Transversals of longest paths*

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Abstract

Let $\mathrm{lpt}(G)$ *be the minimum cardinality of a transversal of longest paths in* G*, that is, a set of vertices that intersects all longest paths in a graph* G*. There are several results in the literature bounding the value of* $\mathrm{lpt}(G)$ *in general or in specific classes of graphs. For instance,* $\mathrm{lpt}(G) = 1$ *if* G *is a connected partial 2-tree, and a connected partial 3-tree* G *is known with* $\mathrm{lpt}(G) = 2$*. We prove that* $\mathrm{lpt}(G) \leq 2$ *for every planar 3-tree* G*; that* $\mathrm{lpt}(G) \leq 3$ *for every connected partial 3-tree* G*; and that* $\mathrm{lpt}(G) = 1$ *if* G *is a connected bipartite permutation graph or a connected full substar graph. Our first two results can be adapted for broader classes, improving slightly some known general results: we prove that* $\mathrm{lpt}(G) \leq k$ *for every connected partial* k*-tree* G *and that* $\mathrm{lpt}(G) \leq \max\{1, \omega(G) - 2\}$ *for every connected chordal graph* G*, where* $\omega(G)$ *is the cardinality of a maximum clique in* G*.*

I. Introduction

It is a well-known fact that, in a connected graph, any two longest paths have a common vertex. In 1966, Gallai raised the following question: *Does every connected graph contain a vertex that belongs to all of its longest paths?* The answer to Gallai's question is already known to be negative. Figure 1 shows the smallest known negative example, on 12 vertices, which was independently found by Walther and Voss [15] and Zamfirescu [16].

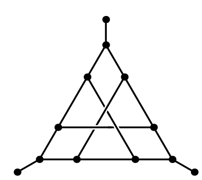

Figure 1: *The classical 12-vertex example that has a negative answer to Gallai's question.*

However, when we restrict ourselves to some specific classes of graphs, for instance trees, the answer to Gallai's question turns out to be positive. De Rezende *et al.* [7] generalized the result above, proving that the answer to Gallai's question is positive for 2-trees, and Chen *et al.* [6]

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extended this result for (connected) series-parallel graphs, also known as partial 2-trees. There are other classes of (connected) graphs which are known to have a positive answer to Gallai's question. Klavžar and Petkovšek [11] proved that this is the case for split graphs, cacti, and graphs whose blocks are Hamilton-connected, almost Hamilton-connected or cycles. Balister *et al.* [2] and Joos [10] proved the same for the class of circular arc graphs. Chen [5] proved the same for graphs with matching number smaller than three, while Cerioli and Lima [4, 12] proved it for P_4-sparse graphs, $(P_5, K_{1,3})$-free graphs, graphs that are the join of two other graphs and starlike graphs, a superclass of split graphs. Finally, Jobson *et al.* [9] proved it for dually chordal graphs and Golan and Shan [8] for $2K_2$-free graphs. A graph is called *chordal* if every induced cycle has length three. Several of the above mentioned classes consist of chordal graphs, so it is tempting to consider Gallai's question for chordal graphs.

A more general approach to Gallai's question is to ask for the size of the smallest *transversal of longest paths* of a graph, that is, the smallest set of vertices that intersects every longest path. Given a graph G, we denote the cardinality of such a set by $\text{lpt}(G)$. In this direction, Rautenbach and Sereni [13] proved that $\text{lpt}(G) \leq \lceil \frac{n}{4} - \frac{n^{2/3}}{90} \rceil$ for every connected graph G on n vertices, and that $\text{lpt}(G) \leq k+1$ for every connected partial k-tree G. The latter result implies that $\text{lpt}(G) \leq \omega(G)$ for every connected chordal graph G, where $\omega(G)$ is the cardinality of a maximum clique of G. This leaves a wide gap, considering that no connected chordal graph G is known with $\text{lpt}(G) > 1$.

In this work, we provide exact results and upper bounds on the value of $\text{lpt}(G)$ when G belongs to some particular classes of graphs. Specifically, we prove that:

- $\text{lpt}(G) = 1$ for every connected bipartite permutation graph G.
- $\text{lpt}(G) = 1$ for every connected full substar graph G.
- $\text{lpt}(G) \leq 2$ for every planar 3-tree G and $\text{lpt}(G) \leq 3$ for every connected partial 3-tree G.

The upper bounds above are either tight or off by one because the graph G in Figure 1 is a partial 3-tree with $\text{lpt}(G) = 2$. We in fact show that $\text{lpt}(G) \leq k$ for every connected partial k-tree G, and derive as a corollary the second part of the third result. Also, the proof of the first part of the third result above can be adjusted to show that $\text{lpt}(G) \leq \max\{1, \omega(G) - 2\}$ for every connected chordal graph G. Both of these more general results are slight improvements on one of the results of Rautenbach and Sereni [13]. A summary of the results on lpt is given in Figure 2.

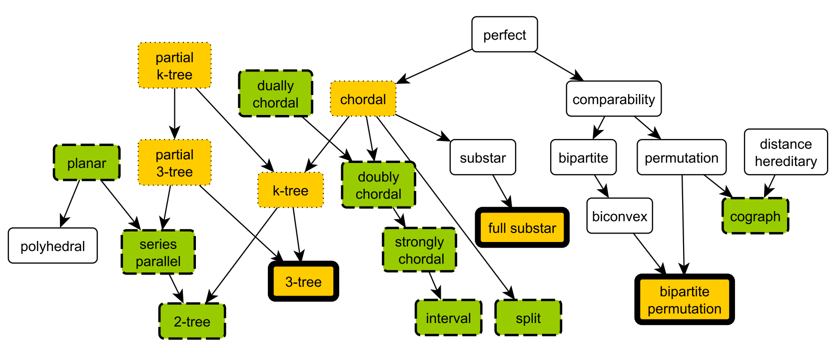

Figure 2: *A map with some classes addressed for* lpt*. For the classes within boxes with dashed borderline, there are previous results in the literature. For the classes within boxes with thick borderline, we present results in this paper. For the classes within boxes with dotted borderline, we also have results [3]. The classes whose boxes are white with thin straight borderline have not yet been studied.*

Before we proceed, let us establish some basic notation. All graphs considered in this paper are simple. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a graph G, respectively. For a vertex u, we denote by $N_G(u)$ the set of neighbors of u in G and by $d_G(u)$ the cardinality of $N_G(u)$. If the context is clear, we write simply $d(u)$ and $N(u)$ respectively. Let P be a path in G. We denote by $|P|$ the length of P, that is, the number of edges in P. Given a path Q such that the only vertex it shares with P is an extreme of both of them, we denote by $P \cdot Q$ the concatenation of P and Q. For a vertex v in P, let P' and P'' be the paths such that $P = P' \cdot P''$ with $V(P') \cap V(P'') = \{v\}$. We refer to these two paths as the *v-tails* of P.

This paper is organized as follows. In Sections II, III, and IV, we consider, respectively, the class of bipartite permutation graphs, the class of full substar graphs, and the class of planar 3-trees. Section IV also contains comments on our results on chordal graphs and on partial 3-trees. In Section V, we state some open problems to be considered in future work.

II. Bipartite permutation graphs

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two parallel lines in the plane. Consider two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ of segments that join a point in $\mathcal{L}_1$ to a point in $\mathcal{L}_2$, such that the extremes of every two segments in $X \cup Y$ are distinct, no two segments in X intersect, and no two segments in Y intersect. We denote by r_i the extreme of x_i in $\mathcal{L}_1$ and by s_i the extreme of y_i in $\mathcal{L}_1$. We may assume that $r_1 < \cdots < r_n$ and $s_1 < \cdots < s_m$, and we write $x_i < x_j$ or $y_i < y_j$ if $i < j$. Let σ be the function that maps the extreme in $\mathcal{L}_1$ of each segment in $X \cup Y$ to its other extreme.

Consider an associated bipartite graph $G = (X, Y, E)$ where $xy \in E$ if and only if the segments x and y intersect each other. We call the tuple $(\mathcal{L}_1, \mathcal{L}_2, X \cup Y, \sigma)$ a *line representation* of G and a graph is called a *bipartite permutation graph* if it has a line representation (Figure 3).

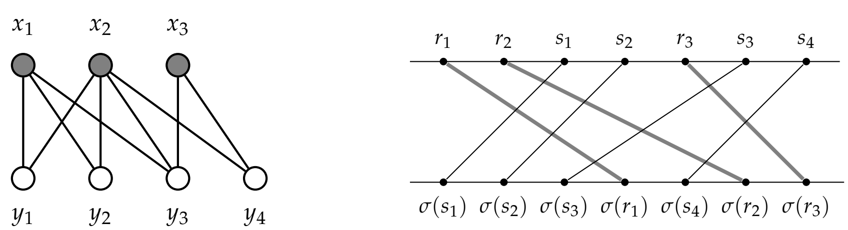

Figure 3: *A bipartite permutation graph and its corresponding line representation.*

In what follows, we assume that $G = (X, Y, E)$ is a connected bipartite permutation graph, with a line representation $(\mathcal{L}_1, \mathcal{L}_2, X \cup Y, \sigma)$ as above.

Consider $x_{i_1}, x_{i_2} \in X$ and $y_{j_1}, y_{j_2} \in Y$ such that $i_1 \leq i_2$ and $j_1 \leq j_2$. Spinrad et al. [14] showed that bipartite permutation graphs satisfy the following properties.

BP1) If x_{i_1} and $x_{i_2} \in N(y_{j_1})$, then $x_k \in N(y_{j_1})$ for $i_1 \leq k \leq i_2$.

BP2) If $x_{i_1}y_{j_2}$ and $x_{i_2}y_{j_1} \in E$, then $x_{i_1}y_{j_1}$ and $x_{i_2}y_{j_2} \in E$.

Using BP1) and BP2) repeatedly, we can generalize BP2) as follows for $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_\ell \leq m$.

BP3) If $x_{i_1}y_{j_\ell}, x_{i_k}y_{j_1} \in E$, then $\{x_{i_1}, \ldots, x_{i_k}, y_{j_1}, \ldots, y_{j_\ell}\}$ induces a complete bipartite graph in G.

We are interested in how longest paths intersect in a bipartite permutation graph. We begin by showing that, for every longest path, there exists a longest path with the same set of vertices that is ordered in some way. Precisely, if $P = a_1 b_1 \cdots a_k b_k$ is a path in G, we say that P is *ordered* if $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$. A similar definition applies when P has even length. As we are interested in vertex intersections, we may restrict attention to such ordered paths.

In what follows, let P be a path in G such that $V(P) \cap X = \{a_1, \ldots, a_k\}$ with $k \geq 1$ and $a_1 < \cdots < a_k$, and $V(P) \cap Y = \{b_1, \ldots, b_\ell\}$ with $\ell \geq 1$ and $b_1 < \cdots < b_\ell$. Let $X_i = \{a_1, \ldots, a_i\}$ and $\overline{X}_i = \{a_{i+1}, \ldots, a_k\}$ for $i = 0, \ldots, k$. Similarly, let $Y_j = \{b_1, \ldots, b_j\}$ and $\overline{Y}_j = \{b_{j+1}, \ldots, b_\ell\}$ for $j = 0, \ldots, \ell$. In particular, $X_0 = Y_0 = \emptyset$. We denote by $d_P(X_i)$ the sum $\sum_{v \in X_i} d_P(v)$ and by $d_P(Y_j)$ the sum $\sum_{w \in Y_j} d_P(w)$.

Proposition 1. *Let i, j be such that $0 \leq i \leq k$, $0 \leq j \leq \ell$, and $0 < i + j < k + \ell$. If $d_P(X_i) \geq d_P(Y_j)$, then there exists an edge from X_i to $\overline{Y}_j$. If $d_P(Y_j) \geq d_P(X_i)$, then there exists an edge from Y_j to $\overline{X}_i$.*

Proof. We will prove only the first claim, as the proof of the second one is analogous. If $j = 0$, then $i > 0$ and the statement holds because $|P| > 1$. Now suppose that $j > 0$ and, by contradiction, that $d_P(X_i) \geq d_P(Y_j)$ and that there is no edge from X_i to $\overline{Y}_j$. Because P is a connected bipartite subgraph of G and $i + j < k$, there must exist at least one edge from Y_j to $\overline{X}_i$, so

$$\begin{aligned}
d_P(Y_j) &= |\{wv \in E(P) : w \in Y_j,\ v \in X_i\}| + |\{wv \in E(P) : w \in Y_j,\ v \in \overline{X}_i\}| \\
&= |\{vw \in E(P) : v \in X_i,\ w \in V(P)\}| + |\{wv \in E(P) : w \in Y_j,\ v \in \overline{X}_i\}| \\
&= d_P(X_i) + |\{wv \in E(P) :\ w \in Y_j,\ v \in \overline{X}_i\}| \\
&> d_P(X_i),
\end{aligned}$$

a contradiction. ☐

Lemma 2. *There exists an ordered path with the same vertex set as P.*

Proof. Without loss of generality, we may assume that $k \geq \ell$ and that, if $k = \ell$, then $i^* \leq j^*$, where a_{i^*} is the extreme of P in X and b_{j^*} is the extreme of P in Y. (If $i^* > j^*$, interchange X and Y.) First, we will show that the following properties hold:

(a) there exists an edge with one end in X_i and the other in $\overline{Y}_{i-1}$ for $i = 1, \ldots, \ell$;

(b) there exists an edge with one end in Y_i and the other in $\overline{X}_i$ for every $i = 1, \ldots, k-1$.

To show (a), observe that $d_P(u) = 1$ for at most two vertices u in X_i (the extremes of P). Thus $d_P(X_i) \geq 2|X_i| - 2 = 2|Y_{i-1}|$. Because $d_P(w) \leq 2$ for every $w \in Y_{i-1}$, we have that $d_P(Y_{i-1}) \leq 2|Y_{i-1}|$. Hence $d_P(X_i) \geq d_P(Y_{i-1})$ and, as $i - 1 < \ell$, such edge exists by Proposition 1.

To show (b), first suppose that $k = \ell$. We have that $d_P(Y_i) \geq d_P(X_i)$ because $i^* \leq j^*$. Indeed, if $a_{i^*} \in X_i$, then $d_P(X_i) = 2|X_i| - 2 = 2|Y_i| - 2 \leq d_P(Y_i)$, and if $a_{i^*} \notin X_i$, then $b_{j^*} \notin Y_i$ as $i^* \leq j^*$. Therefore $d_P(X_i) = 2|X_i| = 2|Y_i| = d_P(Y_i)$. As $i < \ell$, by Proposition 1, such an edge exists. Now suppose that $k = \ell + 1$. Hence $d_P(w) = 2$ for every $w \in Y_i$ and $d_P(Y_i) = 2|Y_i| = 2|X_i| \geq d_P(X_i)$. As $i < k + 1$, Proposition 1 implies (b).

Now let $i < \ell$. By (a), there is a vertex a_q in X_i with a neighbor $b_{r'}$ in $\overline{Y}_{i-1}$. By (b), there is a vertex b_r in Y_i with a neighbor $a_{q'}$ in $\overline{X}_i$. As $a_q \leq a_i < a_{i+1} \leq a_{q'}$ and $b_r \leq b_i \leq b_{r'}$, by Property BP3), both $a_i b_i$ and $b_i a_{i+1}$ are edges (Figure 4(a)). By (a), $a_\ell b_\ell$ is an edge, hence $a_1 b_1 \cdots a_\ell b_\ell$ is a path. This implies the result for the case $k = \ell$. Also, if $k = \ell + 1$, then $b_{k-1} a_k$ is an edge by (b), so $a_1 b_1 \cdots a_{k-1} b_{k-1} a_k$ is a path. ☐

Lemma 2 implies that we can restrict our attention to ordered longest paths from now on. To prove that $\mathrm{lpt}(G) = 1$, we proceed in two steps. First, we will prove that $\mathrm{lpt}(G) \leq 2$. In

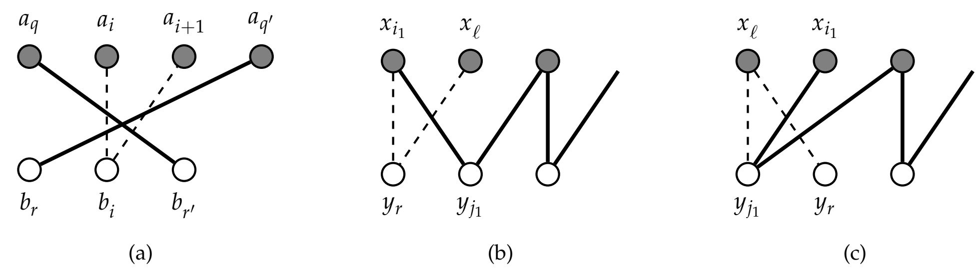

Figure 4: *The solid line segments represent the edges in P, while the dashed ones are other existing edges.*

fact, we prove that, for each edge of G, the set of its ends is a longest path transversal. Finally, we will prove that one element in $\{x_1, y_1\}$ is also a longest path transversal, which implies that $\text{lpt}(G) = 1$.

Let $x_{i_1}y_{j_1}$ and $x_{i_2}y_{j_2}$ be two edges in G. We say that $x_{i_1}y_{j_1}$ *hits* $x_{i_2}y_{j_2}$ if $(i_1 - i_2)(j_1 - j_2) < 0$. If that is not the case, we say they are *parallel*. We say that $|i_1 - i_2|$ is the *distance in X* and that $|j_1 - j_2|$ is the *distance in Y* between such edges. We denote by $dist_X(x_{i_1}y_{j_1}, x_{i_2}y_{j_2})$ and $dist_Y(x_{i_1}y_{j_1}, x_{i_2}y_{j_2})$ these two values respectively.

Lemma 3. *Let $vw \in E$, with $v \in X$ and $w \in Y$. Every ordered longest path contains a vertex of $\{v, w\}$.*

Proof. Suppose by contradiction that there exists an ordered longest path P that does not contain either v or w. We distinguish two cases.

Case 1. There is an edge $x_{i_1}y_{j_1}$ in P that hits vw.

Without loss of generality, assume that $x_{i_1} < v$ and $y_{j_1} > w$. By Property BP2), $x_{i_1}w$ and vy_{j_1} are edges. Let P_1 be the x_{i_1}-tail of P that does not contain y_{j_1}, and P_2 be the y_{j_1}-tail of P that does not contain x_{i_1}. Then $P_1 \cdot x_{i_1}wvy_{j_1} \cdot P_2$ is a path longer than P, a contradiction.

Case 2. Every edge in P is parallel to vw.

Let $x_{i_1}y_{j_1}$ be the edge of P that is "closer" to vw. That is,

$$\begin{aligned} dist_X(x_{i_1}y_{j_1}, vw) &= \min\{dist_X(e, vw) : e \in E(P)\} \quad \text{and} \\ dist_Y(x_{i_1}y_{j_1}, vw) &= \min\{dist_Y(e, vw) : e \in E(P)\}. \end{aligned}$$

Observe that, as P is an ordered path, one of $\{x_{i_1}, y_{j_1}\}$ is an extreme of P. Suppose that x_{i_1} is such an extreme. (A similar proof applies when this is not the case.) Without loss of generality, we may assume that $x_{i_1} > v$ and that P is a path with minimum value of x_{i_1} among all such paths.

Let H be the subgraph of G induced by the vertices $\{x_i : i \geq i_1\} \cup \{y_j : j \geq j_1\}$. Since G is connected and $G \neq H$, there is an edge between H and $G - V(H)$. First suppose that such edge links vertices x_ℓ in H and y_r in $G - V(H)$. Then, by Property BP2), x_{i_1} is adjacent to y_r. Hence $y_r x_{i_1} \cdot P$ is also a path, a contradiction (Figure 4(b)). Now suppose that there is an edge between a vertex x_ℓ in $G - V(H)$ and a vertex y_r in H. Then, by Property BP2), x_ℓ is adjacent to y_{j_1}. So $Q = x_\ell y_{j_1} \cdot (P - x_{i_1})$ is also a longest path. As $V(Q) \setminus V(P) = \{x_\ell\}$, we have that $w \notin V(Q)$. Moreover, if $x_\ell \leq v$, then x_ℓ is adjacent to w by Property BP2), and $wx_\ell \cdot Q$ would be a path longer than P. So $v < x_\ell < x_{i_1}$, and Q is a longest path with all of its edges parallel to vw, which is a contradiction to the way P was chosen (Figure 4(c)). $\square$

Given a collection $\mathcal{C}$ of ordered longest paths, we say that $P \in \mathcal{C}$ is a *left-most* path if, for every other path $Q \in \mathcal{C}$ and for every i, the i-th vertex of P in X is smaller than or equal to the i-th vertex of Q in X, and the same applies with Y instead of X. Such a path exists because all paths in $\mathcal{C}$ are ordered.

Theorem 4. *For every connected bipartite permutation graph G, $\mathrm{lpt}(G) = 1$.*

Proof. Let $G = (X, Y, E)$ with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ as before. Suppose by contradiction that $\mathrm{lpt}(G) > 1$. Then, there exist a longest path P that does not contain y_1 and a longest path Q that does not contain x_1. As G is connected, x_1y_1 is an edge by Property BP2). So, by Lemma 3, $x_1 \in V(P)$ and $y_1 \in V(Q)$. We may assume that $n \geq m$ and that both P and Q are ordered left-most paths. Thus, for every $i \in \{2, \ldots, m\}$, it suffices to prove the following:

(a) y_i is the $(2i-3)$-rd vertex of P, x_{i-1} is the $(2i-2)$-nd vertex of P, x_i is the $(2i-3)$-rd vertex of Q, and y_{i-1} is the $(2i-2)$-nd vertex of Q.

(b) x_iy_i is an edge.

Indeed, if that is the case, we would have a path $R = x_1y_1 \cdots x_my_m$ of length $2m-1$. But $|V(P) \cap Y| = |V(P) \cap \{y_2, \ldots, y_m\}| = m-1$ and thus $|V(P) \cap X| \leq m$, because $y_1 \notin V(P)$ and G is bipartite. Hence $|P| \leq 2m-2 < |R|$, a contradiction (Figure 5).

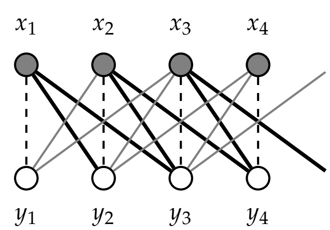

Figure 5: *The solid bold line segments indicate P while the solid thin line segments indicate Q.*

We prove (a) and (b) by induction on i. If $i = 2$, we need to prove that y_2x_1 and x_2y_1 are the first edges of P and Q respectively, and that x_2y_2 is an edge. Remember that $x_1 \in V(P)$. Obviously x_1 is not an extreme of P. So x_1 is the second vertex of P, because P is an ordered longest path. Now we will prove that y_2 is the first vertex of P. If P starts at y_j with $j > 2$, then x_1y_2 is an edge by Property BP1), because y_jx_1 and x_1y_1 are edges. Thus $y_2x_1 \cdot (P - y_j)$ is also a longest path, contradicting the choice of P. A similar reasoning shows that x_2y_1 is the first edge of Q. This implies that x_2y_2 is an edge by Property BP2), finishing the base case of the induction.

Now fix an $i > 2$ and assume that both (a) and (b) are valid for all $j < i$. Then, by the induction hypothesis, $y_{i-1}x_{i-2}$ is the $(2i-5)$-th edge of P. First, we will prove that x_{i-1} is the $(2i-2)$-nd vertex of P. Indeed, suppose that x_j is the $(2i-2)$-nd vertex of P with $j > i-1$. Let P' and P'' be the x_{i-2}-tails of P with y_2 in P'. Then $y_1x_1 \cdots y_{i-2}x_{i-2} \cdot P''$ is an ordered longest path that contains no vertex of $\{x_{i-1}, y_{i-1}\}$, a contradiction to Lemma 3. So x_{i-1} is the $(2i-2)$-nd vertex of P. Now, we will prove that y_i is the $(2i-3)$-rd vertex of P. Suppose that y_j is the $(2i-3)$-rd vertex of P with $j > i$. Then, as $x_{i-1}y_{i-1}$ is an edge by the induction hypothesis, $x_{i-2}y_i$ and $x_{i-1}y_i$ are edges by Property BP3). Hence, by substituting y_j by y_i in P, we obtain a longest path that contradicts the choice of P. A similar argument shows that x_i is the $(2i-3)$-rd vertex of Q and that y_{i-1} is the $(2i-2)$-nd vertex of Q. This implies that x_iy_i is an edge by Property BP2), finishing the proof. □

III. Full substar graphs

We now consider a subclass of chordal graphs, namely the full substar graphs. A *star* is a complete bipartite graph $K_{1,k}$ and, given a tree T, a *substar* is a subgraph of T that is a star. The vertex of degree k is the *center* of the star (if $k = 1$, we pick an arbitrary vertex to be its center). A substar with center x is a *full substar* if $k \geq d_T(x) - 1$. A graph is a *full substar graph* if it is an intersection graph of a set of full substars of a tree. The tree and the set of full substars is an intersection model for the full substar graph (Figure 6).

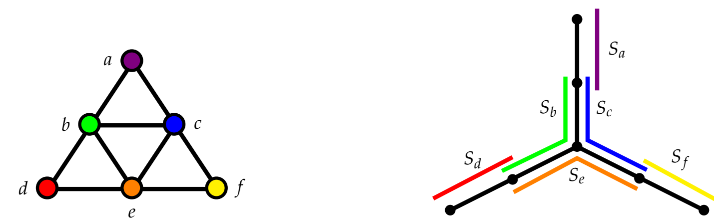

Figure 6: *A full substar graph and its intersection model.*

Given a full substar graph G and its intersection model in a host tree T, we denote by S_x the substar associated with $x \in V(G)$. We use capital letters to refer to the vertices of T and lowercase letters to refer to the vertices of G. For a vertex $X \in V(T)$, let $\mathcal{C}_X$ be the set of vertices of G whose corresponding stars are centered in X, and $\mathcal{C}_Y^X$ be the set of vertices of G whose stars are centered in a vertex that belongs to the component of $T - X$ in which Y lies. In what follows, G is a connected full substar graph and T is the host tree of an intersection model for G.

Lemma 5. *Let $x \in \mathcal{C}_X$. If P is a longest path in G and $x \notin V(P)$, then the following conditions hold:*

(i) $V(P) \subseteq \mathcal{C}_Y^X \cup \mathcal{C}_X$, *for a node* $Y \in N_T(X)$;

(ii) *if* $Y \in V(S_x)$, *then* $V(P) \cap \mathcal{C}_X = \emptyset$; *otherwise* $|V(P) \cap \mathcal{C}_X| \leq 1$.

Proof. First note that no two consecutive vertices in P lie in $\mathcal{C}_X$. Indeed, if two such vertices exist, they would be both adjacent to x and, by adding x to P between these two vertices, we would get a path longer than P, a contradiction. Similarly, no vertex in $\mathcal{C}_X$ is an extreme of P.

Suppose by contradiction that (i) does not hold. So P has vertices u and v whose substars are centered in different components of $T - X$. We may assume that the subpath of P between u and v is as short as possible. If $uv \in E(P)$, then X is in both S_u and S_v, hence x is adjacent to u and v, leading to a contradiction as above. If $uv \notin E(P)$, then there is a $w \in \mathcal{C}_X$ such that $uw, wv \in E(P)$. Because $|E(S_x)| \geq d_T(X) - 1$, one of $\{u, v\}$ is adjacent to x, leading again to a contradiction. This shows that (i) holds.

Let Y be as stated in (i). Suppose that $Y \in V(S_x)$ and, by contradiction, that there is an $x' \in V(P) \cap \mathcal{C}_X$. Because $|P| \geq 1$ and $V(P) \subseteq \mathcal{C}_Y^X \cup \mathcal{C}_X$, there exists $x'v \in E(P)$ such that $v \in \mathcal{C}_Y^X \cup \mathcal{C}_X$. Then $xx', xv \in E(G)$, which leads to a contradiction. Now suppose that $Y \notin V(S_x)$ and, by contradiction, that there are $x', x'' \in V(P) \cap \mathcal{C}_X$. Thus $Y \in V(S_{x'})$ because $|E(S_{x'})| \geq d_T(X) - 1$ and $x \neq x'$. Similarly $Y \in V(S_{x''})$. Let $w', w'' \in V(P)$ be such that $w'x', w''x'' \in E(P)$ and $x', x'' \notin V(P_{w'w''})$, where $P_{w'w''}$ denotes the subpath of P between w' and w''. Possibly $w' = w''$. Also, let $v \in V(P)$ be such that $v \neq w''$ and $x''v \in E(P)$. Note that $Y \in V(S_{w'}), V(S_{w''}), V(S_v)$, hence $x''w'$ and $w''v \in E(G)$. Let $P_{x'}$ be the x'-tail of P that does not contain x'' and P_v be the v-tail of P that does not contain x''. So $P_{x'} \cdot x'xx''w' \cdot P_{w'w''} \cdot w''v \cdot P_v$ is a path longer than P, a contradiction. $\square$

Lemma 6. *If* $\mathrm{lpt}(G) > 1$*, then for any* $X \in V(T)$ *there exists a longest path* P *in* G *and a node* $Y \in N_T(X)$ *such that* $V(P) \subseteq \mathcal{C}_Y^X$*.*

Proof. We divide the proof in the following two cases.

Case 1. $\mathcal{C}_X \neq \emptyset$.

Take $x \in \mathcal{C}_X$ with $|E(S_x)|$ as large as possible. Since $\mathrm{lpt}(G) > 1$, there is a longest path P such that $x \notin V(P)$. By Lemma 5(i), $V(P) \subseteq \mathcal{C}_Y^X \cup \mathcal{C}_X$ for some $Y \in N_T(X)$. If $V(P) \cap \mathcal{C}_X = \emptyset$, then $V(P) \subseteq \mathcal{C}_Y^X$. Otherwise, $Y \notin V(S_x)$ and $V(P) \cap \mathcal{C}_X = \{x'\}$ for some x', by Lemma 5(ii). Note that x' is not an extreme of P, because $xx' \in E(G)$ and $x \notin V(P)$. Also, if $N_G(x') \subseteq N_G(x)$, then we could add x to P between x' and its neighbor in P, a contradiction. Hence $N_G(x') \not\subseteq N_G(x)$. So $d_T(X) - 1 \leq |E(S_{x'})| \leq |E(S_x)| < d_T(X)$, and $|E(S_x)| = |E(S_{x'})| = d_T(X) - 1$. Moreover, there exists $Z \in N_T(X)$ such that $Z \neq Y$, $Z \in V(S_x)$, and $Z \notin V(S_{x'})$. Because $\mathrm{lpt}(G) > 1$, there exists a longest path Q that does not contain x'. By Lemma 5, $V(Q) \subseteq \mathcal{C}_Z^X \cup (\mathcal{C}_X \setminus \{x'\})$, implying that $V(P) \cap V(Q) = \emptyset$, which is a contradiction since G is connected.

Case 2. $\mathcal{C}_X = \emptyset$.

Suppose that every longest path P of G contains vertices whose substars are centered in two different components of $T - X$. Let us argue that $\mathrm{lpt}(G) = 1$. As $\mathcal{C}_X = \emptyset$, there exists $uv \in E(P)$ with S_u and S_v centered in $N_T(X)$. Let $K = \{x \in V(G) : X \in V(S_x)\}$. So $u, v \in K$ and $K \subseteq V(P)$ because K is a clique, and any vertex in K is in every longest path of G, that is, $\mathrm{lpt}(G) = 1$. ☐

We are now ready to prove the main result of this section.

Theorem 7. *If* G *is a connected full substar graph, then* $\mathrm{lpt}(G) = 1$*.*

Proof. Suppose by contradiction that $\mathrm{lpt}(G) > 1$. We create an auxiliary directed graph D on the same vertex set as T and arc set defined in the following way. For every $X \in V(D)$, we have that $XY \in E(D)$ if and only if $Y \in N_T(X)$ and there exists a longest path P such that $V(P) \subseteq \mathcal{C}_Y^X$. By Lemma 6, every node in T has outdegree at least one in D. Let XY be the last arc in a maximal directed path in D. Since T is a tree, YX is also an arc in D. Because both $XY, YX \in E(D)$, there exist two longest paths P and Q in G such that $V(P) \subseteq \mathcal{C}_Y^X$ and $V(Q) \subseteq \mathcal{C}_X^Y$. But $\mathcal{C}_X^Y \cap \mathcal{C}_Y^X = \emptyset$, implying that $V(P) \cap V(Q) = \emptyset$, a contradiction since G is connected. ☐

IV. Planar 3-trees and related classes

A 3-*tree* is defined recursively as follows. The complete graph on three vertices is a 3-tree. Any graph obtained from a 3-tree by adding a new vertex and making it adjacent to all the vertices of an existing triangle is also a 3-tree. Note that a 3-tree is planar if, in its recursive construction, the first triangle is used at most twice and any other triangle is used at most once. In this section, we will show that there is a transversal of size two in planar 3-trees.

Let S be a set of vertices in a graph G. We say that S *separates* vertices u and v if u and v are in different components of $G - S$. Let $X \subseteq V(G)$. We say that S separates X if S separates at least two vertices of X. A path P *crosses* S if S separates $V(P)$ in G. Otherwise S *fences* P. For a set $X \subseteq S$, we say that P crosses S *at* X if $S \cap V(P) = X$. In this case, if $X = \{x\}$, we simply say that P crosses S at x. If P and Q are paths fenced by S, we write $P \sim_S Q$ if there exist vertices $u \in V(P)$ and $v \in V(Q)$ such that u and v are in the same component of $G - S$. Otherwise, we write $P \nsim_S Q$. If the context is clear, we simply write $P \sim Q$ and $P \nsim Q$. Note that, if $V(P) \subseteq S$, then $P \nsim_S Q$ for every path Q. For an integer t, we say that P *t-intersects* S if P intersects S at exactly t vertices. If two paths P and Q intersect S at the same set of vertices, we say they are *S-equivalent*.

Given a path P that contains vertices a and b, we denote by P_a the a-tail of P that does not contain b and by P_b the b-tail of P that does not contain a. Also, if the context is clear, we denote by $\widetilde{P}$ the subpath of P that has a and b as its extremes. Thus $P = P_a \cdot \widetilde{P} \cdot P_b$.

Lemma 8. *Let G be a graph with a clique K. Let $\mathscr{C}$ be the set of all longest paths in G that 2-intersect and cross K, and whose extremes are not separated by K. For every three paths in $\mathscr{C}$, there are two of them that are K-equivalent.*

Proof. Suppose by contradiction that there are longest paths P, Q, and R in $\mathscr{C}$ that are not pairwise K-equivalent. Say $V(P) \cap K = \{a, b\}$, $V(Q) \cap K = \{c, d\}$, and $V(R) \cap K = \{e, f\}$, where $\{a, b\}$, $\{c, d\}$, and $\{e, f\}$ are pairwise distinct but not necessarily pairwise disjoint. We may assume that either $\{a, b\} \cap \{c, d\} = \emptyset$ or $\{a, b\} \cap \{c, d\} = \{b\} = \{d\}$. If $\tilde{P}$ is internally disjoint from Q_c and from Q_d, and $\tilde{Q}$ is internally disjoint from P_a and from P_b, then $P_a \cdot ac \cdot \tilde{Q} \cdot db \cdot P_b$ and $Q_c \cdot ca \cdot \tilde{P} \cdot bd \cdot Q_d$ are paths whose lengths sum more than $|P| + |Q|$, a contradiction (Figure 7(a)). Also, because P is a longest path, it has no extreme in K, so both P_a and P_b have vertices not in K and $P_a \sim P_b$. Analogously, $Q_c \sim Q_d$ and $R_e \sim R_f$. So, $\tilde{P} \sim Q_c \sim Q_d$ or $\tilde{Q} \sim P_a \sim P_b$. Applying the same reasoning to paths P and R, and to paths Q and R, we conclude that $\tilde{P} \sim R_e \sim R_f$ or $\tilde{R} \sim P_a \sim P_b$, and that $\tilde{Q} \sim R_e \sim R_f$ or $\tilde{R} \sim Q_c \sim Q_d$. As P, Q, and R cross K, from the previous, we have that

$$\tilde{P} \not\sim \tilde{Q}, \;\; \tilde{P} \not\sim \tilde{R}, \;\text{ and }\; \tilde{Q} \not\sim \tilde{R}. \tag{1}$$

Without loss of generality, we may assume that $\tilde{P} \sim Q_c \sim Q_d$. (Otherwise, interchange P with Q, and $\{a, b\}$ with $\{c, d\}$.) Now, if $\tilde{P} \sim R_e$, then $\tilde{Q} \not\sim R_e$ by (1), and thus $\tilde{R} \sim Q_c$. But then $\tilde{R} \sim \tilde{P}$, and we contradict (1). Hence, $\tilde{P} \not\sim R_e \sim R_f$, and $\tilde{R} \sim P_a$. Similarly, one can deduce that $\tilde{R} \not\sim Q_c \sim Q_d$. Thus, $\tilde{Q} \sim R_e$ and, again, we can deduce that $\tilde{Q} \not\sim P_a \sim P_b$. We conclude that

$$P_a \sim P_b \not\sim \tilde{Q}, \;\; Q_c \sim Q_d \not\sim \tilde{R}, \;\text{ and }\; R_e \sim R_f \not\sim \tilde{P} \tag{2}$$

(Figure 7(b)). We may assume that either $\{a, b\} \cap \{e, f\} = \emptyset$ or $\{a, b\} \cap \{e, f\} = \{b\} = \{f\}$, and that either $\{c, d\} \cap \{e, f\} = \emptyset$ or $\{c, d\} \cap \{e, f\} = \{d\} = \{f\}$ (the proofs for other cases are analogous). Therefore, by (2), we have three paths, $P_a \cdot ac \cdot \widetilde{Q} \cdot db \cdot P_b$, $Q_c \cdot ce \cdot \widetilde{R} \cdot fd \cdot Q_d$, and $R_e \cdot ea \cdot \widetilde{P} \cdot bf \cdot R_f$, whose lengths sum more than $|P| + |Q| + |R|$, a contradiction. ☐

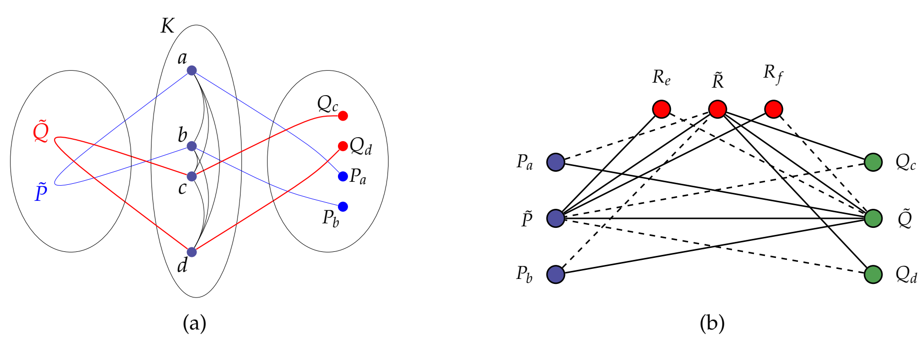

Figure 7: *(a) Paths P and Q from the proof of Lemma 8. (b) Paths P, Q, and R, and a tripartite graph representing the interaction between their parts. The graph has three vertices for each of the paths, one for each part. There are two types of edges: straight edges connect parts that are in different components of $G - K$ and dashed edges connect parts that are in the same component of $G - K$. When the interaction between two parts is not determined, we omit the edge between them.*

A *k-clique* is a clique with k vertices in G. A longest path P in G is an *attractor* for S if P is fenced by S and all S-equivalent longest paths are also fenced by S. We say that P is a *k-attractor* if $|S \cap V(P)| = k$. A graph H is called a *minor* of the graph G if H can be formed from G by deleting edges and vertices and by contracting edges. In this case, we say G has an H-minor.

Lemma 9. *Let G be a 3-tree with a 4-clique K. If $\operatorname{lpt}(G) > 2$, then there is a k-attractor for K with $k \leq 2$.*

Proof. Suppose by contradiction that there exists no k-attractor for K with $k \leq 2$. If there is a longest path in G that does not intersect K, then such path is a 0-attractor, a contradiction. Thus, we may assume that any longest path intersects K at least once.

Case 1. There is a longest path that 1-intersects K.

Because there is no 1-attractor for K, there exists a longest path P crossing K at a vertex v. Let P' and P'' be the two v-tails of P. Let e be an edge of K containing v. As $\operatorname{lpt}(G) > 2$, there exists a longest path Q that does not contain any end of e. If Q 1-intersects K at a vertex w, then, as there is no 1-attractor, we can choose Q crossing K at w. Let Q' and Q'' be the two w-tails of Q. Because both P and Q cross K, we may assume that $P' \nsim Q'$ and $P'' \nsim Q''$. But then $P' \cdot vw \cdot Q'$ and $P'' \cdot vw \cdot Q''$ are paths, one of them longer than P, a contradiction. Hence, there is a longest path Q that 2-intersects K at vertices different from the ends of e, say $\{w, z\}$. As there is no 2-attractor, we can choose Q crossing K at $\{w, z\}$. Because both P and Q cross K, we may assume that $P' \nsim Q_w$, $P'' \nsim \tilde{Q}$, and $P'' \nsim Q_z$. But then $P' \cdot vw \cdot Q_w$ and $P'' \cdot vw \cdot \tilde{Q} \cdot Q_z$ are paths, one of them longer than P, a contradiction.

Case 2. Every longest path intersects K at least twice.

Let $K = \{x, y, w, z\}$. Because $\operatorname{lpt}(G) > 2$, for every edge of K, there exists a longest path that 2-intersects K at the ends of that edge. By Lemma 8, at least four of these paths are such that their extremes are separated by K. Moreover, as there is no 2-attractor in G, there are two longest paths P and Q that cross K such that P 2-intersects K at $\{x, y\}$ and Q 2-intersects K at $\{w, z\}$. Because K separates the extremes of both P and Q, we may assume that $P_x \nsim Q_w$ and $P_y \nsim Q_z$, and also that $\widetilde{Q} \nsim P_x$ or $\widetilde{Q} \nsim P_y$. Now note that $\widetilde{P} \nsim Q_w$ or $\widetilde{P} \nsim Q_z$. Without loss of generality, assume that $\widetilde{P} \nsim Q_w$. Suppose that $\widetilde{Q} \nsim P_y$ (Figure 8(a)). Then one of the paths $P_x \cdot \widetilde{P} \cdot yw \cdot Q_w$ or $P_y \cdot yw \cdot \widetilde{Q} \cdot Q_z$ is longer than P, a contradiction. So, we may assume that $\widetilde{Q} \nsim P_x$. As G has no K_5-minor [1], $\widetilde{P} \nsim \widetilde{Q}$ (Figure 8(b)). Then one of $P_x \cdot \widetilde{P} \cdot yz \cdot \widetilde{Q} \cdot Q_w$ or $P_y \cdot yz \cdot Q_z$ is longer than P, again a contradiction. ☐

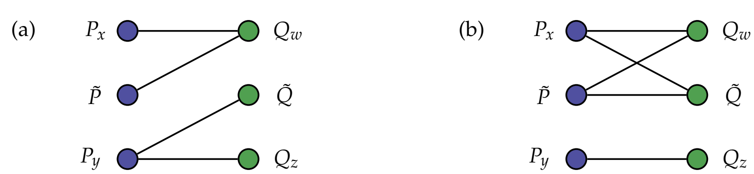

Figure 8: *Each bipartite graph represents the situation of the paths P and Q in one of the cases of the proof of Lemma 9. Each side of the bipartition has three vertices that represent the parts of each path. There is an edge in the graph if the two corresponding paths are internally disjoint.*

Let K be a 4-clique in a graph G. A triangle $\Delta \subseteq K$ is a *triangle attractor* for K if there exists a k-attractor P for K with $k \leq 2$, such that P is fenced by Δ and $P \nsim_\Delta v_\Delta$, where $\{v_\Delta\} = K \setminus \Delta$.

Corollary 10. *Let G be a 3-tree with a 4-clique K. If $\operatorname{lpt}(G) > 2$, then there is a triangle attractor for K.*

Proof. By Lemma 9, as $\mathrm{lpt}(G) > 2$, there is a k-attractor P for K with $k \leq 2$. By the definition of attractor, P is fenced by K, that is, either $V(P) \subseteq K$ or all the vertices of $P - K$ are in a single component H of $G - K$. In the former case, $\mathrm{lpt}(G) = 1$, a contradiction. Thus the latter is true. Because G is a 3-tree, G has no K_5-minor [1]. So there is no path in G internally disjoint from K between H and at least one of the vertices in K, say, vertex v_Δ. Let $\Delta = K - v_\Delta$. Let us argue that $v_\Delta \notin V(P)$. Suppose by contradiction that $v_\Delta \in V(P)$. Then there exists an edge $v_\Delta u \in E(P)$ and, by the previous, we must have that $u \in K$. Because P is a longest path, this implies that P contains all vertices of K, a contradiction, since $|V(P) \cap K| = k \leq 2$. We conclude that $v_\Delta \notin V(P)$. Then P is fenced by Δ and $P \not\sim_\Delta v_\Delta$, so Δ is a triangle attractor for K. □

We can finally prove the main result of this section.

Theorem 11. *For every planar 3-tree G,* $\mathrm{lpt}(G) \leq 2$.

Proof. Suppose by contradiction that $\mathrm{lpt}(G) > 2$. Then, by Corollary 10, for every 4-clique K in G, there exists a triangle attractor Δ for K. We construct a bipartite graph $H = (A, B, E(H))$ as follows. The vertices of A are the 4-cliques of G. The vertices of B are the triangles of G that separate $V(G)$. As G is a planar 3-tree, $|A| = n - 3$ and $|B| = n - 4$, where $n = |V(G)|$. There is an edge between a K in A and a Δ in B if Δ is a triangle attractor for K. By Corollary 10, every K in A has degree at least one. Thus, there exists a vertex Δ in B with degree at least two. So there are 4-cliques K_1 and K_2 in G such that Δ is a triangle attractor for K_1 and K_2. Let $\{u_1\} = K_1 \setminus \Delta$. Let $\{u_2\} = K_2 \setminus \Delta$. Hence, there exist longest paths P and Q in G such that P is a k_1-attractor for K_1 with $k_1 \leq 2$ and $P \not\sim_\Delta u_1$, and Q is a k_2-attractor for K_2 with $k_2 \leq 2$ and $Q \not\sim_\Delta u_2$. Therefore, because G is planar, the number of components of $G - \Delta$ is two, so $V(P) \cap V(Q) \subseteq \Delta$ and, because G is connected, $V(P) \cap V(Q) \neq \emptyset$. Hence both of P and Q intersect Δ at least once.

Suppose that $V(P) \cap K_1 = \{v\}$. This implies that P and Q only intersect each other at v, and that v divides both longest paths in half. Let P' and P'' be the two v-tails of P, and let Q' and Q'' be the two v-tails of Q. We may assume without loss of generality that $u_2 \notin P'$ and $u_1 \notin Q'$. If $Q' \cap K_1 = \{v\}$, then $P' \cdot Q'$ is a longest path that crosses K_1 at v, a contradiction to the fact that P is a 1-attractor for K_1. So Q' 2-intersects K_1 at $\{v, x\}$, with $x \in \Delta \subseteq K_2$. But then $P' \cdot Q'$ is a longest path that crosses K_2 at $\{v, x\}$, a contradiction to the fact that Q is a 2-attractor for K_2. Therefore P 2-intersects K_1, say at ends of an edge xy.

By a similar reasoning, we conclude that Q 2-intersects K_2. First suppose that Q 2-intersects K_2 at the same vertices. Then $|P_x| = |Q_x|$, $|P_y| = |Q_y|$, and $|\tilde{P}| = |\tilde{Q}|$. If $u_1 \notin V(Q_x)$, then $P_y \cdot \tilde{P} \cdot Q_x$ is a longest path that crosses K_1 at $\{x, y\}$, a contradiction because P is a 2-attractor for K_1. Hence $u_1 \in V(Q_x)$ and $u_1 \notin V(\tilde{Q})$. Then $P_x \cdot \tilde{Q} \cdot P_y$ is a longest path that crosses K_1 at $\{x, y\}$, again a contradiction. So we may assume that Q 2-intersects K_2 at the ends of an edge yz with $z \neq x$. But then $P_x \cdot xz \cdot \tilde{Q} \cdot P_y$ and $Q_z \cdot zx \cdot \tilde{P} \cdot Q_y$ are paths, yielding the final contradiction. □

We can generalize the previous result for 3-trees that are not planar, and more broadly for connected chordal graphs, as stated in the next theorem.

Theorem 12. *For every connected chordal graph G,* $\mathrm{lpt}(G) \leq \max\{1, \omega(G) - 2\}$*, where $\omega(G)$ is the cardinality of a maximum clique in G.*

The proof of Theorem 12 is long and technical. We omit it here, and present it in [3]. Note that $\omega(G) = 4$ for a 3-tree G, hence Theorem 12 is indeed a generalization of Theorem 11. Also, $\omega(G) = \mathrm{tw}(G) + 1$ for every chordal graph G, where $\mathrm{tw}(G)$ is the treewidth of G. So Theorem 12 implies that $\mathrm{lpt}(G) \leq \max\{1, \mathrm{tw}(G) - 1\}$ for a connected chordal graph G. Even though it is conceivable that $\mathrm{lpt}(G) = 1$ for every chordal graph G, as far as we know, the previously best known upper bound on $\mathrm{lpt}(G)$ for chordal graphs comes from the more general upper bound of Rautenbach and Sereni [13] that states that $\mathrm{lpt}(G) \leq \mathrm{tw}(G) + 1$ for every connected graph G. Thus Theorem 12 provides a slight improvement for chordal graphs.

A graph that is a subgraph of a 3-tree is called a *partial* 3*-tree*. Given that $\mathrm{lpt}(G) \leq 2$ for every 3-tree G, it is natural to ask whether the same holds for every connected partial 3-tree G. Such a result would be tight, because the graph in Figure 1 is a partial 3-tree. We prove that $\mathrm{lpt}(G) \leq 3$ for every connected partial 3-tree G. In fact, we prove the more general statement that $\mathrm{lpt}(G) \leq k$ for every connected partial k-tree G or, alternatively, the next theorem.

Theorem 13. *For every connected graph G,* $\mathrm{lpt}(G) \leq \mathrm{tw}(G)$*.*

We also omit the proof of this theorem here, and present it in [3]. Note that Theorem 13 is a slight improvement on the result of Rautenbach and Sereni [13] mentioned above.

V. Final remarks

Table 1 summarizes the results on transversals of longest paths.

Class	best upper bound on lpt	References
Arbitrary	$\left\lceil \frac{n}{4} - \frac{n^{2/3}}{90} \right\rceil$	[13]
Partial k-tree	k	Theorem 13
Partial 3-tree	3	Theorem 13
Chordal	$\max\{1, \omega(G) - 2\}$	Theorem 12
k-Tree	$\max\{1, k - 1\}$	Theorem 12
Planar 3-tree	2	Theorem 11
Bip. permutation	1	Theorem 4
Full substar	1	Theorem 7
Dually chordal	1	[9]
Join graph	1	[4, 12]
Split	1	[11]

Table 1: *A summary of the main results on transversals of longest paths.*

In this work, we proved that connected bipartite permutation graphs admit a transversal of size one. The problem remains open for connected biconvex graphs and connected permutation graphs, well-known superclasses of bipartite permutation graphs.

The class of full substar graphs is a particular class of chordal graphs. We proved that these graphs admit a transversal of size one. It would be nice to extend this result for larger classes of chordal graphs, such as substar graphs (intersection graphs of substars of a tree). Another subclass of chordal graphs is the class of 3-trees. It would be interesting to settle whether or not 3-trees admit a transversal of size one.

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