Topological Quantum and Skein-Theoretic Aspects of Braided Fusion Categories

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Thesis for the degree of Philosophiae Doctor (PhD)
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Dedicated to my late grandmother, Jasumati Bhatt.

Abstract

The first part of this thesis is dedicated to the study of anyons and exchange symmetry. We discuss the theory of identical particles and recap the standard algebraic framework for describing the exchange statistics of anyons. The novel component consists of a derivation of the fusion structure of anyons from exchange symmetry. In order to achieve this, we construct a precise notion of exchange symmetry that is compatible with the spatially localised nature of anyons. In particular, given a system of n quasiparticles, we show that the action of a specific n-braid uniquely specifies its superselection sectors. This n-braid satisfies several internal symmetries corresponding to the decompositions of the n-quasiparticle Hilbert space, and its spectrum is related to the topological spins of the quasiparticles.

The second part of this thesis is primarily concerned with skein-theoretic aspects of unitary (braided) fusion categories. Specifically, we consider a fusion rule of the form $q \otimes q \cong \mathbf{1} \oplus \bigoplus_{i=1}^{k} x_i$ in a unitary fusion category $\mathcal{C}$, and extract information using the graphical calculus. For instance, we classify all associated skein relations when $k = 1, 2$ and $\mathcal{C}$ is ribbon. In particular, we also consider the instances where q is antisymmetrically self-dual. Our main results follow from considering the action of a rotation operator on a "canonical basis". Assuming self-duality of the summands x_i, some general observations are made e.g. the real-symmetricity of the F-matrix F_q^{qqq}. We then find explicit formulae for F_q^{qqq} when $k = 2$ and $\mathcal{C}$ is ribbon, and see that the spectrum of the rotation operator distinguishes between the (framed) Kauffman and Dubrovnik link polynomials.

Acknowledgements

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Thank you to my parents, to whom I owe everything. And a special mention goes to my sisters, and Delilah.

Finally, to Laura G. Balcombe – my talisman at sea. Thank you so much for your unwavering support and belief in me. The way would surely have eluded me without you.

It has been a privilege to carry out this work in a serene corner of the world, *millom bakkar og berg ut med havet.*

[1] Warning: do not translate this into Norwegian.

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Notation and Conventions

- In Part I. of this thesis, we follow the physics convention of denoting the 'dual' of an object q by $\bar{q}$.
- In Part II. of this thesis, we follow the mathematics convention of denoting the 'dual' of an object q by q^*.
- For a complex number $z \in \mathbb{C}$, its complex conjugate is written z^*.
- Our diagrams follow the pessimistic (i.e. top-to-bottom) convention.

Introduction

This thesis is divided into two parts:

I. *Exchange Symmetry and Topological Quantum Systems*
II. *Ribbon Categories and Quantum Topology*

Do the adjectives 'quantum' and 'topological' commute? *'Topological quantum'* describes topics lying at the intersection between quantum physics and topology e.g. topological phases of matter[2] (TPMs), topological quantum field theories (TQFTs) and topological quantum computation (TQC). On the other hand, *quantum topology* (and *quantum algebra*)[3] are typically regarded as disciplines within pure mathematics. However, this is not to say that the two parts of this thesis are disconnected.

The origins of quantum algebra and quantum topology are deeply rooted in quantum physics. Both fields have grown to be vast, and their interconnectedness is both overwhelming and captivating. The study of mathematical and physical structures pertaining to topological quantum phenomena has fostered a close and productive relationship between communities of mathematicians and physicists. There is an active exchange of ideas between these fields, each propelling the advance of the other. For instance, the classification programme for TPMs has profited considerably from techniques in category theory, and the development of quantum algebraic structures has largely been inspired by exotic symmetries arising from condensed matter systems. Another example is Atiyah's axiomatisation of TQFTs [Atiyah88]. The list goes on.

The idea of a connection between topology and physics predates the contemporary

[2] The 2016 Nobel Prize in physics was awarded to Thouless, Haldane, and Kosterlitz for their work on TPMs.

[3] The term *quantum mathematics* has also risen to popularity (as a blanket term for areas lying at the interface between mathematics and quantum theory).

notion: Lord Kelvin suggested that different types of knots[4] constitute the building blocks of Nature. While this idea turned out to be incorrect, knots and physics were reunited in a striking manner near the end of the 20^{th} century, when Witten established a connection between the Jones polynomial and Chern-Simons $(2+1)$-TQFTs [Witten89:I]. The eponymous Jones polynomial had only been discovered shortly before this [Jones85, Jones87]. The discoveries of Jones and Witten[5] triggered an avalanche of work on quantum invariants and TQFTs.

Anyons

Anyons are the primary objects of physical interest in this thesis. They are quasiparticles arising in $(2+1)$-dimensional condensed matter systems, and can be sorted into two classes:

(i) *Abelian* anyons i.e. quasiparticles that have fractional exchange statistics with other quasiparticles.[6]

(ii) *Nonabelian* anyons i.e. quasiparticles that possibly have higher-dimensional exchange statistics with other quasiparticles.

The theoretical origins of anyons can be traced back to Leinaas and Myrheim [LM77]. The name 'anyons' was subsequently popularised by Wilczek [Wilc82].

...and Where to Find Them

In 1982, the fractional quantum Hall effect (FQHE) was discovered by Störmer, Tsui and Gossard [STG82].[7] A FQHE system consists of a thin, cold gas of electrons confined between two slabs of semiconductor subject to a strong magnetic field in the perpendicular direction. Moore and Read subsequently argued that the FQHE should play host to anyons [MR91]. Since then, it has been predicted that anyons should manifest in various other $(2+1)$-dimensional condensed matter systems (e.g. topological superconductors).

[4] That is, knotted vortices in the 'aether' (which was believed to exist at that time).

[5] Both Jones and Witten were awarded the Fields Medal in 1990 for this work.

[6] These can be compared to solitons (as characterised by Drazin and Johnson) [DJ89] : objects arising in some medium that (1) are of permanent form, (2) are localised within a region, (3) can interact with other solitons, and emerge from the collision unchanged except for a *phase shift*.

[7] Work for which Störmer and Tsui received the Nobel prize in physics in 1998.

Two well-known classes of anyon theories are the Ising and Fibonacci theories. It is believed that Ising anyons should be realised in the $\nu = \frac{5}{2}$ filling of the FQHE. It is also thought that the low-energy excitations arising in the $\nu = \frac{12}{5}$ filling should closely resemble Fibonacci anyons.
The experimental detection of anyons has proved to be a controversial point in the past, and the existence of nonabelian anyons has yet to be experimentally verified. On the other hand, the existence of abelian anyons is now widely accepted by the physics community, with the strongest evidence yet found in [NLGM20].

Braiding and Fusion

Among the other major themes running through this thesis are those of *braiding* and *fusion*. These are operations that can be performed on a system of anyons. The appropriate algebraic framework for modelling such operations is a *unitary braided fusion category* (UBFC). Anyons further possess a rotational degree of freedom, referred to as *twisting*. In order for our algebraic framework to capture this 'twisting' action, we have to upgrade our UBFC to a *unitary ribbon fusion cateogry* (URFC). Algebraically, this is not a big step up: in fact, the ribbon structure is 'already there' (i.e. encoded in the braiding and fusion operations) in some sense, since a UBFC admits a unique unitary ribbon structure [ENO05, Gal14].
In Chapter 2, we detail the (skeletal) structure of URFCs. We do so mostly by using the same language employed by physicists, but provide some category-theoretic remarks and context along the way. Ultimately, the language is of no consequence, since the mathematical content in which we are interested is the same. In Section 2 of Chapter 4 (i.e. Paper II), URFCs are reintroduced[8] (this time, using some of the same language that is employed by mathematicians). Nonetheless, the equivalence of both presentations will be apparent (especially since we work in the skeleton[9] of the categories in both parts of this thesis). A glossary translating between some of the physical and mathematical jargon is provided in Table 5.1.

One of the primary goals of Part I. is to understand *why* URFCs provide the correct algebraic framework for understanding the exchange statistics of anyons. The possibility of fractional statistics in two spatial dimensions is often sketched as follows:

[8] Some familiarity with the language of tensor categories is assumed.
[9] A definition of the 'skeleton' of a category is given in Section 2 of Chapter 4. 'Skeletal data' is defined in both Chapters 2 and Chapter 4.

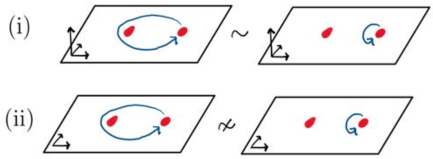


Figure 1: (i) The process of winding one particle around another can be deformed to doing nothing at all in 3 or more spatial dimensions. (ii) If we restrict the number of spatial dimensions to 2, the deformation is no longer possible.

Let $\hat{P}$ be some 1-dimensional unitary operator describing the evolution induced by the anticlockwise exchange of two identical particles. The exchange process corresponding to $\hat{P}^2$ is illustrated on the left-hand side of Figure 1(i)-(ii). In case (i), we have $\hat{P}^2 = 1$, and so the only possibilities are that $\hat{P} = +1$ and $\hat{P} = -1$ (which respectively correspond to bosonic and fermionic exchange statistics). In case (ii), there is no such restriction on $\hat{P}^2$, and so in principle, we could have $\hat{P} = e^{i\theta}$ for arbitrary statistical angle θ.

There is clearly a vast gulf between the above, and the baroque framework of a URFC. In Chapter 1, the exchange statistics of identical particles is explored in further detail, and the above sketch is refined. However, this does not get us much closer to the URFC setting. That the worldlines of particles in two spatial dimensions are given by braids is clear, but where does fusion come from? In many expositions, fusion is often motivated using the flux-charge composite toy models. Fusion is also readily apparent in 2D spin-lattice models such as the toric code. Neither approach is sufficient for explaining the presence of fusion structure in a general, Hamiltonian-free setting.
How were URFCs originally found to provide the correct framework? Chern-Simons TQFTs describe the behaviour of low-energy excitations arising in certain condensed matter systems confined to two spatial dimensions (e.g. fractional quantum Hall systems). These excitations are anyons. A Chern-Simons theory realises a representation of a 'quantum group' at a root of unity, which carries all the structure of a ribbon fusion category. Ribbon categories of this kind are called "quantum group categories".
In Paper I (Chapter 3), we derive fusion structure in a completely different way. The main objective is to derive fusion structure in a general setting (and to further make contact with the URFC framework), by starting from exchange symmetry and using a minimal prescription of extra assumptions. We will see that it is the localised nature of quasiparticles (combined with exchange symmetry) that plays a crucial role in the emergence of fusion structure from exchange symmetry in $(2+1)$-dimensions.

Skein Theory

Another overarching theme of this thesis is the utility of the graphical calculus. In Chapter 4, we exploit this calculus and use a rotation operator to study the properties of certain F-symbols in unitary fusion categories, as well as the skein relations associated to certain fusion rules in URFCs.

Summary of Content

Chapters with an asterisk indicate that the content is the original work of the author.
Chapters marked with a † indicate that the content is an embedded single-author paper.
Chapters marked with a †† indicate that the content is an embedded paper with a coauthor.
Chapters with no marking indicate that the content is chiefly expository.

- **Chapter 1.** We discuss the exchange statistics of identical particles in some detail.
- **Chapter 2.** We give a fairly detailed summary of the algebraic theory of anyons.
- **Chapter 3.**† This is Paper I [Val21] of the thesis, "*Fusion Structure from Exchange Symmetry in (2+1)-Dimensions*".
- **Chapter 4.**†† This is Paper II [PV20] of the thesis, "*Skein-Theoretic Methods for Unitary Fusion Categories*".
- **Chapter 5.*** In Section 5.1, we will use the graphical calculus to look at some examples of entangling operators. In Section 5.2, we provide some explicit examples of calculations for the evaluation of link diagrams in End($\mathbf{1}$): this can be seen as supplementary to the narrative of Chapter 4.

Part I.

Exchange Symmetry and Topological Quantum Systems

1. Identical Particles

In the following, Section 1.1 is based on the exposition in [Simon], Section 1.2 appears in [Val21] and Section 1.4 follows [Pachos].

1.1 Exchange Trajectories and Path Integrals

In classical mechanics, 'particles' are always *distinguishable*: given n particles, each of which has configuration manifold $\mathcal{M}$, they can be distinguished by their trajectories $\{\gamma_i : I \subseteq \mathbb{R} \to T^*\mathcal{M}\}_{i=1}^n$. That is, we can track their positions and momenta over time and distinguish between them accordingly. When we consider the dynamics of particles in quantum systems, this classical book-keeping scheme is rendered useless.

The state space of an n-particle (quantum) system is given by $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$ where $\mathcal{H}_i$ is the Hilbert space of states associated to the i^{th} particle. Let $|\psi\rangle \in \mathcal{H}$ be the state of the system at some fixed time, and suppose the particles lie in $\mathbb{R}^d$ ($d \geq 2$). Let Δ denote the subset of points in $\left(\mathbb{R}^d\right)^n$ where two or more of the $\boldsymbol{x}_i$ coincide. Spatially, the configuration space of the n particles is $\left(\mathbb{R}^d\right)^n - \Delta$. In the position basis,

$$|\psi\rangle = \int_{\left(\mathbb{R}^d\right)^n - \Delta} d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_n \; \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \, |\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\rangle \quad , \quad \boldsymbol{x}_i \in \mathbb{R}^d \tag{1.1.1}$$

Furthermore, if the particles are *identical*, we may identify the states $\{|\boldsymbol{x}_{s(1)}, \ldots, \boldsymbol{x}_{s(n)}\rangle\}_{s \in S_n}$ (where S_n denotes the permutation group) since they are physically indistinguishable. Levying this equivalence relation on the points in $\left(\mathbb{R}^d\right)^n - \Delta$, we obtain the configuration space[1] $\mathcal{M}$ of the n identical particles. Let us consider the (nonrelativistic) propagator from point a to b in $\mathcal{M}$: using the path integral formalism,

$$\langle b| \, \hat{U}(t_f, t_i) \, |a\rangle = \mathscr{N} \int_{(a,t_i)}^{(b,t_f)} \mathcal{D}\left[\gamma(t)\right] e^{\frac{i}{\hbar} S[\gamma(t)]} \tag{1.1.2}$$

[1] $\mathcal{M}$ is also known as the n^{th} unordered configuration space of $\mathbb{R}^d$.

where $\hat{U}(t_f, t_i)$ is the time-evolution operator, $S[\gamma(t)]$ is the classical action of the map $\gamma(t) : [t_i, t_f] \to \mathcal{M}$, and $\mathscr{N}$ is a normalisation constant. The propagator allows us to determine the wavefunction (in the position basis) at time T:

$$|\psi(T)\rangle = \iint_{\mathcal{M}} da \; db \;\; \langle b| \, \hat{U}(T, t_i) \, |a\rangle \, \langle a|\psi(t_i)\rangle \, |b\rangle \tag{1.1.3}$$

Let $\Pi(a, b) := \{\gamma(t) : [t_i, t_f] \to \mathcal{M} \mid \gamma(t_i) = a, \gamma(t_f) = b\}$. Equation (1.1.2) is a sum over all paths in $\Pi(a, b)$. Note that $\Pi(a, a)$ is the space of all loops in $\mathcal{M}$ with basepoint a (i.e. paths permuting the positions of the n identical particles). The homotopy classes of the loops are given by the elements of the fudamental group $\pi_1(\mathcal{M})$ of the configuration space: these encode the topological features of the exchange trajectories.

$$\pi_1(\mathcal{M}) \cong \begin{cases} S_n \; , \;\; d \geq 3 \\ B_n \; , \;\; d = 2 \end{cases} \tag{1.1.4}$$

where the n-strand braid group is defined as

$$B_n = \left\langle \; \sigma_1, \ldots, \sigma_{n-1} \; \middle| \; \begin{array}{c} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \; , \;\; |i - j| \geq 2 \end{array} \; \right\rangle \tag{1.1.5}$$

and the symmetric group of degree n as

$$S_n = \left\langle \; s_1, \ldots, s_{n-1} \; \middle| \; \begin{array}{c} s_i^2 = e \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \; , \;\; |i - j| \geq 2 \end{array} \; \right\rangle \tag{1.1.6}$$

where we use e to denote the identity element of a group.

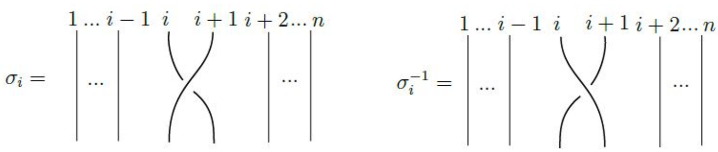


Figure 1.1: Braids are composed from top-to-bottom, and braid words are read from right-to-left. The crossing between strands i and $i+1$ in the braid diagrams corresponding to σ_i and σ_i^{-1} are respectively said to be positive $(+1)$ and negative (-1).

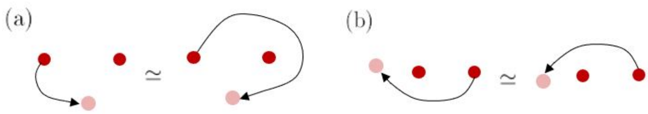


Figure 1.2: Exchange trajectories in $\mathbb{R}^d$ for (a) zero exchanges (left) and two clockwise exchanges (right), (b) single exchanges. Homotopies '$\simeq$' lift the paths through the extra spatial dimension(s) when $d \geq 3$.

Remark 1.1. When $d = 2$, the possible exchange trajectories (as worldlines in $\mathbb{R}^{2+1}$ up to homotopy) of n identical particles are given by n-braids. This is intuitive. When $d \geq 3$, Figure 1.2 illustrates the equivalence of two successive exchanges of a pair of adjacent particles to zero exchanges, and equivalently the insensitivity of a particle exchange to orientation. This corresponds to the epimorphism $\eta : B_n \to S_n$ (whose kernel is the normal subgroup PB_n of n-strand pure braids[2]). By Alexander's theorem[3], this is equivalent to the fact that there are no nontrivial links in more than 3 spatial dimensions.

There is a bijection between the homotopy classes of paths in $\Pi(a, a)$ and $\Pi(a, b)$. The homotopy classes of paths between any two points in $\mathcal{M}$ corresponds to the fundamental groupoid of the configuration space: write $\Pi(a, a) = \{[g]\}_{g\in\pi_1(\mathcal{M})}$ for any $a \in \mathcal{M}$. Then we write $\Pi(a, b) = \{[g]^{(a,b)}\}_{g\in\pi_1(\mathcal{M})}$ where $[g']^{(d,b)} \circ [g]^{(a,c)} = [g'g]^{(a,b)}$ when $c = d$ (else the composition is undefined). Consider the following ansatz for the propagator:

$$\langle b|\, \hat{U}(t_f, t_i)\, |a\rangle = \mathscr{N} \sum_{g\in\pi_1(\mathcal{M})} \chi(g) \sum_{\gamma\in[g]^{(a,b)}} e^{\frac{i}{\hbar}S[\gamma]} \tag{1.1.7}$$

where $\chi : \pi_1(\mathcal{M}) \to U(1)$ is a linear representation. To check whether (1.1.7) is permissible, we need to see if it is consistent under composition. Indeed, for $t_i < t_m < t_f$,

$$\begin{aligned} \langle b|\, \hat{U}(t_f, t_i)\, |a\rangle &= \int_{\mathcal{M}} dq\, \langle b|\, \hat{U}(t_f, t_m)\, |q\rangle \langle q|\, \hat{U}(t_m, t_i)\, |a\rangle \\ &\propto \int_{\mathcal{M}} dq \sum_{g_2\in\pi_1(\mathcal{M})} \chi(g_2) \sum_{\gamma_2\in[g_2]^{(q,b)}} e^{\frac{i}{\hbar}S[\gamma_2]} \sum_{g_1\in\pi_1(\mathcal{M})} \chi(g_1) \sum_{\gamma_1\in[g_1]^{(a,q)}} e^{\frac{i}{\hbar}S[\gamma_1]} \end{aligned}$$

which yields (1.1.7). When $d \geq 3$, there are only two permissible representations

$$\begin{aligned} \chi^{\pm} :\ & S_n \to U(1) \\ & s_i \mapsto \pm 1 \end{aligned} \tag{1.1.8}$$

[2]Pure braids are those whose endpoints have the same order as their starting points.
[3]Any link may be obtained by closing some braid (James W. Alexander II).

whence the propagator corresponding to $\chi^{\pm}$ is given (up to normalisation) by

$$\sum_{\gamma \in \Pi^{\text{even}}} e^{\frac{i}{\hbar}S[\gamma]} \pm \sum_{\gamma \in \Pi^{\text{odd}}} e^{\frac{i}{\hbar}S[\gamma]} \tag{1.1.9}$$

where Π^{even} and Π^{odd} are the subsets of paths in $\Pi(a,b)$ respectively realising an even and odd number of exchanges. Fundamental particles may thus be partitioned into two classes: *bosons* and *fermions*. If two identical bosons are exchanged, the wavefunction is scaled by $+1$; if two identical fermions are exchanged, it is scaled by -1. When $d = 2$,

$$\begin{aligned} \chi : \ B_n &\to U(1) \\ \sigma_i &\mapsto e^{i\alpha} \end{aligned} \tag{1.1.10}$$

whence the propagator is given (up to normalisation) by

$$\sum_{w \in \mathbb{Z}} e^{iw\alpha} \sum_{\gamma \in \Pi_w} e^{\frac{i}{\hbar}S[\gamma]} \tag{1.1.11}$$

where Π_w is the subset of paths (n-braids) in $\Pi(a,b)$ with writhe[4] w.

Remark 1.2. We have been considering how the wavefunction of a system of identical particles evolves under exchanges. The acquired phase[5] is called the *statistical phase*. We have seen that the exchange evolution depends on the topology of the exchange trajectories: in particular, (1.1.7) shows that there can be a relative phase between evolutions along trajectories in distinct homotopy classes, where an evolution along the trivial class (i.e. no exchanges) is taken to induce a trivial statistical phase. When $d \geq 3$, exchange statistics are either bosonic or fermionic (since the braid group representation must factor through the symmetric group). When $d = 2$ we saw that *fractional statistics* may be possible: 'particles' in two-dimensional systems are thus called *any*ons [Wilc82].

Further suppose that there is some K-fold degeneracy in the state of the system i.e.

$$|p\rangle = \sum_{j=1}^{K} a_j \, |p; j\rangle \quad , \quad p \in \mathcal{M} \tag{1.1.12}$$

where $\{|p; j\rangle\}_j$ is an orthonormal basis. By exchange symmetry, this degeneracy must be associated to a global degree of freedom of all n particles. The propagator becomes

$$\langle b; j'| \, \hat{U}(t_f, t_i) \, |a; j\rangle = \mathscr{N} \sum_{g \in \pi_1(\mathcal{M})} \langle j'| \, \rho(g) \, |j\rangle \sum_{\gamma \in [g]^{(a,b)}} e^{\frac{i}{\hbar}S[\gamma]} \tag{1.1.13}$$

[4] We take the *writhe* of a braid to be the sum of the signs of its crossings.

[5] The statistical phase is solely dependent on particle exchanges: the total acquired phase will depend on more than just this (e.g. the dynamical phase).

where $\rho : \pi_1(\mathcal{M}) \to U(K)$ is a linear representation. For $t_i < t_m < t_f$, (1.1.13) is

$$\begin{aligned}
&\sum_i \int_{\mathcal{M}} dq \, \langle b; j'| \, \hat{U}(t_f, t_m) \, |q; i\rangle \, \langle q; i| \, \hat{U}(t_m, t_i) \, |a; j\rangle \\
\propto &\sum_i \int_{\mathcal{M}} dq \sum_{g_2 \in \pi_1(\mathcal{M})} \langle j'| \, \rho(g_2) \, |i\rangle \sum_{\gamma_2 \in [g_2]^{(q,b)}} e^{\frac{i}{\hbar} S[\gamma_2]} \sum_{g_1 \in \pi_1(\mathcal{M})} \langle i| \, \rho(g_1) \, |j\rangle \sum_{\gamma_1 \in [g_1]^{(a,q)}} e^{\frac{i}{\hbar} S[\gamma_1]} \\
\propto &\int_{\mathcal{M}} dq \sum_{g_1, g_2 \in \pi_1(\mathcal{M})} \langle j'| \, \rho(g_2 g_1) \, |j\rangle \sum_{\gamma_2 \in [g_2]^{(q,b)}} \sum_{\gamma_1 \in [g_1]^{(a,q)}} e^{\frac{i}{\hbar} S[\gamma_1 + \gamma_2]}
\end{aligned}$$

whence (1.1.13) is consistent under composition. Thus, the evolution of the system along a trajectory in homotopy class $[g]^{(a,b)} \in \Pi(a, b)$ is given by $\rho(g)$.

Remark 1.3.

(i) When there is no degeneracy present ($K = 1$), the exchange statistics are called *abelian*. When there is degeneracy ($K > 1$), the statistics are called *nonabelian*.

(ii) When $d \geq 3$, the statistical evolutions given by higher-dimensional representations of the symmetric group are referred to as *parastatistics*: this notion appears to be in conflict with the classification[6] of all fundamental particles as either bosons or fermions. Taking into consideration some additional constraints[7], it has been shown that this classification holds [DHR71, DHR74, Müg07]. See also [BHS15]. Indeed, all experimental evidence is in accord with this positon.

1.2 Superselection Sectors

Consider a system with Hilbert space $\mathcal{H}$. A *superselection rule* (SSR) is given by a normal operator $\hat{J} : \mathcal{H} \to \mathcal{H}$ where

$$[\hat{O}, \hat{J}] = 0 \tag{1.2.1}$$

for *all* observables $\hat{O}$ of the system. Let $\mathcal{H}'$ and $\mathcal{H}''$ be any two distinct eigenspaces of $\hat{J}$ (with corresponding eigenvalues $z', z'' \in \mathbb{C}\backslash\{0\}$). For any $|\psi'\rangle \in \mathcal{H}'$ and $|\psi''\rangle \in \mathcal{H}''$,

$$z' \langle \psi''| \, \hat{O} \, |\psi'\rangle = \langle \psi''| \, \hat{O}\hat{J} \, |\psi'\rangle \overset{(1.2.1)}{=} \langle \psi''| \, \hat{J}\hat{O} \, |\psi'\rangle = z'' \langle \psi''| \, \hat{O} \, |\psi'\rangle$$

[6]Some sources treat this statistical property of identical particles as an axiom of quantum mechanics (known as the *symmetrisation postulate*).

[7]Namely pair creation and annihilation, and locality.

for any observable $\hat{O}$ on $\mathcal{H}$. Since z' and z'' are distinct, we see that

$$\langle\psi''|\,\hat{O}\,|\psi'\rangle = 0 \tag{1.2.2}$$

The eigenspaces of $\hat{J}$ are called *superselection sectors*. The defining feature of SSRs is that they preclude the observation of relative phases between states from distinct superselection sectors: let $|\psi\rangle = \alpha\,|\psi'\rangle + \beta\,|\psi''\rangle$ and $|\psi_\theta\rangle = \alpha\,|\psi'\rangle + e^{i\theta}\beta\,|\psi''\rangle$ be normalised states. Then

$$\langle\hat{O}\rangle_\psi = \langle\hat{O}\rangle_{\psi_\theta} = \mathrm{tr}(\hat{O}\hat{\rho}) \tag{1.2.3}$$

where $\hat{\rho} = |\alpha|^2\,|\psi'\rangle\,\langle\psi'| + |\beta|^2\,|\psi''\rangle\,\langle\psi''|$ (i.e. if superpositions ψ_θ were to exist, we would be incapable of physically distinguishing them from a statistical mixture).

Examples of SSRs include spin, electric charge and mass[8]. The spin SSR concerns the superposition of integer and half-integer spins: by the spin-statistics theorem (Theorem 1.4), this is equivalent to the boson-fermion SSR (see Section 1.3).[9] For our purposes, it will be useful to think of the *intrinsic* properties of a particle as corresponding to quantum numbers with an associated SSR. Two particles are *identical* if all of their intrinsic properties match exactly (e.g. all electrons are identical).

1.3 Bosons and Fermions

A permutation of n identical particles is indistinguishable from the original configuration. Following Section 1.1, we may concisely express exchange symmetry by

$$[\hat{O}, \rho(g)] = 0 \tag{1.3.1}$$

for all observables $\hat{O}$ on the n-particle Hilbert space $\mathcal{H}$, and for all $g \in \pi_1(\mathcal{M})$ where $\rho : \pi_1(\mathcal{M}) \to U(\mathcal{H})$ is a unitary linear representation as in (1.1.13). Clearly, if

$$[\hat{O}, \rho(g_i)] = 0 \tag{1.3.2}$$

for all generators g_i of $\pi_1(\mathcal{M})$, then (1.3.1) follows. Recall that for n particles in $\mathbb{R}^d$ where $d \geq 3$, we have $\pi_1(\mathcal{M}) \cong S_n$. The eigenvalues of $\rho(s_i)$ lie in a nonempty subset of $\{\pm 1\}$. We respectively denote the corresponding eigenspaces[10] by $\mathcal{H}_i^\pm$. Since each such

[8] Bargmann's mass SSR arises through demanding the Galilean covariance of the Schrödinger equation: this only pertains to nonrelativistic systems, since Galilean symmetry is superseded by Poincaré symmetry in special relativity.

[9] These two equivalent SSRs are sometimes referred to as the *univalence* SSR.

[10] Suppose $\rho(s_i)$ only has one eigenvalue e.g. $+1$: then we take $\mathcal{H}_i^-$ to be 0-dimensional.

eigenspace defines a superselection sector and the n particles are identical (and are thus either all bosons or all fermions), ρ must be such that $\mathcal{H}_i^{\pm} = \mathcal{H}_j^{\pm}$ for all i, j. We thus have $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ (the subscripts are dropped). This is the boson-fermion SSR.

Theorem 1.4 (Spin-Statistics Theorem).[11]
Particles obeying Bose-Einstein statistics (i.e. bosons) have integer spin, and particles obeying Fermi-Dirac statistics (i.e. fermions) have half-integer spin.

Principle 1.5 (Pauli Exclusion Principle). [12]
No two identical fermions may simultaneously occupy the same quantum state.

Let $\mathcal{H}^{(1)}$ denote some 1-particle state space. For a system of n such identical particles, the state space is generically given by $\mathcal{H}^{(n)} := \bigotimes_{k=1}^{n} \mathcal{H}^{(1)}$. The *Fock space* is the space of all bosonic and fermionic states for an arbitrary number of particles (constructed from $\mathcal{H}^{(1)}$). Indeed, there is a lot of redundancy in $\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$ since this space contains states which are neither symmetric nor antisymmetric under exchanges (and thus do not describe a system of identical particles). The Fock space is written as

$$\mathscr{H}_{\pm} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}_{\pm}^{(2)} \oplus \mathcal{H}_{\pm}^{(3)} \oplus \dots \tag{1.3.3}$$

where $\mathcal{H}_{+}^{(n)}$ and $\mathcal{H}_{-}^{(n)}$ respectively denote the space of bosonic and fermionic n-particle states for $n \geq 2$. Let $\mathcal{H}^{(1)} := \operatorname{span}_{\mathbb{C}}\{|\varphi_i\rangle\}_i$. The vacuum is given by $\mathcal{H}^{(0)} = \operatorname{span}_{\mathbb{C}}\{|\phi\rangle\}$ where $|\phi\rangle$ is the state with 0 particles in state $|\varphi_i\rangle$ for all i. Note that e.g. $|\varphi_1, \varphi_1\rangle \in \mathcal{H}_{+}^{(2)}$ whereas 2-particle state $|\varphi_1, \varphi_2\rangle \notin \mathcal{H}_{\pm}^{(2)}$ is clearly distinguishable under exchanges. Spaces $\mathcal{H}_{+}^{(n)}$ and $\mathcal{H}_{-}^{(n)}$ are respectively the symmetric and antisymmetric parts of $\mathcal{H}^{(n)}$. That is,

$$\mathcal{H}_{+}^{(n)} = \left\{ \sum_{s \in S_n} |s(\psi)\rangle : |\psi\rangle \in \mathcal{H}^{(n)} \right\} \;,\; \mathcal{H}_{-}^{(n)} = \left\{ \sum_{s \in S_n} \operatorname{sgn}(s) \cdot |s(\psi)\rangle : |\psi\rangle \in \mathcal{H}^{(n)} \right\} \tag{1.3.4}$$

where $\operatorname{sgn}(s)$ is the sign of the permutation s (which acts linearly) and where for any basis ket $|\varphi\rangle = |\varphi_{i_1}, \dots, \varphi_{i_n}\rangle$ we have $|s(\varphi)\rangle = |\varphi_{i_{s(1)}}, \dots, \varphi_{i_{s(n)}}\rangle$. In $\mathcal{H}_{-}^{(n)}$, there clearly cannot be more than one particle occupying any given $|\varphi_i\rangle$ (consistent with Principle 1.5).

For a system of n identical bosons or fermions, there is typically no subspace describing

[11] Later, we see an analogous connection for anyons (2.3.4).

[12] This follows from $|\psi\rangle\,|\psi\rangle = -\,|\psi\rangle\,|\psi\rangle$ i.e. there is 0 probability of two identical fermions occupying the same state. There is no such restriction for bosons, hence the existence of Bose-Einstein condensates. The Pauli exclusion principle has many important consequences. For instance, it predicts quantum degeneracy pressure (which plays a role in gravitational collapse), ferromagnetism and the the formation of elements beyond hydrogen.

a subsystem of $k < n$ particles.[13] This is implicit in the structure of the Fock space[14] e.g. $\mathcal{H}_+^{(2)} \not\subset \mathcal{H}_+^{(3)}$. Given $\mathcal{H}^{(1)} = \mathrm{span}_{\mathbb{C}}\{|0\rangle, |1\rangle\}$, states such as $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \in \mathcal{H}_-^{(2)}$ do not describe a physical entanglement since the subsystem for an individual particle is physically inaccessible [Zan02]. Nonetheless, there exist circumstances under which some notion of distinguishability amongst n identical bosons or fermions may be recovered: for instance, when their wavefunctions have (approximately) disjoint compact support. This can happen if the particles are far apart, or separated by sufficiently strong potentials.[15]

1.4 Anyons

As is communicated by the omnipresence of bosons and fermions (amongst other things), our cosmos appears to exist in at least three spatial dimensions. This means that anyons cannot be fundamental particles.

($\mathcal{Q}$) *How can we implement a two-dimensional quantum system?*

Consider a particle moving in a potential of the form

$$V(\boldsymbol{r}) = V_{xy}(x, y) + V_z(z) \tag{1.4.1}$$

where $\boldsymbol{r} = (x, y, z) \in \mathbb{R}^3$. It is easily seen (Appendix A) that the ansatz (1.4.2) solves the Schrödinger equation subject to potential (1.4.1).

$$\psi(\boldsymbol{r}) = \psi_{xy}(x, y)\psi_z(z) \tag{1.4.2}$$

Consider a many-particle system with each constituent subject to the above potential. Though the particles lie in spatial dimension greater than two, their joint wavefunction has planar dynamics entirely decoupled from the z-direction. Anyons may emerge as *spatially localised* properties of this two-dimensional wavefunction, and since they arise from the same wavefunction, they are correlated. Moreover, anyons are not particles in the usual sense, but are *emergent phenomena* of the wavefunction that can be treated as such. For this reason, they are referred to as *quasiparticles*.[16]. These quasiparticles have no internal degrees of freedom, and may thus be considered *identical*.

[13] This is in contrast to anyonic systems which have a well-defined description of state spaces for particle subsystems (since anyons are *spatially localised*).

[14] As a consequence of the mass SSR, observe that the sectors of the Fock space correspond to a SSR for the particle number operator in the nonrelativistic limit.

[15] E.g. the toric code is built from a system of localised spins that fall into the latter category.

[16] By abuse of terminology, we also refer to anyons as particles (for the sake of brevity).

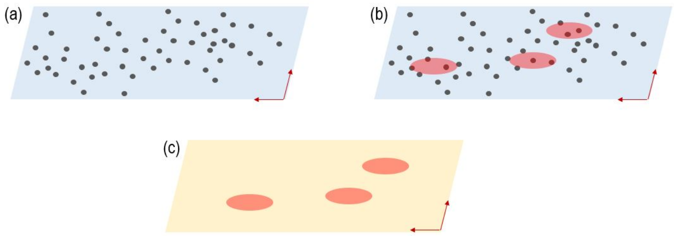


Figure 1.3: (a) Particles of microsystem have planar dynamics described by a two-dimensional wavefunction. (b) Anyons identified as localised excitations. (c) Ignore microsystem and treat quasiparticles as elementary particles in two-dimensional space.

Since the wavefunction is the medium in which these quasiparticles emerge (we will always assume the system to be in the ground state), their stability relies on having a large energy gap separating the wavefunction from the excited states: such a gap can be achieved by a *strongly confining* potential.[17] These ideas translate practically. For instance, quantum Hall systems (a prime candidate for the manufacture of anyons) consist of a thin, cold gas of electrons confined between two slabs of semiconductor subject to a strong magnetic field in the perpendicular direction.

[17] By "strongly confining", we mean a potential whose value is negligible along a small interval of some axis compared to its value outside of it. To illustrate this point more concretely, we can crudely approximate such a potential by an infinite well with potential vanishing in the region $[0, L]$. The eigenenergies of a particle of mass m in the well are given by $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$ for $n \geq 1$. The greater the confinement, the closer L is to zero, and the larger the energy gap between E_1 and E_2.

2. The Algebraic Theory of Anyons

In this chapter, we summarise the standard algebraic framework for modelling theories of anyons. The details that we provide here are not exhaustive (for instance, we do not discuss chiral central charge), but encompass most of the salient points. Parts of our exposition are based on [Bonderson, Kitaev06, Preskill, Simon, Wang].

We begin by describing the fusion rules (and associated properties) of anyons. While the motivation for conditions **(F1)**-**(F5)** may appear unclear, we remind the reader that this is one of the objectives of Chapter 3. Beyond fusion, their are two other key operations that a theory of anyons should describe:

(i) *Braiding.* We wish to understand the exchange statistics of anyons.

(ii) *Twisting.* Anyons possess a spin degree of freedom. Our algebraic model must therefore be able to describe 2π self-rotations of anyons.

Braiding and twisting will correspond to unitary operators acting on the state space of the anyonic system. In order to obtain a concrete description of these operators, we must work to find a state vector description of anyonic systems: we will see that there is some inherent freedom in such a description. Furthermore, when we consider systems of more than two anyons, fixing a basis on the state space will naturally lead us to an operator description of associativity. In our pursuit of this framework, we will make substantial use of a graphical calculus, and we will see that the resulting algebraic description of anyons encodes a wealth of information.

2.1 Fusion

2.1.1 Fusion Rules

The basic input data for a theory of anyons is given by

(i) A finite set of *labels* $\mathfrak{L} = \{q_i\}_{i=0}^{n}$

(ii) A set of finite nonnegative integers $\{N_c^{ab}\}_{a,b,c\in\mathfrak{L}}$ called *fusion coefficients*

Together, these encode the *fusion rules* of a theory. For instance, the fusion rule for labels a and b is written

$$a \times b = \sum_{c\in\mathfrak{L}} N_c^{ab} c \tag{2.1.1}$$

The labels index the possible superselection sectors within a given theory, and the fusion rules describe the possible ways in which two such sectors can be merged. The cardinality of $\mathfrak{L}$ is called the *rank* of the theory. Labels are also referred to as *(topological) charges* and should be thought of as the different 'types' of anyons that may be realised within the theory.[1] From this perspective, the fusion of two anyons should be thought of as bringing them sufficiently close together so that they may be treated as a localised object whose aggregate charge and exchange statistics may be considered.[2] Then 'addition' on the right-hand side of (2.1.1) should be considered as concatenating all *possible* fusion outcomes for a and b. In particular, $N_c^{ab} = k > 1$ means that a and b may be fused to c in k physically distinguishable ways.

The fusion rules for a theory of anyons must satisfy **(F1)**-**(F5)**.

(F1) *Existence of identity:* Any given theory comes equipped with the trivial label $\mathbf{1}$ (typically indexed by 0 in $\mathfrak{L}$)[3] which generically represents the vacuum. We have

$$\mathbf{1} \times a = a \times \mathbf{1} = a \tag{2.1.2}$$

for any $a \in \mathfrak{L}$. In terms of the fusion coefficients,

$$N_b^{0a} = N_b^{a0} = \delta_{ab} \tag{2.1.3}$$

[1] We will use the terms label, charge and anyon interchangeably.
[2] For a familiar analogy, recall that an even number of fermions will collectively behave as a boson.
[3] 0 and $\mathbf{1}$ will be used interchangeably.

(F2) *Existence of fusion channel:* Any pair of anyons may be fused i.e. for any $a, b \in \mathfrak{L}$

$$\sum_c N_c^{ab} \geq 1 \tag{2.1.4}$$

(F3) *Existence of unique dual:* For any $a \in \mathfrak{L}$ there exists *unique* $\bar{a} \in \mathfrak{L}$ such that

$$a \times \bar{a} = \bar{a} \times a = \mathbf{1} + \dots \tag{2.1.5}$$

In terms of the fusion coefficients,

$$N_0^{ab} = N_0^{ba} = \delta_{b\bar{a}} \tag{2.1.6}$$

The label $\bar{a}$ is called the *dual* (or conjugate) charge of a and is considered to be the 'antiparticle' (or 'anti-anyon') of a (since the pair may annihilate to the vacuum).

(F4) *Associativity:* For any $a, b, c \in \mathfrak{L}$ we have

$$(a \times b) \times c = a \times (b \times c) \tag{2.1.7}$$

In terms of the fusion coefficients, for each $d \in \mathfrak{L}$ we have

$$\sum_e N_e^{ab} N_d^{ec} = \sum_f N_d^{af} N_f^{bc} \tag{2.1.8}$$

Associativity tells us that the possible total fusion outcomes do not depend on the order in which pairs of anyons are fused.

(F5) *Commutativity:* For any $a, b \in \mathfrak{L}$ we have

$$a \times b = b \times a \tag{2.1.9}$$

In terms of the fusion coefficients,

$$N_c^{ab} = N_c^{ba} \tag{2.1.10}$$

Commutativity tells us that permuting the order of anyons does not affect the possible total fusion outcomes.

Definition 2.1. The *charge conjugation* matrix C for a theory of anyons is a symmetric binary matrix whose entries are given by $[C]_{ab} = \delta_{b\bar{a}}$ for $a, b \in \mathfrak{L}$.

Terminology 2.2.

(i) A theory of anyons for which $N_c^{ab} \in \{0,1\}$ for all $a,b,c \in \mathfrak{L}$ is called *multiplicity-free.*

(ii) A label a is called *self-dual* (or self-conjugate) if $\bar{a} = a$. A theory for which $C = I$ is called self-dual.

Remark 2.3. Although any theory of anyons has an associated set of fusion rules, the converse is generally not true. In the case that a set of fusion rules does give rise to a consistent theory of anyons, there is typically more than one such associated theory.

2.1.2 Fusion Spaces

The Hilbert space V^{ab} associated to a pair of anyons a and b can be decomposed into its constituent superselection sectors as

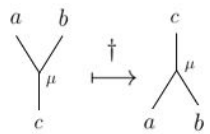
$$V^{ab} = \bigoplus_c V_c^{ab} \quad , \quad \dim(V_c^{ab}) = N_c^{ab} \tag{2.1.11}$$

Its constituent states are called *fusion states* and the spaces are referred to as *fusion spaces*. Given any fusion space V_c^{ab}, we will fix an orthonormal basis $\{|ab \to c; \mu\rangle\}_\mu$ of fusion states.

Figure 2.1: The state $|ab \to c; \mu\rangle$ is diagrammatically represented by a trivalent vertex.

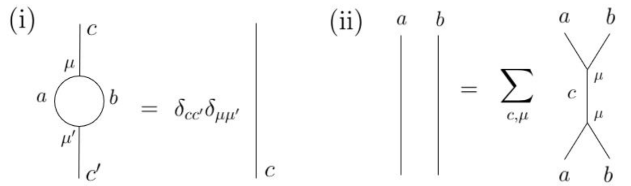
Remark 2.4. (Orientation of edges). We adopt the pessimistic convention for our diagrams. Specifically, any given edge will be accompanied by a label (unless it is obvious what the label should be) and should be interpreted as running from top-to-bottom (i.e. the time axis runs downwards). When a label is self-dual, the orientation does not matter. In some instances, we may opt to append arrows to our edges for clarity.

The dual space of a fusion space V_c^{ab} is written V_{ab}^c and may be interpreted as a '*splitting space*'. Diagrammatically, the splitting state $\langle ab \to c; \mu|$ is given by inverting the trivalent vertex for the fusion state $|ab \to c; \mu\rangle$.



(2.1.12)

Fusion coefficients may thus equivalently be thought of ‘splitting coefficients’. Given an orthonormal basis, we can use the graphical calculus to express the inner product and completeness relation on V^{ab} :



(2.1.13)

Remark 2.5. (Normalisation and gauge freedom).

(i) We will assume that our trivalent vertices implicitly carry a normalisation. Normalisation of trivalent vertices in diagrams is not implicit after Section 2.1.8 unless stated otherwise.

(ii) Observe that there is a U(n) freedom associated to fixing an orthonormal basis on V_c^{ab} (where $n = N_c^{ab}$). This gives rise to some redundancy in our description. This is further discussed in Section 2.4.

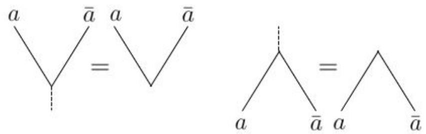


Figure 2.2: The trivial label will be denoted by either a dashed line or nothing. Diagrams for the pair-creation and annihilation of a and $\bar{a}$ are shown above.

Terminology 2.6. Fusion spaces of the form V_c^{ab} or V_{ab}^c are called *triangular spaces.*

2.1.3 Fusing Multiple Charges

Consider a collection $q_1, q_2, \ldots, q_n$ of charges. By **(F4)**, we know that the possible total fusion outcomes are independent of the order in which the charges are fused. The fusion space of this system may thus be written

$$V^{q_1 q_2 \cdots q_n} = \bigoplus_Q V_Q^{q_1 q_2 \cdots q_n} \tag{2.1.14}$$

where Q indexes the possible total fusion outcomes. This space can be understood in terms of the triangular spaces that we have already encountered. In order to do so, we must choose the order in which the charges are fused: this can be seen as fixing a full rooted binary tree with n leaves.

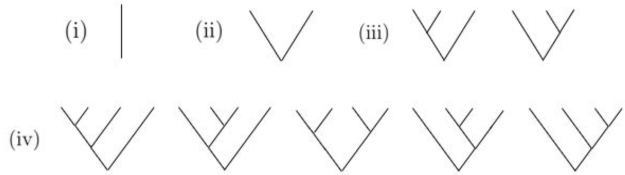


Figure 2.3: The fusion trees in (i)-(iv) respectively depict the ways in which $1, 2, 3$ and 4 particles can be fused. For 5 particles there are 14 trees and for 6 there are 42.

Remark 2.7. The number of fusion trees for n particles is given by the $(n-1)^{th}$ Catalan number[4] C_{n-1} where $C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$. For a derivation, see Appendix B. One might further wonder how many fusion trees there are for other (cellulated) surfaces: we found an answer to this question for the annulus. The number of fusion trees for k particles on the annulus is given by NC_k, where

$$NC_k = 3^{k-2} + \sum_{l=0}^{k-4} 3^l \frac{2(k-l-3)}{(k-l-1)(k-l)} \begin{pmatrix} 2(k-l-2) \\ k-l-2 \end{pmatrix} \quad , \quad k \geq 4 \tag{2.1.15}$$

and $NC_1 = NC_2 = 1, NC_3 = 3$. We call NC_k the k^{th} *necklace Catalan number.* Surprisingly, this was a previously unarchived sequence (which now has an OEIS entry) [A291292].

For instance, consider the fusion space V^{abc}. The two trees in Figure 2.3(iii) respectively specify decompositions

$$V^{abc} \cong \bigoplus_{q,e} V_e^{ab} \otimes V_q^{ec} \quad , \quad V^{abc} \cong \bigoplus_{q,f} V_q^{af} \otimes V_f^{bc} \tag{2.1.16}$$

For the fusion space V^{abcd} e.g. the 1^{st}, 3^{rd} and 4^{th} trees in Figure 2.3(iv) respectively specify decompositions

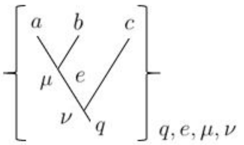
$$V^{abcd} \cong \bigoplus_{q,e} V_e^{abc} \otimes V_q^{ed} \cong \bigoplus_{q,e,f} V_e^{ab} \otimes V_f^{ec} \otimes V_q^{fd} \tag{2.1.17a}$$

$$V^{abcd} \cong \bigoplus_{q,f} V_q^{abf} \otimes V_f^{cd} \cong \bigoplus_{q,e,f} V_e^{ab} \otimes V_q^{ef} \otimes V_f^{cd} \tag{2.1.17b}$$

$$V^{abcd} \cong \bigoplus_{q,f} V_q^{af} \otimes V_f^{bcd} \cong \bigoplus_{q,e,f} V_q^{af} \otimes V_e^{bc} \otimes V_f^{ed} \tag{2.1.17c}$$

[4] The Catalan numbers are ubiquitous. This is concretely reflected by the fact that they are "probably the longest entry in the OEIS" [A000108].

In the diagrammatic formalism the orthonormal basis for e.g the first decomposition of V^{abc} in (2.1.16) is given by



(2.1.18)

where μ runs over indices 1 to N_e^{ab} and ν runs over indices 1 to N_q^{ec}. Fusion diagrams for different decompositions and more charges are drawn analogously.

Terminology 2.8.

(i) Fusion states describing the fusion of a collection of charges in a specified order with all fusion outcomes and trivalent vertices assigned a definite label (e.g. as in (2.1.18)) are sometimes referred to as *fusion channels*.

(ii) Two adjacent particles are said to be in a *direct* fusion channel if they are fused immediately. E.g. for the decomposition specified in (2.1.18), a and b are in a direct fusion channel whereas b and c are not.

Since we always have the freedom to insert the trivial charge anywhere, we must have

$$\dim(V_c^{ab}) = \dim(V_c^{a0b}) = \dim(V_c^{0ab}) = \dim(V_c^{ab0}) \tag{2.1.19}$$

Associativity and (2.1.19) tell us that $N_a^{a0}N_c^{ab} = N_c^{ab}N_b^{0b} = N_c^{ab}$ and so $N_a^{a0} = N_b^{b0} = 1$ for all a, b (which is consistent with **(F1)**). Following the presentation in [Kitaev06], write $V_a^{a0} = \mathrm{span}_{\mathbb{C}}\{|\alpha_a\rangle\}$ and $V_b^{0b} = \mathrm{span}_{\mathbb{C}}\{|\beta_b\rangle\}$ where $|\alpha_a\rangle$ and $|\beta_b\rangle$ are unit vectors. The relation between the spaces in (2.1.19) is characterised by trivial isomorphisms

$$\begin{aligned} \alpha_q : \mathbb{C} &\to V_q^{q0} & \qquad \beta_q : \mathbb{C} &\to V_q^{0q} \\ z &\mapsto z\,|\alpha_q\rangle & z &\mapsto z\,|\beta_q\rangle \end{aligned} \tag{2.1.20}$$

e.g. $V_c^{ab} \overset{\alpha_a}{\xrightarrow{\sim}} V_a^{a0} \otimes V_c^{ab}$ and $V_c^{ab} \overset{\beta_b}{\xrightarrow{\sim}} V_c^{ab} \otimes V_b^{0b}$.

Remark 2.9. (Localised superselection sectors and superpositions). Consider a k-particle subsystem of n-particle system $q_1, q_2, \ldots, q_n$ where $n > 2$ and $1 < k < n$. Suppose the possible total charges of the subsystem are given by $\mathfrak{L}' \subseteq \mathfrak{L}$ and that $|\mathfrak{L}'| > 1$. While the elements of $\mathfrak{L}'$ correspond to superselection sectors locally (i.e. with respect to the k-particle subsystem), they do not define superselection sectors in the context of the larger system. E.g. take

$$V^{q_1\cdots q_n} \cong \bigoplus_{Q,X} V_X^{q_1\cdots q_k} \otimes V_Q^{X q_{k+1}\cdots q_n} \tag{2.1.21}$$

where $\mathfrak{L}'$ consists of the labels X for which $V_Q^{X q_{k+1}\cdots q_n}$ is nonzero. Indeed, the superselection sectors $\{V_X^{q_1\cdots q_k}\}_{X\in\mathfrak{L}'}$ of the subsystem are entangled with the rest of the system in (2.1.21). Crucially, this means that when we consider fusion states associated to the larger system, *it is possible to observe linear superpositions over the superselection sectors of the subsystem.* Typically, interactions between the subsystem and the rest of the system induce transitions between superselection sectors of the subsystem. On the other hand, Q indexes the *global* superselection sectors in (2.1.21). We cannot observe superpositions over the spaces $\{V_Q^{q_1\cdots q_n}\}_Q$ whence *the total charge Q of the whole system is fixed* e.g. if $q_1, \cdots, q_n$ are initialised from the vacuum then the charge of the whole system is trivial. This can be viewed as conservation of charge.

2.1.4 F-Matrices

For an n-particle fusion space $V^{q_1\cdots q_n}$ let $\mathscr{D}_1$ and $\mathscr{D}_2$ be decompositions of this space corresponding to distinct fusion trees. By associativity, we have an isomorphism

$$\mathcal{F} : \mathscr{D}_1 \to \mathscr{D}_2 \tag{2.1.22}$$

By fixing a basis of fusion states on the constituent triangular spaces for $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively, we can obtain a matrix representation of $\mathcal{F}$. Note that this is simply a change of basis matrix on $V^{q_1\cdots q_n}$. Further observe that $\mathcal{F}$ is given by any sequence of so-called *F-moves* that transform between decompositions of the form

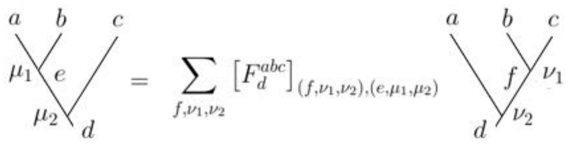
Such transformations are realised by the *F-matrices* of a theory. These are matrices F_d^{abc} which are automorphisms on V_d^{abc} for any $a, b, c, d \in \mathfrak{L}$ where

$$F_d^{abc} : \bigoplus_{e\in\mathfrak{L}} V_e^{ab} \otimes V_d^{ec} \xrightarrow{\sim} \bigoplus_{f\in\mathfrak{L}} V_d^{af} \otimes V_f^{bc} \tag{2.1.23}$$

That is, F_d^{abc} is a *unitary matrix* transforming between orthonormal bases

$$\Big\{ |ab \to e; \mu_1\rangle \, |ec \to d; \mu_2\rangle \Big\}_{e,\mu_1,\mu_2} \quad \text{and} \quad \{|af \to d; \nu_2\rangle \, |bc \to f; \nu_1\rangle\}_{f,\nu_1,\nu_2} \tag{2.1.24}$$

In the graphical calculus,



(2.1.25)

Terminology 2.10.

(i) The distinct fusion trees associated to a fusion space specify distinct bases on the fusion space and are thus also referred to as *fusion bases*.

(ii) The entries of F-matrices are called *F-symbols* (or *$6j$-symbols*).

2.1.5 Coherence I: Pentagon and Triangle Equations

Certain maps between fusion spaces must be 'compatible' with one another. That is, it is sometimes required that all distinct sequences of isomorphisms between two given spaces should coincide: such compatibility requirements are called *coherence conditions*.

Recall the isomorphism $\mathcal{F}$ from (2.1.22). It may be possible that multiple distinct sequences of F-moves realise $\mathcal{F}$. Having fixed a basis on the triangular spaces, the matrix representation of $\mathcal{F}$ must be the same for all such sequences. Remarkably, this coherence condition is fulfilled if all F-symbols satisfy the *pentagon equation* (2.1.29) below, which may be written

$$\sum_{p,r}(F_e^{abr} \otimes \mathrm{id}_{V_r^{cd}})(\mathrm{id}_{V_p^{ab}} \otimes F_e^{pcd}) = \sum_{q,s,t}(\mathrm{id}_{V_e^{as}} \otimes F_s^{bcd})(F_e^{atd} \otimes \mathrm{id}_{V_t^{bc}})(F_q^{abc} \otimes \mathrm{id}_{V_e^{qd}}) \quad (2.1.26)$$

for all $a, b, c, d, e \in \mathfrak{L}$. Fixing the fusion states in the initial and terminal fusion basis, we obtain an entry-wise form of (2.1.26) which is useful for direct calculations. Fix initial state

$$|ab \to p; \alpha\rangle\, |pc \to q; \beta\rangle\, |qd \to e; \lambda\rangle$$

and terminal state

$$|as \to e; \rho\rangle\, |br \to s; \delta\rangle\, |cd \to r; \gamma\rangle$$

This gives us

$$\begin{aligned}
&\sum_{\sigma}\left[F^{abr}_{e}\right]_{(s,\delta,\rho)(p,\alpha,\sigma)}\left[F^{pcd}_{e}\right]_{(r,\gamma,\sigma)(q,\beta,\lambda)} \\
&=\sum_{t,\mu,\nu,\eta}\left[F^{bcd}_{s}\right]_{(r,\gamma,\delta)(t,\mu,\eta)}\left[F^{atd}_{e}\right]_{(s,\eta,\rho)(q,\nu,\lambda)}\left[F^{abc}_{q}\right]_{(t,\mu,\nu)(p,\alpha,\beta)}
\end{aligned} \tag{2.1.27}$$

In the multiplicity-free case, (2.1.27) is simply

$$\left[F^{abr}_{e}\right]_{sp}\left[F^{pcd}_{e}\right]_{rq}=\sum_{t}\left[F^{bcd}_{s}\right]_{rt}\left[F^{atd}_{e}\right]_{sq}\left[F^{abc}_{q}\right]_{tp} \tag{2.1.28}$$

The pentagon equation:

$$\begin{tikzcd}
 & & \bigoplus_{p,r} V^{ab}_{p}\otimes V^{pr}_{e}\otimes V^{cd}_{r} \arrow[drr, "\sum_{r} F^{abr}_{e}\otimes \mathrm{id}_{V^{cd}_{r}}"] & & \\
\bigoplus_{p,q} V^{ab}_{p}\otimes V^{pc}_{q}\otimes V^{qd}_{e} \arrow[urr, "\sum_{p} \mathrm{id}_{V^{ab}_{p}}\otimes F^{pcd}_{e}"] \arrow[dr, "\sum_{q} F^{abc}_{q}\otimes \mathrm{id}_{V^{qd}_{e}}"'] & & & & \bigoplus_{r,s} V^{as}_{e}\otimes V^{br}_{s}\otimes V^{cd}_{r} \\
 & \bigoplus_{q,t} V^{at}_{q}\otimes V^{bc}_{t}\otimes V^{qd}_{e} \arrow[rr, "\sum_{t} F^{atd}_{e}\otimes \mathrm{id}_{V^{bc}_{t}}"'] & & \bigoplus_{s,t} V^{as}_{e}\otimes V^{bc}_{t}\otimes V^{td}_{s} \arrow[ur, "\sum_{s} \mathrm{id}_{V^{as}_{e}}\otimes F^{bcd}_{s}"'] &
\end{tikzcd}$$

commutes for all $a,b,c,d,e\in\mathfrak{L}$.

(2.1.29)

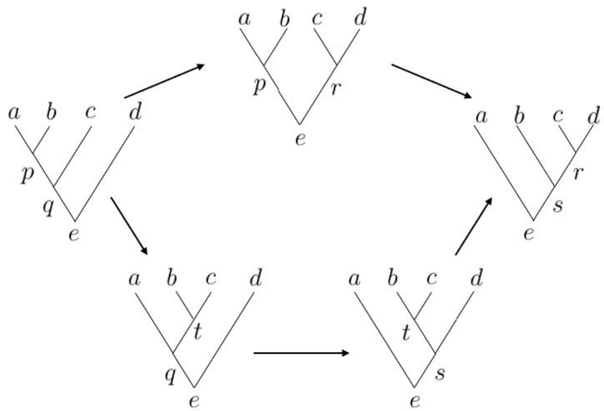


Figure 2.4: An illustration of the fusion trees in (2.1.29).

This pentagon equation has a nice interpretation in terms of associahedra (convex polytopes whose vertices and edges respectively correspond to distinct fusion bases and F-moves between them); see [Kitaev06].

All isomorphisms α and β from (2.1.20) must be compatible with the associativity of fusion (F-moves). This coherence condition is fulfilled if the *triangle equations* (2.1.30) are satisfied.

The triangle equations:

(i)
$$\begin{tikzcd} & V_c^{ab} \arrow[dl, "\alpha_a"'] \arrow[dr, "\beta_b"] & \\ V_a^{a0} \otimes V_c^{ab} \arrow[rr, "F_c^{a0b}"'] & & V_c^{ab} \otimes V_b^{0b} \end{tikzcd}$$

(ii)
$$\begin{tikzcd} & V_c^{ab} \arrow[dl, "\beta_a"'] \arrow[dr, "\beta_c"] & \\ V_a^{0a} \otimes V_c^{ab} \arrow[rr, "F_c^{0ab}"'] & & V_c^{0c} \otimes V_c^{ab} \end{tikzcd}$$

(iii)
$$\begin{tikzcd} & V_c^{ab} \arrow[dl, "\alpha_c"'] \arrow[dr, "\alpha_b"] & \\ V_c^{ab} \otimes V_c^{c0} \arrow[rr, "F_c^{ab0}"'] & & V_c^{ab} \otimes V_b^{b0} \end{tikzcd}$$

commute for all $a, b, c \in \mathfrak{L}$.

(2.1.30)

It can be shown that triangle equations (2.1.30) (ii)-(iii) follow as corollaries of the "fundamental triangle equation" (i) and the pentagon equation [Kitaev06, Lemma E.2.].

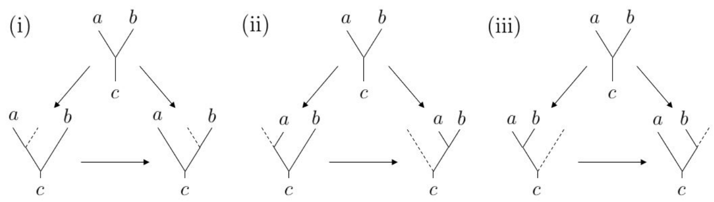


Figure 2.5: An illustration of the fusion trees in (2.1.30).

Remark 2.11. The fundamental triangle equation reads

$$F_c^{a0b}(|\alpha_a\rangle\,|\psi\rangle) = |\psi\rangle\,|\beta_b\rangle \quad , \quad |\psi\rangle \in V_c^{ab} \tag{2.1.31}$$

Considering the matrix form of (2.1.31), $|\alpha_a\rangle$ and $|\beta_b\rangle$ become identified with $1_{\mathbb{C}}$ as per (2.1.20). This means that F_c^{a0b} must be the identity matrix. It follows that all F-matrices of the form F_c^{a0b}, F_c^{0ab} and F_c^{ab0} correspond to the identity map on their respective spaces (and are given by the identity matrix of rank N_c^{ab}), and that the triangle equations will be trivially satisfied.

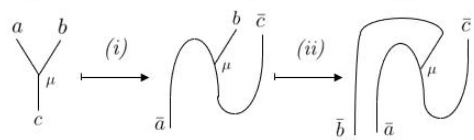
2.1.6 Pivotal Identity and Frobenius-Schur Indicator

For $a, b, c \in \mathfrak{L}$ we define linear ("leg-bending") maps[5] K_c^{ab} and L_c^{ab},

$$K_c^{ab} : V_c^{ab} \longrightarrow V_{\bar{a}a}^{0} \otimes V_c^{ab} \cong V_{\bar{a}c}^{b} \qquad\qquad L_c^{ab} : V_c^{ab} \longrightarrow V_c^{ab} \otimes V_{b\bar{b}}^{0} \cong V_{c\bar{b}}^{a} \tag{2.1.32}$$

These are clearly invertible (whence $N_c^{ab} = N_b^{\bar{a}c} = N_a^{c\bar{b}}$). We have

$$V_c^{ab} \xrightarrow{\left(L_{\bar{a}}^{b\bar{c}}\right)^{-1} \circ K_c^{ab}} V_{\bar{a}}^{b\bar{c}} \xrightarrow{K_{\bar{a}}^{b\bar{c}}} V_{\bar{b}\bar{a}}^{\bar{c}} \tag{2.1.33}$$



[5]In [Bonderson, Kitaev06] and other sources, these maps are respectively written A_c^{ab} and B_c^{ab}. Unlike in [Bonderson], said maps are assumed to be diagonal in [Kitaev06]; here, it will suffice to follow the presentation from the latter.

where the isomorphisms $\gamma_1 := K_{\bar{a}}^{b\bar{c}} \circ \left(L_{\bar{a}}^{b\bar{c}}\right)^{-1} \circ K_c^{ab}$ and $\gamma_2 := L_b^{\bar{c}a} \circ (K_b^{\bar{c}a})^{-1} \circ L_c^{ab}$ correspond to the CPT symmetry of a triangular space. In particular, in [Kitaev06, Theorem E.6.] it is shown that the leg-bending maps are unitary, and that γ_1, γ_2 are unitary and coincide[6]. Applying the map $\gamma_2^{-1} \circ \gamma_1$ to a trivalent vertex, we arrive at the *pivotal identity* shown in Figure 2.6 below.

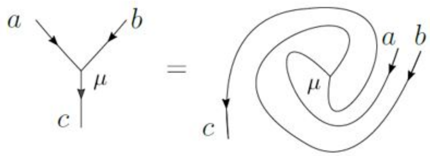
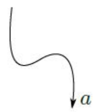


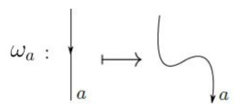
Figure 2.6: The "pivotal identity" tells us that we have equivalence under 2π-rotations of trivalent vertices. The clockwise version of this can be seen by applying $\gamma_1^{-1} \circ \gamma_2$.

Let us consider the following process involving a creation and an annihilation:



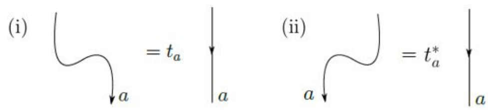
(2.1.34)

Viewing the distortion of the worldline of a as an operator,



(2.1.35)

Noting that the domain is isomorphic to $V_a^{a0} \cong \mathbb{C}$ and the codomain is $V_0^{a\bar{a}} \otimes V_{\bar{a}a}^{0} \cong \mathbb{C}$,



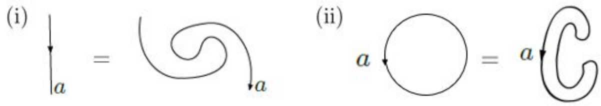
(2.1.36)

where $t_a \in \mathbb{C}$ and (ii) is simply the adjoint of (i). The pivotal identity is useful for proving some properties of t_a. Firstly, note that applying the pivotal identity to a 'cap' gives

(2.1.37)

[6]This is equivalent to the statement that a unitary fusion category admits a pivotal structure. It is conjectured that every fusion category admits a pivotal structure [ENO05].

which allows us to deduce the following identities.

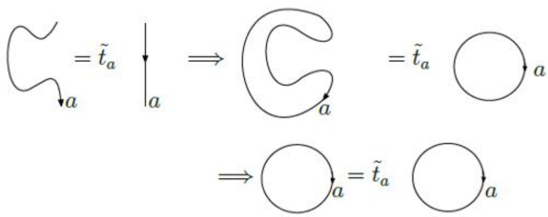


(2.1.38)

Proposition 2.12. (i) $|t_a| = 1$, (ii) $t_{\bar{a}} = t_a^*$

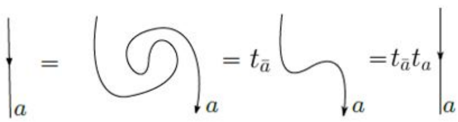
Proof. In the following, we will use the identities (2.1.38).

(i) Let $\tilde{t}_a := |t_a|^2$. Then



The result follows (noting the loop is nonzero since it can be understood as the squared norm of a nonzero vector in $V_0^{a^*a}$).

(ii) Observe that



whence the result follows by (i).

□

The quantity t_a is called the *pivotal coefficient* of a. When $a = \bar{a}$, t_a is the *Frobenius-Schur indicator* and is written $\varkappa_a = t_a$. In particular, note that

$$\varkappa_a = \pm 1 \tag{2.1.39}$$

Remark 2.13.

(i) It is straightforward to show that t_a is gauge-invariant if and only if $a = \bar{a}$ (and so the Frobenius-Schur indicator $\varkappa_a$ is a fixed property of a self-dual anyon a.). When a is non self-dual, it is typical to fix the gauge such that $t_a = 1$. Working in the appropriate gauge, we may thus always straighten distortions of the form (2.1.34) in our calculus, unless a is self-dual with $\varkappa_a = -1$.

(ii) By decorating 'caps' and 'cups' (i.e. creations and annihilations) with an extra degree of freedom (of no physical significance), it is possible to formulate a version of the graphical calculus where distortions of the form (2.1.34) can always be straightened. This scheme is described in [Kitaev06, Section E.2.2].

(iii) Pivotality is equivalent to the existence of of a root of unity t_a for each $a \in \mathfrak{L}$ satisfying the identities in Proposition 2.12 and $t_a^{-1} t_b^{-1} t_c = [F_0^{ab\bar{c}}]_{\bar{a}c} [F_0^{b\bar{c}a}]_{\bar{a}a} [F_0^{\bar{c}ab}]_{\bar{b}b}$ [Wang].

Terminology 2.14. A theory of anyons is called *unimodal* if $\varkappa_a = 1$ for each $a \in \mathfrak{L}$.

Conjecture 2.15. [Wang] For any self-dual $q \in \mathfrak{L}$ such that $N_q^{a\bar{a}} \neq 0$ for some $a \in \mathfrak{L}$, it holds that $\varkappa_q = 1$.[7]

2.1.7 Symmetries of Fusion Coefficients

The symmetries of the fusion coefficients for any given theory of anyons can be summarised by the following three identities for all $a, b, c \in \mathfrak{L}$.

$$N_c^{ab} = N_c^{ba} \tag{2.1.40a}$$

$$N_c^{ab} = N_{\bar{a}}^{b\bar{c}} = N_{\bar{b}}^{\bar{c}a} \tag{2.1.40b}$$

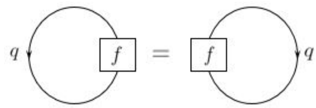
$$N_c^{ab} = N_{\bar{c}}^{\bar{b}\bar{a}} \tag{2.1.40c}$$

Note that (2.1.40a) is just **(F5)**, and that (2.1.40c) corresponds to the isomorphism γ_1 illustrated in (2.1.33). Identity (2.1.40b) can be seen as a simple consequence of **(F3)** and **(F4)** i.e. from $N_{\bar{c}}^{ab} N_0^{\bar{c}c} = N_0^{a\bar{a}} N_{\bar{a}}^{bc}$.

[7] This conjecture is actually believed to hold in the more general setting of pivotal fusion categories.

2.1.8 Sphericality, Quantum Dimension and Normalisation

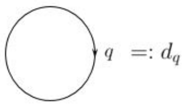
Any theory of anyons satisfies the *spherical property*. In the graphical calculus, this is expressed as



(2.1.41)

for any $q \in \mathfrak{L}$ and permissible diagram f. The spherical property can be attributed to the 'twisting' degree of freedom possessed by anyons.[8]

Thus far, we have assumed that our trivalent vertices carry an implicit normalisation. Let us relax this assumption. Recall the (Hermitian) inner product from (2.1.13)(i). Computing the squared norm of a cup in $V_0^{q^*q}$, we write



(2.1.42)

where $d_q \in \mathbb{R}_{>0}$ is called the *quantum dimension* of q. As an immediate consequence of sphericality, we have that

$$d_q = d_{\bar{q}} \tag{2.1.43}$$

The *total quantum dimension* of a theory of anyons is defined as

$$\mathcal{D} = \sum_{q \in \mathfrak{L}} \sqrt{d_q^2} \tag{2.1.44}$$

Remark 2.16. (Normalisation of trivalent vertices).

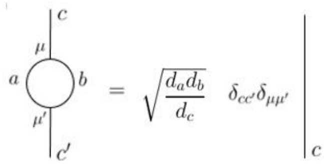
(i) Following (2.1.42), the normalisation factor for any cap or cup associated to q is given by $d_q^{-\frac{1}{2}}$. Similarly, trivalent vertices are assigned a consistent normalisation factor; the convention for this normalisation is

(2.1.45)

[8] Mathematically, this view is justified by observing that a ribbon fusion category is precisely a *spherical* braided fusion category. Furthermore, a unitary braided fusion category admits a unique unitary ribbon structure [ENO05, Gal14].

Further details on this choice of normalisation can be found in Appendix B of Chapter 4.

(ii) Unless stated otherwise, we shall henceforth assume that all trivalent vertices (including caps and cups) appearing in our graphical calculus are unnormalised. For instance, we now have that



(2.1.46)

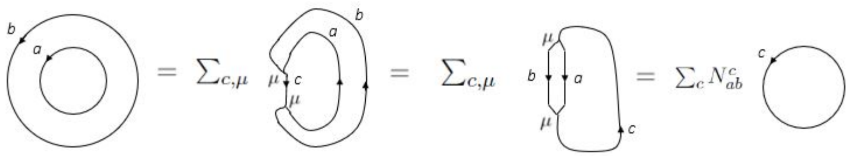
It is easy to check that the F-symbols (2.1.25) are unaffected by our normalisation.

(iii) It is important to note that the annihilation and creation events in distortions of the form (2.1.34) *are not regarded as trivalent vertices*; as a consequence of the pivotal identity, (2.1.34) is just a straight line (up to a pivotal coefficient). For this reason, one should never assign any normalisation factors to such caps and cups. As an example, consider some physical process that corresponds to a knot K in the graphical calculus (oriented and labelled by $q \in \mathfrak{L}$). In order to normalise the knot, we always scale it by a factor of $d_q^{-\mathcal{B}(K)}$ where $\mathcal{B}$ denotes the bridge number.

Proposition 2.17.

$$d_a d_b = \sum_c N_c^{ab} d_c \tag{2.1.47}$$

Proof.



where the first and third equalities respectively use the completeness relation and inner product, and the deformation in the second equality can be seen to follow from the pivotal identity (see Figure 2.7 below). □

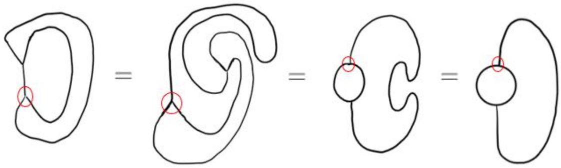


Figure 2.7: Illustration of how pivotality can be applied in the proof of Proposition 2.17. One of the trivalent vertices has been circled for ease of interpretation. A slightly less diagrammatic proof (using the leg-bending maps) can be found in [Wolf]. The depicted deformation is a standard move that is frequently used in the graphical calculus.

From (2.1.47), it is clear that

$$d_0 = 1 \tag{2.1.48}$$

Definition 2.18. The *fusion matrix* N^a of $a \in \mathfrak{L}$ is given by $[N^a]_{bc} = N_c^{ab}$.

Actually, (2.1.47) tells us that we can deduce the quantum dimensions of a theory given its fusion rules. This can be seen as follows. Let k be the rank of the theory and assign $\mathfrak{L}$ an index set $\{0, 1, \ldots, k-1\}$. Letting j denote the index for $b \in \mathfrak{L}$, note that (2.1.47) is the j^{th} row of the eigenvalue equation

$$N^a \boldsymbol{d} = d_a \boldsymbol{d} \tag{2.1.49}$$

where $\boldsymbol{d} = (d_0, \ldots, d_{k-1})^T \in \mathbb{C}^k$. The Frobenius-Perron theorem for nonnegative matrices tells us that a fusion matrix has an eigenvalue $\lambda' \in \mathbb{R}_{>0}$ such that all other eigenvalues λ satisfy $|\lambda| \leq \lambda'$. Furthermore, λ' has algebraic and geometric multiplicity one (i.e. λ' is unique and nondegenerate), and has a corresponding eigenvector $\boldsymbol{x}$ all of whose components lie in $\mathbb{R}_{>0}$ (i.e. a 'positive eigenvector'). Any other positive eigenvector of the fusion matrix is a multiple of $\boldsymbol{x}$. The eigenvalue λ' is called the Frobenius-Perron eigenvalue. Inspecting (2.1.49), we see that d_a is the Frobenius-Perron eigenvalue of N_a, and that each fusion matrix has the (normalised) positive eigenvector $\boldsymbol{d} \backslash \mathcal{D}$ corresponding to its Frobenius-Perron eigenvalue.[9] From this point of view, the triviality of d_0 is an obvious consequence of $N^{\mathbf{1}}$ being the identity matrix. It is easy to see that

$$N^{\bar{a}} = (N^a)^T = C N^a C \tag{2.1.50}$$

[9]Mathematicians call the Frobenius-Perron eigenvalue of N^a the *Frobenius-Perron dimension* of a in order to distinguish it from the quantum dimension d_a. This is since these two quantities may differ by a sign in the more general setting of a (spherical) fusion category. This distinction is not necessary for our purposes as we are implicitly working within the confines of a unitary fusion category: quantum mechanics demands the presence of a Hermitian inner product on the triangular spaces (which ensures that the quantum dimensions are positive), and so the quantum dimension will always coincide with the Frobenius-Perron dimension.

Then (2.1.43) can also be seen by noting that a (fusion) matrix and its transpose have the same spectrum. It is also straightforward to check that the dimension of the fusion space of n charges a in superselection sector b is given by

$$\dim(V_b^{a\cdots a}) = \left[(N^a)^{n-1}\right]_{ab} \tag{2.1.51}$$

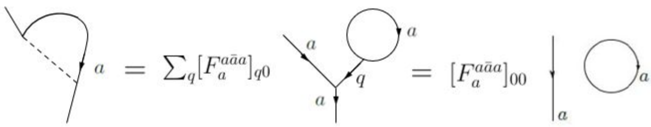
Further note that if a is self-dual, then N^a is symmetric whence all of its eigenvalues are real; it follows that all eigenvalues λ (excluding the Frobenius-Perron eigenvalue) of N^a satisfy $|\lambda| < d_a$. Since N^a is unitarily diagonalisable, then for large n

$$(N^a)^n \sim d_a^n P_{\boldsymbol{d}} \tag{2.1.52}$$

(where $P_{\boldsymbol{d}}$ is the orthogonal projector onto $\boldsymbol{d}\backslash\mathcal{D}$) since all other terms in the diagonalisation become negligible for large n. Plugging (2.1.52) into (2.1.51), we see that

$$\dim(V_b^{a\cdots a}) \sim \frac{d_a^n d_b}{\mathcal{D}^2} \tag{2.1.53}$$

Finally, note the following key relationship between the quantum dimension d_a and the F-symbol $[F_a^{a\bar{a}a}]_{00}$.



from which it follows that

$$[F_a^{a\bar{a}a}]_{00} = \frac{t_a}{d_a} \tag{2.1.54}$$

and that $t_0 = 1$. There are various other identities relating F-symbols and quantum dimensions; for instance, the leg-bending maps can be understood in terms of F-symbols and quantum dimensions (see e.g. [Bonderson, Wolf]).

2.2 Braiding

For a theory of anyons, one of the most valuable (if not *the* most valuable) pieces of data is the exchange statistics amongst its constituent charges. Specifically, we want explicit matrix representations for exchange operators between pairs of anyons. The first thing to note is that in order to obtain matrix representations, we must fix a basis on the pertinent fusion space. Doing so involves

(a) Specifying a fusion basis

(b) Fixing a basis on the triangular spaces

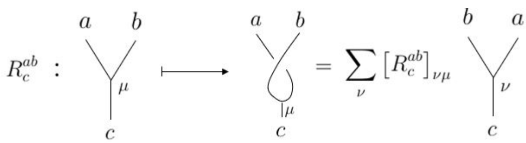
From a practical perspective, point (a) can be seen as fixing a measurement basis (which simply corresponds to specifying the order in which we measure the fusion outcomes of a system of anyons). However, point (b) corresponds to a certain degeneracy in our algebraic description. If our exchange matrices depend on some artificial choice of basis, can they tell us anything about what we might expect to observe? This is further addressed in Section 2.4.
Returning to point (a), we will see that for a given pair of anyons, it suffices to consider their exchange matrix in a fusion basis where the pair is in a direct fusion channel (this is called an *R-matrix*). The exchange matrix for a pair of anyons in an indirect fusion channel can then be found by conjugating their R-matrix by some appropriate sequence of F-matrices.

2.2.1 R-Matrices

The *R-matrix* R^{ab}_c describes the *clockwise* exchange of two charges a and b in a direct fusion channel of total charge c is an isomorphism

$$R^{ab}_c : V^{ab}_c \xrightarrow{\sim} V^{ba}_c \tag{2.2.1}$$

Specifying a basis on the spaces in (2.2.1), the action of R^{ab}_c is given by



(2.2.2)

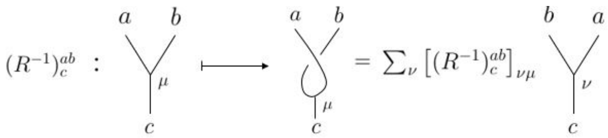
in the graphical calculus. We may also define a matrix

$$R^{ab} : V^{ab} \xrightarrow{\sim} V^{ba} \tag{2.2.3}$$

where $R^{ab} := \bigoplus_c R^{ab}_c$ is block-diagonal with block dimensions $\{N^{ab}_c\}_c$. We similarly let $(R^{-1})^{ab}$ denote the *anticlockwise* exchange of a and b i.e.

$$(R^{ab})^{-1} = (R^{-1})^{ba} \tag{2.2.4}$$

where



(2.2.5)

Remark 2.19.

(i) Since the R-matrices for a theory of anyons correspond to (statistical) evolutions in a closed quantum system, they are *unitary.*

(ii) Braiding with the vacuum is trivial (it is equivalent to doing nothing) i.e.

$$R^{a0}_a = R^{0a}_a = 1 \quad , \quad a \in \mathfrak{L} \tag{2.2.6}$$

(iii) Given any theory of anyons (with multiplicity), it is unknown if there will always exist some gauge (i.e. some permissible choice of basis on the triangular spaces) such that all matrices R^{ab}_c (with $N^{ab}_c > 1$) are diagonal [Wang].

2.2.2 Coherence II: Hexagon Equations

Consider n-particle fusion space $V^{q_1 \cdots q_n}$ where $q_1, \ldots, q_n \in \mathfrak{L}$ and $n \geq 3$. Let s and s' be any two distinct permutations of the string $q_1 \ldots q_n$. Let $\mathcal{D}$ and $\mathcal{D}'$ be any decomposition of V^s and $V^{s'}$ respectively. It may be possible that multiple distinct sequences of F and R-moves realise an isomorphism $\mathcal{B} : \mathcal{D} \to \mathcal{D}'$ (corresponding to the action of some fixed n-braid). Having specified a basis on the constituent triangular spaces of V^s and $V^{s'}$, the resulting matrix representation of $\mathcal{B}$ must be the same for all such sequences. This coherence condition is fulfilled if al of thel F and R-symbols of the theory satisfy the *hexagon equations* (2.2.7) below.

The hexagon equations:

$$\begin{tikzcd}
& \bigoplus\limits_y V_d^{ay} \otimes V_y^{bc} \arrow[rr, "\bigoplus\limits_y \left(\mathrm{id}_{V_y^{bc}} \otimes R_d^{ay}\right) \cdot \mathrm{swap}"] & & \bigoplus\limits_y V_y^{bc} \otimes V_d^{ya} \arrow[dr, "F_d^{bca}"] & \\
\bigoplus\limits_x V_x^{ab} \otimes V_d^{xc} \arrow[ur, "F_d^{abc}"] \arrow[dr, "\bigoplus\limits_x R_x^{ab} \otimes \mathrm{id}_{V_d^{xc}}"'] & & & & \bigoplus\limits_z V_d^{bz} \otimes V_z^{ca} \\
& \bigoplus\limits_x V_x^{ba} \otimes V_d^{xc} \arrow[rr, "F_d^{bac}"'] & & \bigoplus\limits_z V_d^{bz} \otimes V_z^{ac} \arrow[ur, "\bigoplus\limits_z \mathrm{id}_{V_d^{bz}} \otimes R_z^{ac}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \bigoplus\limits_y V_d^{ay} \otimes V_y^{bc} \arrow[rr, "\bigoplus\limits_y \left(\mathrm{id}_{V_y^{bc}} \otimes \left(R^{-1}\right)_d^{ay}\right) \cdot \mathrm{swap}"] & & \bigoplus\limits_y V_y^{bc} \otimes V_d^{ya} \arrow[dr, "F_d^{bca}"] & \\
\bigoplus\limits_x V_x^{ab} \otimes V_d^{xc} \arrow[ur, "F_d^{abc}"] \arrow[dr, "\bigoplus\limits_x \left(R^{-1}\right)_x^{ab} \otimes \mathrm{id}_{V_d^{xc}}"'] & & & & \bigoplus\limits_z V_d^{bz} \otimes V_z^{ca} \\
& \bigoplus\limits_x V_x^{ba} \otimes V_d^{xc} \arrow[rr, "F_d^{bac}"'] & & \bigoplus\limits_z V_d^{bz} \otimes V_z^{ac} \arrow[ur, "\bigoplus\limits_z \mathrm{id}_{V_d^{bz}} \otimes \left(R^{-1}\right)_z^{ac}"'] &
\end{tikzcd}$$

commute for all $a, b, c, d \in \mathfrak{L}$.

(2.2.7)

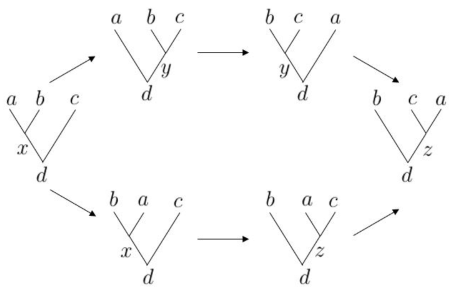


Figure 2.8: An illustration of the fusion trees in (2.2.7).

Note that the only difference between the two hexagon equations is the orientation of the R-moves. Fix initial state $|ab \to x; \alpha\rangle\, |xc \to d; \lambda\rangle$ and terminal state $|bz \to d; \rho\rangle\, |ca \to z; \gamma\rangle$ in (2.2.7). This gives us

$$\sum_{y,\beta,\mu,\sigma} \left[F_d^{bca}\right]_{(z,\gamma,\rho)(y,\beta,\sigma)} \left[R_d^{ay}\right]_{\sigma\mu} \left[F_d^{abc}\right]_{(y,\beta,\mu)(x,\alpha,\lambda)} = \sum_{\delta,\epsilon} \left[R_z^{ac}\right]_{\gamma\epsilon} \left[F_d^{bac}\right]_{(z,\epsilon,\rho)(x,\delta,\lambda)} \left[R_x^{ab}\right]_{\delta\alpha} \tag{2.2.8a}$$

$$\sum_{y,\beta,\mu,\sigma} \left[F_d^{bca}\right]_{(z,\gamma,\rho)(y,\beta,\sigma)} \left[\left(R^{-1}\right)_d^{ay}\right]_{\sigma\mu} \left[F_d^{abc}\right]_{(y,\beta,\mu)(x,\alpha,\lambda)} = \sum_{\delta,\epsilon} \left[\left(R^{-1}\right)_z^{ac}\right]_{\gamma\epsilon} \left[F_d^{bac}\right]_{(z,\epsilon,\rho)(x,\delta,\lambda)} \left[\left(R^{-1}\right)_x^{ab}\right]_{\delta\alpha} \tag{2.2.8b}$$

which in a multiplicity-free theory becomes

$$\sum_{y} \left[F_d^{bca}\right]_{zy} \left[R_d^{ay}\right] \left[F_d^{abc}\right]_{yx} = \left[R_z^{ac}\right] \left[F_d^{bac}\right]_{zx} \left[R_x^{ab}\right] \tag{2.2.9a}$$

$$\sum_{y} \left[F_d^{bca}\right]_{zy} \left[\left(R^{-1}\right)_d^{ay}\right] \left[F_d^{abc}\right]_{yx} = \left[\left(R^{-1}\right)_z^{ac}\right] \left[F_d^{bac}\right]_{zx} \left[\left(R^{-1}\right)_x^{ab}\right] \tag{2.2.9b}$$

Remark 2.20.

(i) We have seen that all the F-matrices for a theory of anyons must be unitary (this follows from the Hermitian inner product structure on our fusion spaces), as well as the R-matrices. The unitarity of both types of matrices is a simple consequence of the postulates of quantum mechanics. From an algebraic perspective, one might wonder the following: given the unitary F-matrices for a theory of anyons (i.e. unitary solutions of the pentagon equations for some set of fusion rules), is it ever possible to find any non-unitary R-matrices that solve the hexagon equations with respect to these F-matrices? The answer is no [Gal14].

(ii) The hexagon equations ensure that the Yang-Baxter equation is satisfied (meaning our strands have braid isotopy in the graphical calculus, which is consistent with what is required for the worldlines of anyons), and essentially that braiding is well-defined in our algebraic model.

2.2.3 B-Matrices

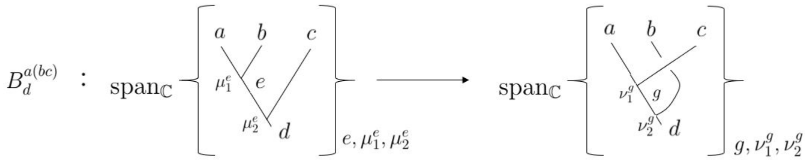
We can also obtain representations of the exchange operator for two adjacent charges a and b by considering its action with respect to a fusion basis in which a and b are in an *indirect* fusion channel. As mentioned above, such a representation can be determined by

(1) Transforming into a fusion basis where the charges are in a *direct* fusion channel

(2) Applying the R-matrix

(3) Transforming back to the original fusion basis

Since steps (1) and (3) correspond to applying a sequence of F-matrices, it is clear that the resulting exchange matrix will always be unitary (as required). Below is the simplest example of such a procedure (where we call the resulting matrix a *B-matrix*).



where

$$\begin{tikzcd} \bigoplus_e V_e^{ab} \otimes V_d^{ec} \arrow[r, "F_d^{abc}"] \arrow[d, "B_d^{a(bc)}"'] & \bigoplus_f V_d^{af} \otimes V_f^{bc} \arrow[d, "R^{bc}"] \\ \bigoplus_g V_g^{ac} \otimes V_d^{gb} \arrow[r, "F_d^{acb}"] & \bigoplus_f V_d^{af} \otimes V_f^{cb} \end{tikzcd} \tag{2.2.10}$$

That is,

$$B_d^{a(bc)} = \left(F_d^{acb}\right)^{\dagger} R^{bc} F_d^{abc} \tag{2.2.11}$$

Similarly, we write

$$(B^{-1})_d^{a(bc)} = \left(F_d^{acb}\right)^{\dagger} (R^{-1})^{bc} F_d^{abc} \tag{2.2.12}$$

Remark 2.21. (Braid representations). The action of braiding on a given system of anyons defines a local "unitary linear representation of the coloured braid groupoid".[10] In the instance where all anyons have the same charge, specifying a fusion basis (and fixing a basis on the triangular spaces) defines s a unitary linear representation of the braid group. For instance, suppose we have a representation $\rho : B_n \to \mathrm{U}(s)$ for the n-particle system $V_x^{q\cdots q}$ of dimension s (where $n \geq 2$). Each $\rho(\sigma_i)$ has the same set (without multiplicity) of eigenvalues $\{r_1, \ldots, r_k\}$. The representation ρ can thus be seen as a restriction of a representation

$$\tilde{\rho} : \mathbb{C}[B_n] \to H_n(Q, k) \to \mathrm{U}(s) \tag{2.2.13}$$

factoring through the generalised Hecke algebra $H_n(Q, k)$ (which is defined to be the quotient of $\mathbb{C}[B_n]$ by the ideal $Q(\sigma_i)$ that is generated by $\Pi_{j=1}^{k}(\sigma_i - r_j)$).

[10] The meaning of this is expounded upon in Appendix A of Chapter 3. The 'locality' of the representation refers to the fact that the action of a single exchange can only be nontrivial for the 'local' part of the fusion space (associated to the exchanged pair).

2.2.4 Abelianity

A charge $a \in \mathfrak{L}$ is called *abelian* if it has a unique fusion channel with any $b \in \mathfrak{L}$. Otherwise, a is called *nonabelian.*[11]

- $a \in \mathfrak{L}$ is called abelian if $\sum_c N_c^{ab} = 1$ for each $b \in \mathfrak{L}$. A theory is called abelian if every charge is abelian; in this case, note that $(\mathfrak{L}, \times)$ is a finite abelian group.[12]
- $a \in \mathfrak{L}$ is called nonabelian if there exists some $b \in \mathfrak{L}$ such that $\sum_c N_c^{ab} > 1$. Clearly, any such b is also nonabelian.

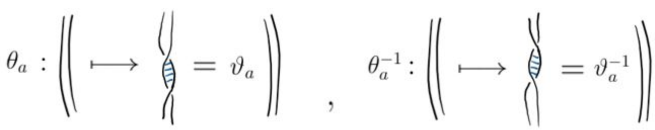
Proposition 2.22.[13] If a is abelian then $d_a = 1$. Otherwise, $d_a \geq \sqrt{2}$.

Proof. If a is abelian then $d_a^2 = d_a d_{\bar{a}} = d_{\mathbf{1}} = 1$. Next, let us show that $\sum_c N_c^{a\bar{a}} = 1$ (i.e. that a is 'invertible') only if a is abelian. If a is invertible, then for any $b \in \mathfrak{L}$ we have that $\sum_e N_b^{\bar{a}e} N_e^{ab} = 1$, whence $\sum_e N_e^{ab} = 1$ for all b (i.e. a is abelian). It follows that for a nonabelian, $d_a^2 = \sum_c N_c^{a\bar{a}} d_c > 1$. The lower bound for the quantum dimension of a nonabelian charge a is only attained when $a \times \bar{a} = \mathbf{1} + x$ where x is abelian. $\square$

Note that Proposition 2.22 does not make use of commutativity **(F5)**. This means that it will also hold in the more general setting of a unitary (spherical) fusion category.

2.3 Twisting

Anyons can perform 2π-rotations. In order to diagrammatically track such operations, let us promote our worldlines to worldribbons. Then 2π twists of the charge $a \in \mathfrak{L}$ are given by the maps



(2.3.1)

where $\vartheta_a \in \mathbb{C}^\times$ is called the *topological spin* of a and corresponds to a 2π clockwise rotation.

[11] The nomenclature derives from the fact that if a has multiple fusion channels with some charge b, then the exchange matrix for a and b will be defined on a space of dimension > 1. Then for exchanges between 'nonabelian' charges, the statistical evolution will typically depend on the order of exchanges.

[12] Indeed, finite abelian groups can give rise to theories of anyons (e.g. $\mathbb{Z}_n$ theories). Some further details can be found in [Bonderson, Simon].

[13] When $d_a < 2$, the Jones index theorem from subfactor theory tells us that $d_a \in \{2\cos(\frac{\pi}{n})\}_{n=3}^{\infty}$.

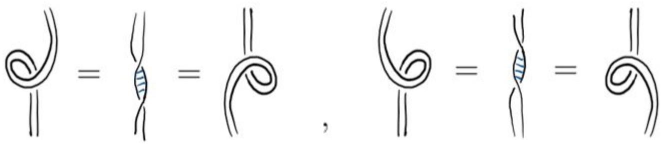
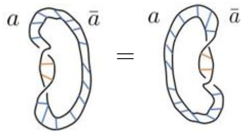


Figure 2.9: Twists may also be seen as above by pulling the ribbon taut.

The topological spins must satisfy the followng two conditions for all labels.[14]



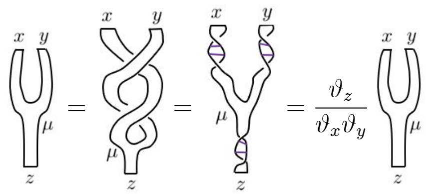
(2.3.2a)

$$\sum_{\lambda} [R^{yx}_z]_{\mu\lambda} [R^{xy}_z]_{\lambda\nu} = \frac{\vartheta_z}{\vartheta_x \vartheta_y} \delta_{\mu\nu} \tag{2.3.2b}$$

In particular, (2.3.2a) says that twists can be pushed around a closed loop and that

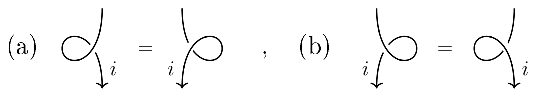
$$\vartheta_a = \vartheta_{\bar{a}} \tag{2.3.3}$$

The *ribbon relation* (2.3.2b) is illustrated by (2.3.4), where the second equality can be seen by pulling taut the ribbon from the tops and bottom. The operator $R^{yx} \circ R^{xy}$ is called the *monodromy operator* of x and y, and is denoted by M^{xy}.

$(R^{yx} \circ R^{xy})$ 

(2.3.4)

It is slightly tedious to draw diagrams with ribbons. We pass back to strands by writing



(2.3.5)

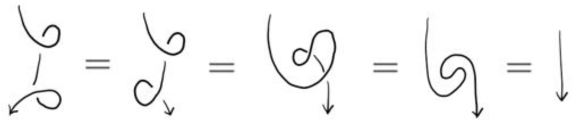
where (a) and (b) respectively correspond to clockwise and anticlockwise twists (compare

[14] From a mathematical perspective, the twisting of anyons is well-motivated. This is since a unitary braided fusion category admits a *unique* unitary ribbon structure [ENO05, Gal14]. The conditions (2.3.2a)-(2.3.2b) come from the definition of ribbon structure. Nonetheless, the discourse of this section can be understood simply by manipulating diagrams: this is the power of the graphical calculus.

to Figure 2.9). Indeed, note that

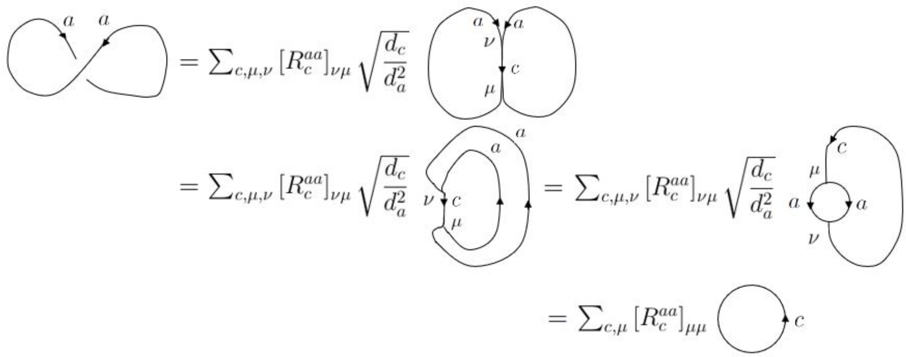
(2.3.6)

whence (2.3.5)(a) is consistent with (2.3.2a) (and similarly for (2.3.5)(b)). Also note that this assignment is consistent with braid isotopy and pivotality i.e.



(2.3.7)

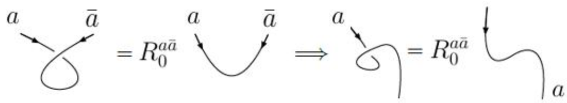
where the 2nd and 3rd equalities follow from braid isotopy, and the 4th from pivotality. We can easily derive a few more identities for the topological spin. Note that



whence we have

$$\vartheta_a = \frac{1}{d_a} \sum_c d_c \operatorname{tr}\left(R^{aa}_c\right) \tag{2.3.8}$$

Also,



from which it follows that

$$\vartheta_a = t_a \left(R_0^{\bar{a}a}\right)^* \tag{2.3.9}$$

We can deduce that $\vartheta_a \in \mathrm{U}(1)$ and $\vartheta_0 = 1$ (as expected).

Definition 2.23. The *T-matrix* of a theory of anyons is given by $[T]_{ab} = \vartheta_a \delta_{ab}$.

Remark 2.24. (Rationality of spins). The Anderson-Moore-Vafa theorem [AM88, Vafa88] tells us that the topological spin is a root of unity for each $a \in \mathfrak{L}$. For a proof, we refer the reader to [Kitaev06, Theorem E.10]. It follows that the monodromy of any pair of charges has finite order.

2.4 Gauge Freedom

There is generally some mathematical redundancy amongst the F and R-symbols describing a theory, arising from the $\mathrm{U}(N_c^{ab})$ freedom when fixing an orthonormal basis on triangular spaces V_c^{ab}. In this context, a change of (orthonormal) basis is referred to as a *gauge transformation*. From a practical perspective, this is important since we should not expect gauge-variant quantities to correspond to physical observations.

Let u_c^{ab} denote a gauge transformation on V_c^{ab}, where

$$|ab \to c; \mu\rangle = \sum_{\mu'} [u_c^{ab}]_{\mu'\mu} \, |ab \to c; \mu'\rangle \tag{2.4.1}$$

Note that gauge-transformed F and R-matrices will also be unitary (since our gauge transformations are unitary). R-matrices transform as[15]

$$\left(R_c^{ab}\right)' = u_c^{ba} R_c^{ab} \left(u_c^{ab}\right)^\dagger \tag{2.4.2}$$

and F-symbols transform as[16]

$$\left[\left(F_d^{abc}\right)'\right]_{(f,\nu_1',\nu_2'),(e,\mu_1',\mu_2')} = \sum_{\mu_1,\mu_2,\nu_1,\nu_2} [u_d^{af}]_{\nu_2'\nu_2} [u_f^{bc}]_{\nu_1'\nu_1} \left[F_d^{abc}\right]_{(f,\nu_1,\nu_2),(e,\mu_1,\mu_2)} [\left(u_e^{ab}\right)^\dagger]_{\mu_1\mu_1'} [(u_d^{ec})^\dagger]_{\mu_2\mu_2'} \tag{2.4.3}$$

In the absence of multiplicities, the above transformations are simply

$$\left(R_c^{ab}\right)' = \frac{u_c^{ba}}{u_c^{ab}} R_c^{ab} \quad , \quad \left[\left(F_d^{abc}\right)'\right]_{fe} = \frac{u_d^{af} u_f^{bc}}{u_e^{ab} u_d^{ec}} \left[F_d^{abc}\right]_{fe} \tag{2.4.4}$$

We also have

$$u_a^{a0} = u_a^{0a} = 1 \quad , \quad a \in \mathfrak{L} \tag{2.4.5}$$

so that gauge transformations respect the triviality of both braiding with the vacuum,

[15] The form of the gauge transformations in [Bonderson] are the transpose of the form in which they appear here. This is since we have opposite conventions for the order of indices in our F, R, and gauge transformation matrices.

[16] See footnote 15.

and F-matrices acting on a triple of labels containing $\mathbf{1}$.

2.4.1 Some Gauge-Invariant Quantities

We will now highlight a few important gauge-invariant quantities.

(1) It is clear that the fusion rules and quantum dimensions are gauge-invariant.

(2) The topological spins are gauge-invariant. This can be seen from

$$\vartheta_a = \frac{1}{d_a}\sum_c d_c \operatorname{tr}\left(u_c^{aa} R_c^{aa} (u_c^{aa})^\dagger\right) = \frac{1}{d_a}\sum_c d_c \operatorname{tr}\left(R_c^{aa}\right)$$

(3) The monodromy operator M^{ab} is gauge-invariant. In order to see this, note that

$$\left(R_c^{ba}\right)' \circ \left(R_c^{ab}\right)' = u_c^{ab} R_c^{ba} R_c^{ab} (u_c^{ab})^\dagger = \frac{\vartheta_c}{\vartheta_a \vartheta_b} u_c^{ab} (u_c^{ab})^\dagger = R_c^{ba} \circ R_c^{ab}$$

(4) It is clear that symbols of the form R_c^{aa} will be gauge-invariant when $N_c^{aa} = 1$.

(5) Note (in a multiplicity-free instance) that symbols of the form $\left[F_b^{abc}\right]_{bb}$ are gauge-invariant. As an immediate consequence, we see that the symbol $[F_a^{aaa}]_{00}$ is gauge-invariant when a is self-dual, whence from (2.1.54) it follows that the Frobenius-Schur indicator $\varkappa_a$ is gauge-invariant.

(6) Consider V_d^{abc} (where all of the fusion outcomes are multiplicity-free), and suppose we have a fixed fusion channel $|ab \to e\rangle\, |ec \to d\rangle$. If we fuse b and c first instead, then the probability $p(bc \to f | ab \to e)$ that b and c will fuse to a charge f is given by $|\left[F_d^{abc}\right]_{fe}|^2$. It is trivial to verify that this probability is gauge-invariant.

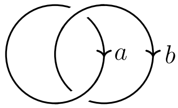
2.5 S-Matrix and Modularity

2.5.1 Properties of the S-Matrix

We will summarise the properties of the S-matrix and highlight its relation to modularity; further details and omitted calculations can be found in [Kitaev06, Appendix E].

We have seen that the unknot corresponds to the quantum dimensions of a theory. The Hopf link corresponds to another useful quantity. For a theory of rank k, the *(topological) S-matrix* is the $k \times k$ matrix given by[17]

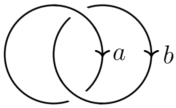
$$[S]_{ab} = s_{ab} = \frac{1}{\mathcal{D}} \tag{2.5.1}$$



and the unnormalised S-matrix is given by $\tilde{S} = \mathcal{D}S$. Noting that s_{ab} is the quantum trace (i.e. the braid closure) of the monodromy of a and $\bar{b}$, we have

$$s_{ab} = \frac{1}{\mathcal{D}} \sum_c \frac{\vartheta_c}{\vartheta_a \vartheta_b} N_c^{a\bar{b}} d_c \tag{2.5.2}$$

and that



$$= \tilde{s}^*_{ab} \tag{2.5.3}$$

Using (2.5.2) and the symmetries of the fusion coefficients, we also see that

$$s_{ab} = s_{ba} \quad , \quad s_{ab} = s_{\bar{a}\bar{b}} \tag{2.5.4}$$

i.e.

$$S^T = S \quad , \quad [C, S] = 0 \tag{2.5.5}$$

As usual, we order the labels in $\mathfrak{L}$ such that the first label is $\mathbf{1}$. It is clear that the 1^{st} row and column of S is the normalised vector of quantum dimensions. By pulling the cup over the cap in the left component of (2.5.1) and evaluating the resulting twists [Kitaev06], we see that

$$s_{ab} = s^*_{\bar{a}b} \tag{2.5.6}$$

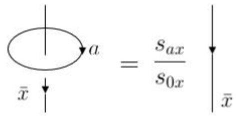
i.e.

$$S^\dagger = SC \tag{2.5.7}$$

It is clear that the S-matrix is gauge-invariant. The following identity is also useful, and

[17] The orientation of our components follows the same convention as in [Kitaev06]. Note that other sources may use different conventions e.g. as in [Bonderson].

is easily verified by applying the quantum trace.



(2.5.8)

This can be used to derive the equation

$$\sum_c N_c^{ab} s_{cx} = \frac{s_{ax} s_{bx}}{s_{0x}} \tag{2.5.9}$$

which can be seen as a row (for label $b \in \mathfrak{L}$) of the eigenvalue equation

$$N^a \boldsymbol{s}_x = \frac{s_{ax}}{s_{0x}} \boldsymbol{s}_x \tag{2.5.10}$$

where the eigenvector $\boldsymbol{s}_x$ is a column of S (for label $x \in \mathfrak{L}$). It follows that any column of the S-matrix is an eigenvector of any fusion matrix (noting that the columns of S may be linearly dependent). For the 1^{st} column of S (i.e. $x = 0$), (2.5.10) is just (2.1.49). Then by the Frobenius-Perron theorem, we also have that

$$\left| \frac{s_{ax}}{s_{0x}} \right| \leq d_a \quad , \quad a \in \mathfrak{L} \tag{2.5.11}$$

where $s_{ax} \backslash s_{0x} = d_a$ if and only if $\boldsymbol{s}_x$ is a (real) multiple of $\boldsymbol{d} (= \mathcal{D} \boldsymbol{s}_0)$.[18]

Remark 2.25. (Self-duality). If a is self-dual, then N^a is real-symmetric (and so it has real eigenvalues). We can then deduce from (2.5.10) that all entries in the row and column (of S) for label a are real. A self-dual theory thus has a real-symmetric S-matrix.

Definition 2.26.

(i) A charge $a \in \mathfrak{L}$ is called *transparent* if its monodromy with any charge x is trivial.

(ii) A theory of anyons is said to have *nondegenerate braiding* if its only transparent charge is $\mathbf{1}$.

(iii) A theory of anyons is called *modular* if its S-matrix is unitary.

Theorem 2.27.[19] A theory of anyons has nondegenerate braiding if and only if it is modular.

[18] It follows that $\boldsymbol{s}_x$ cannot be a (real) multiple of $\boldsymbol{d}$ for x nonabelian.

[19] A proof of this crucial theorem can be found in [Kitaev06, Appendix E].

Remark 2.28.[20] **(Rank of S vs. degeneracy of braiding).** Following Theorem 2.27, it is natural to wonder whether the rank of the S-matrix somehow quantifies the degeneracy of the braiding. The opposite extreme of Theorem 2.27 would be that the S-matrix has rank one if and only if the braiding is fully degenerate (i.e. all charges are transparent): indeed, this is true. By [Kitaev06, Lemma E.13.], x is transparent if and only if

$$N^a \boldsymbol{s}_x = d_a \boldsymbol{s}_x \quad , \quad a \in \mathfrak{L} \tag{2.5.12}$$

which can happen (by the Frobenius-Perron theorem) if and only if $\boldsymbol{s}_x$ is a real (thus positive) multiple of $\boldsymbol{d}$.[21] Thus, S has rank one if and only if all charges are transparent.

Terminology 2.29. The information encoded by the S and T-matrix of a theory is called its *modular data*, and is written $\{S, T\}$.

2.5.2 Modular Theories

From (2.5.10), for each $a \in \mathfrak{L}$ we have

$$N^a S = S D_a \quad , \quad [D_a]_{mn} = \frac{s_{am}}{s_{0m}} \delta_{mn} \tag{2.5.13}$$

When a theory is modular, S is unitary whence

$$N^a = S D_a S^* \tag{2.5.14}$$

Remark 2.30. By the associativity and commutativity of fusion, we have

$$[N^a, N^b] = 0 \quad , \quad a, b \in \mathfrak{L} \tag{2.5.15}$$

from which it follows (using $N^{\bar{a}} = (N^a)^T$) that fusion matrices are normal (and hence, unitarily diagonalisable). It is also clear from (2.5.15) that all of the fusion matrices are simultaneously diagonalisable. For a modular theory, the columns of the S-matrix constitute a simultaneous orthonormal eigenbasis for all of the fusion matrices. For a non-modular theory, the S-matrix is just some degenerate matrix of eigenvectors with respect to any fusion matrix. Given a non-modular theory, we can still define a matrix S' whose columns are a simultaneous orthonormal eigenbasis for all of the fusion matrices:

[20] In category-theoretic jargon, a generic theory of anyons is described by a unitary ribbon fusion category $\mathcal{C}$ (also called a unitary premodular category), while a modular (a.k.a bosonic) theory is described by a unitary modular tensor category. The labels $\mathfrak{L}$ are precisely the isomorphism classes of objects in $\mathcal{C}$, and the Müger centre $\mathcal{Z}_2(\mathcal{C})$ of the category contains all transparent objects. When $\mathcal{Z}_2(\mathcal{C})$ only contains objects isomorphic to the trivial object, $\mathcal{C}$ is called modular; else $\mathcal{C}$ is premodular. If $\mathcal{Z}_2(\mathcal{C})$ contains all objects in $\mathcal{C}$, then $\mathcal{C}$ is called a *symmetric* fusion category.

[21] This tells us that nonabelian charges cannot be transparent.

any such matrix is called a *mock S-matrix* [BGH+20].

By inspecting the element $[N^a]_{bc}$ of (2.5.14), we arrive at the *Verlinde formula*

$$N_c^{ab} = \sum_x \frac{s_{ax}s_{bx}s_{\bar{c}x}}{s_{0x}} \tag{2.5.16}$$

which tells us that all of the fusion coefficients of a theory are encoded in its S-matrix.

Remark 2.31. (Why modularity?)

(i) Modularity can be motivated pragmatically. R-matrices of the form R^{ab} where $a \neq b$ are gauge-variant, and therefore cannot correspond to measurable quantities. On the other hand, monodromies are gauge-invariant. Since the monodromy of any transparent label is trivial, there is no reason to allow for nontrivial transparent labels in our algebraic models (as they cannot be statistically distinguished from the vacuum in practice).

(ii) Modularity comes at a price: there are many non-modular theories of anyons whose description is of practical importance. Let f an abelian charge. R-matrices of the form R^{ff} are gauge-invariant, and assuming modularity has the undesirable effect of discarding theories with transparent objects f such that -1 is an eigenvalue of R^{ff} (e.g. fermions). Unitary modular tensor categories (i.e. modular theories of anyons) are thus limited to describing $(2+1)$-dimensional *bosonic* topological orders. Fermions are typically present in systems of interest (e.g. fractional quantum Hall liquids), and so it is desirable to have an algebraic model that is "almost" modular i.e. where the only nontrivial transparent object is a fermion: this has led to the development of *spin* modular categories [BGH+17].

(iii) It is conjectured that every modular theory of anyons corresponds to a unitary Chern-Simons-Witten TQFT for some pair (G, λ), where G is a compact Lie group and $\lambda \in H^4(BG; \mathbb{Z})$ is a cohomology class [HRW08, MS89, Witten89:II].

Remark 2.32. ("Modularity"). Let $\Theta := \mathcal{D}^{-1}\sum_a d_a^2 \vartheta_a$. A modular theory of anyons satisfies[22],

$$(ST)^3 = \Theta C \quad , \quad S^2 = C \quad , \quad C^2 = I \tag{2.5.17}$$

Note that $C^2 = I$ holds for any theory, and that $S^2 = C$ follows from (2.5.7) and the unitarity of S. A proof of $(ST)^3 = \Theta C$ is given in [Kitaev06, Theorem E.14.].

[22] The relations (2.5.17) bear close semblance to the generators of the modular group (which describes the symmetries of the moduli space of elliptic functions). This is not a coincidence. Modular categories contain lots of structure, and are further explored in [BK01].

Remark 2.33. (Prime decomposition of modular theories). Points (ii)-(v) below follow from [Müg03].

(i) Given one or more theories of anyons as an input, there are many different ways to produce the data for a distinct theory of anyons [Bonderson, Section 5.7]. In particular, given two theories with label sets $\mathfrak{L}_1$ and $\mathfrak{L}_2$, we can define a *product theory*[23] with label set $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2$ where for $a \in \mathfrak{L}$ we write $a = (a_1, a_2)$ (and similarly for b, c etc.), and

$$V_c^{ab} \cong V_{c_1}^{a_1 b_1} \otimes V_{c_2}^{a_2 b_2} \tag{2.5.18}$$

$$\left[F_d^{abc}\right]_{(f,\mu,\nu)(e,\alpha,\beta)} = \left[F_{d_1}^{a_1 b_1 c_1}\right]_{(f_1,\mu_1,\nu_1)(e_1,\alpha_1,\beta_1)} \left[F_{d_2}^{a_2 b_2 c_2}\right]_{(f_2,\mu_2,\nu_2)(e_2,\alpha_2,\beta_2)} \tag{2.5.19}$$

$$\left[R_c^{ab}\right]_{\mu\nu} = \left[R_{c_1}^{a_1 b_1}\right]_{\mu_1\nu_1} \left[R_{c_2}^{a_2 b_2}\right]_{\mu_2\nu_2} \tag{2.5.20}$$

The S-matrix of the product theory is the tensor product of those of the constituent theories. Note that there is no interaction between the constituent theories in this construction.

(ii) A modular theory of anyons is called *prime* if its only nontrivial modular factor theory is itself.

(iii) Any modular theory of anyons can be realised as a finite product of prime modular theories. Furthermore, this factorisation is generally non-unique (unless the only abelian charge is the vacuum).

(iv) A (nontrivial) modular factor theory $\mathcal{B}$ of a modular theory $\mathcal{A}$ is realised as modular subtheory[24] $\mathcal{B} \subset \mathcal{A}$. We write

$$\mathcal{A} = \mathcal{B} \times \tilde{\mathcal{B}} \tag{2.5.21}$$

(where $\tilde{\mathcal{B}}$ must also be a modular factor theory). It follows that a modular theory is prime if its only nontrivial modular subtheory is itself.

(v) It can be shown that the restriction of $\mathcal{A}$ to the subset of charges that have trivial monodromy with all of the charges in modular subtheory $\mathcal{B}$ is also a modular subtheory: this new subtheory of $\mathcal{A}$ is denoted by $\mathcal{B}'$ and is called the *relative commutant* (or *centraliser*) of $\mathcal{B}$ in $\mathcal{A}$. Furthermore, $\tilde{\mathcal{B}} = \mathcal{B}'$ in (2.5.21), and

$$\mathcal{B}'' = \mathcal{B} \tag{2.5.22a}$$

$$\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{B}} \mathcal{D}_{\mathcal{B}'} \tag{2.5.22b}$$

[23] In category-theoretic terms, a 'product theory' is the Deligne product of unitary ribbon categories.

[24] A subtheory $\mathcal{B}$ of $\mathcal{A}$ is a restriction $\mathcal{A}|_{\mathcal{B}}$. Note that a subtheory is produced by restriction to a subset of charges that is closed under fusion and obeys **(F1)-(F5)**.

Remark 2.34. (Minimal data). A braided fusion category $\mathcal{C}$ is uniquely specified by its 'skeletal data' i.e. its associated fusion rules and (gauge class of) F and R-symbols. For categories of large rank, the skeletal data becomes unwieldy. A natural question follows: is this the smallest set of data that uniquely specifies $\mathcal{C}$? It was previously conjectured that a modular category should be uniquely specified by its modular data; this was disproved by counterexample (two modular categories of rank 49 were found to have the same modular data)[25,26] [MS21]. In [BDG$^+$19], the authors showed that the triple $\{S, T, W\}$ (where W is the Whitehead link) distinguishes between the counterexamples in [MS21]; it remains an open question whether this new triple is a faithful invariant of modular categories.

Example 2.35. (Fusion rules for modular theories of rank 2). Let $\mathfrak{L} = \{\mathbf{1}, q\}$. The only nontrivial fusion rule is

$$q \otimes q = \mathbf{1} + mq \quad , \quad m \in \mathbb{N}_0$$

Writing $d := d_q$, we have $d^2 = 1 + md$, whence (by $d > 0$)

$$d = \frac{m + \sqrt{m^2+4}}{2} \tag{2.5.23}$$

We also have

$$N^q = \begin{pmatrix} 0 & 1 \\ 1 & m \end{pmatrix} \quad , \quad \tilde{S} = \begin{pmatrix} 1 & d \\ d & x \end{pmatrix}$$

where x is unknown. We know that N^q has eigenvalues $\{d, -\frac{1}{d}\}$ since $\det(N^q) = -1$. Then by inspecting the equation $N^q \boldsymbol{s}_q = -\frac{1}{d}\boldsymbol{s}_q$, we see that $x = -1$.[27] Then

$$x = -1 = \sum_c N_c^{qq} \vartheta_c \vartheta_q^{-2} d_c = \vartheta_q^{-2}(1 + md\vartheta_q) \implies md = -\vartheta_q - \vartheta_q^{-1} \le 2$$

So using (2.5.23), we see that the only possibilities are $m = 0, 1$. Note that this alone does not guarantee the existence of modular theories for $m = 0, 1$. However, we will see that these fusion rules do indeed give rise to modular theories: $m = 0, 1$ respectively yield the semion and Fibonacci theories. Note that

$$S_{\text{semion}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad S_{\text{Fib}} = \frac{1}{\sqrt{2+\phi}} \begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix} \tag{2.5.24}$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

[25] The authors also proved that arbitrarily many inequivalent modular categories can share the same modular data.

[26] It is currently unknown whether there exists a counterexample of lower rank.

[27] A slicker way: note that $S^2 = C = I$. Then S must have eigenvalues ± 1, so $\mathrm{tr}(S) = 0$.

2.6 Theories of Anyons

2.6.1 Classification

Naturally, we would like to classify theories of anyons (and ideally, obtain their associated F and R-symbols). The most naive approach would be to: (1) take some consistent fusion rules, (2) solve the pentagon and hexagon equations, (3) determine the gauge classes of unitary solutions[28], (4) repeat for new fusion rules. In practice, this approach is unfeasible (largely due to the difficulty of solving the pentagon and hexagon equations as the rank increases). Vastly more refined approaches are used in classification efforts.

Remark 2.36. (Classifying fusion categories). Classifiying theories of anyons can be seen as part of the much larger mathematical programme of classifying fusion categories.[29] This is a challenging task that can be made slightly easier by limiting the search to categories that admit some extra structure or obey certain constraints[30]. For instance,

- Fusion categories $\mathcal{C}$ (over $\mathbb{C}$) of were classified up to rank 2 in [Ostrik03]. This was extended to rank 3 for $\mathcal{C}$ pivotal in [Ostrik15].
- Premodular categories (over $\mathbb{C}$) have been classified up to rank 4 [Ostrik08, Bru16]. This was extended to rank 5 for $\mathcal{C}$ pseudounitary in [BM18].
- Since modular categories admit a prime factorisation [Müg03], it suffices to classify prime modular categories. Unitary modular categories were classified up to rank 4 in [RSW09]; this was extended to rank 5 in [BHRW16:I]. Partial progress has been made for rank 6 [Cream19, Green19]. A classification of unitary premodular categories yields a classification[31] of theories of anyons. Similarly, a classification of unitary modular categories yields a classification of bosonic theories of anyons.
- Spin modular categories were classified up to rank 11 in [BGH$^+$20]. A classification of spin modular categories results in a classification of fermionic theories of anyons (i.e. theories that are almost modular but contain one fermion).

[28] Each gauge class will uniquely specify a theory of anyons.

[29] For the unfamiliar reader, fusion categories can be characterised as follows: take some (not necessarily commutative) fusion rules satisfying **(F1)**-**(F4)**, and solve the pentagon equation. Then fusion categories will correspond to the gauge classes of solutions. Some further details on fusion categories can be found in Part II of this thesis, and a rigorous introduction can be found in [EGNO].

[30] E.g. pivotality, sphericality, braiding, modularity, (pseudo)unitarity, (weak) integrality etc.

[31] Here, 'classification' means knowing which fusion rules will give rise to a theory of anyons. In [RSW09], some skeletal data is also tabulated. While mathematicians are generally not interested in skeletal data, it is important for physical applications.

Remark 2.37. (Towards a "periodic table" of theories). An immediate question: *are there finitely many theories of anyons of a given rank?*

(i) A result known as *Ocneanu rigidity* tells us that a fusion algebra admits finitely many categorifications, and that a fusion category admits finitely many braidings [ENO05]. This means that any given set of fusion rules can only give rise to finitely many theories of anyons.[32]

(ii) It was shown in [BHRW16:II] that there are finitely many modular categories of any given rank. This *rank-finiteness theorem* was strengthened to the case of G-crossed braided fusion categories in [JMNR20]. It follows that for a given rank, there are finitely many sets of fusion rules that can give rise to a theory of anyons (and thus, *finitely many theories of anyons of a given rank*).

(iii) Ocneanu rigidity and rank-finiteness motivate the analogy of building a 'periodic table' of anyonic theories. If we restrict our attention to bosonic theories, this idea holds even greater allure. Recall that modular categories admit a prime factorisation: then elements of our table will be given by prime modular theories.[33]

2.6.2 Some Basic Examples

Given a set of fusion rules with sufficiently low rank[34], solving the pentagon and hexagon equations by hand is a manageable (yet possibly tedious) task. We will describe the data of the theories arising from the $\mathbb{Z}_2$, Fibonacci and Ising fusion rules below. These are simple but important theories that are ubiquitous in the literature. First, let us make a few observations.

- If all of the F-symbols are real and the three R-matrices are diagonal in hexagon equation (2.2.8a), then 'inverse' hexagon equation (2.2.8b) is just its complex conjugate. In such an instance, it follows that (2.2.8b) does not need to be checked

[32] Ocneanu rigidity has the following physical interpretation: the Hamiltonian realising a system of anyons is stable under small deformations. Roughly, due to the discrete nature of the solution set, we cannot 'deform' one theory into another.

[33] There is no canonical convention for the organisation of such a table. One approach for the table of bosonic theories could be to have elements in one-to-one correspondence with the modular data of prime theories. It has been shown that all known examples of modular categories that share the same modular data are related to one another by a construction called ribbon zesting [DGP$^+$20]. Following the periodic table analogy, it has been proposed that a bosonic theory of anyons $\mathcal{A}$ should be called a *modular isotope* of a bosonic theory $\mathcal{B}$ with the same modular data, if it can be obtained by zesting $\mathcal{B}$. However, it is not known whether modular data is an invariant of modular categories up to zesting i.e. whether or not there exist more exotic examples of modular categories (beyond those found in [MS21]) that share the same modular data but are not related by zesting.

[34] Say, less than 4.

once we have solved for (2.2.8a). E.g. in a multiplicity-free theory with real F-matrices, it is sufficient to solve for only one of the equations (2.2.8a)-(2.2.8b).

- Given a theory $\mathcal{A}$, we can obtain its *mirror theory* $\mathcal{A}^P$ by applying a spatial parity transformation. The S, F and R-matrices of $\mathcal{A}^P$ will be given by the Hermitian conjugate of those in $\mathcal{A}$. In order to convince yourself of this, consider the pentagon and hexagon equations (2.1.29) and (2.2.7), and imagine 'viewing them from behind'. Applying a parity transformation flips the 'handedness' or *chirality* of the theory.[35]

In the following, all theories are self-dual, multiplicity-free, and all charges have Frobenius-Schur indicator $+1$ unless stated otherwise. We do not state the twists since they can be deduced using $\vartheta_a = \varkappa_a \left(R_0^{aa}\right)^*$, and we only state the quantum dimension of nonabelian charges (since it is 1 otherwise). Details regarding the gauge group and level of the Chern-Simons theories corresponding to the modular theories below may be found in [RSW09, Simon].[36]

$\mathbb{Z}_2$ fusion rule Let $\mathfrak{L} = \{\mathbf{1}, s\}$ where the only nontrivial fusion rule is

$$s \times s = \mathbf{1}$$

This gives rise to four theories whose nontrivial F and R-symbols are

(i) $F_s^{sss} = -1$, $R^{ss} = i$

(ii) $F_s^{sss} = -1$, $R^{ss} = -i$

(iii) $R^{ss} = -1$

(iv) No nontrivial symbols.

Theories (i)-(ii) are a mirror pair: they are respectively the right and left-handed *semion theories* (where s is called a semion[37]). They have $\varkappa_s = -1$ and are both modular, and their S-matrix is written in (2.5.24). Theories (iii)-(iv) are completely degenerate: they

[35] Many theories of anyons have a field-theoretic description given by a Chern-Simons theory with gauge group G (a compact Lie group) and level $k \in H^4(BG, \mathbb{Z})$ (an integer). Such a theory is denoted by G_k, and its mirror theory is G_{-k}. See also Remark 2.31(iii).

[36] There exists a 'duality' between certain Chern-Simons theories. For instance, the right and left-handed Fibonacci theories are respectively realised by $(G_2)_1$ and $(F_4)_1$ i.e. there is a duality between $(G_2)_1$ and $(F_4)_{-1}$. Some further details can be found in [CHO19].

[37] I.e halfway between a boson and fermion.

respectively describe bosons and fermions. Each entry of their S-matrix is $1/\sqrt{2}$.

Fibonacci fusion rule Let $\mathfrak{L} = \{\mathbf{1}, \tau\}$ where the only nontrivial fusion rule is

$$\tau \times \tau = \mathbf{1} + \tau$$

This gives rise to a mirror pair of theories called the *Fibonacci theories* (where τ is called a Fibonacci anyon[38]), both of which are modular. The nontrivial F and R-symbols of the right-handed theory are given by

$$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \quad , \quad R^{\tau\tau} = \begin{pmatrix} e^{-i\frac{4\pi}{5}} & 0 \\ 0 & e^{i\frac{3\pi}{5}} \end{pmatrix} \tag{2.6.1}$$

where $d_\tau = \phi = \frac{1+\sqrt{5}}{2}$ (golden ratio), and the S-matrix for both theories is in (2.5.24).

Ising fusion rules Let $\mathfrak{L} = \{\mathbf{1}, \sigma, \psi\}$ where the nontrivial fusion rules are

$$\sigma \times \sigma = \mathbf{1} + \psi \quad , \quad \sigma \times \psi = \sigma \quad , \quad \psi \times \psi = \mathbf{1}$$

This gives rise to four pairs of mirror theories, two of which have $\varkappa_\sigma = -1$. All eight theories are modular. The four theories with with $\varkappa_\sigma = 1$ are called *Ising theories* (where σ is called an Ising anyon). We give the nontrivial F and R-symbols for one of each of the Ising pairs:

$$\begin{gathered} R^{\sigma\sigma} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} , \; F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ R_\sigma^{\sigma\psi} = R_\sigma^{\psi\sigma} = -i \; , \; R_0^{\psi\psi} = -1 \; , \; F_\sigma^{\psi\sigma\psi} = F_\sigma^{\sigma\psi\sigma} = -1 \end{gathered} \tag{2.6.2}$$

$$\begin{gathered} R^{\sigma\sigma} = e^{i\frac{7\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} , \; F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ R_\sigma^{\sigma\psi} = R_\sigma^{\psi\sigma} = -i \; , \; R_0^{\psi\psi} = -1 \; , \; F_\sigma^{\psi\sigma\psi} = F_\sigma^{\sigma\psi\sigma} = -1 \end{gathered} \tag{2.6.3}$$

The four theories with with $\varkappa_\sigma = -1$ are called $\mathrm{SU}(2)_2$ *theories*. We give the nontrivial

[38] For n τ-particles, note that $\dim(V_{\mathbf{1}}^{\tau\cdots\tau})$ is given by the n^{th} term of the Fibonacci sequence.

F and R-symbols for one of each of the $\mathrm{SU}(2)_2$ pairs:

$$\begin{gathered}R^{\sigma\sigma} = e^{i\frac{5\pi}{8}}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\ ,\ F^{\sigma\sigma\sigma}_{\sigma} = -\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ R^{\sigma\psi}_{\sigma} = R^{\psi\sigma}_{\sigma} = i\ ,\ R^{\psi\psi}_{0} = -1\ ,\ F^{\psi\sigma\psi}_{\sigma} = F^{\sigma\psi\sigma}_{\sigma} = -1\end{gathered} \tag{2.6.4}$$

$$\begin{gathered}R^{\sigma\sigma} = e^{-i\frac{3\pi}{8}}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}\ ,\ F^{\sigma\sigma\sigma}_{\sigma} = -\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ R^{\sigma\psi}_{\sigma} = R^{\psi\sigma}_{\sigma} = i\ ,\ R^{\psi\psi}_{0} = -1\ ,\ F^{\psi\sigma\psi}_{\sigma} = F^{\sigma\psi\sigma}_{\sigma} = -1\end{gathered} \tag{2.6.5}$$

$d_\sigma = \sqrt{2}$ and the S-matrix for all 8 theories is

$$S = \frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix} \tag{2.6.6}$$

Data for some other theories is tabulated in [Bonderson], and data for prime modular theories up to rank 4 can be found in [RSW09]. Data for various quantum group categories can be found in [AS10]. An extensive catalogue of modular tensor categories can be found at [GHY], and a tabulation of small multiplicity-free fusion rings at [AnyonWiki].

3. Paper I:

Fusion Structure from Exchange Symmetry in (2+1)-Dimensions

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FUSION STRUCTURE FROM EXCHANGE SYMMETRY IN (2+1)-DIMENSIONS

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Abstract. Until recently, a careful derivation of the fusion structure of anyons from some underlying physical principles has been lacking. In [Shi et al., Ann. Phys., 418 (2020)], the authors achieved this goal by starting from a conjectured form of entanglement area law for 2D gapped systems. In this work, we instead start with the principle of exchange symmetry, and determine the minimal prescription of additional postulates needed to make contact with unitary ribbon fusion categories as the appropriate algebraic framework for modelling anyons. Assuming that 2D quasiparticles are spatially localised, we build a functor from the coloured braid groupoid to the category of finite-dimensional Hilbert spaces. Using this functor, we construct a precise notion of exchange symmetry, allowing us to recover the core fusion properties of anyons. In particular, given a system of n quasiparticles, we show that the action of a certain n-braid β_n uniquely specifies its superselection sectors. We then provide an overview of the braiding and fusion structure of anyons in the usual setting of braided $6j$ fusion systems. By positing the duality axiom of [A. Kitaev, Ann. Phys., 321(1) (2006)] and assuming that there are finitely many distinct topological charges, we arrive at the framework of ribbon categories.

1. Introduction

The study and classification of topological phases of matter is a pervasive theme of contemporary physics. Quasiparticles with exotic exchange statistics (called "anyons") are a hallmark of two-dimensional topological phases. The experimental realisation and control of anyons is a much sought-after goal, owing especially to proposed schemes for the robust processing of quantum information [1, 2, 3].

The algebraic theory of anyons (of which various detailed accounts may be found [4, 5, 6, 7, 8]) is considered mature [9, 10]. It is well-understood that the statistical properties of anyons arise due to the distinguished topology of exchange trajectories in two dimensions. In a given theory, anyons are distinguished by their "topological charges" which characterise their mutual statistics. However, it is further expected that these charges possess a fusion structure wherein the 'combination' (or *fusion*) of two anyons effectively results in a single anyon that may possibly exist in a superposition of topological charges. In some expositions, fusion is motivated using flux-charge composite toy models. Fusion structure is also readily apparent in 2D spin-lattice models such as the toric code. However, a careful treatment of the emergence of this fusion structure in a general setting is lacking. We therefore seek to provide a ground-up construction of the braiding and fusion structure of anyons.

Quantum symmetries is an umbrella term for the algebraic structures that are used to describe topological quantum matter. Ribbon fusion categories provide the mathematical framework for studying the statistical behaviour of anyons. Often, anyons are introduced through a discussion of identical particles: the same arguments that

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lead us to conclude that there are only bosons and fermions in three or more spatial dimensions, instead indicate the possibility of fractional statistics in two dimensions. There is an unfortunate gulf between the language of identical particles and that of ribbon categories. Our objective is to clarify the connection between quantum symmetries and the elementary, yet profound principle of *exchange symmetry* in quantum mechanics. Superselection sectors play a key role in our exposition.

A series of 'assumptions' or postulates **A1**-**A3** are given throughout the text. They are proposed as the minimal prescription needed to recover ribbon fusion categories (as an algebraic model for anyons) from exchange symmetry in $(2+1)$-dimensions. Here, **A2**-**A3** are presented in terms slightly more simplified than in the main text. The "associativity condition" in **A3** refers to (6.24).

A1. Two-dimensional quasiparticles are *spatially localised* phenomena.

A2. (i) The Hilbert space of finitely many quasiparticles is finite-dimensional.
(ii) A theory of anyons has finitely many distinct topological charges.

A3. For any topological charge q, there exists a dual charge $\bar{q}$ such that a certain associativity condition is satisfied with respect to their fusion.

The *localisation condition* **A1** is a relevant physical consideration. Less satisfyingly, *finiteness assumption* **A2** appears to be prescribed for mathematical convenience. Some physical motivation is provided for *Kitaev's duality axiom* **A3** in [4]. The main results of this paper are presented in Sections 4 and 5, where we show that **A1** and **A2**(i) are sufficient to recover the core braiding and fusion structure of 2D quasiparticles. In Section 6, we outline how **A2**(ii) and **A3** are required to make contact with ribbon fusion categories as algebraic models for anyons.

1.1. **Relation to existing work.** In attempting to derive fusion structure from some underlying physical principles, our work is similar in spirit to [11] where the authors show that such structure may be recovered from the entanglement area law

$$S(A) = \alpha l - \gamma \tag{1.1}$$

where $S(A)$ is the von Neumann entropy of a simply-connected region A, l is the perimeter of A and γ is a constant correction term (which the authors also show to be equal to $\ln \mathcal{D}$, where $\mathcal{D}$ is the total quantum dimension of the anyon theory).

	Our approach	**Approach in** [11]
Physical principle	Exchange symmetry	Entanglement area law
Construction	Local representations of coloured braid groupoid	Information convex sets

While the construction in [11] may be more fundamental, the narrative of exchange symmetry might be more familiar to the majority of readers. F and R symbols can be recovered from our construction, and we are able to arrive at the usual formalism (of unitary ribbon fusion categories) for modelling theories of anyons. Ultimately, the two approaches will offer different insights and will appeal to different audiences. However, we suggest that they might be viewed as complementing one another. By assuming (1.1) it follows that **A2**(i) implies **A2**(ii) [11, Theorem 4.1], and that for any topological charge q there exists a *unique* dual charge $\bar{q}$ such that they will fuse to the vacuum in a *unique* way [11, Proposition 4.9]. Combining the two approaches, we arrive at an alternative to **A1-A3**:[1]

P1. Two-dimensional quasiparticles are *spatially localised* phenomena.

P2. The Hilbert space of finitely many quasiparticles is finite-dimensional.

P3. The system of quasiparticles satisfies entanglement area law (1.1).

1.2. **Outline of paper.** In Section 2, we recap the notion of superselection rules and identical particles. This is followed by a discussion of the difference between particle exchanges in two and three spatial dimensions. In Section 3, we formulate exchange symmetry via the action of the motion group of a many-particle system, and relate this to the boson-fermion superselection rule for fundamental particles.

In Section 4, we consider the action of braiding on a system of 2D quasiparticles. The localisation condition **A1** means that this action is generally not given by a representation of the braid group; instead, it is given by a local representation of the "coloured" braid groupoid. This action is described in Section 4.1, and we discuss its interpretation as a functor in Appendix A. The heart of our construction is presented in Section 4.2, where we adapt the definition of exchange symmetry from Section 3 to formulate an appropriate commutator via the braiding action. This gives rise to a notion of exchange symmetry on all subsystems of quasiparticles. In Section 4.3, we see how the associated superselection sectors of subsystems fit together to describe the Hilbert space of the whole system.

In Section 5, we present our main results. We show that the superselection sectors of an n-quasiparticle system correspond to the eigenspaces under the action of an n-braid β_n which we call the *superselection braid* (Theorem 5.1). We recover the core fusion structure amongst these superselection sectors by showing that they exhibit the same statistical behaviour as quasiparticles, allowing us to identify them as such (Theorem 5.5). The associativity and commutativity of fusion is deduced in Corollary 5.7. We prove several braid identities pertaining to β_n and see that this braid encodes the structure of all fusion trees for an n-quasiparticle fusion space (Theorem 5.9). We finally show that β_n is the unique braid (up to orientation) whose action specifies the superselection sectors of an n-quasiparticle system (Theorem 5.11).

In Section 6, we review the braiding and fusion structure from Section 5 within the usual setting of braided $6j$ fusion systems, and present the additional postulates required to make contact with the framework of ribbon fusion categories. In Section 6.3, we observe some R-matrix identities that follow from our construction: these reveal some information about the spectrum of β_n, and provide an ansatz for the monodromy operator which is consistent with the categorical ribbon relation.

In Section 7, we give a concise summary of our exposition, and speculate on a possible extension of our construction.

[1] The authors of [11] advocate for using two local entropic constraints [11, A0-A1] in lieu of (1.1).

2. Preliminaries

2.1. **Superselection rules and identical particles.** Consider a system with Hilbert space $\mathcal{H}$. A *superselection rule* (SSR) is given by a normal operator $\hat{J} : \mathcal{H} \to \mathcal{H}$ where

$$[\hat{O}, \hat{J}] = 0 \tag{2.1}$$

for *all* observables $\hat{O}$ of the system. Suppose that $\mathcal{H}'$ and $\mathcal{H}''$ are any two distinct *superselection sectors* (eigenspaces of $\hat{J}$). Then (2.1) tells us that for any $|\psi'\rangle \in \mathcal{H}'$, $|\psi''\rangle \in \mathcal{H}''$ and any observable $\hat{O}$ on $\mathcal{H}$, we have

$$\langle\psi'| \hat{O} |\psi''\rangle = 0 \tag{2.2}$$

The defining feature of SSRs is that they preclude the observation of relative phases between states from distinct superselection sectors: let $|\psi\rangle = \alpha |\psi'\rangle + \beta |\psi''\rangle$ and $|\psi_\theta\rangle = \alpha |\psi'\rangle + e^{i\theta}\beta |\psi''\rangle$ be normalised states. We have

$$\langle\hat{O}\rangle_\psi = \langle\hat{O}\rangle_{\psi_\theta} = tr(\hat{O}\hat{\rho}) \quad \text{for all } \hat{O}, \theta \tag{2.3}$$

where $\hat{\rho} = |\alpha|^2 |\psi'\rangle\langle\psi'| + |\beta|^2 |\psi''\rangle\langle\psi''|$ (i.e. if superpositions ψ_θ were to exist, we would be incapable of physically distinguishing them from a statistical mixture).

Examples of superselection observables[2] include spin, mass[3] and electric charge. Notably, the spin SSR concerns the superposition of integer and half-integer spins: by the spin-statistics theorem, this is equivalent to the boson-fermion SSR. These two equivalent SSRs are sometimes referred to as the univalence SSR.

The *intrinsic* properties of a particle may be characterised as corresponding to quantum numbers with an associated SSR. Two particles are *identical* if all of their intrinsic properties match exactly e.g. all electrons are identical.

2.2. **Particle exchanges.** Consider the exchanges of n identical particles[4] on a connected m-manifold $\mathcal{M}$ for $m \geq 2$. The homotopy classes of exchange trajectories in $\mathcal{M}$ form a group $G_n(\mathcal{M}) \cong \pi_1(\mathcal{U}_n(\mathcal{M}))$ under composition (the fundamental group of the nth unordered configuration space of $\mathcal{M}$). We will call this the *motion group*. We are interested in two cases for $\mathcal{M}$. Firstly, we have $G_n(\mathbb{R}^d) \cong S_n$ (the symmetric group) for $d \geq 3$. Here, a tangle[5] is homotopic to 0 tangles and exchanges are insensitive to orientation (Figure 1).

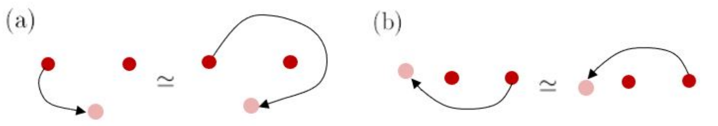

Figure 1. Exchange trajectories in $\mathbb{R}^d$ for (a) a clockwise tangle (right), and (b) single exchanges. When $d \geq 3$, deformations '$\simeq$' lift the strands through the extra spatial dimension(s).

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[2]SSRs for which $\hat{J}$ is an observable.

[3]Bargmann's mass SSR arises through demanding the Galilean covariance of the Schrödinger equation: this only pertains to nonrelativistic systems, since Galilean symmetry is superseded by Poincaré symmetry in special relativity.

[4]It will be assumed that particles are point-like.

[5]We call two successive exchanges of the same orientation on a pair of adjacent particles a *tangle*.

Secondly, for a surface $\mathcal{S}$ we have $G_n(\mathcal{S}) \cong B_n(\mathcal{S})$ (the surface braid group). Given any n points in (the interior of) $\mathcal{S}$, we can take some disc $D \subset \mathcal{S}$ such that all n points lie inside D. We know that $G_n(\mathbb{D}^2) \cong B_n$ where $\mathbb{D}^2$ is the 2-disc and

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \;,\; |i-j| \geq 2 \end{array} \right\rangle \tag{2.4}$$

is the Artin braid group. We will denote the identity element by e. The braid relations for B_n thus also hold in $B_n(\mathcal{S})$ [12]. When considering particle exchanges on a surface $\mathcal{S}$, we henceforth restrict our attention to $B_n(\mathbb{D}^2)$.

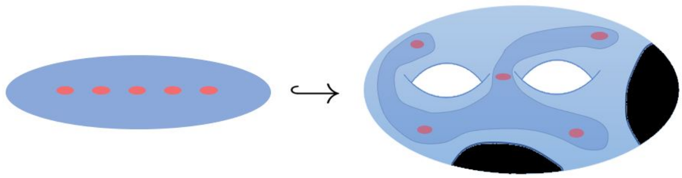


FIGURE 2. Particles are considered as lying in some disc $D \subset \mathcal{S}$. Since we are only interested in the topology of exchange trajectories and $B_n(\mathbb{D}^2) \cong B_n(D)$, we can restrict our attention to particles in $\mathbb{D}^2$.

Remark 2.1. In particular, this means that what we learn about the exchange statistics of particles on a disc is also applicable to particles on surfaces with arbitrary topology.

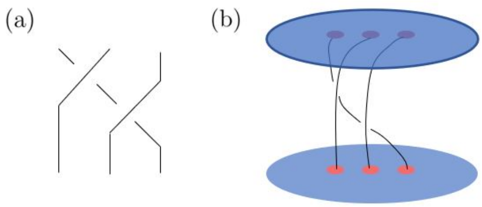


FIGURE 3. A braid diagram with n strands will be interpreted as a worldline diagram for n particles on a disc. *We will let the time axis run downwards.* The above diagram depicts this for the 3-braid $\sigma_2\sigma_1$.

3. Exchange Symmetry in Three or More Spatial Dimensions

A permutation of n identical particles will be indistinguishable from the original configuration: this is called *exchange symmetry* and may be concisely expressed by

$$[\hat{O}, \rho(g)] = 0 \tag{3.1}$$

for all observables $\hat{O}$ on $\mathcal{H}$ (the n-particle Hilbert space), and all g in the n-particle motion group G where $\rho : G \to U(\mathcal{H})$ is the unitary linear representation describing the evolution in $\mathcal{H}$ under the action of G. It is easy to see that if (3.2) holds for all generators g_i of G, then (3.1) follows.

$$[\hat{O}, \rho(g_i)] = 0 \tag{3.2}$$

Recall that S_n is the motion group of n particles in $\mathbb{R}^d$ for $d \geq 3$. We write

$$S_n = \left\langle \; s_1, \ldots, s_{n-1} \; \middle| \; \begin{array}{c} s_i^2 = e \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \;,\; |i-j| \geq 2 \end{array} \right\rangle \tag{3.3}$$

If $\dim(\mathcal{H}) = 1$, it is clear that ρ can only be one of $\rho^{\pm}$ where

$$\begin{aligned} \rho^{\pm} : S_n &\to U(1) \\ s_i &\mapsto \pm 1 \end{aligned} \tag{3.4}$$

If two identical particles are exchanged and their wavefunction is scaled by $+1$, they are called *bosons*; if their wavefunction is scaled by -1, they are called *fermions*.

Letting $\dim(\mathcal{H}) > 1$, it is consistent to expect that statistical evolutions determined by higher-dimensional representations of the symmetric group should be possible. Such exchange statistics are referred to as parastatistics. However, 'paraparticles' have never been observed in nature, and all known fundamental particles may be classified as being either a boson or a fermion. Indeed, the classification of identical particles as being either bosons or fermions is sometimes included as a postulate of quantum mechanics (called the *symmetrisation postulate*). If this postulate is relaxed then it can still be shown (under the pertinent constraints) that the boson-fermion classification will hold [13, 14, 15].

In order for (3.1) to be consistent with the symmetrisation postulate, we must levy some restrictions on ρ when $G = S_n$ and $\dim(\mathcal{H}) > 1$. The eigenvalues of $\rho(s_i)$ belong to a nonempty subset of $\{\pm 1\}$. We respectively denote the corresponding eigenspaces (one of which is possibly zero-dimensional) by $\mathcal{H}_i^{\pm}$. Since each such eigenspace defines a superselection sector and the n particles are identical (and are thus either all bosons or all fermions by the postulate), ρ must be such that $\mathcal{H}_i^{\pm} = \mathcal{H}_j^{\pm}$ for all i, j. We thus have $\mathcal{H} = \mathcal{H}^{+} \oplus \mathcal{H}^{-}$ (i.e. the subscripts are dropped). Under this restriction, we may thus recover the boson-fermion SSR from (3.1).

Remark 3.1. For a system of n bosons or fermions, there is typically no subspace describing a subsystem of $k < n$ particles. This is implicit in the structure of Fock space[6] (here $\mathcal{H}_{(\pm)}^{(k)}$ denotes the space of (anti)symmetric states for k identical particles):

$$\mathfrak{H}_{\pm} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}_{\pm}^{(2)} \oplus \mathcal{H}_{\pm}^{(3)} \oplus \ldots \tag{3.5}$$

E.g. $\mathcal{H}_{+}^{(2)} \not\subset \mathcal{H}_{+}^{(3)}$. For instance, states such as $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \in \mathcal{H}_{-}^{(2)}$ do not describe a physical entanglement, since the subsystem for an individual particle is physically inaccessible [16]. This is in contrast to anyonic systems which have a well-defined description of state spaces for particle subsystems (since anyons are *localised* phenomena). Nonetheless, there exist circumstances under which some notion of distinguishability amongst n identical bosons or fermions may be recovered: for instance, when their wavefunctions have (approximately) disjoint compact support. This can happen if the particles are far apart, or separated by sufficiently strong potentials.

[6] As a consequence of the mass SSR, note that the sectors of Fock space correspond to a SSR for the particle number operator in the nonrelativistic limit.

4. Exchange Symmetry in Two Spatial Dimensions

4.1. **Quasiparticles and braiding.** Although there are no fundamental particles in two spatial dimensions, it is well-known that various two-dimensional systems are theoretically capable of supporting localised excitations with fractional statistics [17, 18, 19, 20]: these emergent phenomena are known as *quasiparticles*[7]; they have no internal degrees of freedom and may thus be considered as *identical*. The localised nature of these excitations is instrumental in the emergence of fusion structure.

A1. Two-dimensional quasiparticles are *spatially localised* phenomena.

Recall that B_n is the motion group of n particles on a disc. Then for a two-quasiparticle system with Hilbert space $\mathcal{V}$, the action of the motion group is is given by a unitary linear representation

$$\rho : B_2 \to U(\mathcal{V}) \tag{4.1}$$

The eigenvalues $\{e^{iu_X}\}_X$ of $\rho(\sigma_1)$ lie in $U(1)$, and we have the corresponding decomposition $\mathcal{V} = \bigoplus_X \mathcal{V}_X$ where eigenspaces $\mathcal{V}_X$ define superselection sectors by exchange symmetry as expressed in (3.1).[8] The possibly arbitrary exchange phase e^{iu_X} is what earns *any*ons their namesake [21].

Remark 4.1. Now consider an n-particle system for $n \geq 2$. **A1** permits us to consider the Hilbert space associated with a subsystem of k adjacent quasiparticles (where $2 \leq k \leq n$). Consequently, the action of the motion *subgroup* B_k on any such subsystem will be independent of the rest of the system. The description of the superselection sectors and exchange statistics given by the action of B_2 on some pair of quasipartilces is thus a property intrinsic to said pair.

Consider a 2-quasiparticle subsystem (of particles labelled q_i and q_{i+1} located at the i^{th} and $i+1^{th}$ positions respectively) of an n-quasiparticle system. Denote the Hilbert space of this subsystem by $\mathcal{V}^{\{q_i,q_{i+1}\}}$ where $\{q_i, q_{i+1}\}$ is an unordered set. Following Remark 4.1, (4.1) defines a fixed action

$$\rho_{\{q_i,q_{i+1}\}} : B_2 \to U(\mathcal{V}^{\{q_i,q_{i+1}\}}) \tag{4.2}$$

on q_i and q_{i+1}, and we write the eigenspace decomposition $\mathcal{V}^{\{q_i,q_{i+1}\}} = \bigoplus_X \mathcal{V}_X^{\{q_i,q_{i+1}\}}$ for $\rho_{\{q_i,q_{i+1}\}}(\sigma_1)$. We label the quasiparticles from 1 to n and let $S_{\{1,\ldots,n\}}$ be the set whose elements are all possible permutations of the string $12\ldots n$. Given some $s \in S_{\{1,\ldots,n\}}$ we write $s = q_1 \ldots q_n$ where q_i is the i^{th} character of string s. We denote the Hilbert space for quasiparticles $q_1 \ldots q_n$ (in that order) by $V^{q_1\cdots q_n}$ or V^s. E.g. $V^{q_1\cdots q_i q_{i+1}\cdots q_n}$ and $V^{q_1\cdots q_{i+1} q_i\cdots q_n}$ are the state spaces assigned to the system in the initial and final time-slices of Figure 4 respectively.

[7] We will use the terms 'quasiparticle' and 'particle' interchangeably.

[8] It is assumed that the superselection sectors are finite-dimensional, and that the number of distinct superselection sectors is finite. This assumption is later codified as **A2** in Section 6.1.

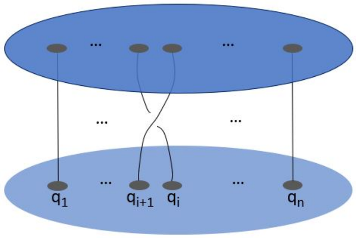


FIGURE 4. The clockwise exchange of quasiparticles q_i and q_{i+1}.

Let $\rho_s\big|_{V^s}(\sigma_i)$ be the unitary linear transformation describing the action of braid $\sigma_i \in B_n$ on the n-quasiparticle system (as shown in Figure 4). For $n > 2$,

$$V^{q_1\cdots q_i q_{i+1}\cdots q_n} \cong V^{q_1\cdots q_{i+1} q_i\cdots q_n} \cong \bigoplus_X \mathcal{V}_X^{\{q_i,q_{i+1}\}} \otimes \bar{V}_X^{(s)} \tag{4.3}$$

where $\bar{V}_X^{(s)}$ denotes the state space for the rest of the system when q_i and q_{i+1} are in superselection sector X. The spaces $V^{q_1\cdots q_i q_{i+1}\cdots q_n}$ and $V^{q_1\cdots q_{i+1} q_i\cdots q_n}$ may be identified under the action of the subgroup $\langle \sigma_1, \ldots, \sigma_{i-2}, \widehat{\sigma}_{i-1}, \sigma_i, \widehat{\sigma}_{i+1}, \sigma_{i+2}, \ldots \sigma_{n-1}\rangle$, but are only equivalent up to isomorphism under the action of B_n. This is because the action of B_n on the system will generally depend upon the order of the quasiparticles for $n > 2$. E.g. the action of $\sigma_1 \in B_3$ on V^{123} will differ from its action on V^{231} (unless $\rho_{\{1,2\}}$ and $\rho_{\{2,3\}}$ are the same). We must therefore distinguish between the spaces $\{V^s\}_{s\in S_{\{1,\ldots,n\}}}$ in order to consider the action of braiding on the whole system.

$$\rho_s\big|_{V^s}(\sigma_i^{\pm 1}) = \bigoplus_X \left[\rho^X_{\{q_i,q_{i+1}\}}(\sigma_1^{\pm 1}) \otimes \mathrm{id}_{\bar{V}_X^{(s)}}\right] \tag{4.4}$$

where $\rho^X_{\{q_i,q_{i+1}\}}$ is the subrepresentation given by restricting $\rho_{\{q_i,q_{i+1}\}}$ to $\mathcal{V}_X^{\{q_i,q_{i+1}\}}$.

Definition 4.2. $\rho_s(\sigma_i^{\pm 1})$ denotes the action of (anti)clockwise exchanging q_i and q_{i+1} on an n-particle Hilbert space. It is therefore *necessary that* $u \in S_{\{1,\ldots,n\}}$ contains the substring $q_i q_{i+1}$ or $q_{i+1} q_i$ for any V^u on which $\rho_s(\sigma_i^{\pm 1})$ is defined. Following from (4.4), that is

$$\rho_s\big|_{V^u}(\sigma_i^{\pm 1}) = \bigoplus_X \left[\rho^X_{\{q_i,q_{i+1}\}}(\sigma_1^{\pm 1}) \otimes \mathrm{id}_{\bar{V}_X^{(u)}}\right] \tag{4.5}$$

The above tells us that the right way to think about the action of braiding on an n-quasiparticle system is as follows: let $\{V^s\}_s$ be defined as above and let $b(s) \in S_{\{1,\ldots n\}}$ be the obvious permutation[9] of s for any $b \in B_n$. We construct an action of the braids $b \in B_n$ as linear transformations between spaces $\{V^s\}_s$. This action is defined through a collection of functions $\{\rho_s\}_s$ such that **(B0)**-**(B5)** hold for any $s \in S_{\{1,\ldots n\}}$ and for all $b, b_1, b_2 \in B_n$.

[9]E.g. $\sigma_i^{\pm 1}(q_1 \ldots q_i q_{i+1} \ldots q_n) = q_1 \ldots q_{i+1} q_i \ldots q_n$. That is, $b(s)$ is the string obtained by reading off the labels of the endpoints of braid b when its starting points are labelled (left-to-right) by the characters of s.

(B0) The domain of ρ_s is the braid group B_n

(B1) The image of b under ρ_s is a linear transformation

$$\rho_s(b) : \bigoplus_{u \in \mathcal{U}_{s,b}} V^u \to \bigoplus_{s' \in S_{\{1,\dots,n\}}} V^{s'} \tag{4.6}$$

where the elements $u \in \mathcal{U}_{s,b} \subseteq S_{\{1,\dots,n\}}$ index the direct summands $\{V^u\}_u \subseteq \{V^{s'}\}_{s'}$ that constitute the domain of $\rho_s(b)$. We have

$$\mathcal{U}_{s,e} := S_{\{1,\dots,n\}} \tag{4.7}$$

(B2) For any $u \in \mathcal{U}_{s,b}$, we have linear isomorphism

$$\rho_s\big|_{V^u}(b) : V^u \xrightarrow{\sim} V^{b(u)} \tag{4.8}$$

and if $u' \notin \mathcal{U}_{s,b}$ then $\rho_s(b)$ is undefined on $V^{u'}$.

(B3) Given b such that $b = b_2 b_1$, then for any u such that $u \in \mathcal{U}_{s,b_1}$ and $b_1(u) \in \mathcal{U}_{b_1(s),b_2}$, we have

$$\rho_s\big|_{V^u}(b_2 b_1) = \rho_{b_1(s)}\big|_{V^{b_1(u)}}(b_2) \circ \rho_s\big|_{V^u}(b_1) \tag{4.9}$$

(B4) $\rho_s\big|_{V^u}(b)$ is a unitary transformation i.e. for $u \in \mathcal{U}_{s,b}$ the map $\rho_s\big|_{V^u}(b)$ has Hermitian adjoint

$$\left(\rho_s\big|_{V^u}(b)\right)^\dagger = \rho_{b(s)}\big|_{V^{b(u)}}(b^{-1}) \tag{4.10}$$

where

$$\rho_{b(s)}\big|_{V^{b(u)}}(b^{-1}) \ \circ \ \rho_s\big|_{V^u}(b) = \mathrm{id}_{V^u} = \rho_s\big|_{V^u}(e) \tag{4.11a}$$

$$\rho_s\big|_{V^u}(b) \ \circ \ \rho_{b(s)}\big|_{V^{b(u)}}(b^{-1}) = \mathrm{id}_{V^{b(u)}} = \rho_{b(s)}\big|_{V^{b(u)}}(e) \tag{4.11b}$$

(B5) $\rho_s\big|_{V^u}(\sigma_i^{\pm 1})$ is defined as in Definition 4.2 for $u \in \mathcal{U}_{s,\sigma_i^{\pm 1}}$

Let us unpack some details. Firstly, what constitutes $\mathcal{U}_{s,b}$? **(B4)** tells us that $\rho_s\big|_{V^u}(b)$ is invertible[10], whence we must have

$$u \in \mathcal{U}_{s,b} \iff b(u) \in \mathcal{U}_{b(s),b^{-1}} \tag{4.12}$$

and therefore

$$u \in \mathcal{U}_{s,\sigma_i^{\pm 1}} \iff \sigma_i(u) \in \mathcal{U}_{\sigma_i(s),\sigma_i^{\mp 1}} \tag{4.13}$$

Combining this with **(B5)**, we deduce that $\mathcal{U}_{s,\sigma_i}$ contains all $u \in S_{\{1,\dots,n\}}$ such that

(i) u contains the substring $q_i q_{i+1}$ or $q_{i+1} q_i$

(ii) u satisfies (4.13)

That is, $\mathcal{U}_{s,\sigma_i}$ contains all u such that u contains the substring $q_i q_{i+1}$ or $q_{i+1} q_i$, and for which said substring does not begin at the $i-1^{th}$ or $i+1^{th}$ character of u.

Clearly, $\mathcal{U}_{s,\sigma_i} = \mathcal{U}_{s,\sigma_i^{-1}}$. **(B3)** tells us that if $u \in \mathcal{U}_{s,b_1}$ and $b_1(u) \in \mathcal{U}_{b_1(s),b_2}$, then $u \in \mathcal{U}_{s,b_1 b_2}$. One can check that **(B3)** together with (4.13) yields (4.12) as required. Also, by combining **(B3)** with our knowledge of $\mathcal{U}_{s,\sigma_i^{\pm 1}}$, we can find $\mathcal{U}_{s,b}$.[11]

[10] In fact, it tells us that $\rho_s\big|_{V^u}(b)$ is a diagonalisable, norm-preserving map for any $u \in \mathcal{U}_{s,b}$.

[11] Note that in this construction of $\mathcal{U}_{s,b}$, one considers all braid words of minimal length for b.

Remark 4.3. (Well-definedness and existence)
We know that for $u \in \mathcal{U}_{s,b}$, the map $\rho_s\big|_{V^u}(b)$ may be parsed into a composition of maps of the form in (4.5). When $n \geq 3$, there exist braids b for which there is more than one way to write b as a product of generators (i.e. as a braid word). This results in $\rho_s\big|_{V^u}(b)$ being given by distinct compositions. In order for the action $\{\rho_s\}_s$ to be well-defined, we require that all distinct compositions for a given $\rho_s\big|_{V^u}(b)$ are equal.[12] This intricate requirement is known as a *coherence condition*: we later see that it is fulfilled by demanding that matrix representations for maps of the form (4.5) satisfy the so-called *hexagon equations* (see Remark 6.3). For the current purposes of our construction, we will just assume that such (nontrivial) actions (satisfying this coherence condition) exist.

For any map $\rho_s\big|_{V^u}(b)$, we have

$$\rho_s\big|_{V^u}(b) = \rho_s\big|_{V^u}(b \cdot e) = \rho_s\big|_{V^u}(b) \circ \rho_s\big|_{V^u}(e) \tag{4.14}$$

whence it is clear that (4.7) must hold and that $\rho_s\big|_{V^u}(e) = \mathrm{id}_{V^u}$. Also note that we always have $s \in \mathcal{U}_{s,b}$, and so we may write

$$\rho_s\big|_{V^s} : B_n \to \mathrm{Hom}\left(V^s, \bigoplus_{s' \in S_{\{1,\ldots,n\}}} V^{s'}\right) \tag{4.15}$$

where $\rho_s\big|_{V^s}(b) : V^s \xrightarrow{\sim} V^{b(s)}$ is a unitary linear transformation.

Take any $b \in B_n$ whose image under the epimorphism $\eta : B_n \to S_n$ (whose kernel is the normal subgroup PB_n of n-strand pure braids) is a permutation of the form

$$\begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ b(1) & \cdots & b(i-1) & j & j+1 & b(i+2) & \cdots & b(n) \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ b(1) & \cdots & b(i-1) & j+1 & j & b(i+2) & \cdots & b(n) \end{pmatrix}$$

Then for all $u \in \mathcal{U}_{s,\sigma_i} \cap \mathcal{U}_{b(s),\sigma_j}$,

$$\rho_s\big|_{V^u}(\sigma_i^{\pm 1}) = \rho_{b(s)}\big|_{V^u}(\sigma_j^{\pm 1}) \tag{4.16}$$

The above construction for the “action” $\{\rho_s\}_s$ of n-braids on the spaces $\{V^s\}_s$ can be thought of as a *unitary linear representation of the braid groupoid for n distinctly coloured strands*. A further discussion of this statement is provided in Appendix A.

4.2. **Exchange symmetry for n quasiparticles.** Recall that superselection sectors arise from exchange symmetry as in (3.1). A subtle but crucial point in this equation is that the n-particle Hilbert space does not depend on the order of the particles. This necessity becomes clearer when we try to write down a (*naive*) version of (3.1) compatible with the braiding action described above:

$$[\hat{O}, \rho_s(b)] = 0 \quad \text{for all} \quad s \in S_{\{1,\ldots,n\}} \quad \text{and all} \quad b \in B_n$$

[12]Of course, there are cases where two distinct compositions may ‘automatically’ be equal by commutativity of constituent maps.

For starters, the image of $\rho_s(b)$ could be in any one of the spaces $\{V^s\}_s$, so the space of observables should be defined on the n-particle Hilbert space "modulo ordering". Let us denote such a space by $\mathcal{V}^{[n]}$ where $[n] := \{1, \ldots, n\}$ is an unordered set. This also makes sense physically, since we should not have different sets of observables depending on the order of the particles (by indistinguishability). This also excludes observables defined on subsystems (which is desirable as we want to consider the exchange symmetry mechanism local to all n quasiparticles). However, in order for the commutator to be well-defined, the braiding action must also be defined on $\mathcal{V}^{[n]}$.

Altogether, the correct adaptation of (3.1) should be given by a commutator of the form

$$[\hat{O}, \rho_{[n]}(g)] = 0 \tag{4.17}$$

for all n-particle observables $\hat{O}$ defined on $\mathcal{V}^{[n]}$ and for all $g \in \mathcal{E}_n \leq B_n$, where

$$\rho_{[n]} : \mathcal{E}_n \to U(\mathcal{V}^{[n]}) \tag{4.18}$$

is some unitary linear representation.

At first, this formulation of exchange symmetry appears rather abstract. In order to obtain a better understanding of what is meant by (4.17)-(4.18), we will outline their construction from the action $\{\rho_s\}_s$. Take $\mathcal{E}_n$ to be the subset of n-braids such that for any $g \in \mathcal{E}_n$, $b \in B_n$ and $s \in S_{\{1,\ldots,n\}}$, we have

$$\rho_{b(s)}(g) \cdot \rho_s(b) \left|\psi\right\rangle = e^{iu_Q} \left|\psi\right\rangle \tag{4.19}$$

where $V^s = \bigoplus_Q V^s_Q$ is the eigenspace decomposition under the unitary operator $\rho_s(g)$ with $\rho_s(g) \left|\psi\right\rangle = e^{iu_Q} \left|\psi\right\rangle$ for any $\left|\psi\right\rangle \in V^s_Q$. Equation (4.19) demands that V^s_Q is the e^{iu_Q}-eigenspace of $\rho_s(g)$ for all s. Since the e^{iu_Q}-eigenspace (of the action of $g \in \mathcal{E}_n$) is stable under the action of all n-braids, it is independent of the order of the particles: we thus denote it by $\mathcal{V}^{[n]}_Q$ where $\mathcal{V}^{[n]} = \bigoplus_Q \mathcal{V}^{[n]}_Q$. The action of g on $\mathcal{V}^{[n]}$ is denoted by $\rho_{[n]}(g)$ where

$$\rho_{[n]}(g) = \sum_Q e^{iu_Q} \hat{P}_Q \tag{4.20}$$

and where $\hat{P}_Q$ is a normalised projector onto $\mathcal{V}^{[n]}_Q$. We will see that $\mathcal{E}_n$ is a subgroup generated by a single n-braid i.e.

$$\mathcal{E}_n = \langle \beta_n \rangle \leq B_n \tag{4.21}$$

and we will therefore call $\beta_n \in B_n$ the *superselection braid*. The n-quasiparticle Hilbert space "modulo ordering" may therefore be understood as the representation space in (4.18), which in turn is constructed through the action $\{\rho_s\}_s$ of n-braids on the spaces $\{V^s\}_s$.[13] From the above, it is clear that $\mathcal{V}^{[n]}_Q \cong V^s_Q$ for any s.

Since the action of the superselection n-braid does not depend on the order of the particles, the braid itself should not favour any single particle over another. This hints that the braid should realise $\binom{n}{2}$ exchanges (i.e. each pair is exchanged once).

[13] Recall from (4.5) that the action $\{\rho_s\}_s$ can be formulated in terms of the pairwise action (4.2). The n-fold exchange symmetry mechanism (4.17) may thus be thought of as emerging from the pairwise exchange symmetries among its constituents.

By the innate symmetry of the representation space $\mathcal{V}^{[n]}$, we expect that the braid word β_n should also satisfy several internal symmetries. Indeed, we will subsequently see that these properties are satisfied, and that the superselection braid is given by

$$\beta_n = \sigma_1\sigma_2 \ldots \sigma_{n-1} \cdot \sigma_1\sigma_2 \ldots \sigma_{n-2} \cdot \ldots \cdot \sigma_1 \quad , \quad n \geq 2 \tag{4.22}$$

and $\beta_1 = e$. Studying the action of this braid reveals the fusion structure amongst quasiparticles and hints at their topological spin structure. This is key in connecting the narrative of exchange symmetry to the framework of braided fusion categories.

4.3. **Superselection sectors for n quasiparticles.** Given a system $V^{q_1\cdots q_n}$ of $n > 2$ quasiparticles, note that exchange symmetry (4.17) is defined with respect to all subsystems of k adjacent quasiparticles (where $2 \leq k \leq n$) i.e.

$$[\hat{O}, \rho_{[k]}(\beta_k)] = 0 \tag{4.23}$$

for all observables $\hat{O}$ on $\mathcal{V}^{[k]}$.[14] We therefore have a hierarchy of exchange symmetries. The next step is to understand how these all fit together. Equation (4.23) tells us that the eigenspaces $\{\mathcal{V}_X^{[k]}\}_X$ of $\rho_{[k]}(\beta_k)$ define superselection sectors. Take the k-particle subsystem $V^{q_1\cdots q_k}$ and write the decomposition into k-particle superselection sectors as $\bigoplus_X V_X^{q_1\cdots q_k} = V^{q_1\cdots q_k}$.

($\mathcal{Q}$) *How are $\{V_X^{q_1\cdots q_k}\}_X$ understood in the context of the full n-particle system?*

Let $k < n$ and write the decomposition into n-particle superselection sectors as $\bigoplus_Q V_Q^{q_1\cdots q_n} = V^{q_1\cdots q_n}$. Suppose the n-particle state is in superselection sector $V_Q^{q_1\cdots q_n}$. The most general way to decompose $V_Q^{q_1\cdots q_n}$ with respect to the k-particle subsystem is

$$V_Q^{q_1\cdots q_n} \cong \bigoplus_X V_X^{q_1\cdots q_k} \otimes V_Q^{X,q_{k+1}\cdots q_n} \tag{4.24}$$

where $V_Q^{X,q_{k+1}\cdots q_n}$ denotes the state space for the rest of the system when $q_1, \ldots, q_k$ are in superselection sector X.

Let us compare (4.3) and (4.24) when $i = 1$ and $k = 2$. In this case, $V_X^{q_1q_2} \cong \mathcal{V}_X^{\{q_1,q_2\}}$ and $\bigoplus_Q V_Q^{X,q_{k+1}\cdots q_n} \cong \bar{V}_X^{(s)}$. Spaces $V_X^{q_1q_2}$ and $V_X^{q_2q_1}$ may be identified with $\mathcal{V}_X^{\{q_1,q_2\}}$ when considered as representation spaces of B_2, but are distinguished between in the context of a larger system (since we usually need to keep track of the particle ordering) and are thus only considered equivalent up to isomorphism.

We can also partition an $n > 3$ particle system into subsystems $V^{q_1\cdots q_k}$ and $V^{q_{k+1}\cdots q_n}$ where we assume $2 \leq k \leq n-2$. Denote the superselection sectors of each by $\{V_X^{q_1\cdots q_k}\}_X$ and $\{V_Y^{q_{k+1}\cdots q_n}\}_Y$. Suppose the n-particle state is in superselection sector $V_Q^{q_1\cdots q_n}$. The most general way to decompose $V_Q^{q_1\cdots q_n}$ with respect to the two subsystems is

$$V_Q^{q_1\cdots q_n} \cong \bigoplus_{X,Y} V_X^{q_1\cdots q_k} \otimes V_Q^{XY} \otimes V_Y^{q_{k+1}\cdots q_n} \tag{4.25}$$

The spaces $\{V_Q^{XY}\}_{X,Y}$ may be thought of as constraining the superselection sectors of the subsystems by relating them to the n-particle superselection sector.

[14] In the instance of subsystems, $[k]$ denotes the unordered set of labels for the k particles.

If $\dim(V_Q^{XY}) = d$, this may be interpreted as the superselection sector Q containing superselection sectors X and Y in "d distinct ways". We may have $d = 0$, but it is also clear that at least one of the spaces $\{V_Q^{XY}\}$ must be nonzero.[15] By comparing (4.24) and (4.25), we see that

$$V_Q^{X,q_{k+1}\cdots q_n} \cong \bigoplus_Y V_Q^{XY} \otimes V_Y^{q_{k+1}\cdots q_n} \tag{4.26}$$

Analogously to (4.24) we can write $V_Q^{q_1\cdots q_n} = \bigoplus_Y V_Q^{q_1\cdots q_k,Y} \otimes V_Y^{q_{k+1}\cdots q_n}$ whence it similarly follows that

$$V_Q^{q_1\cdots q_k,Y} \cong \bigoplus_X V_X^{q_1\cdots q_k} \otimes V_Q^{XY} \tag{4.27}$$

In light of the above, it is easy to check that a "1-quasiparticle Hilbert space" must be canonically isomorphic to $\mathbb{C}$. It is therefore standard practice to omit a 1-quasiparticle Hilbert space in a decomposition.

Remark 4.4. (Superselection sectors of subsystems)
Another salient feature emerges from the hierarchy of superselection sectors in system of n quasiparticles for $n > 2$. To illustrate this, consider decomposition (4.24). While the spaces $\{V_X^{q_1\cdots q_k}\}_X$ still define superselection sectors locally (i.e. with respect to the k-particle subsystem), they do not define superselection sectors in the context of the larger system.[16] This is because the k-particle exchange symmetry mechanism is superseded by the n-particle mechanism. Indeed, the superselection sectors of the subsystem are entangled with the rest of the system in (4.24).[17] Crucially, this means that when we consider the entire system, it is possible to observe linear superpositions over the spaces $\{V_X^{q_1\cdots q_k}\}_X$. It is also possible that interactions between the subsystem and the rest of the system induce transitions between superselection sectors of the subsystem.

5. The Superselection Braid and Fusion Structure

In Section 4.2, we outlined the method for determining the superselection sectors using the action $\{\rho_s\}_s$. The first task is to find the subset $\mathcal{E}_n$ of all n-braids satisfying (4.19). For any candidate braid $g \in B_n$, it suffices to check that (4.19) is satisfied by $b = \sigma_i^{\pm 1}$ for all i. It will be convenient to define the following notation for braids:

$$\sigma_{i_1\ldots i_{k-1}i_k} := \sigma_{i_1}\ldots\sigma_{i_{k-1}}\sigma_{i_k} \quad , \quad b_j := \sigma_{12\ldots j} \quad \text{for all } j \geq 1, \text{ and } b_0 := e \tag{5.1}$$

We argued that a reasonable heuristic for an element of $\mathcal{E}_n$ would be that it exchanges each pair of quasiparticles once. Take the ansatz

$$\beta_n = b_{n-1}b_{n-2}\ldots b_1 \quad , \quad n \geq 2 \tag{5.2}$$

E.g. $\beta_2 = \sigma_1, \beta_3 = \sigma_{121}, \beta_4 = \sigma_{123121}$ etc. We also set $\beta_1 := e$. In Theorem 5.1, we will show that $\beta_n \in \mathcal{E}_n$. Therefore, (the action of) β_n specifies the superselection sectors; in fact, it does so uniquely up to orientation (Theorem 5.11) which proves (4.21) i.e. $\mathcal{E}_n = \langle\beta_n\rangle \leq B_n$. For this reason, we will refer to β_n as the *superselection braid.*

[15] This is equivalent to saying at least one of the spaces $\{V_Q^{X,q_{k+1}\cdots q_n}\}_X$ must be nonzero in (4.24).

[16] When we look at the whole system "from afar" we expect it to be in the ground state. This means that the superselection sector of the whole system should correspond to the vacuum, which later motivates the notion of "dual charges".

[17] Specifically, when X runs over > 1 index and at least two of the spaces $\{V_Q^{X,q_{k+1}\cdots q_n}\}_X$ are nonzero.

5.1. **The superselection braid.**

Theorem 5.1. (Superselection sectors)
We have the eigenspace decomposition $V^s = \bigoplus_Q V_Q^s$ under $\rho_s(\beta_n)$ where

$$\begin{aligned} \rho_s(\beta_n) : V_Q^s &\to V_Q^{\beta_n(s)} \\ |\Psi\rangle &\mapsto e^{iu_Q} |\Psi\rangle \end{aligned}, \quad n \geq 2 \tag{5.3}$$

for any $s \in S_{\{1,\ldots,n\}}$.

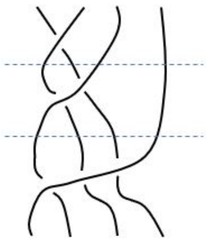


FIGURE 5. β_n has length $\binom{n}{2}$. The above diagram depicts β_4.

Let us recap the rest of the construction from Section 4.2. Theorem 5.1 allows us to identify the spaces $\{V_Q^s\}_s$ as the e^{iu_Q}-eigenspace $\mathcal{V}_Q^{[n]}$ under the action of β_n. Write,

$$\mathcal{V}^{[n]} = \bigoplus_Q \mathcal{V}_Q^{[n]} \tag{5.4}$$

In particular, this corresponds to a unitary representation

$$\rho_{[n]} : \langle \beta_n \rangle \leq B_n \to U(\mathcal{V}^{[n]}) \tag{5.5}$$

where

$$\begin{aligned} \rho_{[n]}(\beta_n) : \mathcal{V}_Q^{[n]} &\to \mathcal{V}_Q^{[n]} \\ |\varphi\rangle &\mapsto e^{iu_Q} |\varphi\rangle \end{aligned} \tag{5.6}$$

That is,

$$\rho_{[n]}(\beta_n) = \sum_Q e^{iu_Q} \hat{P}_Q \tag{5.7}$$

where $\hat{P}_Q$ is a normalised projector onto $\mathcal{V}_Q^{[n]}$. Since the representation space $\mathcal{V}^{[n]}$ is the n-quasiparticle Hilbert space (modulo ordering), exchange symmetry is given by

$$[\hat{O}, \rho_{[n]}(\beta_n)] = 0 \tag{5.8}$$

for all n-particle observables $\hat{O}$ on $\mathcal{V}^{[n]}$. The spaces $\{\mathcal{V}_Q^{[n]}\}_Q$ are superselection sectors of the system, and we have shown by construction that each superselection sector is preserved under the action of of any n-braid. It follows that V_Q^s defines a superselection sector for any (s, Q). In conclusion, the superselection sectors of an n-quasiparticle system are given by the eigenspaces of the action of the braid β_n.

Corollary 5.2. *Given* $|\Psi\rangle \in V_Q^s$ *as in Theorem 5.1,*

$$\rho_s(\beta_n^{-1})\,|\Psi\rangle = e^{-iu_Q}\,|\Psi\rangle \tag{5.9}$$

Proof. Let $\tilde{s} := \beta_n(s)$ (i.e. string s in reverse order). By Theorem 5.1,

$$\begin{aligned} \rho_{\tilde{s}}(\beta_n)\,[\rho_s(\beta_n)\,|\Psi\rangle] &= e^{iu_Q}\,[\rho_s(\beta_n)\,|\Psi\rangle] \\ \Longrightarrow\ \rho_{\tilde{s}}(\beta_n)\,|\Psi\rangle &= e^{iu_Q}\,|\Psi\rangle \\ \Longrightarrow\ [\rho_{\tilde{s}}(\beta_n)]^\dagger\,|\Psi\rangle &= e^{-iu_Q}\,|\Psi\rangle \end{aligned}$$

where the second line is well-defined since it can be shown that $s \in \mathcal{U}_{\tilde{s},\beta_n}$. □

In order to prove Theorem 5.1, we will need the braid identity in Lemma 5.3 (whose proof is given in Appendix B.1).

Lemma 5.3. *Let* $n \geq 2$. *Then for* $i = 1, \ldots, n-1$,

$$\beta_n \sigma_i^{\pm 1} = \sigma_{n-i}^{\pm 1} \beta_n \tag{5.10}$$

Proof of Theorem 5.1.

Take n-quasiparticle space V^s for some chosen $s \in S_{\{1,\ldots,n\}}$. We write the eigenspace decomposition $V^s = \bigoplus_Q V_Q^s$ under $\rho_s(\beta_n)$ where

$$\begin{aligned} \rho_s(\beta_n) : V_Q^s &\to V_Q^{\beta_n(s)} \\ |\Psi\rangle &\mapsto e^{iu_Q}\,|\Psi\rangle \end{aligned} \quad , \ n \geq 2 \tag{5.11}$$

Then for $1 \leq i \leq n-1$,

$$\rho_s(\beta_n \sigma_i^{\pm 1})\,|\Psi\rangle = \rho_{\sigma_i(s)}(\beta_n)\,\left[\rho_s(\sigma_i^{\pm 1})\,|\Psi\rangle\right]$$

and

$$\begin{aligned} \rho_s(\beta_n \sigma_i^{\pm 1})\,|\Psi\rangle &= \rho_s(\sigma_{n-i}^{\pm 1}\beta_n)\,|\Psi\rangle \quad \text{(by Lemma 5.3)} \\ &= e^{iu_Q}\,\left[\rho_{\beta_n(s)}(\sigma_{n-i}^{\pm 1})\,|\Psi\rangle\right] \end{aligned}$$

where $\sigma_i(s)$ swaps the i^{th} and $(i+1)^{th}$ characters of s, and $\beta_n(s)$ reverses the order of the characters in s. Then by (4.16), we have

$$\rho_{\beta_n(s)}(\sigma_{n-i}^{\pm 1})\,|\Psi\rangle = \rho_s(\sigma_i^{\pm 1})\,|\Psi\rangle$$

It follows that the image of V_Q^s under $\rho_s(\sigma_i^{\pm 1})$ is the e^{iu_Q}-eigenspace of $\rho_{\sigma_i(s)}(\beta_n)$, so we write

$$\rho_s(\sigma_i^{\pm 1})\left(V_Q^s\right) =: V_Q^{\sigma_i(s)}$$

The result follows. □

5.2. **Fusion structure.** A composite collection of quasiparticles will exhibit the same statistical behaviour as a single quasiparticle under exchanges: the scheme under which a collection of quasiparticles is considered as a composite is known as *fusion*. In this section, we will carefully show the emergence of this behaviour by considering the action of the superselection braid.

Definition 5.4. We define $t_{k,l}$ to be the braid in B_{k+l} that clockwise exchanges k strands with l strands. Similarly, we define $u_{k,l}$ to be the braid in B_{k+l} that anticlockwise exchanges k strands with l strands. Clearly, $t_{k,l}^{-1} = u_{l,k}$.

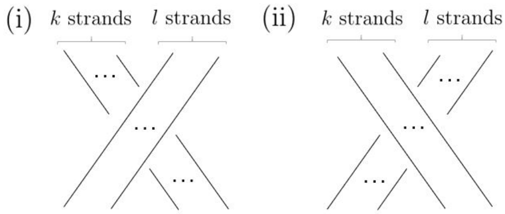


FIGURE 6. (i) $t_{k,l}$, (ii) $u_{k,l}$

For any $a \in \mathbb{N}_0$, we have the homomorphism

$$\begin{aligned} r_a : B_n &\to B_{n+a} \\ \sigma_i &\mapsto \sigma_{i+a} \end{aligned} \tag{5.12}$$

where $r_{a_1} \circ r_{a_2} = r_{a_1+a_2}$. We also have the anti-automorphism

$$\begin{aligned} \chi : B_n &\to B_n \\ \sigma_i &\mapsto \sigma_i \end{aligned} \tag{5.13}$$

which reverses the order of the generators in a braid word. Let $\overleftarrow{b} := \chi(b)$. Note that

$$\begin{aligned} t_{k,l} &= r_0(\overleftarrow{b_l}) \cdot r_1(\overleftarrow{b_l}) \cdot \ldots \cdot r_{k-1}(\overleftarrow{b_l}) \\ &= r_{l-1}(b_k) \cdot \ldots \cdot r_1(b_k) \cdot r_0(b_k) \end{aligned} \tag{5.14}$$

and that $\overleftarrow{t_{k,l}} = t_{l,k}$.

Consider some n-quasiparticle system V_Q^s in fixed superselection sector Q for some $s \in S_{\{1,\ldots,n\}}$ where $n \geq 2$. Partition s into nonempty substrings m_1, m_2 i.e. $V_Q^s = V_Q^{m_1,m_2}$ and denote the length of string m_i by $|m_i|$. We write eigenspace decompositions

$$V^{m_1} = \bigoplus_X V_X^{m_1} \quad , \quad V^{m_2} = \bigoplus_Y V_Y^{m_2} \tag{5.15}$$

under $\rho_{m_1}(\beta_{|m_1|})$ and $\rho_{m_2}(\beta_{|m_2|})$. Similarly to (4.25), we have the decompositions

$$V_Q^{m_1,m_2} \cong \bigoplus_{X,Y} V_X^{m_1} \otimes V_Q^{XY} \otimes V_Y^{m_2} \tag{5.16a}$$

$$V_Q^{m_2,m_1} \cong \bigoplus_{X,Y} V_Y^{m_2} \otimes V_Q^{YX} \otimes V_X^{m_1} \tag{5.16b}$$

Theorem 5.5. (Fusion)
For an n-quasiparticle system V_Q^s with fixed superselection sector Q, consider its decomposition as in (5.16a). Let $(k,l) := (|m_1|, |m_2|)$ and take $(X,Y) = (x,y)$ such that V_Q^{xy} is nonzero. Take arbitrary $|\psi\rangle := |\psi_x\rangle\, |\psi_Q^{xy}\rangle\, |\psi_y\rangle \in V_x^{m_1} \otimes V_Q^{xy} \otimes V_y^{m_2}$ where we have eigenvalues

$$\rho_{m_1}(\beta_k)\,|\psi_x\rangle = e^{iu_x}\,|\psi_x\rangle \;\;,\;\; \rho_{m_2}(\beta_l)\,|\psi_y\rangle = e^{iu_y}\,|\psi_y\rangle \;\;,\;\; \rho_s(\beta_{k+l})\,|\psi\rangle = e^{iu_Q}\,|\psi\rangle$$

Then,

(i) $\rho_s(t_{k,l})\,|\psi\rangle = e^{i(u_Q-u_x-u_y)}\,|\psi\rangle$

(ii) Eigenspaces are preserved under exchanges i.e.

$$\rho_s(t_{k,l}) : V_x^{m_1} \otimes V_Q^{xy} \otimes V_y^{m_2} \xrightarrow{\sim} V_y^{m_2} \otimes V_Q^{yx} \otimes V_x^{m_1} \tag{5.17}$$

(iii) $\rho_{m_2,m_1}(t_{l,k})\,[\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle] = e^{i(u_Q-u_x-u_y)}\,[\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle]$*, and so*

$$\rho_s(t_{l,k}\cdot t_{k,l})\,|\psi\rangle = e^{i2(u_Q-u_x-u_y)}\,|\psi\rangle \tag{5.18}$$

Corollary 5.6.

$$\rho_s(u_{k,l})\,|\psi\rangle = e^{-i(u_Q-u_x-u_y)}\,|\psi\rangle \tag{5.19}$$

Proof.

$$\begin{aligned} &[\rho_{m_2,m_1}(t_{l,k})]^\dagger\,\rho_{m_2,m_1}(t_{l,k})\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle = \rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle \\ \implies &\rho_{m_1,m_2}(u_{k,l})\,\left[e^{i2(u_Q-u_x-u_y)}\,|\psi\rangle\right] = e^{i(u_Q-u_x-u_y)}\,|\psi\rangle \end{aligned}$$

□

Theorem 5.5 tells us that the k and l-quasiparticle composites m_1 and m_2 (in eigenstates of $\rho_{m_1}(\beta_k)$ and $\rho_{m_2}(\beta_l)$ respectively) behave identically to a pair of quasiparticles under exchange: if we fix eigenspaces $V_x^{m_1}$ and $V_y^{m_2}$ such that V_Q^{xy} is nonzero, then composites m_1 and m_2 behave as a pair of quasiparticles in superselection sector Q with exchange phase $e^{i(u_Q-u_x-u_y)}$. The eigenspaces of $\rho_{m_1}(\beta_k)$ and $\rho_{m_2}(\beta_l)$ may thus be considered as representing different 'types' of quasiparticles (since the exchange phase depends on x and y). We will refer to the 'type' of a quasiparticle as its (topological) *charge*. If e.g. $k > 1$, we say that the collection m_1 of quasiparticles *fuses* to a quasiparticle of charge x.[18] It follows that the possible (x,y) for which V_Q^{xy} is nonzero represent the distinct possible *fusion outcomes* here.

Recall from Remark 4.4 that we can have a coherent superposition of distinct fusion outcomes for an entangled subsystem of quasiparticles. Furthermore, since the eigenspaces of any $\rho_\Sigma(\beta_n)$ (where Σ is an unordered set of quasiparticles of cardinality n) can be identified with quasiparticle charges, it follows that the superselection sector of a system can be identified with a (composite) quasiparticle of fixed charge. A complete system of quasiparticles thus has fixed total charge (fusion outcome).

[18]For k=1, note that $u_x = 0$ since the eigenvalue of $\rho_{m_1}(\beta_1)$ is trivial. Let $m_1 = q_j$. In (5.17), we write $x = q_j$ i.e. q_j 'fuses to itself'. Note that $V_{q_j}^{q_j} = V^{q_j}$ since the eigenspace is the whole space, and recall that a 1-quasiparticle Hilbert space is canonically isomorphic to $\mathbb{C}$.

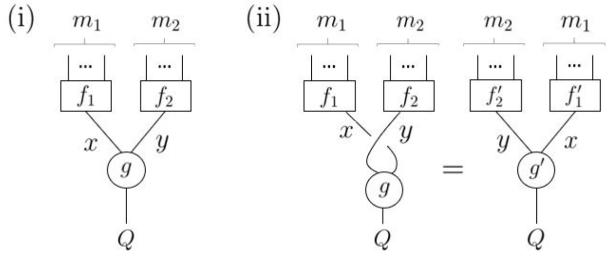


FIGURE 7. (i) The fusion diagram graphically depicting an arbitrary state in $V_x^{m_1} \otimes V_Q^{xy} \otimes V_y^{m_2}$ where $f_1 \in V_x^{m_1}$, $f_2 \in V_y^{m_2}$ and $g \in V_Q^{xy}$. (ii) Composite charges x and y are exchanged in superselection sector Q, so the fusion state acquires phase $e^{i(u_Q - u_x - u_y)}$ relative to (i).

This lends the hitherto abstract factor V_Q^{xy} in (5.17) a more tangible interpretation: $V_x^{m_1} \otimes V_Q^{xy} \otimes V_y^{m_2}$ is the space of states describing the process where collection m_1 fuses to (a quasiparticle of charge) x, collection m_2 fuses to y, and then x and y fuse to Q (see Figure 7(i)). The interpretation of any such tensor decomposition follows analogously. Such Hilbert spaces are thus known as *fusion spaces* and their constituent states are called *fusion states.*

Corollary 5.7. *Fusion is commutative and associative.*

Proof. Commutativity follows from Theorem 5.1: the possible fusion outcomes for an n-quasiparticle system correspond to the eigenspaces of $\rho_{[n]}(\beta_n)$ on $\mathcal{V}^{[n]}$ (whence the order of the n quasiparticles is irrelevant).
Associativity follows from recursive application of Theorem 5.5 i.e. further partitioning m_1 and m_2 and so on until no further partitions can be made: we will view such a recursive choice of partitions as a *full rooted binary tree with n leaves.* This provides us with a *fusion tree* illustrating the order in which n quasiparticles are fused (see Figure 8). Since Q corresponds to an arbitrary eigenspace of $\rho_s(\beta_n)$, it follows that the set of possible fusion outcomes (i.e. the set of possible labels for the root) does not depend on the order in which fusion occurs. □

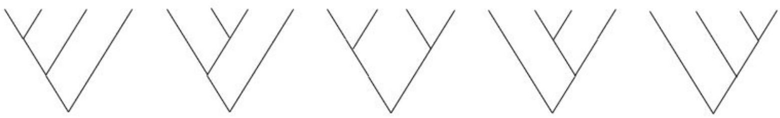


FIGURE 8. All possible fusion trees for 4 particles. For n particles, the number of possible fusion trees is given by $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$ i.e. the $(n-1)^{th}$ Catalan number.

By the associativity and commutativity of fusion, the charge of an unordered collection Σ of quasiparticles can be thought of as a property of any connected region of the system in which solely the excitations in Σ are enclosed. This is one of the reasons that quasiparticle charge is called 'topological' (as opposed to e.g. electric charge which is defined geometrically via the charge density). Similarly to electric charge, we have seen that topological charge may correspond to a superselection rule of a system; but unlike electric charge, we may also observe a superposition of topological charges (for an entangled subsystem).

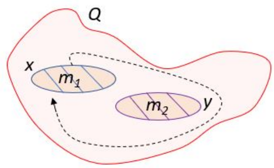


FIGURE 9. Winding a quasiparticle collection m_1 of charge x around collection m_2 of charge y in a region of total charge Q accumulates statistical phase $e^{i2(u_Q-u_x-u_y)}$. This diagram illustrates the same process as on the left-hand side of Figure 7(ii) but with an additional exchange.

Remark 5.8. Take care to note that statistical phases of the form e^{iu_Q} are not a property of charge Q alone, but arise as eigenvalues of some $\rho_s(\beta_n)$ i.e. the phase also depends on the constituent charges fusing to Q. To this end, a better notation for e^{iu_Q} would be $\omega_Q^\Sigma \in U(1)$ where Σ is the unordered set of constituent characters of s. Nonetheless, we have opted for the former notation for sake of presentation.

As indicated by Theorem 5.5, fusion generally does not correspond to a physical process but rather describes how a collection of charges may be considered as a composite charge. Of course, the *measurement* of a fusion outcome is physically significant.

In order to prove Theorems 5.5, we will need the braid identities in Theorem 5.9 (whose proof is given in Appendix B.2). Theorem 5.9 shows that the superselection braid may be defined recursively.[19,20]

Theorem 5.9. (Superselection braid by recursion)
Let $n \geq 2$. For any positive integers k, l such that $k + l = n$, β_n is given by

(i) $[\beta_l \cdot r_l(\beta_k)]\, t_{k,l}$
(ii) $t_{k,l}\, [\beta_k \cdot r_k(\beta_l)]$
(iii) $\beta_l \cdot t_{k,l} \cdot \beta_k$
(iv) $r_l(\beta_k) \cdot t_{k,l} \cdot r_k(\beta_l)$

and $\beta_1 := e$. The terms enclosed in square brackets commute.

[19]Choosing between forms (i)-(iv) at each decision (and permuting the terms in square brackets if desired) parses β_n into a composition of braids of the form $r_d(t_{k,l})$. The braid word (5.2) for β_n is recovered by choosing (ii) at every iteration with $l = 1$.

[20]Note that β_n^{-1} is given by (i)-(iv) but with a superscript '-1' on each t and β. This is easily seen by inverting (i)-(iv).

Proof of Theorem 5.5.

Let $\tilde{v}$ denote the reverse of a string v.

(i) Using Theorem 5.9(ii),

$$\begin{aligned}\rho_s(\beta_n)\,|\psi\rangle &= \rho_{\tilde{m}_1,\tilde{m}_2}(t_{k,l})\ \rho_{m_1,m_2}\left([\beta_k\cdot r_k(\beta_l)]\right)|\psi\rangle\\ &= \rho_{\tilde{m}_1,\tilde{m}_2}(t_{k,l})\left[e^{i(u_x+u_y)}\,|\psi\rangle\right]\end{aligned}$$

Recalling that $\rho_s(\beta_n)\,|\psi\rangle = e^{iu_Q}\,|\psi\rangle$, we deduce that

$$\begin{aligned}\rho_{\tilde{m}_1,\tilde{m}_2}(t_{k,l}) : V_x^{\tilde{m}_1}\otimes V_Q^{xy}\otimes V_y^{\tilde{m}_2} &\to V_Q^{\tilde{s}}\\ |\phi\rangle &\mapsto e^{i(u_Q-u_x-u_y)}\,|\phi\rangle\end{aligned}$$

(ii) We know that

$$\rho_s(t_{k,l}) : V_x^{m_1}\otimes V_Q^{xy}\otimes V_y^{m_2}\to V_Q^{m_2m_1} \tag{5.20}$$

where $V_Q^{m_2,m_1}$ has decomposition (5.16b). We wish to show that the range of (5.20) is restricted as in (5.17). Using Theorem 5.5(i) and Theorem 5.9(iii),

$$\begin{aligned}\rho_s(\beta_n)\,|\psi\rangle &= \rho_{m_2,\tilde{m}_1}(\beta_l)\ \rho_{\tilde{m}_1,m_2}(t_{k,l})\ \rho_{m_1,m_2}(\beta_k)\,|\psi\rangle\\ &= \rho_{m_2,\tilde{m}_1}(\beta_l)\left[e^{i(u_Q-u_y)}\,|\psi\rangle\right]\end{aligned}$$

and since $\rho_s(\beta_n)\,|\psi\rangle = e^{iu_Q}\,|\psi\rangle$, we deduce that

$$\rho_{\tilde{m}_1,m_2}(t_{k,l}) : V^{\tilde{m}_1}\otimes V_Q^{xy}\otimes V_y^{m_2}\to\bigoplus_X V_y^{m_2}\otimes V_Q^{yX}\otimes V_X^{\tilde{m}_1} \tag{5.21}$$

Similarly, by using Theorem 5.5(i) and Theorem 5.9(iv) we may deduce that

$$\rho_{m_1,\tilde{m}_2}(t_{k,l}) : V^{m_1}\otimes V_Q^{xy}\otimes V_y^{\tilde{m}_2}\to\bigoplus_Y V_Y^{\tilde{m}_2}\otimes V_Q^{Yx}\otimes V_x^{m_1} \tag{5.22}$$

Combining (5.21) and (5.22), the result follows.

(iii) By identities (i) and (ii) of Theorem 5.9,

$$\beta_n^2 = t_{l,k}\left[r_l(\beta_k^2)\cdot\beta_l^2\right]t_{k,l} \tag{5.23}$$

whence

$$\begin{aligned}\rho_s(\beta_n^2)\,|\psi\rangle &= e^{i2(u_x+u_y)}\left[\rho_{m_2,m_1}(t_{l,k})\cdot\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle\right]\\ \implies\ \rho_{m_2,m_1}(t_{l,k})\left[\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle\right] &= e^{i2(u_Q-u_x-u_y)}\,|\psi\rangle\\ \implies\ \rho_{m_2,m_1}(t_{l,k})\left[\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle\right] &= e^{i(u_Q-u_x-u_y)}\left[\rho_{m_1,m_2}(t_{k,l})\,|\psi\rangle\right]\end{aligned}$$

where we used parts (i) and (ii) of Theorem 5.5 in the third and first lines respectively.

□

Given the fusion space $V^s = \bigoplus_Q V_Q^s$ (where $s = q_1\ldots q_n \in S_{\{1,\ldots,n\}}$ and Q indexes the superselection sectors), fix a fusion tree (as in Figure 8): each of the $n-1$ fusion vertices[21] corresponds to an eigenspace of $\rho_{s(v)}(\beta_{|s(v)|})$, where for a fusion vertex v we let $s(v)$ denote the substring of s given by the leaves descending from v, and $|s(v)|$ denotes the length of $s(v)$. Note that $2\le|s(v)|\le n$.

[21] By "fusion vertices", we mean vertices in the fusion tree with two or more incident edges i.e. any vertex that is not a leaf. Leaves correspond to initial quasiparticles.

We thus label each fusion vertex v with an eigenspace of $\rho_{s(v)}(\beta_{|s(v)|})$ (recall that such a label represents a fixed topological charge and is called a 'fusion outcome' in this context). Such a labelling is called *admissible* if the corresponding fusion subspace of V^s has nonzero dimension. Note that the root label corresponds to the superselection sector of the system. Observe that fixing a fusion tree specifies a decomposition of V^s in terms of the eigenspaces of $\{\rho_{s(v)}(\beta_{|s(v)|})\}_v$. We write such a decomposition in the form yielded by recursive application of (5.16a) e.g. a fusion tree of the form illustrated in Figure 10 specifies the decomposition

$$V^{q_1q_2q_3q_4} \cong \bigoplus_{X_1,X_2,Q} V^{q_1q_2}_{X_1} \otimes V^{X_1,q_3}_{X_2} \otimes V^{X_2,q_4}_{Q} \tag{5.24}$$

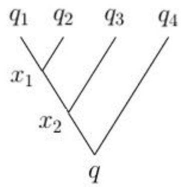


FIGURE 10. The labels x_1, x_2 and q correspond to eigenspaces of $\rho_{q_1q_2}(\beta_2), \rho_{q_1q_2q_3}(\beta_3)$ and $\rho_{q_1q_2q_3q_4}(\beta_4)$ respectively. The triple (x_1, x_2, q) of charges is an admissible labelling of the tree if the fusion subspace $V^{q_1q_2}_{x_1} \otimes V^{x,q_3}_{x_2} \otimes V^{x_2,q_4}_{Q} \subseteq V^{q_1q_2q_3q_4}$ is nonzero.

Theorem 5.9 provides a method for parsing β_n into a composition of braids of the form $r_d(t_{k,l})$. Any such parsing involves making a choice of $n-1$ partitions. From any possible sequence of partitions, we can always recover a fusion tree with which the parsing of β_n is *compatible*. By compatibility, we mean that it is readily apparent how the fusion tree will transform under the action of β_n i.e. β_n can be parsed into a sequence of braids that each have a well-defined action on the decomposed components of the system. The incoming branches of each fusion vertex in the tree are clockwise exchanged and so the initial fusion tree is sent to its mirror image. The braid β_n is thus compatible with all n-leaf fusion trees (as expected).

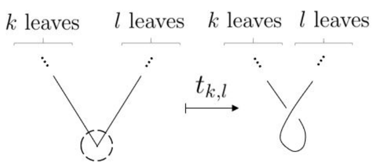
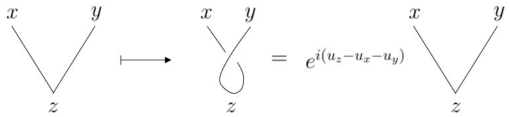


FIGURE 11. $t_{k,l}$ clockwise exchanges the incoming branches of a fusion vertex that has k leaves and l leaves stemming from it.

Remark 5.10. Given $|\psi\rangle \in V^s_Q$, we know that $\rho_s(\beta_n)\,|\psi\rangle = e^{iu_Q}\,|\psi\rangle$. It is illuminating to examine how the phase e^{iu_Q} arises given a decomposition of V^s_Q . Consider any admissibly labelled fusion tree in $V^{q_1\cdots q_n}_Q$ (whence the root has label Q). We know that $\rho_s(\beta_n)$ will clockwise exchange the incoming branches of every fusion vertex. For any fusion vertex, the clockwise exchange is given by



where the phase evolution follows from Theorem 5.5. It is easy to see that the total phase evolution acquired by clockwise exchanging the incoming branches of every fusion vertex will be $e^{i\left[u_Q-(u_{q_1}+\cdots+u_{q_n})\right]}$ (phases associated to internal nodes of the tree will cancel). Finally, observe that the u_{q_i} are zeroes (since they are arguments of eigenvalues under the action of $\beta_1 = e$).

Theorem 5.11. (Uniqueness of the superselection braid)
$\beta_n^{\pm 1}$ are the unique braids under whose action the fusion space decomposes into the superselection sectors of an n-quasiparticle system.

A proof of Theorem 5.11 is outlined in Appendix C.

6. Theories of Anyons

This section primarily serves to connect the narrative of Section 5 with the standard formalism in the literature, by outlining the additional postulates (**A2**-**A3**) required to make contact with the usual algebraic theory of anyons. Our presentation thus omits various details, and is not intended as an introduction. For a more detailed treatment, we refer the reader to [4, 5, 6, 7, 8]. In relation to additional insights arising from consideration of the superselection braid, we highlight Section 6.3.

6.1. **Labels and finiteness.** In any standard theory of anyons, it is assumed that there are finitely many distinct topological charges. A theory of anyons thus comes equipped with a finite set of labels $\mathfrak{L}$ whose cardinality is called the *rank* of the theory. It is also assumed that the representation space in (4.2) is finite which immediately tells us that $\dim(\mathcal{V}_c^{\{a,b\}})$ is finite for any $a, b, c \in \mathfrak{L}$ (from which it easily follows that a fusion space for finitely many quasiparticles is finite-dimensional). We package these two assumptions into the finiteness assumption **A2** below.

Definition 6.1. Given fusion space V_c^{ab} for any $a, b, c \in \mathfrak{L}$, we write $N_c^{ab} := \dim(V_c^{ab})$. The quantities $\{N_c^{ab}\}_{a,b,c\in\mathfrak{L}}$ are called the *fusion coefficients* of the theory.

Since $\dim(\mathcal{V}_c^{\{a,b\}}) = \dim(V_c^{ab}) = \dim(V_c^{ba})$ we have the symmetry

$$N_c^{ab} = N_c^{ba} \quad \text{for all } a, b, c \in \mathfrak{L} \tag{6.1}$$

which is consistent with the commutativity of fusion from Corollary 5.7. The quantity N_c^{ab} may be thought of as counting 'the distinct number of ways charges a and b can fuse to charge c'. Note that $\dim(V^{ab}) = \sum_{c\in\mathfrak{L}} N_c^{ab}$ and that if $N_c^{ab} = 0$ then a and b cannot fuse to c. Consider V_d^{abc} for any $a, b, c, d \in \mathfrak{L}$. By associativity of fusion (Corollary 5.7), the decompositions of a fusion space must be isomorphic

$$\bigoplus_e V_e^{ab} \otimes V_d^{ec} \cong \bigoplus_f V_d^{af} \otimes V_f^{bc} \tag{6.2}$$

and so the fusion coefficients satisfy the associativity relation

$$\sum_{e\in\mathfrak{L}} N_e^{ab} N_d^{ec} = \sum_{f\in\mathfrak{L}} N_d^{af} N_f^{bc} \tag{6.3}$$

A2. A theory of anyons has finitely many distinct topological charges and all fusion coefficients are finite.

Any label set will include the *trivial* label (denoted by 0) which represents (the topological charge of) the vacuum: the fusion of any charge with the vacuum yields the original charge i.e. $N_r^{0q} \propto \delta_{qr}$ for any $q, r \in \mathfrak{L}$. Since we always have the freedom to insert the trivial charge anywhere, we must have

$$\dim(V_c^{ab}) = \dim(V_c^{a0b}) = \dim(V_c^{0ab}) = \dim(V_c^{ab0}) \tag{6.4}$$

Associativity and (6.4) tell us that $N_a^{a0} N_c^{ab} = N_c^{ab} N_b^{0b} = N_c^{ab}$ and so $N_a^{a0} = N_b^{b0} = 1$ for all $a, b \in \mathfrak{L}$. Thus,

$$N_r^{q0} = N_r^{0q} = \delta_{qr} \quad \text{for any } q, r \in \mathfrak{L} \tag{6.5}$$

Following the presentation in [4], write $V_a^{a0} = \mathrm{span}_{\mathbb{C}}\{|\alpha_a\rangle\}$ and $V_b^{0b} = \mathrm{span}_{\mathbb{C}}\{|\beta_b\rangle\}$. The relation between the spaces in (6.4) is characterised by trivial isomorphisms

$$\begin{aligned} \alpha_q : \mathbb{C} &\to V_q^{q0} & \qquad \beta_q : \mathbb{C} &\to V_q^{0q} \\ z &\mapsto z\,|\alpha_q\rangle & z &\mapsto z\,|\beta_q\rangle \end{aligned} \tag{6.6}$$

e.g. $V_c^{ab} \overset{\alpha_a}{\xrightarrow{\sim}} V_a^{a0} \otimes V_c^{ab}$ and $V_c^{ab} \overset{\beta_b}{\xrightarrow{\sim}} V_c^{ab} \otimes V_b^{0b}$. By associativity we see that α_a and β_b are related (see Remark 6.3 and Appendix D). Braiding with the vacuum must be trivial i.e. using the same notation as in (4.2),

$$\rho_{\{q,0\}}(\sigma_1^{\pm 1}) = 1 \quad \text{for all } q \in \mathfrak{L} \tag{6.7}$$

6.2. **Braided** $6j$ **fusion systems.** We write orthonormal bases

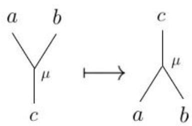
$$V_c^{ab} = \mathrm{span}_{\mathbb{C}}\{|ab \to c; \mu\rangle\}_\mu \quad , \quad V_c^{ba} = \mathrm{span}_{\mathbb{C}}\{|ba \to c; \mu\rangle\}_\mu \tag{6.8}$$

of fusion states given any $a, b, c \in \mathfrak{L}$ where $1 \leq \mu \leq N_c^{ab}$ for $N_c^{ab} \neq 0$.

FIGURE 12. A graphical depiction of the fusion state $|ab \to c; \mu\rangle$. We will implicitly assume that our fusion vertices are normalised.

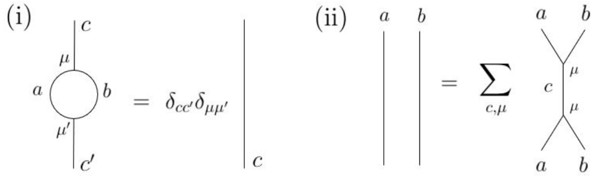
The *dual space* of a fusion space has natural interpretation as a '*splitting space*' i.e.

$$V_c^{ab} \xrightarrow{\dagger} V_{ab}^c := \mathrm{span}_{\mathbb{C}}\{\langle ab \to c; \mu|\}_{\mu=1}^{N_c^{ab}} \ : \tag{6.9}$$



for any $a, b, c \in \mathfrak{L}$. Fusion coefficients may thus also be thought of 'splitting' coefficients. Given an orthonormal basis, we can use the graphical calculus to express the inner product and completeness relation on V^{ab} :

(6.10)



The *R-matrices* of a theory are given by a matrix representation of the unitary operators from (4.2), typically in an eigenbasis: given any $a, b \in \mathfrak{L}$ we have the eigenspace decomposition $\mathcal{V}^{\{a,b\}} = \bigoplus_{Q\in\mathfrak{L}} \mathcal{V}_Q^{\{a,b\}}$ under $\rho_{\{a,b\}}$ where

$$\rho_{\{a,b\}}(\sigma_1^{\pm 1})\,|\psi\rangle = e^{\pm i u_Q}\,|\psi\rangle \tag{6.11}$$

for $|\psi\rangle \in \mathcal{V}_Q^{\{a,b\}}$ with Q such that $N_Q^{ab} \neq 0$. We write R-matrices

$$R_Q^{ab} : V_Q^{ab} \xrightarrow{\sim} V_Q^{ba} \quad , \quad R_Q^{ba} : V_Q^{ba} \xrightarrow{\sim} V_Q^{ab} \tag{6.12}$$

where we let

$$R_Q^{ab} = R_Q^{ba} = \bigoplus_{j=1}^{N_Q^{ab}} [e^{iu_Q}] \tag{6.13a}$$

$$R^{ab} := \bigoplus_{Q\in\mathfrak{L}\,:\,N_Q^{ab}\neq 0} \left[R_Q^{ab}\right] \quad , \quad R^{ba} := \bigoplus_{Q\in\mathfrak{L}\,:\,N_Q^{ba}\neq 0} \left[R_Q^{ba}\right] \tag{6.13b}$$

It is clear that $R^{ab} = R^{ba}$ here.[22] Following (6.7), we have

$$R_q^{q0} = R_q^{0q} = 1 \tag{6.14}$$

for all $q \in \mathfrak{L}$. We let $(R^{-1})^{ab}$ denote the anticlockwise exchange i.e.

$$(R^{ab})^{-1} = (R^{-1})^{ba} \tag{6.15}$$

For an n-quasiparticle fusion space $V^{q_1\cdots q_n}$ (where $q_1, \ldots, q_n \in \mathfrak{L}$) let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be decompositions of this space corresponding to distinct fusion trees. By associativity, we have an isomorphism

$$\mathcal{F} : \mathfrak{D}_1 \to \mathfrak{D}_2 \tag{6.16}$$

Fixing a basis of fusion states, we see that $\mathcal{F} \in \mathrm{Aut}(V^{q_1\cdots q_n})$ is a change of basis matrix. Observe that $\mathcal{F}$ is given by any sequence of so-called *F-moves* that transform between decompositions of the form

and

[22] R-matrices need not always be diagonal and symmetric in their upper indices. However, our construction has implicitly 'fixed a gauge' where this is the case; see Remark 6.2 and (D.6).

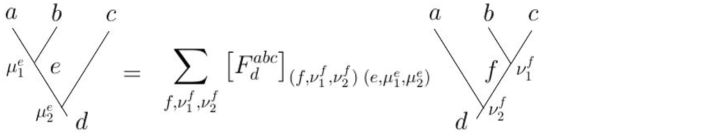
Such transformations are realised by the *F-matrices* of a theory. These are matrices $F_d^{abc} \in \mathrm{Aut}(V_d^{abc})$ for any $a, b, c, d \in \mathfrak{L}$ where

$$F_d^{abc} : \bigoplus_{e\in\mathfrak{L}} V_e^{ab} \otimes V_d^{ec} \xrightarrow{\sim} \bigoplus_{f\in\mathfrak{L}} V_d^{af} \otimes V_f^{bc} \tag{6.17}$$

This is a unitary matrix representing the isomorphism in (6.2). That is, F_d^{abc} transforms between the bases

$$\Big\{ |ab \to e; \mu_1^e\rangle \, |ec \to d; \mu_2^e\rangle \Big\}_{e,\mu_1^e,\mu_2^e} \quad \text{and} \quad \Big\{ |af \to d; \nu_2^f\rangle \, |bc \to f; \nu_1^f\rangle \Big\}_{f,\nu_1^f,\nu_2^f} \tag{6.18}$$

This change of basis is graphically expressed as



(6.19)

Distinct fusion trees specify distinct bases on the fusion space and are therefore also called *fusion bases*. Since R^{ab} is defined for an eigenbasis of V^{ab}, we must fix a fusion basis such that the factors $\{V_Q^{ab}\}_{Q\in\mathfrak{L}}$ appear in the decomposition of the fusion space: for any such fusion basis, we say that 'a and b are in a *direct fusion channel*'. That is, R-matrices can only act on two charges in a direct fusion channel.

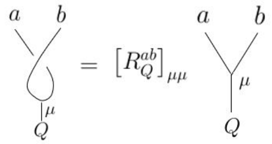
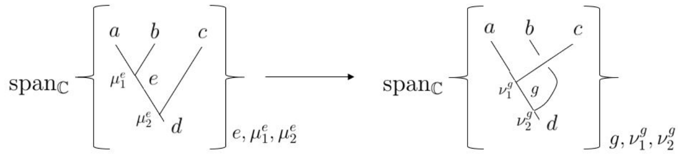


FIGURE 13. Charges a and b are in a direct fusion channel with outcome Q. The above is a graphical expression of the equation $R^{ab}\,|ab \to Q; \mu\rangle = \left[R_Q^{ab}\right]_{\mu\mu} |ab \to Q; \mu\rangle \in \mathrm{span}_{\mathbb{C}}\{|ba \to Q; \mu\rangle\} \subseteq V_Q^{ba}$ where the matrix R^{ab} is defined as in (6.13a) and (6.13b).

We may obtain a (possibly non-diagonal) representation of the exchange operator for two adjacent quasiparticles a and b in a system by considering its action with respect to a fusion basis in which a and b are in an *indirect* fusion channel.[23] Such a representation can be determined by transforming into a fusion basis where the charges are in a *direct* fusion channel, applying the R-matrix and then transforming back to the original fusion basis. Below is the simplest example of such a procedure.

[23]Non-diagonal representations arise since fixing an indirect fusion channel of two charges means that we are not in an eigenbasis of the exchange operator for these charges. Since we are not in an eigenbasis, we cannot apply the R-matrix directly.

$B_d^{a(bc)}$:



where

$$
\begin{tikzcd}
\bigoplus_e V_e^{ab} \otimes V_d^{ec} \arrow[r, "F_d^{abc}"] \arrow[d, "B_d^{a(bc)}"'] & \bigoplus_f V_d^{af} \otimes V_f^{bc} \arrow[d, "R^{bc}"] \\
\bigoplus_g V_g^{ac} \otimes V_d^{gb} \arrow[r, "F_d^{acb}"] & \bigoplus_f V_d^{af} \otimes V_f^{cb}
\end{tikzcd} \tag{6.20}
$$

That is,

$$B_d^{a(bc)} = \left(F_d^{acb}\right)^{\dagger} R^{bc} F_d^{abc} \tag{6.21}$$

where

$$R^{bc} = \bigoplus_{f \in \mathfrak{L} \,:\, N_d^{af} N_f^{bc} \neq 0} R_f^{bc} \tag{6.22}$$

A charge $q \in \mathfrak{L}$ such that $\sum_{u \in \mathfrak{L}} N_u^{qx} = 1$ for all $x \in \mathfrak{L}$ corresponds to an *abelian* anyon (since its exchange statistics with any other charge will always be given by a phase). Otherwise, q corresponds to a *non-Abelian* anyon (since there exists a charge with which its exchange statistics are given by a higher-dimensional unitary transformation). An abelian theory of anyons is one in which there are no non-abelian anyons. Observe that given a fixed fusion basis and an explicit choice of orthonormal basis for a fusion space of n *identical charges*, we obtain a unitary matrix representation of the braid group B_n.

Remark 6.2. (Gauge freedom)
There is generally some redundancy amongst the F and R symbols[24] of a theory: this arises from the $U(N_c^{ab})$ freedom when fixing an orthonormal basis on the spaces $\{V_c^{ab}\}_{a,b,c \in \mathfrak{L}}$. A change of basis[25] is called a *gauge transformation*. We can only attach physical significance to gauge-invariant quantities.
Although R-symbols are generally gauge-variant, gauge transformations are defined to respect the triviality of braiding with the vacuum (i.e. (6.14) is gauge-invariant by construction). A *monodromy* is a composition

$$R^{ba} \circ R^{ab} =: M^{ab} \tag{6.23}$$

It can be shown that monodromies are gauge-invariant, whence it follows that the action of any pure braid is gauge-invariant. We implicitly fixed a gauge where $R^{ab} = R^{ba}$ for all $a, b \in \mathfrak{L}$ in our construction: we will call this the *symmetric gauge*. R-matrices are not necessarily diagonal and symmetric in their upper indices outside of this gauge. Nonetheless, monodromy matrices are always diagonal and symmetric in their upper indices.

[24] F and R symbols refer to the entries of F and R matrices. F-symbols are also called $6j$ *symbols*.
[25] This is not to be confused with a change of *fusion* basis.

Remark 6.3. (Coherence conditions)
Isomorphisms between fusion spaces must be 'compatible' with one another. That is, distinct sequences of isomorphisms (F-moves, R-moves and isomorphisms α and β from (6.6)) between two given spaces should correspond to the same isomorphism. Such compatibility requirements are called *coherence conditions.* Remarkably, all coherence conditions are fulfilled if the triangle, pentagon and hexagon equations are satisfied. Some additional details are provided in Appendix D.

(i) All isomorphisms α and β from (6.6) must be compatible with associativity (F-moves). This coherence condition is fulfilled if the *triangle equations* (D.1) are satisfied.

(ii) Recall the isomorphism $\mathcal{F}$ from (6.16). It may be possible that multiple distinct sequences of F-moves realise $\mathcal{F}$. Given some basis, the matrix representation of $\mathcal{F}$ must be the same for all such sequences. This coherence condition is fulfilled if all F-symbols satisfy the *pentagon equation* (D.2).

(iii) Consider n-quasiparticle space $V^{q_1\cdots q_n}$ where $q_1,\ldots,q_n \in \mathfrak{L}$ and $n \geq 3$. Let s and s' be any two distinct permutations of the string $q_1 \ldots q_n$. Let $\mathcal{D}$ and $\mathcal{D}'$ be any decomposition of V^s and $V^{s'}$ respectively. It may be possible that multiple distinct sequences of F and R moves realise the isomorphism $\mathcal{B} : \mathcal{D} \to \mathcal{D}'$ corresponding to the action of some n-braid. Given some basis, the matrix representation of $\mathcal{B}$ must be the same for all such sequences. This coherence condition is fulfilled if all F and R symbols satisfy the *hexagon equations* (D.7).

For each charge in a theory of anyons, there exists a dual charge wihich with it may fuse to the vacuum. More precisely, we incorporate Kitaev's duality axiom from [4]:

> **A3.** For each $q \in \mathfrak{L}$, there exists some $\bar{q} \in \mathfrak{L}$ and $|\xi\rangle \in V_0^{q\bar{q}}$, $|\eta\rangle \in V_0^{\bar{q}q}$ such that
>
> (6.24) $$\langle \alpha_q \otimes \eta | \, F_q^{q\bar{q}q} \, | \xi \otimes \beta_q \rangle \neq 0$$
>
> where α_q, β_q are as defined in (6.6).

Proposition 6.4. [4, Lemma E.3.] *For $q \in \mathfrak{L}$, there exists unique $\bar{q} \in \mathfrak{L}$ such that*

(6.25) $$N_0^{pq} = N_0^{qp} = \delta_{p\bar{q}}$$

This proposition follows from **A3** and says that any charge has a *unique* dual charge with which it annihilates in a *unique* way. Together with associativity, **A3** tells us that for any $a, b, c \in \mathfrak{L}$ we have $N_{\bar{c}}^{ab} N_0^{\bar{c}c} = N_0^{a\bar{a}} N_{\bar{a}}^{bc}$ and so $N_{\bar{c}}^{ab} = N_{\bar{a}}^{bc}$. We thus have

(6.26) $$N_c^{ab} = N_{\bar{a}}^{b\bar{c}} = N_{\bar{b}}^{\bar{c}a}$$

Corollary 6.5. *Any topological charge $q \in \mathfrak{L}$ may realise a superselection sector.*

Proof. We know that it is possible for a fusion outcome to realise a superselection sector. Suppose there exists a charge $q \in \mathfrak{L}$ such that it is not a fusion outcome for any pair of charges. For any charge b there exists a charge c such that $N_c^{\bar{q}b} \neq 0$. By (6.26) we have $N_c^{\bar{q}b} = N_q^{b\bar{c}}$ which gives a contradiction. ↯

We see that the duality axiom permits any charge to realise a superselection sector. For this reason, labels are often called topological charges and superselection sectors interchangeably in the literature.

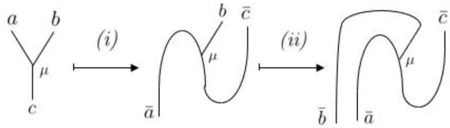
For $a, b, c \in \mathfrak{L}$ we define linear maps K_c^{ab} and L_c^{ab},

$$K_c^{ab} : V_c^{ab} \longrightarrow V_{\bar{a}a}^{0} \otimes V_c^{ab} \cong V_{\bar{a}c}^{b} \qquad\qquad L_c^{ab} : V_c^{ab} \longrightarrow V_c^{ab} \otimes V_{b\bar{b}}^{0} \cong V_{c\bar{b}}^{a}$$

(6.27) 

These are clearly invertible (whence $N_c^{ab} = N_b^{\bar{a}c} = N_a^{c\bar{b}}$). Observe that[26]

$$V_c^{ab} \xrightarrow{\left(L_{\bar{a}}^{b\bar{c}}\right)^{-1} \circ K_c^{ab}} V_{\bar{a}}^{b\bar{c}} \xrightarrow{K_{\bar{a}}^{b\bar{c}}} V_{\bar{b}\bar{a}}^{\bar{c}}$$

(6.28) 

where (i) corresponds to symmetries of the form in (6.26), and the composition of (i) and (ii) tells us that $N_c^{ab} = N_{\bar{c}}^{\bar{b}\bar{a}}$. Together with (6.1), these identities generate all symmetries of the fusion coefficients. Summarising these, for all $a, b, c \in \mathfrak{L}$ we have

(6.29a) $$N_c^{ab} = N_c^{ba}$$

(6.29b) $$N_c^{ab} = N_{\bar{a}}^{b\bar{c}} = N_{\bar{b}}^{\bar{c}a}$$

(6.29c) $$N_c^{ab} = N_{\bar{c}}^{\bar{b}\bar{a}}$$

Definition 6.6. Altogether, a finite label set $\mathfrak{L}$ with fusion coefficients, F-symbols and R-symbols as described above satisfying the triangle, pentagon and hexagon equations is called a *braided* $6j$ *fusion system.*

6.3. **Eigenvalues of the superselection braid.** In Remark 5.10, we examined the action of the superselection braid on any decomposition of the space V_Q^s (where s is any permutation of some n fixed labels). We know that this action results in the same statistical phase independently of the given permutation or decompositon. Our observations from Remark 5.10 look more interesting when recast in terms of R-matrices. Namely, for any choice of labels $1, 2, 3, 4 \in \mathfrak{L}$ such that V_4^{123} is nonzero, the elements of the table below are equal for any choice of e, f, g such that $N_e^{12}N_4^{e3}$, $N_f^{23}N_4^{f1}$ and $N_g^{13}N_4^{g2}$ are nonzero and where there exists a choice of gauge such that the relevant R-matrices may be written as in (6.13a)-(6.13b).

$R_e^{21} \otimes R_4^{e3}$	$R_e^{12} \otimes R_4^{e3}$	$R_4^{3e} \otimes R_e^{12}$	$R_4^{3e} \otimes R_e^{21}$
$R_f^{32} \otimes R_4^{f1}$	$R_f^{23} \otimes R_4^{f1}$	$R_4^{1f} \otimes R_f^{23}$	$R_4^{1f} \otimes R_f^{32}$
$R_g^{31} \otimes R_4^{g2}$	$R_g^{13} \otimes R_4^{g2}$	$R_4^{2g} \otimes R_g^{13}$	$R_4^{2g} \otimes R_4^{31}$

[26]The isomorphisms $K_{\bar{a}}^{b\bar{c}} \circ \left(L_{\bar{a}}^{b\bar{c}}\right)^{-1} \circ K_c^{ab}$ and $L_b^{\bar{c}a} \circ (K_b^{\bar{c}a})^{-1} \circ L_c^{ab}$ correspond to the CPT symmetry of V_c^{ab}. Indeed, in [4, Theorem E.6.] it is shown that these two maps are coincide (and are isometries), which is is equivalent to the statement that a unitary fusion category admits a pivotal structure.

Let r_c^{ab} denote the phase $R_c^{ab} = r_c^{ab} I_k$ (where I_k is the $k \times k$ identity matrix and $k = N_c^{ab}$). Noting that $r_c^{ab} = r_c^{ba}$ in the fixed gauge, the above equivalences may be expressed as

$$r_e^{12} r_4^{e3} = r_f^{23} r_4^{f1} = r_g^{13} r_4^{g2} \tag{6.30}$$

for any choice of e, f, g as specified above. The identity (6.30) characterises the fact that the statistical evolution under the action of the superselection braid is independent of the fusion basis and order of quasiparticles. However, this identity also has the weakness of being gauge-dependent. We easily obtain a gauge-invariant form of (6.30): writing $M_c^{ab} = m_c^{ab} I_k$ (where $m_c^{ab} = m_c^{ba}$ is the monodromy phase),

$$m_e^{12} m_4^{e3} = m_f^{23} m_4^{f1} = m_g^{13} m_4^{g2} \tag{6.31}$$

for any choice of e, f, g as specified above. This gives us the following ansatz: *for every $q \in \mathfrak{L}$ we may assign a quantity $\vartheta_q \in U(1)$ such that*

$$m_c^{ab} = \frac{\vartheta_c}{\vartheta_a \vartheta_b} \quad \text{for all } a, b, c \text{ such that } N_c^{ab} \neq 0 \tag{6.32}$$

Indeed, this ansatz turns out to be correct (see Section 6.5): the quantity ϑ_q is called the topological spin of q and is the phase evolution under a clockwise 2π-rotation of charge q. For a system of charges $q_1, \ldots, q_n$ with overall charge Q, the gauge-invariant statistical evolution under the action of the pure braid β_n^2 is thus given by (6.33) (whose form is consistent with Remark 5.8).

$$\frac{\vartheta_Q}{\vartheta_{q_1} \cdot \ldots \cdot \vartheta_{q_n}} \tag{6.33}$$

6.4. **Fusion algebras and their categorification.**

Definition 6.7. Let $\mathbb{Z}B$ be a free $\mathbb{Z}$-module with finite basis $B = \{b_i\}_{i \in I}$. We equip $\mathbb{Z}B$ with a bilinear product

$$\begin{aligned} \cdot : \mathbb{Z}B \times \mathbb{Z}B &\to \mathbb{Z}B \\ (b_i, b_j) &\mapsto \sum_{k \in I} c_k^{ij} b_k \quad , \quad c_k^{ij} \in \mathbb{N}_0 \end{aligned}$$

such that the following hold for all $i, j, k \in I$:

(i) There exists an element $\mathbb{1} := b_0 \in B$ such that $\mathbb{1} \cdot b_i = b_i \cdot \mathbb{1} = b_i$
(ii) $(b_i \cdot b_j) \cdot b_k = b_i \cdot (b_j \cdot b_k)$
(iii) $\sum_{l \in I} c_l^{ij} > 0$
(iv) There exists an involution $i \mapsto i^*$ of I such that $c_0^{ij} = c_0^{ji} = \delta_{i*j}$

The unital, associative $\mathbb{Z}$-algebra $\mathcal{A} = (\mathbb{Z}B, \cdot\,)$ satisfying the above is called a *fusion algebra*. If we also have (v) then $\mathcal{A}$ is called a *commutative* fusion algebra.

(v) $b_i \cdot b_j = b_j \cdot b_i$

The quantities $\{c_k^{ij}\}_{i,j,k \in I}$ act as the structure constants of a fusion algebra. We can also express properties (i),(ii) and (v) in terms of these constants: (i) $c_j^{i0} = c_j^{0i} = \delta_{ij}$, (ii) $\sum_p c_p^{ij} c_u^{pk} = \sum_r c_u^{ir} c_r^{jk}$ and (v) $c_k^{ij} = c_k^{ji}$. The structure constants clearly have symmetries of the same form as in (6.29b) (and (6.29a) for a commutative algebra). Observing that the *-involution may be extended to an anti-automorphism of $\mathcal{A}$, it easily follows that the structure constants also have symmetry of the form (6.29c).

A commutative fusion algebra $\mathcal{A}$ admits a categorification if there exists a braided $6j$ fusion system with label set $\mathfrak{L}$ and a bijection $\phi : B \to \mathfrak{L}$ such that $c_k^{ij} = N_{\phi(k)}^{\phi(i)\phi(j)}$ for all $i, j, k \in B$. It is possible for a given $\mathcal{A}$ to admit more than one categorification, although only finitely many[27] (up to gauge equivalence and relabelling). The categorification of $\mathcal{A}$ yields a braided fusion category (whose skeletal data is given by the braided $6j$ fusion system). From a physical perspective, we are only interested in categories for which (there exists a choice of gauge where) all associated F and R symbols are unitary; namely, *unitary* braided fusion categories.

6.5. **Ribbon structure.** It is known that a unitary braided fusion category admits a unique unitary ribbon structure [23, 24]. In terms of the R-symbols of the category, this means that for every $q \in \mathfrak{L}$, there exists a quantity $\vartheta_q \in U(1)$ such that the ribbon relation (6.34) is fullfilled. This tells us that given a unitary braided $6j$ fusion system, the ansatz (6.32) is correct and has a unique set of solutions.

$$\sum_{\lambda} \left[R_c^{ba}\right]_{\mu\lambda} \left[R_c^{ab}\right]_{\lambda\nu} = \frac{\vartheta_c}{\vartheta_a \vartheta_b} \delta_{\mu\nu} \tag{6.34}$$

Physically, ϑ_q is the phase evolution induced by a clockwise 2π-rotation of a charge q, and is called its *topological spin.* The topological spins are roots of unity [4, 22] and are gauge-invariant. The ribbon relation allows us to promote quasiparticle worldlines to worldribbons, or equivalently tells us how to evaluate type-I Reidemeister moves on worldlines (Figure 14).

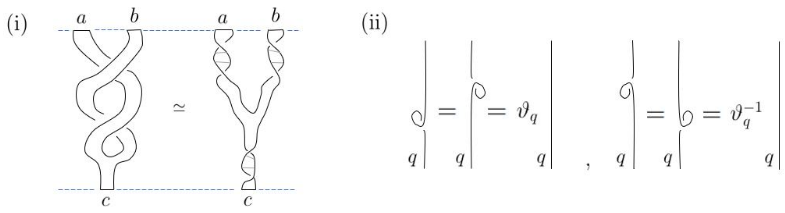


FIGURE 14. (i) The ribbon relation illustrated through the deformation of worldribbons. Boundaries are fixed at the initial and final time slices. (ii) Type-I Reidemeister twists correspond to 2π-rotations.

To summarise, the algebraic structure arising from exchange symmetry in two spatial dimensions (under assumptions **A1**-**A3**) corresponds to a unitary ribbon fusion category (also called a unitary premodular category). A theory of anyons has all of its data contained in a such a category and is determined (up to gauge equivalence) by the skeletal data of the category (fusion coefficients, F-symbols and R-symbols). The underlying fusion algebra encodes the *fusion rules* of the theory.[28] The rank-finiteness theorem for braided fusion categories [25] tells us that there are finitely many theories of anyons for any given rank. Finally, we note that the deduction in Remark 2.1 is verified, for instance, by the toric code modular tensor category which describes quasiparticles on a torus.

[27] This result is known as *Ocneanu rigidty.*

[28] Note that for the basis B of the fusion algebra for an abelian theory of anyons, $(B, \cdot\,)$ defines an abelian group.

6.6. **Modularity.** Pursuing a classification of theories of anyons motivates that of unitary ribbon fusion categories [26]. Levying a nondegeneracy condition on the braiding results in a unitary *modular tensor category*: the extra structure possessed by such categories makes their classification more tractable [27, 28, 29].

Definition 6.8. Suppose monodromy operator M_{xq} is the identity for all labels q. The label x is then said to be *transparent*. The braiding is called *nondegenerate* if the trivial label is the only transparent one.

Modularity can be physically motivated as follows. R-matrices of the form R^{ab} where $a \neq b$ are not gauge-invariant, and therefore cannot correspond to measurable quantities. On the other hand, monodromies are gauge-invariant. Since the monodromy of any transparent label is trivial, there is no reason to allow for nontrivial transparent labels in our algebraic models, as they cannot be distinguished from the vacuum in practice.

However, modularity comes at a price. Let f be such that $N_q^{ff} \in \{0, 1\}$ for all q. R-matrices of the form R^{ff} are gauge-invariant, and so assuming modularity has the undesirable effect of discarding theories with transparent objects f such that -1 is an eigenvalue of R^{ff} (e.g. fermions). Modular tensor categories are thus limited to describing $(2+1)$-dimensional *bosonic* topological orders. Fermions are typically present in systems of interest (e.g. fractional quantum Hall liquids), and so it is desirable to have an algebraic model that is "almost" modular i.e. where the only nontrivial transparent object is a fermion. This has led to the development of *spin* modular categories [30].

7. Concluding Remarks and Outlook

The majority of this paper is devoted to considering the action of braiding on quasiparticle systems. To this end, the "superselection braid" proved to be central to our exposition. We saw that its action uniquely specified the superselection sectors of a system, illuminated the fusion structure amongst them and suggested the ribbon relation. Using exchange symmetry as our guiding physical principle, we showed that postulates **A1**-**A3** suffice to recover unitary ribbon fusion categories as a framework for modelling anyons. Taking into account the results of [11], we also suggested an alternative set of postulates **P1**-**P3** in Section 1.1.

A *motion group* may be defined in a more general context than that found in Section 2.2 in order to describe the 'motions' of a (typically disconnected) nonempty submanifold $\mathcal{N}$ in manifold $\mathcal{M}$ [31]. If $\mathcal{M} = \mathbb{R}^3$ and $\mathcal{N}$ is given by n disjoint loops then the motion group is the *loop braid group* $\mathcal{L}B_n$. Physically, we expect $\mathcal{L}B_n$ to play a similar role in describing the exchange statistics of loop-like excitations in $(3+1)$-dimensions to that of the braid group for point-like excitations in $(2+1)$-dimensions [32]. The next possible generalisation could be to consider the statistics of knotted loops. The representation theory of motion groups and their relation to higher-dimensional TQFTs and topological phases of matter is an active area of research. In the case of loop excitations, various inroads have been made [33, 34, 35, 36, 37, 38]. By formulating exchange symmetry in terms of the local representations of motion groups, the methods presented in this paper might be extended by adapting them to the setting of higher-dimensional excitations.

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Definition A.1. A *groupoid* with base $\mathcal{B}$ is a set G with maps $\alpha, \omega : G \to \mathcal{B}$ and a partially defined binary operation $(\cdot,\cdot) : G \times G \to G$ such that for all $f, g, h \in G$,

(i) gh is defined whenever $\alpha(g) = \omega(h)$, and in this case we have $\alpha(gh) = \alpha(h)$ and $\omega(gh) = \omega(g)$.
(ii) If either of $(fg)h$ or $f(gh)$ is defined then so is the other, and they are equal.
(iii) For each g, there are left and right identity elements respectively denoted by $\lambda_g, \rho_g \in G$, for which we have $\lambda_g g = g = g\rho_g$.
(iv) Each g has an inverse $g^{-1} \in G$ satisfying $g^{-1}g = \rho_g$ and $gg^{-1} = \lambda_g$.

Note that a group is a groupoid G whose base contains a single element.

Consider the set of all possible n-braids where for any braid, each strand is assigned a distinct colour (and we always have the same n colours to choose from). Equivalently, this may be thought of as bijectively assigning a number from $\{1, \dots, n\}$ to each of the n strands in a given braid. Thus, for any n-braid $b \in B_n$, there are now $n!$ distincly 'coloured' versions of it contained in our set.
Under composition (i.e. stacking of braids), it is clear that our set possesses the structure of a groupoid. In this instance, the base is $\mathcal{B} = S_{\{1,\dots,n\}}$ yielding the braid groupoid $B_n(\mathcal{B})$ *for n distinctly coloured strands.*

$g =$  $\in B_3(S_{\{1,2,3\}})$

Figure 15. $\alpha(g) = 123$ and $\omega(g) = 321$

Remark A.2. Given any $s \in S_{\{1,\dots,n\}}$, the subset of all braids $g \in B_n(B)$ such that $\alpha(g) = \omega(g) = s$ defines a subgroup isomorphic to the pure braid group PB_n.

We can equivalently understand a groupoid G with base $\mathcal{B}$ as a category G whose collection of objects $\mathrm{Ob}(G)$ is given by $\mathcal{B}$, and where for any $x, y \in \mathcal{B}$ we have

$$\mathrm{Hom}(x,y) = \{g \in G : \alpha(g) = x, \omega(g) = y\} \tag{A.1}$$

where $g \in \mathrm{Hom}(x,y)$ is a morphism from x to y. Note that all morphisms in the category G are isomorphisms (by invertibility). When $\mathcal{B} = S_{\{1,\dots,n\}}$, Remark A.2 is equivalent to saying that there is a group isomorphism $\mathrm{Aut}(s) \cong PB_n$ for each $s \in \mathcal{B}$.

The categorical framework is convenient for understanding what is meant by a unitary linear representation of $B_n(\mathcal{B})$. In our case, this will be a functor

$$Z : B_n(\mathcal{B}) \to \mathrm{FdHilb} \tag{A.2}$$

where FdHilb is the category of finite Hilbert spaces, and where the image of any morphism under Z is a unitary linear transformation.[29]

[29] FdHilb is equipped with a dagger structure given by the Hermitian adjoint.

Finally, it is worth mentioning the choice of base $\mathcal{B}$ for $B_n(\mathcal{B})$. The coloured braid groupoid $B_n(\mathcal{B})$ is defined for a choice of base $\mathcal{B} = S_{\{l_1,\ldots,l_n\}}$ where $l_i \in \mathfrak{L}$ (for some set of labels $\mathfrak{L}$). When constructing the action $\{\rho_s\}_s$ in Section 4, we do not assume any equalities among the representations $\{\rho_{\{i,j\}}\}_{i,j}$ in order to maintain generality. This means that all n strands in any given braid must be distinctly labelled, and explains the choice of base $\mathcal{B} = S_{\{1,\ldots,n\}}$.

- Suppose $\rho_{\{1,i\}} = \rho_{\{2,i\}}$ for all i. This is equivalent to having $\mathcal{B} = S_{\{1,1,3,4,\ldots,n\}}$ (i.e. $n-1$ colours for n strands, where only 2 strands have the same colour).
- Suppose $\rho_{\{i,j\}}$ coincide for all i,j. This is equivalent to having $\mathcal{B} = S_{\{1,\ldots,1\}}$ (i.e. all strands have the same colour). In this instance, $B_n(\mathcal{B}) \cong B_n$ and (A.2) is a unitary linear representation of B_n.

This suggests that a braided monoidal category $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}) = \mathfrak{L}$ is a sensible way to model a theory of anyons. Indeed, this is the case (anyons are algebraically modelled using braided *fusion* categories). The key step is to identify the existence of a fusion structure amongst the labels in $\mathfrak{L}$: the primary objective of this paper is to show how such structure emerges as a direct consequence of exchange symmetry.

Appendix B. Proofs

B.1. **Proofs from Section 5.1.** In order to prove Lemma 5.3, we must first show the identities in Lemma B.1.

Lemma B.1.

(i) $\beta_n\sigma_{n-1} = \sigma_1\beta_n \quad , \; n \geq 2$
(ii) $b_n\sigma_{n-i} = \sigma_{n+1-i}\; b_n \; , \quad i = 1, \dots, n-1$ where $n \geq 2$

Proof.

(i)
$$\begin{aligned} b_n^2 &= b_{n-1}b_{n-2}\sigma_n\sigma_{n-1}\sigma_n \\ &= b_{n-1}b_{n-2}\sigma_{n-1}\sigma_n\sigma_{n-1} = b_{n-1}^2\sigma_n\sigma_{n-1} \\ &= b_{n-2}^2(\sigma_{n-1}\sigma_{n-2})(\sigma_n\sigma_{n-1}) \\ &= \dots = b_1^2\sigma_{21}\sigma_{32}\dots(\sigma_n\sigma_{n-1}) = \sigma_1 b_n b_{n-1} \end{aligned}$$

whence
$$\begin{aligned} \beta_n\sigma_{n-1} &= b_{n-1}b_{n-2}\sigma_{n-1}\beta_{n-2} = b_{n-1}^2\beta_{n-2} \\ &= \sigma_1 b_{n-1}b_{n-2}\beta_{n-2} = \sigma_1\beta_n \end{aligned}$$

(ii) For $n = 2$, the identity is simply $\sigma_{121} = \sigma_{212}$. Proceeding by induction, assume that the lemma holds for some n. For $2 \leq i \leq n$, we have
$$\begin{aligned} b_{n+1}\sigma_{n+1-i} = b_n\sigma_{n+1}\sigma_{n+1-i} &= b_n\sigma_{n+1-i}\sigma_{n+1} \quad (\text{where } n+1-i \in \{1, \dots, n-1\}) \\ &= \sigma_{n+2-i}b_n\sigma_{n+1} \quad (\text{by induction hypothesis}) \\ &= \sigma_{n+2-i}b_{n+1} \end{aligned}$$

For $i = 1$, we show the result directly:
$$b_n\sigma_{n-1} = b_{n-2}\sigma_{n-1}\sigma_n\sigma_{n-1} = b_{n-2}\sigma_n\sigma_{n-1}\sigma_n = \sigma_n b_n$$

□

Proof of Lemma 5.3.

Let us first show that

$$\beta_n\sigma_i = \sigma_{n-i}\beta_n \tag{B.1}$$

For $n = 2$, (B.1) is simply $\beta_2\sigma_1 = \sigma_1^2 = \sigma_1\beta_2$. Proceeding by induction, assume that (B.1) holds for some n. For $1 \leq i \leq n-1$, we have
$$\begin{aligned} \beta_{n+1}\sigma_i = b_n\beta_n\sigma_i &= b_n\sigma_{n-i}\beta_n \quad (\text{by induction hypothesis}) \\ &= \sigma_{n+1-i}b_n\beta_n \quad (\text{by Lemma B.1(ii)}) \\ &= \sigma_{n+1-i}\beta_{n+1} \end{aligned}$$

For $i = n$, we want to show $\beta_{n+1}\sigma_n = \sigma_1\beta_{n+1}$, which is just Lemma B.1(i). It remains to show that

$$\beta_n\sigma_i^{-1} = \sigma_{n-i}^{-1}\beta_n \tag{B.2}$$

Lemma B.1 implies

$$\beta_n\sigma_{n-1}^{-1} = \sigma_1^{-1}\beta_n \quad , \; n \geq 2 \tag{B.3a}$$

$$b_n\sigma_{n-i}^{-1} = \sigma_{n+1-i}^{-1}b_n \; , \quad i = 1, \dots, n-1 \text{ where } n \geq 2 \tag{B.3b}$$

Using identities (B.3a)-(B.3b), the proof of (B.2) follows similarly to that of (B.1). □

B.2. **Proofs from Section 5.2.** In order to prove Theorem 5.9, we will first need to prove Lemmas B.2-B.5 and Proposition B.6.

Lemma B.2.

$$\beta_k = r_{k-2}(b_1) \cdot \ldots \cdot r_1(b_{k-2}) \cdot r_0(b_{k-1}) \ , \ \ k \geq 2 \tag{B.4}$$

Proof.

$$\begin{aligned}
b_{n+1}b_n &= \sigma_{1\ldots n+1} \cdot \sigma_{1\ldots n} \\
&= \sigma_{1\ldots n} \cdot \sigma_{1\ldots n-1}\sigma_{n+1}\sigma_n = b_n b_{n-1}\sigma_{n+1}\sigma_n \\
&= b_{n-1}b_{n-2}(\sigma_n\sigma_{n-1})(\sigma_{n+1}\sigma_n) \\
&= \ldots = b_2 b_1(\sigma_{32} \cdot \ldots \cdot \sigma_{n,n-1} \cdot \sigma_{n+1,n}) \\
&= \sigma_{12}\sigma_1(\sigma_{32} \cdot \ldots \cdot \sigma_{n,n-1} \cdot \sigma_{n+1,n}) \\
&= \sigma_{21}(\sigma_2 \cdot \sigma_{32} \cdot \ldots \cdot \sigma_{n,n-1} \cdot \sigma_{n+1,n}) \\
&= \sigma_{21}(\sigma_{32} \cdot \sigma_{343} \cdot \sigma_{54} \cdot \ldots \cdot \sigma_{n+1,n}) \\
&= \cdots = (\sigma_{21} \cdot \sigma_{32} \cdot \sigma_{43} \cdot \ldots \cdot \sigma_{n+1,n})\sigma_{n+1} \\
&= \sigma_{2\ldots n+1} \cdot b_{n+1} = r_1(b_n) \cdot b_{n+1}
\end{aligned}$$

from which we see that

$$\begin{aligned}
\beta_k = b_{k-1} \cdot \ldots \cdot b_1 &= (b_{k-1}b_{k-2}) \cdot b_{k-3} \cdot \ldots \cdot b_1 \\
&= r_1(b_{k-2}) \cdot b_{k-1} \cdot b_{k-3} \cdot \ldots \cdot b_1 \\
&= r_1(b_{k-2}) \cdot b_{k-2} \cdot b_{k-3} \cdot \ldots \cdot b_1 \cdot \sigma_{k-1} \\
&= r_1(b_{k-2}) \cdot \beta_{k-1} \cdot \sigma_{k-1} \\
&= \ldots = r_1(b_{k-2}) \cdot \ldots \cdot r_1(b_1) \cdot \beta_2 \cdot (\sigma_2 \cdot \ldots \cdot \sigma_{k-1}) = r_1(\beta_{k-1}) \cdot b_{k-1}
\end{aligned}$$

whence

$$\begin{aligned}
\beta_k &= r_1(\beta_{k-1}) \cdot b_{k-1} \\
&= r_1(r_1(\beta_{k-2}) \cdot b_{k-2}) \cdot b_{k-1} = r_2(\beta_{k-2}) \cdot r_1(b_{k-2}) \cdot b_{k-1} \\
&= \ldots = r_{k-2}(\beta_2) \cdot r_{k-3}(b_2) \cdot \ldots \cdot r_1(b_{k-2}) \cdot r_0(b_{k-1})
\end{aligned}$$

□

Lemma B.3.

$$b_{n-1}\overleftarrow{b_n} = \overleftarrow{b_n} \cdot r_1(b_{n-1}) \tag{B.5}$$

Proof.

$$\begin{aligned}
b_{n-1}\overleftarrow{b_n} &= \sigma_{1\ldots n-1} \cdot \sigma_{n\ldots 1} \\
&= b_{n-2} \cdot \sigma_{n-1}\sigma_n\sigma_{n-1} \cdot \overleftarrow{b_{n-2}} = \sigma_n(b_{n-2} \cdot \sigma_{n-1} \cdot \overleftarrow{b_{n-2}})\sigma_n \\
&= \sigma_n(b_{n-2} \cdot \overleftarrow{b_{n-1}})\sigma_n \\
&= \ldots = \sigma_{n\ldots 3}(b_1 \cdot \overleftarrow{b_2})\sigma_{3\ldots n} \\
&= \sigma_{n\ldots 3}(\sigma_1\sigma_{21})\sigma_{3\ldots n} = (\sigma_{n\ldots 3}\sigma_{21})(\sigma_2\sigma_{3\ldots n})
\end{aligned}$$

□

Lemma B.4. *β_n is a palindrome i.e. $\beta_n = \overleftarrow{\beta_n}$.*

Proof.

$$\begin{aligned}\sigma_n\beta_n = \sigma_n b_{n-1}\beta_{n-1} &= (b_{n-2}\cdot\sigma_n)\cdot\sigma_{n-1}\beta_{n-1}\\ &= \ldots = (b_{n-2}\cdot\sigma_n)\cdot(b_{n-3}\cdot\sigma_{n-1})\cdot\ldots\cdot(b_1\sigma_3)\cdot\sigma_2\beta_2\\ &= (b_{n-2}\cdot\ldots\cdot b_1)(\sigma_n\sigma_{n-1}\cdot\ldots\cdot\sigma_3)\sigma_2\sigma_1 = \beta_{n-1}\overleftarrow{b_n}\end{aligned}$$

whence

$$\begin{aligned}\beta_{n+1} = b_n\beta_n = b_{n-1}(\sigma_n\beta_n) &= b_{n-1}\beta_{n-1}\overleftarrow{b_n}\\ &= b_{n-2}(\sigma_{n-1}\beta_{n-1})\overleftarrow{b_n} = b_{n-2}\beta_{n-2}\overleftarrow{b_{n-1}}\overleftarrow{b_n}\\ &= \ldots = b_2\beta_2\overleftarrow{b_3}\cdot\ldots\cdot\overleftarrow{b_n}\\ &= \sigma_1\sigma_{21}\overleftarrow{b_3}\cdot\ldots\cdot\overleftarrow{b_n} = \overleftarrow{\beta_{n+1}}\end{aligned}$$

□

Lemma B.5.

(i) $\sigma_i\cdot t_{k,l} = t_{k,l}\cdot r_k(\sigma_i)\ ,\quad 1\le i\le l-1\ ,\ k\ge 1\ ,\ l>1$

(ii) $t_{k,l}\cdot\sigma_i = r_l(\sigma_i)\cdot t_{k,l}\ ,\quad 1\le i\le k-1\ ,\ k>1\ ,\ l\ge 1$

Proof. (These identities are graphically obvious; see Figure 16 below)

(i) *Claim:*

For $l>1$ and $j\ge 0$, we have

$$\sigma_i\cdot r_j(\overleftarrow{b_l}) = r_j(\overleftarrow{b_l})\cdot\sigma_{i+1}\quad ,\quad 1+j\le i\le(l-1)+j \tag{B.6}$$

For $l=2$, (B.6) is simply $\sigma_{1+j}(\sigma_{2+j}\sigma_{1+j}) = (\sigma_{2+j}\sigma_{1+j})\sigma_{2+j}$. For $l=3$,

$$\begin{aligned}i=1+j\ &:\ \sigma_{1+j}(\sigma_{3+j}\sigma_{2+j}\sigma_{1+j}) = \sigma_{3+j}(\sigma_{1+j}\sigma_{2+j}\sigma_{1+j}) = (\sigma_{3+j}\sigma_{2+j}\sigma_{1+j})\sigma_{2+j}\\ i=2+j\ &:\ \sigma_{2+j}(\sigma_{3+j}\sigma_{2+j}\sigma_{1+j}) = \sigma_{3+j}(\sigma_{2+j}\sigma_{3+j}\sigma_{1+j}) = (\sigma_{3+j}\sigma_{2+j}\sigma_{1+j})\sigma_{2+j}\end{aligned}\tag{B.7}$$

Let $l\ge 4$. For $2+j\le i\le(l-2)+j$,

$$\begin{aligned}\sigma_i\cdot r_j(\overleftarrow{b_l}) = \sigma_i\cdot\sigma_{l+j\ldots 1+j} &= \sigma_{l+j\ldots i+2}\cdot\sigma_i\sigma_{i+1}\sigma_i\cdot\sigma_{i-1\ldots 1+j}\\ &= \sigma_{l+j\ldots i+2}\cdot\sigma_{i+1}\sigma_i\sigma_{i+1}\cdot\sigma_{i-1\ldots 1+j}\\ &= r_j(\overleftarrow{b_l})\cdot\sigma_{i+1}\end{aligned}$$

For $i=1+j$,

$$\sigma_{1+j}\cdot r_j(\overleftarrow{b_l}) = \sigma_{1+j}\cdot\sigma_{l+j\ldots 1+j} = \sigma_{l+j\ldots 3+j}\cdot\sigma_{1+j}\sigma_{2+j}\sigma_{1+j} = r_j(\overleftarrow{b_l})\cdot\sigma_{2+j}$$

and for $i=(l-1)+j$,

$$\sigma_{(l-1)+j}\cdot r_j(\overleftarrow{b_l}) = \sigma_{(l-1)+j}\sigma_{l+j}\sigma_{(l-1)+j}\cdot\sigma_{(l-2)+j\ldots 1+j} = r_j(\overleftarrow{b_l})\sigma_{l+j}$$

This shows the claim. Recall from (5.14) that $t_{k,l} = \left[r_0(\overleftarrow{b_l})\cdot\ldots\cdot r_{k-1}(\overleftarrow{b_l})\right]$. By applying the claim k times for $j=0,\ldots,k-1$ (in increasing order) to $\sigma_i\cdot t_{k,l}$ for $1\le i\le l-1$, we obtain

$$\sigma_i\cdot\left[r_0(\overleftarrow{b_l})\cdot\ldots\cdot r_{k-1}(\overleftarrow{b_l})\right] = \left[r_0(\overleftarrow{b_l})\cdot\ldots\cdot r_{k-1}(\overleftarrow{b_l})\right]\cdot r_k(\sigma_i) \tag{B.8}$$

(ii) Applying anti-automorphism χ to (i) and relabelling yields the result.

□

Proposition B.6. *Given any positive integers k, l such that $k + l \geq 2$, we have*

(i) $\beta_{k+l} = [r_l(\beta_k) \cdot \beta_l]\, t_{k,l}$

(ii) $\beta_{k+l} = t_{l,k}\, [r_l(\beta_k) \cdot \beta_l]$

where $r_l(\beta_k)$ and β_l commute.

Proof.

(i) By Lemma B.2, we have

$$\beta_{k+l} = r_{k+l-2}(b_1) \cdot r_{k+l-3}(b_2) \cdot \ldots \cdot r_0(b_{k+l-1}) \tag{B.9}$$

and

$$\begin{aligned} r_l(\beta_k) &= r_l\left(r_{k-2}(b_1) \cdot r_{k-3}(b_2) \cdot \ldots \cdot r_0(b_{k-1})\right) \\ &= r_{k+l-2}(b_1) \cdot r_{k+l-3}(b_2) \cdot \ldots \cdot r_l(b_{k-1}) \end{aligned} \tag{B.10}$$

whence it suffices to show that

$$r_{l-1}(b_k) \cdot \ldots \cdot r_0(b_{k+l-1}) = [r_{l-2}(b_1) \cdot \ldots \cdot r_0(b_{l-1})] \cdot \left[r_0(\overleftarrow{b_l}) \cdot \ldots \cdot r_{k-1}(\overleftarrow{b_l})\right] \tag{B.11}$$

where the right-hand side of (B.11) is $\beta_l \cdot t_{k,l}$. We prove (B.11) by induction.

First, perform induction on l for fixed k. The base case $(k, l) = (k, 1)$ is

$$r_0(b_k) = r_0(b_1) \cdot \ldots \cdot r_{k-1}(b_1) \tag{B.12}$$

which is clearly true. Now suppose (B.11) holds for some l given fixed k. Then we want to show that (B.11) also holds for $(k, l+1)$ i.e.

$$r_l(b_k) \cdot \ldots \cdot r_0(b_{k+l}) = [r_{l-1}(b_1) \cdot \ldots \cdot r_0(b_l)] \cdot \left[r_0(\overleftarrow{b_{l+1}}) \cdot \ldots \cdot r_{k-1}(\overleftarrow{b_{l+1}})\right] \tag{B.13}$$

Observe that

$$\begin{aligned} t_{k,l+1} &= \left[\sigma_{l+1} \cdot r_0(\overleftarrow{b_l})\right] \cdot \left[\sigma_{l+2} \cdot r_1(\overleftarrow{b_l})\right] \cdot \ldots \cdot \left[\sigma_{l+k} \cdot r_{k-1}(\overleftarrow{b_l})\right] \\ &= \sigma_{l+1,\ldots,l+k} \cdot \left[r_0(\overleftarrow{b_l}) \cdot \ldots \cdot r_{k-1}(\overleftarrow{b_l})\right] = r_l(b_k) \cdot t_{k,l} \end{aligned}$$

and so the right-hand side of (B.13) is

$$\begin{aligned} \beta_{l+1} \cdot t_{k,l+1} = b_l \beta_l \cdot r_l(b_k) t_{k,l} &= b_l r_l(b_k) \cdot \beta_l t_{k,l} \\ &= b_{k+l} \cdot \beta_l \cdot t_{k,l} \\ &\overset{(B.11)}{=} b_{k+l} \cdot r_{l-1}(b_k) \cdot \ldots \cdot r_0(b_{k+l-1}) \end{aligned}$$

where the final equality follows by the induction hypothesis. Thus, in order to show (B.13), we must show that

$$r_l(b_k) \cdot \ldots \cdot r_0(b_{k+l}) = b_{k+l} \cdot r_{l-1}(b_k) \cdot \ldots \cdot r_0(b_{k+l-1}) \tag{B.14}$$

under the induction hypothesis. Lemma B.1(ii) tells us that $b_n \sigma_i = \sigma_{i+1} b_n$ for any $n \geq 2$ and $1 \leq i \leq n-1$. Applying this result to the right-hand side of (B.14), we see that b_{k+l} acts on each r_j term by r_1 as it moves to its right, yielding the left-hand side. This completes the induction on l.

Next, we perform induction on k for fixed l. The base case $(k,l) = (1,l)$ is

$$r_{l-1}(b_1)\cdot\ldots\cdot r_0(b_l) = [r_{l-2}(b_1)\cdot\ldots\cdot r_0(b_{l-1})]\cdot r_0(\overleftarrow{b_l}) \tag{B.15}$$

which we show via repeated application of Lemma B.3 on the right-hand side.

$$\begin{aligned}
&[r_{l-2}(b_1)\cdot\ldots\cdot r_0(b_{l-1})]\cdot r_0(\overleftarrow{b_l})\\
\overset{(B.5)}{=}&\Big[r_{l-2}(b_1)\cdot\ldots\cdot r_1(b_{l-2})\cdot r_0(\overleftarrow{b_l})\Big]\cdot r_1(b_{l-1})\\
=&[r_{l-2}(b_1)\cdot\ldots\cdot r_2(b_{l-3})]\cdot\Big[r_1(b_{l-2})\cdot r_1(\overleftarrow{b_{l-1}})\sigma_1\Big]\cdot r_1(b_{l-1})\\
\overset{(B.5)}{=}&\Big[r_{l-2}(b_1)\cdot\ldots\cdot r_2(b_{l-3})\cdot r_1(\overleftarrow{b_{l-1}})\Big]\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})\\
=&[r_{l-2}(b_1)\cdot\ldots\cdot r_3(b_{l-4})]\cdot\Big[r_2(b_{l-3})\cdot r_1(\overleftarrow{b_{l-1}})\Big]\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})\\
=&[r_{l-2}(b_1)\cdot\ldots\cdot r_3(b_{l-4})]\cdot\Big[r_2(b_{l-3})\cdot r_2(\overleftarrow{b_{l-2}})\sigma_2\Big]\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})\\
\overset{(B.5)}{=}&\Big[r_{l-2}(b_1)\cdot\ldots\cdot r_3(b_{l-4})\cdot r_2(\overleftarrow{b_{l-2}})\Big]\cdot[r_3(b_{l-3})\cdot\sigma_2]\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})\\
=&\ldots = r_{l-2}(b_1)\cdot r_{l-3}(\overleftarrow{b_3})\cdot[r_{l-2}(b_2)\sigma_{l-3}]\cdot[r_{l-3}(b_3)\sigma_{l-4}]\cdot\ldots\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})\\
=&\Big[r_{l-2}(b_1)\cdot r_{l-2}(\overleftarrow{b_2})\sigma_{l-2}\Big]\cdot[r_{l-2}(b_2)\sigma_{l-3}]\cdot[r_{l-3}(b_3)\sigma_{l-4}]\cdot\ldots\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})\\
\overset{(B.5)}{=}&r_{l-2}(\overleftarrow{b_2})\cdot[r_{l-1}(b_1)\sigma_{l-2}]\cdot[r_{l-2}(b_2)\sigma_{l-3}]\cdot[r_{l-3}(b_3)\sigma_{l-4}]\cdot\ldots\cdot[r_2(b_{l-2})\sigma_1]\cdot r_1(b_{l-1})
\end{aligned}$$

Observe that $\sigma_i r_i(b_{l-i}) = \sigma_i\sigma_{i+1,\ldots,l} = r_{i-1}(b_{l-i+1})$ for $1\le i<l$, whence

$$\begin{aligned}
[r_{l-2}(b_1)\cdot\ldots\cdot r_0(b_{l-1})]\cdot r_0(\overleftarrow{b_l}) &= r_{l-2}(\overleftarrow{b_2})\cdot r_{l-1}(b_1)\cdot[r_{l-3}(b_3)\cdot\ldots\cdot r_0(b_l)]\\
&= \sigma_{l,l-1,l}\cdot[r_{l-3}(b_3)\cdot\ldots\cdot r_0(b_l)]\\
&= r_{l-1}(b_1)\cdot r_{l-2}(b_2)\cdot\ldots\cdot r_0(b_l)
\end{aligned}$$

which proves the base case. Now suppose (B.11) holds for some k given fixed l. Then we want to show that (B.11) also holds for $(k+1,l)$ i.e.

$$r_{l-1}(b_{k+1})\cdot\ldots\cdot r_0(b_{k+l}) = [r_{l-2}(b_1)\cdot\ldots\cdot r_0(b_{l-1})]\cdot\Big[r_0(\overleftarrow{b_l})\cdot\ldots\cdot r_k(\overleftarrow{b_l})\Big] \tag{B.16}$$

Observe that $t_{k+1,l} = t_{k,l}\cdot r_k(\overleftarrow{b_l})$, and so the right-hand side of (B.16) is

$$\begin{aligned}
\beta_l\cdot t_{k+1,l} &= (\beta_l\cdot t_{k,l})\cdot r_k(\overleftarrow{b_l})\\
&\overset{(B.11)}{=} [r_{l-1}(b_k)\cdot\ldots\cdot r_0(b_{k+l-1})]\cdot r_k(\overleftarrow{b_l})
\end{aligned}$$

where the second equality follows by the induction hypothesis. Thus, in order to show (B.16), we must show that

$$r_{l-1}(b_{k+1})\cdot\ldots\cdot r_0(b_{k+l}) = [r_{l-1}(b_k)\cdot\ldots\cdot r_0(b_{k+l-1})]\cdot r_k(\overleftarrow{b_l}) \tag{B.17}$$

under the induction hypothesis. For $l=1$, (B.17) is

$$r_0(b_{k+1}) = r_0(b_k)\cdot r_k(\overleftarrow{b_1}) \tag{B.18}$$

which is clearly true.

Claim:

$$r_{i-1}(b_{k+l-i}) \cdot r_{k+i-1}(\overleftarrow{b_{l-i+1}}) = r_{k+i}(\overleftarrow{b_{l-i}}) \cdot r_{i-1}(b_{k+l-i+1}) \tag{B.19}$$

where $1 \leq i \leq l-1$ and $l \geq 2$. Expanding the left-hand side, we get

$$\begin{aligned} \sigma_{i\ldots k+l-1} \cdot \sigma_{k+l\ldots k+i} &= \sigma_{i\ldots k+l} \cdot \sigma_{k+l-1\ldots k+i} \\ &= \sigma_{i\ldots k+l-2} \cdot \big(\sigma_{k+l-1} \cdot \sigma_{k+l} \cdot \sigma_{k+l-1}\big) \cdot \sigma_{k+l-2\ldots k+i} \\ &= \sigma_{i\ldots k+l-2} \cdot \big(\sigma_{k+l} \cdot \sigma_{k+l-1} \cdot \sigma_{k+l}\big) \cdot \sigma_{k+l-2\ldots k+i} \\ &= \sigma_{k+l} \cdot \big(\sigma_{i\ldots k+l} \cdot \sigma_{k+l-2\ldots k+i}\big) \end{aligned} \tag{B.20}$$

It can be shown that

$$r_{i-1}(b_{k+l-i+1}) \cdot r_{k+i-1}(\overleftarrow{b_{l-i-j+1}}) = \sigma_{k+l-j+1}\left(r_{i-1}(b_{k+l-i+1}) \cdot r_{k+i-1}(\overleftarrow{b_{l-i-j}})\right) \tag{B.21}$$

for $1 \leq j \leq l-i$ which we can recursively apply (for $j = 2$ to $l-i$) to the parenthesised expression in the last line of (B.20) to obtain

$$\sigma_{i\ldots k+l} \cdot \sigma_{k+l-2\ldots k+i} = \sigma_{k+l-1\ldots k+i+1} \cdot \sigma_{i\ldots k+l}$$

This proves the claim (B.19).

We recursively apply (B.19) to the right-hand side of (B.17) for $i = 1$ to $l-1$:

$$\begin{aligned} &[r_{l-1}(b_k) \cdot \ldots \cdot r_0(b_{k+l-1})] \cdot r_k(\overleftarrow{b_l}) \\ \overset{(B.19)}{=} &[r_{l-1}(b_k) \cdot \ldots \cdot r_1(b_{k+l-2})]\, r_{k+1}(\overleftarrow{b_{l-1}}) \cdot r_0(b_{k+l}) \\ \overset{(B.19)}{=} &\ldots \overset{(B.19)}{=} r_{l-1}(b_k) \cdot r_{k+l-1}(\overleftarrow{b_1}) \cdot [r_{l-2}(b_{k+2}) \cdot \ldots \cdot r_0(b_{k+l})] \\ &\quad = r_{l-1}(b_{k+1}) \cdot r_{l-2}(b_{k+2}) \cdot \ldots \cdot r_0(b_{k+l}) \end{aligned}$$

which is the left-hand side of (B.17). This completes the induction on k.

(ii) Applying the anti-automorphism χ to (i), we get

$$\begin{aligned} \overleftarrow{\beta_{k+l}} &= \overleftarrow{t_{k,l}} \left[\overleftarrow{\beta_l} \cdot r_l(\overleftarrow{\beta_k})\right] \\ &= t_{l,k}\, [\beta_l \cdot r_l(\beta_k)] \end{aligned}$$

where the second line follows by Lemma B.4 and $\overleftarrow{t_{k,l}} = t_{l,k}$. It is clear that β_l commutes with $r_l(\beta_k)$. The result follows.

□

Proof of Theorem 5.9.

Expressions (i) and (ii) were already proved in Proposition B.6. From Lemma B.5, it easily follows that for any positive integers[30] k, l, we have

$$\beta_l \cdot t_{k,l} = t_{k,l} \cdot r_k(\beta_l) \tag{B.22a}$$

$$t_{k,l} \cdot \beta_k = r_l(\beta_k) \cdot t_{k,l} \tag{B.22b}$$

Expressions (iii) and (iv) are implied by (i) and (ii) using either one of (B.22a),(B.22b).

□

[30] Lemma B.5 implies (B.22a) and (B.22b) for $l > 1$ and $k > 1$ respectively. However, it is trivial to see that (B.22a) and (B.22b) also hold for $l = 1$ and $k = 1$ respectively.

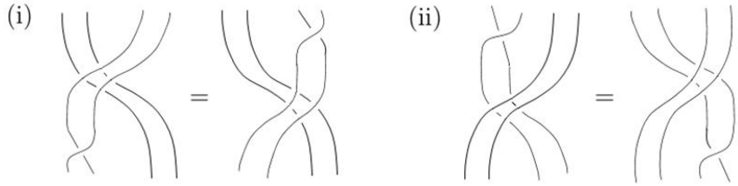


FIGURE 16. The identities in Lemma B.5 are easily seen through braid isotopy. We illustrate these identities for $(k, l) = (2, 2)$.

Proof of Theorem 5.11.

Consider any fusion tree for an n-quasiparticle system. Label each of the $(n-1)$ fusion vertices in the tree with an admissible fusion outcome: in particular, the root is assigned label Q corresponding to a superselection sector of the system.

Any superselection braid Λ_n must be some composition of braids of the form $r_d(t_{k,l}^{\pm 1})$ (since it must be compatible with the fusion trees). Recall that such braids have associated exchange phase of the form in Theorem 5.5(i) (and Corollary 5.6). The statistical phase induced by Λ_n should not depend on the labels of any internal vertices, and should only depend on the root label Q (since the associated eigenspaces should correspond to the n-quasiparticle superselection sectors): we thus denote this phase by $\lambda_n(Q)$. We know $\Lambda_1 = e$ and Λ_2 is uniquely given by σ_1 (up to orientation).

Take an arbitrary fusion vertex v in the tree, and suppose that Λ_n does not contain the braid that exchanges its incoming branches. This introduces the dependence of $\lambda_n(Q)$ on (a) the labels of the immediate children of v (unless they are leaves), and (b) the labels of the parent and sibling of v (unless v is the root). It follows that Λ_n must either (i) exchange every pair of incoming branches once, or (ii) exchange no branches. Since Λ_n does not act trivially for $n > 1$, it must do the former.
By similar considerations, we see that unless the orientation of the branch-exchanging braid acting on a fusion vertex v matches that of the branch-exchanging braids acting on its parent (unless v is the root) and immediate children, then $\lambda_n(Q)$ acquires a dependence on some labels other than Q.

We thus know that Λ_n must exchange every pair of incoming branches once, and that every such exchange must be oriented the same. By construction, all possible superselection braids have the same associated eigenspaces (namely the superselection sectors of the system). The above further tells us that all possible superselection braids whose orientations match have identical associated spectra $\{\lambda_n(Q)\}_Q$ (while all possible superselection braids of the opposite orientation have identical associated spectra $\{\lambda_n^*(Q)\}_Q$).

Next, observe that any Λ_n must contain the braid that exchanges the incident branches of the root node. Thus, any given Λ_n of clockwise orientation must be of one or more of the following forms for any k, l such that $n = k + l$:

(1) $[\Lambda_l \cdot r_l(\Lambda_k)]\, t_{k,l}$
(2) $t_{k,l}\, [\Lambda_k \cdot r_k(\Lambda_l)]$
(3) $\Lambda_l \cdot t_{k,l} \cdot \Lambda_k$
(4) $r_l(\Lambda_k) \cdot t_{k,l} \cdot r_k(\Lambda_l)$

where for any fixed one of the above four forms, the expressions for all possible k, l must be equal. By Theorem 5.9, we know that all four forms are equal and are precisely $\Lambda_n = \beta_n$. [31] $\square$

[31] For Λ_n anticlockwise, simply append a superscript '-1' to each t in (1)-(4). By Theorem 5.9, they are all equivalent to β_n^{-1}.

Appendix D. Coherence Identities

(i) The *triangle equations* are given by

(D.1)

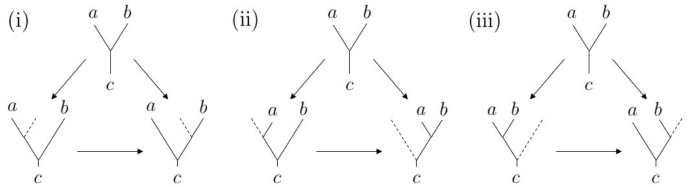
(i)
$$\begin{tikzcd}
& V_c^{ab} \arrow[dl, "\alpha_a"'] \arrow[dr, "\beta_b"] & \\
V_a^{a0}\otimes V_c^{ab} \arrow[rr, "F_c^{a0b}"'] & & V_c^{ab}\otimes V_b^{0b}
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
& V_c^{ab} \arrow[dl, "\beta_a"'] \arrow[dr, "\beta_c"] & \\
V_a^{0a}\otimes V_c^{ab} \arrow[rr, "F_c^{0ab}"'] & & V_c^{0c}\otimes V_c^{ab}
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
& V_c^{ab} \arrow[dl, "\alpha_c"'] \arrow[dr, "\alpha_b"] & \\
V_c^{ab}\otimes V_c^{c0} \arrow[rr, "F_c^{ab0}"'] & & V_c^{ab}\otimes V_b^{b0}
\end{tikzcd}$$

commute for all $a, b, c \in \mathfrak{L}$.

It can be shown that triangle equations (D.1) (ii) and (iii) follow as corollaries of fundamental triangle equation (i) and the pentagon equation [4]. Illustrating the fusion trees in (D.1),



where dashed lines denote the vacuum. Independently of the gauge, symbols F_c^{a0b}, F_c^{0ab} and F_c^{ab0} correspond to the identity map[32]. Then following (6.6), it is clear that the triangle equations will be trivially satisfied.

(ii) We have the *pentagon equation*[33] :

(D.2)

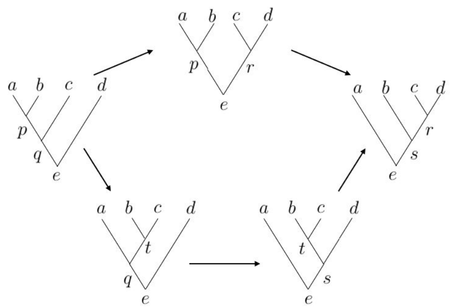
$$\begin{tikzcd}
& \bigoplus_{p,r} V_p^{ab}\otimes V_e^{pr}\otimes V_r^{cd} \arrow[dr, "\sum_r F_e^{abr}\otimes \mathrm{id}_{V_r^{cd}}"] & \\
\bigoplus_{p,q} V_p^{ab}\otimes V_q^{pc}\otimes V_e^{qd} \arrow[ur, "\sum_p \mathrm{id}_{V_p^{ab}}\otimes F_e^{pcd}"] \arrow[d, "\sum_q F_q^{abc}\otimes \mathrm{id}_{V_e^{qd}}"'] & & \bigoplus_{r,s} V_e^{as}\otimes V_s^{br}\otimes V_r^{cd} \\
\bigoplus_{q,t} V_q^{at}\otimes V_t^{bc}\otimes V_e^{qd} \arrow[rr, "\sum_t F_e^{atd}\otimes \mathrm{id}_{V_t^{bc}}"'] & & \bigoplus_{s,t} V_e^{as}\otimes V_t^{bc}\otimes V_s^{td} \arrow[u, "\sum_s \mathrm{id}_{V_e^{as}}\otimes F_s^{bcd}"']
\end{tikzcd}$$

commutes for all $a, b, c, d, e \in \mathfrak{L}$.

[32] In the $6j$ fusion system formalism, this requirement is referred to as the *triangle axiom* [8].

[33] This has a nice interpretation in terms of associahedra (convex polytopes whose vertices and edges respectively correspond to distinct fusion bases and F-moves between them); see [4].

Illustrating the fusion trees in (D.2),



The pentagon equation (D.2) may be written

$$\text{(D.3)}\quad \sum_{p,r}(F_e^{abr}\otimes \mathrm{id}_{V_r^{cd}})(\mathrm{id}_{V_p^{ab}}\otimes F_e^{pcd}) = \sum_{q,s,t}(\mathrm{id}_{V_e^{as}}\otimes F_s^{bcd})(F_e^{atd}\otimes \mathrm{id}_{V_t^{bc}})(F_q^{abc}\otimes \mathrm{id}_{V_e^{qd}})$$

Fixing the fusion states in the initial and terminal fusion basis, we obtain an entry-wise form of (D.3) which is useful for direct calculations. Fix initial state

$$|ab\to p;\alpha\rangle\,|pc\to q;\beta\rangle\,|qd\to e;\lambda\rangle$$

and terminal state

$$|as\to e;\rho\rangle\,|br\to s;\delta\rangle\,|cd\to r;\gamma\rangle$$

This gives us

$$\text{(D.4)}\quad \begin{aligned} &\sum_{\sigma}\left[F_e^{abr}\right]_{(s,\delta,\rho)(p,\alpha,\sigma)}\left[F_e^{pcd}\right]_{(r,\gamma,\sigma)(q,\beta,\lambda)}\\ &=\sum_{t,\mu,\nu,\eta}\left[F_s^{bcd}\right]_{(r,\gamma,\delta)(t,\mu,\eta)}\left[F_e^{atd}\right]_{(s,\eta,\rho)(q,\nu,\lambda)}\left[F_q^{abc}\right]_{(t,\mu,\nu)(p,\alpha,\beta)}\end{aligned}$$

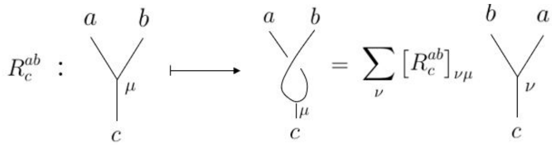
In a *multiplicity-free* theory (a theory where all fusion coefficients are either 0 or 1), (D.4) is simply

$$\text{(D.5)}\quad \left[F_e^{abr}\right]_{sp}\left[F_e^{pcd}\right]_{rq} = \sum_t \left[F_s^{bcd}\right]_{rt}\left[F_e^{atd}\right]_{sq}\left[F_q^{abc}\right]_{tp}$$

The pentagon equation is also known as the *Biedenharn-Elliot identity.*

(iii) R-matrices are transformations between bases of the form in (6.8). In the graphical calculus,

(D.6)



(D.6) is the gauge-free description of an R-matrix. Note that the matrix R^{ab} is block-diagonal with block dimensions $\{N_c^{ab}\}_c$.

(D.7) We have the *hexagon equations*[34] :

$$\begin{tikzcd}
& \bigoplus\limits_y V^{ay}_d \otimes V^{bc}_y \arrow[rr, "\bigoplus\limits_y \left(\mathrm{id}_{V^{bc}_y} \otimes R^{ay}_d\right) \cdot \mathrm{swap}"] & & \bigoplus\limits_y V^{bc}_y \otimes V^{ya}_d \arrow[dr, "F^{bca}_d"] & \\
\bigoplus\limits_x V^{ab}_x \otimes V^{xc}_d \arrow[ur, "F^{abc}_d"] \arrow[dr, "\bigoplus\limits_x R^{ab}_x \otimes \mathrm{id}_{V^{xc}_d}"'] & & & & \bigoplus\limits_z V^{bz}_d \otimes V^{ca}_z \\
& \bigoplus\limits_x V^{ba}_x \otimes V^{xc}_d \arrow[rr, "F^{bac}_d"'] & & \bigoplus\limits_z V^{bz}_d \otimes V^{ac}_z \arrow[ur, "\bigoplus\limits_z \mathrm{id}_{V^{bz}_d} \otimes R^{ac}_z"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \bigoplus\limits_y V^{ay}_d \otimes V^{bc}_y \arrow[rr, "\bigoplus\limits_y \left(\mathrm{id}_{V^{bc}_y} \otimes (R^{-1})^{ay}_d\right) \cdot \mathrm{swap}"] & & \bigoplus\limits_y V^{bc}_y \otimes V^{ya}_d \arrow[dr, "F^{bca}_d"] & \\
\bigoplus\limits_x V^{ab}_x \otimes V^{xc}_d \arrow[ur, "F^{abc}_d"] \arrow[dr, "\bigoplus\limits_x (R^{-1})^{ab}_x \otimes \mathrm{id}_{V^{xc}_d}"'] & & & & \bigoplus\limits_z V^{bz}_d \otimes V^{ca}_z \\
& \bigoplus\limits_x V^{ba}_x \otimes V^{xc}_d \arrow[rr, "F^{bac}_d"'] & & \bigoplus\limits_z V^{bz}_d \otimes V^{ac}_z \arrow[ur, "\bigoplus\limits_z \mathrm{id}_{V^{bz}_d} \otimes (R^{-1})^{ac}_z"'] &
\end{tikzcd}$$

commute for all $a, b, c, d \in \mathfrak{L}$.

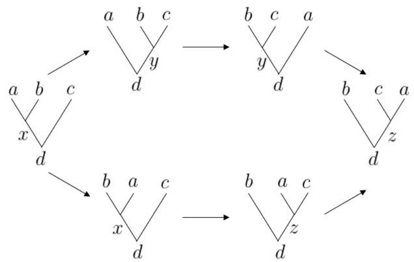


FIGURE 17. An illustration of the fusion trees in (D.7).

[34]We roughly sketch the origin of the hexagon equations. Consider the set F_n of n-leaf fusion trees. Let $\mathcal{F}_n$ be the set whose elements are given by those in F_n but with all possible permutations of the string $q_1 \ldots q_n$ labelling the leaves (so that $|\mathcal{F}_n| = n! \cdot |F_n|$). We define a digraph KR_n to have vertex set $\mathcal{F}_n$ and edges given by all F and (identically oriented) R moves transforming between the elements of $\mathcal{F}_n$. Any pair of adjacent vertices will share precisely one edge. In order to have compatibility between all F and R moves, it suffices to demand that the Yang-Baxter equation is satisfied: we thus only need to consider subgraphs of the form KR_3 i.e. the *Franklin graph*. This graph may be drawn as a dodecagon containing six hexagons and three (automatically commutative) quadrilaterals. The Yang-Baxter equation holds if the dodecagon commutes: imposing the hexagon equations ensures that the hexagons commute, and consequently that the dodecagon commutes. We remark that by restricting the edges of KR_n to only permit R-moves acting on two *leaves* in a direct fusion channel, we obtain the graph corresponding to the n^{th} *permutoassociahedron* [39].

Note that the only difference between the two hexagon equations is the orientation of the R-moves. Fix initial state $|ab \to x;\alpha\rangle\,|xc \to d;\lambda\rangle$ and terminal state $|bz \to d;\rho\rangle\,|ca \to z;\gamma\rangle$ in (D.7). This gives us

$$\begin{aligned}&\sum_{y,\beta,\mu,\sigma} \left[F_d^{bca}\right]_{(z,\gamma,\rho)(y,\beta,\sigma)} \left[R_d^{ay}\right]_{\sigma\mu} \left[F_d^{abc}\right]_{(y,\beta,\mu)(x,\alpha,\lambda)} \\ &= \sum_{\delta,\epsilon} \left[R_z^{ac}\right]_{\gamma\epsilon} \left[F_d^{bac}\right]_{(z,\epsilon,\rho)(x,\delta,\lambda)} \left[R_x^{ab}\right]_{\delta\alpha}\end{aligned} \tag{D.8a}$$

$$\begin{aligned}&\sum_{y,\beta,\mu,\sigma} \left[F_d^{bca}\right]_{(z,\gamma,\rho)(y,\beta,\sigma)} \left[\left(R^{-1}\right)_d^{ay}\right]_{\sigma\mu} \left[F_d^{abc}\right]_{(y,\beta,\mu)(x,\alpha,\lambda)} \\ &= \sum_{\delta,\epsilon} \left[\left(R^{-1}\right)_z^{ac}\right]_{\gamma\epsilon} \left[F_d^{bac}\right]_{(z,\epsilon,\rho)(x,\delta,\lambda)} \left[\left(R^{-1}\right)_x^{ab}\right]_{\delta\alpha}\end{aligned} \tag{D.8b}$$

which in the construction from (6.13a)-(6.13b) becomes

$$\begin{aligned}&\sum_{y,\beta,\mu} \left[F_d^{bca}\right]_{(z,\gamma,\rho)(y,\beta,\mu)} \left[R_d^{ay}\right]_{\mu\mu} \left[F_d^{abc}\right]_{(y,\beta,\mu)(x,\alpha,\lambda)} \\ &\quad = \left[R_z^{ac}\right]_{\gamma\gamma} \left[F_d^{bac}\right]_{(z,\gamma,\rho)(x,\alpha,\lambda)} \left[R_x^{ab}\right]_{\alpha\alpha}\end{aligned} \tag{D.9a}$$

$$\begin{aligned}&\sum_{y,\beta,\mu} \left[F_d^{bca}\right]_{(z,\gamma,\rho)(y,\beta,\mu)} \left[\left(R^{-1}\right)_d^{ay}\right]_{\mu\mu} \left[F_d^{abc}\right]_{(y,\beta,\mu)(x,\alpha,\lambda)} \\ &\quad = \left[\left(R^{-1}\right)_z^{ac}\right]_{\gamma\gamma} \left[F_d^{bac}\right]_{(z,\gamma,\rho)(x,\alpha,\lambda)} \left[\left(R^{-1}\right)_x^{ab}\right]_{\alpha\alpha}\end{aligned} \tag{D.9b}$$

and which in a multiplicity-free theory becomes

$$\sum_y \left[F_d^{bca}\right]_{zy} \left[R_d^{ay}\right] \left[F_d^{abc}\right]_{yx} = \left[R_z^{ac}\right] \left[F_d^{bac}\right]_{zx} \left[R_x^{ab}\right] \tag{D.10a}$$

$$\sum_y \left[F_d^{bca}\right]_{zy} \left[\left(R^{-1}\right)_d^{ay}\right] \left[F_d^{abc}\right]_{yx} = \left[\left(R^{-1}\right)_z^{ac}\right] \left[F_d^{bac}\right]_{zx} \left[\left(R^{-1}\right)_x^{ab}\right] \tag{D.10b}$$

Part II.

Ribbon Categories and Quantum Topology

4. Paper II:

Skein-Theoretic Methods for Unitary Fusion Categories

SKEIN-THEORETIC METHODS FOR UNITARY FUSION CATEGORIES

SACHIN J. VALERA† AND ANUP POUDEL§

Abstract. Unitary fusion categories (UFCs) have gained increased attention due to emerging connections with quantum physics. We consider a fusion rule of the form $q \otimes q \cong \mathbf{1} \oplus \bigoplus_{i=1}^{k} x_i$ in a UFC $\mathcal{C}$, and extract information using the graphical calculus. For instance, we classify all associated skein relations when $k = 1, 2$ and $\mathcal{C}$ is ribbon. In particular, we also consider the instances where q is antisymmetrically self-dual. Our main results follow from considering the action of a rotation operator on a "canonical basis". Assuming self-duality of the summands x_i, some general observations are made e.g. the real-symmetricity of the F-matrix F_q^{qqq}. We then find explicit formulae for F_q^{qqq} when $k = 2$ and $\mathcal{C}$ is ribbon, and see that the spectrum of the rotation operator distinguishes between the Kauffman and Dubrovnik polynomials.

1. Introduction

Fusion categories have played an important role in understanding structures arising from quantum physics, and lie at the heart of quantum algebra and quantum topology. Some fusion categories can be extended to ribbon fusion categories (RFCs): these gadgets are rich in structure, and carry a lot of information. Since ribbon categories are endowed with the topological properties of ribbon graphs, they naturally lend themselves to investigation from a skein-theoretic perspective. For instance, it is known that one can fashion link (in fact, 3-manifold) invariants from RFCs: seminal work in this direction was carried out by Reshetikhin and Turaev [1], followed by Kuperberg who used a skein-theoretic method to obtain new link invariants associated to quantum groups coming from Lie algebras of type A_2, B_2, C_2 and G_2 [2, 3]. In a similar vein, an important class of RFCs known as Temperley-Lieb-Jones categories can be understood using Kauffman and Lins' planar algebra of Jones-Wenzl idempotents at roots of unity [4, 5].

Understanding unitary fusion categories (i.e. fusion categories with a positive dagger structure) is crucial to developing an algebraic framework for describing topological phases of matter (TPMs). Indeed, unitary modular tensor categories (MTCs) have proved to be useful in the program for classifying (bosonic) TPMs and $(2+1)$-dimensional topological quantum field theories (TQFTs) [6, 7, 8]. The connection between link invariants and TQFTs was first observed by Witten when he gave an interpretation of the Jones polynomial in the context of Chern-Simons QFTs [9].

Although the classification of fusion categories is beyond our current capabilities, weaker variants of this problem can be studied by structural embellishment (e.g. imposing pivotality, braiding, (pre)modularity); but even with these modifications, the problem remains out of reach. It has been shown that there are finitely many braided fusion categories of any given rank [10], whence there are finitely many commutative fusion algebras (of a given rank) that admit categorification. The categorifications admitted by a (commutative) fusion algebra can be explicitly calculated by solving the pentagon (and

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hexagon) equations: doing so recovers all of the information contained in the categories. However, solving these equations quickly becomes intractable as the rank grows. This motivates the idea of determining additional general relations between unknowns, in an attempt to reduce the size of the parameter space. In this spirit, much of our exposition revolves around starting with a fusion rule of the form

$$q \otimes q \cong \mathbf{1} \oplus \bigoplus_{i=1}^{k} x_i \tag{1.1}$$

and applying skein-theoretic methods to deduce some properties of the underlying category $\mathcal{C}$ and the associated quantum invariants. Our work is inspired by [11, Theorems 3.1 & 3.2]: using a rotation operator on $\mathrm{End}(q^{\otimes 2})$ for $\mathcal{C}$ ribbon and q symmetrically self-dual[1], the authors discuss the link invariants coming from q invertible and (1.1) for $k = 1, 2$. In each case, they also give some relations between the eigenvalues of the R-matrix R^{qq}.

We systematically recover and extend the results of [11, Theorems 3.1 & 3.2] in Section 3. Our main contributions are contained in Section 4, where we uncover a relationship between the rotation operator and the F-matrix F_q^{qqq} (under certain assumptions), thereby allowing us to deduce some properties of said matrix. We note that understanding F-matrices is particularly important for many physical applications (e.g.in the study of TPMs, topological quantum computation, quantum tensor networks).

1.1. **Outline of the paper.**

In Section 2, we detail the relevant mathematical background. In Section 2.6, we introduce a canonical orthonormal basis of "jumping jacks" on $\mathrm{End}(q^{\otimes 2})$ (for 1.1) that features throughout our main exposition. In Section 2.11, we define some conventions that are followed in the main discourse. The rotation operator (one of the tools most central to this paper) is introduced in Section 2.12, and a supplementary discussion is provided in Appendix C.

In Section 3, we consider the action of the rotation operator on crossings in $\mathrm{End}(q^{\otimes 2})$ so as to ascertain the link invariants associated to the fusion rule (1.1) for $k = 1, 2$. The Jones, Kauffman and Dubrovnik polynomials are recovered, and we find three additional skein relations coming from the antisymmetrically self-dual cases. In Appendix D, the narrative of the Section 3 is reframed in terms of braid group representations. In particular, observing that braid representations associated to fusion rules of the form $q^{\otimes 2} \cong \mathbf{1} \oplus y$ factor through the Iwahori-Hecke and Temperley-Lieb algebras, we derive a skein relation for the framed HOMFLY-PT polynomial (from which we recover the quantum invariants associated to (1.1) for $k = 1$). In Appendix E, we give a few insights into invariants coming from antisymmetrically self-dual objects.

In Section 4, we apply the rotation operator to the canonical basis of jumping jacks on $\mathrm{End}(q^{\otimes 2})$ for a unitary spherical fusion category $\mathcal{C}$. In doing so, we require that the summands in (1.1) are self-dual (Remark 4.2). Theorem 4.3 determines the components of a "bone" morphism (a rotated jumping jack) in the canonical basis: we use this to prove some "bubble-popping" identities (Corollary 4.5) and to make some general observations, a highlight of which is the real-symmetricity of F_q^{qqq} (Corollary 4.6). As a simple application, we deduce the form of F_q^{qqq} when $k = 1$ in (1.1).

[1] For a reminder of the definition of (anti)symmetrically self-dual objects, we direct the reader to Appendix A.

We proceed to apply the results of Section 3 in order to derive explicit formulae for F_q^{qqq} when $\mathcal{C}$ is also ribbon and $k = 2$ (Theorem 4.8), and deduce that q cannot be antisymmetrically self-dual in this instance (Corollary 4.9). It is also observed that the spectrum of the rotation operator distinguishes between the Kauffman and Dubrovnik invariants. In Section 4.3, we investigate some properties of bases for $\mathrm{End}(q^{\otimes 2})$ whose elements are permuted (up to a sign) under the action of the rotation operator. We apply our results to construct such bases when $k = 2$; the diagonalisation of the rotation operator follows as an immediate consequence.

In Section 5, we review the contents of our work with an eye to future extensions.

1.2. **Acknowledgements.** Both authors would like to thank Corey Jones and Dave Penneys for organising the 2019 OSU Quantum Symmetries Summer Reseach Program (with grant support from David Penneys' NSF grant DMS 1654159) where they met. Sachin Valera also wishes to thank Daniel R. Copeland for helpful discussions, and for acquainting him with the rotation operator and the results of [11] (at aforementioned research program) without which this work would not have been possible.

2. Preliminaries

We provide an overview of various concepts that are used throughout this work. For further details on some of these topics, we refer the reader to [12, 13, 14, 15].

2.1. **Tensor categories.** Recall that a *tensor category* $\mathcal{C}$ is a $\Bbbk$-linear, rigid monoidal category with $\mathrm{End}(\mathbf{1}) \cong \Bbbk$ (where $\mathbf{1}$ denotes the unit object). We henceforth let $\Bbbk = \mathbb{C}$. By a *simple* object $X \in \mathcal{C}$, we mean an object X such that every nonzero $f \in \mathrm{End}(X)$ is an isomorphism. For any object $X \in \mathcal{C}$, its left and right dual objects are respectively denoted by X^* and ${}^*X$. Every object $X \in \mathcal{C}$ comes with the $(\mathrm{co})\,\mathrm{ev}_X$ and $(\mathrm{co})\,\mathrm{ev}'_X$ morphisms, which are the left and right *(co)evaluations* respectively.[2]

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1} \qquad \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$$
$$\mathrm{ev}'_X : X \otimes {}^*X \to \mathbf{1} \qquad \mathrm{coev}'_X : \mathbf{1} \to {}^*X \otimes X$$

Dual objects are unique up to unique isomorphism [12, Proposition 2.10.5]. Throughout the rest of this paper, we identify left and right duals, and denote the dual of X by X^*.

2.2. **Pivotality, sphericality and quantum trace.** A *pivotal tensor category* $\mathcal{C}$ is a tensor category with a collection of isomorphisms (called a *pivotal structure*) $a_X : X \xrightarrow{\sim} X^{**}$ natural in X and satisfying $a_{X\otimes Y} = a_X \otimes a_Y$ for all objects $X, Y \in \mathcal{C}$. For any $X \in \mathcal{C}$ and any morphism $f \in \mathrm{End}(X)$, the *left* and *right quantum traces* of f are defined as

$$\widetilde{Tr}_l(f) := \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes f) \circ \mathrm{coev}'_X \in \mathrm{End}(\mathbf{1}) \tag{2.1a}$$

$$\widetilde{Tr}_r(f) := \mathrm{ev}'_X \circ (f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X \in \mathrm{End}(\mathbf{1}) \tag{2.1b}$$

If the left and right quantum traces coincide, then $\mathcal{C}$ is called a *spherical tensor category*. Hence, for a spherical tensor category, we can unambiguously define the *quantum trace* for any object $X \in \mathcal{C}$ and any morphism $f \in \mathrm{End}(X)$. That is,

$$\widetilde{Tr}(f) := \widetilde{Tr}_l(f) = \widetilde{Tr}_r(f) \tag{2.2}$$

and the *quantum dimension* d_X of an object X is is given by (2.3), noting that $d_X = d_{X^*}$.

$$d_X := \widetilde{Tr}(\mathrm{id}_X) \tag{2.3}$$

[2] The compatibility of these morphisms with the monoidal structure is ensured by the *rigidity axioms*.

2.3. **Fusion categories and trivalent vertices.** A *fusion category* $\mathcal{C}$ is a semisimple tensor category with only finitely many simple objects up to isomorphism.

Remark 2.1 (Skeleton of $\mathcal{C}$)**.** Let $\mathrm{Irr}(\mathcal{C})$ denote a set of representatives of isomorphism classes of simple objects in $\mathcal{C}$. Let $X_i \in \mathrm{Irr}(\mathcal{C})$, where $i \in I$ for some index set $I \subseteq \mathbb{Z}_{\geq 0}$ and $X_0 := \mathbf{1}$. We also let i^* denote $j \in I$ such that $X_j = X_i^*$. The cardinality of $\mathrm{Irr}(\mathcal{C})$ is called the *rank* of $\mathcal{C}$. When we restrict to working with objects in $\mathrm{Irr}(\mathcal{C})$, it is understood that we are working in the *skeleton* of $\mathcal{C}$: this is the full subcategory of $\mathcal{C}$ on the subset of objects $\mathrm{Irr}(\mathcal{C})$, and is equivalent to $\mathcal{C}$. A category is called *skeletal* if it contains one object in each isomorphism class. See also Remarks 2.7 and 2.8.

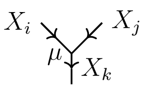
The so-called *fusion rules* for $\mathcal{C}$ are encoded by the *fusion coefficients* $N_k^{ij} \in \mathbb{Z}_{\geq 0}$ where

$$X_i \otimes X_j = \bigoplus_{k\in I} N_k^{ij} X_k, \qquad i, j \in I. \tag{2.4}$$

We also have (where δ_{ij} denotes the Kronecker delta)

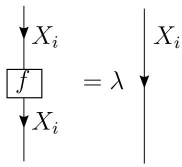
$$N_j^{i0} = N_j^{0i} = \delta_{ij} \tag{2.5}$$

There is a graphical calculus associated with morphisms for any tensor category $\mathcal{C}$. We adopt the *pessimistic* convention i.e. our diagrams are viewed as morphisms going from *top-to-bottom.* Any edge is oriented and labelled by an object $X \in \mathcal{C}$; and for $\mathbf{1} \in \mathcal{C}$, the edge is either invisible or emphasised by a dotted line. Diagrams representing morphisms in the *skeleton* of a fusion category $\mathcal{C}$ have edges labelled by objects $X_i \in \mathrm{Irr}(\mathcal{C})$. A *trivalent vertex* represents a projection from a twofold tensor product onto a summand (or conversely, an inclusion of a summand into such a product). E.g. we have the projections

(2.6) $\mathrm{span}_{\mathbb{C}} \Big\{$ 

$\Big\}_{\mu=1}^{N_k^{ij}} = \mathrm{Hom}(X_i \otimes X_j, X_k)$

where the left-hand side constitutes a *basis* for $\mathrm{Hom}(X_i \otimes X_j, X_k)$. Similarly, flipping the trivalent vertices upside-down in (2.6), we obtain a basis of inclusion morphisms for $\mathrm{Hom}(X_k, X_i \otimes X_j)$. $\mathrm{End}(X_i) \cong \mathbb{C}$, whence diagrammatically, we have



(2.7)

where $f \in \mathrm{End}(X_i)$ and $\lambda \in \mathbb{C}$.

2.4. **Dagger structure, inner product and unitarity.** Let $\mathcal{C}$ be a fusion category. Then $\mathcal{C}$ is called a *dagger* fusion category if it is equipped with an involutive, contravariant functor $\dagger : \mathcal{C} \to \mathcal{C}$ such that it acts as the identity on objects, and satisfies (2.8a)-(2.8d) where for any morphisms $f : X \to Y$, we have $\dagger(f) = f^\dagger : Y \to X$ where $f^\dagger$ is called the *adjoint* of f. For morphisms $f, g \in \mathcal{C}$ and scalars $\lambda_1, \lambda_2 \in \mathbb{C}$, the $\dagger$-functor satisfies

$$(\mathrm{id}_X)^\dagger = \mathrm{id}_X \tag{2.8a}$$

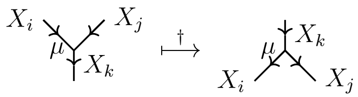
$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \tag{2.8b}$$

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \tag{2.8c}$$

$$(\lambda_1 \cdot f + \lambda_2 \cdot g)^\dagger = \lambda_1^* \cdot f^\dagger + \lambda_2^* \cdot g^\dagger \tag{2.8d}$$

where for (2.8b) we have $f : X \to Y$ and $g : Y \to Z$ for some objects $X, Y, Z \in \mathcal{C}$. Note that λ^* denotes the complex conjugate of $\lambda \in \mathbb{C}$. Considering the skeleton of $\mathcal{C}$, we have

$$\mathrm{Hom}(X_i \otimes X_j, X_k) \overset{\dagger}{\xrightarrow{\sim}} \mathrm{Hom}(X_k, X_i \otimes X_j) \tag{2.9}$$



We can define a sesquilinear form

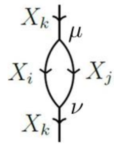
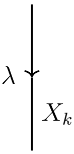
$$\langle g, f\rangle = \mathrm{tr}(fg^\dagger) \tag{2.10}$$

where $f, g \in \mathrm{Hom}(Y, X)$ and $fg^\dagger \in \mathrm{End}(X)$ for any $X, Y \in \mathcal{C}$. Further note that

$$\langle f, g\rangle = \langle g, f\rangle^* \tag{2.11}$$

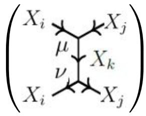
whence, (2.10) actually defines a Hermitian form.[3]

Consider two elements $\boldsymbol{e}_\mu$ and $\boldsymbol{e}_\nu$ of the basis in (2.6). Then

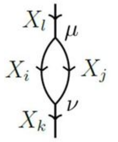
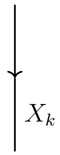
$\boldsymbol{e}_\nu \boldsymbol{e}_\mu^\dagger =$  $= \lambda$  $, \quad \lambda \in \mathbb{C}$

Note that

$$\lambda = \mathrm{tr}(\boldsymbol{e}_\nu \boldsymbol{e}_\mu^\dagger) = \mathrm{tr}(\boldsymbol{e}_\mu^\dagger \boldsymbol{e}_\nu) = \mathrm{tr} \tag{2.12}$$



whence λ vanishes for $\mu \neq \nu$. It follows that (2.6) defines an orthogonal basis with respect to the Hermitian form. We may thus write

(2.13)  $= \lambda_{lijk} \cdot \delta_{lk}\delta_{\mu\nu}$  $, \quad \lambda_{lijk} \in \mathbb{C}$

where the factor of δ_{lk} follows from Schur's lemma.

Proposition 2.2. λ_{kijk} is real.

Proof. Taking the adjoint of (2.13), the result is immediate. □

Remark 2.3 (**Basis for Hom-space**)**.** Consider the space $\mathrm{Hom}(X, Y)$ in the skeleton of $\mathcal{C}$ (where at least one of X or Y is not simple). This space is isomorphic to a direct sum of Hom-spaces of the form $\mathrm{Hom}(\bigotimes_{k=1}^m X_{i_k}, \bigotimes_{l=1}^n X_{j_l})$ where $X_{i_k}, X_{j_l} \in \mathrm{Irr}(\mathcal{C})$. We thus consider spaces of the form

$$\mathrm{Hom}\left(\bigotimes_{k=1}^m X_{i_k}, \bigotimes_{l=1}^n X_{j_l}\right) \cong \bigoplus_{b \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}\left(\bigotimes_{k=1}^m X_{i_k}, b\right) \otimes \mathrm{Hom}\left(b, \bigotimes_{l=1}^n X_{j_l}\right) \tag{2.14}$$

[3]Let $\eta : \mathrm{Hom}(Y, X) \xrightarrow{\sim} \mathrm{Hom}(YX^*, \mathbf{1})$. We may equivalently write $\langle g, f\rangle = \eta(f)(\eta(g))^\dagger \in \mathrm{End}(\mathbf{1}) \cong \mathbb{C}$.

Writing $V_Y^X := \mathrm{Hom}(X,Y)$, further note that

$$V_b^{X_{i_1}\cdots X_{i_m}} \cong \bigoplus_{e_1,\ldots,e_{m-2}\in\mathrm{Irr}(\mathcal{C})} V_{e_1}^{X_{i_1}X_{i_2}} \otimes V_{e_2}^{e_1 X_{i_3}} \otimes \cdots \otimes V_{e_{m-2}}^{e_{m-3}X_{i_{m-1}}} \otimes V_b^{e_{m-2}X_{i_m}} \tag{2.15a}$$

$$V^b_{X_{j_1}\cdots X_{j_n}} \cong \bigoplus_{f_1,\ldots,f_{n-2}\in\mathrm{Irr}(\mathcal{C})} V^b_{f_{n-2}X_{j_n}} \otimes V^{f_{n-2}}_{f_{n-3}X_{j_{n-1}}} \otimes \cdots \otimes V^{f_2}_{f_1 X_{j_3}} \otimes V^{f_1}_{X_{j_1}X_{j_2}} \tag{2.15b}$$

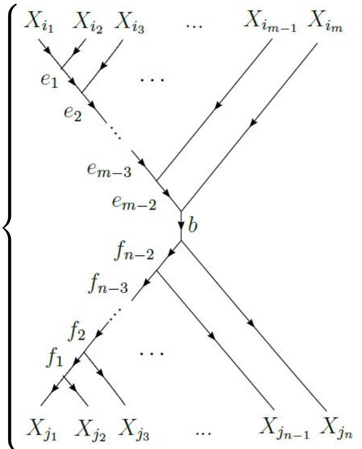
The decompositions in (2.15a) and (2.15b) correspond to a choice of *fusion basis* on the respective Hom-spaces.

FIGURE 1. By a "fusion basis", we mean a parenthesisation of $\bigotimes_k X_{i_k}$. Diagrammatically, a fusion basis corresponds to a full rooted binary tree on a space of the form in (2.15a) or (2.15b). The above trees illustrate the fusion bases for a space of the form V_e^{abcd}. The number of distinct fusion bases for an n-fold product is given by the $(n-1)^{th}$ Catalan number.

Using the basis from (2.6), and fixing fusion bases as in (2.15a) and (2.15b), we obtain the following basis[4]:

(2.16)

$$\mathrm{Hom}\left(\bigotimes_{k=1}^{m} X_{i_k}, \bigotimes_{l=1}^{n} X_{j_l}\right) = \mathrm{span}_{\mathbb{C}}\left\{ \quad \right\}_{\substack{b,e_1,\cdots,e_{m-2},f_1,\cdots,f_{n-2}\\ \mu_1,\cdots,\mu_{m-2},\nu_1,\cdots,\nu_{n-2}}}$$



Let $\{\mathbf{e}_i\}_i$ denote elements of the basis in (2.16). Then we can write

$$g = \sum_i g_i \mathbf{e}_i \quad \text{and} \quad f = \sum_i f_i \mathbf{e}_i \tag{2.17}$$

where $f_i, g_i \in \mathbb{C}$. Let $X' := \bigotimes_{i=1}^m X_i$. We write $\mathbf{e}_i\mathbf{e}_j^\dagger =: K_{ij}\tilde{\mathbf{e}}_{ij}$ where the value of K_{ij} is determined by the following three cases:

(1) $\mathbf{e}_i\mathbf{e}_j^\dagger$ vanishes, in which case $K_{ij} := 0$.
(2) $\mathbf{e}_i\mathbf{e}_j^\dagger \in \mathrm{End}(X')$ and contains no loops, in which case $K_{ij} := 1$.

[4]Note that inpermissible values of the indices do not contribute to the basis. Labels μ_i and ν_j respectively denote the multiplicities of trivalent vertices associated to e_i and f_j (they are not annotated on the basis in (2.16) so as not to clutter the diagram).

(3) $\mathbf{e}_i\mathbf{e}_j^\dagger \in \mathrm{End}(X')$ and contains loops, in which case K_{ij} is a product of some scalars $\lambda_{abca} \in \mathbb{R}$ coming from loops of the form (2.13).

where in cases (2) and (3), $\tilde{\mathbf{e}}_{ij}$ is a basis element of the form (2.16) in $\mathrm{End}(X')$. Then

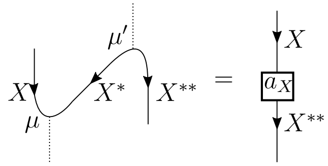
$$\langle g, f\rangle = \mathrm{tr}\left(\sum_{i,j} f_i g_j^* \mathbf{e}_i \mathbf{e}_j^\dagger\right) = \mathrm{tr}\left(\sum_{i,j} f_i g_j^* K_{ij} \tilde{\mathbf{e}}_{ij}\right) = \sum_i f_i g_i^* K_{ii}$$

Remark 2.4 (**Positive dagger structure**)**.** Note that

$$\langle f, f\rangle = \sum_i |f_i|^2 K_{ii} \tag{2.18}$$

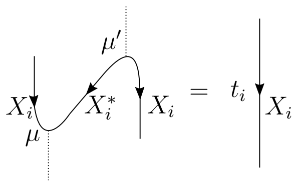
Hence, given $K_{ii} > 0$, our Hermitian form defines a Hermitian inner product. This is ensured by setting $\lambda_{kijk} > 0$ in (2.13). Under this constraint, our category is said to have a *positive dagger structure*. Furthermore, this means that $\mathcal{C}$ is a *unitary* fusion category (see also Remarks 2.7 and 2.8). Also note that basis in (2.16) is orthogonal with respect to this inner product. If $\mathcal{C}$ is also spherical, viewing the quantum dimension of an object as an inner product immediately shows that it must be positive. Throughout this paper, we assume that any category we work with possesses a positive dagger structure.

2.5. **Frobenius-Schur indicator.** Let $\mathcal{C}$ be a unitary pivotal fusion category. Following [17, Proposition 3.9], we identify zig-zag morphisms with the pivotal structure:



(2.19)

Thus, passing to the skeleton yields



(2.20)

where $t_i \in \mathbb{C}^\times$ is called a *pivotal coefficient*. It can be shown [15, Lemma E.3] that (2.20) implies (2.21), whence the indices on the trivalent vertices in (2.19) and (2.20) can be dropped.

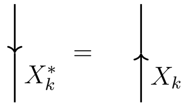
$$N_0^{ij} = N_0^{ji} = \delta_{ij^*} \tag{2.21}$$

It can also be shown (Proposition A.1) that

$$|t_i| = 1 \quad , \quad t_{i^*} = t_i^* \tag{2.22}$$

- If X_i is non self-dual, we will assume that $t_i = 1$. This choice is always possible through a unitary ("gauge") transformation of the trivalent vertices in (2.20).
- If X_i is self-dual, then t_i is called the *Frobenius-Schur* indicator and is written $\varkappa_i$; this quantity is invariant under any unitary transformations of trivalent vertices, and is therefore a fixed property of X_i. Furthermore, (2.22) tells us that $\varkappa_i = \pm 1$. The object X_i is said to be *(anti)symmetrically* self-dual when $\varkappa_i$ is (-1 or) $+1$.

Further details are given in Appendix A. Following (2.20), we can make the identification

(2.23) 

which allows us to slide arrows around cups and caps.

2.6. **Normalisation and partial trace.**

Remark 2.5. A unitary fusion category admits a unique spherical (and corresponding pivotal) structure

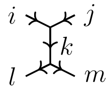
Let $\mathcal{C}$ be a unitary spherical fusion category. We will henceforth use labels $i \in I$ to denote objects $X_i \in \mathrm{Irr}(\mathcal{C})$.

Remark 2.6. (Multiplicity-free) Since the results of this paper pertain to fusion rules without multiplicity, we shall henceforth assume our fusion categories to be *multiplicity-free* i.e. $N_k^{ij} \in \{0,1\}$ for all i, j, k (unless stated otherwise). This obviates the need to index trivalent vertices (e.g. μ can be omitted in (2.6) and (2.9) in this instance).

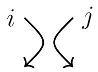
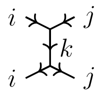
We adopt a normalisation convention where trivalent vertices as in (2.6) are normalised through a scaling of factor $\sqrt[4]{\frac{d_k}{d_i d_j}}$. Further details are provided in Appendix B. Under this normalisation, observe that

$$\lambda_{kijk} = \sqrt{\frac{d_i d_j}{d_k}} \tag{2.24}$$

in (2.13). Following Remark 2.3, we have a *canonical (orthonormal) basis*

(2.25) $\mathrm{Hom}(i \otimes j, l \otimes m) = \mathrm{span}_{\mathbb{C}} \left\{ \left(\sqrt[4]{\frac{d_k^2}{d_i d_j d_l d_m}} \right)$  $\right\}_{k \in I : N_k^{ij} N_k^{lm} \neq 0}$

where we call the graphical components of the basis diagrams *jumping jacks* or *jack morphisms*. Using the canonical basis for $\mathrm{End}(i \otimes j)$, we have the decomposition

(2.26) $\mathrm{id}_{i \otimes j} =$  $= \sum_{k \in I : N_k^{ij} \neq 0} \sqrt{\frac{d_k}{d_i d_j}}$ 

For any morphism $f \in \mathrm{Hom}(i_1 \otimes i_2 \cdots \otimes i_n, j_1 \otimes j_2 \cdots \otimes j_n)$, one can define a *right partial trace* if $i_n = j_n$, and a *left partial trace* if $i_1 = j_1$.

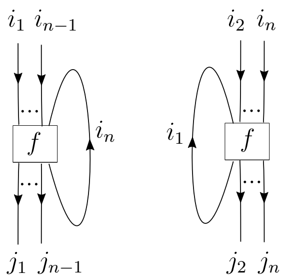
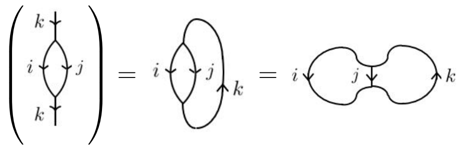


FIGURE 2. The right and left partial traces of f.

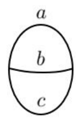
Now suppose that $\mathcal{C}$ is also spherical. We define the *phi-net*

(2.27) $$\Phi(i,j,k) := \widetilde{Tr}$$ 

where the final diagram corresponds to the left and right partial trace of a basis jack in $\mathrm{End}(i^* \otimes k)$. Following (2.24), we know that

$$\Phi(i,j,k) = \sqrt{\frac{d_i d_j}{d_k}} \cdot \widetilde{Tr}(\mathrm{id}_k) = \sqrt{d_i d_j d_k} \tag{2.28}$$

Given a, b, c self-dual, we define the *theta-net*

(2.29) $$\Theta(a,b,c) :=$$ 

where we have left the edges unoriented (since the labels are self-dual). Note that $\Theta(a,b,c) = \Phi(a,b,c)$. Applying the left and right partial traces to (2.26), we get

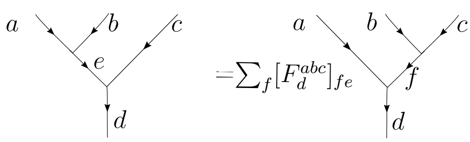
$$d_i d_j = \sum_{k \in I : N_k^{ij} \neq 0} \sqrt{\frac{d_k}{d_i d_j}} \Phi(i^*, k, j) = \sum_k N_k^{ij} d_k \tag{2.30}$$

2.7. F**-matrices.** Recall that a monoidal category $\mathcal{C}$ has associativity isomorphisms $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for any objects $X, Y, Z \in \mathcal{C}$. These isomorphisms satisfy compatibility conditions given by the pentagon and triangle axioms.

For a skeletal fusion category, after making a choice of basis for each $\mathrm{Hom}(i \otimes j, k)$ where $i, j, k \in I$, we obtain a block-diagonal matrix A^{abc} corresponding to each associativity isomorphism $\alpha_{a,b,c}$ where $a, b, c \in I$. Each block in A^{abc} is called an *F-matrix*, and is written F_d^{abc} (where d indexes each block). As a map, F_d^{abc} represents the isomorphism (2.31) and can be interpreted as a change of (fusion) basis on $\mathrm{Hom}(a \otimes b \otimes c, d)$.

$$F_d^{abc} : \bigoplus_e \mathrm{Hom}(a \otimes b, e) \otimes \mathrm{Hom}(e \otimes c, d) \xrightarrow{\sim} \bigoplus_f \mathrm{Hom}(a \otimes f, d) \otimes \mathrm{Hom}(b \otimes c, f) \tag{2.31}$$

where $A^{abc} = \bigoplus_d F_d^{abc}$. In the graphical calculus,

(2.32) 

The entries of an F-matrix are called *F-symbols* (or $6j$-symbols). In terms of the fusion coefficients, associativity is expressed as

$$\sum_e N_e^{ab} N_d^{ec} = \sum_f N_d^{af} N_f^{bc} \tag{2.33}$$

Remark 2.7 (**Skeletal data I**). Given a fusion category $\mathcal{C}$, its *skeletal data* is given by the set of all fusion coefficients and F-symbols; this data completely characterises $\mathcal{C}$. The F-symbols satisfy the pentagon equation coming from the pentagon axiom. If $\mathcal{C}$ has a positive dagger structure, it is easy to see that all associated F-matrices will be unitary (and so $\mathcal{C}$ is called unitary).

2.8. **Braided tensor categories.** Recall that for any two objects X and Y in a *braided tensor category* $\mathcal{C}$, a *braiding* is a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ which is compatible with the associativity isomorphisms: this is ensured by the hexagon axioms, and the braidings consequently satisfy the *Yang-Baxter equation*

$$(c_{Y,Z} \otimes \mathrm{id}_X) \circ (\mathrm{id}_Y \otimes c_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{X,Y}) \circ (c_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z}) \tag{2.34}$$

for any $X, Y, Z \in \mathcal{C}$. This affords us braid isotopy in the graphical calculus.

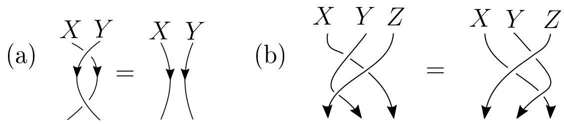
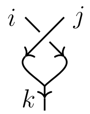
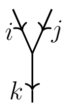
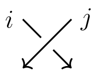
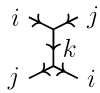


FIGURE 3. (a) $(c_{X,Y})^{-1} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$, (b) Yang-Baxter equation.

For a skeletal braided fusion category, after making a choice of basis for each $\mathrm{Hom}(i \otimes j, k)$ where $i, j, k \in I$, we obtain a block-diagonal matrix R^{ij} corresponding to each braiding isomorphism $c_{i,j}$. In the multiplicity-free case, each block is a 1×1 matrix denoted by R^{ij}_k (where k indexes each block) whose entry is called an *R-symbol*. By abuse of notation, we will use R^{ij}_k to denote the 1×1 matrix and the R-symbol interchangeably. In the graphical calculus, the R-symbols are given by

(2.35)  $:= R^{ij}_k$ 

whence in the graphical calculus, the *R-matrix* is given by

(2.36) $R^{ij} :=$  $\overset{(2.25)}{=} \displaystyle\sum_{k \in I : N^{ij}_k \neq 0} R^{ij}_k \sqrt{\frac{d_k}{d_i d_j}}$ 

Thus, the R-matrix is diagonal; specifically, we have

$$R^{ij} = \bigoplus_{k \in I : N^{ij}_k \neq 0} R^{ij}_k \tag{2.37}$$

In the presence of a braiding, all fusion coefficients clearly satisfy

$$N^{ij}_k = N^{ji}_k \tag{2.38}$$

Remark 2.8 (**Skeletal data II**). Given a braided fusion category $\mathcal{C}$, its *skeletal data* is given by the set of all fusion coefficients, F-symbols and R-symbols; this data completely characterises $\mathcal{C}$.[5] The F-symbols and R-symbols satisfy the hexagon equations coming from the hexagon axioms. If $\mathcal{C}$ has a positive dagger structure, then the category is called unitary: we know that all associated F-matrices will be unitary; furthermore, all associated R-matrices must also be unitary, since every admissible braiding on a unitary fusion category must also be unitary [18, Theorem 3.2].

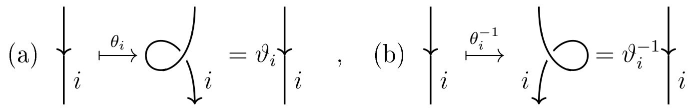
[5] The skeletal data of a ribbon fusion category or modular tensor category is also given by this set.

2.9. **Ribbon structure.** A spherical braided fusion category $\mathcal{C}$ is called a *ribbon fusion* (or *premodular*) *category*. This is a braided fusion category with a *ribbon structure*, which is given by a natural isomorphism $\theta_X : X \xrightarrow{\sim} X$ called the *twist* that satisfies

$$\theta_{X\otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y) \tag{2.39a}$$

$$(\theta_X)^* = \theta_{X^*} \tag{2.39b}$$

for all $X, Y \in \mathcal{C}$, and where * denotes the dual functor on the left-hand side of (2.39b). Graphically, the twist is defined as follows for a skeletal ribbon category:

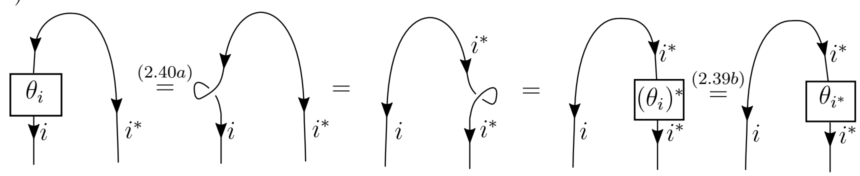
(2.40) 

where $\vartheta_i \in \mathbb{C}^\times$. Note that (2.40b) follows from (2.40a), since by braid isotopy (and pivotality),

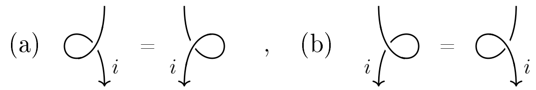
(2.41)

Further note that

(2.42)

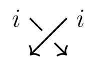


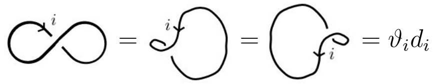
whence we obtain (2.43a). Equation (2.43b) follows similarly.

(2.43) 

From (2.42), the skeletal form of (2.39b) is also made apparent:

$$\vartheta_i = \vartheta_{i^*} \tag{2.44}$$

Taking the left and right partial traces for the crossing , note that

(2.45) 

Resolving the crossing in the first diagram of (2.45) using (2.36), it easy to check that

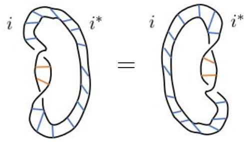
$$\vartheta_i = \frac{1}{d_i} \sum_k R^{ii}_k d_k \tag{2.46}$$

It can also be shown (Appendix A) that for i self-dual,

$$\vartheta_i = \varkappa_i \left(R^{ii}_0\right)^{-1} \tag{2.47}$$

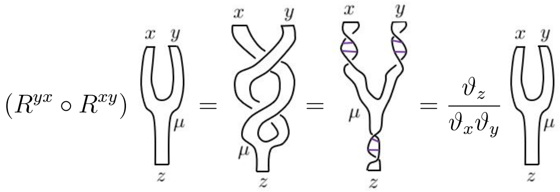
In the graphical calculus for ribbon categories, edges may be promoted from lines to ribbons, and twists are 2π clockwise self-rotations of a ribbon. A labelled edge is assumed to be oriented from top-to-bottom. For instance, (2.44) can be observed from

(2.48)



where the twist is pushed around the closed ribbon. For any $x, y, z \in \mathrm{Irr}(\mathcal{C})$, (2.39a) may be illustrated via the action of the monodromy on a basis element of $\mathrm{Hom}(x \otimes y, z)$:

(2.49)



where we have relaxed the multiplicity-free assumption. Thus,

$$\sum_\lambda [R_z^{yx}]_{\mu\lambda} [R_z^{xy}]_{\lambda\nu} = \frac{\vartheta_z}{\vartheta_x \vartheta_y} \delta_{\mu\nu} \tag{2.50}$$

Graphically, ribbon structure affords our diagrams equivalence under braid isotopy on the 2-sphere. We henceforth refer to braid isotopy on the 2-sphere as *framed isotopy.*

Remark 2.9.

(i) The Anderson-Moore-Vafa theorem [19, 20] tells us that the twist factor ϑ_i is a root of unity for all $i \in I$. For a proof, we refer the reader to [15, Theorem E.10].
(ii) A unitary braided fusion category admits a unique unitary ribbon structure [18].

Proposition 2.10. Let $\mathcal{C}$ be a unitary ribbon fusion category. Then for any $x \in \mathrm{Irr}(\mathcal{C})$,

$$d_x \in \{1\} \cup [\sqrt{2}, \infty) \tag{2.51}$$

Proof. We know that $d_x > 0$ by unitarity. Using (2.30), we have $d_x d_{\mathbf{1}} = d_x$, whence $d_{\mathbf{1}} = 1$. It will be useful to classify x according to whether it satisfies the property[6]

(P0) $\sum_z N_z^{xy} = 1$ for all $y \in \mathrm{Irr}(\mathcal{C})$

Claim: x satisfies (P0) if and only if x is invertible (i.e. $x \otimes x^* = \mathbf{1}$).
If x satisfies (P0), then x is clearly invertible. If x is invertible, then $x^* \otimes x \otimes y = y$ for all $y \in \mathrm{Irr}(\mathcal{C})$. Thus, $\sum_z N_y^{x^* z} N_z^{xy} = 1$, whence $\sum_z N_z^{xy} = 1$ for all y. This shows the claim. It immediately follows that if x satisfies (P0), then so does x^*. Now,

(i) If x satisfies (P0), then $d_x d_{x^*} = d_x^2 = d_{\mathbf{1}} = 1$, whence $d_x = 1$.
(ii) If x does not satisfy (P0), then $d_x d_{x^*} = d_x^2 = d_{\mathbf{1}} + \sum_{y \neq \mathbf{1}} N_y^{xx^*} d_y > 1$, whence $d_x > 1$. The lower bound is attained when $x \otimes x^* = \mathbf{1} \oplus y$ for some y satisfying (P0). Thus, $d_x \geq \sqrt{2}$.

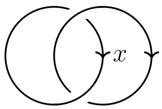
□

[6] In a field-theoretic context, (P0) characterises the *abelianity* of a quasiparticle.

2.10. **Modularity.** Let $\mathcal{C}$ be a braided fusion category. An object X in $\mathcal{C}$ such that

$$c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X\otimes Y} \tag{2.52}$$

for all objects Y in $\mathcal{C}$ is called *transparent*. If all transparent objects in $\mathcal{C}$ are isomorphic to $\mathbf{1}$, then the braiding is called *non-degenerate*.
Further assume that $\mathcal{C}$ is ribbon. We define[7] the matrix $\tilde{S}$ where

(2.53) $[\tilde{S}]_{xy} :=$  x y , $x, y \in \mathrm{Irr}(\mathcal{C})$

i.e. the left and right partial trace of $R^{y^*x} \circ R^{xy^*}$. The *S-matrix* is $S := \dfrac{1}{\mathcal{D}}\tilde{S}$ where

$$\mathcal{D} := \sum_{x\in \mathrm{Irr}(\mathcal{C})} \sqrt{d_x^2} \tag{2.54}$$

is called the *total quantum dimension* of $\mathcal{C}$. A ribbon fusion category $\mathcal{C}$ is called a *modular tensor category* (MTC) if it has a non-degenerate braiding (or equivalently, if the associated S-matrix is invertible).

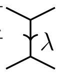
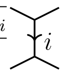
2.11. **Additional conventions.** Throughout much of this paper, we consider a fusion category $\mathcal{C}$ containing a fusion rule of the form

$$q \otimes q = \mathbf{1} \oplus \bigoplus_{i=1}^{k} x_i \tag{2.55}$$

where $q, x_i \in \mathrm{Irr}(\mathcal{C})$ and objects x_i are distinct. In this context, we fix some conventions:

- Unlabelled, unoriented edges are understood to represent edges labelled by the self-dual object q.
- Greek indices (e.g. λ) will be used to denote elements in I for which $N_\lambda^{qq} \neq 0$. Latin indices (e.g. i) will be used to denote elements in $I \setminus \{0\}$ for which $N_i^{qq} \neq 0$.

For instance, (2.36) may be written as follows for $i = j = q$:

(2.56)  $= \sum_\lambda R_\lambda^{qq} \frac{\sqrt{d_\lambda}}{d_q}$  λ $= R_0^{qq} \frac{1}{d_q}$  $+ \sum_i R_i^{qq} \frac{\sqrt{d_i}}{d_q}$  i

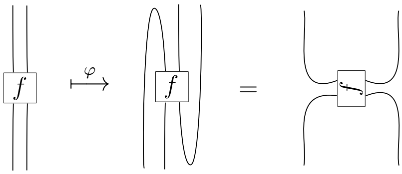
where we call the jumping jacks on the right-hand side of (2.56) *i-jacks*.

2.12. **Rotation operator.** We follow the conventions of Section 2.11, and further assume that $\mathcal{C}$ is *pivotal*. Let $f \in \mathrm{End}(q^{\otimes 2})$ and $f' := \mathrm{id}_q \otimes f \otimes \mathrm{id}_q \in \mathrm{End}(q^{\otimes 4})$. We define the *rotation operator*

$$\varphi : f \mapsto f' \mapsto \underbrace{(\mathrm{id}_q \otimes \mathrm{id}_q \otimes \mathrm{ev}_q)}_{\in\ \mathrm{Hom}(q^{\otimes 4}, q^{\otimes 2}\otimes \mathbf{1})} \circ f' \circ \underbrace{(\mathrm{coev}_q \otimes \mathrm{id}_q \otimes \mathrm{id}_q)}_{\in\ \mathrm{Hom}(\mathbf{1}\otimes q^{\otimes 2}, q^{\otimes 4})} \tag{2.57}$$

[7]We caution the reader that conventions for the orientation of (2.53) vary in the literature.

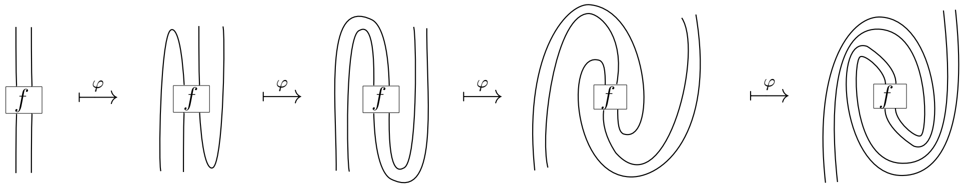
Hence, $\varphi(f) \in \mathrm{Hom}(\mathbf{1} \otimes q^{\otimes 2}, q^{\otimes 2} \otimes \mathbf{1}) = \mathrm{End}(q^{\otimes 2})$. Graphically, φ acts as an anticlockwise $\frac{\pi}{2}$-rotation on a morphism in $\mathrm{End}(q^{\otimes 2})$:



(2.58)

By $\mathbb{C}$-linearity of $\mathcal{C}$ and bilinearity of the bifunctor " $\otimes$ " on morphisms, note that φ is a $\mathbb{C}$-linear operator. Further note that $\varphi^4(f) = f$ (as demonstrated in (2.59), where the final diagram can be straightened to the first diagram).

(2.59)



To the knowledge of the authors, the first instance where the rotation operator was used in a categorical context was in [11]. See Appendix C for a supplementary excursion on the rotation of morphisms.

3. Some Framed Invariants from Ribbon Categories

Let $\mathcal{C}$ be a unitary ribbon fusion category containing a fusion rule of the form

$$q \otimes q = \mathbf{1} \oplus \bigoplus_{i=1}^{k} x_i \tag{3.1}$$

where $q, x_i \in \mathrm{Irr}(\mathcal{C})$ and objects x_i are distinct. Framed, oriented links whose components are labelled by elements of $\mathrm{Irr}(\mathcal{C})$ can be thought of as morphisms in $\mathrm{End}(\mathbf{1})$; the value in $\mathbb{C}$ to which any such link evaluates is invariant under framed isotopy. Restated, given an oriented[8] link diagram D whose components are labelled as such, there is a complex-valued function whose value is constant on the framed isotopy class of D. Such a function coincides with the notion of a *framed link invariant* (when none of the labels are antisymmetrically self-dual).

Let $\Lambda_{\mathcal{C},q}$ denote the framed link invariant for oriented links with all components labelled by $q \in \mathrm{Irr}(\mathcal{C})$ symmetrically self-dual. For q antisymmetrically self-dual, $\Lambda_{\mathcal{C},q}$ denotes the polynomial-valued function obtained from applying the associated skein relation to a link diagram.[9] Our goal is to extract information pertaining to $\Lambda_{\mathcal{C},q}$ when q satisfies (3.1); in particular, we use the rotation operator φ to find relations amongst d_q and the eigenvalues of R^{qq}. We do this for the trivial case (i.e. $q^{\otimes 2} = \mathbf{1}$) and then for cases $k = 1, 2$. When q is symmetrically self-dual (i.e. $\varkappa_q = 1$), it is easy to see that

$$\tilde{\Lambda}_{\mathcal{C},q}(L) = \vartheta_q^{-w(D)} \Lambda_{\mathcal{C},q}(D) \tag{3.2}$$

---

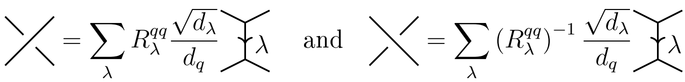
[8]In the instance where all labels are self-dual, D is an unoriented diagram.

[9]This is further explored in Appendix E.

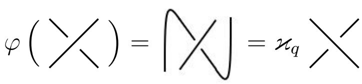
is an oriented link invariant (where L is the oriented link for which D is a diagram, and w is the writhe).

Much of the exposition in this section is already well-established and has been presented in [11] (see Theorems 3.1 & 3.2) where a broader discussion may be found. However, we choose to include this material for its relevance to our main results in Section 4. Furthermore, our approach differs slightly to that taken in [11] and we also treat the instances where q is antisymmetrically self-dual (i.e. $\varkappa_q = -1$), which leads to an extended discussion in Appendix E. In Appendix D, the narrative of this section is approached from the perspective of braid group representations. We follow the conventions fixed in Section 2.11.

R^{qq} is diagonalisable in the canonical basis, so we may resolve crossings as follows:

(3.3) 

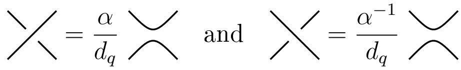
Also,

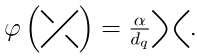
(3.4) 

3.1. **Trivial case.** Here, the fusion rule is given by

$$q \otimes q = \mathbf{1} \tag{3.5}$$

Let $\alpha := R_0^{qq}$. Then from (3.3),

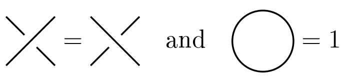
(3.6) 

whence . Using (3.4) and comparing coefficients, we get

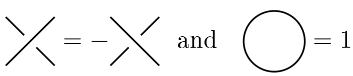
$$\alpha = \varkappa_q \alpha^{-1} \implies \alpha^2 = \varkappa_q.$$

and so we have the following two possibilities:

(1) For $\varkappa_q = 1$, $R_0^{qq} = \pm 1$ with skein relation

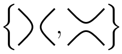
(3.7) 

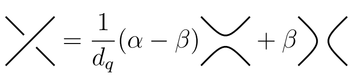
(2) For $\varkappa_q = -1$, $R_0^{qq} = \pm i$ with skein relation

(3.8) 

3.2. **k=1.** Now our fusion rule is of the form

$$q \otimes q = \mathbf{1} \oplus x \tag{3.9}$$

Let $\alpha := R_0^{qq}$ and $\beta := R_x^{qq}$. Using (3.3) and (2.26), we resolve the crossings in the basis  to get

(3.10a) 

$$(3.10b) \qquad = \frac{1}{d_q}(\alpha^{-1} - \beta^{-1}) \qquad + \beta^{-1}$$

whence

$$(3.11) \qquad \varphi\Big(\quad\Big) = \beta \qquad + \frac{1}{d_q}(\alpha - \beta)$$

Then using (3.4) and comparing coefficients with (3.10b), we have

$$(3.12a) \qquad : \quad \varkappa_q \beta^{-1} = \frac{1}{d_q}(\alpha - \beta) \implies \alpha = \varkappa_q d_q \beta^{-1} + \beta$$

$$(3.12b) \qquad : \quad \varkappa_q \beta = \frac{1}{d_q}(\alpha^{-1} - \beta^{-1}) \implies \alpha^{-1} = \varkappa_q d_q \beta + \beta^{-1}$$

which can be solved to get

$$(3.13) \qquad d_q = -\varkappa_q(\beta^2 + \beta^{-2}) \quad , \quad \alpha = -\beta^{-3}$$

We thus have the following two possibilities:

(1) For $\varkappa_q = 1$, $R^{qq} = \mathrm{diag}(-\beta^{-3}, \beta)$ with skein relation

$$(3.14) \qquad = \beta \qquad + \beta^{-1} \qquad \text{and} \qquad = -(\beta^2 + \beta^{-2})$$

i.e. the *Kauffman bracket.*

(2) For $\varkappa_q = -1$, $R^{qq} = \mathrm{diag}(-\beta^{-3}, \beta)$ with skein relation

$$(3.15) \qquad = \beta \qquad - \beta^{-1} \qquad \text{and} \qquad = \beta^2 + \beta^{-2}$$

3.3. **k=2.** Our fusion rule is of the form

$$(3.16) \qquad q \otimes q = \mathbf{1} \oplus x \oplus y$$

Let $\alpha := R_0^{qq}, \beta := R_x^{qq}$ and $\gamma := R_y^{qq}$. Using (3.3) and (2.26), we resolve the crossings in the basis $\Big\{\quad, \quad, \quad x\Big\}$ to get

$$(3.17a) \qquad = A \qquad + B \quad x \quad + C$$

$$(3.17b) \qquad = A' \qquad + B' \quad x \quad + C^{-1}$$

where

$$A := \frac{1}{d_q}(\alpha - \gamma), \;\; B := \frac{\sqrt{d_x}}{d_q}(\beta - \gamma), \;\; C := \gamma$$

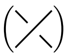
$$A' := \frac{1}{d_q}(\alpha^{-1} - \gamma^{-1}), \;\; B' := \frac{\sqrt{d_x}}{d_q}(\beta^{-1} - \gamma^{-1})$$

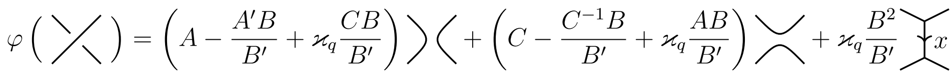
Remark 3.1. We henceforth set $\beta \neq \gamma$. The case $B, B' = 0$ is treated in Section 3.3.1.

Eliminating the x-jacks and rearranging yields

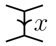
$$(3.18a) \qquad = \left(A - \frac{A'B}{B'}\right) \qquad + \left(C - \frac{C^{-1}B}{B'}\right) \qquad + \frac{B}{B'}$$

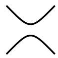
$$(3.18b) \qquad \implies \varphi\Big(\quad\Big) = \left(A - \frac{A'B}{B'}\right) \qquad + \left(C - \frac{C^{-1}B}{B'}\right) \qquad + \varkappa_q \frac{B}{B'}$$

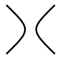
Using (3.17a) to express φ  in our chosen basis,



Applying (3.4) and comparing coefficients with (3.17b), we have

(3.19a)  : $B' = \pm B$

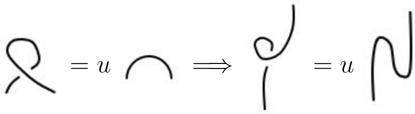
(3.19b)  : $A' = \varkappa_q C + \dfrac{B}{B'}(A - \varkappa_q C^{-1})$

(3.19c)  : $A' = \varkappa_q C + \dfrac{B'}{B}(A - \varkappa_q C^{-1})$

Thus,

(3.20) $$B' = \pm B \quad , \quad A' = \varkappa_q (C \mp C^{-1}) \pm A$$

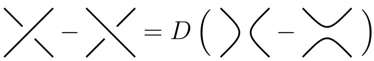
Remark 3.2 (Caveat). When $\varkappa_q = -1$, there is a difference in sign betweeen (vertical) twists and their "horizontal" counterparts (i.e. a $\frac{\pi}{2}$-rotated version). This is taken into account when solving for Cases 3 and 4 below. For instance,



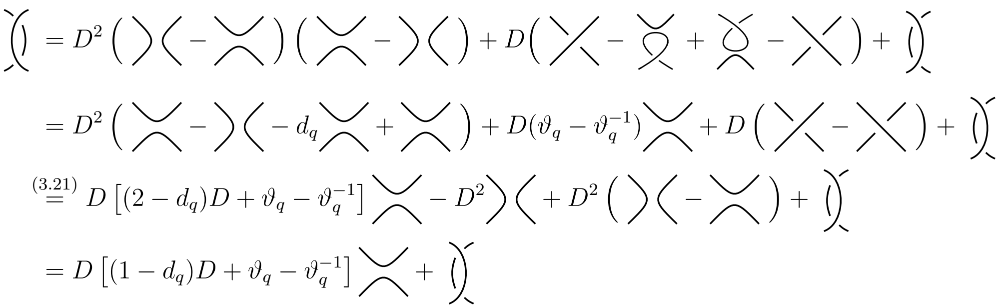
$$\Longrightarrow u = \varkappa_q \vartheta_q$$

There are now four cases to examine (for $B' = \pm B$ and $\varkappa_q = \pm 1$).

<u>Case 1:</u> Let $B' = B$ and $\varkappa_q = 1$. Then (3.18a) becomes

(3.21) 

where $D := C - C^{-1}$. Stacking  on top of  and using (3.21) ,



whence by Reidemeister-II,

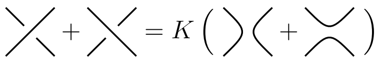
$$\text{(i) } D = \frac{\vartheta_q - \vartheta_q^{-1}}{d_q - 1} \qquad \text{or} \qquad \text{(ii) } D = 0$$

For (i), note that D is well-defined since $d_q > 1$. Then

(3.22) $$d_q = \frac{\alpha^{-1} - \alpha}{\gamma - \gamma^{-1}} + 1 = \frac{\alpha^{-1} - \alpha}{\beta - \beta^{-1}} + 1$$

where the second equality[10] follows from $B' = B$. Since $D \neq 0$, we have $\beta, \gamma \neq \pm 1$ and so (3.22) is also well-defined.

Case 2: Let $B' = -B$ and $\varkappa_q = 1$. Then (3.18a) becomes

(3.23) 

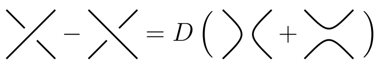
where $K := C + C^{-1}$. Similarly, we get

$$\text{(i) } K = \frac{\vartheta_q + \vartheta_q^{-1}}{d_q + 1} \qquad \text{or} \qquad \text{(ii) } K = 0$$

where for (i),

$$d_q = \frac{\alpha^{-1} + \alpha}{\gamma + \gamma^{-1}} - 1 = \frac{\alpha^{-1} + \alpha}{\beta + \beta^{-1}} - 1 \tag{3.24}$$

Case 3: Let $B' = B$ and $\varkappa_q = -1$. Then (3.18a) becomes

(3.25) 

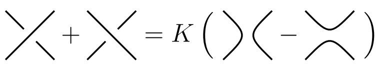
Similarly, we get

$$\text{(i) } D = \frac{\vartheta_q - \vartheta_q^{-1}}{d_q + 1} \qquad \text{or} \qquad \text{(ii) } D = 0$$

where for (i),

$$d_q = \frac{\alpha - \alpha^{-1}}{\gamma - \gamma^{-1}} - 1 = \frac{\alpha - \alpha^{-1}}{\beta - \beta^{-1}} - 1 \tag{3.26}$$

Case 4: Let $B' = -B$ and $\varkappa_q = -1$. Then (3.18a) becomes

(3.27) 

Similarly, we get

$$\text{(i) } K = \frac{\vartheta_q + \vartheta_q^{-1}}{d_q - 1} \qquad \text{or} \qquad \text{(ii) } K = 0$$

where for (i),

$$d_q = -\frac{\alpha^{-1} + \alpha}{\gamma + \gamma^{-1}} + 1 = -\frac{\alpha^{-1} + \alpha}{\beta + \beta^{-1}} + 1 \tag{3.28}$$

Remark 3.3. Cases $D = 0$ and $K = 0$ for $\varkappa_q = \pm 1$ are covered in Section 3.3.1.

For $B' = B$ we have $\beta - \beta^{-1} = \gamma - \gamma^{-1} = D$ whence $\sin(\arg \beta) = \sin(\arg \gamma)$. Since $\beta \neq \gamma$, we have $\arg \beta + \arg \gamma = \pi$. Thus,

$$\beta\gamma = -1 \ \text{ and } \ D = \beta + \gamma \quad , \quad B' = B \tag{3.29}$$

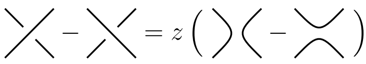
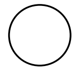
[10] Heuristically, this equality also follows by the symmetry of the construction in β and γ. That is, we could have alternatively used the y-jack in our chosen basis.

For $B' = -B$ we have $\beta + \beta^{-1} = \gamma + \gamma^{-1} = K$ whence $\cos(\arg \beta) = \cos(\arg \gamma)$. Since $\beta \neq \gamma$, we have $\arg \beta + \arg \gamma = 2\pi$. Thus,

$$\beta\gamma = +1 \quad \text{and} \quad K = \beta + \gamma \quad , \quad B' = -B \tag{3.30}$$

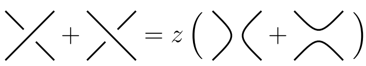
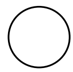
Following the notation in [11], we let $z := \beta + \gamma$ and $a := \vartheta_q$. Summarising these four cases (where $\beta \neq \gamma$ and $z \neq 0$),

(1) For $B' = B$ and $\varkappa_q = 1$, $R^{qq} = \mathrm{diag}(\alpha, \beta, -\beta^{-1})$ with skein relation

(3.31)  and  $= \dfrac{a - a^{-1}}{z} + 1$

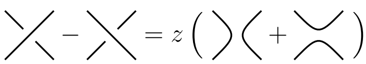
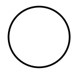
i.e. the *framed Dubrovnik polynomial.*

(2) For $B' = -B$ and $\varkappa_q = 1$, $R^{qq} = \mathrm{diag}(\alpha, \beta, \beta^{-1})$ with skein relation

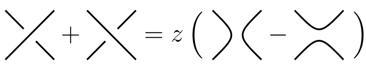
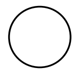
(3.32)  and  $= \dfrac{a + a^{-1}}{z} - 1$

i.e. the *framed Kauffman polynomial.*

(3) For $B' = B$ and $\varkappa_q = -1$, $R^{qq} = \mathrm{diag}(\alpha, \beta, -\beta^{-1})$ with skein relation

(3.33)  and  $= \dfrac{a - a^{-1}}{z} - 1$

(4) For $B' = -B$ and $\varkappa_q = -1$, $R^{qq} = \mathrm{diag}(\alpha, \beta, \beta^{-1})$ with skein relation

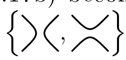
(3.34)  and  $= \dfrac{a + a^{-1}}{z} + 1$

Remark 3.4. Let L denote a link, D a corresponding diagram, and $w(D)$ the writhe of D (given some choice of orientation on D). Let $\Lambda^{(1)}_{(a,z)}$ and $\Lambda^{(2)}_{(a,z)}$ respectively denote the framed Dubrovnik (3.31) and framed Kauffman (3.32) polynomial. Then

$$\Lambda^{(1)}_{(a,z)}(D) = i^{-w(D)}(-1)^{c(L)}\Lambda^{(2)}_{(ia,-iz)}(D) \tag{3.35}$$

noting that the writhe does not depend on the choice of orientation modulo 4. The relation (3.35) was proved by Lickorish in [21].[11]

3.3.1. *Special cases.*

(1) Suppose $B, B' = 0$ (i.e. $\beta = \gamma$). Then (3.17a) and (3.17b) become (3.10a) and (3.10b). That is, the crossings lie in the subspace span . Thus,

	$\varkappa_q = 1$	$\varkappa_q = -1$
Skein relation	(3.14)	(3.15)
R^{qq}	$\mathrm{diag}(-\beta^{-3}, \beta, \beta)$	$\mathrm{diag}(-\beta^{-3}, \beta, \beta)$

(2) Suppose $D = 0$ with $\varkappa_q = \pm 1$. Then

$$(A', B', C^{-1}) = (A, B, C) \implies \alpha, \beta, \gamma \in \{\pm 1\}$$

[11]There is a sign error in the statement of (3.35) in [21] which has been corrected here. The same correction appears in [11].

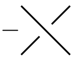
and using (3.17a),(3.17b) and $\beta \neq \gamma$ we get

(3.36) $R^{qq} = (\alpha, \pm 1, \mp 1)$ and  $=$  , $\alpha \in \{\pm 1\}$

(3) Suppose $K = 0$ with $\varkappa_q = \pm 1$. Then

$$(A', B', C^{-1}) = (-A, -B, -C) \implies \alpha, \beta, \gamma \in \{\pm i\}$$

and using (3.17a),(3.17b) and $\beta \neq \gamma$ we get

(3.37) $R^{qq} = (\alpha, \pm i, \mp i)$ and  $= -$  , $\alpha \in \{\pm i\}$

4. Main Results

Let $\mathcal{C}$ be a unitary spherical fusion category containing a fusion rule of the form

$$q \otimes q = \mathbf{1} \oplus \bigoplus_i x_i \tag{4.1}$$

where $q, x_i \in \mathrm{Irr}(\mathcal{C})$, objects x_i are distinct and where $N := \dim(\mathrm{End}(q^{\otimes 2})) \geq 2$.
We write $f_{\mu\nu} := \left[F_q^{qqq}\right]_{\mu\nu}$. For any simple object x in the decomposition of $q^{\otimes 2}$, the symmetries of the fusion coefficients give $N_x^{qq} = N_q^{xq} = N_q^{qx} = N_{x^*}^{qq}$. Firstly, this tells us that the indices of $f_{\mu\nu}$ run over $\mathbf{1}$ and $\{x_i\}_i$. Secondly, this tells us that the set $\{x_i\}_i$ is closed under taking duals: this allows us to define a *(charge) conjugation matrix* $\mathscr{C} := \delta_{\mu\mu^*}$ where μ indexes $\mathbf{1}$ and $\{x_i\}_i$. We follow the conventions from Section 2.11 and let $\sigma(A)$ denote the spectrum of a linear operator A.

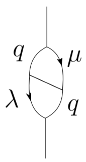
4.1. Rotation operator in the canonical basis.

Lemma 4.1.

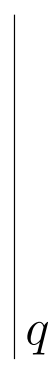
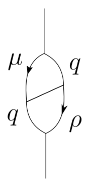
(i) $f_{0\lambda} = \varkappa_q \frac{\sqrt{d_\lambda}}{d_q}$

(ii) $\delta_{\lambda 0} = \varkappa_q \sum_\rho \frac{\sqrt{d_\rho}}{d_q} f_{\rho\lambda}$

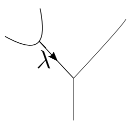
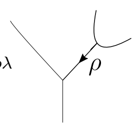
(iii)

$\varkappa_q d_q f_{\mu\lambda}$  $=$ 

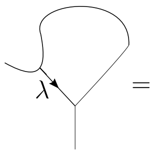
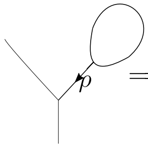
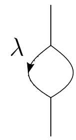
(iv)

$\varkappa_q d_q \delta_{\mu\lambda}$  $= \sum_\rho f_{\rho\lambda}$ 

Proof.

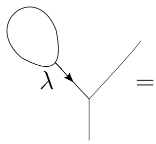
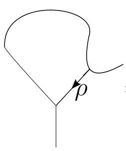
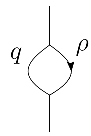
(4.2)  $= \sum_\rho f_{\rho\lambda}$ 

(i) Capping off the rightmost pair of leaves in (4.2) gives

 $= \sum_\rho f_{\rho\lambda}$  $\Longrightarrow \varkappa_q$  $= \sum_\rho f_{\rho\lambda}\delta_{\rho 0} d_q$

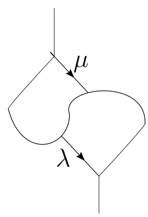
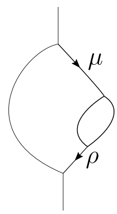
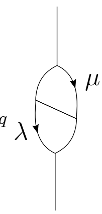
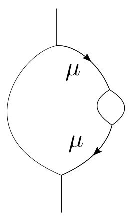
$$\Longrightarrow \varkappa_q \frac{\sqrt{d_\lambda}}{d_q} = \sum_\rho f_{\rho\lambda}\delta_{\rho 0}.$$

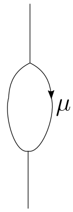
(ii) Capping off the leftmost pair of leaves in (4.2) gives

 $= \sum_\rho f_{\rho\lambda}$  $\Longrightarrow \delta_{\lambda 0} d_q$ $= \varkappa_q \sum_\rho f_{\rho\lambda}$ 

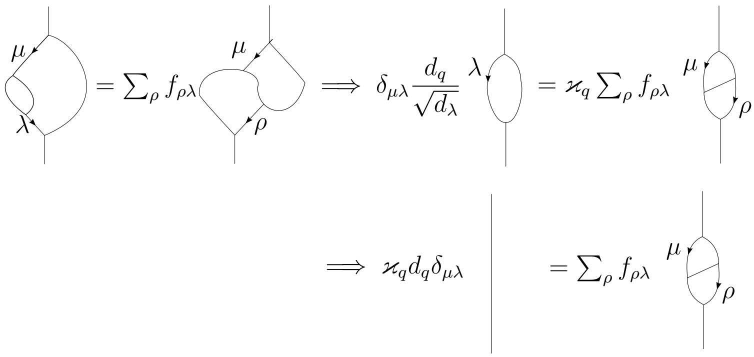
$$\Longrightarrow \delta_{\lambda 0} = \varkappa_q \sum\nolimits_\rho \frac{\sqrt{d_\rho}}{d_q} f_{\rho\lambda}$$

(iii) Stacking the adjoint tree of the right-hand side on (4.2) gives

 $= \sum_\rho f_{\rho\lambda}$  $\Longrightarrow \varkappa_q$  $= f_{\mu\lambda}$ 

$= \dfrac{d_q}{\sqrt{d_\mu}} f_{\mu\lambda}$  $= d_q f_{\mu\lambda}$

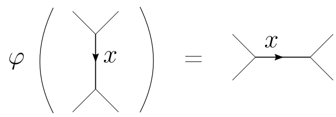
(iv) Stacking the adjoint tree of the left-hand side on (4.2) gives



□

Note that plugging the adjoint of (iii) into (iv) yields $\sum_\rho f_{\rho\lambda} f^*_{\rho\mu} = \delta_{\lambda\mu}$, which agrees with the unitarity of F^{qqq}_q.

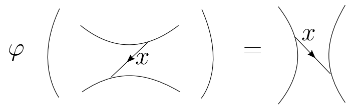
Remark 4.2. (Duality)
In the following, we wish to consider the action of rotation operator φ on our canonical basis. Expanding some arbitrary $h \in \mathrm{End}(q^{\otimes 2})$ in this basis, it is clear that $\varphi^2 = \mathscr{C}$. When considering the image of an x-jack under φ, the directed edge calls for extra caution. For $x \neq \mathbf{1}$ we have



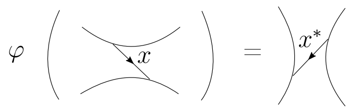
(4.3)

where we call the right-hand side a *bone* morphism. Observing that the jack morphism may equivalently be represented with a slant,

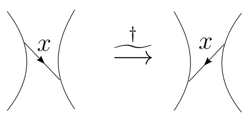
(4.4a)



(4.4b)

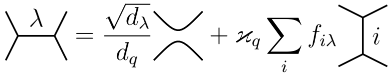


For the x-bone to be well-defined, we must be able to identify (4.4a), (4.4b) and (4.3). The adjunction of (4.4a) yields

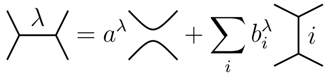


and since the bone is taken to be self-adjoint, we require that $x = x^*$. When considering φ in the canonical basis we must therefore assume that $\mathscr{C} = \mathrm{id}$ i.e. $\{x_i\}_i$ are self-dual in (4.1). This obviates the need to direct any edges in our diagrams.

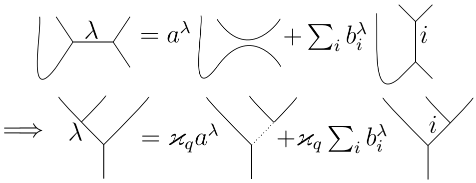
Theorem 4.3. (Bones via jacks)
Given fusion rule (4.1) with x_i self-dual, we have

(4.5)


Proof. Expanding the bone in the canonical basis,

(4.6)


Given a morphism $h \in \mathrm{End}(q^{\otimes 2})$, let $h' := \mathrm{id}_q \otimes h \in \mathrm{End}(q^{\otimes 3})$. Then we define the linear map $\Omega : h \mapsto h' \mapsto (\mathrm{ev}_q \otimes \mathrm{id}_q) \circ h'$. Applying Ω to (4.6), we get



From (2.32), we see that $(a^\lambda, b_i^\lambda) = (\varkappa_q f_{0\lambda}, \varkappa_q f_{i\lambda})$. The result follows from Lemma 4.1(i). □

Corollary 4.4. Let D denote the matrix representation of a rotation operator φ in the canonical basis. Then

(i) $D = \varkappa_q F_q^{qqq}$
(ii) F_q^{qqq} is self-inverse
(iii) $f_{\lambda 0} = f_{0\lambda}$
(iv) The parity of all entries in $\sigma(\varphi)$ cannot be the same

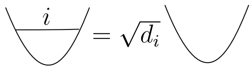
Proof.

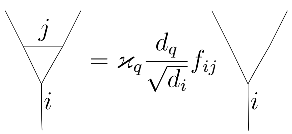
(i) Follows directly from Theorem 4.3.
(ii) $D^2 = \mathscr{C}$ where $\mathscr{C} = \mathrm{id}$ (since $\{x_i\}_i$ are self-dual), whence the result follows by (i).
(iii) For $\lambda = 0$, note that (4.5) is (2.26) and so $f_{i0} = \varkappa_q \frac{\sqrt{d_i}}{d_q}$. The result follows from Lemma 4.1(i).
(iv) Since φ is an involution for $\mathscr{C} = \mathrm{id}$, its spectrum can only consist of ± 1 s. Observe that $|\operatorname{tr}(F_q^{qqq})| < N$ since $|f_{ii}| \leq 1$ and $|f_{00}| = \frac{1}{d_q} < 1$. By (i), $\operatorname{tr}(\varphi) = \varkappa_q \operatorname{tr}(F_q^{qqq})$ whence (iv) follows.

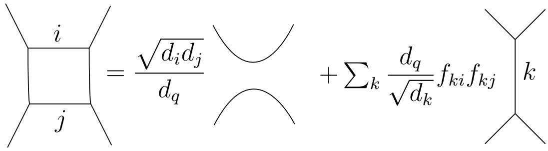
□

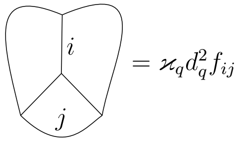
Corollary 4.4(ii) can also be shown by applying linear map $\Omega' : h \mapsto h'' \mapsto (\mathrm{id}_q \otimes \mathrm{ev}_q) \circ h''$ to (4.6), where $h \in \mathrm{End}(q^{\otimes 2})$ and $h'' := h \otimes \mathrm{id}_q$.
Stated differently, Corollary 4.4(iv) says that there are strictly less than N linearly independent formal diagrams in $\mathrm{End}(q^{\otimes 2})$ that are (anti)symmetric under rotation.

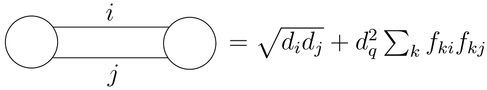
Corollary 4.5. (Bubble-popping)

(i)


(ii)


(iii) 

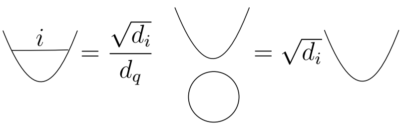
(iv) 

(v) 

See also (4.9).

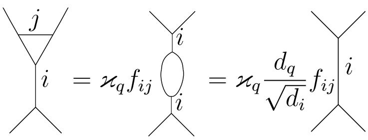
Proof.

(i) Cup off the bottom of an i-bone and use (4.5) to get



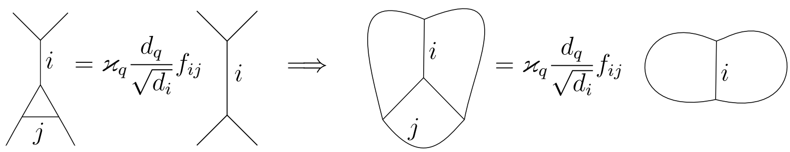
Indeed, capping off both sides agrees with $\Theta(q,i,q) = \Phi(q,i,q) = d_q\sqrt{d_i}$.

(ii) Stacking a j-bone on top of an i-jack and using (4.5), we get

(4.7) 

(iii) Stack an i-bone on top of a j-bone and use (4.5).

(iv)



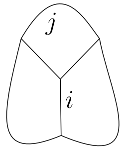
whence the result follows from $\Phi(q,i,q) = d_q\sqrt{d_i}$. Alternatively, this identity coincides with taking the quantum trace of Lemma 4.1(iii) for $(\mu,\lambda) = (i,j)$.

(v) Take the left and right partial traces of (iii) and plug in $\Phi(q,k,q)$.

$\square$

Corollary 4.6. F_q^{qqq} is real-symmetric.

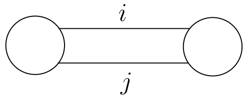
Proof. Corollary 4.4(ii) tells us that F_q^{qqq} is Hermitian. It thus suffices to show that F_q^{qqq} is one of (a) *real* or (b) *symmetric*; nonetheless, we will show both explicitly. Applying the left and right partial traces to (4.7), we obtain

(4.8)


$$= \varkappa_q d_q^2 f_{ij}$$

(a) Inverting the pretzel in Corollary 4.5(iv) via adjunction and comparing the result to (4.8), we see that $f_{ij} = f_{ij}^*$. We know that entries $f_{0\lambda}$ and $f_{\lambda 0}$ are also real from Lemma 4.1(i) and Corollary 4.4(iii).
(b) Note that (4.8) can be deformed to the quantum trace of Lemma 4.1(iii) for $(\mu, \lambda) = (j, i)$. Comparing scalars, we see that $f_{ij} = f_{ji}$. We also know that $f_{0\lambda} = f_{\lambda 0}$ from Corollary 4.4(iii).

□

In light of Corollary 4.6, we may further simplify Corollary 4.5(v) to
(4.9)


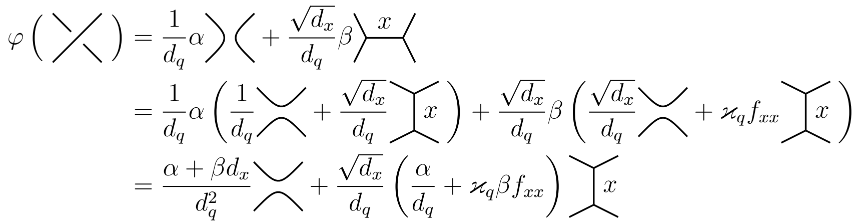
$$= \sqrt{d_i d_j} + \delta_{ij} d_q^2$$

4.2. **Computing some F-symbols.** We now turn our attention to calculating F_q^{qqq} for q self-dual using the rotation operator. If $q \otimes q = \mathbf{1}$ then $F_q^{qqq} = [f_{00}] = \left[\frac{\varkappa_q}{d_q}\right]$. In the case $q \otimes q = \mathbf{1} \oplus x$, we have

$$F_q^{qqq} = \varkappa_q \begin{pmatrix} \dfrac{1}{d_q} & \dfrac{\sqrt{d_q^2 - 1}}{d_q} \\ \dfrac{\sqrt{d_q^2 - 1}}{d_q} & -\dfrac{1}{d_q} \end{pmatrix} \tag{4.10}$$

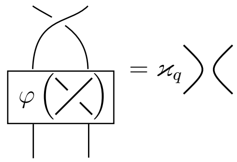
Since x is necessarily self-dual, we may apply the corollaries of Theorem 4.3. Indeed, (4.10) follows almost immediately from Lemma 4.1(i) and Corollary 4.4(iii); all that remains is to find f_{xx}. Applying Corollaries 4.4(i) and (iv), we have $\operatorname{tr}\left(F_q^{qqq}\right) = \varkappa_q \operatorname{tr}(\varphi) = 0$ whence $f_{xx} = -\frac{\varkappa_q}{d_q}$.

If we promote $\mathcal{C}$ to be ribbon, we may also determine f_{xx} by combining the skein theory from Section 3.2 with Theorem 4.3. Resolving as in (3.10a) and rotating,



Comparing coefficients with the -crossing, the cup-cap component corresponds to (2.46), while the x-jack component yields $f_{xx} = \beta^{-2} - \frac{\varkappa_q}{d_q}\alpha\beta^{-1}$. Plugging in the values from (3.13), we get $f_{xx} = -\frac{\varkappa_q}{d_q}$.

Remark 4.7. Another approach to extracting information via the rotation operator is to stack a crossing on its image under φ and then solve for the equation levied by Reidemeister-II. This approach is equivalent to the one taken above i.e. solving



(4.11)

is clearly equivalent to solving $\varphi\left(\times\right) = \varkappa_q \times$. Of course, this is solved with respect to some choice of basis $\mathcal{B}$. For the case $q \otimes q = \mathbf{1} \oplus x$, it is interesting to observe that fixing $\mathcal{B} = \left\{)(, \asymp \right\}$ gave us information pertaining to R^{qq}, while fixing $\mathcal{B}$ canonical gave us information about F_q^{qqq}. Moreover, the information extracted via the latter basis relied on that found using the former basis.

Now suppose that $q \otimes q = \mathbf{1} \oplus x \oplus y$ where $\mathscr{C} = \mathrm{id}$ and $\mathcal{C}$ is ribbon. Following (3.17a),

$$
\begin{aligned}
\varphi\left(\times\right) &= A\,)(+ B \succ\!\!\overset{x}{-}\!\!\prec + C \asymp \\
&\overset{\text{Thm. 4.3}}{=} A\,)(+ \left(\frac{\sqrt{d_x}}{d_q}B + C\right) \asymp + \varkappa_q f_{xx} B\, \mathrm{I}\, x + \varkappa_q f_{yx} B\, \mathrm{I}\, y \\
&\overset{(2.26)}{=} \left(A + \varkappa_q \frac{d_q}{\sqrt{d_y}} f_{yx} B\right))(+ \left(\frac{\sqrt{d_x}}{d_q}B + C - \frac{\varkappa_q}{\sqrt{d_y}} f_{yx} B\right) \asymp \\
&\qquad + \varkappa_q \left(f_{xx} B - \sqrt{\frac{d_x}{d_y}} f_{yx} B\right) \mathrm{I}\, x
\end{aligned}
$$

Solving $\varphi\left(\times\right) = \varkappa_q \times$ with respect to basis $\left\{)(, \asymp , \mathrm{I}\, x \right\}$, we match coefficients with (3.17b) to obtain

(4.12a) $$)(: \quad A = \varkappa_q \left(C^{-1} - \frac{d_q}{\sqrt{d_y}} f_{yx} B\right)$$

(4.12b) $$\asymp : \quad A' = \varkappa_q \left(C + \frac{\sqrt{d_x}}{d_q} B\right) - \frac{1}{\sqrt{d_y}} f_{yx} B$$

(4.12c) $$\mathrm{I}\, x : \quad B' = B\left(f_{xx} - \sqrt{\frac{d_x}{d_y}} f_{yx}\right)$$

Suppose $B, B' \neq 0$. Recall from (3.20) that $B' = \pm B$. Also by Lemma 4.1(i) & (ii),

(4.13) $$-\varkappa_q \frac{\sqrt{d_i}}{d_q} = \sum_j \sqrt{d_j} f_{ji}$$

whence for $i = x$,

(4.14) $$f_{xx} = -\left(\frac{\varkappa_q}{d_q} + \sqrt{\frac{d_y}{d_x}} f_{yx}\right)$$

Combining (4.12c) and (4.14) eventually yields

$$(f_{xx}, f_{yx}) = \left(\frac{\mp\, d_x}{(d_q \mp \varkappa_q)d_q} \pm 1, \frac{\mp\sqrt{d_x d_y}}{(d_q \mp \varkappa_q)d_q}\right) \quad , \quad B' = \pm B \tag{4.15}$$

Setting $i = y$ in (4.13) gives $f_{yy} = -(\frac{\varkappa_q}{d_q} + \sqrt{\frac{d_x}{d_y}} f_{xy})$. By Corollary 4.6, we have $f_{xy} = f_{yx}$ whence

$$f_{yy} = \frac{\mp\, d_y}{(d_q \mp \varkappa_q)d_q} \pm 1 \quad , \quad B' = \pm B \tag{4.16}$$

Theorem 4.8. Let $\mathcal{C}$ be a unitary ribbon fusion category containing a fusion rule

$$q \otimes q = \mathbf{1} \oplus x \oplus y \tag{4.17}$$

where $x, y, q \in \mathrm{Irr}(\mathcal{C})$. If $R^{qq}_x \neq R^{qq}_y$ then $R^{qq}_x R^{qq}_y = \pm 1$ where

- If $R^{qq}_x R^{qq}_y = -1$ then $d_q = \varkappa_q \left(\frac{\vartheta_q - \vartheta_q^{-1}}{R^{qq}_x + R^{qq}_y} + 1\right)$ and the associated skein relation is given by (a) the framed Dubrovnik polynomial (3.31) for $\varkappa_q = 1$ and (b) (3.33) for $\varkappa_q = -1$.
- If $R^{qq}_x R^{qq}_y = 1$ then $d_q = \varkappa_q \left(\frac{\vartheta_q + \vartheta_q^{-1}}{R^{qq}_x + R^{qq}_y} - 1\right)$ and the associated skein relation is given by (c) the framed Kauffman polynomial (3.32) for $\varkappa_q = 1$ and (d) (3.34) for $\varkappa_q = -1$.

If x and y are self-dual then

(i) For $R^{qq}_x R^{qq}_y = -1$, we have

$$F^{qqq}_q = \varkappa_q \begin{pmatrix} \frac{1}{d_q} & \frac{\sqrt{d_x}}{d_q} & \frac{\sqrt{d_y}}{d_q} \\ \frac{\sqrt{d_x}}{d_q} & \frac{-\,d_x}{(\varkappa_q d_q - 1)d_q} + 1 & \frac{-\sqrt{d_x d_y}}{(\varkappa_q d_q - 1)d_q} \\ \frac{\sqrt{d_y}}{d_q} & \frac{-\sqrt{d_x d_y}}{(\varkappa_q d_q - 1)d_q} & \frac{-\,d_y}{(\varkappa_q d_q - 1)d_q} + 1 \end{pmatrix} \tag{4.18}$$

(ii) For $R^{qq}_x R^{qq}_y = 1$, we have

$$F^{qqq}_q = \varkappa_q \begin{pmatrix} \frac{1}{d_q} & \frac{\sqrt{d_x}}{d_q} & \frac{\sqrt{d_y}}{d_q} \\ \frac{\sqrt{d_x}}{d_q} & \frac{d_x}{(\varkappa_q d_q + 1)d_q} - 1 & \frac{\sqrt{d_x d_y}}{(\varkappa_q d_q + 1)d_q} \\ \frac{\sqrt{d_y}}{d_q} & \frac{\sqrt{d_x d_y}}{(\varkappa_q d_q + 1)d_q} & \frac{d_y}{(\varkappa_q d_q + 1)d_q} - 1 \end{pmatrix} \tag{4.19}$$

Corollary 4.9. For a unitary ribbon fusion category $\mathcal{C}$ containing a fusion rule of the form (4.17) with x and y self-dual, we have

(i) q is symmetrically self-dual

(ii) $\sigma(\varphi) = \begin{cases} \{+1,+1,-1\} & , \ \Lambda_{\mathcal{C},q} \text{ is the framed Dubrovnik polynomial} \\ \{+1,-1,-1\} & , \ \Lambda_{\mathcal{C},q} \text{ is the framed Kauffman polynomial} \end{cases}$

Proof.

(i) $\operatorname{tr}\left(F_q^{qqq}\right) = \pm 3$ for $R_x^{qq} R_y^{qq} = \mp 1$ when $\varkappa_q = -1$. The result follows by Corollaries 4.4 (i) and (iv).

(ii) By (i), $\Lambda_{\mathcal{C},q}$ is either the framed Dubrovnik or Kauffman polynomial. For the former, $\operatorname{tr}(F_q^{qqq}) = 1$ and for the latter $\operatorname{tr}(F_q^{qqq}) = -1$. The result follows by Corollary 4.4 (i).

□

4.3. **Some new bases.** As an application of the results thus far, we establish some new bases for $\operatorname{End}(q^{\otimes 2})$ where

$$q \otimes q = \mathbf{1} \oplus x \oplus y \tag{4.20}$$

in a unitary ribbon fusion category $\mathcal{C}$ with x and y self-dual. First, let us make some observations for q such that

$$q \otimes q = \mathbf{1} \oplus \bigoplus_{i=1}^{k} x_i \tag{4.21}$$

with $q, x_i \in \operatorname{Irr}(\mathcal{C})$ and distinct self-dual objects x_i. We restrict our search to bases $\mathcal{B}$ satisfying the following property[12]:

(P1) *The elements of $\mathcal{B}$ are permuted under the action of φ (up to a sign for 1-cycles).*

We will see that bases satisfying this property are closely related to the eigenbasis of φ. Clearly, the matrix representation of φ in such a basis is a symmetric permutation matrix with -1's permitted along the diagonal. The permutation consists of 2-cycles and signed 1-cycles. Then $+1$'s and -1's along the matrix diagonal respectively correspond to 'positive' and 'negative' 1-cycles. Let

$$N := \dim\left(\operatorname{End}(q^{\otimes 2})\right) \ , \ n := \#\{\text{cycles}\}$$
$$b := \#\{\text{positive 1-cycles}\} \ , \ f := \#\{\text{negative 1-cycles}\}$$

Assume $\mathcal{B}$ satisfying **(P1)** exists. Then write $\mathcal{B} = \{D_{ij}\}_{(i,j)}^{(n,l(i))}$ where i indexes the n cycles and $l(i)$ is the length of the i^{th} cycle. We have

$$\varphi(D_{ij}) = \begin{cases} D_{i,\overline{j+1}} & , \quad l(i) = 2 \\ D_{ij} & , \quad i \text{ indexes a positive 1-cycle} \\ -D_{ij} & , \quad i \text{ indexes a negative 1-cycle} \end{cases} \tag{4.22}$$

where $\overline{j}$ denotes j modulo 2. Recall that $\sigma(\varphi)$ must consist of a mixture of ± 1's. Let V_1 and V_{-1} respectively denote the $+1$ and -1 eigenspaces of φ. Then

$$\dim(V_1) = n - f \quad , \quad \dim(V_{-1}) = n - b \tag{4.23}$$

[12] Recall that the method employed for determining $\Lambda_{\mathcal{C},q}$ in Section 3 relied on expressing a crossing as a linear combination of morphisms that were invariant under the action of the rotation operator (up to permutation). This motivates the study of bases satisfying **(P1)**.

whence

$$N = 2n - b - f \quad \text{and} \quad \left\lceil \frac{N}{2} \right\rceil \le n \le N \tag{4.24}$$

where the upper bound is realised when $\mathcal{B}$ is an eigenbasis for φ.

Example 4.10. We can use the above to determine the possible actions (as a signed permutation) of φ on $\mathcal{B}$ given $\sigma(\varphi)$. We will denote such an action by the signed cycle type $(a_1, \ldots, a_n)$ where $|a_i| = l(i)$ and $a_i = \pm 1$ encodes the sign of a 1-cycle. In the following, we exclude the instances where $n = N$ (i.e. eigenbases).

(i) $N = 2$:

(n,b,f)	$(1,0,0)$
$\sigma(\varphi)$	$\{+1,-1\}$
Cycle type	(2)

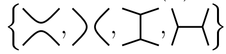
(ii) $N = 3$:

(n,b,f)	$(2,1,0)$	$(2,0,1)$
$\sigma(\varphi)$	$\{+1,+1,-1\}$	$\{+1,-1,-1\}$
Cycle type	$(2,1)$	$(2,-1)$

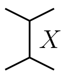
(iii) $N = 4$: (1st instance where there are two distinct cycle types for the same $\sigma(\varphi)$).

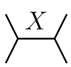
(n,b,f)	$(3,1,1)$	$(3,2,0)$	(3,0,2)	(2,0,0)
$\sigma(\varphi)$	$\{+1+1,-1,-1\}$	$\{+1,+1,+1,-1\}$	$\{+1,-1,-1,-1\}$	$\{+1,+1,-1,-1\}$
Cycle type	$(2,1,-1)$	$(2,1,1)$	$(2,-1,-1)$	$(2,2)$

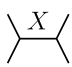
etc.

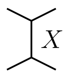
- We already encountered basis  corresponding to Example 4.10(i).
- Let $\mathcal{C}$ be the unitary $(G_2)_2$ ribbon category and take $q \in \mathrm{Irr}(\mathcal{C})$ with fusion rule $q \otimes q = \mathbf{1} \oplus x \oplus y \oplus q$. There exists a basis  on $\mathrm{End}(q^{\otimes 2})$ [3]. This is basis of cycle type $(2,2)$, whence we see from Example 4.10(iii) and Corollary 4.4(i) that F_q^{qqq} is traceless.
- We show by construction that there exist bases corresponding to Example 4.10(ii).

We define

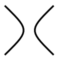
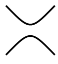
(4.25) $\mathcal{J}_X :=$ 

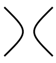
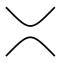
$+$ 

and $\mathcal{J}'_X :=$ 

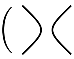
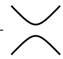
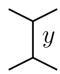
$-$ 

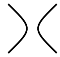
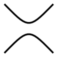
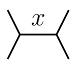
Observe that for (4.20),

(4.26a)  $+$  $= \dfrac{\sqrt{d_x}}{d_q - 1}\, \mathcal{J}_x + \dfrac{\sqrt{d_y}}{d_q - 1}\, \mathcal{J}_y$

(4.26b)  $-$  $= -\dfrac{\sqrt{d_x}}{d_q + 1}\, \mathcal{J}'_x - \dfrac{\sqrt{d_y}}{d_q + 1}\, \mathcal{J}'_y$

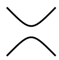
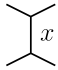
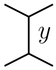
whence

(4.27a) $\sqrt{d_x}\, \mathcal{J}_x + \sqrt{d_y}\, \mathcal{J}'_y = (d_q - 1)\Big($  $+$  $\Big) - 2\sqrt{d_y}$ 

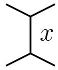
(4.27b) $\sqrt{d_x}\, \mathcal{J}_x - \sqrt{d_y}\, \mathcal{J}'_y = (d_q + 1)\Big($  $-$  $\Big) + 2\sqrt{d_x}$ 

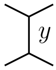
Recall from Corollary 4.9(i) that $\varkappa_q = 1$. Expanding in the canonical basis,

(4.28a)

$\sqrt{d_x}\ \mathcal{J}_x + \sqrt{d_y}\ \mathcal{J}'_y = \left(d_q - \frac{1}{d_q}\right)$  $+ \sqrt{d_x}\left(1 - \frac{1}{d_q}\right)$  $- \sqrt{d_y}\left(1 + \frac{1}{d_q}\right)$ 

(4.28b)

$\sqrt{d_x}\ \mathcal{J}_x - \sqrt{d_y}\ \mathcal{J}'_y = \left(\frac{1}{d_q} - d_q + 2\frac{d_x}{d_q}\right)$  $+ \sqrt{d_x}\left(1 + \frac{1}{d_q} + 2f_{xx}\right)$ 

$+ \left[\sqrt{d_y}\left(1 + \frac{1}{d_q}\right) + 2\sqrt{d_x}f_{yx}\right]$ 

where in (4.28b) we used (4.5). Let

$$\mathcal{J}^+_{xy} := \sqrt{d_x}\ \mathcal{J}_x + \sqrt{d_y}\ \mathcal{J}'_y \quad , \quad \mathcal{J}^-_{xy} := \sqrt{d_x}\ \mathcal{J}_x - \sqrt{d_y}\ \mathcal{J}'_y \tag{4.29}$$

Lemma 4.11. $\mathcal{J}^+_{xy}$ and $\mathcal{J}^-_{xy}$ are linearly independent.

Proof. $\varphi(\mathcal{J}^+_{xy}) = \mathcal{J}^-_{xy}$. Suppose $\mathcal{J}^-_{xy} = z\mathcal{J}^+_{xy}$ for some $z \in \mathbb{C}$. Then $\varphi(\mathcal{J}^+_{xy}) = z\mathcal{J}^+_{xy}$ whence $\mathcal{J}^+_{xy} = \pm\mathcal{J}^-_{xy}$. For $z = +1$ and $z = -1$ we respectively get $\mathcal{J}'_y = 0$ and $\mathcal{J}_x = 0$, both of which yield a contradiction. ↯

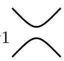
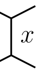
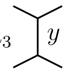
Theorem 4.12. Let q be defined as in (4.20). Then

(i) $\Big\{\mathcal{J}^+_{xy}\ ,\ \mathcal{J}^-_{xy}\ ,\ )(+$  $\Big\}$ defines a basis for $\mathrm{End}(q^{\otimes 2})$ when $\Lambda_{\mathcal{C},q}$ is the framed Dubrovnik polynomial.

(ii) $\Big\{\mathcal{J}^+_{xy}\ ,\ \mathcal{J}^-_{xy}\ ,\ )(-$  $\Big\}$ defines a basis for $\mathrm{End}(q^{\otimes 2})$ when $\Lambda_{\mathcal{C},q}$ is the framed Kauffman polynomial.

Note that the bases in the above theorem satisfy **(P1)** (see Example 4.10(ii)), and that we can permute labels x and y by the symmetry of our construction.

Proof. By Lemma 4.11, it suffices in each case to show that the final basis element is not a linear combination of the first two. For $c_1, c_2 \in \mathbb{C}$,

$c_1\mathcal{J}^+_{xy} + c_2\mathcal{J}^-_{xy} = a_1$  $+ a_2$  $+ a_3$ 

where

$$a_1 := \frac{1}{d_q}\left[(1 - d_q^2)(c_2 - c_1) + 2c_2 d_x\right] \quad , \quad a_2 := \sqrt{d_x}\left[\left(1 + \frac{1}{d_q}\right)c_2 + \left(1 - \frac{1}{d_q}\right)c_1 + 2c_2 f_{xx}\right]$$

$$a_3 := \sqrt{d_y}\left[\left(1 + \frac{1}{d_q}\right)(c_2 - c_1) + 2c_2\sqrt{\frac{d_x}{d_y}}f_{yx}\right]$$

Recall from (4.12c) that

$$\sqrt{\frac{d_x}{d_y}}f_{yx} = f_{xx} - r \tag{4.30}$$

where $r = 1, -1$ when $\Lambda_{\mathcal{C},q}$ is the framed Dubrovnik and framed Kauffman polynomial respectively. Thus,

$$a_3 = \sqrt{\frac{d_y}{d_x}}\, a_2 - 2\sqrt{d_y}(c_1 + rc_2) \tag{4.31}$$

(i) Suppose there exist c_1 and c_2 such that $c_1\mathcal{J}^+_{xy} + c_2\mathcal{J}^-_{xy} = \,)(+ \asymp$. Then $(a_1, a_2, a_3) = \left(1 + \frac{1}{d_q}, \frac{\sqrt{d_x}}{d_q}, \frac{\sqrt{d_y}}{d_q}\right)$. Setting $r = 1$, (4.31) gives $c_1 = -c_2$. Now comparing values for a_1 yields

$$c_1 = \frac{1}{2}\left(\frac{1 + d_q}{d_y}\right) \quad , \quad c_2 = -\frac{1}{2}\left(\frac{1 + d_q}{d_y}\right)$$

whence comparing values for a_2 yields

$$f_{xx} = -\frac{d_y}{d_q(1 + d_q)} - \frac{1}{d_q}$$

where we can manipulate the right-hand side to get

$$\begin{aligned} -\frac{d_y}{d_q(1 + d_q)} - \frac{1}{d_q} &= -1 + \frac{d_x}{d_q(d_q - 1)} - \frac{2d_x}{d_q(d_q^2 - 1)} \\ &\overset{(4.18)}{=} -f_{xx} - \frac{2d_x}{d_q(d_q^2 - 1)} \end{aligned}$$

implying that $f_{xx} = -\frac{d_x}{d_q(d_q^2 - 1)}$. Rearranging the expression for f_{xx} from (4.18),

$$f_{xx} = 1 - \frac{d_x}{(d_q - 1)d_q} = \frac{d_q(d_q^2 - 1) - d_x(d_q + 1)}{d_q(d_q^2 - 1)} = \frac{d_q d_y - d_x}{d_q(d_q^2 - 1)}$$

whence we arrive at a contradiction since $d_q, d_y \neq 0$. $\lightning$

(ii) Suppose there exist c_1 and c_2 such that $c_1\mathcal{J}^+_{xy} + c_2\mathcal{J}^-_{xy} = \,)(- \asymp$. Then $(a_1, a_2, a_3) = \left(\frac{1}{d_q} - 1, \frac{\sqrt{d_x}}{d_q}, \frac{\sqrt{d_y}}{d_q}\right)$. Setting $r = -1$, (4.31) gives $c_1 = c_2$. Now comparing values for a_1 yields

$$c_1 = c_2 = \frac{1}{2}\left(\frac{1 - d_q}{d_x}\right)$$

whence comparing values for a_3 yields

$$f_{yx} = \frac{\sqrt{d_x d_y}}{(d_q - 1)d_q}$$

From (4.19), $f_{yx} = \frac{\sqrt{d_x d_y}}{(d_q + 1)d_q}$ whence we arrive at a contradiction since $d_q \neq 0$. $\lightning$

Corollary 4.13. (Diagonalising φ)
Following the notation from (4.23), we have $\mathrm{End}(q^{\otimes 2}) = V_1 \oplus V_{-1}$.

(i) If $\Lambda_{\mathcal{C},q}$ is the framed Dubrovnik polynomial then

$$V_1 = \mathrm{span}\left\{\mathcal{J}_x\ ,\ )(+ \asymp\right\} \quad , \quad V_{-1} = \mathrm{span}\left\{\mathcal{J}'_y\right\} \tag{4.32}$$

(ii) If $\Lambda_{\mathcal{C},q}$ is the framed Kauffman polynomial then

$$V_1 = \mathrm{span}\left\{\mathcal{J}_x\right\} \quad , \quad V_{-1} = \mathrm{span}\left\{\mathcal{J}'_y\ ,\ )(- \asymp\right\} \tag{4.33}$$

Proof. Given $b_1, b_2, b_3 \in \mathbb{C}$, we have

$$\varphi\left(b_1\mathcal{J}^+_{xy} + b_2\mathcal{J}^-_{xy} + b_3\left()(\pm \asymp\right)\right) = b_1\mathcal{J}^-_{xy} + b_2\mathcal{J}^+_{xy} + b_3\left(\asymp \pm)(\right)$$

The result then easily follows from solving

$$b_1\mathcal{J}^-_{xy} + b_2\mathcal{J}^+_{xy} + b_3\left(\asymp \pm)(\right) = \pm\left(b_1\mathcal{J}^+_{xy} + b_2\mathcal{J}^-_{xy} + b_3\left()(\pm \asymp\right)\right)$$

□

Remark 4.14.

(i) For the $N = 2$ case, $\left\{)(+ \asymp,)(- \asymp\right\}$ trivially defines an eigenbasis for φ.

(ii) By symmetry of our construction, we may permute labels x and y in (4.32) and (4.33). For e.g. (4.32), this tells us that $\mathcal{J}_y \in V_1$ and $\mathcal{J}'_x \in V_{-1}$. Recovering the precise linear relations is a straightforward task.

5. Concluding Remarks and Outlook

Using the rotation operator, we exploited the graphical calculus as a tool for exploring unitary spherical fusion categories (and their braided counterparts). We also used this approach to learn more about the link invariants associated to fusion rules of a particular form. Below, we summarise some of the highlights of the paper and discuss some possible directions for future work.

5.1. **Quantum invariants.**

Using the rotation operator, we extended [11, Theorems 3.1 & 3.2] to cover the antisymmetrically self-dual cases. This produced skein relations (3.15), (3.33) and (3.34): to the knowledge of the authors, these have not previously appeared in the literature. In Appendix E, we briefly investigated some properties of invariants associated to antisymmetrically self-dual objects.

We considered the framed link invariants $\Lambda_{\mathcal{C},q}$ associated to (3.1) for $k = 1, 2$. A natural extension of this narrative would be to solve the problem below.

Problem 1. *What is $\Lambda_{\mathcal{C},q}$ when (i) $k \geq 3$, (ii) the fusion rule for $q^{\otimes 2}$ is not multiplicity-free?*

Partial results are known for (i) when $k = 3$. If $q^{\otimes 2} = \mathbf{1} \oplus x \oplus y \oplus q$, then $\Lambda_{\mathcal{C},q}$ is said to be Kuperberg's G_2 invariant in most "nontrivial" cases [11]. See also [22].

We narrowed our focus to discussing skein-theoretic methods for evaluating link diagrams in $\mathrm{End}(\mathbf{1})$ when all components are labelled by the same self-dual element $q \in \mathrm{Irr}(\mathcal{C})$. More generally, one could ask the same question but for

(i) "Polychromatic" link diagrams (i.e. components may have distinct labels), or

(ii) When the labels are not necessarily self-dual (so that orientation matters).

For instance, when $\mathcal{C}$ is a Temperley-Lieb-Jones (TLJ) category, then any polychromatic link diagram can be evaluated as an element of the Kauffman bracket skein algebra: each component is replaced by the corresponding closed Jones-Wenzl idempotent, and the diagram is evaluated via skein relations (3.14). In TLJ categories, all objects are symmetrically self-dual. An important class of TQFTs known as Jones-Kauffman theories are described by TLJ categories (e.g. Ising and Fibonacci MTCs): here, Jones-Wenzl idempotents may be reinterpreted as anyons [26].

In Appendix D, we studied unitary representations of $\mathbb{C}[B_n]$ that factor through the Iwahori-Hecke and Temperley-Lieb algebras. This resulted in a skein relation (D.17) for the framed HOMFLY-PT polynomial, which specialised to (3.14)-(3.15) in the context of a RFC $\mathcal{C}$ (since $b = \pm a^{-1}$ when $\varkappa_q = \pm 1$). In this vein, we pose the following question.

Problem 2. *Is there some 3-variable link polynomial that specialises to (3.31)-(3.34)?*

At the end of Appendix D, we see that the representation of $\mathbb{C}[B_n]$ associated to $\Lambda_{\mathcal{C},q}$ for $q^{\otimes 2} = \mathbf{1} \oplus x \oplus y$ should factor through the cubic Hecke algebra $H_n(Q, 3)$ and the Temperley-Lieb algebra. This motivates Problem 2 by analogy with the $q^{\otimes 2} = \mathbf{1} \oplus x$ exposition.

5.2. **F-Symbols.**

In Section 4, we considered the action of the rotation operator φ on a basis of jumping jacks for $\mathrm{End}(q^{\otimes 2})$. This was for a unitary spherical fusion category $\mathcal{C}$ containing a fusion rule of the form $q^{\otimes 2} = \mathbf{1} \oplus \bigoplus_i x_i$ with all the x_i self-dual. We deduced that $\varphi = \varkappa_q F_q^{qqq}$ (Theorem 4.3) and that F_q^{qqq} is real-symmetric (Corollary 4.6). For instances where $\mathcal{C}$ admits a braiding and $q^{\otimes 2} = \mathbf{1} \oplus x \oplus y$ (with x and y self-dual), we found formulae for F_q^{qqq} in terms of the quantum dimensions (Theorem 4.8), and concluded that $\varkappa_q \neq -1$ (Corollary 4.9). We also saw that the spectrum of the rotation operator distinguishes between the Dubrovnik and Kauffman invariants (Corollary 4.9). Obvious extensions of this work would entail relaxing various assumptions e.g. as in the problem below.

Problem 3. *Can the results of Section 4 be extended to the case where*

(a) *the simple summands in $q^{\otimes 2}$ are non self-dual?*

(b) *$\mathcal{C}$ is not unitary?*

and can we extend Theorem 4.8 to determine general formulae for F_q^{qqq} when

(c) *$\mathcal{C}$ does not admit a braiding?*

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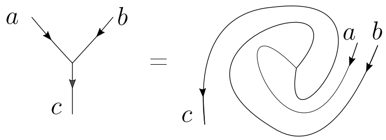
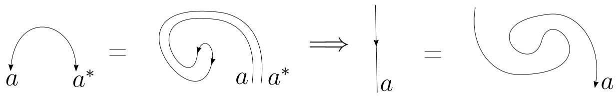
Appendix A. Pivotal Coefficients

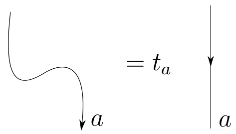
A pivotal structure on a fusion category $\mathcal{C}$ affords our diagrams 2π- rotational isotopy. For example, for a trivalent vertex with a, b and $c \in \mathrm{Irr}(\mathcal{C})$, we have



Applying the identity to a cap, this gives



We write

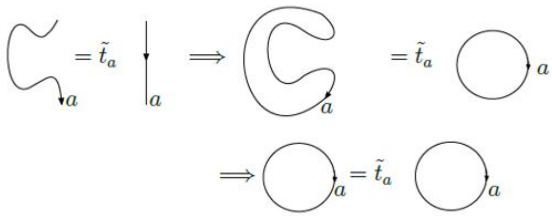


where $t_a \in \mathbb{C}^\times$ by way of the pivotal structure.

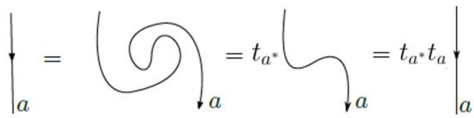
Proposition A.1. *(i)* $|t_a| = 1$, *(ii)* $t_{a^*} = t_a^*$.

Proof.

(i) Let $\tilde{t}_a := |t_a|^2$. Then



(ii) Observe that



whence the result follows by (i).

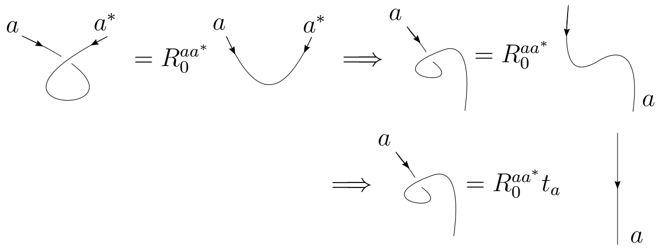
□

Consequently, when $a = a^*$, we have that $t_a = \pm 1$. In this instance, t_a is called the *Frobenius-Schur indicator* and is written $\varkappa_a := t_a$. As stated in Section 2.5, t_a is called the pivotal coefficient of $a \in \mathrm{Irr}(\mathcal{C})$. It is straightforward to show that t_a is gauge-invariant if and only if $a = a^*$.When a is non self-dual it is typical to fix the gauge such that $t_a = 1$.

When a is self-dual,

(i) a is called *symmetrically self-dual* if $\varkappa_a = 1$
(ii) a is called *antisymmetrically self-dual* if $\varkappa_a = -1$

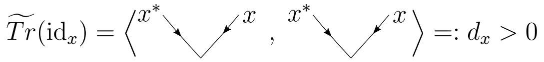
If $\mathcal{C}$ is a unitary ribbon fusion category, note that



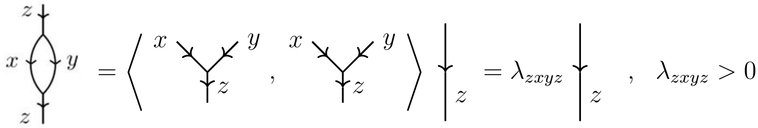
That is, $\vartheta_a^{-1} = R_0^{aa^*} t_a$. In particular, for a self-dual we have $\vartheta_a = \varkappa_a (R_0^{aa})^{-1}$.

Appendix B. Normalisation

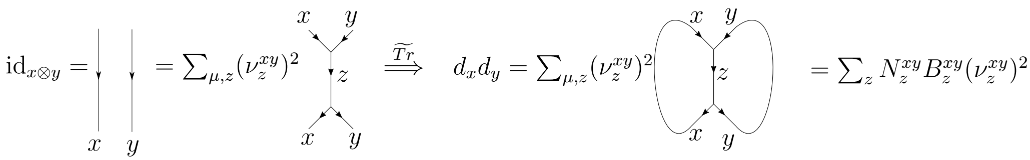
Let $\mathcal{C}$ be a spherical fusion category with positive dagger structure. Recall from Section 2.4 that the Hom-spaces of $\mathcal{C}$ come with a Hermitian inner product $\langle \cdot, \cdot \rangle$. We can write

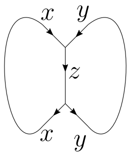
(B.1) 

We want to assign a factor of $\nu_0^{x^*x}$ to the above cup in order to normalise it with respect to the inner product i.e. $\nu_0^{x^*x} = d_x^{-\frac{1}{2}}$. Sphericality gives us $d_x = d_{x^*}$ and so $\nu_0^{x^*x} = \nu_0^{xx^*}$. Using the same notation as in (2.13), we have

(B.2) 

We want to scale the trivalent vertices appearing in (B.2) by a normalising factor $\nu_z^{xy} = \lambda_{zxyz}^{-1/2}$ in a manner that is consistent with the factor $\nu_0^{xx^*}$. Using the dagger structure, note that the normalisation factor for the adjoint trivalent vertex is also ν_z^{xy}. Expanding the identity operator for $\mathrm{End}(x \otimes y)$ in the canonical basis,



where we define $B_z^{xy} :=$ 

Proposition B.1.

(i) B_z^{xy} satisfies the same symmetries as N_z^{xy} for a fusion category i.e.

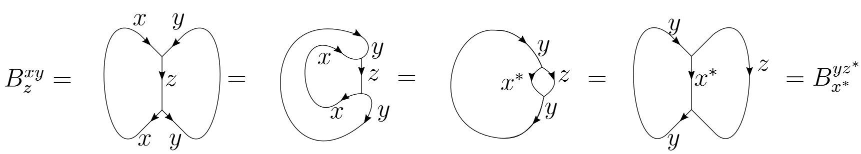
$$B_z^{xy} = B_{y^*}^{z^*x} = B_{x^*}^{yz^*} \quad , \quad B_z^{xy} = B_{z^*}^{y^*x^*}$$

(ii) $B_z^{xy} \in \mathbb{R}_{>0}$

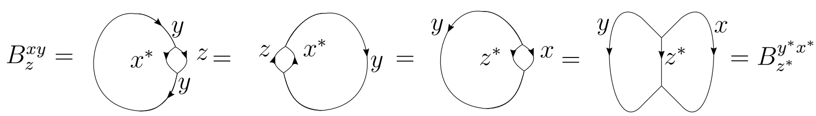
(iii) $B_z^{x0} = B_z^{0x} = \delta_{xz} d_x$

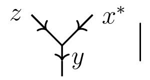
(iv) $B_z^{xy} = d_z (\nu_z^{xy})^{-2}$

Proof.

(i) 

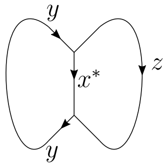
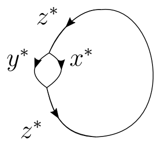
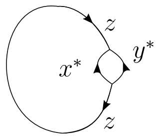
Also,



(ii) From the proof of (i), $B_z^{xy} = \Big|\Big|$  $\Big|\Big|^2 d_y = (\nu_y^{x^* z})^{-2} d_y > 0$

(iii) Follows immediately upon inspection of B_z^{xy}

(iv)

$B_z^{xy} \overset{(i)}{=} B_{x^*}^{yz^*} =$  $=$  $=$  $= (\nu_z^{xy})^{-2} d_z$

□

By (iv), we have that $(\nu_z^{xy})^2 = \dfrac{d_z}{B_z^{xy}}$ where B_z^{xy} must satisfy (i)-(iii) above. Note that

$$(\nu_0^{x^* x})^2 = \frac{d_0}{B_0^{x^* x}} = \frac{1}{B_x^{x0}} = \frac{1}{d_x}$$

which is consistent with our cap and cup normalisation. The simplest such candidate (satisfying (i)-(iii)) for B_z^{xy} is $\sqrt{d_x d_y d_z}$, whose corresponding normalisation is

$$\nu_z^{xy} = \sqrt[4]{\frac{d_z}{d_x d_y}} \tag{B.3}$$

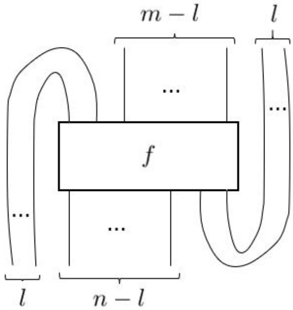
(and consequently $\lambda_{zxyz} = \sqrt{\frac{d_x d_y}{d_z}}$) which is the convention used in the literature.[13] It is easy to see that the F-symbols (2.32) and R-symbols (2.35) are invariant under the prescribed normalisation.

---

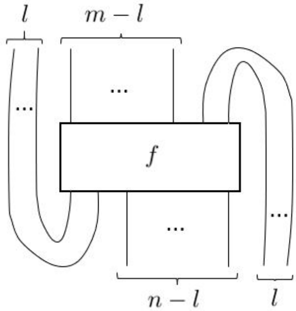
[13]For example, an alternative choice for B_z^{xy} could be $\frac{1}{3}(\sqrt{d_x d_y d_z} + d_x + d_y + d_z - 1)$.

Appendix C. Rotated Morphisms

The rotation operator φ as defined in Section 2 has a simple generalisation to $\mathrm{Hom}(q^{\otimes m}, q^{\otimes n})$ (where q is still assumed to be self-dual, and $m, n \in \mathbb{Z}_{>0}$). Let at least one of m or n be greater than one, and $f \in \mathrm{Hom}(q^{\otimes m}, q^{\otimes n})$. We have



,

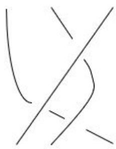


(C.1)

where the left and right diagrams respectively illustrate $\varphi(f)$ for an (anti)clockwise rotation and where $1 \leq l \leq \min\{m, n\}$. A further variant is studied in [23].

Example C.1. Let $m = n = 3$ and $l = 1$ with φ anticlockwise. Then $\varphi^6 = \mathrm{id}$. Let,

$f_0 =$  , $f_1 =$  , $f_2 =$ 

$f_3 =$  , $f_4 =$  , $f_5 =$ 

Observe that $\varphi(f_{\overline{k}}) = \varkappa_q f_{\overline{k+1}}$ where $\overline{k}$ denotes a residue modulo 6, and that $f_{\overline{k+3}} = f_{\overline{k}}$. Suppose there exist $\alpha, \beta \in \mathbb{C}$ such that $f_2 = \alpha f_0 + \beta f_1$. Applying φ, we get

$$f_0 = \alpha f_1 + \beta f_2 = \alpha f_1 + \beta(\alpha f_0 + \beta f_1)$$
$$\implies f_1 = \frac{1 - \alpha\beta}{\alpha + \beta^2} f_0$$

where in the final line we assume that $\beta^3 \neq -1$ and $\alpha \neq \beta^{-1}$. Thus, f_0, f_1 and f_2 are either (a) linearly independent, (b) linearly dependent with $f_2 = \alpha f_0 + \beta f_1$ such that $\beta^3 = -1$ and $\alpha = \beta^{-1}$ or (c) collinear. In the collinear case, note that $f_{\overline{k}}$ is an eigenvector of φ; coupling this with the fact that $\varphi^3(f_{\overline{k}}) = \varkappa_q f_{\overline{k}}$, we have that $\varphi(f_{\overline{k}}) = \omega f_{\overline{k}}$ where ω is a 3^{rd} root of $\varkappa_q$. It follows that $f_{\overline{k+1}} = \varkappa_q \omega f_{\overline{k}}$ (and so all of the morphisms are related to one another by a scaling of some 6^{th} root of unity).

Appendix D. Unitary Representations of the Braid Group

We now review part of the exposition in Section 3 from a slightly different perspective. While much of the discourse here is well-known, we feel that it would be amiss to exclude this material from our presentation. The n-strand braid group is given by

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \ , \ \ |i-j| \geq 2 \end{array} \right\rangle \tag{D.1}$$

whose graphical interpretation is given in Figure 4. We also have the symmetric group

$$S_n = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{c} s_i^2 = e \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \\ s_is_j = s_js_i \ , \ \ |i-j| \geq 2 \end{array} \right\rangle \tag{D.2}$$

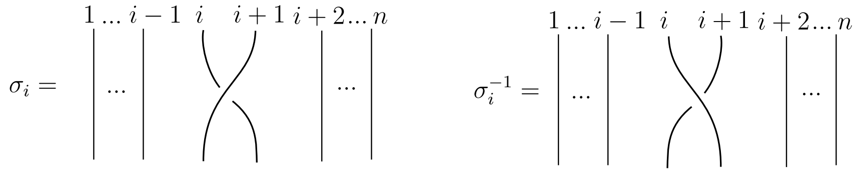


Figure 4. Braid words are read from right-to-left. Braids are drawn and composed from top-to-bottom in accord with our pessimistic convention.

There is an epimorphism $\psi : B_n \to S_n$ where $\psi(\sigma_i^{\pm 1}) = s_i$. The pure braid group PB_n is a normal subgroup of B_n given by $\ker \psi$. That is, ${B_n}/{PB_n} \cong S_n$. There is a closely related quotient for the algebra $\mathbb{C}[B_n]$; namely, we take the ideal $Q(\sigma_i)$ generated by $(\sigma_i - r_1)(\sigma_i - r_2)$ where $r_1, r_2 \in \mathbb{C}^\times$. Then

$$\mathbb{C}[B_n]/Q(\sigma_i) \cong H_n(r_1, r_2) \tag{D.3}$$

where $H_n(r_1, r_2)$ is called the *Iwahori-Hecke algebra*. Indeed, $H_n(\pm 1, \mp 1) \cong \mathbb{C}[S_n]$ (and so the Iwahori-Hecke algebra can be thought of as a *deformation* of $\mathbb{C}[S_n]$). Let $T_1, \ldots, T_{n-1}$ be the generators of the $H_n(r_1, r_2)$. The generators satisfy relations

$$(T_i - r_1)(T_i - r_2) = 0 \tag{D.4a}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \tag{D.4b}$$

$$T_iT_j = T_jT_i \ , \ \ |i-j| \geq 2 \tag{D.4c}$$

where (D.4a) is called the *Hecke relation*. Viewing the Iwahori-Hecke algebra as a vector space, we have $\dim_{\mathbb{C}}(H_n) = n!$. The *generalised Hecke algebra* $H_n(Q, k)$ is given by the quotient of $\mathbb{C}[B_n]$ by the ideal $Q(\sigma_i)$ which is now generated by $\Pi_{j=1}^k(\sigma_i - r_j)$ where $r_j \in \mathbb{C}^\times$ and $k \geq 2$. $H_n(Q, k)$ has the same presentation as the Iwahori-Hecke algebra except that (D.4a) is now replaced with the generalised Hecke relation $\Pi_{j=1}^k(T_i - r_j) = 0$.

Let $\mathcal{C}$ be a ribbon fusion category and take some $q \in \mathrm{Irr}(\mathcal{C})$. Then

$$\mathrm{End}\left(q^{\otimes n}\right) = \bigoplus_X \mathrm{Hom}\left(q^{\otimes n}, X\right) \otimes \mathrm{Hom}\left(X, q^{\otimes n}\right) \tag{D.5}$$

where X indexes all the simple objects appearing in the decomposition of $q^{\otimes n}$. Fixing a fusion basis on $\mathrm{Hom}\left(q^{\otimes n}, x\right)$ for some $X = x$ defines a linear representation

$$\rho : B_n \to U(V_x^{q^n}) \quad , \quad V_x^{q^n} := \mathrm{Hom}\left(q^{\otimes n}, x\right) \tag{D.6}$$

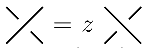
where $U(V_x^{q^n})$ denotes the group of unitary matrices on $V_x^{q^n}$. Let $n \geq 2$. There exists at least one i such that $\rho(\sigma_i) = \mathcal{R}$, where $\mathcal{R}$ is a diagonal matrix whose eigenvalues are some subset of the eigenvalues of R^{qq} (eigenvalues are counted without multiplicity here). Let $\{r_1, \ldots, r_k\}$ denote the eigenvalues of $\mathcal{R}$ where the $r_i \in U(1)$ are distinct and may appear in $\mathcal{R}$ with arbitrary nonzero multiplicity. We define

$$p(Z) = (Z - r_1 I_s) \cdot \ldots \cdot (Z - r_k I_s) \tag{D.7}$$

where $k \leq s := \dim(V_x^{q^n})$, I_s is the $s \times s$ identity matrix and Z is an $s \times s$ matrix with entries in $\mathbb{C}$. It is clear that $p(\mathcal{R}') = 0$ (where $\mathcal{R}'$ is any matrix similar to $\mathcal{R}$); this is an instance of the Cayley-Hamilton theorem. It follows that $p(\rho(\sigma_i)) = 0$ for all i, whence[14]

$$\rho : \mathbb{C}[B_n] \to H_n(Q, k) \to U(V_x^{q^n}) \tag{D.8}$$

i.e. ρ factors through the generalised Hecke algebra $H_n(Q, k)$.

In Section 3.1, we considered a fusion rule $q \otimes q = \mathbf{1}$. Clearly,  for some $z \in \mathbb{C}^\times$. Applying (3.4), we see that $z = \pm 1$ for $\varkappa_q = \pm 1$. For $\varkappa_q = +1$, (D.8) becomes

$$\begin{array}{rl} \rho^{(u)}: & \mathbb{C}[B_n] \to \mathbb{C}[S_n] \to U(1) \\ & \sigma_j \longmapsto s_j \longmapsto u \end{array} \quad , \quad u = \pm 1 \tag{D.9}$$

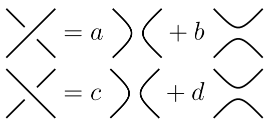
For $\varkappa_q = -1$, (D.8) becomes

$$\begin{array}{rl} \rho^{(u)}: & \mathbb{C}[B_n] \to \mathbb{C}[S_n] \to U(1) \\ & \sigma_j \longmapsto \pm i s_j \longmapsto u \end{array} \quad , \quad u = \pm i \tag{D.10}$$

In Section 3.2, we considered a fusion rule $q \otimes q = \mathbf{1} \oplus y$. This means that our crossings can be written

(D.11a)

(D.11b)



where $a, b, c, d \in \mathbb{C}^\times$. This motivates the idea that the homomorphism (D.8) should also factor through some algebra of cup-cap diagrams and non-intersecting strands (for $n \geq 3$); this is precisely the *Temperley-Lieb algebra* $TL_n(\delta)$: an associative $\mathscr{A}$-algebra (where $\mathscr{A}$ is a commutative ring) with generators $U_1, \ldots, U_{n-1}$ satisfying relations

$$U_i^2 = \delta U_i \quad , \ \delta \in \mathscr{A} \tag{D.12a}$$

$$U_i U_j U_i = U_i \quad , \ |i - j| = 1 \tag{D.12b}$$

$$U_i U_j = U_j U_i \ , \ |i - j| \geq 2 \tag{D.12c}$$

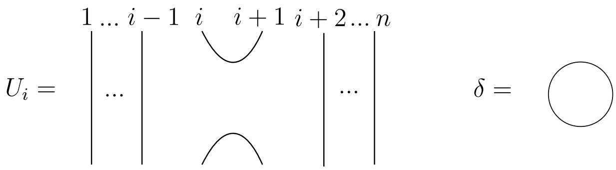


FIGURE 5. Diagrams run from top-to-bottom. The identity element is given by n vertical strands.

[14] In (D.8), ρ is a $\mathbb{C}$-linear extension of ρ in (D.6). Through an abuse of notation, we implicitly assume that the representation in (D.8) is restricted to B_n so as to coincide with (D.6).

$TL_n(\delta)$ is a free $\mathscr{A}$-module of rank C_n where C_n denotes the n^{th} Catalan number. Following (D.11a)-(D.11b), we construct a $\mathbb{C}$-linear map

$$\begin{aligned} \zeta : \mathbb{C}[B_n] &\to TL_n(\delta) \\ \sigma_i &\mapsto a + bU_i \quad , \quad \mathscr{A} = \mathbb{C}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}] \\ \sigma_i^{-1} &\mapsto c + dU_i \end{aligned} \tag{D.13}$$

Proposition D.1. ζ defines an algebra homomorphism if and only if $c = a^{-1}, d = b^{-1}$ and $\delta = -(ab^{-1} + a^{-1}b)$.

We henceforth assume that c, d and δ are as in Proposition D.1. Since our representation

$$\rho : \mathbb{C}[B_n] \xrightarrow{\zeta} TL_n(\delta) \to U(V_x^{q^n}) \tag{D.14}$$

is unitary, the conditions in Proposition D.2 (adapted from [26, p.237]) must hold.

Proposition D.2. Given ρ as in (D.14), we have $U_i^\dagger = U_i$ and $|a| = |b| = 1$.

From (D.8) we know that ρ must also factor through $H_n(Q, k)$. Since we are considering a fusion rule of the form $q^{\otimes 2} = y \oplus z$, we have that

$$\rho : \mathbb{C}[B_n] \to H_n(r_1, r_2) \to U(V_x^{q^n}) \tag{D.15}$$

It is easy to check that (D.14) is compatible with (D.15). Following Proposition D.2,

$$\begin{aligned} U_i^\dagger = U_i \iff & \left[b^{-1}\left(\rho(\sigma_i) - a\right)\right]^\dagger = b^{-1}\left(\rho(\sigma_i) - a\right) \\ \iff & b\left(\rho(\sigma_i)\right)^\dagger - a^* b = b^* \rho(\sigma_i) - b^* a \\ \iff & \left(\rho(\sigma_i) + a^* b^2\right)\left(\rho(\sigma_i) - a\right) = 0 \end{aligned}$$

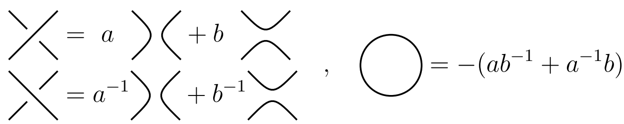
whence

$$\rho : \mathbb{C}[B_n] \to H_n(-a^* b^2, a) \to TL_n(\delta) \to U(V_x^{q^n}) \tag{D.16}$$

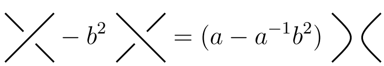
Specifically, we have the following commutative diagram of linear homomorphisms:

$$\begin{tikzcd} \mathbb{C}[B_n] \arrow[r, "\phi"] \arrow[d, "\zeta"] & H_n(-a^{-1}b^2, a) \arrow[dl, "\eta"] \\ TL_n(\delta) \arrow[d, "\rho'"] & \\ U(V_x^{q^n}) & \end{tikzcd} \qquad \text{with} \qquad \begin{aligned} & a, b \in U(1) \\ & \delta = -(ab^{-1} + a^{-1}b) \\ & \rho = \rho' \circ \zeta \end{aligned}$$

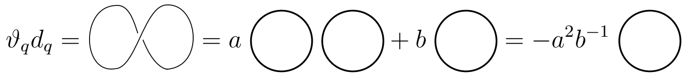
where $\phi(\sigma_i) = T_i$ such that $\ker \phi$ is generated by $(\sigma_i + a^{-1}b^2)(\sigma_i - a)$, and $\eta(T_i) = a + bU_i$. We may thus resolve crossings using skein relations

(D.17) 

Note that the resolution of the crossings in (D.17) implies

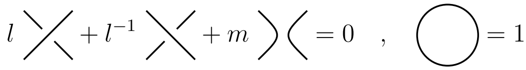
(D.18) 

which is simply the Hecke (skein) relation for $H_n(-a^{-1}b^2, a)$. To recover the boxed result in Section 3.2, we consider relations (D.17) in the setting of a ribbon category. Note that

(D.19)


whence $\vartheta_q = -a^2b^{-1}$. But we also have that $R_0^{qq} = a^{-1}b^2$ whence (2.47) tells us that $b = \pm a^{-1}$ for $\varkappa_q = \pm 1$. Another way of seeing this is by applying (3.4) to (D.17).

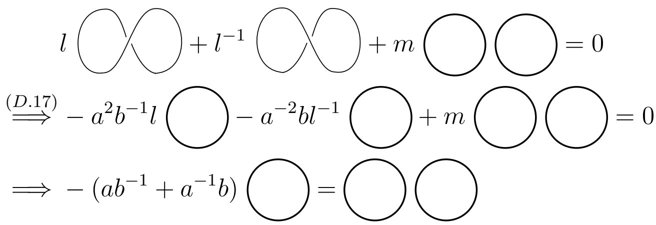
Remark D.3. The skein relations (D.17) correspond to the *framed* HOMFLY-PT polynomial. In order to see this, consider the well-known Lickorish-Millet presentation [27] of the HOMFLY-PT skein relations,

(D.20)


Then setting

$$l = \pm i b^{-1} \quad , \quad m = \mp i(ab^{-1} - a^{-1}b) \tag{D.21}$$

recovers (D.18). Finally, with l and m as in (D.21),

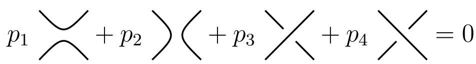


whence we have the rescaled loop value  $= -(ab^{-1} + a^{-1}b)$. Let $\tilde{H}$ denote the framed HOMFLY-PT polynomial, L be a link and D a corresponding link diagram. Then the (unframed) HOMFLY-PT polynomial H is simply

$$H(L) = (-a^{-2}b)^{w(D)}\tilde{H}(D) \tag{D.22}$$

where w denotes the writhe. The HOMFLY-PT invariant can be derived by applying a normalised Markov trace to the Iwahori-Hecke algebra; this trace is characterised by its action on the basis elements of the HOMFLY-PT skein algebra of the annulus[15] [24, 25].

We omit an analogous discussion for the fusion rule $q \otimes q = \mathbf{1} \oplus x \oplus y$, and solely remark that since there must exist p_1, p_2, p_3, p_4 nonzero such that

(D.23)


this indicates that the representation should factor through the Temperley-Lieb algebra, which in turn motivates the construction of a linear map

$$\begin{aligned} \zeta' : \mathbb{C}[B_n] &\to TL_n(\delta) \\ \sigma_i + a\sigma_i^{-1} &\mapsto b + cU_i \end{aligned} \tag{D.24}$$

where $a, b, c \in \mathbb{C}^\times$ and $\delta \in \mathbb{C}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ are such that ζ' defines a homomorphism. It is also clear that the representation should factor through $H_n(Q, 3)$.

[15] Given $H_n(r_1, r_2)$, the HOMFLY-PT skein algebra $\mathcal{H}_n(r_1, r_2)$ is obtained by joining the ends of the strands (where the i^{th} top is respectively connected to the i^{th} bottom). We have $\mathcal{H}_n \cong {H_n}/{[\cdot,\cdot]}$ (i.e. the quotient by the ideal generated by the commutator) and $\dim_{\mathbb{C}}(\mathcal{H}_n)$ is given by the n^{th} partition number.

Appendix E. Invariants coming from $\varkappa_q = -1$

Given a skein relation associated to the fusion rule (3.1) for $\varkappa_q = -1$, we can define a polynomial-valued function $\Lambda_{\mathcal{C},q}$ that acts on any link diagram D. Since invariants $\Lambda_{\mathcal{C},q}$ coming from $\varkappa_q = -1$ are derived from a setting where an isotopy of the form $\xrightarrow{\varphi} \quad = -$ introduces a difference in sign, it is natural to ask whether the invariant $\Lambda_{\mathcal{C},q}$ carries such a sensitivity.

Problem E.1. Let D and D' be link diagrams that are equivalent under framed isotopy, and let $k := \frac{1}{2}|\mathcal{W}(D) - \mathcal{W}(D')|$ where $\mathcal{W}$ denotes the *local writhe*[16]. Is it always true that

$$\Lambda_{\mathcal{C},q}(D') = (-1)^k \Lambda_{\mathcal{C},q}(D) \tag{E.1}$$

given $\Lambda_{\mathcal{C},q}$ for some q with $\varkappa_q = -1$?

E.1. **Understanding the invariant (3.15).** The skein relation (3.15) must be applied *locally* i.e. the link must be without orientation, and the relation should be applied to crossings precisely as they appear.

For instance, while a twist of the form would be resolved as a positive crossing if it were oriented, we only take into account the local form of the crossing (which is negative here). Letting $[\cdot]_\beta$ denote the invariant (3.15),

$$\left[\quad\right]_\beta = \beta^3 \quad , \quad \left[\quad\right]_\beta = -\beta^3 \tag{E.2}$$

whence we see that the categorical distinction between horizontal and vertical twists (Remark 3.2) carries over to the invariant (3.15). Of course, this observation is subsumed by the following:

Proposition E.2. The answer to Problem E.1 is positive when $q^{\otimes 2} = \mathbf{1} \oplus x$.

Proof. In this case, we know that $\Lambda := \Lambda_{\mathcal{C},q}$ is given by (3.15). In the following, we assume that all diagrams are projected onto the plane. It suffices to consider an isotopy $D \to \check{D}$ which is one of: (i) a Reidemeister-0 move (i.e. an ambient isotopy) such that only one crossing has its sign flipped under the deformation (so $k = 1$); (ii) a Reidemeister-II move; (iii) a Reidemeister-III move.

(i) Given a link diagram D containing some crossing , we let D_0 and D_∞ respectively denote the same diagram but with the crossing smoothed to and . We consider $\Lambda(D)$ and $\Lambda(\check{D})$, first applying the skein relation to the crossing whose sign was flipped: suppose (without loss of generality) that the crossing in D was . Then

$$\Lambda(D) = \beta \cdot \Lambda(D_0) - \beta^{-1} \cdot \Lambda(D_\infty)$$
$$\Lambda(\check{D}) = \beta^{-1} \cdot \Lambda(\check{D}_0) - \beta \cdot \Lambda(\check{D}_\infty)$$

Label the boundary points of the crossing in D as $\underset{4\quad 3}{\overset{1\quad 2}{}}$, and so in $\check{D}$ said

[16] The local writhe $\mathcal{W}$ of a link diagram D is defined as the sum over the signs of all crossings in D, where and respectively contribute $+1$ and -1.

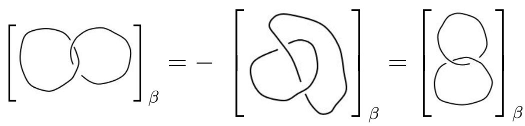
crossing is either  or  . Then it is easy to see that the smoothings $D \to D_0$ and $\check{D} \to \check{D}_\infty$ are locally identical, whence the diagrams D_0 and $\check{D}_\infty$ are also equivalent under ambient isotopy. It follows that $\Lambda(D_0) = \Lambda(\check{D}_\infty)$. Similarly, $\Lambda(\check{D}_0) = \Lambda(D_\infty)$. Thus, $\Lambda(\check{D}) = -\Lambda(D)$.

(ii) Two crossings have their signs flipped under a Reidemeister-II move (so $k = 2$). Thus, (E.1) respects the invariance of Λ under Reidemeister-II i.e. $\Lambda(\check{D}) = \Lambda(D)$.

(iii) Either zero or two crossings have their signs flipped under a Reidemeister-III move (so $k = 0$ or 2). It follows that (E.1) respects the invariance of Λ under Reidemeister-III i.e. $\Lambda(\check{D}) = \Lambda(D)$.

Since $D \to D'$ in Problem E.1 is a composition of moves (i)-(iii), the result follows. $\square$

Thus, $[D]_\beta$ is invariant under framed isotopy *up to a sign* which depends on $\mathcal{W}(D)$ e.g.

(E.3) 

From Figure 6, we see that (E.3) is equal to $\beta^2 d^2 - d - d + \beta^{-2} d^2$ (where $d := \beta^2 + \beta^{-2}$ is the value of the loop).

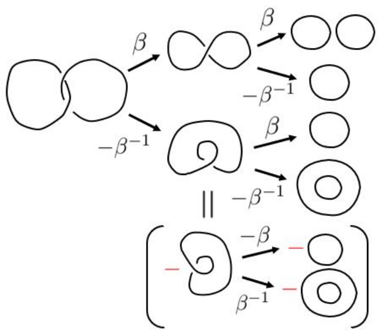


FIGURE 6. When we apply the skein relation (3.15), we must take care to account for any minus signs accumulated if we choose to rotate crossings mid-evaluation e.g. if the parenthesised route is taken above (for which the resulting minus signs are highlighted in red).

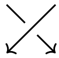
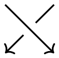
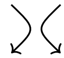
In summary, we have seen that while (3.15) is invariant under Reidemeister-II and III moves, it retains sensitivity to pivotality at some level: when a link diagram is isotoped in a way that locally rotates crossings, this introduces "internal zig-zags" that are detected by the invariant.

Remark E.3.

(i) In light of the above, referring to invariants $\Lambda_{\mathcal{C},q}$ associated to q antisymmetrically self-dual as "framed link invariants" could be considered an abuse of terminology. We therefore propose the term *pivotal framed invariant* for (3.15). Similarly, if some invariant associated to $\varkappa_q = -1$ is sensitive to the local writhe in the same way as (3.15), then it should also be identified as a pivotal framed invariant.[17] A pivotal framed invariant $\Lambda_{\mathcal{C},q}$ may be normalised as in (3.2), thus removing its sensitivity to framing. Nonetheless, the pivotal discrimination remains: we therefore propose that the resulting invariant should similarly be called an (oriented) *pivotal invariant*.

(ii) Although the topological motivation for studying (3.15) and other $\Lambda_{\mathcal{C},q}$ for $\varkappa_q = -1$ may be unclear, we suggest the following counterpoint: in the categorical setting, these invariants arise naturally; they give the evaluation of a link diagram (in $\mathrm{End}(\mathbf{1})$ whose components are all labelled by q) up to a possible factor of -1 coming from zig-zag morphisms. Whether these invariants have more interesting applications outside the categorical context remains to be seen.

(iii) One can obtain the skein relation

(E.4) β  $-\ \beta^{-1}$  $= (\beta^2 - \beta^{-2})$ 

from (3.15), noting that we have further endowed it with pessimistic orientation. This skein relation defines a framed oriented link invariant. Furthermore, the resulting polynomial of a framed oriented link L coincides with the Kauffman bracket polynomial of the framed link $\tilde{L}$ (which is L without orientation). In order to see this, take an oriented link diagram D for L. We can isotope D to obtain a diagram D' whose crossings locally appear with pessimistic orientation. Now observe that applying the oriented skein relation (E.4) to D' is the same as applying the skein relation (D.18) (with $b = a^{-1}$ and $a = \beta$) to the link diagram $\tilde{D}$ (which is the diagram D' without orientation).

[17] E.g. if the answer to Problem E.1 is positive, then we propose that all invariants associated to $\varkappa_q = -1$ should be termed as such.

5. Some Physical Remarks and Examples

Throughout this chapter, we will assume that $\mathcal{C}$ is a unitary ribbon fusion category (URFC), and that any trivalent vertices implicitly carry the normalisation described in Section 2.1.8. In Section 5.1, we will use the graphical calculus to look at some examples of entangling operators. In Section 5.2, we provide some explicit examples of calculations for the evaluation of link diagrams in $\mathrm{End}(\mathbf{1})$: this can be seen as supplementary to the narrative of Chapter 4. The language used in this chapter is a mixture of that found in Part I and Chapter 4, so we will begin by summarising some of the jargon.

In what follows, we will also use the notation

$$V^{a_1 \dots a_m}_{b_1 \dots b_n} := \mathrm{Hom}(\bigotimes_{i=1}^{m} a_i, \bigotimes_{j=1}^{n} b_j) \tag{5.0.1}$$

Remark 5.1. If $q \in \mathrm{Irr}(\mathcal{C})$ is a self-dual abelian charge, then we know from Chapter 4: Section 3.1 (or Appendix D), that q is always one of the following:

(i) q is called a *boson* if $\vartheta_q = +1$

(ii) q is called a *fermion*[3] if $\vartheta_q = -1$

(iii) q is called a *semion* if $\vartheta_q = \pm i$

where in the above, $\vartheta_q = R^{qq}_0$. Semions q always have $\varkappa_q = -1$.

[1] For a unitary spherical fusion category, the Frobenius-Perron dimension coincides with the quantum dimension. If we relax unitarity, these two quantities may differ by a sign. By Proposition 2.22, $\mathrm{FPdim}(x) \in \{1\} \cup [\sqrt{2}, \infty)$ for a unitary spherical fusion category.

[2] And similarly, 'splitting states' for $\mathrm{Hom}(x, \bigotimes_{i=1}^{n} x_i)$.

[3] This nomenclature allows for non-transparent fermions (e.g. the ψ-particle in an Ising theory).

URFC $\mathcal{C}$	**Anyonic system**
Simple objects	Anyons/quasiparticles
Dual object	Antiparticle
Trivial object/$\mathbf{1}$	Vacuum
Label $x \in \mathrm{Irr}(\mathcal{C})$	Topological charge/anyon 'type' $x \in \mathfrak{L}$
Frobenius-Perron dimension[1] $\mathrm{FPdim}(x)$	Quantum dimension d_x
$\mathrm{FPdim}(x) = 1$	x is an abelian anyon
$\mathrm{FPdim}(x) > 1$	x is a nonabelian anyon
$\mathrm{Hom}(\bigotimes_{i=1}^n x_i, x)$, $\mathrm{Hom}(x, \bigotimes_{i=1}^n x_i)$	Fusion/splitting Hilbert space
Morphisms	Physical processes/operators/worldlines
Nonzero (normalised) $\psi \in \mathrm{Hom}(\bigotimes_{i=1}^n x_i, x)$	Fusion state[2]
Parenthesisation of $x_1 \otimes \cdots \otimes x_n$	Choice of fusion basis/order
Associator/F-matrix	Change of fusion basis
Braiding/R-matrix	Particle exchange
Evaluation/coevaluation	Annihilation/creation
Twist factor ϑ_x	Topological spin of x

Table 5.1: Dictionary of terms adapted from [Wang, Table 6.1]

5.1 Entanglement

If the total (topological) charge of a collection of adjacent quasiparticles is fixed (Figure 5.1), this charge is called the *superselection sector* of the charges; else, the whole collection of quasiparticles exist in a superposition of total charges (i.e. a superposition of fusion states).

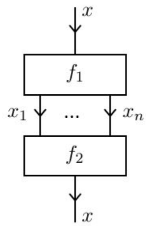


Figure 5.1: $f_1 \in V^{x}_{x_1\dots x_n}$ and $f_2 \in V^{x_1\dots x_n}_{x'}$. The system of particles $x_1, \dots, x_n$ lies in superselection sector $x' = x$ (Schur's lemma). This is conservation of (topological) charge.

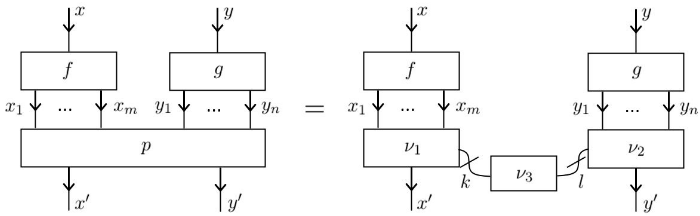
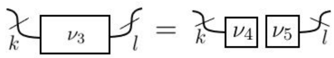


Figure 5.2: $f \in V^{x}_{x_1\dots x_m}$, $g \in V^{y}_{y_1\dots y_n}$ and $p \in V^{x_1\dots x_m y_1\dots y_n}_{x'y'}$. $\not\uparrow_k$ denotes k lines carrying some permissible labels and orientations. ν_j is a (linear combination of) permissible diagram(s) of oriented braided (possibly twisted) strands, and possibly trivalent vertices.

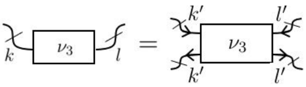
Now take two adjacent systems of quasiparticles $x_1, \dots, x_m$ and $y_1, \dots, y_n$ that are *initialised* in superselection sectors x and y respectively (Figure 5.2). Consider the following cases (which are clearly not exhaustive).

(1) $k = l = 0$. Then $\nu_1 \in V^{x_1\dots x_m}_{x}$, $\nu_2 \in V^{y_1\dots y_n}_{y}$ and $\nu_3 \in \mathbb{C}$.

(2) Only one of k and l is nonzero. Then if k is zero, we have $\nu_1 \in V^{x_1\dots x_m}_{x}$; and if l is zero, we have $\nu_2 \in V^{y_1\dots y_n}_{y}$.

(3)  where k and l are nonzero.

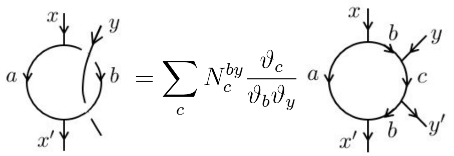
In the above cases, there is no interaction (or trivial interaction) between the two systems, and they respectively remain in superselection sectors x and y.

(4)  where k and l' are nonzero, ν_3 is a $(k'+l')$-braid such that it cannot be partitioned as in (3); and the incoming and outgoing strands on a given side are connected. There are two possibilities:

(a) There is only one permissible label for each of x' and y'.[4] Then the interaction will have caused the systems to respectively transition into superselection sectors x' and y' post-braiding.

(b) Else, ν_3 is an example of an *entangling operator* since we cannot write $p \in V_{x'y'}^{x_1 \dots x_m y_1 \dots y_n}$ as $p = f' \otimes g'$ for some $f' \in V_{x'}^{x_1 \dots x_m}$ and $g' \in V_{y'}^{y_1 \dots y_n}$. Here, *'tangling' between systems results in the entanglement of their fusion states.*

For instance,



(5.1.1)

This shows that the left-hand side is not an inner product for y nontrivial, and so x' is not necessarily x (i.e. it is possible that the total charge of a and b is no longer be constrained to a single possibility, in which case a and b are entangled with y).

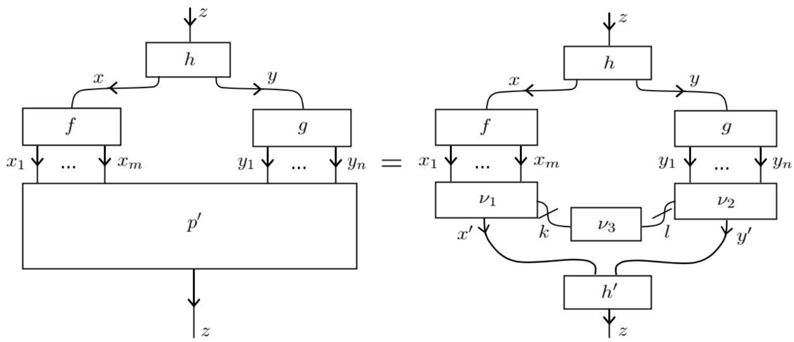


Figure 5.3: Extension of Figure 5.2. $h \in V_{xy}^{z}$, $h' \in V_{z}^{x'y'}$ and $p' \in V_{z}^{x_1 \dots x_m y_1 \dots y_n}$.

If ν_3 is as in cases (1)-(4a) for Figure 5.3, then we may write $p' = f' \otimes h' \otimes g'$ for some $f' \in V_{x'}^{x_1 \dots x_m}$ and $g' \in V_{y'}^{y_1 \dots y_n}$. That is, p' has a nonzero component in precisely one

[4] E.g. this could happen if $m = n = 1$; or if each system is composed of abelian anyons; or if $V_{z}^{x_1 \dots x_m y_1 \dots y_n}$ is 1-dimensional in Figure 5.3. In all these examples, $(x', y') = (x, y)$. However, it is also possible to have examples where $(x', y') \neq (x, y)$ e.g. Example 5.2.

summand of (5.1.2). Note that we are guaranteed $(x', y') = (x, y)$ in cases (1)-(3).

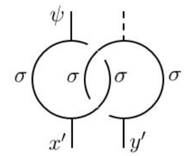
$$V_z^{x_1 \dots x_m y_1 \dots y_n} \cong \bigoplus_{x', y'} V_{x'}^{x_1 \dots x_m} \otimes V_z^{x'y'} \otimes V_{y'}^{y_1 \dots y_n} \tag{5.1.2}$$

If ν_3 is as in case (4b), then the two systems no longer belong to local superselection sectors post-braiding: they can only be said lie in the global superselection sector z. In particular, p' has nonzero components in more than one summand of (5.1.2), and thus corresponds to an entangled fusion state with respect to the two systems. Let us consider some concrete examples.

Example 5.2. Take the Ising theory (2.6.2) with

$$F := F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad , \quad R := R^{\sigma\sigma} = e^{-i\frac{\pi}{8}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{5.1.3}$$

where $\varkappa_\sigma = 1$ and the matrices act on a basis in the order $\mathbf{1}, \psi$. Take the process



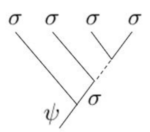
(5.1.4)

and consider the space $V_\psi^{\sigma\sigma\sigma\sigma}$ with the fusion basis fixed as follows:

$$V_\psi^{\sigma\sigma\sigma\sigma} \cong \bigoplus_{x', y'} V_{x'}^{\sigma\sigma} \otimes V_\psi^{x'y'} \otimes V_{y'}^{\sigma\sigma} = \mathrm{span}_{\mathbb{C}}\{|x'y'\rangle\}_{x', y' : N_{x'}^{\sigma\sigma} N_\psi^{x'y'} N_{y'}^{\sigma\sigma} \neq 0} \tag{5.1.5}$$

Note that (x', y') takes values $(\mathbf{1}, \psi)$ or $(\psi, \mathbf{1})$. From (5.1.4), the system is initialised in state $|\psi\mathbf{1}\rangle$. Performing a trivial change of basis,

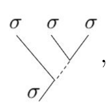
$$\left(F_\psi^{\sigma\sigma\mathbf{1}} \otimes \mathrm{id}_{V_{\mathbf{1}}^{\sigma\sigma}}\right) |\psi\mathbf{1}\rangle =$$



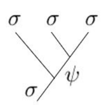
$$\in V_\psi^{\sigma\sigma} \otimes V_\sigma^{\sigma\sigma\sigma}$$

It suffices to consider $V_\sigma^{\sigma\sigma\sigma}$ to compute the action of the braiding. We have

$$V_\sigma^{\sigma\sigma\sigma} = \mathrm{span}_{\mathbb{C}} \Bigg\{$$



,

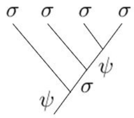


$$\Bigg\} =: \mathrm{span}_{\mathbb{C}} \{|\mathbf{1}\rangle, |\psi\rangle\}$$

Write $\begin{pmatrix} a \\ b \end{pmatrix} = a\,|\mathbf{1}\rangle + b\,|\psi\rangle$. The action of the braiding is given by

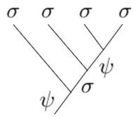
$$FR^2F^{-1}\begin{pmatrix}1\\0\end{pmatrix} = e^{-\frac{i\pi}{4}}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = e^{-\frac{i\pi}{4}}\,|\psi\rangle$$

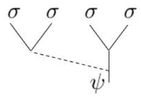
Post-braiding, we thus have



$\in V_{\psi}^{\sigma\sigma} \otimes V_{\sigma}^{\sigma\sigma\sigma}$

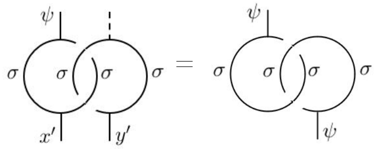
Switching back to the original basis in (5.1.4),



$\xmapsto{\left(F_{\psi}^{\sigma\sigma\psi}\right)^{-1} \otimes \mathrm{id}_{V_{\psi}^{\sigma\sigma}}} \left[F_{\psi}^{\sigma\sigma\psi}\right]^{-1}$


$= \left[F_{\psi}^{\sigma\sigma\psi}\right]^{-1} |\mathbf{1}\psi\rangle$

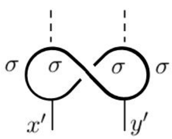
whence (5.1.4) is



(5.1.6)

Thus, the monodromy of the middle pair of Ising anyons results in the 'teleportation' of the fermion from the left pair to the right pair. This is an example of case (4a).

Example 5.3. (Bell state via Ising anyons)
We fix the same Ising theory with F and R as in Example 5.2. Take the process

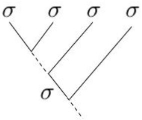


(5.1.7)

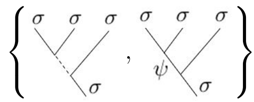
and consider the space $V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma}$ with the fusion basis fixed as follows:

$$V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma} \cong \bigoplus_{x',y'} V_{x'}^{\sigma\sigma} \otimes V_{\mathbf{1}}^{x'y'} \otimes V_{y'}^{\sigma\sigma} = \mathrm{span}_{\mathbb{C}}\{|x'y'\rangle\}_{x',y':N_{x'}^{\sigma\sigma}N_{\mathbf{1}N_{y'}^{\sigma\sigma}}^{x'y'}\neq 0} \tag{5.1.8}$$

Note that (x', y') takes values $(\mathbf{1}, \mathbf{1})$ or (ψ, ψ). From (5.1.7), the system is initialised in state $|\mathbf{11}\rangle$. Performing a trivial change of basis,

$$\left(\mathrm{id}_{V_{\mathbf{1}}^{\sigma\sigma}} \otimes \left[F_{\mathbf{1}}^{\mathbf{1}\sigma\sigma}\right]^{-1}\right)|\mathbf{11}\rangle =$$  $$\in V_{\sigma}^{\sigma\sigma\sigma} \otimes V_{\mathbf{1}}^{\sigma\sigma}$$

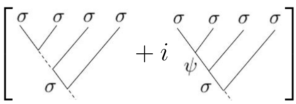
It suffices to consider $V_{\sigma}^{\sigma\sigma\sigma}$ to compute the action of the braiding. We have

$$V_{\sigma}^{\sigma\sigma\sigma} = \mathrm{span}_{\mathbb{C}}$$  $$=: \mathrm{span}_{\mathbb{C}}\left\{|\mathbf{1}\rangle, |\psi\rangle\right\}$$

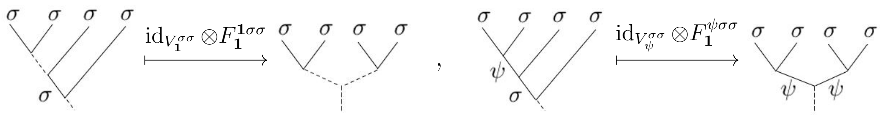
Write $\begin{pmatrix} a \\ b \end{pmatrix} = a\,|\mathbf{1}\rangle + b\,|\psi\rangle$. The action of the braiding is given by

$$F^{-1}R^{-1}F\begin{pmatrix}1\\0\end{pmatrix} = \frac{e^{\frac{i\pi}{8}}}{2}\begin{pmatrix}1-i & 1+i\\ 1+i & 1-i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{e^{\frac{i\pi}{8}}}{\sqrt{2}}\begin{pmatrix}e^{-\frac{i\pi}{4}}\\ e^{\frac{i\pi}{4}}\end{pmatrix}$$

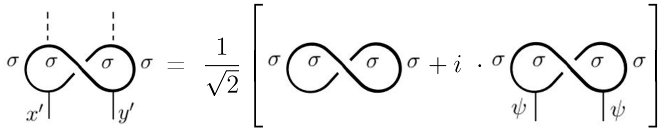
Post-braiding, we thus have

$$\frac{1}{\sqrt{2}}$$  $$\in V_{\sigma}^{\sigma\sigma\sigma} \otimes V_{\mathbf{1}}^{\sigma\sigma}$$

Switching back to the original basis in (5.1.7),



where $F_{\mathbf{1}}^{\mathbf{1}\sigma\sigma} = F_{\mathbf{1}}^{\psi\sigma\sigma} = [1]$. Note that the change of basis on the left is trivial. It follows that (5.1.7) is



(5.1.9)

where the fusion state is

$$\frac{1}{\sqrt{2}}(|\mathbf{11}\rangle + i\,|\psi\psi\rangle) \in V_{\mathbf{1}}^{\sigma\sigma\sigma\sigma} \tag{5.1.10}$$

This is a *Bell state*: the fusion states of the left and right pairs of Ising anyons are *maximally entangled.* Thus, the anticlockwise exchange of the middle pair of anyons realises an example of an entangling operator. We can concretely interpret this exchange as an entangling quantum gate:

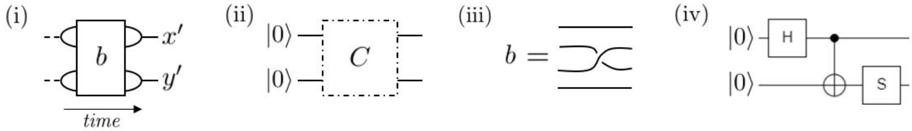


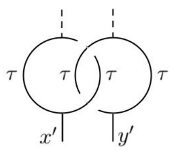
Figure 5.4: (i) b is a 4-braid whose strands are labelled by Ising anyons. Letting $|\mathbf{1}\rangle$ and $|\psi\rangle$ correspond to logical basis states $|0\rangle$ and $|1\rangle$ respectively, (i) can be interpreted as a quantum circuit as in (ii). (ii) is a 2-qubit quantum circuit initialised in state $|0\rangle \otimes |0\rangle$, and with C representing some configuration of quantum gates corresponding to the action of b. (iii) $b = \sigma_2^{-1}$. (iv) The quantum circuit realised by $b = \sigma_2^{-1}$. If the orientation of the exchange is reversed in (iii), we simply append a Pauli-Z gate to the bottom wire in (iv).

This is an example of a process of the form in Figure 5.2 that is not described by cases (1)-(4).

Example 5.4. Take the Fibonacci theory (2.6.1) with

$$F := F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \quad , \quad R := R^{\tau\tau} = \begin{pmatrix} e^{-i\frac{4\pi}{5}} & 0 \\ 0 & e^{i\frac{3\pi}{5}} \end{pmatrix} \tag{5.1.11}$$

where $\varkappa_\sigma = 1$ and the matrices act on a basis in the order $\mathbf{1}, \tau$. Take the process

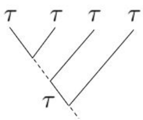


(5.1.12)

and consider the space $V_{\mathbf{1}}^{\tau\tau\tau\tau}$ with the fusion basis fixed as follows:

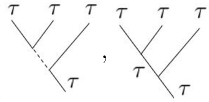
$$V_{\mathbf{1}}^{\tau\tau\tau\tau} \cong \bigoplus_{x',y'} V_{x'}^{\tau\tau} \otimes V_{\mathbf{1}}^{x'y'} \otimes V_{y'}^{\tau\tau} = \mathrm{span}_{\mathbb{C}}\{|x'y'\rangle\}_{x',y':N_{x'}^{\tau\tau}N_{\mathbf{1}}^{x'y'}N_{y'}^{\tau\tau}\neq 0} \tag{5.1.13}$$

Note that (x', y') takes values $(\mathbf{1}, \mathbf{1})$ or (τ, τ). From (5.1.12), the system is initialised in state $|\mathbf{11}\rangle$. Performing a trivial change of basis,

$$\left(\mathrm{id}_{V_{\mathbf{1}}^{\tau\tau}} \otimes \left[F_\tau^{\mathbf{1}\tau\tau}\right]^{-1}\right) |\mathbf{11}\rangle =$$  $$\in V_\tau^{\tau\tau\tau} \otimes V_{\mathbf{1}}^{\tau\tau}$$

It suffices to consider $V_\tau^{\tau\tau\tau}$ to compute the action of the braiding. We have

$$V_\tau^{\tau\tau\tau} = \mathrm{span}_{\mathbb{C}} \left\{ \quad , \quad \right\} =: \mathrm{span}_{\mathbb{C}} \{ |\mathbf{1}\rangle, |\tau\rangle \}$$

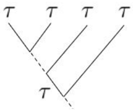


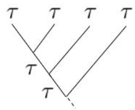
Write $\begin{pmatrix} a \\ b \end{pmatrix} = a\,|\mathbf{1}\rangle + b\,|\psi\rangle$. The action of the braiding is given by

$$F^{-1}R^2F\begin{pmatrix}1\\0\end{pmatrix} = e^{-\frac{i8\pi}{5}}\begin{pmatrix}\phi^{-2}+\phi^{-1}e^{i\frac{4\pi}{5}} & \phi^{-3/2}(1-e^{i\frac{4\pi}{5}}) \\ \phi^{-3/2}(1-e^{i\frac{4\pi}{5}}) & \phi^{-1}+\phi^{-2}e^{i\frac{4\pi}{5}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-\phi^{-2} \\ \phi^{-3/2}\left(e^{-i\frac{8\pi}{5}}-e^{-i\frac{4\pi}{5}}\right)\end{pmatrix}$$

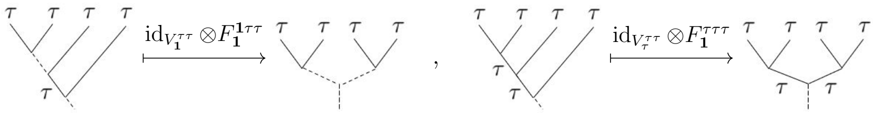
Post-braiding, we thus have

$$\phi^{-2} \quad + \phi^{-3/2}\left(e^{-i\frac{4\pi}{5}} - e^{i\frac{2\pi}{5}}\right) \quad \in V_\tau^{\tau\tau\tau}\otimes V_{\mathbf{1}}^{\tau\tau}$$

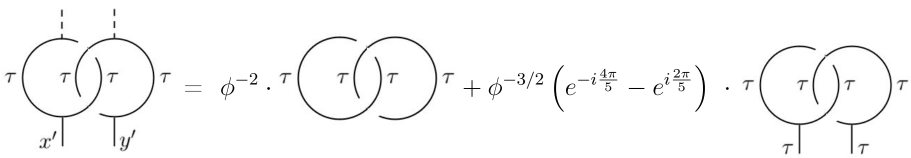




Switching back to the original basis in (5.1.12),



where $F_{\mathbf{1}}^{\mathbf{1}\tau\tau} = F_{\mathbf{1}}^{\tau\tau\tau} = [1]$. Note that the change of basis on the left is trivial. It follows that (5.1.12) is



(5.1.14)

where the fusion state is

$$\phi^{-2}\,|\mathbf{11}\rangle + \phi^{-3/2}\left(e^{-i\frac{4\pi}{5}} - e^{i\frac{2\pi}{5}}\right)|\tau\tau\rangle \in V_{\mathbf{1}}^{\tau\tau\tau\tau} \tag{5.1.15}$$

Thus, the monodromy of the middle pair of Fibonacci anyons entangles the fusion states of the left and right pair. This is an example of case (4b), where 'tangling' two systems results in their entanglement.

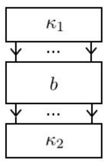
5.2 Evaluation of Link Diagrams

In the physical context, link diagrams in End($\mathbf{1}$) correspond to a process where anyons are pair-created from the vacuum, braided, and then fused straight back to the vacuum. Given such a diagram D, its amplitude $\langle W(D)\rangle$ may be evaluated via skeletal data for $\mathcal{C}$, or by using a skein-theoretic method.[5] In order for $\langle W(D)\rangle$ to be a physically meaningful quantity, it must be gauge-invariant. We will begin by discussing these two approaches, and then we will compute some concrete examples (where in each example, we evaluate the link diagram using both methods).

Procedure 5.5 ($\langle W(D)\rangle$ **via skeleton of** $\mathcal{C}$**).**

(1) Isotope D (using only Reidemeister-II and III moves)[6] so that we obtain some bridge representation D'. That is,

$$D \xrightarrow{\text{isotopy}} D' = \tag{5.2.1}$$



where b is a $2n$-braid, and κ_1 & κ_2 are some configurations of n non-intersecting caps and cups respectively.

(2) κ_1 corresponds to a normalised element of $\mathrm{Hom}(\mathbf{1}, \bigotimes_{k=1}^{2n} X_{i_k})$. Take the adjoint morphism $|\psi_0\rangle \in \mathrm{Hom}(\bigotimes_{k=1}^{2n} X_{i_k}, \mathbf{1})$.

(3) Using some operator $\mathcal{F}$, transform $|\psi_0\rangle$ into a fusion basis compatible with the configuration κ_2 (labels do not matter).

(4) Transform $\mathcal{F}\,|\psi_0\rangle$ using the operator $\rho(b)$ where ρ is a unitary linear action of the braid groupoid on $2n$ strands in the primed fusion basis.

(5) κ_2 corresponds to a normalised element $|\psi_0'\rangle \in \mathrm{Hom}(\bigotimes_{k=1}^{2n} X_{j_k}, \mathbf{1})$. $\langle W(D)\rangle$ is found by applying the linear functional $\langle\psi_0'|$ to the fusion state obtained in (4).

[5] In the instance that the theory of anyons given by $\mathcal{C}$ is realised by a Chern-Simons theory, it is typical to call the diagram D a 'Wilson loop'. The quantity $\langle W(D)\rangle$ is called the expectation value of the Wilson loop, and may also be evaluated via a functional integral using the Chern-Simons action (see e.g. [NSS+08, Section 3]).

[6] In cases where there are components of D spanned by antisymmetrically self-dual anyons, we also need to account for any signs accumulated from straightening zig-zags along these components.

More succinctly, Procedure 5.5 tells us that

$$\langle W(D)\rangle = (-1)^z \langle\psi_0'|\,\rho(b)\mathcal{F}\,|\psi_0\rangle \tag{5.2.2}$$

where $|\psi_0\rangle$ and $|\psi_0'\rangle$ are called vacuum states, and z is the total number of zig-zags straightened along components spanned by antisymmetrically self-dual anyons in (1); note that there may be more than one fusion basis compatible with each of κ_1 and κ_2.

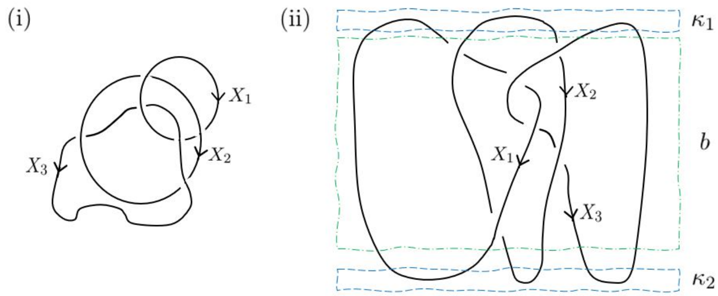


Figure 5.5: For a process given by a link diagram D as in (i), we isotope this into a diagram D' (as in step (1) of Procedure 5.5) illustrated in (ii), where we have outlined the areas of D' corresponding to κ_1, κ_2 and b.

We now consider some concrete examples. For each of these, we assume that each component of the link diagram is spanned by a self-dual anyon, and that the topological charge of anyons spanning distinct components is the same. We will also have that κ_2 is a reflection (in the horizontal) of κ_1, so $\mathcal{F} = \mathrm{id}$ and $|\psi_0'\rangle = |\psi_0\rangle$ in Procedure 5.5. Hence,

$$\langle W(D)\rangle = (-1)^z \langle\psi_0|\,\rho(b)\,|\psi_0\rangle \tag{5.2.3}$$

where $\rho : B_{2n} \to U(V_{\mathbf{1}}^{q^{\otimes 2n}})$ is a unitary linear representation of the braid group, and q is the topological charge of each component. In each example, we compute $\langle W(D)\rangle$ using Procedure 5.5, and also skein-theoretically. For the latter approach, we have

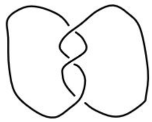
$$\langle W(D)\rangle = (-1)^z d_q^{-\mathcal{B}(L)}\Lambda_{\mathcal{C},q}(D) \tag{5.2.4}$$

where L is the link for which D is a diagram, $\mathcal{B}(L)$ is the bridge number of L, and $\Lambda_{\mathcal{C},q}$ is the operator that evaluates link diagrams (via the skein relation associated to the fusion rule for $q^{\otimes 2}$) as defined as in Chapter 4. The factor $d_q^{-\mathcal{B}(L)}$ accounts for the normalisation of $d_q^{-1/2}$ assigned to caps and cups, of which there are $2\mathcal{B}(L)$.[7]

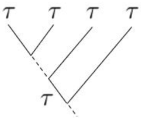
[7]In the context of Procedure 5.5, note that $\mathcal{B}(L) = n$.

In the examples that follow, it is easy to see that $\langle W(D)\rangle$ is a gauge-invariant quantity: when $\Lambda_{\mathcal{C},q}$ is given by one of the skein relations (3.14)-(3.15) or (3.31)-(3.34) (in Chapter 4), the coefficients in the skein relation are gauge-invariant.

Example 5.6. Take the Fibonacci theory (2.6.1), where F and R are as in Example 5.4, and let D be given by the right-handed trefoil



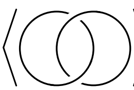
spanned by Fibonacci anyon τ. Then,

$|\psi_0\rangle =$  $\in V_\tau^{\tau\tau\tau}\otimes V_{\mathbf{1}}^{\tau\tau}$

It suffices to consider $V_\tau^{\tau\tau\tau}$ to compute the action of the braiding, which is

$$F^{-1}R^3F = e^{-i\frac{2\pi}{5}}F\begin{pmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{5}}\end{pmatrix}F = e^{-i\frac{2\pi}{5}}\begin{pmatrix}\phi^{-2}+e^{i\frac{\pi}{5}}\phi^{-1} & \phi^{-3/2}(1+e^{-i\frac{\pi}{5}})\\ \phi^{-3/2}(1-e^{i\frac{\pi}{5}}) & \phi^{-1}-e^{-i\frac{\pi}{5}}\phi^{-2}\end{pmatrix}$$

whence $\langle W(D)\rangle = e^{-i\frac{2\pi}{5}}\phi^{-2}+e^{-i\frac{\pi}{5}}\phi^{-1}$. Alternatively, we know that $\Lambda := \Lambda_{\mathcal{C},\tau}$ is the Kauffman bracket (Chapter 4: (3.14)). Then,

$\Lambda(D) = \beta\Big\langle$  $\Big\rangle + \beta^{-1}\Big\langle$  $\Big\rangle$ $(=\beta\tilde{S}_{\tau\tau}+\beta^{-1}\vartheta_\tau^{-2}d_\tau)$

where the Kauffman bracket of the Hopf link is $\beta^6+\beta^2+\beta^{-2}+\beta^{-6}$, whence

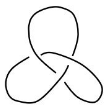
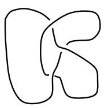
$$\langle D\rangle = \beta^7+\beta^3+\beta^{-1}-\beta^{-9} \tag{5.2.5}$$

Recalling that $\beta = R_\tau^{\tau\tau}$ here, $\langle D\rangle = e^{-i\frac{2\pi}{5}}+e^{-i\frac{\pi}{5}}\phi$. Then using (5.2.4), $\langle W(D)\rangle = d_\tau^{-2}\langle D\rangle$ (which agrees with our skeletal calculation).

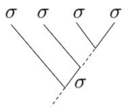
Example 5.7. Take the SU(2)$_2$ theory (2.6.4), where $\varkappa_\sigma = -1$,

$$F := F_\sigma^{\sigma\sigma\sigma} = -\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \quad , \quad R := R^{\sigma\sigma} = e^{i\frac{5\pi}{8}}\begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix} \tag{5.2.6}$$

and the matrices act on a basis in the order $\mathbf{1},\psi$. Let D be given by

$D =$  $\xrightarrow{\text{isotopy}}$  $= D'$

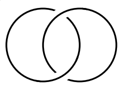
(i.e. the left-handed trefoil) spanned by Ising anyon σ. Then,

$|\psi_0\rangle =$  $\in V_{\mathbf{1}}^{\sigma\sigma} \otimes V_{\sigma}^{\sigma\sigma\sigma}$

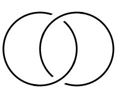
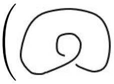
It suffices to consider $V_{\sigma}^{\sigma\sigma\sigma}$ to compute the action of the braiding, which is

$$FRF^{-1}R^{-1}FRF^{-1} = -\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{8}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

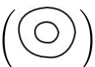
whence $\langle W(D)\rangle = \frac{1}{\sqrt{2}}e^{-i\frac{\pi}{8}}$. Alternatively, $\Lambda := \Lambda_{\mathcal{C},\sigma}$ is given by (3.15) in Chapter 4. Then,

$\Lambda(D) = \beta^{-1}\cdot\Lambda\Bigg($  $\Bigg) - \beta\cdot\Lambda\Bigg($  $\Bigg)$ $(= \beta^{-1}\tilde{S}_{\sigma\sigma} - \beta\vartheta_\sigma^2 d_\sigma)$

where

$\Lambda\Bigg($  $\Bigg) = \beta\cdot\Lambda\Bigg($  $\Bigg) - \beta^{-1}\cdot\Lambda\Bigg($  $\Bigg)$ $(= \beta\vartheta_q d_q - \beta^{-1}\varkappa_q\vartheta_q^{-1}d_q)$

$= \beta\Big[\beta\cdot\Lambda\Big($  $\Big) - \beta^{-1}\cdot\Lambda\Big($  $\Big)\Big]$

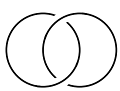
$- \beta^{-1}\Big[\beta\cdot\Lambda\Big($  $\Big) - \beta^{-1}\cdot\Lambda\Big($  $\Big)\Big] = \beta^6 + \beta^2 + \beta^{-2} + \beta^{-6}$

whence $\Lambda(D) = -\beta^9 + \beta + \beta^{-3} + \beta^{-7}$. Recalling that $\beta = R_{\psi}^{\sigma\sigma}$ here, $\Lambda(D) = \sqrt{2}e^{-i\frac{\pi}{8}}$. Then using (5.2.4), $\langle W(D)\rangle = d_\sigma^{-2}\Lambda(D)$ (which agrees with our skeletal calculation).

Example 5.8. Let $\mathcal{C}$ be the $(A_1,5)_{\frac{1}{2}}$ unitary modular category, for which the pertinent skeletal data is tabulated in [RSW09]. We have $\mathrm{Irr}(\mathcal{C}) = \{1,\alpha,\beta\}$ and nontrivial fusion rules

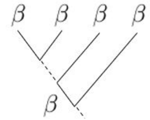
$$\alpha\otimes\alpha = \mathbf{1}\oplus\beta \quad , \quad \alpha\otimes\beta = \alpha\oplus\beta \quad , \quad \beta\otimes\beta = \mathbf{1}\oplus\alpha\oplus\beta$$

where $\varkappa_\alpha = \varkappa_\beta = 1$, $d_\alpha = 2\cos(\frac{\pi}{7})$, $d_\beta = 2\cos(\frac{2\pi}{7})+1$, and $R^{\beta\beta} = \mathrm{diag}(e^{-i\frac{10\pi}{7}}, e^{-i\frac{2\pi}{7}}, e^{-i\frac{5\pi}{7}})$ acts on a basis in the order $\mathbf{1},\alpha,\beta$. Let $R := R^{\beta\beta}$, $F := F_{\beta}^{\beta\beta\beta}$, and let D be given by



i.e. the Hopf link , both of whose components are spanned by a β-particle. Then,

$$|\psi_0\rangle =$$

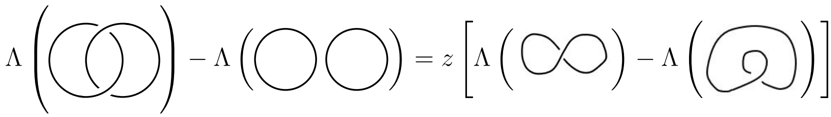


$$\in V_\beta^{\beta\beta\beta} \otimes V_{\mathbf{1}}^{\beta\beta}$$

It suffices to consider $V_\beta^{\beta\beta\beta}$ to compute the action of the braiding, which is given by $F^{-1}R^2F$. Using Lemma 5.1 (i) from Chapter 4, and that F is real-symmetric (Chapter 4: Corollary 5.7), we see that

$$\begin{aligned}\langle W(D)\rangle = \langle \bar{\psi}_0|\, F^{-1}R^2F\,|\bar{\psi}_0\rangle &= \begin{pmatrix} \frac{1}{d_\beta} & \frac{\sqrt{d_\alpha}}{d_\beta} & \frac{1}{\sqrt{d_\beta}} \end{pmatrix} \begin{pmatrix} e^{-i\frac{6\pi}{7}}\frac{1}{d_\beta} & e^{-i\frac{4\pi}{7}}\frac{\sqrt{d_\alpha}}{d_\beta} & e^{i\frac{4\pi}{7}}\frac{1}{\sqrt{d_\beta}} \end{pmatrix}^T \\ &= e^{-i\frac{6\pi}{7}}\frac{1}{d_\beta^2} + e^{-i\frac{4\pi}{7}}\frac{d_\alpha}{d_\beta^2} + e^{i\frac{4\pi}{7}}\frac{1}{d_\beta}\end{aligned}$$

where $|\bar{\psi}_0\rangle$ denotes the component of $|\psi_0\rangle$ in $V_\beta^{\beta\beta\beta}$. Alternatively, we know that $\Lambda := \Lambda_{\mathcal{C},\beta}$ is the framed Dubrovnik polynomial (Chapter 4: (3.31)). Then,



whence $\Lambda(D) = d_\beta^2 + d_\beta z(a - a^{-1})$. Recalling that $a = \vartheta_\beta$ and $z = R_\alpha^{\beta\beta} + R_\beta^{\beta\beta}$ here, we get $\Lambda(D) = -d_\alpha$ $(= \tilde{S}_{\beta\beta})$. Then using (5.2.4), we get $\langle W(D)\rangle = d_\beta^{-2}\Lambda(D)$, which agrees with our skeletal calculation.

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Appendices

A. Decoupled Dynamics

We have a particle of mass m moving in potential

$$V(\boldsymbol{r}) = V_{xy}(x, y) + V_z(z), \quad \boldsymbol{r} \in \mathbb{R}^3 \tag{A.0.1}$$

and the ansatz

$$\psi(\boldsymbol{r}) = \psi_{xy}(x, y)\psi_z(z) \tag{A.0.2}$$

The Hamiltonian is then given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2 + \hat{V}_{xy} + \hat{V}_z \tag{A.0.3}$$

and the Schrödinger equation for eigenenergy $E^{(i)}$ is

$$\nabla^2\psi - \frac{2m}{\hbar^2}(\hat{V}_{xy} + \hat{V}_z - E^{(i)})\psi = 0 \tag{A.0.4}$$

Substituting ansatz (A.0.2),

$$\begin{aligned}
&\nabla^2\psi_{xy}\psi_z - \frac{2m}{\hbar^2}(\hat{V}_{xy} + \hat{V}_z - E^{(i)})\psi_{xy}\psi_z = 0 \\
\implies &\psi_{xy}\nabla^2\psi_z + \psi_z\nabla^2\psi_{xy} - \frac{2m}{\hbar^2}\hat{V}_{xy}\psi_{xy}\psi_z - \frac{2m}{\hbar^2}\hat{V}_z\psi_{xy}\psi_z + \frac{2mE^{(i)}}{\hbar^2}\psi_{xy}\psi_z = 0 \\
\implies &\psi_{xy}\left(\nabla^2\psi_z - \frac{2m}{\hbar^2}\psi_z(\hat{V}_z - E_z^{(i)})\right) + \psi_z\left(\nabla^2\psi_{xy} - \frac{2m}{\hbar^2}\psi_{xy}(\hat{V}_{xy} - E_{xy}^{(i)})\right) = 0
\end{aligned}$$

where $E^{(i)} = E_{xy}^{(i)} + E_z^{(i)}$. This shows that the ansatz holds given that $\psi_{xy}(x, y)$ satisfies the two-dimensional Schrödinger equation subject to $V_{xy}(x, y)$, and that $\psi_z(z)$ satisfies the one-dimensional Schrödinger equation subject to $V_z(z)$.

B. Fusion Trees and Catalan Numbers

Let g_n denote the number of fusion trees with n leaves. A little thought reveals that

$$g_n = \sum_{i=1}^{n-1} g_i \cdot g_{n-i} \quad , \quad g_1 = 1 \tag{B.0.1}$$

We will solve (B.0.1) using generating functions i.e. the coefficients of $f(x) = \sum_{n=0}^{\infty} f_n x^n$ will encode the number of fusion trees. To do so, simply set $f_{n-1} = g_n$. Then (B.0.1) becomes

$$f_n = \sum_{i=0}^{n-1} f_i \cdot f_{n-1-i} \quad , \quad f_0 = 1 \tag{B.0.2}$$

and so

$$f(x) = 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} f_i \cdot f_{n-1-i}\ x^n = 1 + \sum_{n=0}^{\infty} \sum_{i=0}^{n} f_i \cdot f_{n-i}\ x^{n+1} = 1 + x(f(x))^2 \tag{B.0.3}$$

where solving for $f(x)$ yields

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{2}{1 \mp \sqrt{1-4x}}$$

and $f(0) = 1$ we must have $f(x) = \frac{1-\sqrt{1-4x}}{2x}$. We can expand the square root as

$$(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n \tag{B.0.4}$$

which tells us that

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n \tag{B.0.5}$$

That is,

$$f_n = \frac{1}{n+1} \binom{2n}{n} \quad , \quad n \geq 0 \tag{B.0.6}$$

which is the Catalan sequence. Thus, g_n is the $(n-1)^{th}$ Catalan number for $n \geq 1$.

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