wild fibre and $\nu = 1$, so that $m = p^\delta$ for $\delta \in \mathbb{N}_{>0}$.

Now we are ready to prove Lemma 3.2.6.

Proof. Let mP be the wild fibre over the point b of B, $a \geq 1$ the coefficient of P in (3.13). Then

1. The hypotheses of Lemma 3.2.7 are satisfied. Therefore $m = p^\delta$ (with δ positive integer) and $\nu = 1$.
2. Since T has length 2 and is supported only on b, by Lemma 3.2.4 $h^0(mP, \mathcal{O}_{mP}) = 1 + 2 = 3$.

By these two facts, we can apply Lemma 1.8.4 and we get that one of the following equalities must hold:

$$a = \begin{cases} p^\delta - 1 \\ p^\delta - 2 \\ p^\delta - 3 \\ p^\delta - p - 2 \end{cases} \tag{3.21}$$

(i) If $a = p^\delta - p - 2$. Since $a \geq 0$, $\delta \geq 2$. One can use the strategy of Lemma 3.1.4 and get a contradiction with $P_2(S) = 1$ if

$$2(p^\delta - p - 2) \geq p^\delta,$$

that is

$$2 \geq \frac{p^\delta}{p^\delta - p - 2} = \frac{p^\delta - p - 2 + p + 2}{p^\delta - p - 2} = 1 + \frac{p+2}{p^\delta - p - 2}.$$

That is true if $\frac{p+2}{p^\delta - p - 2} \leq 1$, equivalently:

$$p^\delta - 2p - 4 \geq 0.$$

Since $\delta \geq 2$, this last inequality is satisfied when the characteristic p is neither 2 nor 3.

(ii) If $a = p^\delta - s$ with $s = 1, 2$ or 3. The contradiction with $P_2(S) = 1$ can be obtained using the strategy of Lemma 3.1.4 if

$$2(p^\delta - s) \geq p^\delta,$$

equivalently:

$$2 \geq \frac{p^\delta}{p^\delta - s} = \frac{p^\delta - s + s}{p^\delta - s} = 1 + \frac{s}{p^\delta - s}.$$

Therefore one needs $\frac{s}{p^\delta - s} \leq 1$, that is $p^\delta - 2s \geq 0$. This is true if $s = 1$ or 2, while if $s = 3$ it is false in a handful of cases if $p = 2$ or 3, and also when $p = 5$ and $\delta = 1$.

□

Remark 3.2.8. The proof of Lemma 3.2.6 does not go through in characteristic 5 only because of the possible existence of a fibration with one wild fibre $5P$ of multiplicity 5, order 1, and such that $\omega_S = \mathcal{O}(2P)$.

Actually, modifying point (ii) in the proof of Lemma 3.2.6 by checking for which integers n we have

$$n(5^\delta - s) \geq 5^\delta, \tag{3.22}$$

we get the following remark:

Remark 3.2.9. Let S be a minimal surface over an algebraically closed field of characteristic 5 with $P_1(S) = P_n(S) = 1$ for a fixed $n \in \mathbb{N}_{\geq 3}$, $h^1(S, \mathcal{O}_S) = 2$ and Kodaira dimension 1. Assume that the elliptic fibration of S is onto a curve B of genus 1. Then it is not possible to have exactly 1 multiple fibre which is wild.

Having proved Proposition 3.2.3 and having recovered the case of $p = 5$ in Remark 3.2.8 and Remark 3.2.9, Theorem 3.A holds.

As a final remark, one could have proved Proposition 3.2.3 in a slightly different fashion by following the computations in the proof of [KU85, Theorem 5.2] and specializing them to the case at hand, thus getting the statement of Proposition 3.2.10 (which is stronger than Proposition 3.2.3).
The case of elliptic surface in characteristic 5 that did not allow to state Lemma 3.2.6 for $p \geq 5$ is the same that forces in Proposition 3.2.10 to distinguish between $p \geq 5$ and $p \geq 7$, and it is the same case that has been dealt with in Remark 3.2.8 and Remark 3.2.9.

Proposition 3.2.10. *Let $f : S \to B$ be an algebraic elliptic surface in characteristic $p \geq 5$ with $\kappa(S) = 1$, $P_1(S) = 1$, $h^1(S, \mathcal{O}_S) = 2$ then $|mK_S|$ gives the unique structure of the elliptic surface when $m \geq 3$. If $p \geq 7$,then the same holds for $m \geq 2$.*

Proof. Since $g(B) = 0$, $\chi(\mathcal{O}_S) = 0$ and (as seen in (ii) of Lemma 3.1.3) $t = h^0(B, T) = -\deg L = 2$, one is in the situation (III) of the proof of [KU85, Theorem 5.2], that is, one gets that, for $m \in \mathbb{N}$, $|mK_S|$ gives the unique structure of the elliptic surface if, with the notation of Theorem 1.8.3,

$$\mathfrak{D} := \sum_\alpha \left\lfloor \frac{m a_\alpha}{m_\alpha} \right\rfloor \geq 1. \tag{3.23}$$

It is directly stated and proved in [KU85] that if the elliptic fibration $f: S \to B$ has at least one tame fibre then (3.23) is satisfied for $m \geq 2$ (and by their computations, the same is true if there is at least one wild fibre of strange type, i.e a wild fibre with $a_\alpha = m_\alpha - 1$). So one can reduce to the case where the only multiple fibres are not tame and not wild of strange type. By $h^0(B,T) = 2$, there are at most two such fibres.

If there is exactly one wild fibre not of strange type then, following [KU85], only three cases are possible:

(i) $a_1 = m_1 - \nu_1 - 1$,

(ii) $a_1 = m_1 - 2\nu_1 - 1$,

(iii) $a_1 = m_1 - (p+1)\nu_1 - 1$.

As shown in [KU85], in case (i) $\mathfrak{D} = \lfloor m(1 - \frac{1}{p^\gamma} - \frac{1}{p^\gamma \nu_1}) \rfloor$, with γ, $\nu \geq 1$. Here, taking $p \geq 5$, one has $\mathfrak{D} \geq \lfloor m(1 - \frac{1}{5} - \frac{1}{5}) \rfloor = \lfloor m\frac{3}{5} \rfloor$, and so (3.23) is satisfied for $m \geq 2$. Similarly, in case (ii) $\mathfrak{D} = \lfloor m(1 - \frac{2}{p^\gamma} - \frac{1}{p^\gamma \nu_1}) \rfloor$, with γ, $\nu \geq 1$. If $p \geq 5$, then $\mathfrak{D} \geq \lfloor m(1 - \frac{2}{5} - \frac{1}{5}) \rfloor = \lfloor m\frac{2}{5} \rfloor$, and so (3.23) is satisfied for $m \geq 3$. If $p \geq 7$, then $\mathfrak{D} \geq \lfloor m(1 - \frac{2}{7} - \frac{1}{7}) \rfloor = \lfloor m\frac{4}{7} \rfloor$, and so (3.23) is satisfied for $m \geq 2$.
In case (iii), since $m_1 = p^\gamma \nu_1$,

$$\mathfrak{D} = \left\lfloor m \frac{p^\gamma \nu_1 - (p+1)\nu_1 - 1}{p^\gamma \nu_1} \right\rfloor = \left\lfloor m \left(1 - \frac{1}{p^{\gamma-1}} - \frac{1}{p^\gamma} - \frac{1}{p^\gamma \nu_1}\right) \right\rfloor \tag{3.24}$$

with $\gamma \geq 2$ because of $a_1 > 0$. If $p \geq 5$,

$$\mathfrak{D} \geq \left\lfloor m \left(1 - \frac{1}{5} - \frac{1}{5^2} - \frac{1}{5^2}\right) \right\rfloor = \left\lfloor m \frac{18}{25} \right\rfloor \tag{3.25}$$

and therefore (3.23) is satisfied for $m \geq 2$.

If there are exactly two wild fibres, it is shown directly in [KU85] that (3.23) holds when $m \geq 4$, but since in that case the a_α are exactly those of case (i) just above, taking $p \geq 5$ (3.23) is satisfied for $m \geq 2$. □

3.3 Comparison with the Characterisation in Characteristic Zero

By the results of [CH01], in characteristic zero the conditions $P_1(S) = P_2(S) = 1$ and $h^1(S, \mathcal{O}_S) = 2$ are enough to pin down abelian surfaces amongst all other surfaces.

The conditions $P_1(S) = P_4(S) = 1$ that we have fixed (and that were also used by Enriques in his result in [En1905]) imply that $P_2(S) = 1$. Our statement would be stronger if, as in characteristic zero, we could fix $P_1(S) = P_2(S) = 1$ and make no requirement about $P_4(S)$. By the work done here, if one asks only for $P_1(S) = P_2(S) = 1$ there would be only a few specific cases of surfaces that might cause the conclusion of the theorem to fail. We use the conditional tense as we could not construct those surfaces.

Indeed, the condition $P_4(S) = 1$ is used to exclude the following surfaces:

1. an elliptic fibration onto a curve of genus zero when the characteristic of the base field is 5 (see Remark 3.2.8 and Remark 3.2.9);

2. an elliptic fibration over a curve of genus one with exactly one multiple fibre which is not wild (see Remark 3.2.2).

In particular, in the first of the two cases mentioned, since $\chi(S) = 0$ and the elliptic fibration would onto a $\mathbb{P}^1$, Lemma 1.8.5 would imply that $\dim \mathrm{Alb}(S) = 1$, therefore $\dim \mathrm{Alb}(S) < h^1(S, \mathcal{O}_S)$, and this would not happen in characteristic zero. In the second case, i.e. the fibration onto a curve of genus one, Lemma 1.8.5 implies $\mathrm{Alb}(S)$ could be either a surface or a curve.

3.4 Characteristic 2 and 3

The assumption that $p \geq 5$ is needed when $\kappa(S) = 0$ to rule out the possibility of S being birational to an hyperelliptic (or a quasi-hyperelliptic) surface, and to settle the case $\kappa(S) = 1$. In the latter situation, a part of what is done in this work goes through when $p = 2, 3$ (also considering the case of quasi-elliptic surfaces), and the computations made here allow only to say that, if such an S exists, then it must be birational to a (quasi-)elliptic surface fibred over a curve B of genus either 1 or 0. If $g(B) = 1$, then the elliptic fibration has exactly one multiple fibre that is not wild, and if $g(B) = 0$, then the fibration has either one or two multiple fibres, both wild. We were not able to construct such examples of surfaces.

We observe that the cases of $\kappa(S) = 1$ could be ruled out by fixing some more plurigenera: indeed, Catanese and Li showed that for a surface S of Kodaira dimension one there exist an integer $n \leq 8$ such that $P_n(S) \geq 2$ ([CL19, Main Theorem]). Nevertheless, this would not rule out the possibility of S being a quasi-hyperelliptic surface since, as as seen in Table 1.1, those surfaces have all plurigenera equal to one.

3.5 K3 Surfaces

With the methods used to prove Theorem 3.A it is also possible to prove a birational characterisation of K3 surfaces in any characteristic.

Theorem 3.5.1. *Let S be a smooth projective surface over an algebraically closed field of any characteristic. If*

$$P_1(S) = P_2(S) = 1, \qquad \dim \mathrm{H}^1(S, \mathcal{O}_S) = 0, \tag{3.26}$$

then S is birational to a K3 surface.

Proof. Exactly as before, the Kodaira dimension of S is neither $-\infty$ nor, by Corollary 1.6.5, two.

If $\kappa(S) = 0$, then by Table 1.1 S must be birational to a K3.

Therefore we have to rule out the case $\kappa(S) = 1$. By contradiction, assume $\kappa(S) = 1$. Without loss of generality, assume S minimal. Such an S would admit an elliptic fibration (or a quasi-elliptic fibration)

$$f : S \longrightarrow B.$$

For this elliptic fibration we use again the notation introduced in Section 1.8.

As seen in Lemma 3.1.2 and its proof, $h^1(S, \mathcal{O}_S) = h^0(B, L) + h^0(B, T) + h^1(B, \mathcal{O}_B)$. Then the fact that $h^1(S, \mathcal{O}_S) = 0$ yields $h^0(B, L) = h^0(B, T) = g(B) = 0$. By Theorem 1.8.3

$$\begin{aligned} \deg(L^{-1} \otimes \omega_B) &= 2g(B) - 2 + \chi(\mathcal{O}_S) + \mathrm{length}(T) \\ &= -2 + 2 + h^0(B, T) = 0. \end{aligned}$$

Therefore the Riemann-Roch Theorem for curve yields

$$\begin{aligned} h^0(L^{-1} \otimes \omega_B) - h^1(L^{-1} \otimes \omega_B) &= \deg(L^{-1} \otimes \omega_B) - g(B) + 1 \\ &= 0 - 0 + 1. \end{aligned}$$

In particular, since $h^0(L^{-1} \otimes \omega_B) \neq 0$ and $\deg(L^{-1} \otimes \omega_B) = 0$ by Lemma 3.2.1 we get that $L^{-1} \otimes \omega_B \simeq \mathcal{O}_B$.

Since $h^0(B, T) = 0$, there are no wild fibres, and therefore, by Theorem 1.8.3,

$$\omega_S = \mathcal{O}\big(\textstyle\sum (m_\alpha - 1) P_\alpha\big).$$

So, ω_S is of the right form to apply Lemma 3.1.4 with $n = 2$ and $t = 1$: if there were at least one multiple fibre, we would have $P_2(S) \geq 2$, which would contradict the hypotheses in the statement. Therefore there are no multiple fibres and it must be $\omega_S \simeq \mathcal{O}_S$. This is conflict with the absurd hypothesis $\kappa(S) = 1$. Therefore $\kappa(S) \neq 1$ and S must be a K3 surface. □

Roberto Laface brought to my attention that, actually, the characterisation of K3 surfaces is an easy and well-known exercise we solve below for the sake of completeness:

Alternative Proof of Theorem 3.5.1. We assume S minimal. Observe that $K_S^2 = 0$ since, as in the proof of Theorem 3.5.1, $\kappa(S) \neq -\infty, 2$. By the hypotheses we get $\chi(S) = 2$ and, therefore, by the Riemann-Roch Theorem,

$$\begin{aligned} h^0(S, \mathcal{O}_S(-K_S)) - h^1(S, \mathcal{O}_S(-K_S)) + P_2(S) &= \chi(\mathcal{O}_S(-K_S)) \\ &= \chi(S) + \frac{(-K_S)^2 + K_S^2}{2} \\ &= 2 + K_S^2 \\ &= 2 \end{aligned} \tag{3.27}$$

By the above equation it must be $h^0(S, \mathcal{O}_S(-K_S)) \geq 1$. This fact together with $P_1(S) = 1$ implies that the canonical bundle must be trivial, and that $\kappa(S) = 0$. By Table 1.1, S must be birational to a K3 surface. □

Chapter 4

Irregular Surfaces with Small Invariants

In this chapter we consider a problem which is similar to the one we dealt with in the previous chapter, namely, a problem of classification of surfaces with fixed birational invariants.

As seen in Theorem 1.6.3, for a surface S of general type one has that $\chi(S) > 0$. The case $\chi(S) = 1$ is therefore a limit case. In characteristic zero, by the Bogomolov-Miyaoka-Yau inequality, all such surfaces satisfy $K_S^2 \leq 9$, and by an inequality by Debarre it turns out that $p_g(S) = h^1(S, \mathcal{O}_S) \leq 4$. This latter inequality holds in characteristic p if one adds some conditions ($p \neq 2$ and S liftable to $W_2(k)$), as under these assumptions Langer showed that the Bogomolov-Miyaoka-Yau inequality still holds ([La15, Theorem 13]); also, the inequality by Debarre can be substituted by another one that we will state later on (Theorem 4.1.1).

In characteristic zero one sees in the literature that, roughly speaking, a higher geometric genus among the possible five constitutes a constraint on the geometry of the minimal model of the surface S. For example, not all minimal models of the surfaces with $p_g(S) = h^1(S, \mathcal{O}_S) = 0$ have been classified, and among the possible ones there are numerical Godeaux surfaces and Campedelli surfaces. By contrast, Beauville in [Be82] proved that the minimal model of a surface of general type with $p_g(S) = h^1(S, \mathcal{O}_S) = 4$ is always isomorphic to the product of a curve of genus two and one of genus at least two.

In characteristic p, $p_g(S) = h^1(S, \mathcal{O}_S) = 4$ is not necessarily a limit case. However, Wang in [Wa17] showed that if the characteristic is at least 11, and S is mAd, lifts to $W_2(k)$, its Picard variety has no supersingular factors, its Albanese map is separable and $\dim \mathrm{Alb}(S) = 4$, then the minimal model of S is isomorphic to a product of curves of genus two.

The case $p_g(S) = h^1(S, \mathcal{O}_S) = 3$ in characteristic zero was solved independently by Hacon and Pardini in [HP02] and by Pirola in [Pi02]; both articles built on the work of [CCM98]:

Theorem 4.0.1 ([HP02];[Pi02]). *Let S be a smooth minimal surface of general type defined over an algebraically closed field of characteristic zero; assume $p_g(S) = h^1(S, \mathcal{O}_S) = 3$. Then we are in either of the following cases:*

1. *$K_S^2 = 6$ and S is isomorphic to the symmetric product of a smooth curve C of genus 3;*
2. *$K_S^2 = 8$ and $S \simeq (C_1 \times C_2)/(\mathbb{Z}/2\mathbb{Z})$, where C_1, C_2 are smooth curves such that*
   - *$g(C_1) = 2$, C_1 has an elliptic involution σ_1,*
   - *$g(C_2) = 3$, C_2 has a free involution σ_2,*

   and $\mathbb{Z}/2\mathbb{Z}$ acts freely on $C_1 \times C_2$ via the involution $\sigma_1 \times \sigma_2$.

In particular, the second case arises whenever S admits an irrational pencil over a curve of genus 2 ([CCM98, Theorem (3.23)]). In the first case the Albanese morphism is birational onto its image, $\mathrm{Alb}(S)$ is a principally polarised abelian variety and $\mathrm{alb}_S(S)$ is a theta divisor.

Here we try to tackle the analogous problem in characteristic p. We get some partial results that hopefully will be improved in the future. We show the following:

Theorem 4.A. *Let S be a smooth minimal surface of general type over an algebraically closed field; assume S mAd, the Albanese morphism separable and $\mathrm{Pic}^0(S)$ reduced. Assume $p_g(S) = h^1(S, \mathcal{O}_S) = 3$. Consider a resolution of singularities Y of $\mathrm{alb}_S(S)$. Then*

1. *if $\mathrm{alb}_S(S)$ is ample, then Y is a surface of general type with $p_g(Y) = h^1(Y, \mathcal{O}_Y) = 3$ and its Picard variety is reduced;*
2. *if $\mathrm{alb}_S(S)$ is not ample, then $\kappa(Y) = 1$, $\dim \mathrm{Alb}(Y) = 3$, $\chi(Y) = 0$ and Y has one of the following sets of invariants:*

$h^0(Y,\omega_Y)$	$h^1(Y,\omega_Y)$	$\mathrm{Pic}^0(Y)$
2	3	reduced
3	4	non-reduced

   where the first set is the only one we would have in characteristic zero. The surface Y admits a structure of elliptic surface onto a curve of genus two.

   Moreover, there exists a smooth pencil with connected fibres on S onto a curve of genus two. The generic fibre of this pencil has arithmetic genus at least two, and it has genus two if and only if it is smooth. The geometric genus of this fibre is either one or two.
   Furthermore, if the generic fibre is smooth, then the pencil has constant moduli.

In characteristic zero, point 1. and 2. of Theorem 4.A correspond exactly to point 1. and 2. of Theorem 4.0.1 respectively, and ideally in future work one could hope to be able to elaborate the two cases in Theorem 4.A to get a statement more similar to that of Theorem 4.0.1.

In particular, it is possible that the relevant parts of the proof of [CCM98, Theorem (3.23)] could be simply translated into characteristic p starting from a pencil of genus two with constant, smooth modules on S, but this translation would require a technical insight we did not have the time to develop for this work.

In the spirit of the work of [Wa17] concerning the classification of surfaces with $p_g(S) = h^1(S, \mathcal{O}_S) = 4$, if we add some hypotheses we can use results of the Generic Vanishing theory and improve the result of point 1. of Theorem 4.A:

Theorem 4.B. *Let S be a smooth minimal surface of general type over an algebraically closed field k; assume S mAd, the Albanese morphism separable and $\mathrm{Pic}^0(S)$ reduced. Assume $p_g(S) = h^1(S, \mathcal{O}_S) = 3$. We also assume that $\mathrm{alb}_S(S)$ is an ample divisor and that it is normal, that S lifts to $W_2(k)$, and that $\mathrm{Pic}^0(S)$ has no supersingular factors. Then the Albanese morphism is birational onto its image.*

In order to complete the classification, at least in the case with all the hypotheses, one would like to show that $\mathrm{alb}_S(S)$ is a theta divisor.

In Section 4.1 we compute some inequalities concerning a surface of general type with $p_g(S) = h^1(S, \mathcal{O}_S) = 3$. In Section 4.2 we discuss matters related to separability of morphisms that we will need later on. Section 4.3 is dedicated to an introduction to the results of Generic Vanishing needed to prove Theorem 4.B; moreover, we explain the technical hypotheses we need to prove that result. In Section 4.4 we introduce the hypotheses we use to prove both Theorem 4.A and Theorem 4.B; moreover, we prove a Lemma we will have to use for both theorems.

We prove the first half of Theorem 4.A, together with Theorem 4.B, in Section 4.5, while we deal with the second half of Theorem 4.A in Section 4.6. Furthermore, in Section 4.6 we provide some information about the elliptic fibration on the resolution of singularities Y of the non-ample $\mathrm{alb}_S(S)$ that could help to rule out one of the two sets of invariant for Y we have in point 1. of Theorem 4.A.

Finally, in Section 4.7 we compare the solution of the problem in characteristic zero with what we show in positive characteristic, and we also discuss possible directions for future work aiming at improving the results we obtained.

4.1 Inequalities and Invariants

In characteristic zero a minimal surface of general type with $p_g(S) = h^1(S, \mathcal{O}_S) = 3$ has canonical divisor whose intersection number satisfies $6 \leq K_S^2 \leq 9$, and this fact is pivotal for classifying these surfaces both in [HP02] and [Pi02]. The well-known characteristic zero inequalities that lead to $6 \leq K_S^2 \leq 9$ are not available in positive characteristic, so we begin this section by finding bounds (possibly non-optimal) in this setting.

We will need the following inequality, which can be found in [Wa17, Proposition 5.1]:

Theorem 4.1.1. *Let S be a minimal projective surface of general type over an algebraically closed field of positive characteristic. Assume $p_g(S) \geq 2$. Then*

$$K_S^2 \geq 2p_g(S) + h^1(S, \mathcal{O}_S) - 4. \tag{4.1}$$

And so we can proceed with giving the bounds mentioned above.

Lemma 4.1.2. *Let S be a smooth minimal surface of general type over an algebraically closed field of positive characteristic. Assume that $p_g(S) = h^1(S, \mathcal{O}_S) = 3$. Then*

1. $-9 \leq c_2(S) \leq 7$,
2. $5 \leq K_S^2 \leq 21$.

Proof. Since $\chi(S) = 1 - 3 + 3 = 1$, the Noether's Formula becomes

$$12 = K_S^2 + c_2(S). \tag{4.2}$$

By Theorem 4.1.1, since $p_g(S) \geq 2$, we get

$$K_S^2 \geq 2p_g(S) + h^1(\mathcal{O}_S) - 4 = 2 \cdot 3 + 3 - 4 = 5.$$

Using (4.2), we immediately get also $c_2(S) \leq 7$.

In order to get the lower bound for $c_2(S)$, one can bound the Betti numbers. Indeed, by Theorem 1.5.4, we have that $\frac{b_1(S)}{2} \leq h^1(\mathcal{O}_S)$ and that $\frac{b_1(S)}{2}$ is the dimension of the Albanese variety; thus we obtain

$$b_1(S) \in \{0,2,4,6\},$$

and $\mathrm{Pic}^0(S)$ is reduced iff $b_1(S) = 6$. Also, since the Picard number of S is at least one, by Theorem 1.1.2 we get

$$b_2(S) \geq 1.$$

So, finally, $c_2(S) = 2 - 2b_1(S) + b_2(S) \geq 2 - 12 + 1 = -9$. And, again by (4.2), we get $K_S^2 \leq 21$. □

Lemma 4.1.3. *Let S be a smooth minimal surface of general type over an algebraically closed field. Assume*

$$p_g(S) = 3 \qquad \text{and} \qquad h^1(S, \mathcal{O}_S) = 3.$$

Let $f : S \to B$ be an irrational pencil on S (in particular, B is a smooth curve of genus $g(B) > 0$). Then $g(B) \leq \dim \mathrm{Alb}(S) \leq 3$.

Let F be the generic fibre. Then $p_a(F) (:= h^1(F, \mathcal{O}_F)) \geq 2$.
Moreover, if $p_a(F) \neq 2$ and $g(B) \geq 2$, all the fibres of S are singular.

Proof. To prove the condition on $g(B)$, consider the commutative diagram

$$\begin{tikzcd} S \arrow[r, "\mathrm{alb}_S"] \arrow[d, two heads, "f"'] & \mathrm{Alb}(S) \arrow[d, "\varphi"] \\ B \arrow[r, hook] & J(B) \end{tikzcd}$$

where $J(B)$ is the Jacobian of B. Since the diagram commutes, the image of φ, $\mathrm{Im}(\varphi)$, must contain B, and therefore it must generate $J(B)$. Moreover, since φ is a morphism of abelian varieties, $\mathrm{Im}(\varphi)$ is an abelian variety. It follows that $\mathrm{Im}(\varphi) = J(B)$. In particular,

$$\dim \mathrm{Alb}(S) \geq \dim J(B) = g(B).$$

We prove now that $p_a(F) \geq 2$. By the genus formula,

$$h^1(F, \mathcal{O}_F) = 1 + \frac{F^2 + K_S \cdot F}{2} = 1 + \frac{K_S \cdot F}{2}.$$

Immediately, $p_a(F) = 0$ would imply $K_S \cdot F = -2$, which is absurd since S is of general type (otherwise, this case can be ruled out by adjunction).
It cannot be $p_a(F) = 1$. Assume the contrary. Again by the genus formula, $K_S \cdot F = 0$. By Theorem 1.1.1, since $K_S^2 > 0$ and $F^2 = 0$, F is numerically trivial. But F is an integral curve and thus it cannot be numerically trivial.

If the generic fibre is smooth, then by Theorem 1.3.1 we have

$$1 = \chi(S) \geq (g(B) - 1)(g(F) - 1), \tag{4.3}$$

thus we cannot have either of the following:

1. $g(B) \geq 2$ and $g(F) \geq 3$;
2. $g(B) \geq 3$ and $g(F) \geq 2$,

and so the statement follows. □

Lemma 4.1.4. *Let X be a smooth projective variety, T a smooth projective variety such that alb_X admits a factorisation $X \xrightarrow{l} T \xrightarrow{r} \mathrm{alb}_X(X)$, where r is birational. Then $\dim \mathrm{Alb}(X) = \dim \mathrm{Alb}(T)$.*

Proof. One has a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{alb}_X"] \arrow[dr, "l"'] & & \mathrm{alb}_X(X) \arrow[r, hook] & \mathrm{Alb}(X) \arrow[d, bend right, "\alpha"'] \\ & T \arrow[ur, "r"] \arrow[rr, "\mathrm{alb}_T"] & & \mathrm{Alb}(T) \arrow[u, bend right, "\beta"'] \end{tikzcd}$$

where α and β are given by the universal property of the Albanese morphism.

Both α and β are morphisms of abelian varieties. Their images contain $\mathrm{alb}_T(T)$ and $\mathrm{alb}_X(X)$ respectively, and therefore their images must generate $\mathrm{Alb}(T)$ and $\mathrm{Alb}(X)$ respectively. Since α and β are morphisms of abelian varieties, their images must be abelian varieties, therefore both morphisms must be surjective, implying that $\dim \mathrm{Alb}(X) = \dim \mathrm{Alb}(T)$. $\square$

4.2 Separability and Differentials

In this section we prove first a result, that is known to experts, about pullbacks of differentials via separable morphisms. Then we explain how we use it to obtain a certain injective morphism of canonical bundles. Finally, we prove a result we will use in conjunction with the aforementioned injective morphism.

Thus we begin with the known result:

Lemma 4.2.1. *Let S and Y be smooth projective surfaces, $g : S \to Y$ a generically finite morphism. Then the induced map*

$$g^*\Omega^1_Y \longrightarrow \Omega^1_S \tag{4.4}$$

is injective if and only if g is separable.

We prove this lemma in a way that was pointed out to me by C. Liedtke. Before we proceed with the proof, we recall three results we will need in it.

The following lemma about the Kähler differentials of rings and their localisations can be found for example as [Sta19, Lemma 10.130.8].

Lemma 4.2.2. *Let $\varphi : A \to B$ be a morphism of rings. Then*

1. *for any multiplicative set $S \subset B$, we have $S^{-1}\Omega_{B/A} \simeq \Omega_{S^{-1}B/A}$;*
2. *for any multiplicative set $S \subset A$ such that $\varphi(S)$ is a set of invertible elements of B, we have $\Omega_{B/A} \simeq \Omega_{B/S^{-1}A}$.*

Also, we recall the following result of commutative algebra (see for example [Ha77, II Theorem 8.6A] and its proof):

Theorem 4.2.3. *Let K/E be a finite algebraic extension of fields. Then $\Omega_{K/E} \simeq 0$ if and only if K/E is a separable field extension.*

We recall the following characterization of dominant morphisms (see for example [Ha77, I. Exercise 3.3] or [Sta19, Lemma 29.8.6]):

Lemma 4.2.4. *Let $f : X \to W$ be a morphism of integral schemes. Then f is dominant if and only if the local ring map $\varphi^*_x : \mathcal{O}_{W,f(x)} \to \mathcal{O}_{X,x}$ is injective.*

And now we are ready to prove the lemma we started with.

Proof of Lemma 4.2.1. We consider the exact sequence

$$g^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{S/k} \longrightarrow \Omega^1_{S/Y} \longrightarrow 0 \tag{4.5}$$

of sheaves on S (see for example [Ha77, II. Proposition 8.11]).

Let ξ be the generic point of S. We claim that $(\Omega^1_{S/Y})_\xi \simeq 0$ if and only if g is separable. If the claim holds, then we have that $g^*\Omega^1_{Y/k} \to \Omega^1_{S/k}$ is surjective at the generic point. Since S and Y are smooth surfaces, $\Omega^1_{Y/k}$, $\Omega^1_{X/k}$ and $g^*\Omega^1_{Y/k}$ are locally free sheaves of rank 2. It follows that, if g is separable, the kernel of the map we are interested in is a torsion subsheaf of a locally free sheaf, and therefore it is trivial, so that $g^*\Omega^1_Y \to \Omega^1_S$ is injective.

To complete the proof, we need to prove our original claim. We make the computation locally, considering an induced morphism $g : \mathrm{Spec}B \to \mathrm{Spec}A$ of open affine subsets of S and Y. Let $\varphi : A \to B$ be the associated morphism of rings. By Lemma 4.2.4, since g is dominant, all the localisations of φ are injective, and therefore $\varphi : A \to B$ is itself injective.
We get

$$\begin{aligned}(\Omega^1_{S/Y})_\xi \simeq (\Omega^1_{\mathrm{Spec}B/\mathrm{Spec}A})_\xi \simeq (\widetilde{\Omega^1_{B/A}})_\xi \simeq (\Omega^1_{B/A})_{(0)} \\ \simeq \Omega_{B_{(0)}/A} \simeq \Omega_{B_{(0)}/A_{(0)}} \simeq \Omega_{K(S)/K(Y)},\end{aligned}$$

where the second isomorphisms follows for example from [Li06, Proposition 6.1.7] or [Ha77, II. Remark 8.9.2]; the fourth isomorphism follows from the first assertion of Lemma 4.2.2; the fifth from the second assertion of Lemma 4.2.2 keeping in mind that we have constructed B in such a way that all the non-zero elements A are sent to invertible elements of $B_{(0)}$ through the morphism induced by φ. Finally, $B_{(0)} = K(S)$ and $A_{(0)} = K(Y)$ because of [Ha77, II. Exercise 3.6]. □

Remark 4.2.5. In the situation of Lemma 4.2.1, if g is separable, then one gets an injective map

$$\omega_Y \hookrightarrow g_*\omega_S. \tag{4.6}$$

Indeed, having an injective map $g^*\Omega^1_Y \hookrightarrow \Omega^1_S$, we can follow the steps found in [Ti12, Proof of Proposition 5.2.4] to get the desired injective morphism. We rewrite those steps here for the convenience of the reader.

Starting from the injection $g^*\Omega^1_Y \hookrightarrow \Omega^1_S$, we obtain a map $g^*\omega_Y \hookrightarrow \omega_S$ involving the canonical bundes by taking determinants and using on the left-hand side the fact that pullbacks and determinants commute. We can then push-forward through g and use projection formula to get an injection $\omega_Y \otimes g_*\mathcal{O}_S \hookrightarrow g_*\omega_S$. Finally, we find the injection $\omega_Y \hookrightarrow g_*\omega_S$ by considering the composition

$$\omega_Y \hookrightarrow \omega_Y \otimes g_*\mathcal{O}_S \hookrightarrow g_*\omega_S,$$

where the first injection can be obtained by tensoring the inclusion $\mathcal{O}_Y \hookrightarrow g_*\mathcal{O}_S$ with the line bundle ω_Y.

Finally, we prove a last lemma we will use in conjunction with Remark 4.2.5.

Lemma 4.2.6. *Let S be smooth projective surfaces, Y a normal projective surface and let $g : S \to Y$ be a generically finite morphism. Then, for any $\alpha \in \mathrm{Pic}^0(Y)$ and $i \in \mathbb{Z}$,*

$$\mathrm{H}^i(S, \omega_S \otimes g^*\alpha) \simeq \mathrm{H}^i(Y, g_*(\omega_S \otimes g^*\alpha)). \tag{4.7}$$

In particular, $\chi(g_*\omega_S \otimes \alpha) = \chi(\omega_S \otimes g^*\alpha)$.

Proof. We use throughout this proof the fact that $g_*\omega_S \otimes \alpha \simeq g_*(\omega_S \otimes g^*\alpha)$ due to projection formula.

By the Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(Y, R^q g_*(\omega_S \otimes g^*\alpha)) \implies E^{p+q} = \mathrm{H}^{p+q}(S, \omega_S \otimes g^*\alpha) \tag{4.8}$$

which degenerates at page two, we see that

- clearly $\mathrm{H}^0(S, \omega_S \otimes g * \alpha) \simeq \mathrm{H}^0(Y, g_*(\omega_S \otimes g^*\alpha))$;
- since $R^1 g_*(\omega_S \otimes g^*\alpha) = 0$ by Theorem 1.2.3, it follows that

$$\begin{aligned}\mathrm{H}^1(S, \omega_S \otimes g^*\alpha) &\simeq \mathrm{H}^1(Y, g_*\omega_S \otimes \alpha) \oplus \mathrm{H}^0(Y, R^1 g_*(\omega_S \otimes g^*\alpha)) \\ &\simeq \mathrm{H}^1(Y, g_*\omega_S \otimes \alpha);\end{aligned}$$

- finally,

$$\begin{aligned}\mathrm{H}^2(S, \omega_S \otimes g^*\alpha) &\simeq \mathrm{H}^2(Y, R^0 g_*(\omega_S \otimes g^*\alpha)) \oplus \mathrm{H}^1(Y, R^1 g_*(\omega_S \otimes g^*\alpha)) \oplus \\ &\quad \oplus \mathrm{H}^0(Y, R^2(g_*\omega_S \otimes g^*\alpha)) \\ &\simeq \mathrm{H}^2(Y, g_*\omega_S \otimes \alpha).\end{aligned}$$

  Indeed, as above by the Grauert-Riemenschneider Vanishing Theorem we have that $\mathrm{H}^1(Y, R^1 g_*(\omega_S \otimes g^*\alpha)) = 0$. Also, by reasons of dimension, $\mathrm{H}^q(S_y, (\omega_S \otimes g^*\alpha)_y) = 0$ for all $y \in Y$ and $q = 2,3$; therefore $R^2 g_*(\omega_S \otimes g^*\alpha) = 0$ by Theorem 1.2.2, which in turn implies that $\mathrm{H}^0(Y, R^2 g_*(\omega_S \otimes g^*\alpha)) = 0$.

Thus the statement follows. □

4.3 Generic Vanishing Results

In this section we review some results about Generic Vanishing that we will need later on. Sources are, for example, [PP08] and [PP11]. We also give definitions of some of the technical hypotheses we will need to add when working with results related to Generic Vanishing.

Let X be a smooth projective variety, and consider a coherent sheaf $\mathscr{F}$ on X. Then we consider the subsets of $\mathrm{Pic}^0(X)$

$$V^i(\mathscr{F}) := \{\alpha \in \mathrm{Pic}^0(X) | \, h^i(X, \mathscr{F} \otimes \alpha) > 0\}, \tag{4.9}$$

and we call them *cohomological support loci*. These sets are closed in the Zarisky topology by the semicontinuity theorem. We endow them with the reduced induced subscheme structure.

The sheaf $\mathscr{F}$ on X is said to *satisfy Generic Vanishing with index* $-k$ or to *be* gv_{-k} if

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} V^i(\mathscr{F}) \geq i-k \qquad \text{for all } i \geq 0. \tag{4.10}$$

A stronger condition than being gv_0 is that of m-regularity. The sheaf $\mathscr{F}$ is called *m-regular* (or *Mukai-regular*) if

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} V^i(\mathscr{F}) > i \qquad \text{for all } i \geq 0. \tag{4.11}$$

The condition of being m-regular is quite strong on abelian varieties. Indeed we have that [PP08, Lemma 5.1]:

Lemma 4.3.1. *Let A be an abelian variety, $\mathscr{F}$ a coherent sheaf on A. If $\mathscr{F}$ is m-regular, then $\mathscr{F}$ is nef* [1]*, $h^0(A,\mathscr{F}) > 0$ and $\chi(\mathscr{F}) > 0$.*

More than that, actually m-regularity and ampleness are equivalent conditions for line bundles on abelian varieties ([PP08, Example 3.10(1)]).

If X is a smooth projective variety defined over an algebraically closed field of characteristic zero, then it was proved by Green and Lazarsfeld that ω_X satisfies Generic Vanishing with index $\dim(\mathrm{alb}_X(X)) - \dim X$.

For surfaces in positive characteristic, we have the following result by Wang ([Wa17, Theorem 3.1, Remark 3.2]):

Theorem 4.3.2 (Wang)**.** *Let S be a smooth projective surface over an algebraically closed field k of positive characteristic. Let A be an abelian variety and $a : S \to A$ a generically finite morphism. If S lifts to $W_2(k)$, then*

$$\mathrm{H}^k(A, a_*(\omega_S \otimes \alpha) \otimes \gamma) = 0$$

for any $k > 0$, any $\alpha \in \operatorname{Pic}^0(S)$ and general $\gamma \in \operatorname{Pic}^0(A)$.

In particular, under these conditions ω_S is gv_0.

In the statement of the theorem above, $W_2(k)$ is the ring of the second Witt vectors of k; we say that a scheme X over k *lifts to* $W_2(k)$ if there exists a scheme $\tilde{X}$, flat over $\operatorname{Spec}(W_2(k))$, such that

$$X \simeq \tilde{X} \times_{\operatorname{Spec}(W_2(k))} \operatorname{Spec}(k).$$

The condition of liftability to $W_2(k)$ is needed by Wang to apply some results of Deligne and Illusie used in [DI87] to find conditions under which the Hodge spectral sequence degenerates at E_1 in positive characteristic.

[1] To a coherent sheaf $\mathscr{F}$ one can associate a scheme $\mathbb{P}(\mathscr{F}) := \operatorname{Proj}(\oplus_m \operatorname{Sym}^m(\mathscr{F}))$ over A and an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)$. Then $\mathscr{F}$ is *nef* if $\mathcal{O}_{\mathbb{P}(\mathscr{F})}(1)$ is nef.

The cohomological support loci are subvarieties of the Picard variety. Actually, more is known about their structure. In characteristic zero, Green, Lazarsfeld and Simpson proved that the irreducible components of the cohomological support loci are translates of abelian subvarieties of $\mathrm{Pic}^0(X)$ by torsion points. In positive characteristic, again using the results of [DI87], Pink and Roessler ([PR04]) proved an analogous of that theorem with the addition of some hypotheses (see [Wa17, Proposition 2.20]):

Theorem 4.3.3. *Let X be a smooth projective variety defined over a perfect field k of positive characteristic. Assume that X lifts to $W_2(k)$, that $\dim(X) \leq \mathrm{char}(k)$, and that $\mathrm{Pic}^0(X)$ has no supersingular factors. Define for any $i, j, m \in \mathbb{N}$*

$$S_m^{i,j} := \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \Omega_X^j \otimes \alpha) \geq m\}. \tag{4.12}$$

Then the irreducible components of maximal dimension of $S_m^{i,j}$ are translates of abelian subvarieties of $\mathrm{Pic}^0(X)$ by torsion points of $\mathrm{Pic}^0(X)$.

In particular, this is true for $V^1(\omega_X) = S_1^{1,\dim(X)}$.

Now we explain the meaning of having supersingular factors. An elliptic curve over an algebraically closed field of characteristic p is *supersingular* if it has no points of order p ([Oo74, 4.]). An abelian variety is called *supersingular* if it is isogenous to a product of supersingular elliptic curves ([Oo74, Theorem 4.2]). An abelian variety *has a supersingular factor* if it has a non-trivial subquotient which is a supersingular abelian variety ([PR04, 2.]).

In general, abelian varieties have no supersingular factors. More precisely, an ordinary abelian variety (an abelian variety A with $p^{\dim A}$ p-torsion points) has no supersingular factors ([HP16, Lemma 2.3.5]), and the locus of ordinary abelian varieties is open and dense in the moduli space of abelian varieties (see for example [HPZ17]).

4.4 The Problem

We now state the assumptions under which we will work from this point on.

Assumptions 4.4.1. *We assume S to be a smooth minimal surface of general type over an algebraically closed field k. We fix*

$$p_g(S) = 3 \qquad \text{and} \qquad h^1(S, \mathcal{O}_S) = 3.$$

Also, we assume that S is mAd, the Albanese morphism is separable and $\mathrm{Pic}^0(S)$ is reduced.

Remark 4.4.1. As seen in Lemma 4.1.2, K_S^2 is bounded. Therefore Theorem 1.9.1 implies that there exists a prime $\bar{p}$ such that if the characteristic is greater than $\bar{p}$, then $\mathrm{Pic}^0(S)$ is reduced.

This remark implies that requiring $\mathrm{Pic}^0(S)$ reduced is not too restrictive, as it true in general, except for a finite number of prime characteristics.

We also observe the following about the condition of being mAd:

Remark 4.4.2. If S was not mAd, then the image of alb_S would be a curve inducing on S a pencil of genus three, and the generic fibre of this pencil would be a singular curve.

Proof. Indeed, let B the smooth curve which is the normalisation of $\mathrm{alb}_S(S)$. Then we have a diagram

$$\begin{tikzcd}
S \arrow[rr, "\mathrm{alb}_S"] \arrow[dr, "l"'] & & \mathrm{alb}_S(S) \arrow[rr, hook] & & \mathrm{Alb}(S) \arrow[dl, bend right, "\alpha"'] \\
& B \arrow[ur] \arrow[rr] & & J(B) \arrow[ur, bend right, "\beta"'] &
\end{tikzcd}$$

where l is given by the universal property of the normalisation, and α, β are given by the universal property of the Albanese morphism. Both α and β must be surjective, therefore $\dim J(B) = \dim \mathrm{Alb}(S) = 3$, and so $g(B) = 3$. As seen in Lemma 4.1.3, the generic fibre of l must be singular. □

Consider the Albanese morphism $\mathrm{alb}_S : S \to \mathrm{Alb}(S)$. Since we assumed S to be mAd and the Picard variety to be reduced, the image of the Albanese morphism, $\mathrm{alb}_S(S)$, is a divisor on the threefold $\mathrm{Alb}(S)$. We study the two cases: the one where $\mathrm{alb}_S(S)$, is an ample divisor and the case in which $\mathrm{alb}_S(S)$ is not ample.

Before we proceed with the discussion of the two cases, we observe that in both we will consider a resolution of singularity of the surface $\mathrm{alb}_S(S)$.

To explain what we mean by this and why we can do so, we recall the following definition from [Li06, Definition 8.3.39]:

Definition 4.4.3. Let X be a reduced locally Noetherian scheme. A *resolution of singularities of* X (also called a *desingularisation of* X) is a proper birational morphism $\pi : X \to Z$, where Z is regular. Such a π is called a *desingularisation in the strong sense* if it is an isomorphism above every regular point of X.

The result of Hironaka regarding the existence of desingularisations in the strong sense for algebraic varieties in characteristic zero is well-known. In positive characteristic, (reduced) algebraic surfaces admit desingularisation in the strong sense over fields of any characteristic (see [Li06, Theorem 3.44]). Zariski in 1939 was the first to prove the existence of such a desingularisation over any field of characteristic zero; the first proof for surfaces in positive characteristic is due to Abhyankar in 1956. In 1978 Lipman proved the existence of a desingularisation in the strong sense for all excellent reduced Noetherian schemes of dimension 2 (including arithmetic surfaces).

We prove here a lemma we will need to use more than once when dealing with the resolution of singularities of $\mathrm{alb}_S(S)$.

Lemma 4.4.4. *Let S be a mAd smooth projective surface, and assume that $\mathrm{Alb}(S)$ is a threefold. Let X be an abelian surface. If there is a surjective morphism of abelian varieties $f : \mathrm{Alb}(S) \to X$ such that $\mathrm{alb}_S(S)$ is mapped to a curve, then a resolution of singularities Y of $\mathrm{alb}_S(S)$ is a surface of Kodaira dimension one.*

In order to prove this lemma, we are going to need a result that can be found for example in [Li06, 4. Corollary 4.3]:

Proposition 4.4.5. *Let W be an integral scheme, Z a normal locally Noetherian scheme, $g : W \to Z$ a proper birational morphism. Then there exists an open subset V of Z such that $g^{-1}(V) \to V$ is an isomorphism and $Z \setminus V$ has codimension at least two.*

Proof of Lemma 4.4.4. We begin by ruling out $\kappa(Y) = -\infty$. By the universal property of the Albanese morphism, any morphism from Y to an abelian variety must factorise through $\mathrm{alb}_Y : Y \to \mathrm{Alb}(Y)$, and, since $\mathrm{alb}_S(S)$ is a surface, $\mathrm{alb}_Y(Y)$ cannot be a curve. Therefore, since $h^1(Y, \mathcal{O}_Y) \geq 3$, by Theorem 1.6.6, it cannot be $\kappa(Y) = -\infty$.

By Table 1.1, $h^1(Y, \mathcal{O}_Y) \geq 3$ implies that Y cannot have Kodaira dimension zero.

Finally, assume $\kappa(Y) = 2$. By Proposition 1.4.1, the inverse image of any point of X must be a disjoint union of elliptic curves. The restriction of each fibre to $\mathrm{alb}_S(S)$ must be a union of elliptic curves and possibly some points. A classical result of algebraic geometry says that the dimension of every irreducible component of any fibre of a dominant morphism of varieties is at least equal to the difference of the dimensions of the two varieties (see [Ha77, II. Exercise 3.22(b)]). Therefore $\mathrm{alb}_S(S)$ is fibered in abelian subvarieties.

Taking into account Theorem 1.4.2, we have a morphism $Y \to f(\mathrm{alb}_S(S))$, that we can then factorise, by taking the normalisation of $f(\mathrm{alb}_S(S))$ and using the universal property of normalisation, into $j : Y \to C$ for a smooth curve C. The morphism j is a fibration, and we can assume all its fibres are connected (or we take its Stein factorisation).

Let F be the generic fibre of j. Then, calling r the birational map $r : Y \to \mathrm{alb}_S(S)$, we have that the image of $r|_F$ must be an elliptic curve E. Since $r|_F$ is a birational morphism of projective curves and E is smooth, we can apply Proposition 4.4.5 and we see that $r|_F$ must be an isomorphism. Therefore F is an elliptic curve too, and $j : Y \to C$ is an elliptic fibration. Thus we have a contradiction, and $\kappa(Y) = 1$. $\square$

4.5 The Case of Ample Albanese Image

In this section we study what happens when $\mathrm{alb}_S(S)$ is an ample divisor on $\mathrm{Alb}(S)$.

Consider a resolution of singularities $r : Y \to \mathrm{alb}_S(S)$ for $\mathrm{alb}_S(S)$ and a morphism $g : S \to Y$ which factorises alb_S:

$$\begin{tikzcd} S \arrow[rr, "\mathrm{alb}_S"] \arrow[dr, "g"'] & & \mathrm{alb}_S(S) \hookrightarrow \mathrm{Alb}_S(S) \\ & Y \arrow[ur, "r"'] & \end{tikzcd}$$

By adjunction, an ample divisor on an abelian variety is a variety with ample canonical divisor, and therefore by resolving the singularities one gets a smooth variety of general type. So Y above is a smooth surface of general type.

In the next proposition we prove point 1. of Theorem 4.A.

Proposition 4.5.1. *Let S be a surface satisfying the Assumptions 4.4.1, and assume that $\mathrm{alb}_S(S)$ is an ample divisor. Then the resolution of singularities of $\mathrm{alb}_S(S)$ is a surface of general type with $p_g(Y) = h^1(Y,\omega_Y) = 3$ and its Picard variety is reduced.*

Proof. Consider the morphism $g : S \to Y$ introduced above. Then, by Remark 4.2.5, one gets a short exact sequence

$$0 \longrightarrow \omega_Y \longrightarrow g_*\omega_S \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{4.13}$$

which gives rise to a long exact sequence in cohomology

$$0 \longrightarrow \mathrm{H}^0(Y,\omega_Y) \longrightarrow \mathrm{H}^0(Y,g_*\omega_S) \longrightarrow \mathrm{H}^0(Y,\mathcal{Q}) \longrightarrow ... \tag{4.14}$$
$$... \longrightarrow \mathrm{H}^1(Y,\omega_Y) \longrightarrow \mathrm{H}^1(Y,g_*\omega_S) \longrightarrow \mathrm{H}^1(Y,\mathcal{Q}) \longrightarrow ... \tag{4.15}$$
$$... \longrightarrow \mathrm{H}^2(Y,\omega_Y) \longrightarrow \mathrm{H}^2(Y,g_*\omega_S) \longrightarrow \mathrm{H}^2(Y,\mathcal{Q}) \longrightarrow 0. \tag{4.16}$$

Applying Lemma 4.2.6 to rewrite the $\mathrm{H}^i(Y,g_*\omega_S)$'s, and using the hypotheses $p_g(S) = h^1(S,\mathcal{O}_S) = 3$, we get

$$0 \longrightarrow \mathrm{H}^0(Y,\omega_Y) \longrightarrow k^{\oplus 3} \longrightarrow \mathrm{H}^0(Y,\mathcal{Q}) \longrightarrow ... \tag{4.17}$$
$$... \longrightarrow \mathrm{H}^1(Y,\omega_Y) \longrightarrow k^{\oplus 3} \longrightarrow \mathrm{H}^1(Y,\mathcal{Q}) \longrightarrow ... \tag{4.18}$$
$$... \longrightarrow k \longrightarrow k \longrightarrow \mathrm{H}^2(Y,\mathcal{Q}) \longrightarrow 0. \tag{4.19}$$

First, since Y is of general type, $\chi(Y) = h^0(Y,\omega_Y) - h^1(Y,\omega_Y) + 1 \geq 1$ by Theorem 1.6.3. Therefore $h^0(Y,\omega_Y) \geq h^1(Y,\omega_Y)$. Also, by Lemma 4.1.4, $h^1(Y,\omega_Y) \geq \dim \mathrm{Alb}(Y) = \dim \mathrm{Alb}(S) = 3$. Therefore $h^0(Y,\omega_Y) \geq 3$.

From the long exact sequence above we have immediately that $h^0(Y,\omega_Y) \leq 3$, and so it must be $h^0(Y,\omega_Y) = 3$.

Therefore we obtain $3 = h^0(Y,\omega_Y) \geq h^1(Y,\omega_Y) \geq 3$, and thus it follows that also $h^1(Y,\omega_Y) = 3$. □

By adding some hypotheses, we want to prove that the Albanese morphism is birational. We begin by studying the sheaf $\mathcal{Q}$ we have already introduced.

Theorem 4.5.2. *Let S be a surface satisfying the Assumptions 4.4.1. Assume $\mathrm{alb}_S(S)$ is normal and that it is an ample divisor. Let $\mathcal{Q}$ be as in (4.13). Assume moreover that S lifts to $W_2(k)$. Then $r_*\mathcal{Q}$ is gv_0.*

If moreover $\mathrm{Pic}^0(S)$ has no supersingular factors, then $r_*\mathcal{Q} \simeq \mathcal{O}_{\mathrm{Alb}(S)}$.

Proof. As in (4.13), we have the short exact sequence

$$0 \longrightarrow \omega_Y \longrightarrow g_*\omega_S \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{4.20}$$

and we can push it forward:

$$0 \longrightarrow r_*\omega_Y \longrightarrow r_*g_*\omega_S \longrightarrow r_*\mathcal{Q} \longrightarrow R^1 r_*\omega_Y.$$

The Grauert-Riemenschneider Vanishing Theorem ensures that $R^1 r_*\omega_Y \simeq 0$, and therefore, for any line bundle $\alpha \in \mathrm{Pic}^0(\mathrm{Alb}(S))$, we have

$$0 \longrightarrow r_*\omega_Y \otimes \alpha \longrightarrow r_*g_*\omega_S \otimes \alpha \longrightarrow r_*\mathcal{Q} \otimes \alpha \longrightarrow 0.$$

Therefore, using projection formula, we get a long exact sequence

$$\begin{aligned}
0 \longrightarrow \mathrm{H}^0(\mathrm{Alb}(S), r_*(\omega_Y \otimes r^*\alpha)) &\longrightarrow \mathrm{H}^0(\mathrm{Alb}(S), r_*g_*(\omega_S \otimes g^*r^*\alpha)) \longrightarrow \\
&\longrightarrow \mathrm{H}^0(\mathrm{Alb}(S), r_*(\mathcal{Q} \otimes r^*\alpha)) \longrightarrow \ldots \\
\ldots \longrightarrow \mathrm{H}^1(\mathrm{Alb}(S), r_*(\omega_Y \otimes r^*\alpha)) &\longrightarrow \mathrm{H}^1(\mathrm{Alb}(S), r_*g_*(\omega_S \otimes g^*r^*\alpha)) \longrightarrow \\
&\longrightarrow \mathrm{H}^1(\mathrm{Alb}(S), r_*(\mathcal{Q} \otimes r^*\alpha)) \longrightarrow \ldots \\
\ldots \longrightarrow \mathrm{H}^2(\mathrm{Alb}(S), r_*(\omega_Y \otimes r^*\alpha)) &\longrightarrow \mathrm{H}^2(\mathrm{Alb}(S), r_*g_*(\omega_S \otimes g^*r^*\alpha)) \longrightarrow \\
&\longrightarrow \mathrm{H}^2(\mathrm{Alb}(S), r_*(\mathcal{Q} \otimes r^*\alpha)) \longrightarrow 0.
\end{aligned}$$

We can use Lemma 4.2.6 on both $\mathrm{alb}_S : S \to \mathrm{alb}_S(S)$ and $r : Y \to \mathrm{alb}_S(S)$ to rewrite the sequence above as

$$\begin{aligned}
0 \longrightarrow \mathrm{H}^0(Y, \omega_Y \otimes r^*\alpha) \longrightarrow \mathrm{H}^0(S, \omega_S \otimes \mathrm{alb}_S^*\alpha) \longrightarrow \mathrm{H}^0(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow \ldots \\
\ldots \longrightarrow \mathrm{H}^1(Y, \omega_Y \otimes r^*\alpha) \longrightarrow \mathrm{H}^1(S, \omega_S \otimes \mathrm{alb}_S^*\alpha) \longrightarrow \mathrm{H}^1(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow \ldots \\
\ldots \longrightarrow \mathrm{H}^2(Y, \omega_Y \otimes r^*\alpha) \longrightarrow \mathrm{H}^2(S, \omega_S \otimes \mathrm{alb}_S^*\alpha) \longrightarrow \mathrm{H}^2(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow 0.
\end{aligned}$$

Since we know by Proposition 4.5.1 that $\chi(\omega_Y) = \chi(\omega_S)$, we get

$$\begin{aligned}
\chi(r_*(\omega_Y \otimes r^*\alpha)) &= \chi(\omega_Y \otimes r^*\alpha) = \chi(\omega_Y) = \chi(\omega_S) \\
&= \chi(\omega_S \otimes g^*r^*\alpha) = \chi(r_*g_*(\omega_S \otimes g^*r^*\alpha))
\end{aligned}$$

It follows in particular that $\chi(r_*\mathcal{Q} \otimes \alpha) = 0$.

Observe that $\mathrm{alb}_S^* : \mathrm{Pic}^0(\mathrm{Alb}(S)) \to \mathrm{Pic}^0(S) = \mathrm{Pic}^0(\mathrm{Alb}(S))$ is the identity map, and therefore r^* must be injective, too. By Serre Duality, $\mathrm{H}^2(S, \omega_S \otimes \alpha) \simeq \mathrm{H}^0(S, \alpha^{-1})$, and by Proposition 1.5.5 that is zero unless $\alpha \simeq \mathcal{O}_S$. The analogous statement is true for $\mathrm{H}^1(Y, \omega_Y \otimes r^*\alpha)$. Thus, if α is not $\mathcal{O}_S$, we have

$$\begin{aligned}
0 \longrightarrow \mathrm{H}^0(Y, \omega_Y \otimes r^*\alpha) \longrightarrow \mathrm{H}^0(S, \omega_S \otimes \alpha) \longrightarrow \mathrm{H}^0(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow \ldots \\
\ldots \longrightarrow \mathrm{H}^1(Y, \omega_Y \otimes r^*\alpha) \longrightarrow \mathrm{H}^1(S, \omega_S \otimes \alpha) \longrightarrow \mathrm{H}^1(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow 0,
\end{aligned}$$

and $h^2(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) = 0$. Therefore we see that we have $V^2(r_*\mathcal{Q}) = \emptyset$ and $h^0(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) = h^1(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha)$.

If S lifts to $W_2(k)$, then Theorem 4.3.2 implies that ω_S is gv_0, so

$$\operatorname{codim}_{\mathrm{Pic}^0(S)} V^1(\omega_S) \geq 1.$$

This yields, unless α belongs to some surfaces in $\mathrm{Pic}^0(\mathrm{Alb}(S))$,

$$0 \longrightarrow \mathrm{H}^0(Y, \omega_Y \otimes r^*\alpha) \longrightarrow \mathrm{H}^0(S, \omega_S \otimes \alpha) \longrightarrow \mathrm{H}^0(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow ...$$
$$... \longrightarrow \mathrm{H}^1(Y, \omega_Y \otimes r^*\alpha) \longrightarrow 0 \longrightarrow \mathrm{H}^1(\mathrm{Alb}(S), r_*\mathcal{Q} \otimes \alpha) \longrightarrow 0.$$

We deduce that $V^0(r_*\mathcal{Q}) = V^1(r_*\mathcal{Q}) \subset V^1(\omega_S)$, and therefore

$$\operatorname{codim}_{\mathrm{Pic}^0(\mathrm{Alb}(S))} V^i(r_*\mathcal{Q}) \geq 1$$

for $i = 0, 1$, and in particular $r_*\mathcal{Q}$ is gv_0.

In order to show that $r_*\mathcal{Q}$ is trivial, we show that it is m-regular, and assuming $r_*\mathcal{Q}$ non-zero leads to a contradiction with Lemma 4.3.1.

To prove that $r_*\mathcal{Q}$ is m-regular it only remains to show that

$$\operatorname{codim}_{\mathrm{Pic}^0(\mathrm{Alb}(S))} V^1(r_*\mathcal{Q}) \geq 2,$$

and that is to say, $V^1(r_*\mathcal{Q})$ does not contain surfaces. By the exact sequence, we arrive to that conclusion if we show that $V^1(\omega_S)$ does not contain surfaces.

We know that Y has Kodaira dimension two, and therefore, by Lemma 4.4.4, we know that we cannot have a morphism of abelian varieties from $\mathrm{Alb}(S)$ to an abelian surface such that the image of $\mathrm{alb}_S(S)$ is a curve. With this in mind, we follow the reasoning of [Wa17, Lemma 4.5] and adapt it to our case to conclude our proof.

By contradiction, assume that $V^1(\omega_S)$ contains a surface. By Theorem 4.3.3, assuming that $\mathrm{Pic}^0(S)$ has no supersingular factors, we know that there is an abelian surface $X \subset \mathrm{Pic}^0(S)$ such that $X + \tau$ ($\tau \in \mathrm{Pic}^0(S)$ torsion element) is a component of maximal dimension of $V^1(\omega_S)$. Since X is an abelian subvariety of $\mathrm{Pic}^0(S)$, we can take the dual of the inclusion morphism and we get a surjective morphism $f : \mathrm{Alb}(S) \to \hat{X}$. As we have already said, by Lemma 4.4.4 $f(\mathrm{alb}_S(S))$ cannot be a curve. It cannot be a point either, or $\mathrm{alb}_S(S)$ would be an elliptic surface. Therefore $f \circ \mathrm{alb}_S : S \to \hat{X}$ must be surjective.
By projection formula and Lemma 4.2.6, we get, for $\beta \in \mathrm{Pic}^0(\hat{X}) = X$,

$$h^1(S, \omega_S \otimes \tau \otimes \mathrm{alb}^* f^* \beta) = h^1(\hat{X}, f_* \mathrm{alb}_*(\omega_S \otimes \tau) \otimes \beta). \tag{4.21}$$

By Theorem 4.3.2, the right-hand member of the equation above is zero for general $\beta \in X$, but $h^1(S, \omega_S \otimes \tau \otimes \mathrm{alb}^* f^* \beta) \neq 0$ for all $\beta \in X$ by definition of $X + \tau$. Thus we reached a contradiction and $r_*\mathcal{Q}$ is m-regular; thus, by Lemma 4.3.1, $r_*\mathcal{Q}$ must be trivial. □

And now we can easily prove the statement of Theorem 4.B:

Corollary 4.5.3. *Let S be a surface satisfying the Assumptions 4.4.1 and assume in addition that $\mathrm{alb}_S(S)$ is normal and that it is an ample divisor. Assume moreover that S lifts to $W_2(k)$ and that $\mathrm{Pic}^0(S)$ has no supersingular factors. Then $\mathrm{alb}_S : S \to \mathrm{alb}_S(S)$ is birational.*

Proof. By Lemma 4.5.2 we have that $r_*\omega_Y \simeq r_*g_*\omega_S$. Since r is birational, the generic rank of $r_*\omega_Y$ must be one. The generic rank of $r_*g_*\omega_S$ is the generic degree of g, and therefore the generic degree of g must be one. Therefore the generic degree of $r \circ g = \mathrm{alb}_S$ is one. □

We shall discuss how one could try to improve the results of this section in Section 4.7.

4.6 The Case of Non-Ample Albanese Image

In this section we study what happens when $\mathrm{alb}_S(S)$ is not an ample divisor on $\mathrm{Alb}(S)$. We prove point 2. of Theorem 4.A in several separate steps while trying to clarify the geometry of the problem in terms of, for example, how the pencils on S and Y mentioned in point 2. of Theorem 4.A relate to each other.

We begin by studying the geometry of S, and for the moment being we leave Y aside.

Proposition 4.6.1. *Let S be a surface satisfying the Assumptions 4.4.1, and assume that $\mathrm{alb}_S(S)$ is not an ample divisor. Then S admits a pencil $S \to C$ with connected fibres onto a smooth curve C with $g(C) = 2$. The generic fibre F is such that $h^1(F, \mathcal{O}_F) \geq 2$, and F can be smooth only in the case in which equality holds; viceversa, if equality holds, the pencil $S \to C$ is smooth with constant moduli.*

In any case, the normalisation of F has genus either one or two.

Proof. Since $\mathrm{alb}_S(S)$ is not ample, by Lemma 1.4.3 there exist an abelian variety X with a divisor B on it and a morphism of abelian varieties $f : \mathrm{Alb}(S) \to X$ such that $f^{-1}(B) = \mathrm{alb}_S(S)$.

The variety X must be an abelian surface. Indeed,

1. X cannot be a curve. Otherwise B would be a point because $\mathrm{alb}(S)$ is connected. But then $\mathrm{alb}_S(S)$ would be an abelian variety by Proposition 1.4.1, and it would not generate $\mathrm{Alb}(S)$.

2. X cannot be a threefold. Otherwise f would be finite and the pullback of an ample divisor is still ample if the morphism is finite.

Being a prime divisor on X, B has to be a curve. Let $C \to B$ be the normalisation of B. Then one gets a morphism $h : S \to C$ by Proposition 1.10.1. Consider also $S \xrightarrow{s} N \xrightarrow{t} C$, the Stein factorisation of $h : S \to C$. For the convenience of the reader, we remark here that at the end of this proof we will show that actually h has connected fibres. Since S is normal, N is normal too.

$$\begin{tikzcd}
S \arrow[r, "\mathrm{alb}_S"] \arrow[d, "s"'] \arrow[dr, "h"] \arrow[drr] & \mathrm{alb}_S(S) \arrow[r, hook] \arrow[d, two heads, "f|_{\mathrm{alb}_S(S)}"] & \mathrm{Alb}_S(S) \arrow[d, two heads, "f"] \\
N \arrow[r, "t"] & C \arrow[r, two heads] & B \arrow[r, hook] & X.
\end{tikzcd}$$

In order to get information about the pencil with connected fibres $s : S \to N$, we study the genuses of the smooth curves C and N.

First, observe that $g(C), g(N) \leq 2$. Indeed, as seen in Lemma 4.1.3, the fact that $h^1(S, \mathcal{O}_S) = 3$ implies that $g(C), g(N) \leq 3$. Also, as in the proof of Lemma 4.1.3, if for example we had $g(N) = 3$, then we would get a finite map φ of abelian varieties:

$$\begin{tikzcd}
S \arrow[r, "\mathrm{alb}_S"] \arrow[d, two heads] & \mathrm{Alb}(S) \arrow[d, "\varphi"'] \\
N \arrow[r, hook] & J(N).
\end{tikzcd}$$

But the hypothesis of mAd we have on S would then imply, by restriction, that we would get a finite map from a surface ($\mathrm{alb}_S(S)$) to a curve (the image of N in $J(N)$), which is absurd. The case $g(C) = 3$ is settled in an identical fashion.

Now, since we can factorise the map $C \to X$ (with image B) passing through the jacobian $J(C)$ of C, we have that that:

1. $g(C) \neq 0$, because if C where rational $J(C)$ would be a point and it could not surject onto B.

2. $g(C) \neq 1$, otherwise the map from $C = J(C)$ to X would be a morphism of abelian varieties, and therefore its image (i.e. B) would be an abelian variety. But B cannot be an elliptic curve because it is ample and therefore generates X.

Thus it must be $g(C) = 2$. Since the map $t : N \to C$ is finite, we have that $2 \geq g(N) \geq g(C) = 2$, and therefore $g(N) = 2$.

Now, the finite morphism $t : N \to C$ can be factorised into a purely inseparable one (say, $N \to N'$) followed by a separable one ($N' \to C$). Since $g(N') \neq 0, 1$, by [Ha77, IV, Example 2.5.4] it follows that $N' \to C$ must be an isomorphism. But then $t : N \to C$ would be purely inseparable. This in particular implies that the fibres of h are connected.

By Lemma 4.1.3, the generic fibre of the map s has arithmetic genus at least two, and it is smooth if and only if its arithmetic genus is exactly two. Moreover, by Theorem 1.3.2, we can find a bound on the genus of the normalisation of the generic fibre F. Indeed, we have

$$\begin{aligned} c_2(S) &\geq e(N)e(F) = 4(1-g(N))(1-g(F)) \\ &= -4(1-g(F)) = 4g(F) - 4. \end{aligned}$$

Using Lemma 4.1.2, we get

$$7 \geq 4g(F) - 4,$$

which implies that $g(F) \leq 2$. Also, $g(F)$ cannot be zero, or F would be a (singular) rational curve and it would be contracted by alb_S, which is impossible since S is mAd. □

Remark 4.6.2. Notice that the map h in the proof has connected fibres, but, as discussed in § 1.10, since the characteristic is positive, h does not necessarily satisfy $h_*\mathcal{O}_S \simeq \mathcal{O}_C$, while s would satisfy $s_*\mathcal{O}_S \simeq \mathcal{O}_N$ by construction.

Remark 4.6.3. Reasoning as we did in the proof of the preceding proposition, one can see that the mAd condition on S implies that S cannot have pencils onto curves of genus higher than two.

We now turn our attention to the resolution of singularities of $\mathrm{alb}_S(S)$.

Proposition 4.6.4. *Let S be a surface satisfying the Assumptions 4.4.1, and assume that $\mathrm{alb}_S(S)$ is not an ample divisor.*
Then Y, the (minimal) resolution of singularities of $\mathrm{alb}_S(S)$, is a surface with $\kappa(Y) = 1$, $\dim \mathrm{Alb}(Y) = 3$, and Y must have one of the following sets of invariants:

- $h^0(Y,\omega_Y) = 2$, $h^1(Y,\mathcal{O}_Y) = 3$, $\chi(Y) = 0$, $\mathrm{Pic}^0(Y)$ *reduced;*
- $h^0(Y,\omega_Y) = 3$, $h^1(Y,\mathcal{O}_Y) = 4$, $\chi(Y) = 0$, $\mathrm{Pic}^0(Y)$ *non-reduced.*

Moreover, Y has an elliptic fibration onto a smooth curve E with $g(E) = 2$.

Proof. As in the case when $\mathrm{alb}_S(S)$ is an ample divisor, consider a smooth surface Y which is a resolution of singularities of $r : Y \to \mathrm{alb}_S(S)$. Also, consider the commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\mathrm{alb}_S"] \arrow[dr, "l"'] & & \mathrm{alb}_S(S) \arrow[r, hook] & \mathrm{Alb}_S(S) \\ & Y \arrow[ur, "r"'] & & \end{tikzcd}$$

Again as in the proof of Proposition 4.6.1, since $\mathrm{alb}_S(S)$ is not ample, there must be an abelian surface X with a divisor B on it and a morphism of abelian varieties $f : \mathrm{Alb}(S) \to X$ such that $f^{-1}(B) = \mathrm{alb}_S(S)$. Therefore, we are in the conditions of Lemma 4.4.4, and $\kappa(Y) = 1$. In the proof of Lemma 4.4.4 we also built on Y an elliptic fibration (a priori Y could have had a quasi-elliptic fibration but not an elliptic one).

As in the proof of Proposition 4.5.1, by Remark 4.2.5 we get a short exact sequence

$$0 \longrightarrow \omega_Y \longrightarrow l_*\omega_S \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{4.22}$$

which in turn, applying Lemma 4.2.6 to the induced long exact sequence, yields the long exact sequence

$$0 \longrightarrow \mathrm{H}^0(Y,\omega_Y) \longrightarrow k^{\oplus 3} \longrightarrow \mathrm{H}^0(Y,\mathcal{Q}) \longrightarrow ... \tag{4.23}$$

$$... \longrightarrow \mathrm{H}^1(Y,\omega_Y) \longrightarrow k^{\oplus 3} \longrightarrow \mathrm{H}^1(Y,\mathcal{Q}) \longrightarrow ... \tag{4.24}$$

$$... \longrightarrow k \longrightarrow k \longrightarrow \mathrm{H}^2(Y,\mathcal{Q}) \longrightarrow 0. \tag{4.25}$$

This sequence implies in particular that $h^0(Y,\mathcal{O}_Y) \leq 3$. By Theorem 1.6.3, since $\kappa(Y) = 1$ we have $\chi(Y) \geq 0$. Therefore

$$4 = 3+1 \geq h^0(Y,\mathcal{O}_Y)+1 \geq h^1(Y,\mathcal{O}_Y) \geq 3. \tag{4.26}$$

Rephrasing, we obtain that $2 \leq h^0(Y,\mathcal{O}_Y) \leq 3 \leq h^1(Y,\mathcal{O}_Y) \leq 4$. Of the four cases that we get by considering all the possible combinations of values for $h^0(Y,\mathcal{O}_Y)$ and $h^1(Y,\mathcal{O}_Y)$, we have to exclude the one in which $h^0(Y,\mathcal{O}_Y) = 2$ and $h^1(Y,\mathcal{O}_Y) = 4$ because in that case $\chi(Y) < 0$. The remaining three cases are:

- $h^0(Y,\omega_Y) = 2$, $h^1(Y,\mathcal{O}_Y) = 3$, $\chi(Y) = 0$, $\mathrm{Pic}^0(Y)$ reduced;
- $h^0(Y,\omega_Y) = 3$, $h^1(Y,\mathcal{O}_Y) = 3$, $\chi(Y) = 1$, $\mathrm{Pic}^0(Y)$ reduced;
- $h^0(Y,\omega_Y) = 3$, $h^1(Y,\mathcal{O}_Y) = 4$, $\chi(Y) = 0$, $\mathrm{Pic}^0(Y)$ non-reduced.

Finally, consider the elliptic fibration $\pi : Y \to E$. Recall that by Lemma 4.1.4 $\dim \mathrm{Alb}(S) = \dim \mathrm{Alb}(Y)$. Then by Lemma 1.8.5 we have that $\dim \mathrm{Alb}(S)$ is either equal to $g(E)$ or to $g(E)+1$, and if the former case is verified then $\mathrm{alb}_Y(Y)$ is a curve. Since the map $\varphi : \mathrm{Alb}(S) \to \mathrm{Alb}(Y)$ is finite, the mAd hypothesis on S implies that $\mathrm{alb}_Y(Y)$ must be a surface. Therefore $\dim \mathrm{Alb}(S) = g(E)+1$, implying $g(E) = 2$.

Lemma 1.8.5 also implies that no fibre of $\pi : Y \to E$ is contracted by alb_Y. The universal property of the Albanese morphism and the isogeny between $\mathrm{Alb}(S)$ and $\mathrm{Alb}(Y)$ ensure that r cannot contract fibres of $Y \to E$ either. So the singular fibres of π must be multiples of elliptic curves (type I_0 fibres) by the classification of Kodaira and Néron (see § 1.8). Thus, by (1.19), $c_2(Y) = \chi(Y) = 0$. This excludes the second set of invariants of W we listed above. □

We now use the same notation introduced in Proposition 4.6.1 and Proposition 4.6.4, and we explain the connection between the two situations. Also, we connect them with Lemma 4.4.4.

Stating with the notation of Proposition 4.6.1, by the universal property of the normalisation, we get a map $h' : Y \to C$ which makes the following diagram commutative:

$$
\begin{tikzcd}
S \arrow[rr, "\mathrm{alb}_S"] \arrow[dr, "l"] \arrow[ddr, "h"'] & & \mathrm{alb}_S(S) \arrow[r, hook] \arrow[dd, two heads, "f|_{\mathrm{alb}_S(S)}"] & \mathrm{Alb}_S(S) \arrow[dd, two heads, "f"] \\
 & Y \arrow[ur, "r"] \arrow[d, "h'"] & & \\
 & C \arrow[r] & B \arrow[r, hook] & X
\end{tikzcd}
$$

By Proposition 1.4.1, the inverse image through $f|_{\mathrm{alb}_S(S)}$ of a point on B is a disjoint union of elliptic curves on $\mathrm{alb}_S(S)$. Since r is birational, the inverse image through $f|_{\mathrm{alb}_S(S)} \circ r$ of a generic point on B is still a disjoint union of elliptic curves. By exploiting the commutativity of the diagram and the fact that h has connected fibres one can see that the map h' has connected fibres and that the generic fibre is an elliptic curve.

In positive characteristic elliptic fibrations are defined not only by the condition of connected fibres, but by the condition that the push-forward of the structure sheaf on the surface be isomorphic to the structure sheaf on the base curve. Therefore, by taking the Stein factorisation of h', one gets the elliptic fibration $\pi : Y \to E$ of Proposition 4.6.4 on Y (that satisfies $\pi_* \mathcal{O}_Y \simeq \mathcal{O}_E$):

$$
\begin{tikzcd}
S \arrow[rr, "\mathrm{alb}_S"] \arrow[dr, "l"] & & \mathrm{alb}_S(S) \arrow[r, hook] \arrow[dd, two heads, "f|_{\mathrm{alb}_S(S)}"] & \mathrm{Alb}_S(S) \arrow[dd, two heads, "f"] \\
 & Y \arrow[ur, "r"] \arrow[d, "h'"] \arrow[dl, "\pi"'] & & \\
E \arrow[r] & C \arrow[r] & B \arrow[r, hook] & X
\end{tikzcd}
$$

Notice that the map $E \to C$ is either purely inseparable (if $h'_* \mathcal{O}_Y \not\simeq \mathcal{O}_C$) or an isomorphism (if h' already gave the structure of elliptic fibration).

As we did in previous chapters for elliptic fibrations, we can write $R^1 \pi_* \mathcal{O}_Y = L \oplus T$, with L invertible sheaf and T torsion sheaf. We can find out more information about L and T for each of the cases determined by the different possible invariants of Y computed in Proposition 4.6.4.

Proposition 4.6.5. *With the notation already introduced in this section, we have to be in one of the following three cases:*

i. $h^0(Y, \omega_Y) = 2$, $h^1(Y, \mathcal{O}_Y) = 3$, $\chi(Y) = 0$, $\mathrm{Pic}^0(Y)$ *reduced. Then*

1. *either* $T \simeq 0$ *and* $L \simeq \mathcal{O}_E$,
2. *or* $h^0(E, T) = 1$, $h^0(E, L) = 0$ *and* $\deg L = -1$.

ii. $h^0(Y,\omega_Y)=3$, $h^1(Y,\mathcal{O}_Y)=4$, $\chi(Y)=0$, $\mathrm{Pic}^0(Y)$ *non-reduced.*
Then $h^0(E,T)=2$, $h^0(E,L)=0$ *and* $\deg L=-2$.

In particular, in all cases but i.(1.) there are wild fibres. So case i.(1) is the only one that we have in characteristic zero.

Proof. We make the computations in each of the cases given by Proposition 4.6.4. As in the proof of Lemma 3.1.2, we use the decomposition

$$\mathrm{H}^1(Y,\mathcal{O}_Y)\simeq \mathrm{H}^0(E,T)\oplus \mathrm{H}^0(E,L)\oplus \mathrm{H}^1(E,\mathcal{O}_E) \tag{4.27}$$

given by the degeneration of the Leray spectral sequence.

i. By (4.27) we get that $h^0(E,T)+h^0(E,L)=1$. In this situation, by Theorem 1.8.3, $\deg L=-\chi(Y)-h^0(E,T)=-h^0(E,T)$. So,
   - If $h^0(E,T)=0$ and $h^0(E,L)=1$, then $\deg L=0$ and $L\simeq\mathcal{O}_E$ by Lemma 3.2.1.
   - Otherwise $h^0(E,T)=1$ and $h^0(E,L)=0$.
ii. Again by (4.27) we obtain $h^0(E,T)+h^0(E,L)=2$. By the Canonical Bundle Formula, $\deg L=-\chi(Y)-h^0(E,T)=-h^0(E,T)\le 0$. By Lemma 3.2.1, if $h^0(E,L)\neq 0$, then $\deg L=-h^0(E,T)=0$. But then, by the same Lemma, $L\simeq\mathcal{O}_E$, implying $h^0(E,L)=1$, and we get a contradiction $0+1=h^0(E,T)+h^0(E,L)=2$. It follows that it must be $h^0(E,L)=0$ and $h^0(E,T)=2$.

And thus we proved the statement of the Proposition. □

Recall that wild fibres are a subset of the multiple fibres of an elliptic fibration. It would not be unreasonable to hope that the geometry of the problem could help us exclude multiple fibres, and observe that in that situation, by the Canonical Bundle Formula and the fact that $L\simeq\mathcal{O}_E$ by i.(1) above, we would have $\omega_Y\simeq\pi^*\omega_E$, and therefore ω_Y would be the pull-back of two points on E. This, in turn, thanks to the geometry of the problem, could help to study the morphism l.

4.7 Comparison with Characteristic Zero

In this section, we explain the strategy used for proving point 1. of Theorem 4.0.1 in [HP02], as this could give insights about what one could hope to prove for improving the solution of the problem when $\mathrm{alb}_S(S)$ is ample in characteristic p. In doing so, we observe also some of the differences between the problem in characteristic p and the problem in characteristic zero.

As we mentioned at the beginning of this chapter, in the case of non-ample $\mathrm{alb}_S(S)$ it is possible that the strategy of [CCM98, Theorem (3.23)] for proving that S is the quotient of a product of curves of genus two could be repeated, *mutatis mutandis*, in positive characteristic if one had a pencil on S with constant moduli (smooth curves of genus two) onto a curve of genus two. The existence of such a pencil would be granted by point 2. of Theorem 4.A if one could eliminate the cases of pencils with singular generic fibre.

In [HP02] the authors consider a smooth minimal complex projective surface W of general type with $p_g(W) = h^1(W, \mathcal{O}_W) = 3$. Therefore, being in characteristic zero, $\mathrm{Alb}(S)$ is a threefold. Then, they know that ([HP02, Proposition 2.5]):

* The inequalities $6 \leq K_W^2 \leq 9$ hold by the Bogomolov-Miyaoka-Yau inequality and an inequality by Debarre; compare to the inequalities we obtained in positive characteristic in Lemma 4.1.2.

  Recall by [Li12, Theorem 7.4] that if a minimal surface of general type has negative c_2, then the albanese morphism has one-dimensional image with generic fibre a singular rational curve (being thus inseparably uniruled), which shows how negative c_2 is linked to a feature of the geometry of positive characteristic (also, if $c_2 = 0$, then the surface is inseparably dominated by a surface of special type). Observe that assuming $c_2(S) > 0$ in positive characteristic would also imply $5 \leq K_S^2 \leq 11$, yielding bounds much closer to those of characteristic zero.

* W must be mAd. As seen in Remark 4.4.2, if W were not mAd it would have a pencil of genus three with singular generic fibre, and this cannot happen in characteristic zero.

* If W has a pencil of genus ≥ 2, then S is a surface as in point 2. of Theorem 4.0.1; this was a result of [CCM98].

Then the authors of [HP02] go through the following steps:

- they assume that W has no irrational pencil of genus ≥ 2;
- with this assumption, they show that $V^1(\omega_W)$ has dimension zero, and more specifically that $V^1(\omega_W) = \{\mathcal{O}_W\}$;
- $V^1(\omega_W) = \{\mathcal{O}_W\}$ enables them to use a result by Hacon yielding that $\mathrm{Alb}(W)$ is principally polarised and that $\mathrm{alb}_W(W)$ is a theta divisor;
- at least in characteristic zero, any abelian threefold with an irreducible principal polarisation is the Jacobian of a curve C of genus three, and the theta divisor is the canonical model of the symmetric product of C; this implies that to prove point 1. of Theorem 4.0.1 the authors just need to show that alb_W is birational;
- knowing that $K^2_{\mathrm{alb}_W(W)} = 6$ (because $\mathrm{alb}_W(W)$ is the symmetric product of a curve of genus three) and the aforementioned bounds on K_W^2 enables them to show the birationality of alb_W through a simple computation.

In positive characteristic, in the case of ample albanese image, one would want to show that the Albanese variety is a principally polarised abelian variety and that the image of the (birational) Albanese morphism is a theta divisor. One could try to use [BLNP12, Corollary 3.2] and reduce the problem, as in characteristic zero, to studying $V^1(\omega_S)$:

Theorem 4.7.1 ([BLNP12]). *Let X be a smooth projective algebric variety defined over an algebraically closed field satisfying:*

a. $\chi(\omega_X) = 1$*;*

b. $\dim V^1(\omega_X) = 0$ *for all* $i > 0$*;*

c. $\dim X < \dim \mathrm{Alb}(X)$ *and* alb_X *is generically finite.*

Then $\mathrm{Alb}(X)$ *is a principally polarised abelian variety and the Albanese morphism maps X birationally onto a theta divisor.*

In our case $V^2(\omega_S) = \{\mathcal{O}_S\}$, so in order to apply this result one would have to show only that $V^1(\omega_S)$ is made of points, which is slightly less of what the authors of [HP02] had to show in order to use the older result by Hacon. Moreover, the above result would yield the birationality of the Albanese morphism, and no additional computations would be needed.

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