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WELL-POSEDNESS FOR A DISPERSIVE SYSTEM OF THE WHITHAM–BOUSSINESQ TYPE*

E. DINVAY†, S. SELBERG†, AND A. TESFAHUN†

Abstract. We regard the Cauchy problem for a particular Whitham–Boussinesq system modeling surface waves of an inviscid incompressible fluid layer. We are interested in well-posedness at a very low level of regularity. We derive dispersive and Strichartz estimates and implement them together with a fixed point argument to solve the problem locally. Hamiltonian conservation guarantees global well-posedness for small initial data in the one dimensional settings.

Key words. surface waves, Cauchy problem, Boussinesq system, Strichartz's estimate

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1. Introduction.

We consider the following Whitham-type system posed on $\mathbb{R}^{1+1}$:

$$\begin{cases} \partial_t \eta + \partial_x v = -K_1^2 \partial_x(\eta v), \\ \partial_t v + K_1^2 \partial_x \eta = -K_1^2 \partial_x (v^2/2), \end{cases} \tag{1.1}$$

where

$$K_1 := K_1(D) = \sqrt{\tanh(D)/D} \quad \text{with } D = -i\partial_x. \tag{1.2}$$

The operator K_1 is a Fourier multiplier operator with the symbol $\xi \mapsto \sqrt{\tanh \xi/\xi}$. It is bounded and invertible in $L^2(\mathbb{R})$; more precisely, it is a linear isomorphism from $L^2(\mathbb{R})$ to $H^{1/2}(\mathbb{R})$. Its inverse K_1^{-1} is equivalent to the Bessel potential $J^{1/2}$ defined by the symbol $\xi \mapsto (1+\xi^2)^{1/4}$. Functions η, v are assumed to be real valued. Note that $K_1^2 \partial_x = i \tanh D$ and so system (1.1) has a semilinear nature.

We complement (1.1) with the initial data

$$\eta(0) = \eta_0 \in H^s(\mathbb{R}), \qquad v(0) = v_0 \in H^{s+1/2}(\mathbb{R}), \tag{1.3}$$

where $H^s = (1-\partial_x^2)^{-s/2} L^2(\mathbb{R})$ is the standard notation for the Sobolev space of order s. Such an initial value problem describes the evolution with time of surface waves of a liquid layer. The model approximates the two dimensional water wave problem for an inviscid incompressible potential flow. The variables η and v denote the surface elevation and fluid velocity, respectively. For some discussion on its precise physical meaning we refer the reader to the work by Dinvay, Dutykh, and Kalisch [10], where the system (1.1) appeared for the first time. Formally, v equals $i \tanh D$-derivative of the velocity potential trace on surface associated with the irrotational velocity field. In the long wave Boussinesq regime v coincides with the horizontal fluid velocity at the surface.

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†Department of Mathematics, University of Bergen, PO Box 7803, 5020 Bergen, Norway (Evgueni.Dinvay@uib.no, Sigmund.Selberg@uib.no, achenef@gmail.com).

The system (1.1) possesses a Hamiltonian structure [10]. To our knowledge, there are at least two conserved quantities associated with this system. The first one,

$$\mathcal{H}(\eta, v) = \frac{1}{2}\int_{\mathbb{R}} \left(\eta^2 + vK_1^{-2}v + \eta v^2\right) dx, \tag{1.4}$$

has the meaning of total energy. The second one,

$$\mathcal{I}(\eta, v) = \int_{\mathbb{R}} \eta K_1^{-2} v dx,$$

has the meaning of momentum. The system (1.1) has a Hamiltonian structure of the form

$$\partial_t(\eta, v)^T = \mathcal{J}\nabla\mathcal{H}(\eta, v)$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -i\tanh D \\ -i\tanh D & 0 \end{pmatrix},$$

which in particular guarantees conservation of the energy functional $\mathcal{H}$. It is worth noticing that system (1.1) can be derived at least formally in the long wave asymptotic regime from the Zakharov–Craig–Sulem formulation of the water wave problem [19], also known to be Hamiltonian. The Hamiltonian structure of the Zakharov–Craig–Sulem formulation is canonical, in the sense that the corresponding skew-adjoint matrix $\mathcal{J} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It is interesting to notice that model (1.1) also enjoys a canonical Hamiltonian structure, which is directly comparable with the one of the full water wave system when using variables (η, ψ), where ψ is such that $v = i\tanh D\psi$. Numerical simulations done in [10] show how insignificantly values of functional $\mathcal{H}$ differ from the corresponding energy levels of the full water problem.

We also consider a system posed on $\mathbb{R}^{2+1}$ of the following Whitham–Boussinesq type:

$$\begin{cases} \partial_t\eta + \nabla\cdot\mathbf{v} = -K_2^2\nabla\cdot(\eta\mathbf{v}), \\ \partial_t\mathbf{v} + K_2^2\nabla\eta = -K_2^2\nabla\left(|\mathbf{v}|^2/2\right), \end{cases} \tag{1.5}$$

where $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is a curl free vector field, i.e., $\nabla\times\mathbf{v} = 0$, and

$$K_2 := K_2(D) = \sqrt{\tanh|D|/|D|} \quad (D = -i\nabla)$$

with the corresponding symbol $K_2(\xi) = \sqrt{\tanh(|\xi|)/|\xi|}$. We complement (1.5) with the initial data

$$\eta(0) = \eta_0 \in H^s\left(\mathbb{R}^2\right), \qquad \mathbf{v}(0) = \mathbf{v}_0 \in \left[H^{s+1/2}\left(\mathbb{R}^2\right)\right]^2. \tag{1.6}$$

This is a two dimensional analogue of system (1.1) describing evolution with time of surface waves of a liquid layer in the three dimensional physical space. As above the variables η and $\mathbf{v}$ denote the surface elevation and the fluid velocity, respectively. The system enjoys the Hamiltonian structure

$$\partial_t(\eta, \mathbf{v})^T = \mathcal{J}\nabla\mathcal{H}(\eta, \mathbf{v})$$

with the skew-adjoint matrix

$$\mathcal{J} = \begin{pmatrix} 0 & -K_2^2\partial_{x_1} & -K_2^2\partial_{x_2} \\ -K_2^2\partial_{x_1} & 0 & 0 \\ -K_2^2\partial_{x_2} & 0 & 0 \end{pmatrix},$$

which in particular guarantees conservation of the energy functional

$$\mathcal{H}(\eta, \mathbf{v}) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\eta^2 + \left|K_2^{-1}\mathbf{v}\right|^2 + \eta|\mathbf{v}|^2\right) dx. \tag{1.7}$$

Equations (1.1) were first proposed and studied numerically in [10]. Later in [9] the first proof of local well-posedness based on an energy method and a compactness argument was given. System (1.1) is an alternative to other weakly nonlinear dispersive models describing two-wave propagation [10]. Those models are in good agreement with experiments [7]. They also have many peculiarities of the full water wave problem. The existing results on well-posedness theory, however, are not completely satisfactory. To our knowledge, apart from the model under consideration, there is only one local well-posedness result so far for the regarded system in [10] which has been proved by Pei and Wang [22]. To achieve this the authors imposed an additional nonphysical condition $\eta \geqslant C > 0$. The initial value problem regarded in [22] is probably ill-posed for large data if one removes the positivity assumption $\eta > 0$, as a heuristic argument given in [18] shows. Recently, Kalisch and Pilod [17] proved local well posedness for a surface tension regularization of the system from [22]. They were able to exclude the positivity assumption $\eta > 0$. However, the maximal time of existence for their regularization is bounded by the capillary parameter. One does not need any regularization or special nonphysical conditions to claim the well posedness for (1.1), (1.3).

In fact (1.1) can be regarded itself as a regularization of the system introduced by Hur and Pandey [15]. The latter was also investigated numerically in [10] and compared with other models of Whitham–Boussinesq type. Admitting formally $\tanh D \sim D$ for small frequencies and substituting D instead of $\tanh D$ to the nonlinear part of equations (1.1), one comes to the system regarded in [15]. Hur and Pandey have proved the Benjamin–Feir instability [15] of periodic travelling waves for their system, which makes it valuable. If one in addition formally discards the term $\eta\partial_x u$ in the system given in [15], then a new alternative system turns out to be locally well-posed and features wave breaking [16]. However, the latter does not belong to the class of Boussinesq–Whitham models since nonlinear nondispersive terms have been neglected.

We would like to pay special attention to a system that was not considered in [10] but was introduced by Duchêne, Israwi, and Talhouk [11]. They modified the bilayer Green–Naghdi model improving the frequency dispersion. In fact, their system is also linearly fully dispersive, which makes it a close relative to system (1.1). Note that their system is Hamiltonian as well. Moreover, they have justified the Green–Naghdi modification proving the well-posedness, consistency, and convergence to the full water wave problem in the Boussinesq regime [11]. In addition, the consistency of Hamiltonian structure is shown, so that energy levels of the approximate model can be compared with the full water energy. Existence of solitary waves for their system is also proved in [12]. Returning to the system regarded by Pei and Wang [22], we should notice that a question of existence of solitary waves for it is closed as well [21]. Finally, we point out that well-posedness of the modified Green–Naghdi model is satisfactory, in the sense that it needs neither surface tension nor any nonphysical initial condition. All this together makes it a promising system. And indeed, as noticed in [11], their modification gives more reliable results when it comes to large-frequency Kelvin–Helmholtz instabilities than other models of the Green–Naghdi type.

On the contrary, system (1.1) has a couple of advantages compared with the modified Green–Naghdi model [11]. First, it is derived, though not rigorously, from the Zakharov–Craig–Sulem formulation, and as a result one knows the relation between variables (η, v) and those describing the full potential fluid flow [10]. As to the modification discussed, it is presented in variables where the first one has the meaning of the surface elevation and so coincides with η. Its dual variable is called the layer-averaged horizontal velocity [11]. In the Boussinesq regime it definitely coincides with the same object associated with the full Euler equations. However, one cannot guarantee that it will be the case in shorter wave regimes, whereas for Whitham type models one might anticipate a good agreement which is confirmed by experiments [7]. Here we must admit that neither the Whitham–Boussinesq system (1.1) nor the modified Green–Naghdi system are tested by Carter [7]. So it might be only a matter of time before the modified Green–Naghdi velocity is given an exact physical meaning. In other words, we expect that this velocity will be associated with the full water problem notions. The second issue is that it does not seem obvious how the modified Green–Naghdi system can be generalized to a three dimensional model, whereas for system (1.1) it is straightforward.

Let us formulate the main results. The first one is an improvement of the local existence claimed in [9].

THEOREM 1 (local existence in one dimension). *Let* $s > -1/10$. *Given any* $R > 0$ *there exists a time* $T = T(R) > 0$ *such that for any initial data* $(\eta_0, v_0) \in X^s := H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})$ *with norm* $\|\eta_0\|_{H^s} + \|v_0\|_{H^{s+1/2}} \le R$, *there exists a solution* (η, v) *in the space* $X^s_T := C([0,T]; H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R}))$ *of the Cauchy problem* (1.1), (1.3). *Moreover, the solution is unique in a subspace of* X^s_T *and it depends continuously on the initial data.*

THEOREM 2 (local existence in two dimensions). *Let* $s > 1/4$. *Given any* $R > 0$ *there exists a time* $T = T(R) > 0$ *such that for any initial data* $(\eta_0, \mathbf{v}_0) \in X^s := H^s(\mathbb{R}^2) \times (H^{s+1/2}(\mathbb{R}^2))^2$ *with* $\nabla \times \mathbf{v}_0 = 0$ *and with norm* $\|\eta_0\|_{H^s} + \|\mathbf{v}_0\|_{(H^{s+1/2})^2} \le R$, *there exists a solution* $(\eta, \mathbf{v})$ *in the space* $X^s_T := C([0,T]; H^s(\mathbb{R}^2) \times (H^{s+1/2}(\mathbb{R}^2))^2)$ *of the Cauchy problem* (1.5), (1.6). *Moreover, the solution is unique in a subspace of* X^s_T *and it depends continuously on the initial data.*

Remark 1. For $s > 0$ in one dimension and $s > 1/2$ in two dimensions the solution is unique in the whole space X^s_T. Moreover, the flow map is real analytic for such values of s.

Theorem 1 does not rely on the noncavitation hypothesis $1+\eta > 0$, since smallness of waves is implied in the model. It can be seen as a drawback compared with the model from [11]. However, as mentioned above, it is difficult to say for now which one of these two competing models is a better approximation to the Euler equations. Instead of the noncavitation, there is another condition that we have to impose to prove the following global result. The meaning of this new condition is that the total energy should be positive and not too big. We point out that this condition is imposed at the energy level of regularity and is independent on the regularity s of the initial data.

THEOREM 3 (global existence in one dimension). *Assume that $s \geqslant 0$ and consider the local solution from Theorem 1. There exists $\delta > 0$ such that if*

$$\|\eta_0\|_{L^2(\mathbb{R})} + \|v_0\|_{H^{1/2}(\mathbb{R})} \leqslant \delta,$$

then the solution extends to a global-in-time solution

$$(\eta, v) \in C\left(\mathbb{R}; H^s(\mathbb{R}) \times H^{s+1/2}(\mathbb{R})\right).$$

In the sections below, we first diagonalize systems (1.1) and (1.5) and reformulate the local theorems in the new variables. Then we demonstrate how the local result can be obtained in less general settings applying an elegant classical PDE technique based on the standard Sobolev embedding. This also demonstrates the necessity of dispersive estimates for going down to the energy level of regularity $s = 0$ in one dimension. Note that the domain of the Hamiltonian functional (1.4) is $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$. After that we obtain estimates of Strichartz type studying asymptotic behavior of a particular oscillatory integral (see Lemma 9 and its proof below). This is an improvement compared with dispersive estimates obtained in [3]. In fact we have L^∞-norm decay dominated by L^1-norm locally in frequency, which gives us localized Strichartz estimates. Whereas the decay in [3] is dominated by weighted Sobolev spaces, though frequency independent. With the new estimates in hand we can apply the fixed point argument in a ball of the Bourgain space associated with the water wave dispersion. This gives us the local existence theorems, Theorems 1 and 2.

The last step is to prove the global well-posedness theorem, 3. For $s = 0$ it comes straightforwardly from the energy (1.4) conservation via the continuity argument and the local result. For $s > 0$ we prove the persistence of regularity. Surprisingly, it is not enough just to have the dispersive Strichartz estimates to claim the persistence. Thankfully, our velocity variable v is bounded in $H^{1/2}$-norm and so we are able to use the following limiting case of the Sobolev embedding theorem.

LEMMA 1 (Brezis–Gallouet inequality). *Suppose $f \in H^s(\mathbb{R}^d)$ with $s > d/2$. Then*

$$\|f\|_{L^\infty} \leqslant C_{s,d}\left(1 + \|f\|_{H^{d/2}}\sqrt{\log(2 + \|f\|_{H^s})}\right). \tag{1.8}$$

Inequality (1.8) was first put forward and proved for a domain in $\mathbb{R}^d$ with $d = 2$ in the work by Brezis and Gallouet [5]. It was extended to the other Sobolev spaces in [6]. An implementation of this inequality for deriving a global a priori estimate can be found, for example, in the work by Ponce [23] on the global well-posedness of the Benjamin–Ono equation. We apply a similar trick here, and so that we repeat the formulation of Lemma 1 as it is given in [23]. This provides us with the persistence of regularity that in turn concludes the proof of Theorem 3.

Let us finally give some explanations for the choice of strategy, focusing on the one dimensional case. The local well-posedness for $s > 0$ follows from the standard technique related to semilinear equations. It requires only Duhamel's formula and suitable product estimates for the right-hand side (RHS) of (1.1) in the Sobolev-based space $X^s = H^s \times H^{s+1/2}$. The global bound in X^0 follows from the Hamiltonian conservation, since $\mathcal{H}(\eta, v) \approx \|(\eta, v)\|^2_{X^0}$ provided $\|(\eta, v)\|_{X^0}$ is small. Hence the global well-posedness in X^s with $s > 0$ follows from the local result and an a priori bound obtained from the persistence of regularity and the Brezis–Gallouet inequality.

The main focus of the work is on lowering the regularity threshold for the local well-posedness through the use of dispersive estimates. One anticipates that even the weak dispersive properties of system (1.1) can lower the threshold at least to the limit case $s = 0$. This together with the global bound automatically gives us the global well-posedness in X^0. However, the weakness of dispersion means that the time-decaying $L^1 \to L^\infty$-boundedness of the semigroup, associated with the linearized system, does not hold. As a result the standard strategy based on Strichartz estimates is unavailable. So instead, we obtain the decay estimate on each component of the dyadic Littlewood–Paley decomposition with a sharp dependence on the dyadic number. From this local decay we deduce bilinear estimates in the Bourgain space associated with the water wave dispersion relation. The local well-posedness is deduced from Duhamel's formula with the help of these bilinear estimates.

The main peculiarity of the two dimensional case is that with this technique we are able to prove the local well-posedness in $X^s = H^s \times H^{s+1/2} \times H^{s+1/2}$ only for $s > 1/4$. It still leaves a gap from the energy space X^0, too big to claim global existence. Moreover, even in one dimension it is not clear so far if the problem is globally well-posed for some $s \in (-1/10, 0)$.

Another interesting thing one can notice is that in the two dimensional case we were able to get the maximal gain of $d/8$ derivatives with respect to the naive estimate based only on the unitary property of the semigroup. This is optimal in view of the known smoothing of $\exp(it|D|^{1/2})$ that is essentially the semigroup under consideration. We refer to [1, 2] for more details. It is interesting to notice that in the one dimensional case we obtained the gain of 1/10 derivatives that turns out to be the same for the full water wave problem [1]. The question remains open if one can improve the result and lower the threshold from $s > -1/10$ to the optimal $s > -1/8$ in one dimension.

2. Diagonalization of (1.1) and (1.5) and reformulations of the local existence theorems.

We diagonalize (1.1) as follows. Defining the new variables

$$u_1^+ = \frac{K_1\eta + v}{2K_1}, \qquad u_1^- = \frac{K_1\eta - v}{2K_1}$$

we have

$$\eta = u_1^+ + u_1^-, \quad v = K_1(u_1^+ - u_1^-). \tag{2.1}$$

Then we can write the equation for $u_1^\pm$ as follows:

$$\begin{aligned} 2K_1\partial_t u_1^\pm &= K_1\eta_t \pm v_t \\ &= -K_1\partial_x v - K_1^3\partial_x(\eta v) \mp K_1^2\partial_x\eta \mp K_1^2\partial_x(v^2/2) \\ &= \mp iDK_1(K_1\eta \pm v) - iDK_1^2[K_1(\eta v) \pm v^2/2]. \end{aligned}$$

Thus,

$$i\partial_t u_1^\pm = \pm DK_1 u_1^\pm + \frac{DK_1}{2}[K_1(\eta v) \pm v^2/2]. \tag{2.2}$$

The nonlinear terms can also be written in terms of $u_1^\pm$ as

$$\eta v = (u_1^+ + u_1^-)K_1(u_1^+ - u_1^-), \qquad v^2 = [K_1(u_1^+ - u_1^-)]^2. \tag{2.3}$$

Now let

$$m_1(D) = DK_1(D).$$

From (2.2)–(2.3) we see that the system (1.1) transforms to

$$\begin{cases} (i\partial_t - m_1(D))u_1^+ = B_1^+(u_1^+, u_1^-), \\ (i\partial_t + m_1(D))u_1^- = B_1^-(u_1^+, u_1^-), \end{cases} \tag{2.4}$$

where

$$4B_1^\pm(u_1^+, u_1^-) = DK_1\left[2K_1\left\{(u_1^+ + u_1^-)K_1(u_1^+ - u_1^-)\right\} \pm [K_1(u_1^+ - u_1^-)]^2\right]. \tag{2.5}$$

The initial data (1.3) transforms to

$$u_1^\pm(0) = f_1^\pm := \frac{K_1\eta_0 \pm v_0}{2K_1} \in H^s(\mathbb{R}), \tag{2.6}$$

where we used the fact that $K_1(\xi) \sim \langle\xi\rangle^{-1/2}$, and hence

$$\|K_1^{-1}v_0\|_{H^s(\mathbb{R})} \sim \|\langle D\rangle^{1/2}v_0\|_{H^s(\mathbb{R})} = \|v_0\|_{H^{s+1/2}(\mathbb{R})}.$$

Here and below we use the notation $\langle\xi\rangle = \sqrt{1+\xi^2}$, so $\langle D\rangle = J$ is the Bessel potential of order -1.

To diagonalize (1.5) we define

$$u_2^\pm = \frac{K_2|D|\eta \mp i\nabla\cdot\mathbf{v}}{2K_2|D|}.$$

Hence

$$\eta = u_2^+ + u_2^- \quad \mathbf{v} = -i|D|^{-1}K_2\nabla(u_2^+ - u_2^-), \tag{2.7}$$

where we used the fact that $\mathbf{v}$ is curl free, which in turn implies $\nabla\nabla\cdot\mathbf{v} = \Delta\mathbf{v} = -|D|^2\mathbf{v}$. Then the equations for $u_2^\pm$ are written as follows:

$$\begin{aligned} 2K_2|D|\partial_t u_2^\pm &= K_2|D|\eta_t \mp i\nabla\cdot\mathbf{v}_t \\ &= \mp iK_2|D|(K_2|D|\eta \mp i\nabla\cdot\mathbf{v}) + i|D|^2K_2^2[K_2|D|^{-1}(i\nabla)\cdot(\eta\mathbf{v}) \pm (|\mathbf{v}|^2)/2]. \end{aligned}$$

Thus,

$$i\partial_t u_2^\pm = \pm|D|K_2u_2^\pm - \frac{|D|K_2}{2}[iK_2R\cdot(\eta\mathbf{v}) \mp |\mathbf{v}|^2/2)], \tag{2.8}$$

where $R = (R_1, R_2)$ with $R_j = \partial_j/|D|$ being the Riesz transforms. Now setting

$$m_2(D) := |D|K_2(D)$$

and combining (2.7)–(2.8) we see that the system (2.9) transforms to

$$\begin{cases} (i\partial_t - m_2(D))u_2^+ = B_2^+(u_2^+, u_2^-), \\ (i\partial_t + m_2(D))u_2^- = B_2^-(u_2^+, u_2^-), \end{cases} \tag{2.9}$$

where

$$4B_2^\pm(u_2^+, u_2^-) = -|D|K_2\left[2K_2R\left\{(u_2^+ + u_2^-)K_2R(u_2^+ - u_2^-)\right\} \mp \left|K_2R(u_2^+ - u_2^-)\right|^2\right]. \tag{2.10}$$

The initial data (1.6) transforms to

$$u_2^\pm(0) = f_2^\pm := \frac{K_2|D|\eta_0 \mp i\nabla\cdot\mathbf{v}_0}{2K_2|D|} \in H^s(\mathbb{R}), \tag{2.11}$$

where we used the fact that $K_2(\xi) \sim \langle\xi\rangle^{-1/2}$.

Now let us reformulate Theorems 1 and 2 in terms of the new variables as follows.

Theorem 4. *Let $s > -1/10$. Given any $R > 0$ there exists a time $T = T(R) > 0$ such that for any initial data $(f_1^+, f_1^-) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with norm $\|f_1^+\|_{H^s(\mathbb{R})} + \|f_1^-\|_{H^s(\mathbb{R})} \le R$, the Cauchy problem (2.4)–(2.6) has a solution*

$$(u_1^+, u_1^-) \in C\left([0,T]; H^s(\mathbb{R}) \times H^s(\mathbb{R})\right).$$

Moreover, the solution is unique in a subset of this space and depends continuously on the data.

Theorem 5. *Let $s > 1/4$. Given any $R > 0$ there exists a time $T = T(R) > 0$ such that for any initial data $(f_2^+, f_2^-) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with norm $\|f_2^+\|_{H^s(\mathbb{R}^2)} + \|f_2^-\|_{H^s(\mathbb{R}^2)} \le R$, the Cauchy problem (2.9)–(2.11) has a solution*

$$(u_2^+, u_2^-) \in C\left([0,T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\right).$$

Moreover, the solution is unique in a subset of this space and depends continuously on the data.

The system (2.4)–(2.6) can be written in the form of integral equations as

$$u_1^\pm(t) = e^{\mp itm_1(D)} f_1^\pm \mp i \int_0^t e^{\mp i(t-s)m_1(D)} B_1^\pm(u_1^+, u_1^-)(s)\, ds. \tag{2.12}$$

Similarly, the system (2.9)–(2.11) can be written in the form of integral equations as

$$u_2^\pm(t) = e^{\mp itm_2(D)} f_2^\pm \mp i \int_0^t e^{\mp i(t-s)m_2(D)} B_2^\pm(u_2^+, u_2^-)(s)\, ds. \tag{2.13}$$

Applying the contraction argument to (2.12) together with the Sobolev embedding one can prove Theorem 4 for $s > 0$ and Theorem 5 for $s > 1/2$, as shown in the next section. However, to prove Theorem 4 for $s > -1/10$ and Theorem 5 for $s > 1/4$ we need to derive dispersive estimates on the semigroups $S_{m_d}(\pm t) := e^{\mp itm_d(D)}$, where

$$m_1(\xi) = \xi K_1(\xi) = \xi\sqrt{\frac{\tanh\xi}{\xi}} \qquad (\xi \in \mathbb{R}),$$

$$m_2(\xi) = |\xi| K_2(\xi) = |\xi|\sqrt{\frac{\tanh|\xi|}{|\xi|}} \qquad (\xi \in \mathbb{R}^2).$$

3. Nondispersive estimates.

3.1. Local well-posedness for $s > 0$ in one dimension. In this section we prove the local well-posedness in $H^s \times H^{s+1/2}$ with $s > 0$ for system (1.1) applying a fixed-point argument. It is only a particular case of Theorem 1 (or of the equivalent theorem, 4). In this sense, the section has mainly an illustrative character. However, the proof is elegant and does not need any use of dispersive techniques. The idea is close to the one used in [4], for instance. This allows us to think about system (1.1) as a fully dispersive bidirectional relative to the Benjamin–Bona–Mahony equation.

Regard the Whitham operator $K = \sqrt{\tanh D/D}$ and introduce the space $X^s = H^s \times H^{s+1/2}$ equipped with the norm

$$\|(f,g)\|_{X^s}^2 = \|f\|_{H^s}^2 + \|K^{-1}g\|_{H^s}^2, \tag{3.1}$$

which is obviously equivalent to the standard one. Denote by X_T^s the space of continuous functions defined on $[0,T]$ with values in X^s, equipped with the supremum-norm. Define matrices

$$\mathcal{K} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ K & -K \end{pmatrix}, \quad \mathcal{K}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & K^{-1} \\ 1 & -K^{-1} \end{pmatrix}.$$

Clearly, that $\mathcal{K}$ is isometric from $H^s \times H^s$ to X^s for any $s \in \mathbb{R}$, i. e. $\|\mathcal{K}(f,g)^T\|_{X^s} = \|(f,g)\|_{H^s\times H^s}$. Regard the unitary group

$$\mathcal{S}(t) = \mathcal{K}\begin{pmatrix} e^{-itm} & 0 \\ 0 & e^{itm} \end{pmatrix}\mathcal{K}^{-1},$$

where $m = m(D) = \sqrt{D\tanh D}\,\mathrm{sgn}\, D$. Note that for any $s,t \in \mathbb{R}$, $u \in X^s$ holds $\|\mathcal{S}(t)u\|_{X^s} = \|u\|_{X^s}$ and consequently $\|\mathcal{S}(t)u\|_{X_T^s} = \|u\|_{X_T^s}$ for any $T > 0$. These follow from the isometricity of operators $\mathcal{K}$, $\mathcal{K}^{-1}$ and that symbols of eigenvalues of $\mathcal{S}(t)$ have absolute value equal to one. For any fixed $u_0 = (\eta_0, v_0)^T \in X^s$ function $\mathcal{S}(t)u_0$ solves the linear initial-value problem associated with (1.1). Regard a mapping $\mathcal{A} : X_T^s \to X_T^s$ defined by

$$\mathcal{A}(\eta,v) = \mathcal{A}(\eta,v;u_0)(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-t')(-i\tanh D)\begin{pmatrix} \eta v \\ v^2/2 \end{pmatrix}(t')dt'. \tag{3.2}$$

Then the Cauchy problem for system (1.1) with the initial data u_0 may be rewritten equivalently as an equation in X_T^s of the form

$$u = \mathcal{A}(u;u_0), \tag{3.3}$$

where $u = (\eta,v)^T \in X_T^s$. Below the latter integral equation is solved locally in time by making use of Picard iterations.

LEMMA 2 (particularly the case of Theorem 1). *Let $s > 0$, $u_0 = (\eta_0, v_0)^T \in X^s$, and $T = (7C_s\|u_0\|_{X^s})^{-1}$ with some constant $C_s > 0$ depending only on s. Then there exists a unique solution $u = (\eta,v)^T \in X_T^s$ of Problem* (3.3).

Moreover, for any $R > 0$ there exists $T = T(R) > 0$ such that the flow map associated with (3.3) *is a real analytic mapping of the open ball $B_R(0) \subset X^s$ to X_T^s.*

Proof. The idea is to show that the restriction of $\mathcal{A}$ on some closed ball B_M centered at $\mathcal{S}(t)u_0$ is a contraction mapping. The key ingredient is the product estimate $\|\eta v\|_{H^s} \lesssim \|\eta\|_{H^s}\|v\|_{H^{s+1/2}}$ that can be found, for example, in [14]. Obviously, there exists a positive constant C_s such that

$$\|(\eta v, v^2/2)\|_{X^s} \leqslant C_s\|(\eta,v)\|_{X^s}^2$$

and

$$\|(\eta_1 v_1 - \eta_2 v_2, v_1^2/2 - v_2^2/2)\|_{X^s} \leqslant C_s \|(\eta_1 - \eta_2, v_1 - v_2)\|_{X^s} (\|(\eta_1, v_1)\|_{X^s} + \|(\eta_2, v_2)\|_{X^s}).$$

Thus for any $T, M > 0$ and $u, u_1, u_2 \in B_M \subset X_T^s$ it holds that

$$\|\mathcal{A}(u) - \mathcal{S}(t)u_0\|_{X_T^s} \leqslant \int_0^T \|(\eta v, v^2/2)\|_{X^s} \leqslant C_s T \|u\|_{X_T^s}^2,$$
$$\|\mathcal{A}(u_1) - \mathcal{A}(u_2)\|_{X_T^s} \leqslant C_s T \|u_1 - u_2\|_{X_T^s} (\|u_1\|_{X_T^s} + \|u_2\|_{X_T^s}),$$

and so taking $M = 2\|u_0\|_{X^s}$ and T as in the lemma formulation we conclude that $\mathcal{A}$ is a contraction in the closed ball B_M. The first statement of the lemma follows from the contraction mapping principle.

We turn our attention to smoothness of the flow map. Let $R > 0$, $T = (7C_s R)^{-1}$, and $B = B_R(0)$ be an open ball in X^s. Define $\Lambda : B \times X_T^s \to X_T^s$ as

$$\Lambda(u_0, u) = u - \mathcal{A}(u; u_0)$$

that is obviously a smooth map. Its Fréchet derivative with respect to the second variable is defined by

$$d_u \Lambda(u_0, u)h = h + i \int_0^t \mathcal{S}(t - t') \tanh D \begin{pmatrix} v & \eta \\ 0 & v \end{pmatrix} h(t') dt',$$

where $u = (\eta, v)^T$ and $h \in X_T^s$. If $u_1 \in X_T^s$ is the solution of Problem (3.3) corresponding the initial data $u_0 \in B$, then $\Lambda(u_0, u_1) = 0$. Moreover, it satisfies the estimate

$$\|u_1(t)\|_{X^s} \leqslant \|u_0\|_{X^s} + C_s \int_0^t \|u_1(t')\|_{X^s}^2 dt'$$

and so

$$\int_0^t \|u_1(t')\|_{X^s}^2 dt' \leqslant \frac{t\|u_0\|_{X^s}^2}{1 - C_s t \|u_0\|_{X^s}}$$

for any t. The latter is used to estimate operator $I - d_u\Lambda(u_0, u_1)$ as

$$\begin{aligned} \|h - d_u \Lambda(u_0, u_1)h\| &\leqslant C_s \sup_{t \in [0,T]} \int_0^t \|u_1(t')\|_{X^s} \|h(t')\|_{X^s} dt' \\ &\leqslant C_s \sup_{t \in [0,T]} \left(t \int_0^t \|u_1(t')\|_{X^s}^2 dt' \right)^{1/2} \|h\|_{X_T^s} \\ &\leqslant \frac{C_s T \|u_0\|_{X^s}}{\sqrt{1 - C_s T \|u_0\|_{X^s}}} \|h\|_{X_T^s} \leqslant \frac{1}{\sqrt{42}} \|h\|_{X_T^s}, \end{aligned}$$

which is true for any $h \in X_T^s$. As a result operator $d_u\Lambda(u_0, u_1)$ is invertible and so the second assertion of the lemma follows from the implicit function theorem. ∎

The next and most difficult step is to extend the statement of the lemma to the case $s \leqslant 0$ as well. Even extension to the limiting case $s = 0$ is not trivial. On the one hand, it seems possible to do it without the dispersive estimates, applying the energy method, for example. Indeed, we have the Hamiltonian conservation that can provide us with a necessary a priori bound (see Lemma 4 below). However, at such a level of regularity with $s = 0$ the regularization of system (1.1) can be a serious issue. In other words, one cannot guarantee that the a priori estimate will still be valid for the regularized problem. Moreover, we can hardly hope for more than a weak solution after implementing the compactness argument. So we turn our attention to the harmonic analysis methods, since we can eventually achieve a more general result with the dispersive estimates obtained below in the next sections.

3.2. Local well-posedness for $s > 1/2$ in two dimensions. The proof is essentially the same. Now the change of variables has the form

$$\mathcal{K} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -iKR_1 & iKR_1 \\ -iKR_2 & iKR_2 \end{pmatrix},$$

where $K = \sqrt{\tanh|D|/|D|}$. Then $\mathcal{K}$ is an isometric operator from $H^s \times H^s$ to the subspace X^s of $H^s \times (H^{s+1/2})^2$ with the curl free second coordinate and endowed with the norm $\|\mathcal{K}^{-1}(\eta, \mathbf{v})^T\|_{H^s \times H^s}$. This $\mathcal{K}$ defines a continuous group $\mathcal{S}(t)$ as above. For any fixed $u_0 = (\eta_0, \mathbf{v}_0)^T \in X^s$ function $\mathcal{S}(t)u_0$ solves the linear initial-value problem associated with (1.5) in $X^s_T = C([0,T]; X^s)$. Considering the map $\mathcal{A} : X^s_T \to X^s_T$ defined by

$$\mathcal{A}(\eta, \mathbf{v}; u_0)(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t - t') \begin{pmatrix} K^2\nabla \cdot (\eta\mathbf{v}) \\ K^2\nabla\left(|\mathbf{v}|^2/2\right) \end{pmatrix} (t')dt' \tag{3.4}$$

we reduce the Cauchy problem for system (1.5) with the initial data u_0 to (3.3) in X^s_T again, with the only difference that now $u = (\eta, \mathbf{v})^T \in X^s_T$ is a three component vector.

LEMMA 3 (particularly the case of Theorem 2). *Let $s > 1/2$, $u_0 \in X^s$, and $T = (7C_s\|u_0\|_{X^s})^{-1}$ with some constant $C_s > 0$ depending only on s. Then there exists a unique solution $u \in X^s_T$ of problem* (3.3).

Moreover, for any $R > 0$ there exists $T = T(R) > 0$ such that the flow map associated with (3.3) *is a real analytic mapping of the open ball $B_R(0) \subset X^s$ to X^s_T.*

As above the key ingredient is the same product estimate that in the case $d = 2$ is valid only provided $s > 1/2$, and so we omit the proof.

3.3. A priori estimates for $s \geqslant 0$ in one dimension. First, we prove the following global bound in the energy space X^0.

LEMMA 4. *There exists a constant $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, if a pair $u(t) = (\eta(t), v(t)) \in L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ having initial condition $\|u_0\|_{L^2 \times H^{1/2}} \leqslant \epsilon/2$ solves system* (1.1)*, then its norm remains bounded $\|u(t)\|_{L^2 \times H^{1/2}} \leqslant \epsilon$ for any time t.*

Proof. We use a continuity argument. Without loss of generality we prove the statement with the X^0-norm defined in (3.1), which is equivalent to the $L^2 \times H^{1/2}$-norm. For $u = (\eta, v)$, define

$$\|u\|^2 := \frac{1}{2}\|u\|^2_{X^0} = \frac{1}{2}\|\eta\|^2_{L^2} + \frac{1}{2}\|K^{-1}v\|^2_{L^2}.$$

Then there exists $C > 0$ such that

$$\|u\|^2(1 - C\|u\|) \leqslant \mathcal{H}(u) \leqslant \|u\|^2(1 + C\|u\|),$$

where $u = u(t)$ is a solution of (1.1) defined on some interval. Take $\epsilon_0 = (2C)^{-1}$, any $0 < \epsilon \leqslant \epsilon_0$, and a solution with $u_0 = u(0)$ having $\|u_0\| \leqslant \epsilon/2$. By continuity $\|u\| \leqslant \epsilon$ on some $[0, T_\epsilon]$ and so

$$\|u\| \leqslant \sqrt{2\mathcal{H}(u)} = \sqrt{2\mathcal{H}(u_0)} \leqslant \sqrt{\frac{1 + C\epsilon/2}{2}}\epsilon < \epsilon,$$

which means that the continuous function $\|u(t)\|$ cannot touch the level ϵ with time. ∎

Proving the next lemma, we will employ a sharper variant of the bilinear estimates used at the beginning of the proof of Lemma 2. Recall the notation $\|(\eta, v)\|_{X^s}$ defined by (3.1).

Lemma 5 (persistence of regularity). *Suppose $s > 0$ and a pair $\eta(t) \in H^s$, $v(t) \in H^{s+1/2}$ solves problem* (1.1), (1.3). *Then if $s < 1/2$, the following holds true,*

$$\|(\eta, v)(t)\|_{X^s} \leqslant \|(\eta_0, v_0)\|_{X^s} + C_s \int_0^t \left(\|v\|_{H^{1/2}} + \|v\|_{L^\infty}\right) \|(\eta, v)\|_{X^s},$$

and if $s \geqslant 1/2$, then

$$\|(\eta, v)(t)\|_{X^s} \leqslant \|(\eta_0, v_0)\|_{X^s} + C_s \int_0^t \|v\|_{H^{s+1/4}} \|(\eta, v)\|_{X^s},$$

where constant C_s depends only on s.

Proof. Estimating $\mathcal{A}(t)$ given by (3.2) in X^s-norm defined by (3.1), one deduces from (3.3) the following inequality:

$$\|(\eta, v)(t)\|_{X^s} \leqslant \|(\eta_0, v_0)\|_{X^s} + \int_0^t \left\| \begin{pmatrix} \tanh D(\eta v) \\ \tanh D(v^2/2) \end{pmatrix} (t') \right\|_{X^s} dt'.$$

It is left to calculate the integrand. Provided $s \in (0, 1/2)$ by the Leibniz rule [20] we have

$$\|J^s \tanh D(\eta v)\|_{L^2} \lesssim \|J^s \eta\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|\eta\|_{L^{p_2}} \|J^s v\|_{L^{q_2}}, \tag{3.5}$$

where setting $p_1 = 2$, $q_1 = \infty$, $p_2 = 2/(1 - 2s)$, $q_2 = 1/s$ and using the Sobolev embedding we obtain

$$\|J^s \tanh D(\eta v)\|_{L^2} \lesssim \|\eta\|_{H^s} \left(\|v\|_{L^\infty} + \|v\|_{H^{1/2}}\right).$$

Similarly, but now for any $s \in (0, \infty)$ we have

$$\left\|J^s K^{-1} \tanh D v^2\right\|_{L^2} \lesssim \left\|J^{s+1/2} v^2\right\|_{L^2} \lesssim \|v\|_{L^\infty} \left\|J^{s+1/2} v\right\|_{L^2} \lesssim \|v\|_{L^\infty} \left\|K^{-1} v\right\|_{H^s}. \tag{3.6}$$

This implies the first inequality in the statement valid for $s \in (0, 1/2)$.

Regarding the case $s = 1/2$ and setting $p_2 = q_2 = 4$ with the same $p_1 = 2$, $q_1 = \infty$ in the Leibniz inequality (3.5), after implementation the Sobolev embedding, obtain

$$\|J^s \tanh D(\eta v)\|_{L^2} \lesssim \|\eta\|_{H^s} \|v\|_{H^{s+1/4}}.$$

This inequality is obvious for $s > 1/2$ since H^s is an algebra under the pointwise product, and so is true for any $s \geqslant 1/2$. Taking into account (3.6) we deduce the second inequality of the lemma. ∎

In order to use the persistence of regularity lemma, 5, one needs two Gronwall inequalities. One of them is considered to be standard. For completeness, we give here a proof of the other Gronwall type inequality, which is less standard and will be used below.

LEMMA 6 (Gronwall inequality). *Let $y(t) > 1$ be a continuous function defined on some interval $[0, T]$ with $y(0) = y_0$. Suppose that for any $t \in [0, T]$ it holds that*

$$y(t) \leqslant y_0 + C \int_0^t y \log y.$$

Then

$$y(t) \leqslant \exp\left(e^{Ct} \log y_0\right).$$

Proof. One can easily calculate

$$\frac{d}{dt} \log\log\left(y_0 + C\int_0^t y\log y\right) = \frac{Cy\log y}{\left(y_0 + C\int_0^t y\log y\right)\log\left(y_0 + C\int_0^t y\log y\right)} \leqslant C,$$

where we have used the dominance of $y(t)$ by the integral expression. The fundamental theorem of calculus provides us with the claim. ∎

The persistence of regularity based on the energy estimate lemma, 5, transforms to the following a priori estimates.

LEMMA 7. *Suppose $s > 0$ and a pair $u(t) = (\eta(t), v(t)) \in X^s$ solves system* (1.1) *on some time interval with $u(0) = u_0$ small enough with respect to X^0-norm in the sense of Lemma* 4. *Then if $s < 1/2$, the following holds true,*

$$\|u(t)\|_{X^s} \leqslant \exp\left(Ce^{Ct}\right),$$

and if $s \geqslant 1/2$, then

$$\|u(t)\|_{X^s} \leqslant \|u_0\|_{X^s} \exp\left(C\int_0^t \|v\|_{H^{s+1/4}}\right),$$

where constant C depends only on s, $\|u(0)\|_{X^0}$, and $\|u(0)\|_{X^s}$.

Proof. Suppose $s \in (0, 1/2)$ and $u(t) = (\eta(t), v(t)) \in X^s$ solves system (1.1) on some time interval. Let its initial data u_0 be small with respect to X^0-norm in the sense of Lemma 4. Then $u(t)$ stays bounded in X^0, and so $\|v(t)\|_{H^{1/2}}$ is bounded by the same constant independent on the time interval. Hence from the Brezis–Gallouet limiting embedding (1.8) one deduces

$$\|v(t)\|_{L^\infty} \lesssim 1 + \log\left(2 + \|v(t)\|_{H^{s+1/2}}\right)$$

and applying Lemma 5 obtains

$$\|u\|_{X^s} \leqslant \|u_0\|_{X^s} + C\int_0^t \left(1 + \log\left(2 + \|u\|_{X^s}\right)\right)\|u\|_{X^s}.$$

Introducing $y(t) = 2 + \|u(t)\|_{X^s}$ we arrive at the assumption of the Gronwall inequality, Lemma 6. As a result we have the estimate

$$2 + \|u\|_{X^s} \leqslant \exp\left(e^{2Ct}\log\left(2 + \|u_0\|_{X^s}\right)\right),$$

which is the first claim.

In the case $s \geqslant 1/2$ we make use of the second inequality in Lemma 5 and a more standard Gronwall inequality [25]. ∎

4. Dispersive estimate for $S_{m_d}(\pm t)f$. First we establish a lower bound for the first and second derivatives of the function $m(r) = r\sqrt{\tanh(r)/r}$. These estimates will be used later to derive dispersive estimates for the free waves $S_{m_d}(\pm t)f$ using a stationary phase method.

Throughout the next three sections we use the following notation: The Greek letter λ denotes a dyadic number, i.e., this variable ranges over numbers of the form 2^k for $k \in \mathbb{Z}$. In estimates we use $A \lesssim B$ as shorthand for $A \le CB$ and $A \ll B$ for $A \le C^{-1}B$, where $C \gg 1$ is a positive constant which is independent of dyadic numbers such as λ and time T, whereas $A \sim B$ means $B \lesssim A \lesssim B$.

Lemma 8. *Set $m(r) = rK(r)$, where $K(r) = \sqrt{\tanh(r)/r}$. Then for $r > 0$,*

$$0 < m'(r) \sim \langle r \rangle^{-1/2}, \tag{4.1}$$

$$0 < -m''(r) \sim r\langle r \rangle^{-5/2}. \tag{4.2}$$

Proof. First note that

$$K'(r) = \frac{r\mathrm{sech}^2(r) - \tanh(r)}{2r^2K(r)},$$

$$K''(r) = -\frac{\tanh(r)\mathrm{sech}^2(r)}{rK(r)} - \frac{\left(r\mathrm{sech}^2(r) - \tanh(r)\right)}{r^3K(r)} - \frac{\left(r\mathrm{sech}^2(r) - \tanh(r)\right)^2}{4r^4K^3(r)},$$

which imply

$$\begin{aligned} m'(r) &= K(r) + rK'(r) = \frac{K(r)}{2} + \frac{\mathrm{sech}^2(r)}{2K(r)} > 0, \\ m''(r) &= 2K'(r) + rK''(r) \\ &= -\frac{\tanh(r)\mathrm{sech}^2(r)}{K(r)} - \frac{\left(r\mathrm{sech}^2(r) - \tanh(r)\right)^2}{4r^3K^3(r)} \\ &= -\frac{1}{4r}\left[4r^2K\mathrm{sech}^2(r) + K^{-3}(r)\left(K^2(r) - \mathrm{sech}^2(r)\right)^2\right]. \end{aligned}$$

Now let us estimate $m'(r)$. One can assume without loss of generality that $r > 0$. Since

$$K(r) = \sqrt{\tanh(r)/r} \sim \langle r \rangle^{-1/2} \quad \text{and} \quad \mathrm{sech}(r) \sim e^{-r}$$

we have

$$m'(r) \sim \langle r \rangle^{-1/2} + \langle r \rangle^{1/2}e^{-2r} \sim \langle r \rangle^{-1/2}. \tag{4.3}$$

Next we estimate $m''(r)$. We can write

$$K^2(r) - \mathrm{sech}^2(r) = E(r)\mathrm{sech}^2(r),$$

where

$$E(r) = \frac{e^{2r} - e^{-2r} - 4r}{4r}.$$

Now if $0 < r < 1/2$ we have

$$E(r) = \frac{1}{2r}\sum_{n=0}^{\infty}\frac{(2r)^{2n+3}}{(2n+3)!} = 4Cr^2,$$

where $C := C(r) = \sum_{n=0}^{\infty} \frac{(2r)^{2n}}{(2n+3)!} < \infty$. If $r \geq 1/2$, we have

$$E(r) = \frac{e^{2r}}{4r}[1 - e^{-4r} - 4re^{-2r}] \sim \frac{e^{2r}}{r}.$$

Therefore,

$$E(r) \sim \begin{cases} r^2 & \text{if } 0 < r < 1/2, \\ r^{-1}e^{2r} & \text{if } r \geq 1/2. \end{cases} \tag{4.4}$$

Then using (4.3) and (4.4) we obtain

$$\begin{aligned} |m''(r)| &= \frac{1}{4|r|}\left[4r^2K(r)\mathrm{sech}^2(r) + K^{-3}(r)E^2(r)\mathrm{sech}^4(r)\right] \\ &\sim |r|^{-1}\left[r^2\langle r\rangle^{-\frac{1}{2}}e^{-2r} + \langle r\rangle^{\frac{3}{2}}E^2(r)e^{-4r}\right] \\ &\sim |r|\langle r\rangle^{-5/2}. \end{aligned}$$

□

Next we use the estimates on the derivatives of $m(r)$ in Lemma 8 and stationary phase method to derive a frequency localized dispersive estimate for the free waves $S_m(\pm t)f$. To this end, we consider an even function $\chi \in C_0^\infty((-2,2))$ such that $\chi(s) = 1$ if $|s| \leq 1$. Let

$$\beta(s) = \chi(s) - \chi(2s), \qquad \beta_\lambda(s) := \beta(s/\lambda),$$

where $\lambda \in 2^{\mathbb{Z}}$ is dyadic. Thus, $\operatorname{supp}\beta_\lambda \subset \{s \in \mathbb{R} : \lambda/2 \leqslant |s| \leqslant 2\lambda\}$. Now define the frequency projection P_λ by

$$\widehat{P_\lambda f}(\xi) = \begin{cases} \chi(|\xi|)\widehat{f}(\xi) & \text{if } \lambda = 1, \\ \beta_\lambda(|\xi|)\widehat{f}(\xi) & \text{if } \lambda > 1. \end{cases}$$

We write $f_\lambda := P_\lambda f$. Then $f = \sum_{\lambda \geq 1} f_\lambda$.

The following is the key dispersive estimate that will be crucial in the proof of Theorems 4 and 5.

Lemma 9 (localized dispersive estimate). *Let $\lambda > 1$ and $d \in \{1,2\}$. Then we have the estimate*

$$\|S_{m_d}(\pm t)f_\lambda\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \lambda^{3d/4}|t|^{-d/2}\|f\|_{L_x^1(\mathbb{R}^d)}.$$

Interpolating this with the trivial bound (by Plancherel)

$$\|S_{m_d}(\pm t)f_\lambda\|_{L_x^2(\mathbb{R}^d)} \leq \|f\|_{L_x^2(\mathbb{R}^d)},$$

we obtain the following.

Corollary 1. *Assuming $\lambda > 1$, $d \in \{1,2\}$, and $2 \leq r \leq \infty$, we have*

$$\|S_{m_d}(\pm t)f_\lambda\|_{L_x^r(\mathbb{R}^d)} \lesssim \left(\lambda^{3d/4}|t|^{-d/2}\right)^{1-2/r}\|f\|_{L_x^{r'}(\mathbb{R}^d)}.$$

The remainder of this section is devoted to the proof of Lemma 9. It suffices to prove the estimate for positive times:

$$\|S_{m_d}(t)f_\lambda\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \lambda^{3d/4}t^{-d/2}\|f\|_{L_x^1(\mathbb{R}^d)} \quad (t > 0). \tag{4.5}$$

One can write

$$[S_{m_d}(t)f_\lambda](x) = \mathcal{F}_x^{-1}\left[e^{itm_d(\xi)}\beta_\lambda(|\xi|)\hat{f}\right](x) = (I_{\lambda,t} * f)(x),$$

where

$$\begin{aligned} I_{\lambda,t}(x) &= \mathcal{F}_x^{-1}\left[e^{itm_d(\xi)}\beta_\lambda(|\xi|)\right](x) \\ &= \int_{\mathbb{R}^d} e^{ix\cdot\xi+itm_d(\xi)}\beta_\lambda(|\xi|)\,d\xi = \lambda^d\int_{\mathbb{R}^d} e^{i\lambda x\cdot\xi+itm_d(\lambda\xi)}\beta(|\xi|)\,d\xi. \end{aligned} \tag{4.6}$$

Then by Young's inequality

$$\|S_{m_d}(t)f_\lambda\|_{L_x^\infty(\mathbb{R}^d)} \le \|I_{\lambda,t}\|_{L_x^\infty(\mathbb{R}^d)}\|f\|_{L_x^1(\mathbb{R}^d)}, \tag{4.7}$$

so (4.5) reduces to proving

$$\|I_{\lambda,t}\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \lambda^{3d/4}t^{-d/2}. \tag{4.8}$$

But clearly,

$$\|I_{\lambda,t}\|_{L_x^\infty(\mathbb{R}^d)} \lesssim \lambda^d,$$

so in view of (4.8) it is enough to consider the case where

$$\lambda^{3d/4}t^{-d/2} \ll \lambda^d \Leftrightarrow t \gg \lambda^{-1/2}. \tag{4.9}$$

The proof of (4.8) in this case is given in the following two subsections, first for space dimension $d = 1$ and then for $d = 2$.

4.1. Proof of (4.8) when $d = 1$. In one dimension we have

$$I_{\lambda,t}(x) = \lambda\int_{\mathbb{R}} e^{it\phi_\lambda(\xi)}\beta(|\xi|)\,d\xi,$$

where

$$\phi_\lambda(\xi) := \lambda\xi x/t + m_1(\lambda\xi) = \lambda\xi x/t + \lambda\xi K_1(\lambda\xi).$$

Note that $m_1(\xi) = m(\xi)$, where m is as in Lemma 8. Now since the function ϕ_λ is odd we can write

$$I_{\lambda,t}(x) = 2\lambda\int_0^\infty \cos(t\phi_\lambda(\xi))\beta(\xi)\,d\xi = 2\lambda\int_{1/2}^2 \cos(t\phi_\lambda(\xi))\beta(\xi)\,d\xi.$$

Since

$$\phi'_\lambda(\xi) = \lambda\left[x/t + m'(\lambda\xi)\right], \tag{4.10}$$
$$\phi''_\lambda(\xi) = \lambda^2 m''(\lambda\xi), \tag{4.11}$$

we see from Lemma 8 that

$$0 < -\phi''_\lambda(\xi) = -\lambda^2 m''(\lambda\xi) \sim \lambda^3\langle\lambda\rangle^{-5/2} \sim \lambda^{1/2} \tag{4.12}$$

for $\xi \in [1/2, 2]$. Here we used also the assumption $\lambda \ge 1$.

4.1.1. Nonstationary contribution. This is the case when either (i) $x \geq 0$ or (ii) $x < 0$ and $-x/t \ll \lambda^{-1/2}$ or $-x/t \gg \lambda^{-1/2}$. Then since $m'(\lambda\xi))$ is positive and comparable to $\langle \lambda\xi \rangle^{-1/2}$ (Lemma 8), we see from (4.10) that

$$|\phi'_\lambda(\xi)| \gtrsim \lambda^{1/2} \tag{4.13}$$

for $\xi \in [1/2, 2]$. Integration by parts yields

$$\begin{aligned} I_{\lambda,t}(x) &= 2\lambda t^{-1} \int_{1/2}^{2} \frac{d}{d\xi} [\sin(t\phi_\lambda(\xi))] \, [\phi'_\lambda(\xi)]^{-1} \, \beta(\xi) \, d\xi \\ &= -2\lambda t^{-1} \int_{1/2}^{2} \sin(t\phi_\lambda(\xi)) \, [\phi'_\lambda(\xi)]^{-2} \, [\beta'(\xi)\phi'_\lambda(\xi) - \beta(\xi)\phi''_\lambda(\xi)] \, d\xi, \end{aligned} \tag{4.14}$$

and hence (4.12) and (4.13) allow us to estimate

$$\begin{aligned} |I_{\lambda,t}(x)| &\leq 2\lambda t^{-1} \int_{1/2}^{2} |\phi'_\lambda(\xi)|^{-2} \, [|\beta'(\xi)||\phi'_\lambda(\xi)| + |\beta(\xi)||\phi''_\lambda(\xi)|] \, d\xi \\ &\lesssim \lambda t^{-1} \left[\lambda^{-1/2} + \lambda^{-1} \cdot \lambda^{1/2}\right] \\ &\sim \lambda^{1/2} t^{-1} \\ &\ll \lambda^{3/4} t^{-1/2}, \end{aligned} \tag{4.15}$$

where the last step follows by the assumption (4.9). This concludes the proof of the desired estimate (4.8) with $d = 1$ in the nonstationary case.

4.1.2. Stationary contribution: $x < 0$ and $-x/t \sim \lambda^{-1/2}$. In this case, we see from (4.10) that $\phi'_\lambda(\xi)$ may vanish, but this can happen for at most one point $\xi \in [1/2, 2]$, since $\xi \mapsto \phi'_\lambda(\xi)$ is strictly decreasing for $\xi > 0$. (Indeed, $\phi''_\lambda(\xi)$ is negative, by Lemma 8.) We consider first the case where there exists such a point in $[1/2, 2]$.

So suppose first that $\phi'_\lambda(\xi_0) = 0$ for some $\xi_0 \in [1/2, 2]$. Define

$$\delta = t^{-1/2}\lambda^{-1/4}.$$

Note that $\delta \ll 1$ by (4.9). Assuming for the moment that $1/2 \leq \xi_0 - \delta$ and $\xi_0 + \delta \leq 2$, we decompose the integral as

$$I_{\lambda,t}(x) = 2\lambda \left(\int_{1/2}^{\xi_0-\delta} + \int_{\xi_0-\delta}^{\xi_0+\delta} + \int_{\xi_0+\delta}^{2} \right) \cos(t\phi_\lambda(\xi))\beta(\xi) \, d\xi. \tag{4.16}$$

To estimate the first integral, we use integration by parts to get

$$\begin{aligned} \left| \int_{1/2}^{\xi_0-\delta} \cos(t\phi_\lambda(\xi))\beta(\xi) \, d\xi \right| &\leq t^{-1} \left| \left[\sin(t\phi_\lambda(\xi)) \frac{\beta(\xi)}{\phi'_\lambda(\xi)} \right]_{\xi=1/2}^{\xi=\xi_0-\delta} \right| \\ &\quad + t^{-1} \left| \int_{1/2}^{\xi_0-\delta} \sin(t\phi_\lambda(\xi)) \left(\frac{\beta'(\xi)}{\phi'_\lambda(\xi)} - \frac{\beta(\xi)\phi''_\lambda(\xi)}{[\phi'_\lambda(\xi)]^2} \right) d\xi \right|. \end{aligned}$$

Since ϕ'_λ is positive and decreasing in the interval $[1/2, \xi_0 - \delta]$, and since ϕ''_λ is negative, we can continue the estimate by

$$\begin{aligned}
&\lesssim t^{-1}\left(\frac{1}{\phi_\lambda'(\xi_0-\delta)}+\int_{1/2}^{\xi_0-\delta}\frac{-\phi_\lambda''(\xi)}{[\phi_\lambda'(\xi)]^2}\,d\xi\right)\\
&=t^{-1}\left(\frac{1}{\phi_\lambda'(\xi_0-\delta)}+\int_{1/2}^{\xi_0-\delta}\frac{d}{d\xi}\left(\frac{1}{\phi_\lambda'(\xi)}\right)\,d\xi\right)\\
&\le 2t^{-1}\frac{1}{\phi_\lambda'(\xi_0-\delta)}.
\end{aligned}$$

But by the mean value theorem and (4.12),

$$|\phi_\lambda'(\xi)|=|\phi_\lambda'(\xi)-\phi_\lambda'(\xi_0)|\sim\lambda^{1/2}|\xi-\xi_0|\quad\text{for }\xi\in[1/2,2],$$

so we we conclude that

$$\left|\int_{1/2}^{\xi_0-\delta}\cos(t\phi_\lambda(\xi))\beta(\xi)\,d\xi\right|\lesssim t^{-1}\lambda^{-1/2}\delta^{-1}=t^{-1/2}\lambda^{-1/4},$$

by the definition of δ above. The third integral in (4.16) can be estimated in a similar way and satisfies the same estimate, while the second integral (4.16) is trivially estimated as

$$\int_{\xi_0-\delta}^{\xi_0+\delta}\cos(t\phi_\lambda(\xi))\beta(\xi)\,d\xi\lesssim\delta=t^{-1/2}\lambda^{-1/4}.$$

Summing up the three contributions, we conclude that the desired estimate holds,

$$|I_{\lambda,t}(x)|\lesssim\lambda^{3/4}t^{-1/2},$$

in the stationary case under the assumptions that $\phi_\lambda'(\xi_0)=0$ for some $\xi_0\in[1/2,2]$ and that $1/2\le\xi_0-\delta$ and $\xi_0+\delta\le 2$. If $1/2>\xi_0-\delta$ or $\xi_0+\delta>2$, the above argument is easily modified. For example, if $\xi_0+\delta>2$, we split the integral as $\int_{1/2}^{\xi_0-\delta}+\int_{\xi_0-\delta}^{2}$ instead; the first integral is then treated as above and the second is trivially $O(\delta)$.

It remains to prove the estimate when the function ϕ_λ' has no zero in $[1/2,2]$, so it is either positive or negative everywhere in that interval. Since the arguments for these two cases are similar, we just treat the case where $\phi_\lambda'<0$ in $[1/2,2]$. Then we split the integral as

$$I_{\lambda,t}(x)=2\lambda\left(\int_{1/2}^{1/2+\delta}+\int_{1/2+\delta}^{2-\delta}+\int_{2-\delta}^{2}\right)\cos(t\phi_\lambda(\xi))\beta(\xi)\,d\xi.$$

The first and third integrals are trivially dominated in absolute value by δ, while for the second integral we use integration by parts, estimating

$$\begin{aligned}
\left|\int_{1/2+\delta}^{2-\delta}\cos(t\phi_\lambda(\xi))\beta(\xi)\,d\xi\right|&\lesssim t^{-1}\left(\frac{1}{-\phi_\lambda'(1/2+\delta)}+\int_{1/2+\delta}^{2-\delta}\frac{-\phi_\lambda''(\xi)}{[\phi_\lambda'(\xi)]^2}\,d\xi\right)\\
&=t^{-1}\left(\frac{1}{-\phi_\lambda'(1/2+\delta)}+\int_{1/2+\delta}^{2-\delta}\frac{d}{d\xi}\left(\frac{1}{\phi_\lambda'(\xi)}\right)\,d\xi\right)\\
&\le 2t^{-1}\frac{1}{-\phi_\lambda'(1/2+\delta)}.
\end{aligned}$$

Here we used the fact that ϕ'_λ is negative and decreasing in the interval $[1/2, 2]$ and that ϕ''_λ is negative. Using the mean value theorem and the estimate (4.12) on the second derivative, we find moreover that

$$-\phi'_\lambda(1/2+\delta) \geq \phi'_\lambda(1/2) - \phi'_\lambda(1/2+\delta) \sim \lambda^{1/2}\delta,$$

so we conclude that

$$\left| \int_{1/2+\delta}^{2-\delta} \cos(t\phi_\lambda(\xi))\beta(\xi)\, d\xi \right| \lesssim t^{-1}\lambda^{-1/2}\delta^{-1} = t^{-1/2}\lambda^{-1/4},$$

as desired.

4.2. Proof of (4.8) when $d = 2$. In two dimensions we have

$$I_{\lambda,t}(x) = \lambda^2 \int_{\mathbb{R}^2} e^{i\lambda x\cdot\xi + itm_2(\lambda\xi)}\beta(\xi)\, d\xi,$$

which is the inverse Fourier transform of the radial function $\lambda^2 e^{itm_2(\lambda\xi)}\beta(\xi)$, and hence $I_{\lambda,t}(x)$ is also radial. So we may set $x = (|x|, 0)$. Then in polar coordinates we have

$$I_{\lambda,t}(x) = \lambda^2 \int_0^\infty \int_0^{2\pi} e^{i\lambda r|x|\cos\theta} e^{itm_2(\lambda r)} r\beta(r)\, d\theta dr.$$

We can write

$$\begin{aligned}\int_0^{2\pi} e^{i\lambda r|x|\cos\theta}\, d\theta &= \int_0^{\pi} \left(e^{i\lambda r|x|\cos\theta} + e^{-i\lambda r|x|\cos\theta}\right) d\theta \\ &= 2\int_{-1}^{1} e^{i\lambda r|x|s}\left(1-s^2\right)^{-1/2} ds \\ &= 2\pi J_0(\lambda r|x|),\end{aligned}$$

where $J_k(r)$ is the Bessel function:

$$J_k(r) = \frac{(r/2)^k}{\Gamma(k+1/2)\sqrt{\pi}} \int_{-1}^{1} e^{irs}\left(1-s^2\right)^{k-1/2} ds \quad \text{for } k > -1/2.$$

Thus,

$$I_{\lambda,t}(x) = 2\pi\lambda^2 \int_{1/2}^{2} e^{itm(\lambda r)} J_0(\lambda r|x|)\tilde{\beta}(r)\, dr, \tag{4.17}$$

where $\tilde{\beta}(r) = r\beta(r)$ and $m(r) = m_2(r)$.

We shall use the following properties of $J_k(r)$ for $k > -1/2$ and $r > 0$ (see [13, Appendix B] and [24]):

$$J_k(r) \leq Cr^k, \tag{4.18}$$

$$J_k(r) \leq Cr^{-1/2}, \tag{4.19}$$

$$\partial_r\left[r^{-k}J_k(r)\right] = -r^{-k}J_{k+1}(r). \tag{4.20}$$

Moreover, we can write

$$J_0(s) = e^{is}h(s) + e^{-is}\bar{h}(s) \tag{4.21}$$

for some function h satisfying the estimate

$$|\partial_r^j h(r)| \leq C_j \langle r\rangle^{-1/2-j} \quad \text{for all } j \geq 0. \tag{4.22}$$

We treat the cases $|x| \lesssim \lambda^{-1}$ and $|x| \gg \lambda^{-1}$ separately.

4.2.1. Case 1: $|x| \lesssim \lambda^{-1}$. By (4.18) and (4.20) we have for all $r \in (1/2, 2)$ the estimate

$$\left|\partial_r^j J_0(\lambda r|x|)\right| \lesssim 1 \quad \text{for } j = 0, 1. \tag{4.23}$$

We integrate by parts (4.17) to obtain

$$\begin{aligned} I_{\lambda,t}(x) &= -2\pi i\lambda t^{-1} \int_{1/2}^{2} \frac{d}{dr}\left\{e^{itm(\lambda r)}\right\}[m'(\lambda r)]^{-1} J_0(\lambda r|x|)\tilde{\beta}(r)\,dr \\ &= 2\pi i\lambda t^{-1} \int_{1/2}^{2} e^{itm(\lambda r)}[m'(\lambda r)]^{-1}\partial_r\left[J_0(\lambda r|x|)\tilde{\beta}(r)\right]\,dr \\ &\quad - 2\pi i\lambda t^{-1} \int_{1/2}^{2} e^{itm(\lambda r)}[m'(\lambda r)]^{-2}\lambda m''(\lambda r) J_0(\lambda r|x|)\tilde{\beta}(r)\,dr. \end{aligned}$$

Then applying Lemma 8 and (4.23) we obtain

$$|I_{\lambda,t}(x)| \lesssim \lambda t^{-1}\left(\lambda^{1/2} + \lambda^2 \cdot \lambda^{-3/2}\right) \lesssim \lambda^{3/2} t^{-1}. \tag{4.24}$$

4.2.2. Case 2: $|x| \gg \lambda^{-1}$. Using (4.21) in (4.17) we write

$$I_{\lambda,t}(x) = 2\pi\lambda^2\left\{\int_{1/2}^{2} e^{it\phi_\lambda^+(r)} h(\lambda r|x|)\tilde{\beta}(r)\,dr + \int_{1/2}^{2} e^{-it\phi_\lambda^-(r)}\bar{h}(\lambda r|x|)\tilde{\beta}(r)\,dr\right\},$$

where

$$\phi_\lambda^\pm(r) = \lambda r|x|/t \pm m(\lambda r).$$

Set $F_\lambda(|x|, r) = h(\lambda r|x|)\tilde{\beta}(r)$. In view of (4.22) we have

$$|F_\lambda(|x|, r)| + |\partial_r F_\lambda(|x|, r)| \lesssim (\lambda|x|)^{-1/2} \tag{4.25}$$

for all $r \in (1/2, 2)$, where we also used the fact $\lambda|x| \gg 1$.

Now we write

$$I_{\lambda,t}(x) = I_{\lambda,t}^+(x) + I_{\lambda,t}^-(x),$$

where

$$\begin{aligned} I_{\lambda,t}^+(x) &= 2\pi\lambda^2 \int_{1/2}^{2} e^{it\phi_\lambda^+(r)} F_\lambda(|x|, r)\,dr, \\ I_{\lambda,t}^-(x) &= 2\pi\lambda^2 \int_{1/2}^{2} e^{-it\phi_\lambda^-(r)} \bar{F}_\lambda(|x|, r)\,dr. \end{aligned}$$

Observe that

$$\partial_r\phi_\lambda^\pm(r) = \lambda\left[|x|/t \pm m'(\lambda r)\right], \qquad \partial_r^2\phi_\lambda^\pm(r) = \pm\lambda^2 m''(\lambda r),$$

and hence by Lemma 8,

$$|\partial_r\phi_\lambda^+(r)| \gtrsim \lambda^{1/2}, \qquad |\partial_r^2\phi_\lambda^\pm(r)| \sim \lambda^{1/2} \tag{4.26}$$

for all $r \in (1/2, 2)$, where we also used the fact that m' is positive.

Estimate for $I^+_{\lambda,t}(x)$. It is easy to estimate $I^+_{\lambda,t}(x)$ since $\partial_r\phi^+_\lambda(r)$ is never zero. Indeed, using integration by parts we have

$$\begin{aligned} I^+_{\lambda,t}(x) &= -2\pi i t^{-1}\lambda^2 \int_{1/2}^2 \partial_r\left[e^{it\phi^+_\lambda(r)}\right]\left[\partial_r\phi^+_\lambda(r)\right]^{-1} F_\lambda(|x|,r)\,dr \\ &= 2\pi i t^{-1}\lambda^2 \int_{1/2}^2 e^{it\phi^+_\lambda(r)}\left\{\frac{\partial_r F_\lambda(|x|,r)}{\partial_r\phi^+_\lambda(r)} - \frac{\partial_r^2\phi^+_\lambda(r)F_\lambda(|x|,r)}{\left[\partial_r\phi^+_\lambda(r)\right]^2}\right\}dr. \end{aligned}$$

Then using (4.25) and (4.26) we have

$$|I^+_{\lambda,t}(x)| \lesssim t^{-1}\lambda^2\cdot\lambda^{-1/2}\cdot(\lambda|x|)^{-1/2} \lesssim \lambda^{3/2}t^{-1}. \tag{4.27}$$

Estimate for $I^-_{\lambda,t}(x)$. We treat the nonstationary and stationary cases separately. In the nonstationary case, where $|x|/t \ll \lambda^{-1/2}$ or $|x|/t \gg \lambda^{-1/2}$, we have

$$|\partial_r\phi^-_\lambda(r)| \gtrsim \lambda^{1/2},$$

and hence $I^-_{\lambda,t}(x)$ can be estimated in exactly the same way as $I^+_{\lambda,t}(x)$ above. It satisfies

$$|I^-_{\lambda,t}(x)| \lesssim \lambda^{3/2}t^{-1}. \tag{4.28}$$

It remains to consider the stationary case, where $|x|/t \sim \lambda^{-1/2}$. Note that $\partial_r\phi^-_\lambda(r)$ is strictly increasing for $r>0$, since $\partial_r^2\phi^-_\lambda(r) = -\lambda^2 m''(\lambda r)$ is strictly positive, by Lemma 8. Thus there is at most one point $r_0\in[1/2,2]$ at which $\partial_r\phi^-_\lambda$ vanishes. Setting as before

$$\delta = t^{-1/2}\lambda^{-1/4},$$

we limit our attention to the case where there is such a point r_0 in $[1/2+\delta, 2-\delta]$; the remaining cases are treated by straightforward modifications of the following argument, much as in the one dimensional case in subsection 4.1.2.

We decompose

$$I^-_{\lambda,t}(x) = 2\pi\lambda^2\left(\int_{1/2}^{r_0-\delta} + \int_{r_0-\delta}^{r_0+\delta} + \int_{r_0+\delta}^2\right) e^{-it\phi^-_\lambda(r)}\bar{F}_\lambda(|x|,r)\,dr. \tag{4.29}$$

Integrating by parts we write the first integral as

$$\begin{aligned} &\int_{1/2}^{r_0-\delta} e^{-it\phi^-_\lambda(r)}\bar{F}_\lambda(|x|,r)\,dr \\ &\quad= \underbrace{it^{-1}\left[e^{-it\phi^-_\lambda(r)}\frac{\bar{F}_\lambda(|x|,r)}{\partial_r\phi^-_\lambda(r)}\right]_{r=1/2}^{r_0-\delta}}_{=:A} \underbrace{-\,it^{-1}\int_{1/2}^{r_0-\delta} e^{-it\phi^-_\lambda(r)}\partial_r\left(\frac{\bar{F}_\lambda(|x|,r)}{\partial_r\phi^-_\lambda(r)}\right)dr}_{=:B}. \end{aligned}$$

Using (4.25) and noting that for $r\in[1/2, r_0-\delta]$, $\partial_r\phi^-_\lambda(r)$ is negative and increasing, while $\partial_r^2\phi^-_\lambda(r)$ is positive, we find

$$|A| \lesssim t^{-1}(\lambda|x|)^{-1/2}\frac{1}{|\partial_r\phi^-_\lambda(r_0-\delta)|}$$

and

$$\begin{aligned}|B| &\lesssim t^{-1}(\lambda|x|)^{-1/2}\left[\int_{1/2}^{r_0-\delta}\frac{1}{|\partial_r\phi_\lambda^-(r)|}\,dr+\int_{1/2}^{r_0-\delta}\frac{\partial_r^2\phi_\lambda^-(r)}{[\partial_r\phi_\lambda^-(r)]^2}\,dr\right]\\ &= t^{-1}(\lambda|x|)^{-1/2}\left[\int_{1/2}^{r_0-\delta}\frac{1}{|\partial_r\phi_\lambda^-(r)|}\,dr+\int_{1/2}^{r_0-\delta}\partial_r\left(\frac{1}{-\partial_r\phi_\lambda^-(r)}\right)\,dr\right]\\ &\lesssim t^{-1}(\lambda|x|)^{-1/2}\frac{1}{|\partial_r\phi_\lambda^-(r_0-\delta)|}.\end{aligned}$$

But using (4.26) and the mean value theorem, we see that

$$|\partial_r\phi_\lambda^-(r_0-\delta)|=|\partial_r\phi_\lambda^-(r_0-\delta)-\partial_r\phi_\lambda^-(r_0)|\sim\lambda^{1/2}\delta=\lambda^{1/4}t^{-1/2}.$$

Using also the assumption $|x|/t\sim\lambda^{-1/2}$, we conclude that

$$\begin{aligned}\left|\int_{1/2}^{r_0-\delta}e^{it\phi_\lambda^-(r)}\bar F_\lambda(|x|,r)\,dr\right| &\le |A|+|B|\\ &\lesssim t^{-1}(\lambda|x|)^{-1/2}\lambda^{-1/4}t^{1/2}\\ &\sim t^{-1}(\lambda^{1/2}t)^{-1/2}\lambda^{-1/4}t^{1/2}\\ &= t^{-1}\lambda^{-1/2}.\end{aligned}$$

The third integral in (4.29) can also be estimated in a similar way and satisfies the same estimate, while the second integral can be simply estimated as

$$\left|\int_{r_0-\delta}^{r_0+\delta}e^{it\phi_\lambda^-(r)}\bar F_\lambda(|x|,r)\,dr\right|\lesssim(\lambda|x|)^{-1/2}\delta\lesssim t^{-1}\lambda^{-1/2}.$$

Therefore, combining the above computations with (4.29) we have

$$\left|I_{\lambda,t}^-(x)\right|\lesssim\lambda^{3/2}t^{-1}, \tag{4.30}$$

concluding the stationary case.

In summary, from (4.27), (4.28), and (4.30) we obtain

$$|I_{\lambda,t}(x)|\le\sum_\pm|I_{\lambda,t}^\pm(x)|\lesssim\lambda^{3/2}t^{-1},$$

which is the desired estimate (4.8) with $d=2$.

5. Function spaces, linear and bilinear estimates.

5.1. Function spaces. The mixed space-time Lebesgue space $L_t^qL_x^r$ on $\mathbb{R}^{d+1}$ is defined with the norm

$$\|u\|_{L_t^qL_x^r}=\|\|u(t,\cdot)\|_{L_x^r}\|_{L_t^q}=\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^d}|u(t,x)|^r\,dx\right)^{\frac{q}{r}}dt\right)^{\frac1q}$$

for $1\le q,r<\infty$ with an obvious modification when $q=\infty$ or $r=\infty$. We write $L_T^qL_x^r$ when the time variable is restricted to the interval $[0,T]$.

Define the Bourgain space $X^{s,b}_{\pm}$ on $\mathbb{R}^{d+1}$ by the norm

$$\|u\|_{X^{s,b}_{\pm}} = \left\| \langle\xi\rangle^s \langle\tau \pm m_d(\xi)\rangle^b \widetilde{u}(\xi,\tau)\right\|_{L^2_{\tau,\xi}},$$

where $\widetilde{u}$ denotes the space-time Fourier transform given by

$$\widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{d+1}} e^{-i(t\tau + x\cdot\xi)} u(t,x)\, dtdx.$$

The restriction to the time slab $(0,T)\times\mathbb{R}^d$ of the Bourgain space, denoted by $X^{s,b}_{\pm}(T)$, is a Banach space when equipped with the norm

$$\|u\|_{X^{s,b}_{\pm}(T)} = \inf\left\{ \|v\|_{X^{s,b}_{\pm}} :\ v = u \text{ on } (0,T)\times\mathbb{R}^d \right\}.$$

5.2. Linear estimates. Let us recall some of the properties of these spaces. We have

$$\sup_{0\le t\le T} \|u(t)\|_{H^s} \le C\, \|u\|_{X^{s,b}_{\pm}(T)} \quad \text{for } b > 1/2. \tag{5.1}$$

For $-1/2 < b' \le b < 1/2$ and $0 < T < 1$ we have

$$\|u\|_{X^{s,b'}_{\pm}(T)} \le CT^{b-b'}\, \|u\|_{X^{s,b}_{\pm}(T)}, \tag{5.2}$$

where C is independent on T. The proof for (5.1) and (5.2) can, for instance, be found in [25]. We recall also that for $2 \le q \le \infty$ and $b > 1/2$,

$$\|u\|_{L^q_t L^2_x} \le C\, \|u\|_{X^{0,b}_{\pm}}, \tag{5.3}$$

as can be seen by writing the LHS as $\|e^{\pm itm_d(D)}u\|_{L^q_T L^2_x}$, applying Plancherel in x, then using Minkowski's integral inequality to switch the order of the norms to $L^2_\xi L^q_t$, and finally applying Sobolev embedding in t.

It is well known (see, e.g., [8]) that for any $s \in \mathbb{R}$ and $b > 1/2$ one has

$$\|S_{m_d}(\pm t)f\|_{X^{s,b}_{\pm}(T)} \le C\|f\|_{H^s}, \tag{5.4}$$

$$\left\| \int_0^t S_{m_d}(\pm(t-t'))F(t')\, dt' \right\|_{X^{s,b}_{\pm}(T)} \le C\|F\|_{X^{s,b-1}_{\pm}(T)}, \tag{5.5}$$

where the constant $C > 0$ depends only on b.

We need the following Bernstein inequality, which is valid for $1 \le p \le r \le \infty$ (see, for instance, [25, Appendix A]):

$$\|P_\lambda f\|_{L^r(\mathbb{R}^d)} \le C\lambda^{\frac{d}{p}-\frac{d}{r}}\, \|P_\lambda f\|_{L^p(\mathbb{R}^d)}. \tag{5.6}$$

Another useful tool is the Hardy–Littlewood–Sobolev inequality (see [25, Appendix A]), which asserts that

$$\left\| |\cdot|^{-\alpha} * f \right\|_{L^a(\mathbb{R})} \le C\, \|f\|_{L^b(\mathbb{R})} \tag{5.7}$$

whenever $1 < b < a < \infty$ and $0 < \alpha < 1$ obey the scaling condition

$$\frac{1}{b} = \frac{1}{a} + 1 - \alpha.$$

Lemma 10 (localized Strichartz estimates). *Let $\lambda > 1$ and $d \in \{1, 2\}$. Assume that $2 < q < \infty$ and $2 \le r \le \infty$ satisfy*

$$\frac{2}{q} = \frac{d}{2}\left(1 - \frac{2}{r}\right).$$

Then we have the estimate

$$\|S_{m_d}(\pm t) f_\lambda\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \lambda^{(3d/8)(1-2/r)} \|f_\lambda\|_{L_x^2(\mathbb{R}^d)}. \tag{5.8}$$

Moreover, if $b > 1/2$, we have

$$\|u_\lambda\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \lambda^{(3d/8)(1-2/r)} \|u_\lambda\|_{X_\pm^{0,b}}. \tag{5.9}$$

Proof. By the standard TT^*-argument, (5.8) is equivalent to the estimate

$$\|K_\lambda \star F\|_{L_t^q L_x^r(\mathbb{R}^{d+1})} \lesssim \lambda^{(3d/4)(1-2/r)} \|F\|_{L_t^{q'} L_x^{r'}(\mathbb{R}^{d+1})}, \tag{5.10}$$

where $1/q + 1/q' = 1$ and $1/r + 1/r' = 1$, and where

$$K_\lambda(x, t) = \int_{\mathbb{R}^d} e^{ix\cdot\xi \pm itm_d(\xi)} \tilde{\beta}_\lambda(\xi)\, d\xi \tag{5.11}$$

with $\tilde{\beta}_\lambda = \beta_\lambda^2$. Here $\star$ denotes the space-time convolution. Then by Corollary 1 (with β replaced by β^2, which does not affect the validity of the corollary) we have the estimate

$$\|K_\lambda(\cdot, t) * f\|_{L_x^r(\mathbb{R}^d)} \lesssim \left(\lambda^{3d/4}|t|^{-d/2}\right)^{1-2/r} \|f\|_{L_x^{r'}(\mathbb{R}^d)}.$$

Combining this with the Hardy–Littlewood–Sobolev inequality in the t variable, with $(a, b) = (q, q')$ and $\alpha = (d/2)(1 - 2/r)$, we estimate

$$\begin{aligned}
\|K_\lambda \star F\|_{L_t^q L_x^r} &= \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^d} K_\lambda(x - y, t - s) F(y, s)\, dy\, ds \right\|_{L_t^q L_x^r} \\
&\le \left\| \int_{\mathbb{R}} \left\| \int_{\mathbb{R}^d} K_\lambda(x - y, t - s) F(y, s)\, dy \right\|_{L_x^r} ds \right\|_{L_t^q} \\
&\lesssim \left\| \int_{\mathbb{R}} \left(\lambda^{3d/4}|t - s|^{-d/2}\right)^{1-2/r} \|F(\cdot, s)\|_{L_x^{r'}}\, ds \right\|_{L_t^q} \\
&\lesssim \lambda^{(3d/4)(1-2/r)} \left\| \|F\|_{L_x^{r'}} \right\|_{L_t^{q'}},
\end{aligned}$$

proving (5.10) and hence (5.8). By a standard argument, the latter implies (5.9) (see, for example, the proof of [25, Lemma 2.9]). ∎

5.3. Bilinear estimates. Set $K = K(D) = K_d(D)$ for $d = 1, 2$ and note that the Fourier symbol equals in both dimensions

$$K(\xi) = \sqrt{\frac{\tanh|\xi|}{|\xi|}} \sim \langle\xi\rangle^{-1/2},$$

and hence

$$\|K f_\lambda\|_{L_x^2} \lesssim \lambda^{-\frac{1}{2}} \|f_\lambda\|_{L_x^2}, \quad \||D|K^2 f_\lambda\|_{L_x^2} \lesssim \|f_\lambda\|_{L_x^2}, \quad \||D|K f_\lambda\|_{L_x^2} \lesssim \lambda^{\frac{1}{2}} \|f_\lambda\|_{L_x^2}. \tag{5.12}$$

We first note the following consequence of the localized Strichartz estimates in Lemma 10.

LEMMA 11. *Let $b > 1/2$ and dyadic $\lambda_1, \lambda_2 \geq 1$. For $d = 1, 2$ and $1 < p \leqslant 2 < r \leqslant \infty$ satisfying*

$$\frac{1}{2} + \frac{1}{r} = \frac{1}{p}$$

we have the estimate

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_t^2 L_x^p} \lesssim \min(\lambda_1, \lambda_2)^{(3d/8)(1-2/r)} \|u_{\lambda_1}\|_{X_\pm^{0,b}} \|v_{\lambda_2}\|_{X_\pm^{0,b}}$$

provided $p < 2$ in the case $d = 2$.

For $d = 2$ and $2 < r < \infty$, we have for all $T > 0$,

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_T^2 L_x^2} \lesssim T^{1/r} \min(\lambda_1, \lambda_2)^{3/4+1/(2r)} \|u_{\lambda_1}\|_{X_\pm^{0,b}} \|v_{\lambda_2}\|_{X_\pm^{0,b}}.$$

In both estimates, the signs in the $X_\pm$ norms can be chosen independently of each other.

Proof. By symmetry we may assume $1 \leq \lambda_1 \leq \lambda_2$. Consider first the case $d = 1$. By Hölder's inequality and (5.3),

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_t^2 L_x^p} \leq \|u_{\lambda_1}\|_{L_t^q L_x^r} \|v_{\lambda_2}\|_{L_t^{q_1} L_x^2} \lesssim \|u_{\lambda_1}\|_{L_t^q L_x^r} \|v_{\lambda_2}\|_{X_\pm^{0,b}},$$

where q is taken as in Lemma 10 and $1/q + 1/q_1 = 1/2$. So it only remains to check that

$$\|u_{\lambda_1}\|_{L_t^q L_x^r} \lesssim \lambda_1^{3/8(1-2/r)} \|u_{\lambda_1}\|_{X_\pm^{0,b}},$$

but this holds by Lemma 10 if $\lambda_1 > 1$, while if $\lambda_1 = 1$ we can use the Bernstein inequality (5.6) followed by (5.3) to obtain

$$\|u_{\lambda_1}\|_{L_t^q L_x^r} \lesssim \|u_{\lambda_1}\|_{L_t^q L_x^2} \lesssim \|u_{\lambda_1}\|_{X_\pm^{0,b}}.$$

Similarly, one obtains the first estimate for $p < d = 2$.

Now consider the case $p = d = 2$. We apply Hölder's inequality and (5.3) to write

$$\|u_{\lambda_1} v_{\lambda_2}\|_{L_T^2 L_x^2} \leq \|u_{\lambda_1}\|_{L_T^2 L_x^\infty} \|v_{\lambda_2}\|_{L_T^\infty L_x^2} \lesssim \|u_{\lambda_1}\|_{L_T^2 L_x^\infty} \|v_{\lambda_2}\|_{X_\pm^{0,b}}.$$

To estimate $\|u_{\lambda_1}\|_{L_T^2 L_x^\infty}$ we want to use Lemma 10, so we let $2 < r < \infty$ and define q by $2/q = 1 - 2/r$. Thus $1/2 = 1/q + 1/r$, so applying Hölder in t, the Bernstein inequality in x, and finally Lemma 10, we get

$$\|u_{\lambda_1}\|_{L_T^2 L_x^\infty} \leq T^{1/r} \lambda_1^{2/r} \|u_{\lambda_1}\|_{L_T^q L_x^r} \lesssim T^{1/r} \lambda_1^{2/r} \lambda_1^{(3/4)(1-2/r)} \|u_{\lambda_1}\|_{X_\pm^{0,b}},$$

proving the claimed estimate in the case $\lambda_1 > 1$. If $\lambda_1 = 1$, we can apply the Bernstein inequality and (5.3), instead of Lemma 10, and again we get the desired estimate. ☐

We now present the key bilinear space-time estimates needed for the proof of local well-posedness.

LEMMA 12. *Let $1/2 < b < 1$ and $0 < T < 1$. Assume that $s_d > -1/10$ if $d = 1$ and $s_d > 1/4$ if $d = 2$. Then we have the estimates*

$$\left\| |D| K^2 (u \cdot Kv) \right\|_{X_\pm^{s_d, b-1}(T)} \lesssim T^{1-b} \|u\|_{X_\pm^{s_d,b}} \|v\|_{X_\pm^{s_d,b}}, \tag{5.13}$$

$$\left\| |D| K (Ku \cdot Kv) \right\|_{X_\pm^{s_d, b-1}(T)} \lesssim T^{1-b} \|u\|_{X_\pm^{s_d,b}} \|v\|_{X_\pm^{s_d,b}}, \tag{5.14}$$

where the signs in all the $X_\pm$ norms can be chosen independently on each other.

Proof of (5.13). In view of (5.2) the estimate (5.13) reduces to proving

$$\left\| |D| K^2 \left(u \cdot K v\right)\right\|_{L_T^2 H_x^{s_d}} \lesssim \|u\|_{X_\pm^{s_d,b}} \|v\|_{X_\pm^{s_d,b}},$$

which by duality can be reduced to

(5.15)
$$\left| \int_0^T \int_{\mathbb{R}^d} |D| K^2 \langle D\rangle^{s_d} \left(\langle D\rangle^{-s_d} u \cdot \langle D\rangle^{-s_d} K v\right) w \; dxdt \right| \lesssim \|u\|_{X_\pm^{0,b}} \|v\|_{X_\pm^{0,b}} \|w\|_{L^2_{t,x}}.$$

Decomposing $u = \sum_{\lambda_1 \geq 1} u_{\lambda_1}$ and $v = \sum_{\lambda_2 \geq 1} v_{\lambda_2}$ we have

(5.16)
$$\text{LHS (5.15)} \lesssim \sum_{\lambda,\lambda_1,\lambda_2 \geq 1} \left| \int_0^T \int_{\mathbb{R}^d} |D| K^2 \langle D\rangle^{s_d} P_\lambda \left(\langle D\rangle^{-s_d} u_{\lambda_1} \cdot \langle D\rangle^{-s_d} K v_{\lambda_2}\right) w_\lambda \; dxdt \right|.$$

Setting

$$a_{\lambda_1} := \|u_{\lambda_1}\|_{X_\pm^{0,b}}, \quad b_{\lambda_2} := \|v_{\lambda_2}\|_{X_\pm^{0,b}}, \quad c_\lambda := \|w_\lambda\|_{L^2_{t,x}}$$

we have

$$\|u\|_{X_\pm^{0,b}} \sim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}}, \quad \|v\|_{X_\pm^{0,b}} \sim \|(b_{\lambda_2})\|_{l^2_{\lambda_2}}, \quad \|w\|_{L^2_{t,x}} \sim \|(c_\lambda)\|_{l^2_\lambda},$$

and hence the estimate (5.15) reduces to proving

$$\text{RHS (5.16)} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}. \tag{5.17}$$

To this end, we note that by Lemma 11 we have, for $\varepsilon > 0$ arbitrarily small and $1 < p \leqslant 2$ with $1/r = 1/p - 1/2$,

$$\|P_\lambda \left(u_{\lambda_1} v_{\lambda_2}\right)\|_{L_T^2 L_x^p} \lesssim \min(\lambda_1, \lambda_2)^{3d/8(1-2/r)+\varepsilon} \|u_{\lambda_1}\|_{X_\pm^{0,b}} \|v_{\lambda_2}\|_{X_\pm^{0,b}}. \tag{5.18}$$

We remark that in dimension $d = 1$, the lemma would actually allow us to take $\varepsilon = 0$, but the proof below works for sufficiently small, positive $\varepsilon > 0$ in both dimensions. Note apart from estimate (5.18) that all the nontrivial terms of RHS (5.16) can be separated into the three groups I_1, I_2, I_3 with $\lambda \lesssim \lambda_1 \sim \lambda_2, \lambda_1 \ll \lambda_2 \sim \lambda$ and $\lambda_2 \ll \lambda_1 \sim \lambda$, respectively. It is a straightforward consequence of taking λ-projection of the product of λ_1, λ_2-projections. For each of these groups we can make a particular choice of p applying estimate (5.18).

Thus using Cauchy–Schwarz, the Bernstein inequality (5.6), (5.12), and (5.18) we obtain

(5.19)
$$\begin{aligned} \text{RHS (5.16)} &\lesssim \sum_{\lambda,\lambda_1,\lambda_2 \geq 1} \left\| |D| K^2 \langle D\rangle^{s_d} P_\lambda \left(\langle D\rangle^{-s_d} u_{\lambda_1} \cdot \langle D\rangle^{-s_d} K v_{\lambda_2}\right)\right\|_{L^2_{T,x}} \|w_\lambda\|_{L^2_{T,x}} \\ &\lesssim \sum_{\substack{\lambda,\lambda_1,\lambda_2 \geq 1 \\ P_\lambda(\dots)\neq 0}} \lambda^{s_d + d/p - d/2} \min(\lambda_1, \lambda_2)^{3d/8(1-2/r)+\varepsilon} \lambda_1^{-s_d} \lambda_2^{-1/2 - s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda \\ &\lesssim I_1(d) + I_2(d) + I_3(d), \end{aligned}$$

where

$$I_1(d) = \sum_{\substack{\lambda,\lambda_1,\lambda_2\geq 1\\ \lambda\lesssim\lambda_1\sim\lambda_2}} \lambda^{s_d+d/p-d/2}\lambda_2^{3d/8(1-2/r)+\varepsilon-1/2-2s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda,$$

$$I_2(d) = \sum_{\substack{\lambda,\lambda,\lambda_2\geq 1\\ \lambda_1\ll\lambda_2\sim\lambda}} \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \lambda_1^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda,$$

$$I_3(d) = \sum_{\substack{\lambda,\lambda_1,\lambda\geq 1\\ \lambda_2\ll\lambda_1\sim\lambda}} \lambda_2^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda.$$

We first estimate $I_1(1)$. Notice

$$\lambda^{s_1+1/p-1/2}\lambda_2^{3/8(1-2/r)+\varepsilon-1/2-2s_1} \lesssim \lambda^{1/p-1+3/8(1-2/r)+\varepsilon-s_1}$$

provided $3/8(1-2/r)+\varepsilon-1/2-2s_1 < 0$ or equivalently if $s_1 > -(1+6/r)/16+\varepsilon/2$. Consequently, we can apply the Cauchy–Schwarz inequality first in $\lambda_1 \sim \lambda_2$ and then in λ to estimate $I_1(1)$ as

$$I_1(1) \lesssim \left(\sum_{\lambda\geq 1} \lambda^{(2/r-1)/8+\varepsilon-s_1} c_\lambda\right) \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}$$

provided $s_1 > (2/r-1)/8+\varepsilon$ and $s_1 > -(1+6/r)/16+\varepsilon/2$. The lowest possible bound $s_1 > -1/10$ is obtained by taking $r = 10$, since $\varepsilon > 0$ is arbitrary small (and can be actually set to zero according to Lemma 11). This corresponds to $p = 5/3$. To estimate the other sums it is enough to stick to $p = 2$ everywhere below.

Next we estimate $I_1(2)$ with $p = 2$. Since $s_2 > 1/4$ by assumption, we have $\lambda^{s_2}\lambda_2^{\varepsilon+1/4-2s_2} \lesssim (\lambda/\lambda_2)^{s_2}$. Then we apply the Cauchy–Schwarz inequality first in λ and then in $\lambda_1 \sim \lambda_2$ to obtain the desired estimate:

$$I_1(2) \lesssim \sum_{\lambda_1\sim\lambda_2} \left(\sum_{\lambda\lesssim\lambda_2} (\lambda/\lambda_2)^{s_2} c_\lambda\right) a_{\lambda_1} b_{\lambda_2} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}.$$

Next we estimate $I_3(d)$. Since $s_d > 3d/8-1/2$, we have $\varepsilon+3d/8-1/2-s_d < 0$ for $\varepsilon > 0$ small enough. Applying the Cauchy–Schwarz inequality first in $\lambda_1 \sim \lambda$ and then in λ_2, we get

$$I_3(d) \lesssim \left(\sum_{\lambda_2\geq 1} \lambda_2^{3d/8+\varepsilon-1/2-s_d} b_{\lambda_2}\right) \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(c_\lambda)\|_{l^2_\lambda} \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda}.$$

Finally, we note that in $I_2(d)$, we can discard the small factor $(\lambda_1/\lambda_2)^{1/2}$ and reduce to the same estimate as for $I_3(d)$. This completes the proof of (5.13). ∎

Proof of (5.14). We follow the same argument as in the proof of (5.13). By duality and dyadic decomposition, (5.14) reduces to proving

$$S \lesssim \|(a_{\lambda_1})\|_{l^2_{\lambda_1}} \|(b_{\lambda_2})\|_{l^2_{\lambda_2}} \|(c_\lambda)\|_{l^2_\lambda},$$

where

$$S = \sum_{\lambda,\lambda_1,\lambda_2 \ge 1} \left| \int_0^T \int_{\mathbb{R}^d} |D| K \langle D \rangle^{s_d} P_\lambda \left(\langle D \rangle^{-s_d} K u_{\lambda_1} \cdot \langle D \rangle^{-s_d} K v_{\lambda_2} \right) w_\lambda \; dx dt \right|.$$

By Cauchy–Schwarz, (5.12), and (5.18) we obtain

$$\begin{aligned} S &\lesssim \sum_{\substack{\lambda,\lambda_1,\lambda_2 \ge 1 \\ P_\lambda(\dots) \neq 0}} \lambda^{\frac{1}{2}+s_d} \min(\lambda_1,\lambda_2)^{3d/8+\varepsilon} (\lambda_1\lambda_2)^{-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda \\ &= \sum_{\substack{\lambda,\lambda_1,\lambda_2 \ge 1 \\ P_\lambda(\dots) \neq 0}} \left(\frac{\lambda}{\lambda_1} \right)^{1/2} \lambda^{s_d} \min(\lambda_1,\lambda_2)^{3d/8+\varepsilon} \lambda_1^{-s_d} \lambda_2^{-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda, \end{aligned}$$

and comparing with the corresponding sum (5.19) from the proof of (5.13), we see that the only difference is that we now have an extra factor $(\lambda/\lambda_1)^{1/2}$. This factor is bounded except for the case $\lambda_1 \ll \lambda_2 \sim \lambda$, so it is enough to consider $I_2(d)$ with this factor inserted:

$$\begin{aligned} I_2'(d) &= \sum_{\substack{\lambda,\lambda,\lambda_2 \ge 1 \\ \lambda_1 \ll \lambda_2 \sim \lambda}} \left(\frac{\lambda}{\lambda_1} \right)^{1/2} \left(\frac{\lambda_1}{\lambda_2} \right)^{1/2} \lambda_1^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda \\ &\lesssim \sum_{\substack{\lambda,\lambda,\lambda_2 \ge 1 \\ \lambda_1 \ll \lambda_2 \sim \lambda}} \lambda_1^{3d/8+\varepsilon-1/2-s_d} a_{\lambda_1} b_{\lambda_2} c_\lambda. \end{aligned}$$

But the RHS was already estimated in the proof of (5.13) (the estimate for $I_3(d)$). This completes the proof of (5.14). ∎

6. Proof of Theorems 4, 5, and 3.

6.1. Proof of Theorems 4 and 5. We solve the integral equations (2.12) and (2.13) by contraction mapping techniques as follows. Define the mapping

$$(u_d^+, u_d^-) \mapsto (\Phi_+(u_d^+, u_d^-), \Phi_-(u_d^+, u_d^-))$$

by

$$\Phi_\pm(u_d^+, u_d^-)(t) := S_{m_d}(\pm t) f_d^\pm - i \int_0^t S_{m_d}(\pm(t-s)) B_d^\pm(u_d^+, u_d^-)(s)\, ds.$$

Let

$$R_d = \|f_d^+\|_{H^{s_d}} + \|f_d^-\|_{H^{s_d}}.$$

We look for a solution in the set

$$\mathcal{D}(R_d) = \left\{ (u_d^+, u_d^-) \in X_+^{s_d,b}(T) \times X_-^{s_d,b}(T) \colon \|u_d^+\|_{X_+^{s_d,b}(T)} + \|u_d^-\|_{X_-^{s_d,b}(T)} \le 4CR_d \right\},$$

where $b \in (1/2, 1)$ and C is as in (5.4), (5.5). Now for $(u_d^+, u_d^-) \in \mathcal{D}(R_d)$ we have by

(5.4), (5.5), and Lemma 12,

$$\|\Phi_+(u_d^+,u_d^-)\|_{X_+^{s_d,b}(T)} + \|\Phi_-(u_d^+,u_d^-)\|_{X_-^{s_d,b}(T)} \leq 2CR_d + C'T^{1-b}R_d^2 \leq 4CR_d,$$

where the last inequality certainly holds provided that

$$T = \left(\frac{1}{16CC'(1+R_d)}\right)^{\frac{1}{1-b}}.$$

Moreover, for (u_d^+,u_d^-) and (v_d^+,v_d^-) in $\mathcal{D}(R_d)$ with the same data, one can show similarly the difference estimate

$$\begin{aligned}\sum_{\pm} &\|\Phi_\pm(u_d^+,u_d^-) - \Phi_\pm(v_d^+,v_d^-)\|_{X_\pm^{s_d,b}(T)} \\ &\leq C'T^{1-b}\left(\sum_{\pm}\|u_d^\pm - v_d^\pm\|_{X_\pm^{s_d,b}(T)}\right)\left(\sum_{\pm}\left(\|u_d^\pm\|_{X_\pm^{s_d,b}(T)} + \|v_d^\pm\|_{X_\pm^{s_d,b}(T)}\right)\right) \\ &\leq 8CC'R_dT^{1-b}\left(\sum_{\pm}\|u_d^\pm - v_d^\pm\|_{X_\pm^{s_d,b}(T)}\right).\end{aligned}$$

With T chosen as above, the constant $8CC'R_dT^{1-b}$ is strictly less than one, and hence (Φ_+,Φ_-) is a contraction on $\mathcal{D}(R_d)$ and therefore it has a unique fixed point $(u_d^+,u_d^-) \in \mathcal{D}(R_d)$ solving the integral equation on $\mathbb{R}^d \times [0,T]$. Uniqueness in the whole space $X_+^{s_d,b}(T) \times X_-^{s_d,b}(T)$ and continuous dependence on the initial data can be shown in a similar way, by the difference estimates. This concludes the proof of Theorems 4 and 5.

Then we use the transformation (2.1) to obtain the solution

$$(\eta,v) \in C\left([0,T]; H^{s_1}(\mathbb{R}) \times H^{s_1+1/2}(\mathbb{R})\right)$$

of the original system (1.1)–(1.3). Similarly, we use the transformation (2.7) to obtain the solution

$$(\eta,\mathbf{v}) \in C\left([0,T]; H^{s_2}\left(\mathbb{R}^2\right) \times \left(H^{s_2+1/2}\left(\mathbb{R}^2\right)\right)^2\right)$$

of the original system (1.5)–(1.6). Thus we obtain also Theorems 1 and 2.

6.2. Proof of Theorem 3. Here we assume $d = 1$. For $s = 0$ one can easily extend the local result globally making use of Lemma 4. With the global bound of the lemma we can reapply the local result, Theorem 1, as many times as we want, thus proving Theorem 3 with $\delta = \epsilon_0/2$ for $s = 0$. The proof for positive s is done iteratively. In other words, assuming the result for some $s' \geq 0$ we prove for $s \in (s', s'+1/4]$. The argument is essentially the persistence of regularity based on the a priori estimate lemma, 7, where we use the notation $\|(\eta,v)\|_{X^s}$ defined by (3.1). Indeed, the first estimate in Lemma 7 allows us to reapply the local result and extend the solution to any time interval if $0 < s < 1/2$. In the case $s \geqslant 1/2$ extension is carried out iteratively making use of the second inequality in Lemma 7.

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