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Metainferential Reasoning on Strong Kleene Models

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Abstract

Barrio et al. (*Journal of Philosophical Logic*, *49*(1), 93–120, 2020) and Pailos (*Review of Symbolic Logic*, *13*(2), 249–268, 2020) develop an approach to define various metainferential hierarchies on strong Kleene models by transferring the idea of distinct standards for premises and conclusions from inferences to metainferences. In particular, they focus on a hierarchy named the $\mathbb{ST}$-hierarchy where the inferential logic at the bottom of the hierarchy is the non-transitive logic ST but where each subsequent metainferential logic 'says' about the former logic that it is transitive. While Barrio et al. (2020) suggests that this hierarchy is such that each subsequent level 'in some intuitive sense, more classical than' the previous level, Pailos (2020) proposes an extension of the hierarchy through which a 'fully classical' metainferential logic can be defined. Both Barrio et al. (2020) and Pailos (2020) explore the hierarchy in terms of semantic definitions and every proof proceeds by a rather cumbersome reasoning about those semantic definitions. The aim of this paper is to present and illustrate the virtues of a proof-theoretic tool for reasoning about the $\mathbb{ST}$-hierarchy and the other metainferential hierarchies definable on strong Kleene models. Using the tool, this paper argues that each level in the $\mathbb{ST}$-hierarchy is non-classical to an equal extent and that the 'fully classical' metainferential logic is actually just the original non-transitive logic ST 'in disguise'. The paper concludes with some remarks about how the various results about the $\mathbb{ST}$-hierarchy could be seen as a guide to help us imagine what a non-transitive metalogic for ST would tell us about ST. In particular, it teaches us that ST is from the perspective of ST as metatheory not only non-transitive but also transitive.

Keywords Non-transitive logic · Metainferential hierarchies · Labelled sequent calculus · Nested sequent calculus · Strict-tolerant metatheory

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1 Introduction

[1] and [2] develop an approach to define various metainferential hierarchies on strong Kleene models by transferring the idea of mixed inferences to the metainferential level. In particular, their investigations reveal that a particular hierarchy named $\mathbb{ST}$ is definable where the inferential logic at the bottom of the hierarchy is the non-transitive logic ST advocated by [3] and [4] but where each subsequent metainferential logic 'says' about the former logic that it is transitive. While [1] suggest that this hierarchy provides metainferential logics where each subsequent level is 'in some intuitive sense, more classical than' the previous level, [2] proposes an extension of the hierarchy through which a 'fully classical' metainferential logic can be defined. Both [1] and [2] explore the hierarchies from a semantic perspective and every proof proceeds by a rather cumbersome reasoning about those semantic definitions.

The primary aim of this paper is to develop and illustrate the use of a proof-theoretic tool obtained by combining ideas from nested sequent calculi with labelled sequent calculi for reasoning about $\mathbb{ST}$ and the other metainferential hierarchies definable on strong Kleene models. To that purpose, Section 2 presents the approach to metainferential hierarchies on strong Kleene models developed by [1] and Section 3 presents a "labelled nested" sequent calculus based on the definitions provided in Section 2. This tool is then employed to make some remarks about [1]'s metainferential hierarchy $\mathbb{ST}$ and [2]'s 'fully classical' metainferential logic. In particular, it is shown in Section 4 that each level in the $\mathbb{ST}$ hierarchy is non-classical to an equal extent, a result which is extended in Section 5 to the 'fully classical' metainferential logic presented by [2]. Moreover, it is also shown that every metainference of the 'fully classical' metainferential logic is equivalent to an inference of the original non-transitive logic ST, and that the former is thus the latter 'in disguise'. Finally, the paper proposes in Section 6 that the hierarchy $\mathbb{ST}$ can fruitfully be understood as a tool to help us imagine what ST would tell us about ST if ST is used as metatheory where the most interesting observation being that ST is from the perspective of ST both transitive and non-transitive.

2 Language and Models

This section presents the language and models that will form the basis for the proof theory.

Definition 2.1 (The language) Let $\mathcal{L}$ be a propositional language based on a countable set of propositional variables, a nullary connective λ, a unary connective $\neg$ and the binary connective $\vee$. Let $\text{FORML}_{\mathcal{L}}$ be the set of formulas of $\mathcal{L}$.

We shall use upper case Latin letters A, B etc as metalinguistic variables for formulas in general and lower case Latin letters p, q etc as metalinguistic variables for propositional variables.

In addition to having formulas that are assigned values and can satisfy certain standards on strong Kleene models, we are interested in metainferences as objects that can satisfy certain appropriate standards on strong Kleene models, that is, as objects that can feature in a satisfaction relation. Following [1] we will define a hierarchy of metainferential objects as follows using the notation $[\ldots \Rightarrow \ldots]$:

Definition 2.2 (The metainferential objects)

- If $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ are formulas of $\mathcal{L}$, then $[A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m]$ is a metainferential object of level 0
- If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are metainferential objects of level k then $[X_1, \ldots, X_n \Rightarrow Y_1, \ldots, Y_m]$ is a metainferential object of level $k + 1$.

Metainferential objects can thus be seen as binary connectives that applies to sets of objects. They are however not part of $\mathcal{L}$ even if they contain objects from $\text{FORM}_{\mathcal{L}}$. Moreover, while one might be tempted to add numerals to the objects in order to identify its level, this is not necessary since X_i in $[X_1, \ldots, X_n \Rightarrow Y_1, \ldots, Y_m]$ will be an object of the previous level if the level is > 0 or a formula if the level is 0.

Definition 2.3 (Strong Kleene valuations) A function $\mathcal{V} : \text{FORML}_{\mathcal{L}} \rightarrow \{1, \frac{1}{2}, 0\}$ is a strong Kleene valuation just in case $\mathcal{V}(\lambda) = \frac{1}{2}$ and the following conditions are satisfied for every complex formula:

$$\mathcal{V}(A \vee B) = \begin{cases} 1 & \mathcal{V}(A) = 1 \text{ or } \mathcal{V}(B) = 1 \\ 0 & \mathcal{V}(A) = 0 \text{ and } \mathcal{V}(B) = 0 \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad \mathcal{V}(\neg A) = \begin{cases} 1 & \mathcal{V}(A) = 0 \\ 0 & \mathcal{V}(A) = 1 \\ \frac{1}{2} & \mathcal{V}(A) = \frac{1}{2} \end{cases}$$

Following [5], a formula on trivalent models can be either strictly or tolerantly satisfied. This is made precise as follows:

Definition 2.4 (Satisfaction of formulas)

- $\mathcal{V} \Vdash_s A$ if and only if $\mathcal{V}(A) = 1$
- $\mathcal{V} \Vdash_t A$ if and only if $\mathcal{V}(A) \in \{1, \frac{1}{2}\}$

[1] extends the notion of satisfaction from formulas to metainferential objects using a hierarchy of metainferential standards based on the strict-tolerant distinction. Informally presented, the idea is as follows:

- A formula can satisfy one out of two standards, s and t.
- A metainference of level 0 can satisfy one or more out of four standards, st, ts, ss and tt.
- A metainference of level 1 can satisfy one or more out of sixteen standards:
  - $stst$, $tsst$, $ssst$, $ttst$
  - $stts$, $tsts$, $ssts$, $ttts$
  - $stss$, $tsss$, $ssss$, $ttss$

  - $sttt, tstt, sstt, tttt$
- A metainference of level 2 can satisfy one or more out of 256 standards which we shall not list.

An inductive definition can thus be given as follows:

Definition 2.5 (The standards)
- s and t are formula standards.
- if x and y are formula-standards, then xy is a metainferential standard of level 0.
- if x and y are metainferential standards of level n, then xy is a metainferential standard of level $n+1$.

Following [1], the notion of satisfaction can now be extended as follows:

Definition 2.6 (Satisfaction of metainferences) If $[\Gamma \Rightarrow \Delta]$ is a metainferential object of level n and xy a metainferential standard of level n, then,

$$\mathcal{V} \Vdash_{xy} [\Gamma \Rightarrow \Delta] \text{ iff some } X \in \Gamma, \mathcal{V} \nVdash_x X \text{ or some } Y \in \Delta, \mathcal{V} \Vdash_y Y$$

Finally, validity for the various inferential and metainferential logics is now defined as follows:

Definition 2.7 (Validity) $\Gamma \vDash_{xy} \Delta$ iff for every $\mathcal{V}$, $\mathcal{V} \Vdash_{xy} [\Gamma \Rightarrow \Delta]$

Unsurprisingly, the various inferential and metainferential logics definable on strong Kleene models recently discussed in the literature fall out of this definition. We shall in general refer to a particular logic through its standard, e.g. the logic st or the logic $tsst$. There is however one logic definable on strong Kleene models that is not captured by this approach and which on occasion is discussed in the literature, e.g. by [6], namely that definable using $\leq$ as follows: $\Gamma \vDash \Delta$ iff every $\mathcal{V}$ is such that $\min(\mathcal{V}(A) \in \Gamma) \leq \max(\mathcal{V}(B) \in \Delta)$. This is an acceptable limitation considering the aim of this paper.

3 The HST Calculus

This section presents a sequent calculus representing metainferential hierarchies on strong Kleene valuations based on the definitions provided in the previous section. The *hierarchical strict-tolerant* calculus will be a labelled sequent calculus in the sense that the rules will not manipulate formulas directly as in the case of a standard sequent calculus, but rather labelled formulas and labelled metainferential objects. To that purpose we shall introduce one label for each standard to thereby obtain labelled formulas (e.g. $s{:}A$) and labelled metainferential objects of the form $x{:}[\Gamma \Rightarrow \Delta]$ where Γ and Δ will be formulas if it is a metainferential object of level 0 and metainferential object of level n if it is a metainferential object of level $n+1$. Since the calculus will thus contain expressions that look like standard sequents nested within each others, it can also rightly be described as a sequent calculus for nested sequents. It is thus a labelled nested sequent calculus.

While the calculus is straightforwardly modified to also include the addition of so-called "antivalidities" as introduced into the debate by [7] and thus also capture the arguments presented by [7], such modifications are purposely left out to keep things simpler and more straightforward. The reader with an interest in such issues is invited to make the appropriate amendments.

Definition 3.1 (Typed sequent expression)

- If A is a formula and $x \in \{s, t\}$, then $x{:}A$ is a sequent expression of type $(0, x)$
- If $x{:}X_1, \ldots, x{:}X_n$ are sequent expressions of type (α, x) and $y{:}Y_1, \ldots, y{:}Y_m$ are of sequent expressions of type (α, y) then $xy{:}[X_1, \ldots, X_n \Rightarrow Y_1, \ldots, Y_m]$ is a sequent expression of type $(\alpha + 1, xy)$.

For a sequent expression of type (α, x), we refer to α as the level and x as the standard. We shall use $x : X$ to refer to an arbitrary sequent expression of any type. Moreover, we let $\mathbb{X}_{(\alpha,x)}$ and $\mathbb{Y}_{(\alpha,y)}$ designate finite multisets of sequent expressions of type (α, x) and (α, y) respectively. We also let $\mathbb{X}$ designate the multiset obtained by removing labels from the members of $\mathbb{X}_{(\alpha,x)}$, i.e. $\mathbb{X} = \{X \mid x{:}X \in \mathbb{X}_{(\alpha,x)}\}$. Note also that it follows from the notation that a metainference of level n is represented by a sequent expression of level $n + 1$.

The following are examples of typed sequent expressions.

$$st{:}[A, \neg A \vee B \Rightarrow B] \qquad tt{:}[A \vee B \Rightarrow A, B]$$

$$tsst{:}[[A \Rightarrow A], [\Rightarrow \neg A \vee B] \Rightarrow [A \Rightarrow B]]$$

$$ttst{:}[\Rightarrow [\Rightarrow A], [A \Rightarrow]]$$

$$tssttss{:}[[[A \Rightarrow A], [\Rightarrow \neg A \vee B] \Rightarrow [A \Rightarrow B]] \Rightarrow [\Rightarrow [\Rightarrow A], [A \Rightarrow]]]$$

As the examples suggest, it will be increasingly difficult to read the sequent expressions in order to decipher their level and thus their meaning. Luckily the construction is compositional and we are in general only interested in the inductive steps from level n to level $n + 1$.

Definition 3.2 (The HST calculus) Let HST be the sequent calculus obtained with the following rules where sequents are pairs of multisets of typed sequent expressions.

Initial sequents:

$$s{:}p, \Gamma \Rightarrow \Delta, s{:}p \qquad s{:}p, \Gamma \Rightarrow \Delta, t{:}p \qquad t{:}p, \Gamma \Rightarrow \Delta, t{:}p$$

Rules for sequent expressions of level 0:

$$\frac{x{:}A, \Gamma \Rightarrow \Delta \quad x{:}A, \Gamma \Rightarrow \Delta}{x{:}A \vee B, \Gamma \Rightarrow \Delta}\vee\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, x{:}A, x{:}B}{\Gamma \Rightarrow \Delta, x{:}A \vee B}\vee\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, x{:}A}{y{:}\neg A, \Gamma \Rightarrow \Delta}\neg\mathrm{L}(x \neq y) \qquad \frac{x{:}A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, y{:}\neg A}\neg\mathrm{R}(x \neq y)$$

$$\frac{}{s{:}\lambda, \Gamma \Rightarrow \Delta}\lambda\mathrm{L} \qquad \frac{}{\Gamma \Rightarrow \Delta, t{:}\lambda}\lambda\mathrm{R}$$

Rules for sequent expressions of level > 0:

$$\frac{\Gamma \Rightarrow \Delta, x{:}X \quad \text{for every } x{:}X \in \mathbb{X}_{(\alpha,x)} \qquad y{:}Y, \Gamma \Rightarrow \Delta \quad \text{for every } y{:}Y \in \mathbb{Y}_{(\alpha,y)}}{xy{:}[\mathbb{X} \Rightarrow \mathbb{Y}], \Gamma \Rightarrow \Delta}[\Rightarrow]\text{L}$$

$$\frac{\mathbb{X}_{(\alpha,x)}, \Gamma \Rightarrow \Delta, \mathbb{Y}_{(\alpha,y)}}{\Gamma \Rightarrow \Delta, xy{:}[\mathbb{X} \Rightarrow \mathbb{Y}]}[\Rightarrow]\text{R}$$

This calculus is well-behaved from the perspective of structural proof theory as elucidated by [8] and [9]. In particular, we have the following lemmas and theorems where each proof is obtainable through a straight-forward adaptation of the corresponding proof in [9].

Definition 3.3 (Derivation height) The height of a HST-derivation $\mathcal{D}$, $\mathcal{H}(\mathcal{D})$ is defined inductively as follows.

- If $\mathcal{D}$ is an initial sequent or conclusion of a zero-premise rule, then $\mathcal{H}(\mathcal{D}) = 0$.
- If $\mathcal{D}$ is obtained with an α-premise rule from derivations $\mathcal{D}_i$ for $0 \leq i < \alpha$, then $\mathcal{H}(\mathcal{D}) = \sup_{i<\alpha}(\mathcal{H}(\mathcal{D}_i) + 1)$

We'll say that a rule is height-preservingly admissible (HP-admissible) just in case whenever there is a derivation of the premise-sequent with height n then there is a derivation of the conclusion-sequent with height $\leq n$.

Lemma 3.4 (Weakening and contraction) *The following rules are HP-admissible in HST:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \qquad \frac{x{:}X, x{:}X, \Gamma \Rightarrow \Delta}{x{:}X, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, x{:}X, x{:}X}{\Gamma \Rightarrow \Delta, x{:}X}$$

Proof The proofs are obtained by slight modifications on the proofs of proposition 4.4 and theorem 4.12 in [9]. □

Lemma 3.5 (Inversion) *The inversion of each primitive HST-rule is HP-admissible in HST.*

Proof Proof is a modification of proposition 4.11 in [9]. □

Definition 3.6 (Formula complexity) The complexity of a $\mathcal{L}$-formula A,|A|, is defined inductively as follows:

- If A is an atomic formula, then $| A |= 0$
- If A is of the form $\neg B$, then $| A |=| B | +1$
- If A is of the form $B \vee C$, then $| A |=| B | + | C | +1$.

Definition 3.7 (Expression weight) Suppose that $x{:}X$ is a sequent expression. Then the weight of X, $\mathcal{W}(X)$, is defined as follows: if X is a formula A, then $\mathcal{W}(A)$ =|A| and if X is a metainferential object $[\Gamma \Rightarrow \Delta]$, then $\mathcal{W}([\Gamma \Rightarrow \Delta]) = \sum_{Y \in \Gamma}(\mathcal{W}(Y)) + \sum_{Y \in \Delta}(\mathcal{W}(Y)) + 1$.

Theorem 3.8 (Cut) *The following rule is admissible in HST:*

$$\frac{\Gamma \Rightarrow \Delta, x{:}X \qquad x{:}X, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof By double induction on the weight of X and the sum of the heights of the derivations of the premise-sequents. See theorem 4.13 in [9]. □

Definition 3.9 (Sequent validity) A sequent $\Gamma \Rightarrow \Delta$ is valid just in case there is no valuation $\mathcal{V}$ such that

- for every typed sequent expression $x{:}X \in \Gamma$, $\mathcal{V} \Vdash_x X$
- for every typed sequent expression $y{:}Y \in \Delta$, $\mathcal{V} \nVdash_y Y$

Theorem 3.10 (Completeness) *A sequent $\Gamma \Rightarrow \Delta$ is valid if and only if $\Gamma \Rightarrow \Delta$ is derivable in HST.*

Proof The right-to-left direction proceeds as usual by induction on the height of a derivation. The left-to-right direction proceeds as usual via the construction of a reduction tree for every underivable sequent from which a countermodel for that sequent is extracted. We present here a few details from the latter proof.

Assume that $\Gamma \Rightarrow \Delta$ is underivable. It follows that we can construct a tree above it by applying the rules of the HST calculus backwards until each branch ends with a sequent containing only labelled propositional variables and labelled λ's. At least one branch will be such that the leaf is not an initial sequent or a zero-premise rule of the HST calculus. We pick such a branch $\mathcal{B}$ and define a function $\mathcal{V}$ from the set of propositional variables of $\mathcal{L}$ to $\{1, 0, \frac{1}{2}\}$ as follows where $\Gamma' \Rightarrow \Delta'$ is the leaf-sequent of $\mathcal{B}$:

- for every $s{:}p$ in Γ', $\mathcal{V}(p) = 1$,
- for every $t{:}p$ in Δ', $\mathcal{V}(p) = 0$,
- otherwise, $\mathcal{V}(p) = \frac{1}{2}$.

The definition of $\mathcal{V}$ is extended to complex formulas and λ in accordance with definition 2.3. The satisfaction relation $\Vdash$ is defined in accordance with definitions 2.4 and 2.6. We can now show by induction on the complexity of a formula that for every $\Gamma'' \Rightarrow \Delta'' \in \mathcal{B}$:

- If $x{:}A \in \Gamma''$ then $\mathcal{V} \Vdash_x A$.
- If $x{:}A \in \Delta''$ then $\mathcal{V} \nVdash_x A$.

It is left to show by an induction on the set of standards from definition 2.5 that also the following statements hold for every $\Gamma'' \Rightarrow \Delta'' \in \mathcal{B}$:

- If $xy{:}[\Gamma' \Rightarrow \Delta'] \in \Gamma''$ then $\mathcal{V} \Vdash_{xy} [\Gamma' \Rightarrow \Delta']$.
- If $xy{:}[\Gamma' \Rightarrow \Delta'] \in \Delta''$ then $\mathcal{V} \nVdash_{xy} [\Gamma' \Rightarrow \Delta']$.

With $\Gamma \Rightarrow \Delta \in \mathcal{B}$, it follows that for every typed sequent expression $x{:}X \in \Gamma$, $\mathcal{V} \Vdash_x X$ and for every typed sequent expression $y{:}Y \in \Delta$, $\mathcal{V} \nVdash_y Y$. $\mathcal{V}$ is thus a countermodel for the sequent $\Gamma \Rightarrow \Delta$. □

Corollary 3.11 *$\Rightarrow xy{:}[\Gamma \Rightarrow \Delta]$ is derivable if and only if $\Gamma \vDash_{xy} \Delta$*

Before we dive into the perhaps more serious applications of this proof-theoretic tool, we shall first provide a few illustrations of its immediate usefulness. As a first curiosity we shall show that the inferential logic defined with the standard st is nontransitive using the admissibility of cut.

Lemma 3.12 (Nontriviality) *For every label xy, sequent $\Rightarrow xy{:}[\ \Rightarrow\ ]$ is not derivable.*

Proof The empty sequent follows by inversion, but the empty sequent is excluded by design. □

Lemma 3.13 (st is inconsistent) *The sequents $\Rightarrow st{:}[\ \Rightarrow \lambda]$ and $\Rightarrow st{:}[\lambda \Rightarrow\ ]$ are derivable.*

Proposition 3.14 (st is nontransitive) *The sequent $st{:}[\ \Rightarrow \lambda], st{:}[\lambda \Rightarrow\ ] \Rightarrow st{:}[\ \Rightarrow\ ]$ is underivable.*

Proof If that sequent is derivable then theorem 3.8 and lemma 3.13 together imply that $\Rightarrow st{:}[\ \Rightarrow\ ]$ is derivable but this is excluded by lemma 3.12. □

With inspiration from [10], we have thus shown the nontransitivity of a logic typically defined proof-theoretically by rejecting cut using the admissibility of cut.

In fact, we actually have a tool which can be used as a "metasequent" calculus for the four logics st, ts, tt and ss. In the case of st, for example, the following sequents are derivable:

$$\Rightarrow st{:}[A, \Gamma \Rightarrow \Delta, A]$$

$$st{:}[\Gamma \Rightarrow \Delta, A] \Rightarrow st{:}[\neg A, \Gamma \Rightarrow \Delta]$$

$$st{:}[A, \Gamma \Rightarrow \Delta] \Rightarrow st{:}[\Gamma \Rightarrow \Delta, \neg A]$$

$$st{:}[A, \Gamma \Rightarrow \Delta], st{:}[B, \Gamma \Rightarrow \Delta] \Rightarrow st{:}[A \vee B, \Gamma \Rightarrow \Delta]$$

$$st{:}[\Gamma \Rightarrow \Delta, A, B] \Rightarrow st{:}[\Gamma \Rightarrow \Delta, A \vee B]$$

One can now use cut to obtain every st-validity, where then each application of cut corresponds to one application of a rule in the standard two-sided sequent calculus for st such as that presented by [3].

Moreover, we can use the sequent calculus to illustrate the relationships between the logics st, ts, ss and tt familiar from the literature. For example, we can show that a certain instance of transitivity holds in st just in case a corresponding instance reflexivity holds in ts with the following derivations:

$$\dfrac{\dfrac{\dfrac{s{:}A, t{:}A \Rightarrow s{:}A}{t{:}A \Rightarrow s{:}A, st{:}[A \Rightarrow\ ]}[\Rightarrow]\mathrm{R} \qquad \dfrac{t{:}A \Rightarrow s{:}A, t{:}A}{t{:}A \Rightarrow s{:}A, st{:}[\ \Rightarrow A]}[\Rightarrow]\mathrm{R}}{stst{:}[[\ \Rightarrow A], [A \Rightarrow\ ] \Rightarrow\ ], t{:}A \Rightarrow s{:}A}[\Rightarrow]\mathrm{L}}{stst{:}[[\ \Rightarrow A], [A \Rightarrow\ ] \Rightarrow\ ] \Rightarrow ts{:}[A \Rightarrow A]}[\Rightarrow]\mathrm{R}$$

$$\dfrac{\dfrac{\dfrac{t{:}A, st{:}[A \Rightarrow\ ] \Rightarrow t{:}A}{st{:}[\ \Rightarrow A], st{:}[A \Rightarrow\ ] \Rightarrow t{:}A}[\Rightarrow]\mathrm{R} \qquad \dfrac{s{:}A, st{:}[\ \Rightarrow A] \Rightarrow t{:}A}{s{:}A, st{:}[\ \Rightarrow A], st{:}[A \Rightarrow\ ] \Rightarrow}[\Rightarrow]\mathrm{R}}{ts{:}[A \Rightarrow A], st{:}[\ \Rightarrow A], st{:}[A \Rightarrow\ ] \Rightarrow}[\Rightarrow]\mathrm{L}}{ts{:}[A \Rightarrow A] \Rightarrow stst{:}[[\ \Rightarrow A], [A \Rightarrow\ ] \Rightarrow\ ]}[\Rightarrow]\mathrm{R}$$

To provide further familiar facts about the logics (and also for some propositions in the next section), the following lemma for transforming labels will be useful.

Lemma 3.15 (label transformation) *(a) If A is a formula, then following rules are HP-admissible:*

$$\dfrac{t{:}A, \Gamma \Rightarrow \Delta}{s{:}A, \Gamma \Rightarrow \Delta}\ t/sL \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, s{:}A}{\Gamma \Rightarrow \Delta, t{:}A}\ s/tR$$

(b) If X is a sequent expression of level 1, then the following rules are HP-admissible:

$$\dfrac{xt{:}X, \Gamma \Rightarrow \Delta}{xs{:}X, \Gamma \Rightarrow \Delta}\ xt/xsL \qquad\qquad \dfrac{sx{:}X, \Gamma \Rightarrow \Delta}{tx{:}X, \Gamma \Rightarrow \Delta}\ sx/txL$$

$$\dfrac{\Gamma \Rightarrow \Delta, tx{:}X}{\Gamma \Rightarrow \Delta, sx{:}X}\ tx/sxR \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, xs{:}X}{\Gamma \Rightarrow \Delta, xt{:}X}\ xs/xtR$$

Proof The proofs straightforwardly follow the general strategy in structural proof theory to show the height-preserving admissibility of a rule.

Regarding (a), we prove t/sL and s/tR simultaneously. We here focus on t/sL.

Base case: Assume $0 \vdash t{:}A, \Gamma \Rightarrow \Delta$. If A is a propositional variable and $t{:}A \in \Delta$, then $s{:}A, \Gamma \Rightarrow \Delta$ is also an initial sequent. If $\Gamma \Rightarrow \Delta$ is an initial sequent or an instance of zero-premise rule, then $s{:}A, \Gamma \Rightarrow \Delta$ is also an initial sequent or an instance of a zero-premise rule and thus $0 \vdash s{:}A, \Gamma \Rightarrow \Delta$ holds.

Inductive step: Assume $n{+}1 \vdash t{:}A, \Gamma \Rightarrow \Delta$. If $t{:}A$ is principal and of the form $\neg B$, it follows that $n \vdash \Gamma \Rightarrow \Delta, s{:}B$. By the inductive hypothesis we obtain $n \vdash \Gamma \Rightarrow \Delta, t{:}B$ and by one application of $\neg L$ we obtain $n{+}1 \vdash t{:}\neg B, \Gamma \Rightarrow \Delta$. The case for $\vee$L is similar. If $t{:}A$ is not principal, the sequent is obtained with some k-premise rule $\mathcal{R}$ and it is thus the case that $n \vdash t{:}A, \Gamma^i \Rightarrow \Delta^i$ for every $i < k$. We apply the inductive hypothesis to obtain $n \vdash s{:}A, \Gamma^i \Rightarrow \Delta^i$ and one application of $\mathcal{R}$ delivers $n{+}1 \vdash s{:}A, \Gamma \Rightarrow \Delta$.

Regarding (b) we proceed as follows, focusing only on the case of tx/sxR. Assume $n{+}1 \vdash \Gamma' \Rightarrow \Delta', tx{:}[\Gamma \Rightarrow \Delta]$. If $tx{:}[\Gamma \Rightarrow \Delta]$ is principal, then $n \vdash t{:}\Gamma, \Gamma' \Rightarrow \Delta', x{:}\Delta$. By (a) we obtain $n \vdash s{:}\Gamma, \Gamma' \Rightarrow \Delta', x{:}\Delta$ and finally $n{+}1 \vdash \Gamma' \Rightarrow \Delta', sx{:}[\Gamma \Rightarrow \Delta]$ by the relevant rule for introducing the desired expression. If

$tx{:}[\Gamma \Rightarrow \Delta]$ is not principal, then the sequent in question is obtained with some k-premise rule $\mathcal{R}$ and it is thus the case that $n \vdash \Gamma^i \Rightarrow \Delta^i, tx{:}[\Gamma \Rightarrow \Delta]$ for every $i < k$. We apply the inductive hypothesis to obtain $n \vdash \Gamma^i \Rightarrow \Delta^i, sx{:}[\Gamma \Rightarrow \Delta]$ and then apply $\mathcal{R}$ to obtain $n{+}1 \vdash \Gamma' \Rightarrow \Delta', sx{:}[\Gamma \Rightarrow \Delta]$. □

This lemma has two immediate corollaries. The first concerns the relationship between valid inferences in the four logics as familiar from [5].

Corollary 3.16 (Relationships between st, ts, ss and tt) *The following sequents are derivable:*

$$ts{:}X \Rightarrow tt{:}X \quad ts{:}X \Rightarrow ss{:}X \quad tt{:}X \Rightarrow st{:}X \quad ss{:}X \Rightarrow st{:}X$$

The second concerns the relationship between inferences of st and ts on the one hand, and formulas that are tolerantly and strictly satisfied on the other hand.

Corollary 3.17 *The following rules are admissible were $x \neq y$:*

$$\begin{array}{c} xy{:}[A_1, \ldots, A_n \Rightarrow_1 B_1, \ldots, B_m], \Gamma \Rightarrow \Delta \\ \hline\hline y{:}\neg A_1 \vee \ldots \vee \neg A_n \vee B_1 \vee \ldots \vee B_m, \Gamma \Rightarrow \Delta \end{array}$$

$$\begin{array}{c} \Gamma \Rightarrow \Delta, xy{:}[A_1, \ldots, A_n \Rightarrow_1 B_1, \ldots, B_m] \\ \hline\hline \Gamma \Rightarrow \Delta, y{:}\neg A_1 \vee \ldots \vee \neg A_n \vee B_1 \vee \ldots \vee B_m \end{array}$$

Proof By inversion and straightforward applications of the relevant rules. □

An immediate consequence of that corollary is the following result by [11] and [12]:

Proposition 3.18 *The following rule is admissible:*

$$\begin{array}{c} st{:}[A_1^1, \ldots, A_n^1 \Rightarrow B_1^1, \ldots, B_m^1], \ldots\ st{:}[A_1^k, \ldots, A_{n'}^k \Rightarrow B_1^k, \ldots, B_{m'}^k] \Rightarrow st{:}[A_1, \ldots, A_{n''} \Rightarrow B_1, \ldots, B_{m''}] \\ \hline\hline \Rightarrow tt{:}[\neg A_1^1 \vee \ldots \vee \neg A_n^1 \vee B_1^1 \vee \ldots \vee B_m^1, \ldots, \neg A_1^k \vee \ldots \vee \neg A_{n'}^k \vee B_1^k \vee \ldots \vee B_{m'}^k \Rightarrow \neg A_1 \vee \ldots \neg A_{n''} \vee B_1 \vee \ldots \vee B_{m''}] \end{array}$$

The corresponding result for ts as presented by for example [13] is obviously also available:

Proposition 3.19 *The following rule is admissible:*

$$\begin{array}{c} ts{:}[A_1^1, \ldots, A_n^1 \Rightarrow B_1^1, \ldots, B_m^1], \ldots\ ts{:}[A_1^k, \ldots, A_{n'}^k \Rightarrow B_1^k, \ldots, B_{m'}^k] \Rightarrow ts{:}[A_1, \ldots, A_{n''} \Rightarrow B_1, \ldots, B_{m''}] \\ \hline\hline \Rightarrow ss{:}[\neg A_1^1 \vee \ldots \vee \neg A_n^1 \vee B_1^1 \vee \ldots \vee B_m^1, \ldots, \neg A_1^k \vee \ldots \vee \neg A_{n'}^k \vee B_1^k \vee \ldots \vee B_{m'}^k \Rightarrow \neg A_1 \vee \ldots \neg A_{n''} \vee B_1 \vee \ldots \vee B_{m''}] \end{array}$$

What is interesting here is not the fact that these propositions hold about the four logics st, ts, tt and ss, but the ease with which we have obtained them using proof analysis.

4 Approximating Classicality with the ST-Hierarchy?

The results in the previous section concerned only inferences and metainferences, not inferences of metainferences and so forth. Given the generality of our calculus, it should be clear that we can also use it to prove facts about "higher-order" metainferences. To illustrate that we shall have a look at what we can say about the $\mathbb{ST}$-hierarchy of metainferential standards presented by [1]. The basic idea with the hierarchy is to "reproduce" the "st-phenomenon" at a metainferential level by defining a hierarchy of metainferential standards where the standard for being a premise in a sound inference is stricter than the standard for being a conclusion. For our purposes, we can replicate the hierarchy of standards with labels using the following definition:

Definition 4.1 (The $\mathbb{ST}$-hierarchy) The set of labels $\mathbb{ST}$ is defined inductively as follows:

– $st \in \mathbb{ST}$
– If $xy \in \mathbb{ST}$ then $yxxy \in \mathbb{ST}$

The standard for metainferences of level 1 is thus $tsst$ as opposed to the standard $stst$, and for metainferences of level 2 we have the standard $sttstsst$ as opposed to the standard $tssttsst$. We will use ST_n to refer to the nth level in $\mathbb{ST}$, so that ST_0 is st, ST_1 is $tsst$, and so on.

What is interesting about $\mathbb{ST}$ according to [1] is that "in some intuitive sense, TS/ST is classical to a greater degree than ST", and moreover that we obtain at each ST_{n+1} a "metainferential" logic which is supposedly more similar to classical logic than ST_n because stage n is according to stage $n+1$ transitive. To illustrate this, consider the following observation about $tsst$:

Proposition 4.2 ($tsst$ concerns a transitive logic) *The following sequent is derivable:*

$$\Rightarrow tsst{:}[[\Gamma \Rightarrow \Delta, A], [A, \Gamma' \Rightarrow \Delta'] \Rightarrow [\Gamma, \Gamma' \Rightarrow \Delta, \Delta']]$$

Proof We have for every formula A a derivation of the following form:

$$\dfrac{\dfrac{\dfrac{\vdots}{t{:}\Gamma \Rightarrow t{:}\Gamma} \quad \dfrac{\vdots}{s{:}\Delta \Rightarrow s{:}\Delta} \quad \dfrac{\vdots}{s{:}A \Rightarrow t{:}A}}{ts{:}[\Gamma \Rightarrow \Delta, A], t{:}\Gamma \Rightarrow s{:}\Delta, t{:}A}\,[\Rightarrow]\mathrm{L} \quad \dfrac{\vdots}{t{:}\Gamma' \Rightarrow t{:}\Gamma'} \quad \dfrac{\vdots}{s{:}\Delta' \Rightarrow s{:}\Delta'}}{\dfrac{ts{:}[\Gamma \Rightarrow \Delta, A], ts{:}[A, \Gamma' \Rightarrow \Delta'], t{:}\Gamma, t{:}\Gamma' \Rightarrow s{:}\Delta, s{:}\Delta'}{ts{:}[\Gamma \Rightarrow \Delta, A], ts{:}[A, \Gamma' \Rightarrow \Delta'] \Rightarrow ts{:}[\Gamma, \Gamma' \Rightarrow \Delta, \Delta']}\,[\Rightarrow]\mathrm{R}}\,[\Rightarrow]\mathrm{L}$$

We then proceed as follows using label transformation:

$$\dfrac{\dfrac{\dfrac{ts{:}[\Gamma \Rightarrow \Delta, A], ts{:}[A, \Gamma' \Rightarrow \Delta'] \Rightarrow ts{:}[\Gamma, \Gamma' \Rightarrow \Delta, \Delta']}{ts{:}[\Gamma \Rightarrow \Delta, A], ts{:}[A, \Gamma' \Rightarrow \Delta'] \Rightarrow ss{:}[\Gamma, \Gamma' \Rightarrow \Delta, \Delta']}\,\mathrm{tx/sxR}}{ts{:}[\Gamma \Rightarrow \Delta, A], ts{:}[A, \Gamma' \Rightarrow \Delta'] \Rightarrow st{:}[\Gamma, \Gamma' \Rightarrow \Delta, \Delta']}\,\mathrm{xs/xtR}}{\Rightarrow tsst{:}[[\Gamma \Rightarrow \Delta, A], [A, \Gamma' \Rightarrow \Delta'] \Rightarrow [\Gamma, \Gamma' \Rightarrow \Delta, \Delta']]}$$

□

While reasoning from st to st is nontransitive, reasoning from ts to st is transitive. To extend this observation to each standard in $\mathbb{ST}$ we shall use the following lemma:

Lemma 4.3 (Cut-elimination for $\mathbb{ST}$) *Suppose that* $xy \in \mathbb{ST}$. *Then the following rule is admissible:*

$$\frac{\Gamma \Rightarrow \Delta, x{:}X \qquad y{:}X, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Proof Base case (=ST) (Simplified by omitting contexts):

$$\dfrac{\Rightarrow s{:}A \qquad \dfrac{t{:}A \Rightarrow}{s{:}A \Rightarrow}\,\text{t/sL}}{\Rightarrow}\,\text{cut}$$

Inductive step (Simplified by omitting contexts and some branching): Assume that it holds for xy. Then it also holds for $yxxy$ by applying inversion and then the inductive hypothesis:

$$\dfrac{\dfrac{xy{:}[\Gamma \Rightarrow \Delta] \Rightarrow}{\Rightarrow x{:}\Gamma} \qquad \dfrac{\dfrac{\Rightarrow yx{:}[\Gamma \Rightarrow \Delta]}{y{:}\Gamma \Rightarrow x{:}\Delta} \qquad \dfrac{xy{:}[\Gamma \Rightarrow \Delta] \Rightarrow}{y{:}\Delta \Rightarrow}}{y{:}\Gamma \Rightarrow}}{\Rightarrow}$$

□

Proposition 4.4 (Every stage in $\mathbb{ST}$ is a reflexive logic) *If* $xy \in \mathbb{ST}$, *then* $\Rightarrow xy{:}[X \Rightarrow X]$ *is derivable*

Proof

$$\dfrac{\dfrac{x{:}X \Rightarrow x{:}X \qquad y{:}X \Rightarrow y{:}X}{x{:}X \Rightarrow y{:}X}\,\text{Lemma 4.3}}{\Rightarrow xy{:}[X \Rightarrow X]}\,[\Rightarrow]\text{R}$$

□

Proposition 4.5 (Every stage in $\mathbb{ST}$ concerns a transitive logic) *If* $yxxy \in \mathbb{ST}$, *then* $\Rightarrow yxxy : [[\Gamma \Rightarrow \Delta, X], [X, \Gamma \Rightarrow \Delta] \Rightarrow [\Gamma \Rightarrow \Delta]]$ *is derivable*

Proof

$$\dfrac{\dfrac{\dfrac{x{:}\Gamma \Rightarrow y{:}\Gamma \qquad x{:}\Delta \Rightarrow y{:}\Delta \qquad \dfrac{x{:}\Gamma \Rightarrow y{:}\Gamma \qquad x{:}\Delta \Rightarrow y{:}\Delta \qquad x{:}X \Rightarrow y{:}X}{x{:}X, yx{:}[X, \Gamma \Rightarrow \Delta], x{:}\Gamma, \Rightarrow y{:}\Delta}\,[\Rightarrow]\text{L}}{yx{:}[\Gamma \Rightarrow \Delta, X], yx{:}[X, \Gamma \Rightarrow \Delta], x{:}\Gamma, \Rightarrow y{:}\Delta}\,[\Rightarrow]\text{L}}{yx{:}[\Gamma \Rightarrow \Delta, X], yx{:}[X, \Gamma \Rightarrow \Delta] \Rightarrow xy{:}[\Gamma \Rightarrow \Delta]}\,[\Rightarrow]\text{R}}{\Rightarrow yxxy{:}[[\Gamma \Rightarrow \Delta, X], [X, \Gamma \Rightarrow \Delta] \Rightarrow [\Gamma \Rightarrow \Delta]]}\,[\Rightarrow]\text{R}$$

□

Again, while these observations are already made by [1], our proofs thereof are obtained using proof analysis. In particular, the key ingredients are our cut-elimination theorem in 3.8 and the label transformation lemmas. Our proofs are thus arguably more elegant and easier to read than those presented involving semantic reasoning by [1].

In addition to establishing the fact that each stage is a reflexive "metainferential" logic which concerns a transitive "metainferential" logic, we can also establish that each stage is inconsistent, again using proof analysis.

Lemma 4.6 (Inconsistency is inheritable in $\mathbb{ST}$) *For every $xy \in \mathbb{ST}$, if xy is inconsistent then $yxxy$ is inconsistent.*

Proof The following pieces of reasoning are admissible:

$$\dfrac{\dfrac{\dfrac{\Rightarrow xy{:}[X \Rightarrow\ ]}{x{:}X \Rightarrow}\ \text{Inv. of } [\Rightarrow]\text{R}}{yx{:}[\ \Rightarrow X] \Rightarrow}\ [\Rightarrow]\text{L}}{\Rightarrow yxxy{:}[[\ \Rightarrow X] \Rightarrow\ ]}\ [\Rightarrow]\text{R} \qquad \dfrac{\Rightarrow xy{:}[\ \Rightarrow X]}{\Rightarrow yxxy{:}[\ \Rightarrow [\ \Rightarrow X]]}\ [\Rightarrow]\text{R}$$

□

With st being inconsistent, the following proposition follows:

Proposition 4.7 *For every $xy \in \mathbb{ST}$, xy is inconsistent.*

Finally, we obtain thus the following:

Proposition 4.8 (Non-classicality of every stage) *For every $yxxy \in \mathbb{ST}$ the following sequents are derivable for some expression X:*

$$\Rightarrow yxxy{:}[\ \Rightarrow [\ \Rightarrow X]] \qquad \Rightarrow yxxy{:}[\ \Rightarrow [X \Rightarrow\ ]]$$

$$\Rightarrow yxxy{:}[[\Gamma \Rightarrow \Delta, X], [X, \Gamma \Rightarrow \Delta]\ \Rightarrow [\Gamma \Rightarrow \Delta]]$$

The conclusion should thus not be that each level is classical to a greater extent than the previous level as we transcend up in the hierarchy as if the next level takes us closer to classical logic (even if we never reach classical logic), but rather that each stage is non-classical to the same extent. This should not be too surprising considering how each stage in the hierarchy is a st-ish logic for the previous stage obtained by what amounts to a strict-tolerant standard for that stage.

5 A "fully classical" Metainferential Logic?

Following the observation that no stage in the $\mathbb{ST}$-hierarchy is classical, [2] presents a way to "recovers every classically valid metainference of every level". This consists in defining a collection of metainferences $\mathbb{ST}\omega$ of any level in such a way that we can understand it "as the union of" each $x \in \mathbb{ST}$. In this way then, we are supposed to obtain a "fully classical" (metainferential) logic. In the concluding remarks in [2], it is observed that there still is plenty work to do in relation to these logics and truth theories. For example, it seems not easy to imagine a proof theory for them [2].

As it turns out, it is straightforward to extend the HST calculus to a sequent calculus for $\mathbb{ST}\omega$. Let us thus proceed with the definitions. In [2], a definition of $\mathbb{ST}\omega$ is provided which is equivalent to the following:

Definition 5.1 Suppose that X is a metainference of level $j > 0$. Then $X \in \mathbb{ST}\omega$ if and only if for every $\mathcal{V}$, $\mathcal{V} \Vdash_{\mathrm{ST}_j} X$.

This definition can fruitfully be split into two stages as follows:

Definition 5.2

(a) Suppose that X is a metainference of level $j > 0$ and $\mathcal{V}$ is a strong Kleene valuation. Then $\mathcal{V} \Vdash_{\omega} X$ if and only if $\mathcal{V} \Vdash_{\mathrm{ST}_j} X$.
(b) $X \in \mathbb{ST}\omega$ if and only if for every strong Kleene valuation $\mathcal{V}$, $\mathcal{V} \Vdash_{\omega} X$.

We proceed now to define the extended sequent calculus. To that purpose we first extend the definition of sequent expressions.

Definition 5.3 (Extended typed sequent expression) If $x : X$ is a sequent expression of type (α, x) where $\alpha > 0$, then $x : X$ and $\omega : X$ are extended sequent expressions.

It follows that $\omega{:}A$ where A is a formula is *not* an extended sequent expression whereas $\omega{:}[\Gamma \Rightarrow \Delta]$ where $[\Gamma \Rightarrow \Delta]$ is a metainference of any level > 0 is an extended sequent expression.

Definition 5.4 (Extended HST calculus) Let EHST be the sequent calculus obtained by expanding the HST calculus with the following rules where $x \in \mathbb{ST}$, $x \neq st$ and X is a metainference of level x:

$$\frac{x{:}X, \Gamma \Rightarrow \Delta}{\omega{:}X, \Gamma \Rightarrow \Delta}\,\omega\mathrm{L} \qquad \frac{\Gamma \Rightarrow \Delta, x{:}X}{\Gamma \Rightarrow \Delta, \omega{:}X}\,\omega\mathrm{R}$$

The adequacy of the rules for ω is immediate by considering clause (a) in definition 5.2. The various lemmas and theorems for HST transfers to EHST by appropriately extending the various definitions.

Lemma 5.5 (Weakening and contraction) *The following rules are HP-admissible in EHST:*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'} \qquad \frac{x{:}X, x{:}X, \Gamma \Rightarrow \Delta}{x{:}X, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, x{:}X, x{:}X}{\Gamma \Rightarrow \Delta, x{:}X}$$

Lemma 5.6 (Inversion) *The inversion of each primitive EHST-rule is HP-admissible in EHST.*

Theorem 5.7 (Cut) *The following rule is admissible in EHST:*

$$\frac{\Gamma \Rightarrow \Delta, x{:}X \qquad x{:}X, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

As above in the HST-calculus, cut-elimination does not imply that a metainferential logic defined with the calculus is transitive. A reasonable question to ask now is thus whether ω really is a "fully classical" metainferential logic?

Proposition 5.8 *Reasoning in ω about metainferential levels in $\mathbb{ST}$ is not transitive.*

Proof By proposition 4.6 it follows that for any metainferential level, there is an X such that:

$$\Rightarrow \omega{:}[\ \Rightarrow X] \qquad \Rightarrow \omega{:}[X \Rightarrow\ ]$$

By cut-elimination, it follows that the sequent

$$\omega{:}[\Gamma \Rightarrow \Delta, X], \omega{:}[X, \Gamma \Rightarrow \Delta] \Rightarrow \omega{:}[\Gamma \Rightarrow \Delta]$$

is *not* derivable. □

It follows that ω is not "more" classical than anything in $\mathbb{ST}$. In fact, we can show that ω is st in disguise.

Definition 5.9 (Notation) Let $X \longleftrightarrow Y$ mean that the following rules are admissible:

$$\frac{X, \Gamma \Rightarrow \Delta}{Y, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, X}{\Gamma \Rightarrow \Delta, Y}$$

With the rules for introducing a metainference having the same shape as the rules for introducing a formula of the form $(A_0 \wedge \ldots \wedge A_n) \supset (B_0 \vee \ldots \vee B_m)$ where $\wedge$ and $\supset$ are defined as $\neg(\neg A \vee \neg B)$ and $\neg A \vee B$ respectively, it is relatively evident that we can engage in a process of flatting metainferences. For example, it follows by corollary 3.17 that the following claims hold where $\Gamma \supset \Delta$ abbreviates $(A_0 \wedge \ldots \wedge A_n) \supset (B_0 \vee \ldots \vee B_m)$ where $A_i \in \Gamma$ and $B_i \in \Delta$:

$$st{:}[\Gamma \Rightarrow \Delta] \longleftrightarrow t{:}\Gamma \supset \Delta$$

$$ts{:}[\Gamma \Rightarrow \Delta] \longleftrightarrow s{:}\Gamma \supset \Delta$$

With that established we proceed to observe the following:

$$tsst{:}[[\Gamma_1 \Rightarrow \Delta_1], \ldots, [\Gamma_n \Rightarrow \Delta_n] \Rightarrow [\Gamma_1' \Rightarrow \Delta_1'], \ldots, [\Gamma_m' \Rightarrow \Delta_m']]$$

$$\longleftrightarrow$$

$$st{:}[\Gamma_1 \supset \Delta_1, \ldots, \Gamma_n \supset \Delta_n \Rightarrow \Gamma_1' \supset \Delta_1', \ldots, \Gamma_m' \supset \Delta_m']$$

We shall here illustrate how to establish the left rule. First, we note that for each $[\Gamma_i \Rightarrow \Delta_i]$:

$$\cfrac{tsst{:}[[\Gamma_1 \Rightarrow \Delta_1], \ldots, [\Gamma_n \Rightarrow \Delta_n] \Rightarrow [\Gamma_1' \Rightarrow \Delta_1'], \ldots, [\Gamma_m' \Rightarrow \Delta_m']] \Rightarrow}{\cfrac{\Rightarrow ts : [\Gamma_i \Rightarrow \Delta_i]}{\Rightarrow s : \Gamma_i \supset \Delta_i}}$$

Correspondingly, we obtain that for each $[\Gamma_i' \Rightarrow \Delta_i']$:

$$\dfrac{\dfrac{tsst{:}[[\Gamma_1 \Rightarrow \Delta_1], \ldots, [\Gamma_n \Rightarrow \Delta_n] \Rightarrow [\Gamma_1' \Rightarrow \Delta_1'], \ldots, [\Gamma_m' \Rightarrow \Delta_m']] \Rightarrow}{st : [\Gamma_i' \Rightarrow \Delta_i'] \Rightarrow}}{t : \Gamma_i' \supset \Delta_i' \Rightarrow}$$

The desired conclusion is now obtained by the introduction an st-metainference.

Let us thus provide some definitions and a more general result:

Definition 5.10 (Hierarchical reduction) If $[\Gamma \Rightarrow \Delta]$ is a metainference of level $n + 1$, then $[\Gamma_\supset \Rightarrow \Delta_\supset]$ is the metainference of level n obtained by replacing each metainference $[\Gamma' \Rightarrow \Delta']$ of level 0 in $[\Gamma \Rightarrow \Delta]$ with the formula $\Gamma' \supset \Delta'$.

The above transformation of a $tsst$-metainference to a st-inference illustrates this definition in action.

Proposition 5.11 *For every $yxxy \in \mathbb{ST}$, $yxxy{:}[\Gamma \Rightarrow \Delta] \longleftrightarrow xy{:}[\Gamma_\supset \Rightarrow \Delta_\supset]$*

Proof By induction on the levels. □

Definition 5.12 If $[\Gamma \Rightarrow \Delta]$ is a metainference of level n then $[\Gamma^0_\supset \Rightarrow \Delta^0_\supset]$ is the result of performing an hierarchical reduction n times on $[\Gamma \Rightarrow \Delta]$.

A metainference of the form $[\Gamma^0_\supset \Rightarrow \Delta^0_\supset]$ is thus simply an inference.

Theorem 5.13 (ω is st in disguise)

$$\omega{:}[\Gamma \Rightarrow \Delta] \longleftrightarrow st{:}[\Gamma^0_\supset \Rightarrow \Delta^0_\supset]$$

Proof Iterated applications of proposition 5.11. □

To make sense of this theorem, it is useful to observe that ω does not, despite its label, represent a limit. Instead, it is simply a collection of metainferences of various levels. Every metainference of ω will be of a particular finite level, and can thus be reduced according to proposition 5.11. Moreover, and with that in mind, this result shouldn't actually be particularly surprising considering how the $\mathbb{ST}$-hierarchy and ω is defined and how for example $tsst$ is the metainferential analogue of st since ts is a stricter standard than st in the same way as s is a stricter standard than t.

6 Imitating *st* as Metatheory

Since we have used the proof-theoretic tool developed in this paper to illustrate problems with the interpretation of the $\mathbb{ST}$-hierarchy proposed by [1] and [2], it seems appropriate to use the concluding remarks of this paper to engage in some speculation about whether we can utilise the results obtained with the tool to provide an alternative interpretation of the $\mathbb{ST}$-hierarchy, and whether this could be used to make sense of st.

The flatting of the $\mathbb{ST}$-hierarchy into st suggests that $\mathbb{ST}$-hierarchy is merely a metainferential twist on the st-phenomenon, as if the $\mathbb{ST}$-hierarchy doesn't tell us anything that we couldn't already express within st, and one could thus argue that the hierarchy is somehow superfluous.

On the other hand, precisely because the $\mathbb{ST}$-hierarchy is representable within st, it is perhaps not too incredulous to suggest that the $\mathbb{ST}$-hierarchy can actually tell us something about how it would be to reason within st about st, that is, how it would be to use st as metatheory for st, and thus what st looks like from the perspective of st. Contrary to the received wisdom and thus awkwardly enough, this amounts to looking at the material conditional of st to learn more about st. While some initial scepticism is certainly warranted, we have shown that we can represent the $\mathbb{ST}$-hierarchy with the material conditional in st, so if the $\mathbb{ST}$-hierarchy tells us something about st as a metatheory for st, then surely valid inferences about the material conditional in st would tell us what st as metatheory would tell us about st.

With that in mind we can reason as follows. Under the assumption that the valid inferences about the material conditional in st represent the metainferences that hold of st within st, st is transitive according to st. In other words, according to st it is the case that the inferences from anything to the liar and from the liar to anything together imply that anything follows from anything. This observation corresponds to that made by [14] with regard to a validity predicate defined in st along the lines of the material conditional. From the perspective of a classical metatheory according to which st is non-transitive, that claim is false, and [14] considered their observation as presenting a problem for such an approach to defining a validity predicate in st. However, we are no longer supposed to think of st from the perspective of classical logic. Instead, we are considering st from the perspective of st, and what if st really is transitive when st is the metatheory for st? After all, st is non-transitive from within a classical metatheory because assuming otherwise leads to inconsistency. With st as metatheory, however, the inconsistency is not an issue, and it follows that st can be transitive according to st.

Continuing down the rabbit hole then, we also note that st tells us about st that anything implies the liar and that the liar implies anything. Now, does it follow, since those facts imply that anything implies anything, that anything implies anything? On the one hand, we can find formulas A and B such that $\neg(A \supset B)$ follows from no premises, a fact which is reasonably interpreted as that it is not the case that anything implies anything. On the other hand, we have cases such as $\neg(\lambda \supset \lambda)$ and $(\lambda \supset \lambda)$ which are both valid according to st. Taking each statement to represent an inference as suggested above, there are thus inferences that are both valid and not valid in st according to st. From the perspective of st then, it seems reasonable to think of transitivity of entailment in the same way; that it is (metainferentially) valid, but there are counterexamples. While a classically minded referee would certainly protest at this point since being valid and having a counterexample are supposed to be 'at least contraries', such a protest would just illustrate the uphill battle faced by an advocate of paraconsistent metatheory as generalised to include metainferences. Indeed, could the moral be that you're free to apply transitivity when reasoning within *st* about *st* as long as you're willing to accept that the conclusion you draw is a dialetheia?

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