

S_3 -inspired three-Higgs-doublet models: A class with a complex vacuumA. Kunčinas^{*} and P. Osland[†]*Department of Physics and Technology, University of Bergen, Postboks 7803, N-5020 Bergen, Norway*O. M. Ogreid[‡]*Western Norway University of Applied Sciences, Postboks 7030, N-5020 Bergen, Norway*M. N. Rebelo[§]*Centro de Física Teórica de Partículas—CFTP and Departamento de Física, Instituto Superior Técnico—IST, Universidade de Lisboa, Avenida Rovisco Pais, P-1049-001 Lisboa, Portugal, and CERN, Theoretical Physics Department, CH-1211 Geneva 23, Switzerland*

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In this paper we analyze in detail an S_3 -symmetric three-Higgs-doublet model with a specific vacuum configuration. This analysis allows us to illustrate important features of models with several Higgs doublets, such as the possibility of having spontaneous CP violation. We start with a real potential and pick a particularly interesting complex vacuum configuration, which does not violate CP before adding soft breaking terms to the potential. We study the role played by different soft symmetry breaking terms. These are essential for our choice of vacuum in order to remove unwanted massless scalars that arise from the spontaneous breaking of an accidental continuous symmetry. We list scalar sector and scalar-gauge sector couplings for the particular case we consider in detail in this work. Results presented in this paper will be useful for model building, in particular for implementations of models with S_3 symmetry and spontaneous CP violation, extensions of the fermionic sector with realistic Yukawa couplings and for dark matter studies.

DOI: [10.1103/PhysRevD.101.075052](https://doi.org/10.1103/PhysRevD.101.075052)**I. INTRODUCTION**

In the Standard Model there is only one Higgs doublet, leading, through the Higgs mechanism [1–4], to the existence of one Higgs boson. Such a particle was discovered in 2012 at the LHC [5,6]. The question remains of whether there are additional Higgs bosons in nature. Multi-Higgs extensions of the Standard Model (SM) are very well motivated. In particular, a lot of work has been done in the context of models with two Higgs doublets; for reviews see [7,8] as well as in the context of models with three or more Higgs doublets [9]. As the complexity of the Higgs sector grows, the number of free parameters increases [10].

Symmetries play an important role in controlling the number of free parameters, therefore increasing the

predictability of such extensions. In these scenarios some of the problems of the SM, such as the need for new sources of CP violation, can be addressed. These models have a rich phenomenology and will be tested at the LHC and future colliders. An important feature of multi-Higgs extensions of the SM is the possibility of having spontaneous CP violation. It was shown by T. D. Lee [11] that models with two Higgs doublets can violate CP spontaneously. Imposing additional symmetries may eliminate the possibility of having spontaneous CP violation. On the other hand, continuous symmetries broken by vacuum expectation values (VEVs) lead to the existence of massless scalars [12–14]. These are ruled out by experiment. There are also strong experimental constraints that have to be taken into consideration when extending the Higgs sector of the SM [15].

In this paper, we revisit the different vacuum solutions for the S_3 -symmetric potential with three Higgs doublets which were studied previously in Ref. [16]. The S_3 -symmetric scalar sector with three Higgs doublets was studied in the past by several authors, starting in 1977 by Pakvasa and Sugawara [17] who worked with irreducible representations consisting of a doublet and singlet of S_3 and also analyzed couplings to fermions. Derman and Tsao [18,19] shortly afterwards discussed several properties of these models in terms of the defining representation of S_3 .

^{*}Anton.Kuncinas@protonmail.com[†]Per.Osland@uib.no[‡]omo@hvl.no[§]rebelo@tecnico.ulisboa.pt

On leave of absence from CFTP/IST, University of Lisboa.

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In Ref. [16], special attention was paid to the possibility of having spontaneous CP violation. However, it was also pointed out that several potentially interesting vacua led to the existence of massless scalars [16,20]. These massless states are due to the spontaneous breaking of accidental continuous symmetries resulting from constraints imposed on the region of parameters arising from the minimization conditions, with a few rare exceptions that will be pointed out in the paper. The identification of such symmetries is of great relevance and has been dealt with in the context of general three-Higgs-doublet models (3HDMs) by several authors [21,22] (see also Ref. [23]).

Here, we work with the irreducible representations, doublet and singlet, and we introduce soft breaking terms (terms bilinear in the fields) in the potential, which break the symmetries leading to massless scalars. First, we consider all possible forms for the soft breaking terms, and we separate the vacuum solutions according to whether they are real or complex. Next, we classify them according to the number of zero vacuum expectation values and their positions. The rest of our analysis centers on the study of a specific vacuum solution of the unbroken S_3 symmetry that suffers from unwanted massless bosons. This we call the C -III-c solution.

Before introducing soft breaking terms, the C -III-c vacuum displays very curious properties. It is the only vacuum allowing for a nontrivial phase, which is not determined by any of the parameters of the potential and remains free. However, there is a term in the potential that is sensitive to this phase, denoted λ_7 below. This term accounts for a coupling between two fields from the S_3 doublet and two fields from the S_3 singlet. Since the C -III-c vacuum and its generalizations when soft breaking terms are included have a vanishing singlet VEV, this dependence on the phase will enter the mass-squared matrix for the singlet states.

This phase will also enter in trilinear couplings. When soft breaking terms are introduced, this phase will also have an impact on other parts of the potential.

This paper is organized as follows. In Sec. II we explain the general framework together with a brief discussion of some of the vacuum solutions obtained in [16]. In Sec. III we discuss the origin of the massless states and in Sec. IV we review some properties of the C -III-c model. In Sec. V we list all possible forms for the vacua in terms of zero VEVs, and we discuss the effects of each allowed soft breaking term. Next, in Sec. VI we focus our attention on the discussion of a class of S_3 -inspired 3HDMs with a complex vacuum characterized by having zero VEV for the S_3 singlet and with the two other VEVs being arbitrary complex. Previously, we called this vacuum C -III-c before introducing soft breaking terms. In Ref. [16] we had shown that without soft breaking terms this vacuum did not violate CP spontaneously, despite being complex. We now show what effect the introduction of the different soft breaking terms can have on the CP properties of this complex vacuum. Finally, in the last section, we present our summary. Results presented in this paper will be important for model building, in particular for implementations of models with S_3 symmetry and spontaneous CP violation, extensions of the fermionic sector with realistic Yukawa couplings and for dark matter studies.

II. FRAMEWORK

The S_3 symmetry is a symmetry for the permutation of three objects, in this case three Higgs doublet fields, ϕ_1 , ϕ_2 , and ϕ_3 .

The scalar potential expressed in terms of the S_3 irreducible representation singlet and doublet fields, respectively, (h_S) and (h_1, h_2), can be written as

$$V = V_2 + V_4, \quad (2.1)$$

with [24–26]

$$V_2 = \mu_0^2 h_S^\dagger h_S + \mu_1^2 (h_1^\dagger h_1 + h_2^\dagger h_2), \quad (2.2a)$$

$$\begin{aligned} V_4 = & \lambda_1 (h_1^\dagger h_1 + h_2^\dagger h_2)^2 + \lambda_2 (h_1^\dagger h_2 - h_2^\dagger h_1)^2 + \lambda_3 [(h_1^\dagger h_1 - h_2^\dagger h_2)^2 + (h_1^\dagger h_2 + h_2^\dagger h_1)^2] \\ & + \lambda_4 [(h_S^\dagger h_1)(h_1^\dagger h_2 + h_2^\dagger h_1) + (h_S^\dagger h_2)(h_1^\dagger h_1 - h_2^\dagger h_2) + \text{H.c.}] + \lambda_5 (h_S^\dagger h_S)(h_1^\dagger h_1 + h_2^\dagger h_2) \\ & + \lambda_6 [(h_S^\dagger h_1)(h_1^\dagger h_S) + (h_S^\dagger h_2)(h_2^\dagger h_S)] + \lambda_7 [(h_S^\dagger h_1)(h_S^\dagger h_1) + (h_S^\dagger h_2)(h_S^\dagger h_2) + \text{H.c.}] + \lambda_8 (h_S^\dagger h_S)^2, \end{aligned} \quad (2.2b)$$

where we are taking all coefficients to be real. Therefore, there is no explicit CP violation.

For the $SU(2)$ doublets we use the notation

$$h_i = \begin{pmatrix} h_i^+ \\ (w_i + \eta_i + i\chi_i)/\sqrt{2} \end{pmatrix}, \quad i = 1, 2, \quad h_S = \begin{pmatrix} h_S^+ \\ (w_S + \eta_S + i\chi_S)/\sqrt{2} \end{pmatrix}. \quad (2.3)$$

In some cases, it is also convenient to extract an overall phase.

The irreducible representations can be related to the defining set of Higgs doublets ϕ_1 , ϕ_2 , and ϕ_3 , by

$$\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \quad (2.4)$$

Early papers studying this potential were written in terms of the defining representation [18,19]. Notice that Eq. (2.4) chooses a particular direction in the space of doublets of the defining representation. There is no freedom in the direction of h_S , where they all play an equal role. However, for h_1 , any permutation of $(1, -1, 0)$ is equally valid, implying the corresponding changes in the definition of h_2 . There is no physics content in any of these arbitrary choices, but they translate differently in terms of consistency conditions when going from the defining representation to the irreducible representation. This should be clear from the discussion in Sec. 4 of Ref. [16]. Further comments will be presented below.

A full classification of all possible vacua, together with the necessary constraints on the parameters of the potential coming from the minimization conditions, was given in Ref. [16]. In Appendix A we present a summary of some of their properties. There are 11 vacua with all VEVs real (Table VI), which were denoted R followed by a further specification, and 17 vacua with at least one complex VEV (Table VII), for which the letter C is used. Each vacuum is labeled by R or C followed by a Roman number and possibly another alphabetic index. The Roman number indicates how many constraints on the parameters are required by the minimization conditions. The additional alphabetic index is used to distinguish different (real or complex) vacua with the same number of constraints.

It is instructive to summarize some properties of the possible vacua of the S_3 -symmetric potential. Table VI in Appendix A lists all possible real vacuum solutions, with λ_a defined by

$$\lambda_a = \lambda_5 + \lambda_6 + 2\lambda_7, \quad (2.5a)$$

$$\lambda_b = \lambda_5 + \lambda_6 - 2\lambda_7 \quad (2.5b)$$

(the quantity λ_b is to be used in the discussion of the complex vacua).

All real vacuum solutions other than $R-0$ and $R-I-1$ violate the S_3 symmetry spontaneously. For these solutions, the residual symmetries were discussed by Derman and Tsao [19].

The constraints given in Table VI come from the stationary-point conditions¹

¹While all consistent solutions correspond to local minima of the potential, it is not yet known whether they all correspond to global minima. This issue has been addressed in the context of 2HDMs [27,28], whereas for 3HDMs [29] it is not yet fully analyzed. Such results would give an important contribution to the understanding of 3HDMs.

$$2\mu_0^2 w_S + \lambda_4(3w_1^2 - w_2^2)w_2 + \lambda_a(w_1^2 + w_2^2)w_S + 2\lambda_8 w_S^3 = 0, \quad (2.6a)$$

$$[2\mu_1^2 + 2(\lambda_1 + \lambda_3)(w_1^2 + w_2^2) + 6\lambda_4 w_2 w_S + \lambda_a w_S^2]w_1 = 0, \quad (2.6b)$$

$$2\mu_1^2 w_2 + 2(\lambda_1 + \lambda_3)(w_1^2 + w_2^2)w_2 + 3\lambda_4(w_1^2 - w_2^2)w_S + \lambda_a w_2 w_S^2 = 0, \quad (2.6c)$$

which were discussed in Ref. [16]. We see that in order to classify all possible solutions to these equations, one must consider all configurations of the vacuum parameters w_i , in particular, all possible situations where two of the VEVs vanish as well as all the possible situations where only one VEV vanishes, and finally the situation where none of the VEVs vanishes (R -III). For each configuration of vacuum parameters, we determine the resulting constraints on the parameters of the potential, in each case solving for μ_0^2 and μ_1^2 (if possible). In particular, since both Eqs. (2.6b) and (2.6c) contain μ_1^2 , in order to solve them simultaneously we will get an additional constraint either on the parameters of the VEVs or on the parameters of the potential. For all vacua other than R -III, this translates into restrictions on the parameter space as well as restrictions on the allowed VEVs.

For $w_1 = 0$, the derivative given by Eq. (2.6b) is automatically zero and there is no clash. Otherwise, the terms proportional to λ_4 in Eqs. (2.6b) and (2.6c) must be restricted, requiring

$$\lambda_4(3w_2^2 - w_1^2)w_S = 0. \quad (2.7)$$

This can be achieved by having $\lambda_4 = 0$ or $w_1 = \pm\sqrt{3}w_2$ or $w_S = 0$. For $w_S = 0$, an additional condition arises from Eq. (2.6a),

$$\lambda_4(3w_1^2 - w_2^2)w_2 = 0, \quad (2.8)$$

which implies that either $\lambda_4 = 0$ or $w_2 = \pm\sqrt{3}w_1$ or else $w_2 = 0$.

The cases R -I-2 satisfy both Eqs. (2.7) and (2.8) without requiring $\lambda_4 = 0$. In terms of the defining representation they are obviously equivalent. In terms of the irreducible representation they obey Eq. (2.8) by having $w_2 = 0$ in case a, $w_2 = \sqrt{3}w_1$ in case b, and $w_2 = -\sqrt{3}w_1$ in case c. This difference is the result of having chosen a particular direction for h_1 , as pointed out above. In this case, the residual symmetry is S_2 . Another interesting set of cases with residual symmetry S_2 are the R -II-1 vacua, which are also equivalent in terms of the defining representation and obey the consistency conditions with $w_1 = 0$ in case a, $w_1 = -\sqrt{3}w_2$ in case b, and $w_1 = \sqrt{3}w_2$ in case c. The vacua R -II-2, R -II-3, and R -III all require $\lambda_4 = 0$.

TABLE I. Real vacua, for the unbroken S_3 case, with massless states and degeneracies indicated. The first entry in the parentheses refers to the charged sector, and the second one to the neutral sector.

Vacuum	Name	λ_4	Symmetry	Number of massless states	Degeneracies
R_{00x}	R -I-1	✓		None	(1, 2)
R_{0x0}	R -II-2	0	O(2)	(None, 1)	None
R_{x00}	R -I-2a	✓		None	None
R_{0xy}	R -II-1a	✓		None	None
R_{x0y}		0	O(2)	(None, 1)	None
R_{xy0}	R -I-2b,2c	✓		None	None
R_{xy0}	R -II-3	0	O(2)	(None, 1)	None
R_{xyz}	R -II-1b,1c	✓		None	None
R_{xyz}	R -III	0	O(2)	(None, 1)	None

A particularly interesting vacuum is the one identified as case C -III- c ,

$$(w_1, w_2, w_S) = (\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0), \quad (2.9)$$

with \hat{w}_1 and \hat{w}_2 real and positive. The three stationary-point constraints are

$$\mu_1^2 = -(\lambda_1 + \lambda_3)(\hat{w}_1^2 + \hat{w}_2^2), \quad (2.10a)$$

$$\lambda_2 + \lambda_3 = 0, \quad (2.10b)$$

$$\lambda_4 = 0. \quad (2.10c)$$

In this solution, λ_4 is required to be zero. It should be pointed out that λ_4 plays a very important role in this potential. Whenever $\lambda_4 \neq 0$, the only independent Higgs-family symmetries of the potential are the global U(1) symmetry and the symmetry under which h_1 changes sign. The combination of these two symmetries also yields a potential that is symmetric under the simultaneous change of sign of both h_2 and h_S . On the other hand, whenever $\lambda_4 = 0$, the potential acquires an additional (continuous) independent O(2) symmetry between the fields h_1 and h_2 . Combining this symmetry with the two symmetries that were already present without a vanishing λ_4 , one finds that now, with $\lambda_4 = 0$, the potential is also symmetric under the independent sign changes of either h_2 or h_S .

This O(2) symmetry allows for the rotation between h_1 and h_2 in such a way that, together with an appropriate overall rephasing, this vacuum can be transformed into [30]

$$(w_1, w_2, w_S) = (\hat{w} e^{i\sigma/2}, \hat{w} e^{-i\sigma/2}, 0), \quad (2.11)$$

so that it can easily be shown [30,31] that it does not violate CP spontaneously.

The continuous O(2) symmetry is spontaneously broken by the C -III- c vacuum, giving rise to a massless neutral scalar field. However, we have one more massless neutral scalar for which there is no explanation in terms of a

spontaneously broken continuous symmetry. We shall comment on its origin below. In all, for the C -III- c case, we have *two* massless neutral scalars that need to be removed in order to construct a physical model. One way to achieve this is by adding one or more terms that break the S_3 symmetry softly.

III. GOLDSTONE BOSONS

Several of the possible vacuum solutions of the S_3 -symmetric potential have massless scalars. These result from the spontaneous breakdown of accidental continuous symmetries that arise when we impose the constraints required for these solutions. In Tables I and II we list the number of massless scalars for each case, together with whether λ_4 is required to be zero.

For $\lambda_4 = 0$ the potential acquires an additional O(2) symmetry between the two members of the S_3 doublet. When this symmetry is broken by the vacuum, one massless scalar state appears. In some cases, λ_7 is also required to be zero, together with $\lambda_4 = 0$, and the potential acquires an additional U(1) symmetry that we denote by $U(1)_{h_S}$. This corresponds to the freedom of rephasing h_S independently from h_1 and h_2 . Once again, an additional massless scalar state appears when this symmetry is spontaneously broken. In the C -III- c case, the condition $\lambda_4 = 0$ is accompanied by $\lambda_2 + \lambda_3 = 0$. This last condition does not increase the symmetry, and therefore it may not be exactly preserved at all energy scales.² However, there are two massless states in the C -III- c case, as discussed in Sec. IV. Note that there is no vacuum that requires $\lambda_2 + \lambda_3 = 0$ or $\lambda_7 = 0$ without also having $\lambda_4 = 0$.

In the C -V case, all of these are required to be zero, in this case we can independently rephase any of the doublets h_1 , h_2 , and h_S , and therefore there are three U(1) symmetries which we denote by $U(1)_{h_1}$, $U(1)_{h_2}$, and $U(1)_{h_S}$. The spontaneous breakdown of the resulting symmetry

²We thank Pedro Ferreira for illustrating this feature in a private discussion.

TABLE II. Complex vacua, for the unbroken S_3 case, with massless states and degeneracies indicated, not taking into account would-be Goldstone bosons. The first entry in the parentheses refers to the charged sector, the second one to the neutral sector. Degeneracies only refer to massive pairs. In the footnotes below, L indicates that a linear expression in its arguments vanishes.

Vacuum	Name	λ_4	Symmetry	Number of massless states	Degeneracies
C_{0xy}	C-III-a	✓		None	None
C_{x0y}	C-III-b	0	O(2)	(None, 1)	None
C_{x0y}	C-IV-a	0 ^a	$O(2) \otimes U(1)_{h_s}$	(None, 2)	None
C_{xy0}	C-I-a	✓		None	(None, 2)
C_{xy0}	C-III-c	0 ^b	O(2)	(None, 2)	None
C_{xyz}	C-III-d,e	✓		None	None
C_{xyz}	C-III-f,g	0	O(2)	(None, 1)	None
C_{xyz}	C-III-h,i	✓		None	None
C_{xyz}	C-IV-b	0	O(2)	(None, 1)	None
C_{xyz}	C-IV-c	^c	...	(None, 1)	None
C_{xyz}	C-IV-d	0 ^a	$O(2) \otimes U(1)_{h_s}$	(None, 2)	None
C_{xyz}	C-IV-e	0	O(2)	(None, 1)	None
C_{xyz}	C-IV-f	^c	...	(None, 1)	None
C_{xyz}	C-V	0 ^{a,b}	$O(2) \otimes U(1)_{h_1} \otimes U(1)_{h_2} \otimes U(1)_{h_s}$	(None, 3)	None

^aAlso $\lambda_7 = 0$.

^bAlso $\lambda_2 + \lambda_3 = 0$.

^c $L(\lambda_2 + \lambda_3, \lambda_4), L(\lambda_2 + \lambda_3, \lambda_7)$.

$O(2) \otimes U(1)_{h_1} \otimes U(1)_{h_2} \otimes U(1)_{h_s}$ is responsible for the three massless states that appear in addition to the would-be Goldstone boson. Note that with an overall phase rotation of the three doublets one can always reduce an independent rephasing of the three doublets to an independent rephasing of any two of them. Additional U(1) symmetries only arise in cases with $\lambda_4 = 0$.

Tables I and II illustrate the fact that there are several vacua of the same generic form, when expressed, for instance, in terms of VEVs of the irreducible representations, as specified in the first column, which have different physical implications. This is due to the fact that the same generic solution can be obtained for different regions of parameters. Different regions may also lead to different accidental symmetries.

IV. THE C-III-c MODEL WITHOUT SOFT BREAKING TERMS

The C-III-c model (based on the “ σ vacuum”) has some peculiar properties. As mentioned above, it has two massless states in the neutral sector (apart from the would-be Goldstone boson). Removing them is the main purpose of introducing soft S_3 -breaking terms. This will be done in the next section. As pointed out before, in the C-III-c case the condition $\lambda_4 = 0$ is accompanied by $\lambda_2 + \lambda_3 = 0$. This last condition does not increase the symmetry. If, in addition, we were to have $\lambda_7 = 0$, then new continuous symmetries of the potential would exist just as in case C-V.

One may wonder then why, in this case, one has two massless scalars rather than only one. The reason has to do with the fact that there is no λ_7 term in the mass terms of the scalar fields coming from the S_3 doublet. This fact is accidental; it results from this vacuum configuration having $\hat{w}_S = 0$. The mass terms for these fields of the S_3 doublet mimic the existence of a larger symmetry under which h_1 and h_2 may be independently rephased, since we have $\lambda_2 + \lambda_3 = 0$ and $\lambda_4 = 0$. The vacuum is not invariant under this rephasing of h_1 and h_2 . However, rephasing of h_s independently from the other doublets leaves the vacuum invariant. Invariance under an overall rephasing of the three scalars implies that only one of these former two U(1) transformations is independent.

The technique proposed in Ref. [21] is a useful tool to search for symmetries, in the context of three-Higgs-doublet models, that are not explicit. That method applied to this case confirms the nonexistence of an additional continuous symmetry. In Secs. IVA and IV C we shall be confronted with the fact that the mass splitting of the neutral scalars in the S_3 singlet sector and some trilinear couplings will depend on σ , the relative phase of the two VEVs, which seems to be unrelated to the coefficients of the potential. This apparent paradox is addressed in Sec. IV D.

Examples of the connection between symmetries and mass degeneracies in two- and three-Higgs-doublet models with vanishing VEVs can be found in Ref. [32].

A. Masses

Since $\lambda_4 = 0$ and $\hat{w}_S = 0$, the S_3 doublet and the S_3 singlet do not mix in the mass terms. In the charged sector, we have

$$m_{H^\pm}^2 = 2\lambda_2 v^2, \quad (4.1)$$

$$m_{S^\pm}^2 = \mu_0^2 + \frac{1}{2}\lambda_5 v^2, \quad (4.2)$$

where $v^2 = \hat{w}_1^2 + \hat{w}_2^2$ and H^\pm and S^\pm refer to the charged states of the doublet and singlet sector, respectively.

In the neutral sector of the S_3 doublet, there is only one massive (CP -even) state,

$$m_h^2 = 2(\lambda_1 - \lambda_2)v^2, \quad (4.3)$$

which would have to be identified with the SM-like Higgs, since it appears in the doublet where the would-be Goldstone bosons are. There is no further mixing with the other fields. The S_3 singlet sector has two massive states (S_1 and S_2),

$$m_{S_1}^2 = \mu_0^2 + \frac{1}{2}(\lambda_5 + \lambda_6)v^2 - \lambda_7 \cos \sigma v^2, \quad (4.4a)$$

$$m_{S_2}^2 = \mu_0^2 + \frac{1}{2}(\lambda_5 + \lambda_6)v^2 + \lambda_7 \cos \sigma v^2. \quad (4.4b)$$

Thus, the phase σ , which is left undetermined by the potential, is related to the mass splitting of the neutral scalars in the S_3 singlet sector.

B. Gauge couplings

For the different couplings, we should define the degenerate fields carefully. Thus, rather than adopting the decomposition (2.3), we take

$$\begin{aligned} h_1 &= e^{i\sigma/2} \begin{pmatrix} h_1^+ \\ (\hat{w} + \eta_1 + i\chi_1)/\sqrt{2} \end{pmatrix}, \\ h_2 &= e^{-i\sigma/2} \begin{pmatrix} h_2^+ \\ (\hat{w} + \eta_2 + i\chi_2)/\sqrt{2} \end{pmatrix}, \end{aligned} \quad (4.5)$$

and

$$h_S = \begin{pmatrix} S^+ \\ (S_1 + iS_2)/\sqrt{2} \end{pmatrix}, \quad (4.6)$$

with $\hat{w}^2 = v^2/2$. Since the S_3 singlet has a vanishing VEV, it is straightforward to transform to the Higgs basis [33,34]. A convenient choice is to leave the singlet fields as they are and take

$$h_1^\pm = (G^\pm - H^\pm)/\sqrt{2}, \quad h_2^\pm = (G^\pm + H^\pm)/\sqrt{2}, \quad (4.7)$$

$$\eta_1 = (h - H)/\sqrt{2}, \quad \eta_2 = (h + H)/\sqrt{2}, \quad (4.8)$$

$$\chi_1 = (G^0 - A)/\sqrt{2}, \quad \chi_2 = (G^0 + A)/\sqrt{2}. \quad (4.9)$$

This choice fixes the definitions of the degenerate, massless bosons H and A (any orthogonal basis would be equally good).

The covariant derivatives induce gauge couplings, and those linear in the gauge fields are

$$\begin{aligned} \mathcal{L}_{VHH} &= ieA^\mu [(H^+ \overleftrightarrow{\partial}_\mu H^-) + (S^+ \overleftrightarrow{\partial}_\mu S^-)] - \frac{g}{2 \cos \theta_W} Z^\mu \{ (H \overleftrightarrow{\partial}_\mu A) + (S_1 \overleftrightarrow{\partial}_\mu S_2) - i \cos 2\theta_W [(H^+ \overleftrightarrow{\partial}_\mu H^-) + (S^+ \overleftrightarrow{\partial}_\mu S^-)] \} \\ &+ \frac{ig}{2} \{ W^\mu [(H \overleftrightarrow{\partial}_\mu H^-) + i(A \overleftrightarrow{\partial}_\mu H^-) + (S_1 \overleftrightarrow{\partial}_\mu S^-) + i(S_2 \overleftrightarrow{\partial}_\mu S^-)] - \text{H.c.} \}, \end{aligned} \quad (4.10)$$

where θ_W is the weak mixing angle. Furthermore, H and A denote the CP -even and -odd massless states. Next, the terms bilinear in gauge fields are

$$\mathcal{L}_{VVH} = \frac{g^2 v}{4 \cos^2 \theta_W} Z_\mu Z^\mu h + \frac{g^2 v}{2} W_\mu^\dagger W^\mu h, \quad (4.11)$$

and

$$\begin{aligned} \mathcal{L}_{VVHH} &= \left[\left(eA + \frac{g \cos 2\theta_W}{2 \cos \theta_W} Z \right)_\mu \left(eA + \frac{g \cos 2\theta_W}{2 \cos \theta_W} Z \right)^\mu + \frac{g^2}{2} W_\mu^\dagger W^\mu \right] (H^- H^+ + S^- S^+) \\ &+ \left[\frac{g^2}{8 \cos^2 \theta_W} Z_\mu Z^\mu + \frac{g^2}{4} W_\mu^\dagger W^\mu \right] (h^2 + H^2 + A^2 + S_1^2 + S_2^2) \\ &+ \frac{eg}{2} \{ A_\mu W^\mu [H^- (H + iA) + S^- (S_1 + iS_2)] + \text{H.c.} \} \\ &- \frac{g^2 \sin^2 \theta_W}{2 \cos \theta_W} \{ Z_\mu W^\mu [H^- (H + iA) + S^- (S_1 + iS_2)] + \text{H.c.} \}. \end{aligned} \quad (4.12)$$

Note that $\mathcal{L}_{V\!V\!H}$ contains no term linear in the singlet fields since it has a vanishing VEV.

C. Trilinear couplings

The nonzero trilinear couplings are (as coefficients of the potential)

$$hhh: v(\lambda_1 - \lambda_2), \quad (4.13a)$$

$$hAA: v(\lambda_1 - \lambda_2), \quad (4.13b)$$

$$hHH: v(\lambda_1 - \lambda_2), \quad (4.13c)$$

$$hH^+H^-: 2v(\lambda_1 + \lambda_2), \quad (4.13d)$$

$$hS^+S^-: v\lambda_5, \quad (4.13e)$$

$$hS_1S_1: \frac{v}{2}[\lambda_5 + \lambda_6 + 2\cos\sigma\lambda_7], \quad (4.13f)$$

$$hS_2S_2: \frac{v}{2}[\lambda_5 + \lambda_6 - 2\cos\sigma\lambda_7], \quad (4.13g)$$

$$AS_1S_1: v\sin\sigma\lambda_7, \quad (4.13h)$$

$$AS_2S_2: -v\sin\sigma\lambda_7, \quad (4.13i)$$

$$HS_1S_2: -2v\sin\sigma\lambda_7, \quad (4.13j)$$

$$H^+S^-S_1: -iv\sin\sigma\lambda_7, \quad (4.13k)$$

$$H^+S^-S_2: -v\sin\sigma\lambda_7. \quad (4.13l)$$

We have left out here couplings involving the would-be Goldstone bosons. Couplings involving H^-S^+ are obtained from those involving H^+S^- by complex conjugation. The AAA , HHH , hhA , hhH , AAH , H^+H^-A , H^+H^-H , S^+S^-A , and S^+S^-H couplings all vanish. The dependence on the phase σ only appears in couplings involving λ_7 and two fields from the S_3 singlet sector.

D. The phase σ

The phase σ governs the mass splitting between the two neutral fields coming from the S_3 singlet Higgs doublet, and it also appears in some of the trilinear couplings. This phase is always associated with λ_7 since in the C -III-c case this is the only coupling in the potential that is sensitive to a phase. Since the minimization conditions do not constrain this phase, and this phase was also not included in the Lagrangian, it looks as if there are physical quantities that depend on parameters that are not physical. This apparent paradox can be solved by noticing that σ can be promoted to a parameter of the potential by just rephasing the fields h_1 and h_2 in such a way that their VEVs become real. What is special in this case is that there is a parameter that can appear either in the scalar potential or in the specification of

the vacuum, depending on the choice of scalar basis. This parameter has physical implications and, in particular, contributes to the mass splitting of some scalars. This corresponds to the transformation given by Eqs. (4.5). The C -III-c vacuum differs from all other complex S_3 vacua, since in those cases the phases appear in the minimization conditions and therefore cannot be considered as free parameters [16].

As mentioned above, in a basis where $\hat{w}_1 = \hat{w}_2$, it is convenient to define $\sigma \equiv \sigma_1 - \sigma_2$. We recall that this phase σ is not determined by the potential. However, it parametrizes correlations among certain physical couplings, and among those couplings and the mass splitting in the neutral S_3 singlet sector. The mass splitting and couplings given above, by Eqs. (4.4) and (4.13), assume a basis where $\hat{w}_1 = \hat{w}_2 = v/\sqrt{2}$.

Clearly,

$$\Delta^2 \equiv m_{S_2}^2 - m_{S_1}^2 \quad (4.14)$$

is a physical quantity, expressed as $2\cos\sigma\lambda_7v^2$ in the adopted basis. Next, if we denote the AS_1S_1 coupling ($v\sin\sigma\lambda_7$) by the abbreviation \hat{g} , then we can identify the modulus of λ_7 as a physical quantity,³

$$|\lambda_7| = \sqrt{\left(\frac{\Delta^2}{2v^2}\right)^2 + \left(\frac{\hat{g}}{v}\right)^2}, \quad (4.15)$$

whereas σ (in the chosen basis) parametrizes the ratio of the two physical quantities Δ^2 and \hat{g} by

$$\tan\sigma = \frac{2\hat{g}v}{\Delta^2}. \quad (4.16)$$

V. THE SOFTLY BROKEN POTENTIAL

We now replace the potential (2.1) by

$$V = V_2 + V'_2 + V_4, \quad (5.1)$$

allowing for terms V'_2 that softly break the S_3 symmetry. The most general form of these terms is⁴

$$V'_2 = \mu_2^2(h_1^\dagger h_1 - h_2^\dagger h_2) + \frac{1}{2}\nu_{12}^2(h_1^\dagger h_2 + \text{H.c.}) + \frac{1}{2}\nu_{01}^2(h_S^\dagger h_1 + \text{H.c.}) + \frac{1}{2}\nu_{02}^2(h_S^\dagger h_2 + \text{H.c.}). \quad (5.2)$$

The vacua studied in Ref. [16] will then be modified. In the following we shall briefly discuss some general properties of the different vacua that result from the inclusion of

³It can be readily seen that this result does not depend on the basis chosen for the definition of H and A .

⁴As for the unbroken case, we restrict ourselves to terms with real coefficients. The analysis of the S_3 -symmetric model with complex parameters and with the addition of complex breaking terms would give an important contribution to the understanding of such models.

TABLE III. Real vacua compatible with the most general soft S_3 -breaking terms, Eq. (5.2), together with the minimization conditions.

Label	w_1, w_2, w_S	Constraints
R_{00x}	$(0, 0, w_S)$	$\mu_0^2 = -\lambda_8 w_S^2,$ $\nu_{01}^2 = \nu_{02}^2 = 0$
R_{0x0}	$(0, w, 0)$	$\mu_1^2 = \mu_2^2 - (\lambda_1 + \lambda_3)w^2,$ $\nu_{12}^2 = 0, \nu_{02}^2 = w^2 \lambda_4$
R_{x00}	$(w, 0, 0)$	$\mu_1^2 = -\mu_2^2 - (\lambda_1 + \lambda_3)w^2,$ $\nu_{12}^2 = \nu_{01}^2 = 0$
R_{0xy}	$(0, w, w_S)$	$\mu_0^2 = -\frac{1}{2}\nu_{02}^2 \frac{w}{w_S} + \frac{1}{2}\lambda_4 \frac{w^3}{w_S} - \frac{1}{2}\lambda_a w^2 - \lambda_8 w_S^2,$ $\mu_1^2 = \mu_2^2 - \frac{1}{2}\nu_{02}^2 \frac{w_S}{w} - (\lambda_1 + \lambda_3)w^2 + \frac{3}{2}\lambda_4 w w_S - \frac{1}{2}\lambda_a w_S^2,$ $\nu_{12}^2 = -\nu_{01}^2 \frac{w_S}{w}$
R_{x0y}	$(w, 0, w_S)$	$\mu_0^2 = -\frac{1}{2}\nu_{01}^2 \frac{w}{w_S} - \frac{1}{2}\lambda_a w^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -\mu_2^2 - \frac{1}{2}\nu_{01}^2 \frac{w_S}{w} - (\lambda_1 + \lambda_3)w^2 - \frac{1}{2}\lambda_a w_S^2,$ $\nu_{12}^2 = -\nu_{02}^2 \frac{w_S}{w} - 3\lambda_4 w w_S$
R_{xy0}	$(w_1, w_2, 0)$	$\mu_1^2 = -\nu_{12}^2 \frac{w_1^2 + w_2^2}{4w_1 w_2} - (\lambda_1 + \lambda_3)(w_1^2 + w_2^2),$ $\mu_2^2 = \nu_{12}^2 \frac{w_1^2 - w_2^2}{4w_1 w_2},$ $\nu_{01}^2 = [-\nu_{02}^2 + (w_2^2 - 3w_1^2)\lambda_4] \frac{w_2}{w_1}$
R_{xyz}	(w_1, w_2, w_S)	$\mu_0^2 = -\frac{1}{2}\nu_{01}^2 \frac{w_1}{w_S} - \frac{1}{2}\nu_{02}^2 \frac{w_2}{w_S} - \frac{1}{2}\lambda_4 \frac{w_2(3w_1^2 - w_2^2)}{w_S} - \frac{1}{2}\lambda_a (w_1^2 + w_2^2) - \lambda_8 w_S^2,$ $\mu_1^2 = -\nu_{12}^2 \frac{w_1^2 + w_2^2}{4w_1 w_2} - \frac{w_S}{4w_1} \nu_{01}^2 - \frac{w_S}{4w_2} \nu_{02}^2 - \frac{3}{4}\lambda_4 \frac{(w_1^2 + w_2^2)w_S}{w_2} - (\lambda_1 + \lambda_3)(w_1^2 + w_2^2) - \frac{1}{2}\lambda_a w_S^2,$ $\mu_2^2 = \nu_{12}^2 \frac{w_1^2 - w_2^2}{4w_1 w_2} - \frac{w_S}{4w_1} \nu_{01}^2 + \frac{w_S}{4w_2} \nu_{02}^2 - \frac{3}{4}\lambda_4 \frac{w_S(3w_2^2 - w_1^2)}{w_2}$

soft S_3 -breaking terms, employing a more generic terminology to label them. The labeling will specify how many and which VEVs vanish, and our focus will be on massless states and mass degeneracies. Our approach is to fix the zero VEVs in all possible positions and derive the resulting constraints.

A. Real vacua

We summarize in Table III the different real vacua for the softly broken S_3 -symmetric potential. In the following, we list some further properties, commenting also on the degeneracies that arise in the limit of no soft S_3 -breaking terms. This classification is based on considering all vacua with two, or one or with no vanishing VEVs, where the labels are self-explanatory. In Table VI we did not include the case corresponding to R_{x0y} because $w_2 = 0$ only appeared in the consistency conditions together with $w_S = 0$. However, this could obviously be a limit of the general case R -III.

Below, we briefly comment on some of the properties of the different categories of real vacua, allowing for soft breaking terms.

1. $(\mathbf{0}, \mathbf{0}, w_S)$

Soft breaking terms ν_{01}^2 and ν_{02}^2 do not survive minimization. If no soft breaking terms are present, there is

mass degeneracy among the charged scalars, as well as two pairs of mass degenerate neutral scalars. If either ν_{12}^2 or μ_2^2 is present, there is no mass degeneracy.

2. $(\mathbf{0}, w, \mathbf{0})$

The soft breaking term ν_{12}^2 does not survive minimization. All masses are nondegenerate with or without soft breaking terms. If no soft breaking terms are present (R -II-2), this vacuum requires $\lambda_4 = 0$ and it has one massless state. If ν_{02}^2 is the only soft breaking term, it requires $\lambda_4 \neq 0$, and we still have one neutral massless state. This massless state results from the condition relating ν_{02}^2 and λ_4 in Table III. If either μ_2^2 or ν_{01}^2 is present, then all neutral scalars become massive.

3. $(w, \mathbf{0}, \mathbf{0})$

Soft breaking terms ν_{12}^2 and ν_{01}^2 do not survive minimization. There are no massless scalars and no mass degeneracy with or without the soft breaking terms.

4. $(\mathbf{0}, w, w_S)$

All soft breaking terms survive minimization. There are no massless scalars and no mass degeneracy with or without the soft breaking terms. Note that the soft terms ν_{12}^2 and ν_{01}^2 are proportional; i.e., they have to coexist.

5. $(\mathbf{w}, \mathbf{0}, \mathbf{w}_S)$

All soft breaking terms survive minimization. There is no mass degeneracy with or without soft breaking terms. If no soft breaking term is present (requiring for consistency $\lambda_4 = 0$), one of the neutral masses vanishes. If there is at least one soft breaking term present, then all neutral scalars become massive.

6. $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{0})$

All soft breaking terms survive minimization. There is no mass degeneracy with or without soft breaking terms. If no soft breaking term is present (requiring for consistency

$\lambda_4 = 0$), one of the neutral masses vanishes. If there is at least one soft breaking term present, then all neutral scalars become massive. Note that the μ_2^2 term cannot be the only soft term.

7. $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_S)$

All soft breaking terms survive minimization. There is no mass degeneracy with or without soft breaking terms. If no soft breaking term is present (requiring for consistency $\lambda_4 = 0$), one of the neutral masses vanishes. If there is at least one soft breaking term present, then all neutral scalars become massive.

TABLE IV. Complex vacua compatible with the most general soft S_3 -breaking terms, Eq. (5.2), together with the minimization conditions. The following abbreviations are introduced: $S_{\pm} = \sin^2 \sigma_1 \hat{w}_1^2 \pm \sin^2 \sigma_2 \hat{w}_2^2$.

w_1, w_2, w_S	Constraints
$(0, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S) C_{0xy}$	$\begin{aligned} \mu_0^2 &= -\frac{1}{2} \lambda_b \hat{w}_2^2 - \lambda_8 \hat{w}_S^2, \\ \mu_1^2 &= \mu_2^2 - (\lambda_1 + \lambda_3) \hat{w}_2^2 + \cos \sigma_2 \hat{w}_2 \hat{w}_S \lambda_4 - \frac{1}{2} \lambda_b \hat{w}_S^2, \\ \nu_{02}^2 &= \hat{w}_2 (\hat{w}_2 \lambda_4 - 4 \cos \sigma_2 \hat{w}_S \lambda_7), \\ \nu_{12}^2 &= \nu_{01}^2 = 0 \end{aligned}$
$(\hat{w}_1 e^{i\sigma_1}, 0, \hat{w}_S) C_{x0y}$	$\begin{aligned} \mu_0^2 &= -\frac{1}{2} \lambda_b \hat{w}_1^2 - \lambda_8 \hat{w}_S^2, \\ \mu_1^2 &= -\mu_2^2 - (\lambda_1 + \lambda_3) \hat{w}_1^2 - \frac{1}{2} \lambda_b \hat{w}_S^2, \\ \nu_{12}^2 &= -2 \cos \sigma_1 \hat{w}_1 \hat{w}_S \lambda_4, \\ \nu_{01}^2 &= -4 \cos \sigma_1 \hat{w}_1 \hat{w}_S \lambda_7, \\ \nu_{02}^2 &= -\lambda_4 \hat{w}_1^2 \end{aligned}$
$(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0) C_{xy0}$	$\begin{aligned} \mu_1^2 &= -(\lambda_1 - \lambda_2) (\hat{w}_1^2 + \hat{w}_2^2), \\ \mu_2^2 &= -(\hat{w}_1^2 - \hat{w}_2^2) (\lambda_2 + \lambda_3), \\ \nu_{12}^2 &= -4 \hat{w}_1 \hat{w}_2 \cos(\sigma_2 - \sigma_1) (\lambda_2 + \lambda_3), \\ \nu_{01}^2 &= -2 \hat{w}_1 \hat{w}_2 \cos(\sigma_2 - \sigma_1) \lambda_4, \\ \nu_{02}^2 &= -(\hat{w}_1^2 - \hat{w}_2^2) \lambda_4 \end{aligned}$
$(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S) C_{xyz}$ provided.	$\begin{aligned} \mu_1^2 &= \{2 \hat{w}_S^2 S_+ \mu_0^2 - 4 \hat{w}_1^2 \hat{w}_2^2 (\hat{w}_1^2 + \hat{w}_2^2) \sin^2(\sigma_2 - \sigma_1) (\lambda_1 - \lambda_2) \\ &\quad - \hat{w}_S^2 [2 \hat{w}_1^2 \hat{w}_2^2 \sin^2(\sigma_2 - \sigma_1) - (\hat{w}_1^2 + \hat{w}_2^2) S_+] \lambda_b + 2 \hat{w}_S^4 \lambda_8\} / \\ &\quad [4 \hat{w}_1^2 \hat{w}_2^2 \sin^2(\sigma_2 - \sigma_1)], \\ \mu_2^2 &= -\{2 \hat{w}_S^2 S_- \mu_0^2 + 4 \hat{w}_1^2 \hat{w}_2^2 (\hat{w}_1^2 - \hat{w}_2^2) \sin^2(\sigma_2 - \sigma_1) (\lambda_2 + \lambda_3) \\ &\quad + 4 \cos \sigma_2 \hat{w}_1^2 \hat{w}_2^2 \hat{w}_S \sin^2(\sigma_2 - \sigma_1) \lambda_4 + \hat{w}_S^2 S_- [(\hat{w}_1^2 + \hat{w}_2^2) \lambda_b + 2 \hat{w}_S^2 \lambda_8]\} / \\ &\quad [4 \hat{w}_1^2 \hat{w}_2^2 \sin^2(\sigma_2 - \sigma_1)], \\ \nu_{12}^2 &= -\{2 \sin \sigma_1 \sin \sigma_2 \hat{w}_S^2 \mu_0^2 + 4 \hat{w}_1^2 \hat{w}_2^2 \cos(\sigma_2 - \sigma_1) \sin^2(\sigma_2 - \sigma_1) (\lambda_2 + \lambda_3) \\ &\quad + 2 \cos \sigma_1 \hat{w}_1^2 \hat{w}_2 \hat{w}_S \sin^2(\sigma_2 - \sigma_1) \lambda_4 + \sin \sigma_1 \sin \sigma_2 (\hat{w}_1^2 + \hat{w}_2^2) \hat{w}_S^2 \lambda_b \\ &\quad + 2 \sin \sigma_1 \sin \sigma_2 \hat{w}_S^4 \lambda_8\} / [\hat{w}_1 \hat{w}_2 \sin^2(\sigma_2 - \sigma_1)], \\ \nu_{01}^2 &= -\{2 \sin \sigma_2 \hat{w}_S \mu_0^2 + \hat{w}_1^2 \hat{w}_2 \sin[2(\sigma_2 - \sigma_1)] \lambda_4 + \sin \sigma_2 \hat{w}_S (\hat{w}_1^2 + \hat{w}_2^2) \lambda_b \\ &\quad + 4 \cos \sigma_1 \hat{w}_1^2 \hat{w}_S \sin(\sigma_2 - \sigma_1) \lambda_7 + 2 \sin \sigma_2 \hat{w}_S^3 \lambda_8\} / [\hat{w}_1 \sin(\sigma_2 - \sigma_1)], \\ \nu_{02}^2 &= \{2 \sin \sigma_1 \hat{w}_S \mu_0^2 + \hat{w}_2 (\hat{w}_2^2 - \hat{w}_1^2) \sin(\sigma_2 - \sigma_1) \lambda_4 + \sin \sigma_1 \hat{w}_S (\hat{w}_1^2 + \hat{w}_2^2) \lambda_b \\ &\quad - 4 \cos \sigma_2 \hat{w}_2^2 \hat{w}_S \sin(\sigma_2 - \sigma_1) \lambda_7 + 2 \sin \sigma_1 \hat{w}_S^3 \lambda_8\} / [\hat{w}_2 \sin(\sigma_2 - \sigma_1)] \end{aligned}$
$(\hat{w}_1 e^{i\sigma}, \pm \hat{w}_2 e^{i\sigma}, \hat{w}_S) C_{xyz}$ provided $\sin \sigma \neq 0$.	$\begin{aligned} \mu_0^2 &= -\frac{1}{2} (\hat{w}_1^2 + \hat{w}_2^2) \lambda_b - \lambda_8 \hat{w}_S^2, \\ \mu_1^2 &= \frac{-1}{4 \hat{w}_1 \hat{w}_2} \{4 \hat{w}_1 \hat{w}_2 (\hat{w}_1^2 + \hat{w}_2^2) (\lambda_1 + \lambda_3) \\ &\quad + 2 \hat{w}_1 \hat{w}_S [(\hat{w}_1^2 + \hat{w}_2^2) \cos \sigma \lambda_4 + \hat{w}_2 \hat{w}_S \lambda_b] \\ &\quad \pm (\hat{w}_1^2 + \hat{w}_2^2) \nu_{12}^2\}, \\ \mu_2^2 &= \frac{1}{4 \hat{w}_1 \hat{w}_2} \{2 \lambda_4 \hat{w}_1 \hat{w}_S (\hat{w}_1^2 - 3 \hat{w}_2^2) \cos \sigma \pm (\hat{w}_1^2 - \hat{w}_2^2) \nu_{12}^2\}, \\ \nu_{01}^2 &= \mp 2 \hat{w}_1 (\lambda_4 \hat{w}_2 + 2 \lambda_7 \hat{w}_S \cos \sigma), \\ \nu_{02}^2 &= \lambda_4 (\hat{w}_2^2 - \hat{w}_1^2) - 4 \lambda_7 \hat{w}_2 \hat{w}_S \cos \sigma. \end{aligned}$

B. Complex vacua

Below, we briefly comment on some of the properties of the different categories of complex vacua, allowing for soft breaking terms.

1. $(\mathbf{0}, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S)$

Soft breaking terms ν_{12}^2 and ν_{01}^2 do not survive minimization. There are no massless neutral scalars and no mass degeneracy with or without the soft breaking terms.

2. $(\hat{w}_1 e^{i\sigma_1}, \mathbf{0}, \hat{w}_S)$

All soft breaking terms survive minimization.

Let us first assume $\sigma_1 \neq \pm \frac{\pi}{2}$ (C-IV-a). If there are no soft breaking terms (requiring $\lambda_4 = \lambda_7 = 0$), we have two massless neutral scalars. With λ_4 and λ_7 both equal to zero, the scalar potential acquires an $O(2) \otimes U(1)_{h_S}$ symmetry, where $U(1)_{h_S}$ refers to the independent rephasing of h_S . Vacua that break these two continuous symmetries lead to two massless neutral scalars.

It is impossible to have ν_{12}^2 or ν_{02}^2 as the only soft breaking term. If the μ_2^2 term is the only soft breaking term, we have one massless neutral scalar. If the ν_{01}^2 term is present, there are no massless neutral scalars.

Possible situations where we have only two soft breaking terms are when we have ν_{12}^2 and ν_{02}^2 (this situation requires $\lambda_4 \neq 0$) or when we have μ_2^2 and ν_{01}^2 (this situation requires $\lambda_7 \neq 0$). In both these situations there are no massless neutral scalars.

Possible situations where we have three soft breaking terms are when we have ν_{12}^2 , μ_2^2 , and ν_{02}^2 or when we have ν_{12}^2 , ν_{01}^2 , and ν_{02}^2 . In either of these two situations there are no massless neutral scalars. If all four soft breaking terms are present, there are no massless neutral scalars.

Next, let us assume $\sigma_1 = \pm \frac{\pi}{2}$ (C-III-b). In this case we immediately get $\nu_{12}^2 = \nu_{01}^2 = 0$, so the only possible soft breaking terms are μ_2^2 and ν_{02}^2 . If there are no soft breaking terms (this requires $\lambda_4 = 0$), we have one massless scalar. If the μ_2^2 or ν_{02}^2 term is present, there are no massless scalars.

3. $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, \mathbf{0})$

All soft breaking terms survive minimization.

Let us first assume $\sigma_1 - \sigma_2 \neq \pm \frac{\pi}{2}$ (C-III-c). If there are no soft breaking terms (this requires $\lambda_2 + \lambda_3 = 0$ and $\lambda_4 = 0$), we have two massless neutral scalars. Not all combinations of soft breaking terms are allowed, but if at least one soft breaking term is present, we have no massless neutral scalars.

Next, let us assume $\sigma_1 - \sigma_2 = \pm \frac{\pi}{2}$. In this case we immediately get $\nu_{12}^2 = \nu_{01}^2 = 0$, so the only possible soft breaking terms are μ_2^2 and ν_{02}^2 . If $\hat{w}_2 \neq \pm \hat{w}_1$ and no soft breaking terms are present, we have two massless scalars. If

either of the soft breaking terms are present, we have no massless neutral scalars.

Finally, if $\sigma_1 - \sigma_2 = \pm \frac{\pi}{2}$ and $\hat{w}_2 = \pm \hat{w}_1$ (C-I-a), there are no soft breaking terms and also no massless neutral scalars.

4. $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S)$

All four soft breaking terms survive minimization. The case $\sin(\sigma_2 - \sigma_1) = 0$ requires special attention and is listed separately in Table IV. Finally, if $\sigma_1 - \sigma_2 \neq 0$, then all soft terms are constrained by the parameters of the unbroken potential, together with the VEVs (moduli \hat{w}_1 , \hat{w}_2 , \hat{w}_S , and the phases σ_1 and σ_2).

VI. THE C-III-c MODEL WITH SOFT S_3 -BREAKING

The C-III-c vacuum was characterized [16] as a vacuum with $\hat{w}_S = 0$ and with the other two VEVs arbitrary complex, $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$. It is worth stressing that the C-I-a vacuum $(\hat{w}_1, \pm i\hat{w}_1, 0)$ is not a particular case of C-III-c. This is clearly seen by comparing the constraints arising from the stationary-point equations. Whereas there are three constraints attached to C-III-c [see Eq. (2.10)], there is only one constraint attached to C-I-a, namely $\mu_1^2 = -(\lambda_1 - \lambda_2)v^2$. The C-III-c vacuum requires $\lambda_4 = 0$, $\lambda_2 + \lambda_3 = 0$, and $\mu_1^2 = -(\lambda_1 - \lambda_2)v^2$ and can, in fact, be simplified to $(\hat{w} e^{i\sigma}, \hat{w}, 0)$ due to the $O(2)$ symmetry resulting from $\lambda_4 = 0$.

The C-I-a case was studied by Derman and Tsao [19] and by Branco, Gérard, and Grimus [31]. It has the property of being geometrical in the sense that it is complex with the phases fixed by the symmetry rather than by the parameters of the potential. In the defining representation this vacuum appears as $(x, x e^{2\pi i/3}, x^{-2\pi i/3})$. It was shown [31] that this vacuum does not violate CP in spite of being complex.

One of the constraints of C-III-c is $\lambda_4 = 0$. Whenever $\lambda_4 = 0$ the potential acquires an additional $O(2)$ symmetry that is continuous. This symmetry is broken by the VEVs, and therefore there will be massless scalars. There will, in fact, be two massless scalars, as discussed above. One way of avoiding massless scalars is to include soft breaking terms. Soft breaking terms combining h_S with one of the h_i are not consistent with $\lambda_4 = 0$.

It was shown [16,30] that the presence of the $O(2)$ symmetry allows one to transform the C-III-c vacuum into $(\hat{w} e^{i\sigma/2}, \hat{w}^{-i\sigma/2}, 0)$, and therefore it can readily be shown that it also preserves CP . Actually, the introduction of soft breaking terms can only introduce CP violation in the mass part of the potential, not in the interactions. Thus, if the mass-squared matrices split into CP -even and CP -odd parts, then CP is conserved.

The C-III-c vacuum with soft breaking terms is denoted as C_{xy0} in Table IV. The minimization constraints are

$$\mu_1^2 = -(\hat{w}_1^2 + \hat{w}_2^2)(\lambda_1 - \lambda_2), \quad (6.1a)$$

$$\mu_2^2 = -(\hat{w}_1^2 - \hat{w}_2^2)(\lambda_2 + \lambda_3), \quad (6.1b)$$

$$\nu_{12}^2 = -4\hat{w}_1\hat{w}_2 \cos(\sigma_2 - \sigma_1)(\lambda_2 + \lambda_3), \quad (6.1c)$$

$$\nu_{01}^2 = -2\hat{w}_1\hat{w}_2 \cos(\sigma_2 - \sigma_1)\lambda_4, \quad (6.1d)$$

$$\nu_{02}^2 = -(\hat{w}_1^2 - \hat{w}_2^2)\lambda_4. \quad (6.1e)$$

It is clear from these expressions that μ_2^2 and ν_{12}^2 can be different from zero only if $\lambda_2 + \lambda_3 \neq 0$. Likewise, ν_{01}^2 and ν_{02}^2 can only be present for $\lambda_4 \neq 0$.

We shall here present three avenues to the introduction of modifications to C -III-c:

- (i) Models with VEVs $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$, $\lambda_4 = 0$, and $\lambda_2 + \lambda_3 \neq 0$,
- (ii) Models with VEVs $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$, $\lambda_4 \neq 0$, and $\lambda_2 + \lambda_3 = 0$,
- (iii) Models with VEVs $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$, $\lambda_4 \neq 0$, and $\lambda_2 + \lambda_3 \neq 0$.

A. Models with VEVs $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$, $\lambda_4 = 0$, and $\lambda_2 + \lambda_3 \neq 0$

It is clear from expressions (5.2), (6.1d), and (6.1e) that the condition $\lambda_4 = 0$ is not consistent with having soft breaking terms involving h_5 and either one of the h_i . Soft breaking terms involving h_1 and h_2 are possible only if we relax the condition $\lambda_2 + \lambda_3 = 0$.

The introduction of soft breaking terms of S_3 also breaks the $O(2)$ symmetry that resulted from having λ_4 equal to zero. As a result, in this case we cannot use this symmetry to write this set of vacua with equal moduli for the first two entries.

In general, CP is spontaneously broken in this case, provided that $\cos(\sigma_2 - \sigma_1) \neq 0$. It can readily be seen that, if $\cos(\sigma_2 - \sigma_1) = 0$, then ν_{12}^2 must be zero and only the soft breaking term proportional to μ_2^2 survives. The vacuum will have the form $(\pm i\hat{w}_1, \hat{w}_2, 0)$, the initial symmetry $h_1 \rightarrow -h_1$ is not broken, and CP is conserved by the vacuum, since the following condition [31]

$$U_{ij} \langle 0 | \Phi_j | 0 \rangle^* = \langle 0 | \Phi_i | 0 \rangle \quad (6.2)$$

is satisfied for

$$U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.3)$$

It can be checked that by going to the Higgs basis, according to the method proposed in Ref. [30], that CP is conserved when the following four conditions are met:

$$\bullet \sin[2(\sigma_2 - \sigma_1)] = 0, \quad (6.4a)$$

$$\bullet \lambda_7(\sin 2\sigma_1 - \sin 2\sigma_2) = 0, \quad (6.4b)$$

$$\bullet \lambda_7(\hat{w}_1^2 \sin 2\sigma_1 + \hat{w}_2^2 \sin 2\sigma_2) = 0, \quad (6.4c)$$

$$\bullet \lambda_7(\hat{w}_1^2 \sin 2\sigma_2 + \hat{w}_2^2 \sin 2\sigma_1) = 0. \quad (6.4d)$$

Clearly, if $\sigma_1 = \sigma_2 = 0$, there is no CP violation. Actually, these conditions are *sufficient*, not *necessary* for CP conservation. This distinction will be illustrated by Case 4 discussed in Appendixes B and C.

If $\cos(\sigma_2 - \sigma_1) \neq 0$ and $\hat{w}_1 = \hat{w}_2$, the term in μ_2^2 is forced to be zero and only the soft breaking term in ν_{12}^2 survives. The vacuum will have the form $(\hat{w} e^{i\sigma_1}, \hat{w} e^{i\sigma_2}, 0)$ which can be rephased into $(\hat{w} e^{i\sigma}, \hat{w} e^{-i\sigma}, 0)$. The potential will have symmetry for $h_1 \leftrightarrow h_2$ and CP is conserved with the following choice of U in Eq. (6.2):

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.5)$$

In Case 4 of Table V one might expect CP to be violated. However, this is not the case as will be shown in Appendix C. To prove it one can go to the Higgs basis where only one of the fields acquires a nonzero real VEV and use the freedom to rephase the fields with zero VEV in order to make all the coefficients of the potential real [30]. This set of transformations changes the form of the potential but does not change the physics. In its final version both the potential and the VEVs are real.

B. Models with VEVs $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$, $\lambda_4 \neq 0$, and $\lambda_2 + \lambda_3 = 0$

These are also models denoted by C_{xy0} in Table IV. In the particular case of $\cos(\sigma_2 - \sigma_1) = 0$ and $\hat{w}_1 = \hat{w}_2$, no soft breaking term survives; we fall into the case C -I-a, and there is no spontaneous CP violation.

We now find that CP is violated unless Eqs. (6.4b)–(6.4d) are satisfied, together with

$$\bullet \hat{w}_1^2 \sin(2\sigma_1 - \sigma_2) + (2\hat{w}_1^2 - \hat{w}_2^2) \sin \sigma_2 = 0, \quad (6.6a)$$

$$\bullet \hat{w}_2^2 \sin(2\sigma_1 - \sigma_2) + (2\hat{w}_2^2 - \hat{w}_1^2) \sin \sigma_2 = 0, \quad (6.6b)$$

$$\bullet \hat{w}_1^2 \sin(2\sigma_1 - \sigma_2) - 3\hat{w}_2^2 \sin \sigma_2 = 0, \quad (6.6c)$$

$$\bullet \hat{w}_2^2 \sin(2\sigma_1 - \sigma_2) - 3\hat{w}_1^2 \sin \sigma_2 = 0, \quad (6.6d)$$

which replace Eq. (6.4a). Clearly, if $\sigma_1 = \sigma_2 = 0$, there is no CP violation. There are two special cases worth considering:

$$(i) \cos(\sigma_2 - \sigma_1) = 0, \text{ and } \hat{w}_1 \neq \hat{w}_2,$$

$$(ii) \cos(\sigma_2 - \sigma_1) \neq 0, \text{ and } \hat{w}_1 = \hat{w}_2.$$

In the first case, the vacuum can be written as $(\pm i\hat{w}_1, \hat{w}_2, 0)$, where we chose $\sigma_2 = 0$ by the freedom to rephase. In this case, the term proportional to ν_{01}^2 does not survive, and the potential is symmetric under $h_1 \rightarrow -h_1$. CP is therefore conserved since Eq. (6.3) is satisfied.

TABLE V. Summary of softly broken C -III-c-like vacua. Here, “SBT” stands for “soft breaking terms.” When the two moduli are equal, we denote it \hat{w} . In the last column we listed the symmetry responsible for no spontaneous CP violation.

Case	Constraints	Allowed SBT	Vacuum	CP
1	$\lambda_4 = 0, \lambda_2 + \lambda_3 = 0$ C -III-c	None	$(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$ $\equiv (\hat{w} e^{i\sigma/2}, \hat{w} e^{-i\sigma/2}, 0)$	Conserving $O(2)$
2	$\lambda_4 = 0, \lambda_2 + \lambda_3 \neq 0$ $\cos(\sigma_2 - \sigma_1) = 0, \hat{w}_1 \neq \hat{w}_2$	μ_2^2	$(\pm i \hat{w}_1, \hat{w}_2, 0)$	Conserving $h_1 \rightarrow -h_1$
3	$\lambda_4 = 0, \lambda_2 + \lambda_3 \neq 0$ $\cos(\sigma_2 - \sigma_1) \neq 0, \hat{w}_1 = \hat{w}_2$	ν_{12}^2	$(\hat{w} e^{i\sigma_1}, \hat{w} e^{i\sigma_2}, 0)$ $\equiv (\hat{w} e^{i\sigma/2}, \hat{w} e^{-i\sigma/2}, 0)$	Conserving $h_1 \leftrightarrow h_2$
4	$\lambda_4 = 0, \lambda_2 + \lambda_3 \neq 0$ No other conditions	μ_2^2, ν_{12}^2	$(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$	Conserving
5	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 = 0$ $\cos(\sigma_2 - \sigma_1) = 0, \hat{w}_1 = \hat{w}_2$ C -I-a	None	$(\pm i \hat{w}, \hat{w}, 0)$	Conserving $h_1 \rightarrow -h_1$
6	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 = 0$ $\cos(\sigma_2 - \sigma_1) = 0, \hat{w}_1 \neq \hat{w}_2$	ν_{02}^2	$(\pm i \hat{w}_1, \hat{w}_2, 0)$	Conserving $h_1 \rightarrow -h_1$
7	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 = 0$ $\cos(\sigma_2 - \sigma_1) \neq 0, \hat{w}_1 = \hat{w}_2$	ν_{01}^2	$(\hat{w} e^{i\sigma}, \hat{w}, 0)$	Violating
8	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 = 0$ No other conditions	ν_{01}^2, ν_{02}^2	$(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$	Violating
9	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 \neq 0$ $\cos(\sigma_2 - \sigma_1) = 0, \hat{w}_1 \neq \hat{w}_2$	μ_2^2, ν_{02}^2	$(\pm i \hat{w}_1, \hat{w}_2, 0)$	Conserving $h_1 \rightarrow -h_1$
10	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 \neq 0$ $\cos(\sigma_2 - \sigma_1) \neq 0, \hat{w}_1 = \hat{w}_2$	ν_{12}^2, ν_{01}^2	$(\hat{w} e^{i\sigma}, \hat{w}, 0)$	Violating
11	$\lambda_4 \neq 0, \lambda_2 + \lambda_3 \neq 0$ $\sigma_2 - \sigma_1 \neq 0, \hat{w}_1 \neq \hat{w}_2$	All	$(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$	Violating

In the second case, only the term proportional to ν_{01}^2 survives, and in general CP will be violated (unless both phases vanish). This phenomenon illustrates the fact that soft breaking terms can introduce spontaneous CP violation [35].

C. Models with VEVs $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$, $\lambda_4 \neq 0$, and $\lambda_2 + \lambda_3 \neq 0$

In this general case, going to the Higgs basis, we see that the full set of conditions, including Eqs. (6.4) and (6.6), must be satisfied. Obviously, for $\sigma_2 - \sigma_1 = 0$, these equations can be verified after a suitable rephasing. Otherwise, CP is violated.

Table V summarizes the results obtained in this section. In Cases 3, 7, and 10 the vacuum can be written in the same form, but Case 3 is CP conserving, while the other two are CP violating.

In Appendix B we collect some information on the mass spectra of these different models.

VII. SUMMARY

We have presented a detailed discussion of some 3HDM vacua obtained from an S_3 -symmetric potential with soft

symmetry breaking. The S_3 -symmetric potential, for certain VEV alignments, is plagued with massless states and degeneracies, whose origins have been identified. In fact, all possible vacua, except (x, x, x) , expressed in the defining representation, break the S_3 symmetry spontaneously. We have shown that some vacuum solutions require conditions on the parameters of the potential that lead to accidental continuous symmetries that in turn are broken by the vacuum. Allowing for soft S_3 breaking terms, all states become massive.

The case of a vanishing singlet VEV, without the introduction of soft breaking terms, which we denoted as C -III-c is particularly interesting. It exhibits an unfamiliar feature: the minimum of the potential allows for a relative phase between the two nonzero VEVs, whose value is not constrained by the minimization conditions [16]. For all other complex S_3 vacua the phases always appear in the minimization conditions and therefore cannot be considered as free parameters [16]. In the C -III-c case the phase appearing in the VEVs is an additional free parameter which determines the mass splitting in the neutral S_3 singlet sector. This phase also shows up in certain couplings. Whenever this phase is chosen to be

zero the two neutral scalars from the S_3 singlet sector are degenerate in mass. It is possible to remove this phase from the VEVs by a rephasing giving rise to a CP conserving potential with λ_7 complex. In this basis all free parameters appear in the potential.

This work focuses on the C -III-c vacuum as well as on its versions with soft breaking terms. It is beyond the scope of this paper to discuss other possible vacua with soft breaking terms. However, models with \hat{w}_S nonzero may also provide interesting possibilities from the phenomenological point of view.

Furthermore, we have seen that the form of the vacuum does not determine whether CP is violated spontaneously. One and the same form of the vacuum may conserve or violate CP , depending on which soft S_3 breaking terms (and corresponding constraints) are present.

The results presented in this paper should be useful for model building, providing guidelines for various interesting scenarios. The S_3 -symmetric potential with three Higgs doublets has been analyzed by several authors in the past few years with many different aims [36], such as looking for realistic Yukawa couplings [37–40], looking for dark matter candidates [41–44], looking for CP violation [45,46], as well as many other studies.

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APPENDIX A: REAL AND COMPLEX VACUA OF THE S_3 -SYMMETRIC POTENTIAL

For convenience, we include Tables VI and VII that summarize some of the properties of the possible real and complex vacuum solutions [16].

TABLE VI. Possible real vacua (partly after Derman and Tsao [19]). This classification of vacua [16] uses the notation R -X-y, explained in the text. The VEVs of the ϕ_i are denoted by ρ_i .

Vacuum	ρ_1, ρ_2, ρ_3	w_1, w_2, w_S	Comment
R -0	0, 0, 0	0, 0, 0	Not interesting
R -I-1	x, x, x	0, 0, w_S	$\mu_0^2 = -\lambda_8 w_S^2$
R -I-2a	$x, -x, 0$	$w, 0, 0$	$\mu_1^2 = -(\lambda_1 + \lambda_3)w_1^2$
R -I-2b	$x, 0, -x$	$w, \sqrt{3}w, 0$	$\mu_1^2 = -\frac{4}{3}(\lambda_1 + \lambda_3)w_2^2$
R -I-2c	0, $x, -x$	$w, -\sqrt{3}w, 0$	$\mu_1^2 = -\frac{4}{3}(\lambda_1 + \lambda_3)w_2^2$
R -II-1a	x, x, y	0, w, w_S	$\mu_0^2 = \frac{1}{2}\lambda_4 \frac{w_2^3}{w_S} - \frac{1}{2}\lambda_a w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -(\lambda_1 + \lambda_3)w_2^2 + \frac{3}{2}\lambda_4 w_2 w_S - \frac{1}{2}\lambda_a w_S^2$
R -II-1b	x, y, x	$w, -w/\sqrt{3}, w_S$	$\mu_0^2 = -4\lambda_4 \frac{w_2^3}{w_S} - 2\lambda_a w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -4(\lambda_1 + \lambda_3)w_2^2 - 3\lambda_4 w_2 w_S - \frac{1}{2}\lambda_a w_S^2$
R -II-1c	y, x, x	$w, w/\sqrt{3}, w_S$	$\mu_0^2 = -4\lambda_4 \frac{w_2^3}{w_S} - 2\lambda_a w_2^2 - \lambda_8 w_S^2,$ $\mu_1^2 = -4(\lambda_1 + \lambda_3)w_2^2 - 3\lambda_4 w_2 w_S - \frac{1}{2}\lambda_a w_S^2$
R -II-2	$x, x, -2x$	0, $w, 0$	$\mu_1^2 = -(\lambda_1 + \lambda_3)w_2^2, \lambda_4 = 0$
R -II-3	$x, y, -x - y$	$w_1, w_2, 0$	$\mu_1^2 = -(\lambda_1 + \lambda_3)(w_1^2 + w_2^2), \lambda_4 = 0$
R -III	ρ_1, ρ_2, ρ_3	w_1, w_2, w_S	$\mu_0^2 = -\frac{1}{2}\lambda_a (w_1^2 + w_2^2) - \lambda_8 w_S^2,$ $\mu_1^2 = -(\lambda_1 + \lambda_3)(w_1^2 + w_2^2) - \frac{1}{2}\lambda_a w_S^2,$ $\lambda_4 = 0$

TABLE VII. Complex vacua. Notation: $\epsilon = 1$ and -1 for C-III-d and C-III-e, respectively; $\xi = \sqrt{-3 \sin 2\rho_1 / \sin 2\rho_2}$, $\psi = \sqrt{[3 + 3 \cos(\rho_2 - 2\rho_1)] / (2 \cos \rho_2)}$. Imposing the vacuum constraints [16], the vacua labeled with an asterisk (*) are in fact real.

	Irreducible Representation w_1, w_2, w_S	Defining Representation ρ_1, ρ_2, ρ_3
C-I-a	$\hat{w}_1, \pm i\hat{w}_1, 0$	$x, xe^{\pm \frac{2\pi i}{3}}, xe^{\mp \frac{2\pi i}{3}}$
C-III-a	$0, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$y, y, xe^{i\tau}$
C-III-b	$\pm i\hat{w}_1, 0, \hat{w}_S$	$x + iy, x - iy, x$
C-III-c	$\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0$	$xe^{i\rho} - \frac{y}{2}, -xe^{i\rho} - \frac{y}{2}, y$
C-III-d,e	$\pm i\hat{w}_1, \epsilon \hat{w}_2, \hat{w}_S$	$xe^{i\tau}, xe^{-i\tau}, y$
C-III-f	$\pm i\hat{w}_1, i\hat{w}_2, \hat{w}_S$	$re^{i\rho} \pm ix, re^{i\rho} \mp ix, \frac{3}{2} re^{-i\rho} - \frac{1}{2} re^{i\rho}$
C-III-g	$\pm i\hat{w}_1, -i\hat{w}_2, \hat{w}_S$	$re^{-i\rho} \pm ix, re^{-i\rho} \mp ix, \frac{3}{2} re^{i\rho} - \frac{1}{2} re^{-i\rho}$
C-III-h	$\sqrt{3}\hat{w}_2 e^{i\sigma_2}, \pm \hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$xe^{i\tau}, y, y$ $y, xe^{i\tau}, y$
C-III-i	$\sqrt{\frac{3(1+\tan^2 \sigma_1)}{1+9\tan^2 \sigma_1}} \hat{w}_2 e^{i\sigma_1},$ $\pm \hat{w}_2 e^{-i \arctan(3 \tan \sigma_1)}, \hat{w}_S$	$x, ye^{i\tau}, ye^{-i\tau}$ $ye^{i\tau}, x, ye^{-i\tau}$
C-IV-a*	$\hat{w}_1 e^{i\sigma_1}, 0, \hat{w}_S$	$re^{i\rho} + x, -re^{i\rho} + x, x$
C-IV-b	$\hat{w}_1, \pm i\hat{w}_2, \hat{w}_S$	$re^{i\rho} + x, -re^{-i\rho} + x, -re^{i\rho} + re^{-i\rho} + x$
C-IV-c	$\sqrt{1 + 2 \cos^2 \sigma_2} \hat{w}_2,$ $\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$re^{i\rho} + r\sqrt{3(1 + 2 \cos^2 \rho)} + x,$ $re^{i\rho} - r\sqrt{3(1 + 2 \cos^2 \rho)} + x, -2re^{i\rho} + x$
C-IV-d*	$\hat{w}_1 e^{i\sigma_1}, \pm \hat{w}_2 e^{i\sigma_1}, \hat{w}_S$	$r_1 e^{i\rho} + x, (r_2 - r_1) e^{i\rho} + x, -r_2 e^{i\rho} + x$
C-IV-e	$\sqrt{-\frac{\sin 2\sigma_2}{\sin 2\sigma_1}} \hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$re^{i\rho_2} + re^{i\rho_1} \xi + x, re^{i\rho_2} - re^{i\rho_1} \xi + x,$ $-2re^{i\rho_2} + x$
C-IV-f	$\sqrt{2 + \frac{\cos(\sigma_1 - 2\sigma_2)}{\cos \sigma_1}} \hat{w}_2 e^{i\sigma_1},$ $\hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$re^{i\rho_1} + re^{i\rho_2} \psi + x,$ $re^{i\rho_1} - re^{i\rho_2} \psi + x, -2re^{i\rho_1} + x$
C - V*	$\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, \hat{w}_S$	$xe^{i\tau_1}, ye^{i\tau_2}, z$

APPENDIX B: MASSES IN THE SOFTLY BROKEN C-III-c MODELS

Regardless of the softly broken S_3 parameters, the charged mass eigenstates are of the same form in every softly broken C-III-c model. Therefore, only the neutral states are analyzed here. Also, since $\hat{w}_S = 0$, the masses will only depend on the relative phase,

$$\sigma \equiv \sigma_1 - \sigma_2, \quad (\text{B1})$$

not on σ_1 and σ_2 separately.

1. Models with $\lambda_4 = 0$ and $\lambda_2 + \lambda_3 \neq 0$

These are the models discussed in Sec. VI A. The softly broken parameters consistent with the $\lambda_4 = 0$ constraint are μ_2^2 and ν_{12}^2 . Because of the fact that $\lambda_4 = 0$ and $\hat{w}_S = 0$, there is no mixing between the S_3 doublet and singlet at the level of masses. These models might provide possible dark matter candidates. Since the S_3 singlet possesses zero

vacuum value, it could be associated with an inert doublet. Moreover, the $\lambda_4 = 0$ constraint results in a discrete symmetry $\mathbb{Z}_2: h_S \rightarrow -h_S$, which could stabilize the scalar dark matter sector.

a. Case 2. C-III-c- μ_2^2

CP is conserved, and the squared masses are given by

$$m_{H_{(1,2)}}^2 = (\lambda_1 + \lambda_3) v^2 \mp \Delta, \quad (\text{B2a})$$

$$m_A^2 = 2(\lambda_2 + \lambda_3) v^2, \quad (\text{B2b})$$

$$m_{S_{(1,2)}}^2 = \mu_0^2 + \frac{1}{2} (\lambda_5 + \lambda_6) v^2 \mp \lambda_7 (\hat{w}_1^2 - \hat{w}_2^2), \quad (\text{B2c})$$

where

$$\Delta^2 = (\lambda_1 + \lambda_3)^2 v^4 - 16(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3) \hat{w}_1^2 \hat{w}_2^2. \quad (\text{B3})$$

In the limit $\lambda_2 + \lambda_3 \rightarrow 0$, both $m_{H_1}^2$ and m_A^2 vanish linearly.

b. Case 3. C-III-c- ν_{12}^2

CP is conserved, and the squared masses are given by

$$m_{H_{(1,3)}}^2 = (\lambda_1 + \lambda_3 \mp \Delta)v^2, \quad (\text{B4a})$$

$$m_{H_2}^2 = 2(\lambda_2 + \lambda_3)v^2, \quad (\text{B4b})$$

$$m_{S_{(1,2)}}^2 = \mu_0^2 + \frac{1}{2}(\lambda_5 + \lambda_6)v^2 \mp \lambda_7 \cos \sigma v^2, \quad (\text{B4c})$$

where

$$\begin{aligned} \Delta^2 &= (\lambda_1 - \lambda_2)^2 + (\lambda_2 + \lambda_3)^2 \\ &\quad + 2(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3) \cos(2\sigma). \end{aligned} \quad (\text{B5})$$

The above terminology is determined by $m_{H_1}^2 < m_{H_2}^2 < m_{H_3}^2$, valid for $\mu_1^2 < 0$. In the limit $\lambda_2 + \lambda_3 \rightarrow 0$, both $m_{H_1}^2$ and $m_{H_2}^2$ vanish linearly.

c. Case 4. C-III-c- $\mu_2^2 - \nu_{12}^2$

CP is conserved, illustrating that the conditions (6.4) are not *necessary*. This is explained in Appendix C. The squared masses are given by

$$m_{H_{(1,2)}}^2 = (\lambda_1 + \lambda_3)v^2 \mp \Delta_H, \quad (\text{B6a})$$

$$m_{H_3}^2 = 2(\lambda_2 + \lambda_3)v^2, \quad (\text{B6b})$$

$$m_{S_{(1,2)}}^2 = \mu_0^2 + \frac{1}{2}(\lambda_5 + \lambda_6)v^2 \mp \lambda_7 \Delta_S, \quad (\text{B6c})$$

where

$$\Delta_H^2 = (\lambda_1 + \lambda_3)^2 v^4 - 16(\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3) \sin^2 \sigma \hat{w}_1^2 \hat{w}_2^2, \quad (\text{B7})$$

$$\Delta_S^2 = v^4 - 4 \sin^2 \sigma \hat{w}_1^2 \hat{w}_2^2. \quad (\text{B8})$$

The above terminology is determined by $m_{H_1}^2 < m_{H_2}^2 < m_{H_3}^2$, valid for $\mu_1^2 < 0$. In the limit $\lambda_2 + \lambda_3 \rightarrow 0$, both $m_{H_1}^2$ and $m_{H_3}^2$ vanish linearly.

2. Models with $\lambda_4 \neq 0$ and $\lambda_2 + \lambda_3 = 0$

These are the models discussed in Sec. VIB. They actually contain states of negative squared mass, but are described here as limits to be avoided in any discussion of realistic versions of Cases 9, 10, and 11.

The softly broken parameters consistent with the $\lambda_3 + \lambda_2 = 0$ constraint are ν_{01}^2 and ν_{02}^2 . Because of the fact that $\lambda_4 \neq 0$, there is mixing between the S_3 doublet and singlet.

a. Case 6. C-III-c- ν_{02}^2

Because of the mixing between the doublet and singlet sectors, the neutral mass-squared matrix is now 6×6 . Transforming to the Higgs basis and removing the would-be Goldstone boson, it is reduced to a 5×5 matrix spanned by the fields

$$\{\eta_1^{\text{HB}}, \eta_2^{\text{HB}}, \eta_3^{\text{HB}}, \chi_2^{\text{HB}}, \chi_3^{\text{HB}}\}. \quad (\text{B9})$$

The 5×5 mass-squared matrix is block diagonal, a 3×3 “ η ” block and a 2×2 “ χ ” block [see Eq. (2.3)], reflecting the fact that CP is conserved. The η block has the following properties:

$$\det(\mathcal{M}_\eta^2) = 8(\lambda_2 - \lambda_1)\lambda_4^2 \hat{w}_1^2 \hat{w}_2^4, \quad (\text{B10})$$

$$\begin{aligned} \text{Tr}(\mathcal{M}_\eta^2) &= \mu_0^2 + 2(\lambda_1 - \lambda_2)v^2 \\ &\quad + \frac{1}{2}(\lambda_5 + \lambda_6)v^2 + \lambda_7(\hat{w}_2^2 - \hat{w}_1^2). \end{aligned} \quad (\text{B11})$$

For the product of the three squared masses to be positive, we must have $\lambda_2 > \lambda_1$. For the χ block we may solve explicitly for the squared masses in terms of a square root:

$$m_{A_{(1,2)}}^2 = \frac{1}{4}(2\mu_0^2 + \lambda_a \hat{w}_1^2 + \lambda_b \hat{w}_2^2 \mp \Delta), \quad (\text{B12})$$

where

$$\Delta^2 = 16\lambda_4^2 \hat{w}_1^2 v^2 + (2\mu_0^2 + \lambda_a \hat{w}_1^2 + \lambda_b \hat{w}_2^2)^2. \quad (\text{B13})$$

We note that the product of these two masses is negative,

$$m_{A_1}^2 m_{A_2}^2 = -\lambda_4^2 \hat{w}_1^2 v^2 < 0, \quad (\text{B14})$$

so Case 6 must be abandoned. Actually, with $\lambda_1 - \lambda_2$ positive (as follows from μ_1^2 being negative), also the determinant (B10) is negative, signaling the fact that not one, but *two* masses squared are negative.

b. Case 7. C-III-c- ν_{01}^2

In this case, the 5×5 matrix spanned by the fields (B9) has the following structure:

$$\mathcal{M}^2 = \begin{pmatrix} \times & 0 & \times & 0 & \times \\ 0 & 0 & \times & 0 & 0 \\ \times & \times & \times & \times & \times \\ 0 & 0 & \times & 0 & \times \\ \times & 0 & \times & \times & \times \end{pmatrix}. \quad (\text{B15})$$

While individual elements (denoted “ \times ”) also depend on μ_0^2 , λ_5 , λ_6 , and λ_7 , the product of all five masses squared is very simple,

$$M_1^2 M_2^2 M_3^2 M_4^2 M_5^2 = \det(\mathcal{M}^2) = \frac{1}{2}(\lambda_1 - \lambda_2)\lambda_4^4 \sin^4 \sigma v^{10}. \quad (\text{B16})$$

However, the sum of all masses squared depends on these additional parameters,

$$\text{Tr}(\mathcal{M}^2) = 2\mu_0^2 + 2(\lambda_1 - \lambda_2)v^2 + (\lambda_5 + \lambda_6)v^2. \quad (\text{B17})$$

A necessary condition for positive squared masses is $\lambda_1 > \lambda_2$.

One might be tempted to conclude from Eq. (B16) that *four* of the squared masses vanish as $\lambda_4 \rightarrow 0$. This is not necessarily the case, as illustrated by the following example. Let a toy model have the two squared masses:

$$m_{a,b}^2 = \left[\lambda_0 \mp \sqrt{\lambda_0^2 - \lambda_4^2} \right] v^2. \quad (\text{B18})$$

While the product is given by $m_a^2 m_b^2 = \lambda_4^2 v^4$, the individual masses squared are for $\lambda_4/\lambda_0 \rightarrow 0$ given as

$$m_a^2 \simeq \frac{\lambda_4^2}{2\lambda_0} v^2, \quad m_b^2 \simeq 2\lambda_0 v^2. \quad (\text{B19})$$

As mentioned above, in Case 7 and for $\lambda_1 > \lambda_2$, the overall determinant is positive, allowing for an even number of negative squared masses. A numerical exploration shows that two of them are negative, as for Case 6.

c. Case 8. C-III-c- $\nu_{01}^2 - \nu_{02}^2$

The Higgs basis rotation and reduction to a 5×5 matrix yields the following structure:

$$\mathcal{M}^2 = \begin{pmatrix} \times & 0 & \times & 0 & \times \\ 0 & 0 & \times & 0 & \times \\ \times & \times & \times & \times & \times \\ 0 & 0 & \times & 0 & \times \\ \times & \times & \times & \times & \times \end{pmatrix}, \quad (\text{B20})$$

with

$$\det(\mathcal{M}^2) = 8(\lambda_1 - \lambda_2)\lambda_4^4 \sin^4 \sigma \hat{w}_1^4 \hat{w}_2^4 v^2, \quad (\text{B21})$$

$$\text{Tr}(\mathcal{M}^2) = 2\mu_0^2 + (2\lambda_1 - 2\lambda_2 + \lambda_5 + \lambda_6)v^2. \quad (\text{B22})$$

For positive squared masses we must have $\lambda_1 > \lambda_2$.

The mass eigenstates are all mixtures of the gauge fields of Eq. (B9), and CP is violated.

The overall determinant is positive, allowing for an even number of negative squared masses. A numerical exploration shows that two of them are negative, as for Case 6.

3. Models with $\lambda_4 \neq 0$ and $\lambda_2 + \lambda_3 \neq 0$

These are the models discussed in Sec. VI C. In these models there is mixing between the S_3 doublet and singlet.

a. Case 9. C-III-c- $\mu_2^2 - \nu_{02}^2$

The 5×5 neutral mass-squared matrix is block diagonal, a 3×3 η block and a 2×2 χ block, reflecting the fact that CP is conserved. The following properties can be extracted:

$$\det(\mathcal{M}_\eta^2) = 8(\lambda_1 - \lambda_2)[(\lambda_2 + \lambda_3)(2\mu_0^2 + \lambda_b \hat{w}_1^2 + \lambda_a \hat{w}_2^2) - \lambda_4^2 \hat{w}_2^2] \hat{w}_1^2 \hat{w}_2^2, \quad (\text{B23})$$

$$\text{Tr}(\mathcal{M}_\eta^2) = \mu_0^2 + 2(\lambda_1 + \lambda_3)v^2 + \frac{1}{2}(\lambda_b \hat{w}_1^2 + \lambda_a \hat{w}_2^2). \quad (\text{B24})$$

The masses squared of the χ sector are given by

$$m_{A(1,2)}^2 = \frac{1}{4} \left[2\mu_0^2 + 4 \left(\lambda_2 + \lambda_3 + \frac{1}{4} \lambda_a \right) \hat{w}_1^2 + 4 \left(\lambda_2 + \lambda_3 + \frac{1}{4} \lambda_b \right) \hat{w}_2^2 \mp \Delta \right], \quad (\text{B25})$$

where

$$\Delta^2 = \left[2\mu_0^2 + 4 \left(\lambda_2 + \lambda_3 + \frac{1}{4} \lambda_a \right) \hat{w}_1^2 + 4 \left(\lambda_2 + \lambda_3 + \frac{1}{4} \lambda_b \right) \hat{w}_2^2 \right]^2 - 16[(2\mu_0^2 + \lambda_a \hat{w}_1^2 + \lambda_b \hat{w}_2^2)(\lambda_2 + \lambda_3) - \lambda_4^2 \hat{w}_1^2] v^2. \quad (\text{B26})$$

For the χ sector we have

$$\det(\mathcal{M}_\chi^2) = [(\lambda_2 + \lambda_3)(2\mu_0^2 + \lambda_a \hat{w}_1^2 + \lambda_b \hat{w}_2^2) - \lambda_4^2 \hat{w}_1^2] v^2. \quad (\text{B27})$$

Necessary conditions to have all squared masses positive is then

$$(\lambda_2 + \lambda_3)(2\mu_0^2 + \lambda_b \hat{w}_1^2 + \lambda_a \hat{w}_2^2) - \lambda_4^2 \hat{w}_2^2 > 0, \quad (\text{B28})$$

$$(\lambda_2 + \lambda_3)(2\mu_0^2 + \lambda_a \hat{w}_1^2 + \lambda_b \hat{w}_2^2) - \lambda_4^2 \hat{w}_1^2 > 0. \quad (\text{B29})$$

b. Case 10. C-III-c- $\nu_{12}^2 - \nu_{01}^2$

In this case, the 5×5 neutral mass-squared matrix takes the form

$$\mathcal{M}_0^2 = \begin{pmatrix} \times & 0 & \times & \times & \times \\ 0 & \times & \times & 0 & 0 \\ \times & \times & \times & \times & \times \\ \times & 0 & \times & \times & \times \\ \times & 0 & \times & \times & \times \end{pmatrix}, \quad (\text{B30})$$

with

$$\det(\mathcal{M}^2) = \sin^2 \sigma (\lambda_1 - \lambda_2) [A_4 (\mu_0^2)^2 + A_2 \mu_0^2 + A_0] v^6, \quad (\text{B31})$$

where

$$A_4 = 8(\lambda_2 + \lambda_3)^2, \quad (\text{B32a})$$

$$A_2 = 4(\lambda_2 + \lambda_3)[2(\lambda_2 + \lambda_3)(\lambda_5 + \lambda_6) - \lambda_4^2] v^2, \quad (\text{B32b})$$

$$A_0 = \frac{1}{2} \{ [2(\lambda_2 + \lambda_3)(\lambda_5 + \lambda_6) - \lambda_4^2]^2 - \cos^2 \sigma [4(\lambda_2 + \lambda_3)\lambda_7 - \lambda_4^2]^2 \} v^4 \quad (\text{B32c})$$

$$= \frac{1}{2} \sin^2 \sigma \lambda_4^4 v^4 - 2(\lambda_2 + \lambda_3) \lambda_4^2 (\lambda_5 + \lambda_6 - 2\lambda_7 \cos^2 \sigma) v^4 + 2(\lambda_2 + \lambda_3)^2 [(\lambda_5 + \lambda_6)^2 - 4\lambda_7^2 \cos^2 \sigma] v^4, \quad (\text{B32d})$$

and

$$\text{Tr}(\mathcal{M}^2) = 2\mu_0^2 + (2\lambda_1 + 2\lambda_2 + 4\lambda_3 + \lambda_5 + \lambda_6) v^2. \quad (\text{B33})$$

The mass eigenstates are a mixture of all five gauge fields, and CP is violated.

c. Case 11. C-III-c- μ_2^2 - ν_{12}^2 - ν_{01}^2 - ν_{02}^2

In this case, all elements of the 5×5 neutral mass-squared matrix are nonzero, CP is violated, and the determinant is rather complex,

$$\det(\mathcal{M}^2) = 8 \sin^2 \sigma (\lambda_1 - \lambda_2) [A_4 (\mu_0^2)^2 + A_2 \mu_0^2 + A_0] \hat{w}_1^2 \hat{w}_2^2 v^2, \quad (\text{B34})$$

with

$$A_4 = 4(\lambda_2 + \lambda_3)^2, \quad (\text{B35a})$$

$$A_2 = 2(\lambda_2 + \lambda_3)[2(\lambda_2 + \lambda_3)(\lambda_5 + \lambda_6) - \lambda_4^2] v^2, \quad (\text{B35b})$$

$$A_0 = \sin^2 \sigma \lambda_4^4 \hat{w}_1^2 \hat{w}_2^2 - (\lambda_2 + \lambda_3) \lambda_4^2 [(\lambda_5 + \lambda_6) v^4 - 2(v^4 - 4 \sin^2 \sigma \hat{w}_1^2 \hat{w}_2^2) \lambda_7] + (\lambda_2 + \lambda_3)^2 [(\lambda_5 + \lambda_6)^2 v^4 - 4(v^4 - 4 \sin^2 \sigma \hat{w}_1^2 \hat{w}_2^2) \lambda_7^2] \quad (\text{B35c})$$

TABLE VIII. Contributions to sums of masses squared.

Case	CPC	μ_0^2	μ_1^2	$\frac{1}{2}(\lambda_2 + \lambda_3)$	$\frac{1}{2}(\lambda_5 + \lambda_6)$	λ_7
1	✓	1 + 1	-2 + 0	0	1 + 1	$\pm a \mp a$
2	✓	1 + 1	-2 + 0	4 + 4	1 + 1	$\pm b \mp b$
3	✓	1 + 1	-2 + 0	4 + 4	1 + 1	$\pm a \mp a$
4	✓	1 + 1	-2 + 0	4 + 4	1 + 1	$\pm c \mp c$
5	✓	1 + 1	-2 + 0	0	1 + 1	$-b + b$
6	✓	1 + 1	-2 + 0	0	1 + 1	$-b + b$
7	...	2	-2	0	2	0
8	...	2	-2	0	2	0
9	✓	1 + 1	-2 + 0	4 + 4	1 + 1	$-b + b$
10	...	2	-2	8	2	0
11	...	2	-2	8	2	0

$$a = \cos \sigma v^2, \quad b = (\hat{w}_1^2 - \hat{w}_2^2), \quad c = \Delta_S.$$

$$= (\lambda_2 + \lambda_3) [(\lambda_2 + \lambda_3) \lambda_a - \lambda_4^2] \lambda_b v^4 + \sin^2 \sigma [4(\lambda_2 + \lambda_3) \lambda_7 - \lambda_4^2]^2 \hat{w}_1^2 \hat{w}_2^2, \quad (\text{B35d})$$

whereas the trace has the familiar value, given by Eq. (B33).

4. Sums of masses squared

The contributions to the trace of the mass-squared matrix in the neutral sector, i.e., the sum of all squared masses of the neutral scalars, can be expressed in terms of μ_0^2 , μ_1^2 , $\lambda_2 + \lambda_3$, $\lambda_5 + \lambda_6$, and λ_7 as

$$\text{Tr}(\mathcal{M}^2) = \#\mu_0^2 + \#\mu_1^2 + \#\frac{1}{2}(\lambda_2 + \lambda_3)v^2 + \#\frac{1}{2}(\lambda_5 + \lambda_6)v^2 + \#\lambda_7 v^2, \quad (\text{B36})$$

where we use the minimization condition $\mu_1^2 = -(\lambda_1 - \lambda_2)v^2$ and trade λ_3 for μ_1^2 and $(\lambda_2 + \lambda_3)$.

We summarize in Table VIII the coefficients denoted “#” above. Where CP is conserved, contributions to the CP -even and CP -odd parts are given separately, as $x + y$. The singlet sector contributes to the μ_0^2 , $(\lambda_5 + \lambda_6)$, and λ_7 terms (the latter cancel among CP -even and -odd terms), whereas the doublet sector contributes to the μ_1^2 and $(\lambda_2 + \lambda_3)$ terms.

Expressed in these terms, the structure is remarkably simple and regular.

APPENDIX C: CP CONSERVATION IN CASE 4

For Case 4, when the two soft breaking terms μ_2^2 and ν_{12}^2 are present, the vacuum configuration $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$ does not result in a CP -violating model. This can be shown explicitly by constructing a basis transformation that results in both a real potential and a real vacuum, thereby eliminating the possibility of having spontaneous CP violation. We start by simultaneously rephasing all three doublets to get a vacuum configuration of the form $(\hat{w}_1 e^{i\sigma}, \hat{w}_2, 0)$, leaving all parameters of the potential

$$V = \mu_0^2 h_S^\dagger h_S + \mu_1^2 (h_1^\dagger h_1 + h_2^\dagger h_2) + \mu_2^2 (h_1^\dagger h_1 - h_2^\dagger h_2) \quad \text{with}$$

$$+ \frac{1}{2} \nu_{12}^2 (h_2^\dagger h_1 + h_1^\dagger h_2) \quad (\text{C1})$$

$$+ \lambda_1 (h_1^\dagger h_1 + h_2^\dagger h_2)^2 + \lambda_2 (h_1^\dagger h_2 - h_2^\dagger h_1)^2$$

$$+ \lambda_3 [(h_1^\dagger h_1 - h_2^\dagger h_2)^2 + (h_1^\dagger h_2 + h_2^\dagger h_1)^2]$$

$$+ \lambda_5 (h_S^\dagger h_S) (h_1^\dagger h_1 + h_2^\dagger h_2) + \lambda_6 [(h_S^\dagger h_1) (h_1^\dagger h_S)$$

$$+ (h_S^\dagger h_2) (h_2^\dagger h_S)] + \lambda_7 [(h_S^\dagger h_1) (h_S^\dagger h_1)$$

$$+ (h_S^\dagger h_2) (h_S^\dagger h_2) + \text{H.c.}] + \lambda_8 (h_S^\dagger h_S)^2 \quad (\text{C2})$$

unchanged. Consider now the change of basis given by

$$\begin{pmatrix} \bar{h}_1 \\ \bar{h}_2 \\ \bar{h}_S \end{pmatrix} = e^{i\psi} \begin{pmatrix} \cos \theta & e^{-i\xi} \sin \theta & 0 \\ -e^{i\chi} \sin \theta & e^{i(\chi-\xi)} \cos \theta & 0 \\ 0 & 0 & e^{i\tau} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \quad (\text{C3})$$

$$\theta = \arctan\left(\frac{\hat{w}_2}{\hat{w}_1}\right), \quad (\text{C4})$$

$$\chi = -\arctan\left(\frac{v^2}{(\hat{w}_2^2 - \hat{w}_1^2) \tan \sigma}\right), \quad (\text{C5})$$

$$\xi = -\sigma, \quad (\text{C6})$$

$$\psi = -\sigma, \quad (\text{C7})$$

$$\tau = \frac{\pi}{4} + \frac{\sigma}{2} - \frac{1}{2} \arctan\left(\frac{v^2}{(\hat{w}_2^2 - \hat{w}_1^2) \tan \sigma}\right). \quad (\text{C8})$$

This takes us to the Higgs basis, with the real vacuum configuration $(v, 0, 0)$ along with a transformed potential of the form

$$\begin{aligned} \bar{V} = & \bar{\gamma}_0 (\bar{h}_S^\dagger \bar{h}_S) + \bar{\gamma}_1 (\bar{h}_1^\dagger \bar{h}_1) + \bar{\gamma}_2 (\bar{h}_2^\dagger \bar{h}_2) + \bar{\gamma}_3 [(\bar{h}_1^\dagger \bar{h}_2) + (\bar{h}_2^\dagger \bar{h}_1)] \\ & + \frac{\bar{\Lambda}_1}{2} (\bar{h}_1^\dagger \bar{h}_1)^2 + \frac{\bar{\Lambda}_2}{2} (\bar{h}_2^\dagger \bar{h}_2)^2 + \bar{\Lambda}_3 (\bar{h}_1^\dagger \bar{h}_1) (\bar{h}_2^\dagger \bar{h}_2) + \bar{\Lambda}_4 (\bar{h}_1^\dagger \bar{h}_2) (\bar{h}_2^\dagger \bar{h}_1) \\ & + \frac{\bar{\Lambda}_5}{2} [(\bar{h}_2^\dagger \bar{h}_1)^2 + (\bar{h}_1^\dagger \bar{h}_2)^2] + \bar{\Lambda}_6 [(\bar{h}_1^\dagger \bar{h}_1) (\bar{h}_1^\dagger \bar{h}_2) + (\bar{h}_1^\dagger \bar{h}_1) (\bar{h}_2^\dagger \bar{h}_1)] \\ & + \bar{\Lambda}_7 [(\bar{h}_2^\dagger \bar{h}_2) (\bar{h}_1^\dagger \bar{h}_2) + (\bar{h}_2^\dagger \bar{h}_2) (\bar{h}_2^\dagger \bar{h}_1)] + \bar{\Lambda}_8 (\bar{h}_S^\dagger \bar{h}_S) [(\bar{h}_1^\dagger \bar{h}_1) + (\bar{h}_2^\dagger \bar{h}_2)] \\ & + \bar{\Lambda}_9 [(\bar{h}_S^\dagger \bar{h}_1) (\bar{h}_1^\dagger \bar{h}_S) + (\bar{h}_S^\dagger \bar{h}_2) (\bar{h}_2^\dagger \bar{h}_S)] + \frac{\bar{\Lambda}_{10}}{2} [(\bar{h}_S^\dagger \bar{h}_1)^2 + (\bar{h}_1^\dagger \bar{h}_S)^2] \\ & + \frac{\bar{\Lambda}_{11}}{2} [(\bar{h}_S^\dagger \bar{h}_2)^2 + (\bar{h}_2^\dagger \bar{h}_S)^2] + \frac{\bar{\Lambda}_{12}}{2} [(\bar{h}_S^\dagger \bar{h}_1) (\bar{h}_S^\dagger \bar{h}_2) + (\bar{h}_1^\dagger \bar{h}_S) (\bar{h}_2^\dagger \bar{h}_S)] \\ & + \bar{\Lambda}_{13} (\bar{h}_S^\dagger \bar{h}_S)^2, \end{aligned} \quad (\text{C9})$$

where all the $\bar{\gamma}_i$ and $\bar{\Lambda}_i$ become real when imposing Eq. (6.1), thus implying a CP conserving model.

APPENDIX D: CONTINUOUS SYMMETRIES OF THE POTENTIAL

Symmetries of multi-Higgs-doublet models are only manifest in particular bases. Reference [21] provides a very useful prescription to identify the symmetries present in specific implementations of three-Higgs-doublet models when written in bases where these symmetries are not manifest. It is well known that a \mathbb{Z}_2 symmetry acting on one

Higgs doublet in models with two Higgs doublets may manifest itself as a symmetry for the interchange of the two doublets. Here we show how a continuous $O(2)$ symmetry of an S_3 -symmetric potential corresponding to the C -III-c vacuum can explicitly appear as a different continuous symmetry.

Let us consider the C -III-c vacuum configuration $(\hat{w}_1 e^{i\sigma_1}, \hat{w}_2 e^{i\sigma_2}, 0)$ in the framework of the S_3 -symmetric 3HDM potential without soft breaking terms. The stationary-point equations require $\lambda_4 = 0$ and $\lambda_3 = -\lambda_2$, so the quartic part of the potential simplifies to

$$V_4 = \lambda_1 (h_1^\dagger h_1 + h_2^\dagger h_2)^2 - \lambda_2 [(h_1^\dagger h_1 - h_2^\dagger h_2)^2 + 4(h_1^\dagger h_2)(h_2^\dagger h_1)] + \lambda_5 (h_S^\dagger h_S) (h_1^\dagger h_1 + h_2^\dagger h_2)$$

$$+ \lambda_6 [(h_S^\dagger h_1) (h_1^\dagger h_S) + (h_S^\dagger h_2) (h_2^\dagger h_S)] + \lambda_7 [(h_S^\dagger h_1)^2 + (h_S^\dagger h_2)^2 + (h_1^\dagger h_S)^2 + (h_2^\dagger h_S)^2]$$

$$+ \lambda_8 (h_S^\dagger h_S)^2. \quad (\text{D1})$$

Using the basis-independent checks given in [21], we find that our potential satisfies the constraints for both $O(2)$ and $U(1)_1$ symmetries. The $O(2)$ symmetry is manifest in the basis in which we start out because the potential is symmetric now under the change of basis given by

$$\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} \rightarrow O \begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix}, \quad (\text{D2})$$

where

$$O \in \left\{ \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ \sin \alpha & -\cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \quad (\text{D3})$$

The $U(1)_1$ symmetry does not manifest itself in this basis, so it must be a hidden symmetry which manifests itself in another basis. Let us therefore change into another basis using

$$\begin{pmatrix} h_1 \\ h_2 \\ h_S \end{pmatrix} = B \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad (\text{D4})$$

where

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (\text{D5})$$

The transformed potential in the new basis is given by

$$V_2 = \mu_0^2 \phi_3^\dagger \phi_3 + \mu_1^2 (\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2), \quad (\text{D6})$$

$$\begin{aligned} V_4 = & \lambda_1 (\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2)^2 - \lambda_2 [(\phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2)^2 \\ & + 4(\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1)] + \lambda_5 (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2) \\ & + \lambda_6 [(\phi_3^\dagger \phi_1)(\phi_1^\dagger \phi_3) + (\phi_3^\dagger \phi_2)(\phi_2^\dagger \phi_3)] \\ & + 2\lambda_7 [(\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_3) + (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2)] + \lambda_8 (\phi_3^\dagger \phi_3)^2. \end{aligned} \quad (\text{D7})$$

Applying the transformation (D5) to the vacuum (2.11), it is seen to become real.

In this new basis, the potential is manifestly symmetric under the $U(1)_1$ transformation given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \rightarrow \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{-i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \quad (\text{D8})$$

This $U(1)_1$ symmetry is not an additional continuous symmetry, it is just the way the original $O(2)$ symmetry manifests itself after the change of basis. If the potential possesses a symmetry in the original basis represented by a matrix S , then after changing to a new basis, the potential will possess a symmetry represented by the matrix $B^\dagger S B$. Applying this to the matrices in (D3), we find that the $O(2)$ matrices are transformed into the following two matrices in the new basis

$$\left\{ \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{-i\alpha} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & e^{i\alpha} & 0 \\ e^{-i\alpha} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \quad (\text{D9})$$

thus showing that this $U(1)_1$ symmetry is just the original $O(2)$ symmetry expressed in the new basis.

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- [1] P. W. Higgs, Broken symmetries, massless particles and gauge fields, *Phys. Lett.* **12**, 132 (1964).
 [2] P. W. Higgs, Broken Symmetries and the Masses of Gauge Bosons, *Phys. Rev. Lett.* **13**, 508 (1964).
 [3] F. Englert and R. Brout, Broken Symmetry and the Mass of Gauge Vector Mesons, *Phys. Rev. Lett.* **13**, 321 (1964).
 [4] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Global Conservation Laws and Massless Particles, *Phys. Rev. Lett.* **13**, 585 (1964).
 [5] G. Aad *et al.* (ATLAS Collaboration), Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC, *Phys. Lett. B* **716**, 1 (2012).
 [6] S. Chatrchyan *et al.* (CMS Collaboration), Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC, *Phys. Lett. B* **716**, 30 (2012).
 [7] J. F. Gunion, H. E. Haber, G. L. Kane, and S. Dawson, *The Higgs Hunter's Guide* (Westview Press, Boulder, CO, 2000).
 [8] G. C. Branco, P. M. Ferreira, L. Lavoura, M. N. Rebelo, M. Sher, and J. P. Silva, Theory and phenomenology of two-Higgs-doublet models, *Phys. Rep.* **516**, 1 (2012).
 [9] I. P. Ivanov, Building and testing models with extended Higgs sectors, *Prog. Part. Nucl. Phys.* **95**, 160 (2017).
 [10] K. Olaussen, P. Osland, and M. A. Solberg, Symmetry and mass degeneration in multi-Higgs-doublet models, *J. High Energy Phys.* **07** (2011) 020.

- [11] T. D. Lee, A theory of spontaneous T violation, *Phys. Rev. D* **8**, 1226 (1973).
- [12] Y. Nambu, Quasiparticles and Gauge Invariance in the Theory of Superconductivity, *Phys. Rev.* **117**, 648 (1960).
- [13] J. Goldstone, Field theories with superconductor solutions, *Nuovo Cimento* **19**, 154 (1961).
- [14] J. Goldstone, A. Salam, and S. Weinberg, Broken symmetries, *Phys. Rev.* **127**, 965 (1962).
- [15] M. Tanabashi *et al.* (Particle Data Group), Review of particle physics, *Phys. Rev. D* **98**, 030001 (2018).
- [16] D. Emmanuel-Costa, O. M. Ogreid, P. Osland, and M. N. Rebelo, Spontaneous symmetry breaking in the S_3 -symmetric scalar sector, *J. High Energy Phys.* **02** (2016) 154; Erratum, *J. High Energy Phys.* **08** (2016) 169.
- [17] S. Pakvasa and H. Sugawara, Discrete symmetry and Cabibbo angle, *Phys. Lett.* **73B**, 61 (1978).
- [18] E. Derman, Flavor unification, τ decay and b decay within the six quark six lepton Weinberg-Salam model, *Phys. Rev. D* **19**, 317 (1979).
- [19] E. Derman and H. S. Tsao, $SU(2) \times U(1) \times S(n)$ flavor dynamics and a bound on the number of flavors, *Phys. Rev. D* **20**, 1207 (1979).
- [20] A. Kuncinas, Properties of S_3 -symmetric Three-Higgs-Doublet models, Master thesis, University of Bergen, 2019, <http://bora.uib.no/handle/1956/20467>.
- [21] I. de Medeiros Varzielas and I. P. Ivanov, Recognizing symmetries in a 3HDM in a basis-independent way, *Phys. Rev. D* **100**, 015008 (2019).
- [22] N. Darvishi and A. Pilaftsis, Classifying accidental symmetries in Multi-Higgs Doublet models, [arXiv:1912.00887](https://arxiv.org/abs/1912.00887).
- [23] H. Ishimori, T. Kobayashi, H. Ohki, H. Okada, Y. Shimizu, and M. Tanimoto, An introduction to non-Abelian discrete symmetries for particle physicists, *Lect. Notes Phys.* **858**, 1 (2012).
- [24] J. Kubo, H. Okada, and F. Sakamaki, Higgs potential in minimal $S(3)$ invariant extension of the standard model, *Phys. Rev. D* **70**, 036007 (2004).
- [25] T. Teshima, Higgs potential in S_3 invariant model for quark/lepton mass and mixing, *Phys. Rev. D* **85**, 105013 (2012).
- [26] D. Das and U. K. Dey, Analysis of an extended scalar sector with S_3 symmetry, *Phys. Rev. D* **89**, 095025 (2014); *Phys. Rev. D* **91**, 039905 (2015).
- [27] P. Ferreira, R. Santos, and A. Barroso, Stability of the tree-level vacuum in two Higgs doublet models against charge or CP spontaneous violation, *Phys. Lett. B* **603**, 219 (2004).
- [28] A. Barroso, P. M. Ferreira, and R. Santos, Charge and CP symmetry breaking in two Higgs doublet models, *Phys. Lett. B* **632**, 684 (2006).
- [29] I. Ivanov, Properties of the general NHD. II. Higgs potential and its symmetries, *J. High Energy Phys.* **07** (2010) 020.
- [30] O. M. Ogreid, P. Osland, and M. N. Rebelo, A Simple Method to detect spontaneous CP Violation in multi-Higgs models, *J. High Energy Phys.* **08** (2017) 005.
- [31] G. C. Branco, J. M. Gerard, and W. Grimus, Geometrical T violation, *Phys. Lett. B* **136**, 383 (1984).
- [32] H. E. Haber, O. M. Ogreid, P. Osland, and M. N. Rebelo, Symmetries and mass degeneracies in the scalar sector, *J. High Energy Phys.* **01** (2019) 042.
- [33] J. F. Donoghue and L. F. Li, Properties of charged Higgs bosons, *Phys. Rev. D* **19**, 945 (1979).
- [34] H. Georgi and D. V. Nanopoulos, Suppression of flavor changing effects from neutral spinless meson exchange in gauge theories, *Phys. Lett. B* **82**, 95 (1979).
- [35] G. C. Branco and M. N. Rebelo, The Higgs mass in a model with two scalar doublets and spontaneous CP violation, *Phys. Lett.* **160B**, 117 (1985).
- [36] N. Chakrabarty, High-scale validity of a model with Three-Higgs-doublets, *Phys. Rev. D* **93**, 075025 (2016).
- [37] D. Das, U. K. Dey, and P. B. Pal, S_3 symmetry and the quark mixing matrix, *Phys. Lett. B* **753**, 315 (2016).
- [38] J. C. Gómez-Izquierdo and M. Mondragón, B–L Model with S_3 symmetry: Nearest neighbor interaction textures and broken $\mu \leftrightarrow \tau$ symmetry, *Eur. Phys. J. C* **79**, 285 (2019).
- [39] N. Chakrabarty and I. Chakraborty, Flavour-alignment in an S_3 -symmetric Higgs sector and its RG-behaviour, [arXiv:1903.09388](https://arxiv.org/abs/1903.09388).
- [40] D. Das and I. Saha, Alignment limit in three Higgs-doublet models, *Phys. Rev. D* **100**, 035021 (2019).
- [41] A. C. B. Machado and V. Pleitez, A model with two inert scalar doublets, *Ann. Phys. (Amsterdam)* **364**, 53 (2016).
- [42] E. C. F. S. Fortes, A. C. B. Machado, J. Montaña, and V. Pleitez, Scalar dark matter candidates in a two inert Higgs doublet model, *J. Phys. G* **42**, 105003 (2015).
- [43] C. Espinoza, E. A. Garcés, M. Mondragón, and H. Reyes-González, The S_3 symmetric model with a dark scalar, *Phys. Lett. B* **788**, 185 (2019).
- [44] S. Mishra, Majorana dark matter and neutrino mass with S_3 symmetry, [arXiv:1911.02255](https://arxiv.org/abs/1911.02255).
- [45] J. M. Gerard, Fermion mass spectrum in $SU(2)_L \times U(1)$, *Z. Phys. C* **18**, 145 (1983).
- [46] E. Barradas-Guevara, O. Félix-Beltrán, and E. Rodríguez-Jáuregui, CP breaking in $S(3)$ flavoured Higgs model, [arXiv:1507.05180](https://arxiv.org/abs/1507.05180).