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Automorphism groups of pseudo H-type algebras[☆]

Kenro Furutani[a,1], Irina Markina[b,*]

[a] *Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba (278-8510), Japan*

[b] *Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway*

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ABSTRACT

In the present paper we determine the group of automorphisms of pseudo H-type Lie algebras, that are two step nilpotent Lie algebras closely related to the Clifford algebras $\mathrm{Cl}(\mathbb{R}^{r,s})$.

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* Corresponding author.

E-mail addresses: furutani_kenro@ma.noda.tus.ac.jp (K. Furutani), irina.markina@uib.no (I. Markina).

1 New affiliation: Advanced Mathematical Institute, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka City 558-8585, Japan.

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1. Introduction

The pseudo H-type Lie algebras are two step nilpotent Lie algebras $\mathfrak{n}_{r,s}(U) = (U \oplus \mathbb{R}^{r,s}, [\,.\,,.\,])$ endowed with a non-degenerate scalar product $\langle\,.\,,.\,\rangle_U + \langle\,.\,,.\,\rangle_{\mathbb{R}^{r,s}}$, where U is the orthogonal complement to the center $\mathbb{R}^{r,s}$ and the commutation relations are defined by

$$\langle\, J_z u, v \,\rangle_U = \langle\, z, [u,v] \,\rangle_{\mathbb{R}^{r,s}}, \quad u, v \in U,\ z \in \mathbb{R}^{r,s}.$$

Here $J_z \in \mathrm{End}(U)$, $J_z^2 = -\langle\, z, z \,\rangle_{\mathbb{R}^{r,s}} \mathrm{Id}_U$ is the defining map for the representation (J, U) of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$. These Lie algebras are the natural generalisation of the H (eisenberg)-type algebras $\mathfrak{n}_{r,0}(U)$, introduced in [30,31], and they are related to the Clifford algebras $\mathrm{Cl}(\mathbb{R}^{r,s})$ generated by a vector space endowed with the quadratic form of an arbitrary signature (r, s). The pseudo H-type Lie algebras were introduced in [10,27] and studied in [11,12,23–25]. These types of algebras arise in study of parabolic subgroups with square integrable nilradicals [46], as maximal transitive prolongation of super Poincaré algebras [1,2] and the nilpotent part of 2-gradings for semisimple Lie algebras [26,28]. These algebras are some special examples of metric Lie algebras, studied in [4,15,19,20,22]. The pseudo H-type Lie groups is a fruitful source for study of geometry with non-holonomic constrains or nilmanifolds [16,18,21,34], symmetric spaces and harmonic spaces [7,8,14,39], differential operators on Lie groups [6,9,38,40].

The main goal of the present paper is to describe the automorphism groups $\mathrm{Aut}(\mathfrak{n}_{r,s}(U))$ of pseudo H-type algebras $\mathfrak{n}_{r,s}(U)$ depending on the integer parameters (r, s) and the structure of the representation U of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{r,s})$. The automorphism groups preserving metric on $\mathfrak{n}_{r,0}(U)$ were studied in [41,42] and the general automorphism groups of $\mathfrak{n}_{r,0}(U)$ were first described in [44], see also [5,32,45]. Some attempt for study of $\mathrm{Aut}(\mathfrak{n}_{0,1}(U))$ was done in [17]. An automorphism group $\mathrm{Aut}(\mathfrak{n}_{r,s}(U))$ is decomposed into an abelian subgroup of dilatations, the group $\mathrm{Hom}(U, \mathbb{R}^{r,s})$, the group generated by $\mathrm{Pin}(r, s)$, and a group $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ that acts trivially on the center $\mathbb{R}^{r,s}$, see Section 3.2. We determine the group $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ in terms of classical groups over $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$. The structure of the paper is the following. In Sections 2 and 3 we recall the necessary material about Clifford algebras, pseudo H-type Lie algebras, and classical groups. Section 4 is dedicated to the determination of $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$. The main result is contained in Table 3. In Appendix one can find a recapture over the relation between the isomorphic pseudo H-type Lie algebras and their groups of automorphisms.

2. Clifford algebras

2.1. Definition of Clifford algebras

We denote by $\mathbb{R}^{r,s}$ the space $\mathbb{R}^m$, $r + s = m$, with the non-degenerate quadratic form $Q_{r,s}(z) = \sum_{i=1}^{r} z_i^2 - \sum_{j=1}^{s} z_{r+j}^2$, $z \in \mathbb{R}^m$ of the signature (r, s). The non-degenerate bi-linear form obtained from $Q_{r,s}$ by polarization is denoted by $\langle \cdot , \cdot \rangle_{r,s}$ and we call it a *scalar product*. A vector $z \in \mathbb{R}^{r,s}$ is called *positive* if $\langle z, z \rangle_{r,s} > 0$, *negative* if $\langle z, z \rangle_{r,s} < 0$, and *null* if $\langle z, z \rangle_{r,s} = 0$. We use the orthonormal basis $\{z_1, \dots, z_r, z_{r+1}, \dots, z_{r+s}\}$ for $\mathbb{R}^{r,s}$, where $\langle z_i, z_i \rangle_{r,s} = 1$ for $i = 1, \dots, r$, $\langle z_j, z_j \rangle_{r,s} = -1$ for $j = r+1, \dots, r+s$ and $\langle z_i, z_j \rangle = 0$ for $i \neq j$.

Let $\mathrm{Cl}_{r,s}$ be the real Clifford algebra generated by $\mathbb{R}^{r,s}$, that is the quotient of the tensor algebra

$$\mathcal{T}(\mathbb{R}^{r+s}) = \mathbb{R} \oplus \left(\mathbb{R}^{r+s}\right) \oplus \left(\overset{2}{\otimes} \mathbb{R}^{r+s}\right) \oplus \left(\overset{3}{\otimes} \mathbb{R}^{r+s}\right) \oplus \cdots ,$$

divided by the two-sided ideal $I_{r,s}$ which is generated by the elements of the form $z \otimes z + \langle z, z \rangle_{r,s}$, $z \in \mathbb{R}^{r+s}$. The explicit determination of the Clifford algebras is given in [3] and they are isomorphic to the matrix algebras $M(n, \mathbb{F})$ or $M(n, \mathbb{F}) \oplus M(n, \mathbb{F})$, for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, where the size n is determined by r and s, see for instance [36].

Given an algebra homomorphism $\widehat{J}\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(U)$, we call the space U a Clifford module and the operator J_ϕ a *Clifford action* or a *representation map* of an element $\phi \in \mathrm{Cl}_{r,s}$. If there is a map

$$\begin{aligned} J\colon\ \mathbb{R}^{r,s} &\to \mathrm{End}(U) \\ z &\mapsto \quad J_z, \end{aligned}$$

satisfying $J_z^2 = -\langle z, z \rangle_{r,s} \mathrm{Id}_U$ for an arbitrary $z \in \mathbb{R}^{r,s}$, then J can be uniquely extended to the algebra homomorphism $\widehat{J}$ by the universal property, see, for instance [29,35,36]. We recommend to read [33] for a wonderful introduction to the Clifford algebras $\mathrm{Cl}_{r,s}$. Even though the representation matrices of the Clifford algebras $\mathrm{Cl}_{r,s}$, and the Clifford modules U are given over the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we refer to $\mathrm{Cl}_{r,s}$ as a real algebra and U as a real vector space.

If $r - s \not\equiv 3 \,(\mathrm{mod}\ 4)$, then $\mathrm{Cl}_{r,s}$ is a simple algebra. In this case, there is only one irreducible module $U = V^{r,s}_{irr}$. If $r - s \equiv 3 \,(\mathrm{mod}\ 4)$, then the algebra $\mathrm{Cl}_{r,s}$ is not simple, and there are two non-equivalent irreducible modules. They can be distinguished by the action of the ordered volume form $\Omega^{r,s} = \prod_{k=1}^{r+s} z_k$. In fact, the elements $\phi = \frac{1}{2}\left(\mathbf{1} \mp \Omega^{r,s}\right)$ act as an identity operator on the Clifford module, so $J_{\Omega^{r,s}} \equiv \pm \mathrm{Id}_U$. Thus we denote by $V^{r,s}_{irr;\pm}$ two non-equivalent irreducible Clifford modules on which the action of the volume form is given by $J_{\Omega^{r,s}} = \prod_{k=1}^{r+s} J_{z_k} \equiv \pm \mathrm{Id}$.

Proposition 2.1.1. *[36, Proposition 4.5] Clifford modules are completely reducible; any Clifford module U can be decomposed into irreducible modules:*

$$U = \begin{cases} \overset{p}{\oplus}\, V^{r,s}_{irr}, & \textit{if} \quad r - s \not\equiv 3 \ (mod\ 4), \\ \left(\overset{p_+}{\oplus}\, V^{r,s}_{irr;+}\right) \oplus \left(\overset{p_-}{\oplus}\, V^{r,s}_{irr;-}\right), & \textit{if} \quad r - s \equiv 3 \ (mod\ 4). \end{cases} \tag{2.1}$$

The numbers p, p_+, p_- are uniquely determined by the dimension of U.

The first type of module U in (2.1) is called *isotypic* and the second one is called *non-isotypic*. The Clifford algebras possess the periodicity properties [3]:

$$\mathrm{Cl}_{r,s}\otimes \mathrm{Cl}_{0,8}\cong \mathrm{Cl}_{r,s+8},\quad \mathrm{Cl}_{r,s}\otimes \mathrm{Cl}_{8,0}\cong \mathrm{Cl}_{r+8,s},\quad \mathrm{Cl}_{r,s}\otimes \mathrm{Cl}_{4,4}\cong \mathrm{Cl}_{r+4,s+4}\,. \tag{2.2}$$

The Clifford algebras $\mathrm{Cl}_{\mu,\nu}$ for $(\mu,\nu)\in\{(8,0),(0,8),(4,4)\}$ are isomorphic to the matrix algebra $M(16,\mathbb{R})$.

2.2. Admissible modules

Definition 2.2.1. [10] A module U of the Clifford algebra $\mathrm{Cl}_{r,s}$ is called admissible if there is a scalar product $\langle\cdot\,,\cdot\rangle_U$ on U such that

$$\langle J_z x, y\rangle_U + \langle x, J_z y\rangle_U = 0,\quad \text{for all }\; x,y\in U \text{ and } z\in\mathbb{R}^{r,s}. \tag{2.3}$$

We write $(U,\langle\cdot\,,\cdot\rangle_U)$ for an admissible module to emphasise the scalar product $\langle\cdot\,,\cdot\rangle_U$ and call it an *admissible scalar product*. We collect properties of admissible modules in several propositions.

Proposition 2.2.2. *Let $\mathrm{Cl}_{r,s}$ be the Clifford algebra generated by the space $\mathbb{R}^{r,s}$.*

(1) *If $\langle\cdot\,,\cdot\rangle_U$ is an admissible scalar product on a Clifford module U for $\mathrm{Cl}_{r,s}$, then $K\langle\cdot\,,\cdot\rangle_U$ is also admissible for any constant $K\neq 0$. We can assume that $K=\pm 1$ by normalisation of the scalar products.*
(2) *Let $(U,\langle\cdot\,,\cdot\rangle_U)$ be an admissible module for $\mathrm{Cl}_{r,s}$ and let $(U_1,\langle\cdot\,,\cdot\rangle_{U_1})$ be such that U_1 is a submodule of U and $\langle\cdot\,,\cdot\rangle_{U_1}$ is a non-degenerate restriction of $\langle\cdot\,,\cdot\rangle_U$ to U_1. Then the orthogonal complement $U_1{}^{\perp}=\{x\in U \mid \langle x,y\rangle_U=0, \text{ for all } y\in U_1\}$ with the scalar product obtained by the restriction of $\langle\cdot\,,\cdot\rangle_U$ to $U_1{}^{\perp}$ is also an admissible module.*
(3) *Condition (2.3) and the property $J_z^2=-\langle z,z\rangle_{r,s}\mathrm{Id}_U$ imply*

$$\langle J_z x, J_z y\rangle_U = \langle z,z\rangle_{r,s}\langle x,y\rangle_U\,. \tag{2.4}$$

(4) *If $s>0$, then any admissible module $(U,\langle\cdot\,,\cdot\rangle_U)$ of $\mathrm{Cl}_{r,s}$ is neutral, i.e.,* $\dim U=2l$, *$l\in\mathbb{N}$, and U is isometric to $\mathbb{R}^{l,l}$, see* [10, Proposition 2.2].
(5) *If $s=0$, then any Clifford module of $\mathrm{Cl}_{r,0}$ is admissible and it is isometric either to $\mathbb{R}^{l,0}$ or to $\mathbb{R}^{0,l}$, see [29, Theorem 2.4].*

Proposition 2.2.3 describes the relation between irreducible and admissible modules. An admissible module of the minimal possible dimension is called a *minimal admissible module*.

Proposition 2.2.3. *[10, Theorem 3.1][24, Proposition 1] Let $\mathrm{Cl}_{r,s}$ be the Clifford algebra generated by the space $\mathbb{R}^{r,s}$.*

(1) *If $s = 0$, then any irreducible Clifford module is minimal admissible with respect to a positive definite or a negative definite scalar product.*

(2) *If $r-s \equiv 0, 1, 2 \mod 4$, $s > 0$, then a unique irreducible module $V^{r,s}_{irr}$ is not necessary admissible. The following situations are possible:*
   (2-1) *The irreducible module $V^{r,s}_{irr}$ is minimal admissible or,*
   (2-2) *The irreducible module $V^{r,s}_{irr}$ is not admissible, but the direct sum $V^{r,s}_{irr} \oplus V^{r,s}_{irr}$ is minimal admissible.*

(3) *If $r - s \equiv 3 \mod 4$, $s > 0$, then for two non-equivalent irreducible modules $V^{r,s}_{irr;\pm}$ the following can occur:*
   (3-1) *If $r \equiv 3 \mod 4$, $s \equiv 0 \mod 4$, or*
   $r \equiv 1 \mod 8$, $s \equiv 6 \mod 8$, *or*
   $r \equiv 5 \mod 8$, $s \equiv 2 \mod 8$
   then each irreducible module $V^{r,s}_{irr;\pm}$ is minimal admissible.
   (3-2) *Otherwise none of the irreducible modules $V^{r,s}_{irr;\pm}$ is admissible.*
      (3-2-1) *If $r \equiv 1 \mod 8$, $s \equiv 2 \mod 8$, or*
      $r \equiv 5 \mod 8$, $s \equiv 6 \mod 8$
      then $V^{r,s}_{irr;+} \oplus V^{r,s}_{irr;+}$, $V^{r,s}_{irr;-} \oplus V^{r,s}_{irr;-}$ are minimal admissible modules, and the module $V^{r,s}_{irr;+} \oplus V^{r,s}_{irr;-}$ is not admissible.
      (3-2-2) *If s is odd, then the module $V^{r,s}_{irr;+} \oplus V^{r,s}_{irr;-}$ is minimal admissible and neither $V^{r,s}_{irr;+} \oplus V^{r,s}_{irr;+}$ nor $V^{r,s}_{irr;-} \oplus V^{r,s}_{irr;-}$ is admissible.*

2.3. System of involutions $PI_{r,s}$ and common 1-eigenspace $E_{r,s}$

2.3.1. Mutually commuting isometric involutions

Recall that a linear map Λ defined on a vector space U with a scalar product $\langle \cdot, \cdot \rangle_U$ is called *symmetric* with respect to the scalar product $\langle \cdot, \cdot \rangle_U$, if $\langle \Lambda x, y \rangle_U = \langle x, \Lambda y \rangle_U$. We say that Λ is *positive* if it maps positive vectors to positive vectors and negative vectors to negative vectors and Λ is *negative* if it reverses the positivity and negativity of the vectors. Let J_{z_i} be representation maps for an orthonormal basis $\{z_1, \ldots, z_{r+s}\}$ of $\mathbb{R}^{r,s}$. The simplest positive involutions, written as a product of the maps J_{z_i}, have one of the following forms:

$$\begin{cases} \text{type (1)}: \ \mathcal{P}_1 = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}} J_{z_{i_4}}, & \text{all } z_{i_k} \text{ are positive,} \\ \text{type (2)}: \ \mathcal{P}_1 = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}} J_{z_{i_4}}, & \text{all } z_{i_k} \text{ are negative,} \\ \text{type (3)}: \ \mathcal{P}_2 = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}} J_{z_{i_4}}, & \text{two } z_{i_l} \text{ are positive and two are negative,} \\ \text{type (4)}: \ \mathcal{P}_3 = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}}, & \text{all three } z_{i_k} \text{ are positive,} \\ \text{type (5)}: \ \mathcal{P}_4 = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}}, & \text{one } z_{i_l} \text{ is positive and two are negative.} \end{cases}$$

For a given minimal admissible module $V^{r,s}_{min}$, we denote by $PI_{r,s}$ a set consisting of possible maximal number of mutually commuting symmetric positive involutions of types (1)-(5) such that none of them is a product of other involutions in $PI_{r,s}$. The set $PI_{r,s}$

Table 1
Dimensions of minimal admissible modules.

8	$16^{\pm}$	$32^{\pm}$	$64^{\pm}$	$64^{\pm}_{\times 2}$	$128^{\pm}$	$128^{\pm}$	$128^{\pm}$	$128^{\pm}_{\times 2}$	$256^{\pm}$
7	16^{N}	32^{N}	$\mathbf{64}^{N}$	$64^{\pm}$	$\mathbf{128}^{N}$	$\mathbf{128}^{N}$	$\mathbf{128}^{N}$	$128^{\pm}$	256^{N}
6	16^{N}	$16^{N}_{\times 2}$	32^{N}	$32^{\pm}$	$\mathbf{64}^{N}$	$\mathbf{64}^{N}_{\times 2}$	$\mathbf{128}^{N}$	$128^{\pm}$	256^{N}
5	$\mathbf{16}^{N}$	16^{N}	16^{N}	$16^{\pm}$	$\mathbf{32}^{N}$	$\mathbf{64}^{N}$	$\mathbf{128}^{\pm}$	128^{N}	$\mathbf{256}^{N}$
4	$8^{\pm}$	$8^{\pm}$	$8^{\pm}$	$8^{\pm}_{\times 2}$	$16^{\pm}$	$32^{\pm}$	$64^{\pm}$	$64^{\pm}_{\times 2}$	$128^{\pm}$
3	$\mathbf{8}^{N}$	$\mathbf{8}^{N}$	$\mathbf{8}^{N}$	$8^{\pm}$	16^{N}	32^{N}	$\mathbf{64}^{N}$	$64^{\pm}$	$\mathbf{128}^{N}$
2	$\mathbf{4}^{N}$	$\mathbf{4}^{N}_{\times 2}$	$\mathbf{8}^{N}$	$8^{\pm}$	16^{N}	$16^{N}_{\times 2}$	32^{N}	$32^{\pm}$	$\mathbf{64}^{N}$
1	$\mathbf{2}^{N}$	$\mathbf{4}^{N}$	$\mathbf{8}^{N}$	$8^{\pm}$	$\mathbf{16}^{N}$	16^{N}	16^{N}	$16^{\pm}$	$\mathbf{32}^{N}$
0	$1^{\pm}$	$2^{\pm}$	$4^{\pm}$	$4^{\pm}_{\times 2}$	$8^{\pm}$	$8^{\pm}$	$8^{\pm}$	$8^{\pm}_{\times 2}$	$16^{\pm}$
s/r	0	1	2	3	4	5	6	7	8

is not unique, while the number of involutions $p_{r,s} = \#\{PI_{r,s}\}$ in $PI_{r,s}$ is unique for the given signature (r, s). The set $PI_{r,s}$ can be ordered, if necessary, in such a way that at most one involution of the type (4) or (5) is included in $PI_{r,s}$ and it is the last one. We denote by $PI^*_{r,s}$ the reduced system of involution, that contains only involutions of type (1)-(3). In the case when there are no involutions of type (4) or (5), we have $PI_{r,s} = PI^*_{r,s}$ and we write $PI_{r,s}$ if no confusion arises.

We define the subspace $E_{r,s}$ of a minimal admissible module $V^{r,s}_{min}$ by

$$\begin{aligned} E_{r,s} = \{v \in V^{r,s}_{min} \mid \ & P_i v = v, \quad i \leq p_{r,s}, \\ & r - s \neq 3 \mod 4, \text{ or } r - s \equiv 3 \mod 4 \text{ with odd } s\}, \end{aligned}$$

$$E_{r,s} = \{v \in V^{r,s}_{min} \mid \ P_i v = v, \quad i \leq p_{r,s} - 1, \quad r - s \equiv 3 \mod 4 \text{ with even } s\}.$$

We call $E_{r,s}$ the "*common 1-eigenspace*" for the system of involutions $PI_{r,s}$. The space $E_{r,s}$ is the minimal subspace of $V^{r,s}_{min}$ that is invariant under the action of all the involutions from $PI_{r,s}$. The system of involutions $PI_{r,s}$ does not depend on the scalar product on the admissible modules $V^{r,s}_{min} = (V^{r,s}_{min}, \langle \cdot , \cdot \rangle_{V^{r,s}_{min}})$ and $V^{r,s}_{min} = (V^{r,s}_{min}, -\langle \cdot , \cdot \rangle_{V^{r,s}_{min}})$. Nevertheless, the restrictions of the admissible scalar products on the respective $E_{r,s}$ will have opposite signs. Namely,

(1) the restriction of the admissible scalar product to $E_{r,s}$ is sign definite for $r \equiv 0, 1, 2 \mod 4$ and $s \equiv 0 \mod 4$ or for $r \equiv 3 \mod 4$ and arbitrary s;
(2) otherwise the restriction of the admissible scalar product to the common 1-eigenspaces $E_{r,s}$ is neutral,

see Table 1 and [25, Section 2.6] for details of the proof.

The dimensions of minimal admissible modules need to be determined only for basic cases

$$\begin{aligned}&(r,s) \text{ for } 0\le r\le 7 \text{ and } 0\le s\le 3,\\ &(r,s) \text{ for } 0\le r\le 3 \text{ and } 4\le s\le 7, \text{ and}\\ &(r,s)\ \in\{(8,0),(0,8),(4,4)\}.\end{aligned}\tag{2.5}$$

We use periodicity property (2.2) to find the dimension of a minimal admissible module $\dim(V_{min}^{r+\mu,s+\nu}) = \dim(V_{min}^{r,s})\cdot\dim(V_{min}^{\mu,\nu}) = 16\dim(V_{min}^{r,s})$ provided that $V_{min}^{r,s}$ is minimal admissible and $(\mu,\nu)\in\{(8,0),(0,8),(4,4)\}$. Moreover $\dim(V_{min}^{r,s}) = 2^{r+s-p_{r,s}}$. We describe the number and the dimension of minimal admissible modules $V_{min}^{r,s}$ in Table 1.

We make the following comments to Table 1:

(1) We use the bold characters when $\dim(V_{\min}^{r,s}) = 2\dim(V_{irr}^{r,s})$, see Proposition 2.2.3, statements (2-2) and (3-2).
(2) Writing the subscript "×2" we mean that the Clifford algebra has two minimal admissible modules corresponding to the non-equivalent irreducible modules, see Proposition 2.2.3, statements (3-1) and (3-2-1).
(3) The upper index "N" means that the scalar product restricted to the common 1-eigenspace $E_{r,s}$ is neutral.
(4) The upper index "±" indicates that the scalar product restricted to the common 1-eigenspace $E_{r,s}$ of the system $PI_{r,s}$ is sign definite.

From now on we use ± or N as the upper index and write $V_{min}^{r,s;+}$ ($V_{min}^{r,s;-}$) or $V_{min}^{r,s;N}$ if the restriction of the admissible scalar product on $E_{r,s}$ is positive (negative) definite or neutral. We also use the lower index ± to distinguish the minimal admissible modules, corresponding to a choice of non-equivalent irreducible modules that were mentioned in Proposition 2.2.3, statements (3-1) and (3-2-1).

According to these agreements, any admissible module can be decomposed into the orthogonal sum of minimal admissible modules, see Proposition 2.2.2, statement (2). We distinguish the following possibilities.

If $r-s\not\equiv 3 \mod 4$ and s is arbitrary or $r-s\equiv 3 \mod 4$ and s is odd, then

$$U = \Big(\overset{p^+}{\oplus} V_{min}^{r,s;+}\Big)\bigoplus\Big(\overset{p^-}{\oplus} V_{min}^{r,s;-}\Big).\tag{2.6}$$

If $r-s\equiv 3 \mod 4$ and s is even, then

$$U = \Big(\overset{p_+^+}{\oplus} V_{min;+}^{r,s;+}\Big)\bigoplus\Big(\overset{p_+^-}{\oplus} V_{min;+}^{r,s;-}\Big)\bigoplus\Big(\overset{p_-^+}{\oplus} V_{min;-}^{r,s;+}\Big)\bigoplus\Big(\overset{p_-^-}{\oplus} V_{min;-}^{r,s;-}\Big).\tag{2.7}$$

Since the involutions in $PI_{r,s}$ are symmetric, the eigenspaces of involutions are mutually orthogonal. The involutions commute, therefore, they decompose the eigenspaces of other involutions into smaller (eigen)-subspaces. We give an example, that is crucial for the paper.

Example 1. The set $PI_{\mu,\nu}$ for $(\mu, \nu) \in \{(8, 0), (0, 8), (4, 4)\}$ is given by

$$T_1 = J_{\zeta_1} J_{\zeta_2} J_{\zeta_3} J_{\zeta_4},\, T_2 = J_{\zeta_1} J_{\zeta_2} J_{\zeta_5} J_{\zeta_6},\, T_3 = J_{\zeta_1} J_{\zeta_2} J_{\zeta_7} J_{\zeta_8},\, T_4 = J_{\zeta_1} J_{\zeta_3} J_{\zeta_5} J_{\zeta_7}.$$

The module $V^{\mu,\nu}_{min}$ is decomposed into 16 one dimensional common eigenspaces of the four involutions T_i. Let $v \in E_{\mu,\nu}$ and $|\langle v, v \rangle_{V^{\mu,\nu}_{min}}| = 1$. Then other common eigenspaces are spanned by $J_{\zeta_i} v$, $i = 1, \dots, 8$, and $J_{\zeta_1} J_{\zeta_j} v$, $j = 2, \dots, 8$. Hence we have

$$V^{\mu,\nu}_{min} = E_{\mu,\nu} \bigoplus_{i=1}^{8} J_{\zeta_i}(E_{\mu,\nu}) \bigoplus_{j=2}^{8} J_{\zeta_1} J_{\zeta_j}(E_{\mu.\nu}). \tag{2.8}$$

The value $\langle v, v \rangle_{V^{\mu,\nu}_{min}}$ can be ± 1 according to the admissible scalar product, however we may assume $\langle v, v \rangle_{V^{\mu,\nu}_{min}} = 1$, see [25, Example 1, Lemma 3.2.5].

3. Pseudo *H*-type Lie algebras

3.1. Definitions of pseudo *H*-type Lie algebras and their Lie groups

Let $(U, \langle \cdot , \cdot \rangle_U)$ be an admissible module of a Clifford algebra $\mathrm{Cl}_{r,s}$. We define a vector valued skew-symmetric bi-linear form

$$\begin{aligned} [\cdot\,,\cdot]\colon\ U \times U &\to \mathbb{R}^{r,s} \\ (x, y) &\longmapsto [x, y] \end{aligned}$$

by the relation

$$\langle J_z x, y \rangle_U = \langle z, [x, y] \rangle_{r,s} . \tag{3.1}$$

Definition 3.1.1. *[10]* The space $U \oplus \mathbb{R}^{r,s}$ endowed with the Lie bracket

$$[(x, z), (y, w)] = (0, [x, y])$$

is called a pseudo H-type Lie algebra and it is denoted by $\mathfrak{n}_{r,s}(U)$.

A pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(U)$ is 2-step nilpotent, the space $\mathbb{R}^{r,s}$ is the center, and the direct sum $U \oplus \mathbb{R}^{r,s}$ is orthogonal with respect to $\langle \cdot , \cdot \rangle_U + \langle \cdot , \cdot \rangle_{r,s}$.

The Baker-Campbell-Hausdorff formula allows us to define the Lie group structure on the space $U \oplus \mathbb{R}^{r,s}$ by

$$(x, z) * (y, w) = \big(x + y, z + w + \frac{1}{2}[x, y]\big).$$

The Lie group is denoted by $N_{r,s}(U)$ and is called pseudo H-type Lie group. Note that the scalar product $\langle \cdot , \cdot \rangle_U$ is implicitly included in the definitions of the H-type Lie algebra and the corresponding Lie group. In general, the Lie algebra structure might change if we replace the admissible scalar product on U, see [4,19,20].

3.2. General structure of the group $\mathrm{Aut}(\mathfrak{n}_{r,s}(U))$

In the present section, all the matrix groups are considered over the field $\mathbb{R}$. Let $\mathfrak{n} = (U \oplus \mathfrak{z}, [\,.\,,.\,])$ be a real 2-step nilpotent graded Lie algebra with the center $\mathfrak{z}$ and $\mathrm{Aut}(\mathfrak{n})$ be a group of automorphisms of this Lie algebra. We use the identification $U \cong \mathbb{R}^n$ and $\mathfrak{z} \cong \mathbb{R}^m$. An automorphism has to preserve the center and therefore an element $\Phi \in \mathrm{Aut}(\mathfrak{n})$ has to be of the form

$$\Phi = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}, \quad A \in \mathrm{GL}(n,\mathbb{R}), \quad C \in \mathrm{GL}(m,\mathbb{R}), \quad B \in \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m),$$

and $C([u,v]) = [Au,Av]$. The subgroup

$$\mathcal{B}(\mathfrak{n}) = \left\{ \begin{pmatrix} t\,\mathrm{Id}_n & 0 \\ B & t^2\,\mathrm{Id}_m \end{pmatrix}, \quad B \in \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m), \quad t \neq 0 \right\}$$

is a normal subgroup of $\mathrm{Aut}(\mathfrak{n})$. The factor group

$$\mathrm{Aut}(\mathfrak{n})/\mathcal{B}(\mathfrak{n}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},\ A \in \mathrm{SL}(n,\mathbb{R}),\ C \in \mathrm{GL}(m,\mathbb{R}),\ C([u,v]) = [Au,Av] \right\}$$

is a subgroup of $\mathrm{Aut}(\mathfrak{n})$ and it will be denoted by $\mathcal{C}(\mathfrak{n}) := \mathrm{Aut}(\mathfrak{n})/\mathcal{B}(\mathfrak{n})$. Thus the group $\mathrm{Aut}(\mathfrak{n})$ is a semi-direct product of $\mathcal{B}(\mathfrak{n})$ and $\mathcal{C}(\mathfrak{n})$, and it is enough to determine the group $\mathcal{C}(\mathfrak{n})$.

Let us assume now, that $\mathfrak{n}$ is a pseudo H-type Lie algebra $\mathfrak{n}_{r,s}(U) = U \oplus \mathfrak{z}$ with $\mathfrak{z} = \mathbb{R}^{r,s}$, $r+s=m$. If we write $A \oplus C$ for an element of $\mathcal{C}(\mathfrak{n})$, then (3.1) implies

$$\langle J_z Ax, Ay \rangle_U = \langle z, [Ax,Ay] \rangle_{r,s} = \langle z, C[x,y] \rangle_{r,s} = \langle J_{C^\tau(z)}x, y \rangle_U .$$

Thus the condition $C([u,v]) = [Au,Av]$ is equivalent to $A^\tau J_z A = J_{C^\tau(z)}$, where the transpositions A^τ and C^τ are taken with respect to the corresponding scalar products on U and on $\mathbb{R}^{r,s}$. The group

$$\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) = \left\{ A \oplus \mathrm{Id}_m,\ A \in \mathrm{SL}(n,\mathbb{R}),\ A^\tau J_z A = J_z \text{ for any } z \in \mathbb{R}^{r,s} \right\} \tag{3.2}$$

is a normal subgroup in $\mathcal{C}(\mathfrak{n})$.

Lemma 3.2.1. *[24, Theorem 2] The subgroup of the maps* $C \in \mathrm{GL}(m,\mathbb{R})$ *such that* $A \oplus C \in \mathcal{C}(\mathfrak{n}_{r,s}(U))$ *is contained in* $\mathrm{O}(r,s)$*,* $r+s=m$*.*

Due to Lemma 3.2.1, we conclude that

$$\mathcal{C}(\mathfrak{n}_{r,s}(U)) = \Big\{A \oplus C,\ A \in \mathrm{SL}(n, \mathbb{R}),\ C \in \mathrm{O}(r, s),\ A^{\tau} J_z A = J_{C^{\tau}(z)} \quad z \in \mathbb{R}^{r,s}\Big\}.$$

In the next step we show that the map

$$\mathcal{C}(\mathfrak{n}_{r,s}(U)) \to \mathrm{O}(r, s): \quad A \oplus C \mapsto C$$

is surjective. To achieve the goal we recall the definition of the group $\mathrm{Pin}(r, s)$. For the beginning we introduce two useful involutions. The tensor algebra $\mathcal{T}(\mathbb{R}^{r,s})$ has an involution given on simple elements by the reversal of order:

$$(v_1 \oplus \ldots \oplus v_k)^T = v_k \oplus \ldots \oplus v_1.$$

Since the map preserves the ideal $I_{r,s}$ it descends to a map $(\cdot)^T\colon\ \mathrm{Cl}_{r,s} \to \mathrm{Cl}_{r,s}$. The map

$$\mathbb{R}^{r,s} \ni z \mapsto -z \in \mathbb{R}^{r,s} \subset \mathrm{Cl}_{r,s}$$

is extended to the Clifford algebra automorphism $\alpha\colon \mathrm{Cl}_{r,s} \to \mathrm{Cl}_{r,s}$ by the universal property of the Clifford algebras. The norm mapping $N\colon \mathrm{Cl}_{r,s} \to \mathrm{Cl}_{r,s}$ is defined by $N(\phi) = \phi \cdot \alpha(\phi^T)$. It is easy to see that $N(z) = \langle z, z \rangle_{r,s}$ for any $z \in \mathbb{R}^{r,s}$. More about the properties of the maps $* \mapsto *^T$ and N can be found in [36, Page 15].

We denote by $\mathrm{Cl}^{\times}_{r,s}$ the group of invertible elements in $\mathrm{Cl}_{r,s}$ and in particular $\mathbb{R}^{r,s\times} = \{z \in \mathbb{R}^{r,s}|\ \langle z, z \rangle_{r,s} \neq 0\}$. The representation $\widetilde{\mathrm{Ad}}\colon \mathbb{R}^{r,s\times} \to \mathrm{End}(\mathbb{R}^{r,s})$, is defined as

$$\widetilde{\mathrm{Ad}}_z(w) = -zwz^{-1} = \left(w - 2\frac{\langle w, z \rangle_{r,s}}{\langle z, z \rangle_{r,s}} z\right) \in \mathbb{R}^{r,s} \quad \text{for} \quad w \in \mathbb{R}^{r,s},\quad z \in \mathbb{R}^{r,s\times}.$$

The map $\widetilde{\mathrm{Ad}}_z\colon \mathbb{R}^{r,s} \to \mathbb{R}^{r,s}$ is the reflection of the vector $w \in \mathbb{R}^{r,s}$ with respect to the hyperplane orthogonal to the vector $z \in \mathbb{R}^{r,s}$. Then it extends to the twisted adjoint representation $\widetilde{\mathrm{Ad}}\colon\ \mathrm{Cl}^{\times}_{r,s} \to \mathrm{GL}(\mathrm{Cl}_{r,s})$ by setting

$$\mathrm{Cl}^{\times}_{r,s} \ni \varphi \longmapsto \widetilde{\mathrm{Ad}}_{\varphi}, \quad \widetilde{\mathrm{Ad}}_{\varphi}(\phi) = \alpha(\varphi)\phi\varphi^{-1}, \quad \phi \in \mathrm{Cl}_{r,s}\,. \tag{3.3}$$

The map $\widetilde{\mathrm{Ad}}_z$ for $z \in \mathbb{R}^{r,s\times}$, leaving the space $\mathbb{R}^{r,s} \subset \mathrm{Cl}_{r,s}$ invariant, is also an isometry: $\langle \widetilde{\mathrm{Ad}}_z(w), \widetilde{\mathrm{Ad}}_z(w) \rangle_{r,s} = \langle w, w \rangle_{r,s}$. Moreover, the properties of preserving the space $\mathbb{R}^{r,s}$ and the bilinear symmetric form $\langle\, .\,, .\, \rangle_{r,s}$ are fulfilled by the group

$$\mathrm{P}(\mathbb{R}^{r,s}) = \{v_1 \cdots v_k \in \mathrm{Cl}^{\times}_{r,s} \mid\ \langle v_i, v_i \rangle_{r,s} \neq 0\}. \tag{3.4}$$

The map $\widetilde{\mathrm{Ad}}\colon \mathrm{P}(\mathbb{R}^{r,s}) \to \mathrm{O}(r, s)$ is a surjective homomorphism [36, Theorem 2.7]. It particularly implies $\widetilde{\mathrm{Ad}}_{\varphi^{-1}} = \widetilde{\mathrm{Ad}}^{T}_{\varphi}$. The subgroups of $P(\mathbb{R}^{r,s}) \subset \mathrm{Cl}^{\times}_{r,s}$ defined by

$$\begin{aligned}
\mathrm{Pin}(r,s) &= \{v_1\cdots v_k \in \mathrm{Cl}_{r,s}^{\times} \mid \ \langle v_i, v_i\rangle_{r,s} = \pm 1\},\\
\mathrm{Spin}(r,s) &= \{v_1\cdots v_k \in \mathrm{Cl}_{r,s}^{\times} \mid \ k \text{ is even}, \langle v_i, v_i\rangle_{r,s} = \pm 1\},
\end{aligned}$$

are called pin and spin groups, respectively.

Proposition 3.2.2. *[3,36] The map* $\widetilde{\mathrm{Ad}}\colon \mathrm{Pin}(r,s) \to \mathrm{O}(r,s)$ *is the double covering map.*

Proposition 3.2.3. *Let* $J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(U)$ *be a Clifford algebra representation and* $\varphi \in \mathrm{Pin}(r,s)$*. Then the map* $\mathcal{P}\colon \mathrm{Pin}(r,s) \to \mathcal{C}(\mathfrak{n}_{r,s}(U))$ *defined by*

$$\varphi \mapsto \mathcal{P}(\varphi) = \begin{pmatrix} J_\varphi & 0 \\ 0 & (-1)^n N(\varphi)\widetilde{\mathrm{Ad}}_\varphi \end{pmatrix},$$

is a group homomorphism.

Proof. First we show that $\mathcal{P}(\varphi) \in \mathcal{C}(\mathfrak{n}_{r,s}(U))$; that is

$$J_\varphi^\tau J_z J_\varphi = J_{(-1)^n N(\varphi)\widetilde{\mathrm{Ad}}_\varphi^\tau}.$$

Let $\varphi \in \mathrm{Pin}(r,s)$. Then $\widetilde{\mathrm{Ad}}_\varphi^\tau \in \mathrm{O}(r,s)$. Moreover, $\widetilde{\mathrm{Ad}}_\varphi^\tau(z) = \widetilde{\mathrm{Ad}}_{\varphi^{-1}}(z) = \alpha(\varphi^{-1})z\varphi$. Thus for any $\varphi = \prod_{k=1}^n x_k \in \mathrm{Pin}(r,s)$ we obtain

$$\varphi^T = (x_1\cdot\ldots\cdot x_n)^T = (x_n\cdot\ldots\cdot x_1) \quad \text{and} \quad N(\varphi) = \varphi\cdot\alpha(\varphi^T) = \prod_{k=1}^n \langle x_k, x_k\rangle_{r,s}.$$

Then since $x_k^{-1} = \frac{\alpha(x_k)}{\langle x_k, x_k\rangle_{r,s}}$, $k = 1,\ldots,n$, and $J_{x_k}^\tau = -J_{x_k}$ we have $\alpha(\varphi^{-1}) = \frac{\varphi^T}{N(\varphi)}$ and $J_{\varphi^T} = (-1)^n J_\varphi^\tau$. Thus

$$J_{\widetilde{\mathrm{Ad}}_\varphi^\tau(z)} = J_{\widetilde{\mathrm{Ad}}_{\varphi^{-1}}(z)} = J_{\alpha(\varphi^{-1})} J_z J_\varphi = \frac{(-1)^n}{N(\varphi)} J_\varphi^\tau J_z J_\varphi.$$

This proves the proposition. □

Proposition 3.2.4. *Let* $J\colon \mathrm{Cl}_{r,s} \to \mathrm{End}(U)$ *be a Clifford algebra representation and* $\varphi \in \mathrm{Pin}(r,s)$*. Both lines in the following diagram*

$$\begin{array}{ccccccccc}
\{\mathrm{Id}\} & \longrightarrow & \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) & \xrightarrow{\iota} & \mathcal{C}(\mathfrak{n}_{r,s}(U)) & \xrightarrow{\psi} & \mathrm{O}(r,s) & \longrightarrow & \{\mathrm{Id}\} \\
 & & \uparrow & & \mathcal{P}\uparrow & & \uparrow\mathrm{Id} & & \\
\{\mathrm{Id}\} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Pin}(r,s) & \xrightarrow{\widetilde{\mathrm{Ad}}} & \mathrm{O}(r,s) & \longrightarrow & \{\mathrm{Id}\}
\end{array} \tag{3.5}$$

are short exact sequences. The kernel $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ *is defined in* (3.2)*.*

Proof. It is a well known fact that the second line is a short exact sequence, see, for instance [36]. Let $C \in \mathrm{O}(r,s)$ and φ be any element of $\mathrm{Pin}(r,s)$ such that $\widetilde{\mathrm{Ad}}_\varphi = C$. Then for any $C \in \mathrm{O}(r,s)$ there is $\psi^{-1}(C) = \mathcal{P}(\varphi) \in \mathcal{C}(\mathfrak{n}_{r,s}(U))$, given by Proposition 3.2.3. It shows that the map ψ is surjective. Note that diagram (3.5) is not necessarily commutative. □

Lemma 3.2.5. *Let*

$$\{\mathrm{Id}\} \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\psi} H \longrightarrow \{\mathrm{Id}\} \tag{3.6}$$

be a short exact sequence of groups. We assume that K is a subgroup in G such that $\psi|_K$ is surjective. Then there is a group homomorphism $\rho\colon N \rtimes_\phi K \to G$ with $\ker\rho = \{(n,n^{-1}) \mid\ n \in K \cap N\}$.

Proof. Since N is a normal subgroup of G, the subgroup K acts on N by conjugation

$$\phi\colon K \to \mathrm{Aut}(N), \quad \phi_k(n) = knk^{-1}, \quad \text{for} \quad n \in N,\ k \in K.$$

We have the surjective group homomorphism

$$\rho : N \rtimes_\phi K \ni (n,k) \mapsto nk \in G.$$

In fact, $\rho((n,k)\cdot(n',k')) = nkn'k^{-1}kk' = \rho((n,k))\rho((n',k'))$. The kernel of ρ is

$$\ker\rho = \{(n,k) \mid\ nk = e,\ n \in N,\ k \in K\} = \{(n,n^{-1}) \mid\ n \in K \cap N\},$$

where e is the unit element in G. Consequently, $(N \rtimes_\phi K)/\ker\rho \cong G$. □

We set $G = \mathcal{C}(\mathfrak{n}_{r,s}(U))$, $K = \mathcal{P}\big(\mathrm{Pin}(r,s)\big)$ and $N = \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ in Lemma 3.2.5. Then $\ker\rho = \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) \cap \mathcal{P}\big(\mathrm{Pin}(r,s)\big)$. Now we determine the order of the intersection $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) \cap \mathcal{P}\big(\mathrm{Pin}(r,s)\big)$.

Theorem 3.2.6. *In the notations above, we have*

(1) $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) \cap \mathcal{P}\big(\mathrm{Pin}(r,s)\big) = \{\pm\mathrm{Id}\oplus\mathrm{Id}\}$ *in the following cases*
  (1a) *r is even, s is arbitrary;*
  (1b) $r = 1 \mod 4$, $s = 1,2 \mod 4$*;*
  (1c) $r = 3 \mod 4$, $s = 0,3 \mod 4$ *and the admissible module is isotypic;*
(2) $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) \cap \mathcal{P}\big(\mathrm{Pin}(r,s)\big) = \{\pm\mathrm{Id}\oplus\mathrm{Id};\ \pm J_{\Omega^{r,s}}\oplus\mathrm{Id}\}$ *in the following cases*
  (2a) $r = 1 \mod 4$, $s = 0,3 \mod 4$*;*
  (2b) $r = 3 \mod 4$, $s = 1,2 \mod 4$*;*
  (2c) $r = 3 \mod 4$, $s = 0,3 \mod 4$ *and the admissible module is non-isotypic.*

Proof. To prove the theorem we need to find $\phi \in \mathrm{Pin}(r,s)$ such that

$$\Psi(\phi) = (-1)^n N(\phi)\widetilde{\mathrm{Ad}}_\phi = \mathrm{Id}\,.$$

Then $\pm J_\phi \oplus \mathrm{Id}$ will belong to $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) \cap \mathcal{P}\big(\mathrm{Pin}(r,s)\big)$. Note that $\Psi(\pm\mathbf{1}) = \mathrm{Id}$. We also note that

$$N(\Omega^{r,s}) = \alpha(\Omega^{r,s})(\Omega^{r,s})^T = (-1)^s \quad \text{and} \quad z\Omega^{r,s} = (-1)^{r+s-1}\Omega^{r,s}z \text{ for any } z \in \mathbb{R}^{r,s}.$$

Thus

$$(-1)^{r+s}N(\Omega^{r,s})\widetilde{\mathrm{Ad}}_{\Omega^{r,s}}z = (-1)^{2r+4s}(-1)^{r-1}z\Omega^{r,s}(\Omega^{r,s})^{-1} = (-1)^{r-1}z.$$

Hence $\Psi(\Omega^{r,s}) = \mathrm{Id}$ for odd values of r and arbitrary values of s. Thus, if r is even, then for arbitrary s elements in $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)) \cap \mathcal{P}\big(\mathrm{Pin}(r,s)\big)$ are $\pm\,\mathrm{Id} \oplus \mathrm{Id}$. Moreover, in this case all the modules are isotypic. This shows (1a).

Before we proceed, we remind some properties of the volume form:

$$(\Omega^{r,s})^2 = \begin{cases} (-1)^s, & \text{if} \quad r+s = 3,4 \mod 4, \\ (-1)^{s+1}, & \text{if} \quad r+s = 1,2 \mod 4. \end{cases} \tag{3.7}$$

We need to check the values $r = 1,3 \mod 4$.

Let $r = 1 \mod 4$. In this case all admissible modules are isotypic. Moreover (3.7) implies

$$(\Omega^{r,s})^2 = \begin{cases} 1, & \text{if} \quad s = 1,2 \mod 4, \\ -1, & \text{if} \quad s = 0,3 \mod 4. \end{cases}$$

Thus, if $r = 1 \mod 4$ and $s = 1,2 \mod 4$, then we have $J_{\Omega^{r,s}} = \pm\,\mathrm{Id}$ and it proves (1b). In the case $r = 1 \mod 4$ and $s = 0,3 \mod 4$ we obtain (2a).

Let $r = 3 \mod 4$. In this case we need to distinguish isotypic and non-isotypic admissible modules. The property (3.7) implies

$$(\Omega^{r,s})^2 = \begin{cases} 1, & \text{if} \quad s = 0,3 \mod 4, \\ -1, & \text{if} \quad s = 1,2 \mod 4. \end{cases}$$

Thus if $r = 3 \mod 4$ and $s = 1,2 \mod 4$ we obtain (2b). If $r = 3 \mod 4$, $s = 0,3 \mod 4$ and module is isotypic then $J_{\Omega^{r,s}} = \pm\,\mathrm{Id}$, that shows (1c). In the case $r = 3 \mod 4$, $s = 0,3 \mod 4$ with a non-isotypic module we obtain (2c).

At the end we notice that the cases (1a), (1c), and (2c) contain a result of [44]. □

We conclude that any element of $\mathcal{C}(\mathfrak{n}_{r,s}(U))$ has the form $AJ_\varphi \oplus (-1)^n N(\varphi)\widetilde{\mathrm{Ad}}_\varphi$. The only thing that we left to find is a subgroup of $\mathrm{SL}(n,\mathbb{R})$ containing maps A such that

$$A^\tau J_z A = J_z \quad \text{for all} \quad z \in \mathbb{R}^{r,s}. \tag{3.8}$$

3.3. Relation between the structure of involutions $PI_{r,s}$ and $\mathrm{Aut}^0\big(\mathfrak{n}_{r,s}(U)\big)$

In this section we show that $\mathrm{Aut}^0\big(\mathfrak{n}_{r,s}(U)\big)$ is closely related to the structure of the set of involutions $PI^*_{r,s}$ of types (1)-(3) acting on U. The proof of the following lemma is obtained from (3.8) by induction and can be found in [24, Lemma 3] for the product of any number of J_{z_k}.

Lemma 3.3.1. *Let $\{z_i\}_{i=1}^{r+s}$ be an orthonormal basis for $\mathbb{R}^{r,s}$ and let $\Phi = A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$. Then the following relations hold:*

$$AJ_{z_k}J_{z_l} = J_{z_k}J_{z_l}A, \qquad AJ_{z_k}J_{z_l}J_{z_m}J_{z_n} = J_{z_k}J_{z_l}J_{z_m}J_{z_n}A, \tag{3.9}$$

$$A^\tau J_{z_j}J_{z_k}J_{z_l}A = J_{z_j}J_{z_k}J_{z_l}. \tag{3.10}$$

Lemma 3.3.2. *Let $\{z_i\}_{i=1}^{r+s}$ be an orthonormal basis for $\mathbb{R}^{r,s}$ and U an admissible module. If a linear map $A\colon U \to U$ satisfies the conditions*

$$\begin{array}{ll} A^\tau J_{z_{k_0}}A = J_{z_{k_0}} \quad \textit{for one index} \quad k_0 \in \{1,\dots r+s\}, \quad \textit{and} \\ AJ_{z_{k_0}}J_{z_l} = J_{z_{k_0}}J_{z_l}A \quad \textit{for all indices} \quad l = 1,\dots r+s, \end{array} \tag{3.11}$$

then $\Phi = A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$.

Proof. We only need to show (3.8) for all $z = z_l$ for $l = 1,\dots r+s$. If (3.11) is fulfilled, then

$$A^\tau J_{z_l}A = \pm A^\tau J_{z_{k_0}}J_{z_{k_0}}J_{z_l}A = \pm A^\tau J_{z_{k_0}}AJ_{z_{k_0}}J_{z_l} = \pm J^2_{z_{k_0}}J_{z_l} = J_{z_l}. \quad \square$$

Corollary 3.3.3. $\Phi = A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ *if and only if* (3.11) *holds.*

Let $(V^{r,s}_{min}, J)$ be a minimal admissible module of $\mathrm{Cl}_{r,s}$. Let $P\colon V^{r,s}_{min} \to V^{r,s}_{min}$ be an involution from the set $PI^*_{r,s}$ that is the product of four generators. We denote by E^k_P, $k \in \{1,-1\}$ the eigenspace of the involution P with the eigenvalue $k = \pm 1$. In order to denote the intersection of eigenspaces of several involutions $P_l \in PI^*_{r,s}$, $l = 1,\dots,N = \#(PI^*_{r,s})$, we use the multi-index $I = (k_1,\dots,k_N)$, $k_l = \pm 1$ and write $E^I = \bigcap_{l=1}^N E^{k_l}_{P_l}$. Assume that $\Phi = A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min}))$. Then
1. $A = \oplus A_I$, where $A_I\colon E^I \to E^I$ for any choice of $I = (k_1,\dots,k_N)$;
2. if $J_{z_j},\ J_{z_j}J_{z_k},\ J_{z_j}J_{z_k}J_{z_m}\colon E^I \to E^I$ for some I, then

$$A_IJ_{z_j} = J_{z_j}(A^\tau_I)^{-1}, \quad A_IJ_{z_j}J_{z_k}J_{z_m} = J_{z_j}J_{z_k}J_{z_m}(A^\tau_I)^{-1}$$

$$A_I J_{z_j} J_{z_k} = J_{z_j} J_{z_k} A_I,$$

and

$$A_I^\tau J_{z_j} = J_{z_j}(A_I)^{-1}, \quad A_I^\tau J_{z_j} J_{z_k} J_{z_m} = J_{z_j} J_{z_k} J_{z_m}(A_I)^{-1},$$
$$A_I^\tau J_{z_j} J_{z_k} = J_{z_j} J_{z_k} A_I^\tau.$$

Proof. The first statement follows from the fact that $AP_l = P_l A$ for all $l = 1, \dots, N$.
The second statement is the direct consequence of (3.8) and Lemma 3.3.1. □

Thus the construction of the map $A\colon V^{r,s}_{min} \to V^{r,s}_{min}$ can be reduced to the construction of the maps $A_I\colon E^I \to E^I$ and setting $A = \oplus A_I$. Theorem 3.3.4 states that, under some conditions, the construction of all maps A_I can be obtained from the map $A_1\colon E^1 \to E^1$, where we denote $E^1 = \bigcap_{l=1}^N E^1_{P_l}$. Note that E^1 is exactly the subspace $E^*_{r,s} \subset V^{r,s}_{min}$ that is the common 1-eigenspace of involutions from $PI^*_{r,s}$ that are of types (1)-(3).

Theorem 3.3.4. *Under the previous notation we assume that*
(a) there exist maps $G_I\colon E^1 \to E^I$ for all multi-indices I of the form either $G_I = J_{z_i}$ or $J_{z_i} J_{z_k}$ for some $i, k = 1, \dots, r+s$, and
(b) there exists a linear map $A_1\colon E^1 \to E^1$ such that if $J_{z_j}, J_{z_j} J_{z_k}, J_{z_j} J_{z_k} J_{z_m}\colon E^1 \to E^1$, then the map A_1 satisfies

$$A_1 J_{z_j} = J_{z_j}(A_1^\tau)^{-1}, \quad A_1 J_{z_j} J_{z_k} J_{z_m} = J_{z_j} J_{z_k} J_{z_m}(A_1^\tau)^{-1}, \tag{3.12}$$

and the same for any other product of odd number of generators J_{z_l}, leaving the space E^1 invariant; also

$$A_1 J_{z_j} J_{z_k} = J_{z_j} J_{z_k} A_1, \tag{3.13}$$

and the same for any other product of even number of generators J_{z_l}, leaving the space E^1 invariant.

Then the map $A\colon V^{r,s}_{min} \to V^{r,s}_{min}$, $A = \oplus A_I$ with $A_I\colon E^I \to E^I$ such that

$$A_I = \begin{cases} G_I (A_1^{-1})^\tau G_I^{-1}, & \text{if } \ G_I = J_{z_i} \ \text{ for some } \ i = 1, \dots, r+s, \\ G_I A_1 G_I^{-1}, & \text{if } \ G_I = J_{z_i} J_{z_k} \ \text{ for some } \ i, k = 1, \dots, r+s, \end{cases} \tag{3.14}$$

uniquely defines the automorphism $\Phi = A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min}))$.

Proof. The spaces E^I are mutually orthogonal because all the involutions in $PI^*_{r,s}$ are symmetric. Thus $V^{r,s}_{min} = \oplus E^I$, where the direct sum is orthogonal. For the convenience we also write the maps defining A_I^τ:

$$A_I^\tau = \begin{cases} G_I A_1^{-1} G_I^{-1}, & \text{if } G_I = J_{z_i}, \\ G_I A_1^\tau G_I^{-1}, & \text{if } G_I = J_{z_i} J_{z_k}. \end{cases} \tag{3.15}$$

Then we set $A = \oplus A_I$. We only need to check the condition $AJ_{z_{j_0}} A^\tau = J_{z_{j_0}}$ for an arbitrary z_{j_0} in the orthonormal basis for $\mathbb{R}^{r,s}$.

We choose $y \in V^{r,s}_{min} = \oplus E^I$. Then we write $y = \oplus y_I$ with $y_I \in E^I$. Thus we distinguish the cases when the map G_I is the product of an odd or an even number of maps J_{z_i}. Moreover, we find a multi-index K for the multi-index I, such that $G_K^{-1} J_{z_{j_0}} G_I$ leaves the space E^1 invariant. Since G_K can also be the product of an even or an odd number of J_{z_k}, we distinguish the following cases: $AJ_{z_{j_0}} A^\tau y_I = A_K J_{z_{j_0}} A_I^\tau y_I$

$$= \begin{cases} G_K (A_1^{-1})^\tau G_K^{-1} J_{z_{j_0}} G_I A_1^{-1} G_I^{-1} y_I & \text{if } G_I = J_{z_i},\ G_K = J_{z_l}, \\ G_K A_1 G_K^{-1} J_{z_{j_0}} G_I A_1^{-1} G_I^{-1} y_I & \text{if } G_I = J_{z_i},\ G_K = J_{z_k} J_{z_l}, \\ G_K (A_1^{-1})^\tau G_K^{-1} J_{z_{j_0}} G_I A_1^\tau G_I^{-1} y_I & \text{if } G_I = J_{z_i} J_{z_m},\ \ G_K = J_{z_l}, \\ G_K A_1 G_K^{-1} J_{z_{j_0}} G_I A_1^\tau G_I^{-1} y_I & \text{if } G_I = J_{z_i} J_{z_m},\ \ G_K = J_{z_k} J_{z_l}, \end{cases}$$

by definitions (3.14) and (3.15) of A_K and A_I^τ. We only check the first condition, since the others can be verified similarly. The condition that $G_K^{-1} J_{z_{j_0}} G_I$ leaves the space E^1 invariant, reads as $(A_1^{-1})^\tau G_K^{-1} J_{z_{j_0}} G_I A_1^{-1} = G_K^{-1} J_{z_{j_0}} G_I$. Indeed from (3.12) we have

$$(A_1^{-1})^\tau G_K^{-1} J_{z_{j_0}} G_I A_1^{-1} = (A_1^{-1})^\tau J_{z_l}^{-1} J_{z_{j_0}} J_{z_i} A_1^{-1} = J_{z_l}^{-1} J_{z_{j_0}} J_{z_i} = G_K^{-1} J_{z_{j_0}} G_I.$$

We calculate

$$G_K (A_1^{-1})^\tau G_K^{-1} J_{z_{j_0}} G_I A_1^{-1} G_I^{-1} y_I = G_K G_K^{-1} J_{z_{j_0}} G_I G_I^{-1} = J_{z_{j_0}}.$$

Thus, $AJ_{z_{j_0}} A^\tau y_I = J_{z_{j_0}} y_I$.

Now we show the uniqueness. Let us assume that $G_I, G_{\tilde{I}} \colon E^1 \to E^I$ and both $G_I, G_{\tilde{I}}$ are products of even numbers of J_{z_k}. Then $A_1 G_I = G_I A_1$, and $A_1 G_{\tilde{I}} = G_{\tilde{I}} A_1$. It implies

$$A_I \circ A_{\tilde{I}}^{-1} = G_I A_1 G_I^{-1} G_{\tilde{I}} A_1^{-1} G_{\tilde{I}}^{-1} = G_I G_I^{-1} G_{\tilde{I}} A_1 A_1^{-1} G_{\tilde{I}}^{-1} = \mathrm{Id},$$

because $G_I^{-1} G_{\tilde{I}}$ is the product of an even number of J_{z_k} that allows to apply (3.13).

Let now $G_I, G_{\tilde{I}} \colon E^1 \to E^I$ and both of them are products of odd numbers of generators. Then by making use of (3.14) and (3.13) we obtain

$$A_I \circ A_{\tilde{I}}^{-1} = G_I (A_1^\tau)^{-1} G_I^{-1} G_{\tilde{I}} A_1^\tau G_{\tilde{I}}^{-1} = G_I (A_1^\tau)^{-1} A_1^\tau G_I^{-1} G_{\tilde{I}} G_{\tilde{I}}^{-1} = \mathrm{Id}$$

since $G_I^{-1} G_{\tilde{I}}$ is the product of an even number of generators.

Finally, if $G_I \colon E^1 \to E^I$ is the product of an odd number of generators and $G_{\tilde{I}} \colon E^1 \to E^I$ is the product of an even number of generators, then we obtain

$$A_I \circ A_{\tilde{I}}^{-1} = G_I (A_1^\tau)^{-1} G_I^{-1} G_{\tilde{I}} A_1^{-1} G_{\tilde{I}}^{-1} = G_I G_I^{-1} G_{\tilde{I}} A_1 A_1^{-1} G_{\tilde{I}}^{-1} = \mathrm{Id}$$

Table 2
Classification result for $\mathfrak{n}_{r,s}(V^{r,s}_{min})$.

8	≅								
7	d	d	d	≇					
6	d	≅	≅	h					
5	d	≅	≅	h					
4	≅	h	h	h					
3	d	≇	≇		d	d	d	≇	
2	≅	h		≇	d	≅	≅	h	
1	≅		d	≇	d	≅	≅	h	
0		≅	≅	h	≅	h	h	h	≅
s/r	0	1	2	3	4	5	6	7	8

by (3.14). Here we used the fact that $G_I^{-1}G_{\tilde{I}}$ is the product of an odd number of generators and then applied (3.12). □

3.4. *Classification of pseudo H-type Lie algebras $\mathfrak{n}_{r,s}(U)$*

We start from the necessary condition for isomorphisms between two H-type Lie algebras.

Theorem 3.4.1. *[24, Theorem 2]. Let $(V^{r,s}, \langle\cdot\,,\cdot\rangle_{V^{r,s}})$ and $(V^{\tilde{r},\tilde{s}}, \langle\cdot\,,\cdot\rangle\, V^{\tilde{r},\tilde{s}})$ be admissible modules of the Clifford algebras $\mathrm{Cl}_{r,s}$ and $\mathrm{Cl}_{\tilde{r},\tilde{s}}$, respectively. Assume that $r+s=\tilde{r}+\tilde{s}$, $\dim(V^{r,s})=\dim(V^{\tilde{r},\tilde{s}})$, and that the Lie algebras $\mathfrak{n}_{r,s}(V^{r,s})$ and $\mathfrak{n}_{\tilde{r},\tilde{s}}(V^{\tilde{r},\tilde{s}})$ are isomorphic. Then, either $(r,s)=(\tilde{r},\tilde{s})$ or $(r,s)=(\tilde{s},\tilde{r})$.*

The classification of pseudo H-type algebras $\mathfrak{n}_{r,s}(V^{r,s}_{min})$, constructed from the minimal admissible modules was done in [24]. We summarise the results of the classification in Table 2.

Here “d” stands for “double”, meaning that $\dim V^{r,s}_{min} = 2\dim V^{s,r}_{min}$ and “h” (half) means that $\dim V^{r,s}_{min} = \frac{1}{2}\dim V^{s,r}_{min}$. The corresponding pairs are trivially non-isomorphic due to the different dimension of minimal admissible modules. The symbol ≅ denotes the Lie algebra $\mathfrak{n}_{r,s}(V^{r,s}_{min})$ having isomorphic counterpart $\mathfrak{n}_{s,r}(V^{s,r}_{min})$, the symbol ≇ shows that the Lie algebra $\mathfrak{n}_{r,s}(V^{r,s}_{min})$ is not isomorphic to $\mathfrak{n}_{s,r}(V^{s,r}_{min})$.

The result of the classification for the cases when the Lie algebras have the same signature (r,s) of the scalar product on the center and arbitrary admissible modules is contained in [25, Theorems 4.1.1-4.1.3]. We summarise the result here.

Theorem 3.4.2. *Let $U=(U,\langle\cdot\,,\cdot\rangle_U)$ and $\tilde{U}=(\tilde{U},\langle\cdot\,,\cdot\rangle_{\tilde{U}})$ be admissible modules of a Clifford algebra $\mathrm{Cl}_{r,s}$.*

1. *If $r=0,1,2 \mod 4$, then $\mathfrak{n}_{r,s}(U)\cong\mathfrak{n}_{r,s}(\tilde{U})$, if and only if $\dim(U)=\dim(\tilde{U})$.*

2. *Let* $r = 3 \mod 4$ *and* $s = 0 \mod 4$ *and let the admissible modules be decomposed into the direct sums of the type* (2.7). *Then the Lie algebras* $\mathfrak{n}_{r,s}(U)$ *and* $\mathfrak{n}_{r,s}(\tilde{U})$ *are isomorphic, if and only if,*

$$p = p_+^+ + p_-^- = \tilde{p}_+^+ + \tilde{p}_-^- = \tilde{p} \quad and \quad q = p_+^- + p_-^+ = \tilde{p}_+^- + \tilde{p}_-^+ = \tilde{q}, \qquad or$$
$$p = p_+^+ + p_-^- = \tilde{p}_+^- + \tilde{p}_-^+ = \tilde{q} \quad and \quad q = p_+^- + p_-^+ = \tilde{p}_+^+ + \tilde{p}_-^- = \tilde{p}.$$

3. *Let* $r = 3 \mod 4$ *and* $s = 1, 2, 3 \mod 4$ *and let* U *and* $\tilde{U}$ *be decomposed into the direct sums* (2.6) *Then* $\mathfrak{n}_{r,s}(U) \cong \mathfrak{n}_{r,s}(\tilde{U})$, *if and only if*

$$p = p^+ = \tilde{p}^+ = \tilde{p} \quad and \quad q = p^- = \tilde{p}^- = \tilde{q}, \qquad or$$
$$p = p^+ = \tilde{p}^- = \tilde{q} \quad and \quad q = p^- = \tilde{p}^+ = \tilde{p}.$$

According to Theorem 3.4.2 in the cases $r = 3 \mod 4$ and $s = 0 \mod 4$, we can substitute $\overset{p_-^+}{\oplus} V_{min;-}^{r,s;+}$ by $\overset{p_+^-}{\oplus} V_{min;+}^{r,s;-}$ if $p_-^+ = p_+^-$. Analogously, we replace $\overset{p_-^-}{\oplus} V_{min;-}^{r,s;-}$ by $\overset{p_+^+}{\oplus} V_{min;+}^{r,s;+}$ if $p_-^- = p_+^+$. Hence we reduce the decompositions of an admissible module to the sums containing only $V_{min;+}^{r,s;\pm}$. Moreover, we omit the subscript "+" below and simply write $V_{min}^{r,s;\pm}$. Thus, if $r = 3 \mod 4$, then the type of the Lie algebra $\mathfrak{n}_{r,s}(U)$ depends only on the decomposition

$$U = \big(\overset{p}{\oplus} V_{min}^{r,s;+}\big) \bigoplus \big(\overset{q}{\oplus} V_{min}^{r,s;-}\big), \tag{3.16}$$

where the numbers p, q are defined in items 2 and 3 of Theorem 3.4.2. We call admissible modules with decompositions (3.16) *isotypic* if one of the numbers p or q vanishes. Otherwise the admissible module is called *non-isotypic* of type (p, q).

Now we state the classification when the Lie algebras have opposite signatures (r, s) and (s, r) of the scalar products on the centers and arbitrary admissible modules, see [25, Theorems 4.6.2]. We formulate here the revised version of the result obtained in [25, Theorem 4.6.2].

Theorem 3.4.3. *Let* $r = 0, 1, 2 \mod 4$ *and* $s = 0, 1, 2 \mod 4$. *Then* $\mathfrak{n}_{r,s}(U^{r,s}) \cong \mathfrak{n}_{s,r}(U^{s,r})$ *if* $\dim(U^{r,s}) = \dim(U^{s,r})$.

Let $r = 3 \mod 8$, $s = 0, 4, 5, 6 \mod 8$ *or* $r = 7 \mod 8$, $s = 0, 1, 2 \mod 8$. *Then* $\mathfrak{n}_{r,s}(U^{r,s}) \cong \mathfrak{n}_{s,r}(U^{s,r})$ *if* $\dim(U^{r,s}) = \dim(U^{s,r})$ *and* $U^{r,s} = \big(\overset{p}{\oplus} V_{min}^{r,s;+}\big) \bigoplus \big(\overset{p}{\oplus} V_{min}^{r,s;-}\big)$.

Let $r \equiv 3 \,(\mathrm{mod}\ 8)$ *and* $s \equiv 1, 2, 7 \,(\mathrm{mod}\ 8)$. *Then* $\mathfrak{n}_{r,s}(U^{r,s})$ *is never isomorphic to* $\mathfrak{n}_{s,r}(U^{s,r})$.

3.5. Periodicity of $\mathrm{Aut}(\mathfrak{n}_{r,s}(U))$ *in parameters* (r, s)

To obtain $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ for all the range of the parameters (r, s), we need only to describe basic cases (2.5), since the rest follows from the theorems on periodicity.

Proposition 3.5.1. *[25, Propositions 4.2.1 and 4.2.2] Let* $(U^{r,s}_{min}, \langle\cdot\,,\cdot\rangle_{U^{r,s}_{min}})$ *be a minimal admissible module of* $\mathrm{Cl}_{r,s}$ *and* J_{z_i}, $i = 1,\dots,r+s$ *the Clifford actions of an orthonormal basis* $\{z_i\}$. *Let also* $(V^{\mu,\nu}_{min}, \langle\cdot\,,\cdot\rangle_{V^{\mu,\nu}_{min}})$ *be a minimal admissible module of* $\mathrm{Cl}_{\mu,\nu}$ *for* $(\mu,\nu) \in \{(8,0),(0,8),(4,4)\}$ *and* J_{ζ_i}, $i = 1,\dots,8$ *the Clifford actions of an orthonormal basis* $\{\zeta_i\}$. *Then*

$$U^{r,s}_{min} \otimes V^{\mu,\nu}_{min} = (U^{r,s}_{min} \otimes E_{\mu,\nu}) \bigoplus_{i=1}^{8} \big(U^{r,s}_{min} \otimes J_{\zeta_i}(E_{\mu,\nu})\big) \bigoplus_{j=2}^{8} \big(U^{r,s}_{min} \otimes J_{\zeta_1} J_{\zeta_j}(E_{\mu,\nu})\big) \quad (3.17)$$

is a minimal admissible module $U^{r+\mu,s+\nu}_{min}$ *of the Clifford algebra* $\mathrm{Cl}_{r+\mu,s+\nu}$.

Conversely, if $U^{r+\mu,s+\nu}_{min}$ *is a minimal admissible module of the algebra* $\mathrm{Cl}_{r+\mu,s+\nu}$, *then the common 1-eigenspace* E_0 *of the involutions* T_i, $i = 1,2,3,4$ *from Example 1 can be considered as a minimal admissible module* $U^{r,s}_{min}$ *of the algebra* $\mathrm{Cl}_{r,s}$. *The action of the Clifford algebra* $\mathrm{Cl}_{r,s}$ *on* E_0 *is the restricted action of* $\mathrm{Cl}_{r+\mu,s+\nu}$ *obtained by the natural inclusion* $\mathrm{Cl}_{r,s} \subset \mathrm{Cl}_{r+\mu,s+\nu}$.

According to the correspondence between minimal admissible modules stated in Proposition 3.5.1, there is a natural injective map

$$\mathcal{B}\colon\ \mathcal{C}(\mathfrak{n}_{r,s}(U^{r,s}_{min})) \to \mathcal{C}(\mathfrak{n}_{r+\mu,s+\nu}(U^{r+\mu,s+\nu}_{min})). \quad (3.18)$$

Conversely, automorphisms of the form $A \oplus C \in \mathcal{C}(\mathfrak{n}_{r+\mu,s+\nu}(U^{r+\mu,s+\nu}_{min}))$ *with the property that* $C(\zeta_j) = \zeta_j$, $j = 1,\dots,8$, *defines an automorphism* $A_{|E_0} \oplus C_{|\mathbb{R}^{r,s}}$ *of the algebra* $\mathfrak{n}_{r,s}(E_0)$, *where the space* E_0 *is the common 1-eigenspace of the involutions* T_j, $j = 1,2,3,4$, *viewed as a minimal admissible module of* $\mathrm{Cl}_{r,s}$.

Corollary 3.5.2. *Let* $U^{r,s}$ *and* $U^{r+\mu,s+\nu} = U^{r,s} \otimes V^{\mu,\nu}_{min}$ *be admissible modules. Then*

$$\mathrm{Aut}^0(\mathfrak{n}_{r+\mu,s+\nu}((U^{r+\mu,s+\nu})) = \mathcal{B}\big(\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U^{r,s}))\big),$$

that is the group $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U^{r,s}))$ *is invariant under the map* $\mathcal{B}$ *defined in* (3.18).

Proof. The proof follows from Proposition (3.2.4). □

Finally, we state the result of the periodicity of isomorphisms for the Lie algebras.

Theorem 3.5.3. *[25, Theorem 4.6.1] The Lie algebras* $\mathfrak{n}_{r,s}(U^{r,s})$ *and* $\mathfrak{n}_{s,r}(U^{s,r})$ *are isomorphic if and only if the Lie algebras* $\mathfrak{n}_{r+\mu,s+\nu}(U^{r+\mu,s+\nu})$ *and* $\mathfrak{n}_{s+\nu,r+\mu}(U^{s+\nu,r+\mu})$ *are isomorphic for* $(\mu,\nu) \in \{(8,0),(0,8),(4,4)\}$.

3.6. Definition of classical groups

We aim to determine a subgroup $\mathbb{A}$ of the group $SL(n,\mathbb{R})$ such that if $A \in \mathbb{A}$, then $A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(U)$. In what follows we will identify $\mathbb{A}$ and $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$. The maps

$A\colon U \to U$ are linear maps over the field of real numbers. From the other side the admissible modules U can carry complex or quaternion structures such that the map A commutes with them. Thus, the map A has to be linear with respect to these additional algebras. We recall some useful embeddings of the algebras $\mathbb{C}$, $\mathbb{H}$ into the space of real matrices.

The use of the notation $*^T$ for the transpose matrix and the reverse ordered element in the Clifford algebras will not cause any confusion.

We write $\lambda = a + b\mathbf{i}$, $\mathbf{i}^2 = -1$, for $\lambda \in \mathbb{C}$ and $h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ for $h \in \mathbb{H}$. Recall that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{3.19}$$

We describe here the embeddings of the algebras $\mathbb{F} = \mathbb{C}, \mathbb{H}$ and square matrices $M(n, \mathbb{F})$ into the set of real square matrices $M(n, \mathbb{R})$ and complex square matrices $M(n, \mathbb{C})$, respectively. We define an embedding

$$\begin{aligned} \rho_{\mathbb{C}}\colon \qquad \mathbb{C} \qquad &\to M(2, \mathbb{R}) \\ \lambda = a + b\mathbf{i} &\mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \end{aligned} \tag{3.20}$$

Then one has

$$A = \rho_{\mathbb{C}}(A_{\mathbb{C}}) = \rho_{\mathbb{C}}\Big(\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}\Big) = \begin{pmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} \\ b_{11} & a_{11} & b_{12} & a_{12} \\ a_{21} & -b_{21} & a_{22} & -b_{21} \\ b_{21} & a_{21} & b_{22} & a_{22} \end{pmatrix}$$

for $\lambda_{kl} = a_{kl} + b_{kl}\mathbf{i}$. The map $\rho_{\mathbb{C}}$ is the algebra homomorphism:

$$\begin{gathered} \rho_{\mathbb{C}}(A_{\mathbb{C}}B_{\mathbb{C}}) = \rho_{\mathbb{C}}(A_{\mathbb{C}})\rho_{\mathbb{C}}(B_{\mathbb{C}}) = AB, \\ \rho_{\mathbb{C}}(\overline{\lambda}) = \big(\rho_{\mathbb{C}}(\lambda)\big)^T, \quad \lambda \in \mathbb{C}, \\ \rho_{\mathbb{C}}(\overline{A_{\mathbb{C}}}^T) = \big(\rho_{\mathbb{C}}(A_{\mathbb{C}})\big)^T = A^T, \quad A_{\mathbb{C}} \in M(n, \mathbb{C}), \end{gathered}$$

where superscript A^T denotes the transposition of A. Note also that if we denote by $\operatorname{diag}_n L$ a block-diagonal real matrix with the blocks L on the diagonal, then

$$\operatorname{diag}_n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rho_{\mathbb{C}}(A_{\mathbb{C}}) \operatorname{diag}_n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \rho_{\mathbb{C}}(\bar{A}_{\mathbb{C}}). \tag{3.21}$$

A quaternion number can be expressed by using complex numbers by

$$h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \lambda + \mathbf{j}\mu, \quad \lambda = a + b\mathbf{i}, \quad \mu = c + d\mathbf{i},$$

with the conjugation $\bar{h} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} = \bar{\lambda} - \mathbf{j}\bar{\mu}$. Thus we define

$$\begin{aligned} \rho_{\mathbb{H}}: \quad & \mathbb{H} \quad \to M(2,\mathbb{C}) \\ & h=\lambda+\mathbf{j}\mu \mapsto \begin{pmatrix} \lambda & -\overline{\mu} \\ \mu & \overline{\lambda} \end{pmatrix}. \end{aligned}$$

Consider the space $\mathbb{H}^n$ as a right quaternion space. Thus, $A_{\mathbb{H}}(vh) = (A_{\mathbb{H}}v)h$ for $h \in \mathbb{H}$, for the quaternion column vector $v \in \mathbb{H}^n$, and for the quaternion matrix $A_{\mathbb{H}}$. A column vector $h = (h_1, \ldots, h_n)^T \in \mathbb{H}^n$ with $h_l = \lambda_l + \mathbf{j}\mu_l$ will be represented by the column vector $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)^T \in \mathbb{C}^{2n}$. Then the quaternion matrix $Q_{\mathbb{H}} \in M(n,\mathbb{H})$ written as $Q_{\mathbb{H}} = \Lambda_{\mathbb{C}} + \mathbf{j}\Psi_{\mathbb{C}}$ with $\Lambda_{\mathbb{C}}, \Psi_{\mathbb{C}} \in M(n,\mathbb{C})$ will be represented as

$$\rho_{\mathbb{H}}(Q_{\mathbb{H}}) = \begin{pmatrix} \Lambda_{\mathbb{C}} & -\overline{\Psi_{\mathbb{C}}} \\ \Psi_{\mathbb{C}} & \overline{\Lambda_{\mathbb{C}}} \end{pmatrix} \in M(2n,\mathbb{C}).$$

This representation is convenient by the following reason: if $\mathbb{H} \ni h = \lambda + \mathbf{j}\mu$ is given as a column vector $\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$, then multiplication from the left by a complex matrix representation of a quaternion produces a new column vector representing the correct quaternion. The map $\rho_{\mathbb{H}}$ is also the algebra homomorphism:

$$\begin{aligned} \rho_{\mathbb{H}}(A_{\mathbb{H}}B_{\mathbb{H}}) &= \rho_{\mathbb{H}}(A_{\mathbb{H}})\rho_{\mathbb{H}}(B_{\mathbb{H}}), \\ \rho_{\mathbb{H}}(\overline{h}) &= \overline{(\rho_{\mathbb{H}}(h))}^T, \quad h \in \mathbb{H}, \\ \rho_{\mathbb{H}}(\overline{B_{\mathbb{H}}}^T) &= \overline{(\rho_{\mathbb{H}}(B_{\mathbb{H}}))}^T, \quad B_{\mathbb{H}} \in M(n,\mathbb{H}). \end{aligned}$$

We recall the following definitions of the classical groups that will be used in the sequel. The **general linear group** $\mathrm{GL}(n,\mathbb{F})$ of degree n over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is

$$\mathrm{GL}(n,\mathbb{F}) := \{M \in M(n,\mathbb{F}) \mid M \text{ is invertible}\}.$$

The **general orthogonal group** $\mathrm{O}(n,\mathbb{F})$ over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is

$$\mathrm{O}(n,\mathbb{F}) := \{M \in \mathrm{GL}(n,\mathbb{F}) \mid M^T M = \mathrm{Id}_n\},$$

where Id_n is the $(n \times n)$ identity matrix. In the case $\mathbb{F} = \mathbb{R}$ we also use the pseudo-orthogonal group $\mathrm{O}(p,q)$

$$\mathrm{O}(p,q) := \{M \in \mathrm{GL}(p+q,\mathbb{R}) \mid M^T \mathrm{Id}_{p,q} M = \mathrm{Id}_{p,q}\}, \qquad \mathrm{Id}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}.$$

All the groups over $\mathbb{R}$ preserving a symmetric bilinear form of index (p,q) are isomorphic to $\mathrm{O}(p,q)$. The groups over $\mathbb{C}$ preserving a symmetric bilinear form of index (p,q) are isomorphic to $\mathrm{O}(n,\mathbb{C})$ with $n = p+q$, see [43, Chapter 3.1].

The **symplectic group** $\mathrm{Sp}(2n,\mathbb{F})$ of degree $2n$ over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is

$$\mathrm{Sp}(2n,\mathbb{F}) := \{M \in \mathrm{GL}(2n,\mathbb{F}) \mid M^T \Omega_n M = \Omega_n\}, \qquad \Omega_n = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

All the groups preserving a skew-symmetric bilinear form are isomorphic to $\mathrm{Sp}(2n, \mathbb{F})$.

The **general unitary group** $\mathrm{U}(p,q)$ of degree n is

$$\mathrm{U}(p,q) := \{M \in \mathrm{GL}(n,\mathbb{C}) \mid \overline{M}^T \mathrm{Id}_{p,q} M = \mathrm{Id}_{p,q}\}.$$

The subgroup $\mathrm{U}(p,0) \subset \mathrm{U}(p,q)$ is denoted by $\mathrm{U}(p)$. Note that from a qualitative point of view, consideration of skew-Hermitian forms (up to isomorphism) provides no new groups, since the multiplication by $\mathbf{i}$ renders a skew-Hermitian form Hermitian, and vice versa. Thus only the Hermitian case needs to be considered.

Now we turn to define the groups over the algebra $\mathbb{H}$. Under the identification described above

$$\begin{aligned}
\mathrm{GL}(n,\mathbb{H}) &= \{M \in \mathrm{GL}(2n,\mathbb{C}) \mid \ \Omega_n M = \overline{M}\Omega_n, \quad \det M \neq 0\} \\
\mathrm{SL}(n,\mathbb{H}) &= \{M \in \mathrm{GL}(n,\mathbb{H}) \mid \ \det M = 1\}, \\
\mathrm{Sp}(p,q) &= \{M \in \mathrm{GL}(n,\mathbb{H}) \mid \ \overline{M}^T \mathrm{Id}_{p,q} M = \mathrm{Id}_{p,q}, \ p+q=n\} \\
&= \Big\{M \in \mathrm{GL}(2n,\mathbb{C}) \mid \ \overline{M}^T \operatorname{diag}\begin{pmatrix} \mathrm{Id}_{p,q} & 0 \\ 0 & \mathrm{Id}_{p,q}\end{pmatrix} M = \operatorname{diag}\begin{pmatrix} \mathrm{Id}_{p,q} & 0 \\ 0 & \mathrm{Id}_{p,q}\end{pmatrix}\Big\}.
\end{aligned}$$

The group $\mathrm{Sp}(p,q)$ is called **quaternionic unitary group**. If $p=0$ or $q=0$, then $\mathrm{Sp}(0,p) \cong \mathrm{Sp}(p,0)$ is denoted by $\mathrm{U}(p,\mathbb{H})$ and called **hyperunitary group.** The reason for the notation $\mathrm{Sp}(p,q)$ is that this group can be represented, as a subgroup of $\mathrm{Sp}(2n,\mathbb{C})$ preserving an Hermitian form of signature $(2p,2q)$ for $p+q=n$.

The last group is the **quaternionic orthogonal group** denoted by $\mathrm{O}^*(2n) = \mathrm{O}(n,\mathbb{H})$ and it is defined by

$$\begin{aligned}
\mathrm{O}^*(2n) = \mathrm{O}(n,\mathbb{H}) &= \{M \in \mathrm{GL}(n,\mathbb{H}) \mid \ \bar{M}^T \operatorname{diag}_n \mathbf{j} M = \ \operatorname{diag}_n \mathbf{j}\} \\
&= \{M \in \mathrm{GL}(2n,\mathbb{C}) \mid \ \bar{M}^T \Omega_n M = \Omega_n\}.
\end{aligned}$$

Here $\mathbf{j}$ is the quaternionic unit represented by $\rho_{\mathbb{H}}(\mathbf{j}) \in \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \in M(2,\mathbb{C})$. The definition of $\mathrm{O}^*(2n) = \mathrm{O}(n,\mathbb{H})$ can be given equivalently as

$$\begin{aligned}
\mathrm{O}^*(2n) = \mathrm{O}(n,\mathbb{H}) &= \{M \in \mathrm{GL}(n,\mathbb{H}) \mid \ \bar{M}^T \operatorname{diag}_n \mathbf{i} M = \ \operatorname{diag}_n \mathbf{i}\} \\
&= \{M \in \mathrm{GL}(n,\mathbb{H}) \mid \ \bar{M}^T \operatorname{diag}_n \mathbf{k} M = \ \operatorname{diag}_n \mathbf{k}\}.
\end{aligned}$$

This is true due to the fact that by conjugation with some $h, \tilde{h} \in \mathrm{Sp}(1) = \mathrm{U}(1,\mathbb{H})$ we can get $h\mathbf{i}h^{-1} = \mathbf{j}$ and analogously $\tilde{h}\mathbf{k}\tilde{h}^{-1} = \mathbf{j}$. The group $\mathrm{O}^*(2n) = \mathrm{O}(n,\mathbb{H})$ can be viewed as a subgroup of $\mathrm{O}(2n,\mathbb{C})$ that preserves an Hermitian form of index (n,n). Particularly, if $n=1$, then one needs to check the condition

$$\begin{pmatrix} \bar{\lambda} & \bar{\mu} \\ -\mu & \lambda \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & -\bar{\mu} \\ \mu & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with $\lambda = a + \mathbf{i}b$, $\mu = c + \mathbf{i}d$. It leads to the solution of the system

$$\begin{cases} \mathrm{Im}(\bar{\lambda}\mu) = 0 \\ \lambda^2 + \mu^2 = 1 \end{cases} \quad \Longrightarrow \quad \begin{cases} ad = bc \\ ab + cd = 0 \\ a^2 - b^2 + c^2 - d^2 = 1 \end{cases} \quad \Longrightarrow \quad \begin{cases} a^2 + c^2 = 1 \\ b = d = 0. \end{cases}$$

Thus

$$\begin{pmatrix} \lambda & -\bar{\mu} \\ \mu & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} = \alpha = a + \mathbf{i}c \quad \text{and} \quad a^2 + c^2 = |\alpha|^2 = 1,$$

and we conclude that $\mathrm{O}^*(2) \cong \mathrm{U}(1)$.

4. Determination of $\mathbf{Aut^0}(\mathfrak{n}_{r,s}(U))$

4.1. Integral basis

Definition 4.1.1. We fix the standard orthonormal basis $\{z_k\}$ for $\mathbb{R}^{r,s}$. Then we call a basis $\{x_i\}$ of the minimal admissible module V_{min}, an integral basis with respect to the orthonormal basis $\{z_k\}$, if it satisfies the conditions that

- the basis $\{x_i\}$ is orthonormal with respect to the admissible scalar product,
- for any z_k and x_i, there exists a unique x_j such that either $J_{z_k}(x_i) = x_j$ or $J_{z_k}(x_i) = -x_j$.

One way to construct such a basis is given by taking a suitable vector $v \in E_{r,s}$ and choosing an orthonormal basis for $V^{r,s}_{min}$ from the vectors

$$\{v, \pm J_{z_k}v, \ldots, \pm J_{z_{k_1}}J_{z_{k_2}}\ldots J_{z_{k_l}}v, \ldots, \pm J_{z_1}J_{z_2}\ldots J_{z_{r+s}}v,\ 1 \le k_1 < \ldots < k_l \le r+s\}.$$

The choice of the integral basis is not unique. For the construction of an integral basis for the H-type Lie algebras $\mathfrak{n}_{r,0}(U)$, see [13]. The presence of an integral basis on a Lie algebra guaranties the existence of a lattice of the corresponding Lie group, see [37]. Nevertheless, once we fix an integral basis, we denote by η the matrix of the admissible scalar product. Thus either $\eta = \mathrm{Id}_{2n}$ or $\eta = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & -\mathrm{Id}_n \end{pmatrix}$ according to the ordering from positive vectors to negative vectors of a fixed integral basis. The construction of an integral basis can be found in [23].

Recall that $J^{\tau}_{z_i}$ is the transposition with respect to an admissible scalar product and $J^T_{z_i}$ the transposition with respect to the standard Euclidean scalar product. The relation between two transpositions is given by $J^{\tau}_{z_i} = \eta J^T_{z_i}\eta$.

Lemma 4.1.2. *If $J^{\tau}_{z_i} = -J_{z_i}$, $J^2_{z_i} = \pm\mathrm{Id}$, $i = 1, 2, 3$, $J_{z_i}J_{z_j} = -J_{z_j}J_{z_i}$, $i, j = 1, 2, 3$, $i \neq j$, and $\eta^T = \eta$, $\eta^2 = \mathrm{Id}$ is non-degenerate bi-linear form, then*

(1.) $(\eta J_{z_i})^T = -\eta J_{z_i}$;
(2.) $(\eta J_{z_k})^2 = -\operatorname{Id}$;
(3.) $\eta J_{z_i} = \begin{cases} -J_{z_i}\eta & \textit{if} \quad J_{z_i}^2 = \operatorname{Id} \\ J_{z_i}\eta & \textit{if} \quad J_{z_i}^2 = -\operatorname{Id} \end{cases}$;

Proof. (1.) We obtain $(\eta J_{z_i})^T = J_{z_{z_i}}^T \eta^T = -\eta J_{z_i}$ from $\eta J_{z_i} = -J_{z_i}^T \eta$.

(2.) We consider four cases:

(a) Let $J_{z_i}^2 = -\operatorname{Id}$ and x_j an element of the integral basis such that $\langle x_j, x_j \rangle > 0$. Then $\eta J_{z_i} x_j = J_{z_i} x_j$ and $(\eta J_{z_i})^2(x_j) = \eta J_{z_i}^2 x_j = -x_j$.

(b) Let $J_{z_i}^2 = -\operatorname{Id}$ and $\langle x_j, x_j \rangle < 0$. Then $\eta J_{z_i} x_j = -J_{z_i} x_j$ and

$$(\eta J_{z_i})^2(x_j) = -\eta J_{z_i}^2 x_j = \eta x_j = -x_j.$$

(c) Let $J_{z_i}^2 = \operatorname{Id}$ and $\langle x_j, x_j \rangle > 0$. Then $\eta J_{z_i} x_j = -J_{z_i} x_j$ and

$$(\eta J_{z_i})^2(x_j) = -\eta J_{z_i}^2 x_j = -\eta x_j = -x_j.$$

(d) Let $J_{z_i}^2 = \operatorname{Id}$ and $\langle x_j, x_j \rangle < 0$. Then $\eta J_{z_i} x_j = J_{z_i} x_j$ and

$$(\eta J_{z_i})^2(x_j) = \eta J_{z_i}^2 x_j = \eta x_j = -x_j.$$

(3.) The property $\eta J_{z_i} \eta J_{z_i} = -\operatorname{Id}$ implies $J_{z_i} \eta J_{z_i}^2 = -\eta J_{z_i}$. □

4.2. Description of the procedure of determination of $\operatorname{Aut}^0(\mathfrak{n}_{r,s}(U))$

In this section, we describe step by step the procedure of determination of $\operatorname{Aut}^0(\mathfrak{n}_{r,s}(U))$.

Step 1. We determine the groups $\operatorname{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}))$ for the basic cases (2.5). According to Corollary 3.5.2 it provides the groups for all range of (r,s). Thus, the next steps are explained only for basic cases.

Step 2. We determine the groups $\operatorname{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min}))$ for minimal admissible modules.

2.1 We find the sets $PI_{r,s}$ of involutions of all types (1)-(5) and their subsets $P^*_{r,s} \subset PI_{r,s}$ that are involutions of types (1)-(3). We write P_k for the operators from $PI_{r,s}$. We denote by $E^*_{r,s}$ the common 1-eigenspace of involutions from $P^*_{r,s}$ and $E_{r,s}$ the common 1-eigenspace of involutions from $PI_{r,s}$. We find operators that commute with all involutions from $P^*_{r,s}$. These operators will leave the space $E^*_{r,s}$ invariant. Among these operators we denote by **I** the almost complex structure, and by **I**, **J**, **K** the almost quaternion structure, i.e. the operators satisfying (3.19) and being the product of an even number of J_{z_k}. We use the notation **Q** for a negative operator

$\mathbf{Q} = J_{z_i}J_{z_j}$ such that $\mathbf{Q}^2 = \mathrm{Id}$. Apart from mentioned operators it could be at most one more, denoted by $\mathbf{\Pi}$ that is the product of an even number of J_{z_k} commuting with all involutions from $P^*_{r,s}$. All these operators will be indicated for each case in tables. We denote by A an operator on $P^*_{r,s}$ such that $A \oplus \mathrm{Id} \in \mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min}))$ and satisfying (3.8).

2.2 We choose an integral basis generated from a vector $v \in E_{r,s}$, $\langle v, v \rangle_{E_{r,s}} = 1$. Here we emphasise that $E_{r,s} \subset E^*_{r,s}$ is the common 1-eigenspace of all types of involutions from $PI_{r,s}$. The details of the construction of the integral basis can be found in [24]. The basis of $E^*_{r,s}$ will be indicated for each case. We use the black colour to denote the basis vectors x_k such that $\langle x_k, x_k \rangle_{E^*_{r,s}} = 1$ and use the red colour for the basis vectors x_l such that $\langle x_l, x_l \rangle_{E^*_{r,s}} = -1$.

2.3 In this step we distinguish 6 possible collections of operators $\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{Q}, \mathbf{\Pi}$ on $E^*_{r,s}$ that leave it invariant.

2.3.1 **The set $E^*_{r,s}$ has neither complex, quaternion structure, no operator Q.** In this case the operator $A\colon E^*_{r,s} \to E^*_{r,s}$ is real. In the presence of an operator $\mathbf{\Pi}$ we check the condition (3.8), that we write in the form:

$$A^T \eta\, \mathbf{\Pi}\, A = \eta\, \mathbf{\Pi}\,. \tag{4.1}$$

These are the cases

$$(r, s) \in \{(1, 0), (0, 1), (7, 0), (0, 7), (8, 0), (0, 8), (3, 4), (4, 3), (4, 4)\}.$$

2.3.2 **The set $E^*_{r,s}$ has a complex structure, but neither quaternion structure, no operator Q.** Since A commutes with $\mathbf{I}$ we conclude that $A \in \mathrm{GL}(k, \mathbb{C})$, where $k = \dim_{\mathbb{C}}(E^*_{r,s})$. If there is no operator $\mathbf{\Pi}$ on $E^*_{r,s}$, then $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min})) = \mathrm{GL}(k, \mathbb{C})$. Otherwise we check the condition (4.1). There are two options: if the map $\eta\,\mathbf{\Pi}$ is complex liner ($\eta\,\mathbf{\Pi}$ commutes with $\mathbf{I}$), then

$$\mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min})) \cong \mathrm{Sp}(k, \mathbb{C}) \quad \text{or} \quad \mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min})) \cong \mathrm{U}(k).$$

If the operator $\eta\,\mathbf{\Pi}$ is not complex linear, then $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min})) \cong \mathrm{O}(k, \mathbb{C})$. These are the cases

$$(r, s) \in \{(2, 0), (0, 2), (6, 0), (0, 6), (2, 4), (4, 2), (3, 5), (5, 3), (7, 1), (1, 7)\}.$$

2.3.3 **The set $E^*_{r,s}$ has a quaternion structure, and has no operator Q.** Since A commutes with $\mathbf{I}, \mathbf{J}, \mathbf{K}$ we conclude $A \in \mathrm{GL}(k, \mathbb{H})$, where $k = \dim_{\mathbb{H}}(E^*_{r,s})$. All the operators $\eta\,\mathbf{\Pi}$ will be quaternion linear and by checking (4.1), we make the conclusions in the cases

$$(r, s) \in \{(3, 0), (0, 3), (4, 0), (0, 4), (5, 0), (0, 5), (4, 1), (1, 4), (5, 2), (2, 5),$$
$$(6, 1), (1, 6), (6, 2), (2, 6), (6, 3), (3, 6), (7, 2), (2, 7)\}.$$

2.3.4 **The set $E^*_{r,s}$ has an operator $\mathbf{Q}$ and neither has complex no quaternion structure.** In the presence of the operator $\mathbf{Q}$ we decompose $E^*_{r,s}$ into eigenspaces of the involution $\mathbf{Q}$ that we denote by $N_{\pm}$. Thus $E^*_{r,s} = N_+ \oplus N_-$. Since A commutes with $\mathbf{Q}$, we get $A = A_+ \oplus A_-$, where $A_{\pm}\colon N_{\pm} \to N_{\pm}$. We check (4.1) and make the conclusion. Since in this case there are no other conditions on $A_{\pm}$ the group $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min}))$ will be given by the direct product of subgroups of $\mathrm{GL}(k,\mathbb{R})$ with $k = \dim(N_{\pm})$. These are the cases

$$(r,s) \in \{(1,1),(3,3)\}.$$

2.3.5 **The set $E^*_{r,s}$ has a complex structure and an operator $\mathbf{Q}$ but does not have a quaternion structure.** We start from the decompositions $E^*_{r,s} = N_+ \oplus N_-$ and $A = A_+ \oplus A_-$. In all these cases we have $\mathbf{Q}\,\mathbf{I} = -\mathbf{I}\,\mathbf{Q}$ and therefore we define $A_- = -\mathbf{I}A_+\mathbf{I}$. If it needs, we check (4.1) on N_+ and make the conclusions. These are the cases

$$(r,s) \in \{(2,2),(3,2),(2,3),(2,1),(1,2)\}.$$

2.3.6 **The set $E^*_{r,s}$ has a quaternion structure and an operator $\mathbf{Q}$.** We start from the decompositions $E^*_{r,s} = N_+ \oplus N_-$ and $A = A_+ \oplus A_-$. The result depends on the situation whether N_+ carries the complex or quaternion structure. These are the cases

$$(r,s) \in \{(3,1),(1,3),(5,1),(1,5),(7,3),(3,7)\}.$$

2.4 Having in hands the operator $A\colon E^*_{r,s} \to E^*_{r,s}$, we can extend it to the operator $\overline{A}\colon V^{r,s}_{min} \to V^{r,s}_{min}$. The operator $\overline{A}$ is completely and uniquely determined by the operator A according to Theorem 3.3.4. To match the notation of the present description and Theorem 3.3.4 we note that $E^*_{r,s} = E^1$ and $A = A^1$ in Theorem 3.3.4. The operators G_I used for the construction of $\overline{A}$ are indicated for all the cases in tables. We emphasise that we present only some of the operators G_I, since the extension of $\overline{A}$ from A does not depend on the choice of a specific operator G_I, but only on its existence. The map $\overline{A}$ will satisfy (3.8) by Theorem 3.3.4. Thus the group $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}_{min}))$ is already defined in steps 2.3.1-2.3.6.

Step 3. We determine the groups $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(V^{r,s}))$ for arbitrary admissible modules $V^{r,s} = \oplus V^{r,s}_{min}$. It follows from the following procedure. We decompose the module $V^{r,s}$ into the orthogonal direct sum (3.16) of minimal admissible modules following the classification of Theorem 3.4.2. We write $V^{r,s} \supset E = \sum_{l=1}^{p} \oplus (E^*_{r,s})_l$, where $(E^*_{r,s})_l \subset (V^{r,s}_{min})_l$. In each $(E^*_{r,s})_l$ will be chosen a vector v_l, with $\langle v_l, v_l\rangle_{(E_{r,s})_l} = \pm 1$, generating an orthonormal basis on $(V^{r,s}_{min})_l$. We draw the attention of the reader to the fact that $\langle v_l, v_l\rangle_{(E_{r,s})_l} = 1$ if $(E_{r,s})_l \in (V^{r,s;+}_{min})_l$, $\langle v_l, v_l\rangle_{(E_{r,s})_l} = -1$ if $(E_{r,s})_l \in (V^{r,s;-}_{min})_l$ and

Table 3
Groups $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$.

8	$\mathrm{GL}(p,\mathbb{R})$								
7	$\mathrm{O}(p,p)$	$\mathrm{U}(p,p)$	$\mathrm{Sp}(p,p)$	$\mathrm{Sp}(p,q)\times\mathrm{Sp}(p,q)$					
6	$\mathrm{O}(2p,\mathbb{C})$	$\mathrm{O}^*(2p)$	$\mathrm{GL}(p,\mathbb{H})$	$\mathrm{Sp}(p,q)$					
5	$\mathrm{O}^*(4p)$	$\mathrm{O}^*(2p)\times\mathrm{O}^*(2p)$	$\mathrm{O}^*(2p)$	$\mathrm{U}(p,q)$					
4	$\mathrm{GL}(p,\mathbb{H})$	$\mathrm{O}^*(2p)$	$\mathrm{O}(p,\mathbb{C})$	$\mathrm{O}(p,q)$	$\mathrm{GL}(p,\mathbb{R})$				
3	$\mathrm{Sp}(p,p)$	$\mathrm{U}(p,p)$	$\mathrm{O}(p,p)$	$\mathrm{O}(p,q)\times\mathrm{O}(p,q)$	$\mathrm{O}(p,p)$	$\mathrm{U}(p,p)$	$\mathrm{Sp}(p,p)$	$\mathrm{Sp}(p,q)\times\mathrm{Sp}(p,q)$	
2	$\mathrm{Sp}(2,\mathbb{C})$	$\mathrm{Sp}(2p,\mathbb{R})$	$\mathrm{GL}(2p,\mathbb{R})$	$\mathrm{O}(2p,2q)$	$\mathrm{O}(2p,\mathbb{C})$	$\mathrm{O}^*(2p)$	$\mathrm{GL}(p,\mathbb{H})$	$\mathrm{Sp}(p,q)$	
1	$\mathrm{Sp}(2p,\mathbb{R})$	$\mathrm{Sp}(2p,\mathbb{R})\times\mathrm{Sp}(2p,\mathbb{R})$	$\mathrm{Sp}(4p,\mathbb{R})$	$\mathrm{U}(2p,2q)$	$\mathrm{O}^*(4p)$	$\mathrm{O}^*(2p)\times\mathrm{O}^*(2p)$	$\mathrm{O}^*(2p)$	$\mathrm{U}(p,q)$	
0		$\mathrm{Sp}(2p,\mathbb{R})$	$\mathrm{Sp}(2p,\mathbb{C})$	$\mathrm{Sp}(p,q)$	$\mathrm{GL}(p,\mathbb{H})$	$\mathrm{O}^*(2p)$	$\mathrm{O}(p,\mathbb{C})$	$\mathrm{O}(p,q)$	$\mathrm{GL}(p,\mathbb{R})$
	0	1	2	3	4	5	6	7	8

always $\langle v_l, v_l \rangle_{(E_{r,s})_l} = 1$ for $(E_{r,s})_l \in (V^{r,s;N}_{min})_l$. We write $v = \oplus_{l=1}^{l=p} v_l$ for the generating vector on $E \subset V^{r,s}$. The result for $E \subset V^{r,s}$ is the direct sum of the results for $(E^*_{r,s})_l \subset (V^{r,s}_{min})_l$, $l = 1, \dots, p$, that will allow us to make the conclusion in each case.

We list the final result and then we proceed to consider case by case.

In the following sections, we will write the calculation in the order that was described in steps 2.3.1-2.3.6. We write J_k for J_{z_k} for shortness.

4.3. Modules over $\mathbb{R}$

4.3.1. $\dim_{\mathbb{R}}(E^*_{r,s}) = 1$: cases $\mathfrak{n}_{7,0}(U)$, $\mathfrak{n}_{3,4}(U)$, $\mathfrak{n}_{8,0}(U)$, $\mathfrak{n}_{4,4}(U)$, $\mathfrak{n}_{0,8}(U)$

$V^{7,0}_{\min}$									dim = 8
$E^{\pm}_{P_1}$	+				-				dim = 4
$E^{\pm}_{P_2}$	+		-		+		-		dim = 2
$E^{\pm}_{P_3}$	+ $E^*_{7,0}$	-	+	-	+	-	+	-	dim = 1
Basis for $E^*_{7,0}$	v	...	...	...	...	...	...	...	$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_3J_4J_6$ $P_4 = J_1J_2J_3$ $\mathbf{\Pi} = J_1J_2J_3$
G_I		J_3	J_7	J_6	J_5	J_4	J_2	J_1	

There are four types of minimal admissible modules:

$$V^{7,0;+}_{min;+}, \quad V^{7,0;-}_{min;+}, \quad V^{7,0;+}_{min;-}, \quad V^{7,0;-}_{min;-}.$$

According to the classification Theorem 3.4.2, we can reduce the consideration to the non-isotypic (p, q)-module

$$U = \big(\overset{p}{\oplus} V^{7,0;+}_{min;+}\big) \oplus \big(\overset{q}{\oplus} V^{7,0;-}_{min;+}\big). \tag{4.2}$$

We consider non-isotypic (p, q)-module (4.2) and a vector space $E = \big(\overset{p}{\oplus} (E^*_{7,0})^+\big) \oplus \big(\overset{q}{\oplus} (E^*_{7,0})^-\big)$, with $(E^*_{7,0})^+ \subset V^{7,0;+}_{min;+}$ and $(E^*_{7,0})^- \subset V^{7,0;-}_{min;+}$. Note that $\mathbf{\Pi}$ acts as Id on E and $\eta = \mathrm{Id}_{p,q}$. The unique condition that needs to be checked is

$$A^T \eta\, \mathbf{\Pi}\, A = \eta\, \mathbf{\Pi} \quad \Longleftrightarrow \quad A^T\, \mathrm{Id}_{p,q}\, A = \mathrm{Id}_{p,q}\,.$$

We conclude $\mathrm{Aut}^0(\mathfrak{n}_{7,0}(U)) = \mathrm{O}(p, q)$.

Structure of the minimal admissible modules and the involutions for $\mathfrak{n}_{3,4}(U)$ are similar to $\mathfrak{n}_{7,0}(U)$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{3,4}(U)) \cong \mathrm{O}(p, q)$ for a non-isotypic (p, q)-module U.

$V_{min}^{8,0}$																	dim = 16
$E_{P_1}^{\pm}$	+								-								dim = 8
$E_{P_2}^{\pm}$	+				-				+				-				dim = 4
$E_{P_3}^{\pm}$	+		-		+		-		+		-		+		-		dim = 2
$E_{P_4}^{\pm}$	$+\ E_{8,0}^{*}$	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-	dim = 1
Basis for $E_{8,0}^{*}$	v	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_2J_7J_8$ $P_4 = J_1J_3J_5J_7$
G_I		J_1J_2	J_8	J_7	J_6	J_5	J_1J_3	J_1J_4	J_4	J_3	J_1J_5	J_1J_6	J_1J_7	J_1J_8	J_2	J_1	

The tables for $(r, s) \in \{(0, 8), (4, 4)\}$ are the same. There are no operators $\mathbf{\Pi}$ leaving the space $E = \overset{p}{\oplus} E_{r,s}^{*}$, $(r, s) \in \{(8, 0), (0, 8), (4, 4)\}$, invariant. This means that there are no restrictions on the group of automorphisms acting on an admissible module. We conclude that $\mathrm{Aut}^0(U) = \mathrm{GL}(p, \mathbb{R})$ for $U = \overset{p}{\oplus} V_{min}^{r,s;+}$ and for $(r, s) \in \{(8, 0), (0, 8), (4, 4)\}$.

4.3.2. $\dim_{\mathbb{R}}(E_{r,s}^{*}) = 2$*: cases* $\mathfrak{n}_{1,0}(U)$*,* $\mathfrak{n}_{0,1}(U)$*;* $\mathfrak{n}_{0,7}(U)$*,* $\mathfrak{n}_{4,3}(U)$

$\mathfrak{n}_{1,0}$		dim=2
Basis	$x_1 = v$ $x_2 = J_1 v$	

$\mathfrak{n}_{0,1}$		dim=2
Basis	$x_1 = v$ $x_2 = J_1 v$	

Let $U = \overset{p}{\oplus} V_{min}^{1,0;+}$. In this case $A \in \mathrm{Aut}^0(\mathfrak{n}_{1,0}(U))$ has to fulfil the relation $A^T J_1 A = J_1$ for $J_1 = \mathrm{diag}_p \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We conclude $\mathrm{Aut}^0(\mathfrak{n}_{1,0}(U)) \cong \mathrm{Sp}(2p, \mathbb{R})$

Let $U = \overset{p}{\oplus} V_{min}^{0,1;N}$. Then $A^T \eta J_1 A = \eta J_1$, where $\eta J_1 = \mathrm{diag}_p \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It follows that $\mathrm{Aut}^0(\mathfrak{n}_{0,1}(U)) = \mathrm{Sp}(2p, \mathbb{R})$ as in the previous case.

$V_{min}^{0,7}$								dim = 16
$E_{P_1}^{\pm}$	+				-			dim = 8
$E_{P_2}^{\pm}$	+		-		+		-	dim = 4
$E_{P_3}^{\pm}$	$+\ E_{0,7}^{*}$	-	+	-	+	-	+	-	dim = 2
Basis for $E_{0,7}^{*}$	$x_1 = v$ $x_2 = J_1J_2J_7 v$	$\dots$ $\dots$	$\dots$ $\dots$	$\dots$ $\dots$	$\dots$ $\dots$	$\dots$ $\dots$	$\dots$ $\dots$	$\dots$ $\dots$	$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_3J_5J_7$ $\mathbf{\Pi} = J_1J_2J_7$
G_I		J_7	J_6	J_5	J_4	J_3	J_2	J_1	

We need to check the condition

$$A^T \eta\, \mathbf{\Pi}\, A = \eta\, \mathbf{\Pi} \quad \Longleftrightarrow \quad A^T \,\mathrm{diag}_p \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \mathrm{diag}_p \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.3}$$

In the basis $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ for $E_{0,7}^{*} \subset V_{min}^{0,7;N}$ condition (4.3) becomes

$$A^T \,\mathrm{diag}_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A = \mathrm{diag}_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We conclude that $\mathrm{Aut}^0(\mathfrak{n}_{0,7}(U)) \cong \mathrm{O}(p, p)$ for $U = \overset{p}{\oplus} V_{min}^{0,7;N}$.

For the case $\mathfrak{n}_{4,3}(U)$ the system of involutions and operators are similar to $\mathfrak{n}_{0,7}(U)$. We conclude that $\mathrm{Aut}^0(\mathfrak{n}_{4,3}(U)) = \mathrm{O}(p,p)$.

4.4. Modules over $\mathbb{C}$

In this section we first consider the cases when the operators $\eta\,\mathbf{\Pi}_k$ are complex linear, or in other words they commute with the almost complex structure $\mathbf{I}$. In this case the group of automorphisms is related to unitary transformations. The last part of the cases is related to the situations when the operators $\eta\,\mathbf{\Pi}_k$ are not complex linear.

4.4.1. $\dim_{\mathbb{C}}(E^*_{r,s}) = 1$*: cases* $\mathfrak{n}_{7,1}(U)$, $\mathfrak{n}_{3,5}(U)$; $\mathfrak{n}_{6,0}(U)$, $\mathfrak{n}_{2,4}(U)$

$V^{7,1}_{min}$									dim = 16
$E^{\pm}_{P_1}$	+				-				dim = 8
$E^{\pm}_{P_2}$	+		-		+		-		dim = 4
$E^{\pm}_{P_3}$	+ $E^*_{7,1}$	-	+	-	+	-	+	-	dim = 2
Basis for $E^*_{7,1}$	$x_1 = v$	…	…	…	…	…	…	…	$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_3J_5J_7$
	$x_2 = \mathbf{I}v$	…	…	…	…	…	…	…	$P_4 = J_1J_2J_3$ $\mathbf{I} = J_1J_2J_3J_8$ $\mathbf{\Pi} = J_1J_2J_3$
G_I		J_3	J_6	J_7	J_4	J_5	J_2	J_1	

We have $E^*_{7,1} = E^{+1}_{\mathbf{\Pi}} \oplus E^{-1}_{\mathbf{\Pi}}$, with $E^{+1}_{\mathbf{\Pi}} = \mathrm{span}\{v\}$ and $E^{-1}_{\mathbf{\Pi}} = \mathrm{span}\{\mathbf{I}v\}$. We let $U = (\overset{p}{\oplus}\, V^{7,1;+}_{min}) \oplus (\overset{q}{\oplus}\, V^{7,1;-}_{min})$. Since $\eta\,\mathbf{\Pi}$ is complex linear, we need to check

$$A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi} \quad\Longleftrightarrow\quad \bar{A}^T_{\mathbb{C}}\,\mathrm{Id}_{p,q}\,A_{\mathbb{C}} = \mathrm{Id}_{p,q}\,. \tag{4.4}$$

Here we used the embedding (3.20) and denoted by $A_{\mathbb{C}}$ the matrix with complex entries such that $\rho_{\mathbb{C}}(A_{\mathbb{C}}) = A$. It shows that $A \in \mathrm{U}(p,q)$ and $\mathrm{Aut}^0(\mathfrak{n}_{7,1}(U)) \cong \mathrm{U}(p,q)$.

The table and calculations for $\mathfrak{n}_{3,5}(U)$ are analogous to $\mathfrak{n}_{7,1}(U)$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{3,5}(U)) = \mathrm{U}(p,q)$ for $U = (\overset{p}{\oplus}\, V^{3,5;+}_{min}) \bigoplus (\overset{q}{\oplus}\, V^{3,5;-}_{min})$.

We consider now cases when the operators $\eta\,\mathbf{\Pi}$ are not complex linear.

$V^{6,0}_{min}$					dim = 8
$E^{\pm}_{P_1}$	+		-		dim = 4
$E^{\pm}_{P_2}$	+ $E^*_{6,0}$	-	+	-	dim = 2
Basis for $E^*_{6,0}$	$x_1 = v$	…	…	…	$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_3J_5$
	$x_2 = \mathbf{I}v$	…	…	…	$\mathbf{I} = J_1J_2$ $\mathbf{\Pi} = J_1J_3J_5$
G_I		J_5	J_3	J_1	

We have $E^*_{6,0} = E^+_{\mathbf{\Pi}} \oplus E^-_{\mathbf{\Pi}}$ with $E^+_{\mathbf{\Pi}} = \operatorname{span}\{v\}$, $E^-_{\mathbf{\Pi}} = \operatorname{span}\{\mathbf{I}v\}$ and $A \in \mathrm{GL}(1, \mathbb{C})$. We also have that $\mathbf{\Pi}\,\mathbf{I} = -\mathbf{I}\,\mathbf{\Pi}$ with $\mathbf{\Pi} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We obtain

$$A^T\,\mathbf{\Pi}\,A = \mathbf{\Pi} \quad \Longleftrightarrow \quad \mathbf{\Pi}\,\rho_{\mathbb{C}}(\bar{A}^T_{\mathbb{C}})\,\mathbf{\Pi}\,\rho_{\mathbb{C}}(A_{\mathbb{C}}) = \mathrm{Id}\,. \tag{4.5}$$

By making use of (3.21), we conclude that $A^T_{\mathbb{C}} A_{\mathbb{C}} = \mathrm{Id}$. For an admissible module $U = (\overset{p}{\oplus}\, V^{6,0;+}_{min})$ we obtain $\mathrm{Aut}^0(\mathfrak{n}_{6,0}(U)) = \mathrm{O}(p, \mathbb{C})$.

Calculations and the table for $\mathfrak{n}_{2,4}(U)$ are similar to the case $\mathfrak{n}_{6,0}(U)$. Thus $\mathrm{Aut}^0(\mathfrak{n}_{2,4}(U)) = \mathrm{O}(p, \mathbb{C})$.

4.4.2. $\dim_{\mathbb{C}}(E^*_{r,s}) = 2$*: cases* $\mathfrak{n}_{1,7}(U)$, $\mathfrak{n}_{5,3}(U)$; $\mathfrak{n}_{2,0}(U)$, $\mathfrak{n}_{0,2}(U)$; $\mathfrak{n}_{0,6}(U)$, $\mathfrak{n}_{4,2}(U)$

$V^{1,7}_{min}$									dim = 32
$E^{\pm}_{P_1}$	+				-				dim = 16
$E^{\pm}_{P_2}$	+		-		+		-		dim = 8
$E^{\pm}_{P_3}$	+ $E^*_{1,7}$	-	+	-	+	-	+	-	dim = 4
Basis for $E^*_{1,7}$	$x_1 = v$	...	...	...	...	...	...	...	$P_1 = J_2J_3J_4J_5$
	$x_2 = \mathbf{I}v$	...	...	...	...	...	...	...	$P_2 = J_2J_3J_6J_7$ $P_3 = J_2J_4J_6J_8$
	$x_3 = J_1v$	...	...	...	...	...	...	...	$\mathbf{I} = J_1J_6J_7J_8$
	$x_4 = \mathbf{I}J_1v$	...	...	...	...	...	...	...	$\mathbf{\Pi} = J_1$
G_I		J_8	J_7	J_6	J_5	J_4	J_3	J_2	

First we consider a minimal admissible module. We have $A \in \mathrm{GL}(2, \mathbb{C})$ and $\eta\,\mathbf{\Pi}\,\mathbf{I} = \mathbf{I}\eta\,\mathbf{\Pi}$. Thus the complex linear map $\eta\,\mathbf{\Pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is skew-Hermitian. As it was noticed, from a qualitative point of view, consideration of skew-Hermitian forms (up to isomorphism) provides no new classical groups, since the multiplication by $\mathbf{i}$ renders a skew-Hermitian form Hermitian, and vice versa. The form $\mathbf{i}\eta\,\mathbf{\Pi}$ is Hermitian of the signature $(1, 1)$ and the condition $\bar{A}^T_{\mathbb{C}}\mathbf{i}\eta\,\mathbf{\Pi}\,A_{\mathbb{C}} = \mathbf{i}\eta\,\mathbf{\Pi}$ leads to $\mathrm{Aut}^0(\mathfrak{n}_{1,7}(V^{1,7;N}_{min})) \cong \mathrm{U}(1, 1)$. It shows that $\mathrm{Aut}^0(\mathfrak{n}_{1,7}(U)) \cong \mathrm{U}(p, p)$ for $U = (\overset{p}{\oplus}\, V^{1,7;N}_{min})$.

The calculations and the table for $\mathfrak{n}_{5,3}(U)$ are similar to $\mathfrak{n}_{1,7}(U)$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{5,3}(U)) \cong \mathrm{U}(p, p)$.

$V^{2,0}_{min}$		dim=4
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}x_1$	$\mathbf{I} = J_1J_2$
	$x_3 = J_1v$	$\mathbf{\Pi} = J_1$
	$x_4 = \mathbf{I}x_3$	

$V^{0,2}_{min}$		dim = 4
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}v$	$\mathbf{I} = J_1J_2$
	$x_3 = J_1v$	$\mathbf{\Pi} = J_1$
	$x_4 = \mathbf{I}v$	

We make calculations for $U = V^{2,0;+}_{min}$. We have $A \in \mathrm{GL}(2, \mathbb{C})$, $\mathbf{\Pi}\,\mathbf{I} = -\mathbf{I}\,\mathbf{\Pi}$, and

$$\mathbf{\Pi} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \mathrm{diag}_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -\mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix}.$$

The condition $A^T\,\mathbf{\Pi}\,A = \mathbf{\Pi}$ is equivalent to

$$\mathrm{diag}_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rho_{\mathbb{C}}(\bar{A}^T_{\mathbb{C}})\,\mathrm{diag}_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix} \rho_{\mathbb{C}}(A_{\mathbb{C}}) = \begin{pmatrix} 0 & -\mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix}.$$

Observation (3.21) implies that

$$A^T_{\mathbb{C}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_{\mathbb{C}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathrm{Aut}^0(\mathfrak{n}_{2,0}(V^{2,0;+}_{\min})) \cong \mathrm{Sp}(2, \mathbb{C}).$$

We obtain that $\mathrm{Aut}^0(\mathfrak{n}_{2,0}(U)) \cong \mathrm{Sp}(2p, \mathbb{C})$ for $U = \overset{p}{\oplus} V^{2,0;+}_{min}$.

Let now $U = \overset{p}{\oplus} V^{0,2;N}_{min}$. For the neutral metric η we obtain

$$\eta\,\mathbf{\Pi} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Thus by calculations for $A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi}$ as above we get $\mathrm{Aut}^0(\mathfrak{n}_{0,2}(U)) = \mathrm{Sp}(2p, \mathbb{C})$.

$V^{0,6}_{\min}$					$\dim = 16$
$E^{\pm}_{P_1}$	+		-		$\dim = 8$
$E^{\pm}_{P_2}$	$+\ E^*_{0,6}$	-	+	-	$\dim = 4$
Basis for $E^*_{0,6}$	$x_1 = v$	...	...	...	$P_1 = J_1J_2J_3J_4$
	$x_2 = \mathbf{I}v$	...	...	...	$P_2 = J_1J_2J_5J_6$
	$x_3 = J_1J_3J_5v$	...	...	...	$\mathbf{I} = J_1J_2$
	$x_4 = \mathbf{I}x_3$	...	...	...	$\mathbf{\Pi} = J_1J_3J_5$
G_I		J_5	J_3	J_1	

We start from $U = V^{0,6}_{\min}$. Note that $A \in \mathrm{GL}(2; \mathbb{C})$, $\eta\,\mathbf{\Pi}\,\mathbf{I} = -\mathbf{I}\eta\,\mathbf{\Pi}$ and

$$\eta\,\mathbf{\Pi} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \mathrm{diag}_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix}.$$

Therefore,

$$A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi} \quad \Longleftrightarrow \quad A^T_{\mathbb{C}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_{\mathbb{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is symmetric of signature $(1,1)$. Thus $\mathrm{Aut}^0(\mathfrak{n}_{0,6}(V^{0,6;N}_{min})) \cong \mathrm{O}(1,1,\mathbb{C}) \cong \mathrm{O}(2,\mathbb{C})$. We obtain $\mathrm{Aut}^0(\mathfrak{n}_{0,6}(U)) \cong \mathrm{O}(2p,\mathbb{C})$ for $U = \overset{p}{\oplus} V^{0,6;N}_{min}$.

The calculations and the table for $\mathfrak{n}_{4,2}(U)$ are similar to $\mathfrak{n}_{0,6}(U)$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{4,2}(U)) = \mathrm{O}(2p,\mathbb{C})$.

4.5. Modules over $\mathbb{H}$

4.5.1. $\dim_{\mathbb{H}}(E^*_{r,s}) = 1$*: cases* $\mathfrak{n}_{4,0}(U)$, $\mathfrak{n}_{0,4}(U)$, $\mathfrak{n}_{6,2}(U)$, $\mathfrak{n}_{2,6}(U)$, $\mathfrak{n}_{6,1}(U)$, $\mathfrak{n}_{1,6}(U)$, $\mathfrak{n}_{5,2}(U)$, $\mathfrak{n}_{2,5}(U)$, $\mathfrak{n}_{5,0}(U)$, $\mathfrak{n}_{1,4}(U)$, $\mathfrak{n}_{3,0}(U)$, $\mathfrak{n}_{3,6}(U)$, $\mathfrak{n}_{7,2}(U)$

$V^{4,0}_{min}$			dim = 8
$E^{\pm}_{P_1}$	+ $E^*_{4,0}$	-	dim = 4
Basis for $E^*_{4,0}$	$x_1 = v$	$\dots$	$P_1 = J_1J_2J_3J_4$
	$x_2 = \mathbf{I}v$	$\dots$	$\mathbf{I} = J_1J_2$
	$x_3 = \mathbf{J}v$	$\dots$	$\mathbf{J} = J_2J_3$
	$x_4 = \mathbf{K}v$	$\dots$	$\mathbf{K} = J_3J_1$
G_I		J_1	

The table for $\mathfrak{n}_{0,4}(V^{0,4}_{min})$ is analogous, with $\mathbf{I} = J_1J_2$, $\mathbf{J} = J_2J_3$, $\mathbf{K} = J_1J_3$.

$V^{6,2}_{min}$									dim = 32
$E^{\pm}_{P_1}$	+				-				dim = 16
$E^{\pm}_{P_2}$	+		-		+		-		dim = 8
$E^{\pm}_{P_3}$	+ $E^*_{6,2}$	-	+	-	+	-	+	-	dim = 4
Basis for $E^*_{6,2}$	$x_1 = v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_2J_7J_8$
	$x_2 = \mathbf{I}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{I} = J_1J_2$
	$x_3 = \mathbf{J}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{J} = J_1J_3J_5J_7$
	$x_4 = \mathbf{K}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{K} = J_2J_3J_5J_7$
G_I		J_7	J_5	J_1J_3	J_3	J_1J_5	J_1J_7	J_1	

The table for $\mathfrak{n}_{2,6}$ is similar. In all 4 cases there are no conditions except the requirement to commute with the quaternion structure. We conclude that $\mathrm{Aut}^0(\mathfrak{n}_{4,0}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{0,4}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{6,2}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{2,6}(U)) = \mathrm{GL}(p,\mathbb{H})$.

$V^{1,6}_{min}$					dim = 16
$E^{\pm}_{P_1}$	+		-		dim = 8
$E^{\pm}_{P_2}$	+ $E^*_{1,6}$	-	+	-	dim = 4
Basis for $E^*_{1,6}$	$x_1 = v$	$\dots$	$\dots$	$\dots$	$P_1 = J_2J_3J_4J_5$ $P_2 = J_2J_3J_6J_7$ $P_3 = J_1J_2J_3$
	$x_2 = \mathbf{I}v$	$\dots$	$\dots$	$\dots$	$\mathbf{I} = J_1J_2J_4J_6$
	$x_3 = \mathbf{J}v$	$\dots$	$\dots$	$\dots$	$\mathbf{J} = J_2J_3$
	$x_4 = \mathbf{K}v$	$\dots$	$\dots$	$\dots$	$\mathbf{K} = J_1J_3J_4J_6$ $\mathbf{\Pi} = J_1$
G_I		J_6	J_4	J_2	

Observe that $P_3 = -\operatorname{Id}$ on $E^*_{1,6}$ according to the agreement that $E^*_{1,6} \subset V^{1,6;N}_{min;+}$ with the volume form $\Omega^{1,6}$ acting as identity on $V^{1,6;N}_{min;+}$. We consider $U = \overset{p}{\oplus} V^{1,6;N}_{min;+}$. Since $\eta\,\mathbf{\Pi} = \operatorname{diag}_p \mathbf{j}$ and

$$A^T \eta\,\mathbf{\Pi}\, A = \eta\,\mathbf{\Pi} \quad \Longleftrightarrow \quad \bar{A}^T_{\mathbb{H}} \operatorname{diag}_p \mathbf{j} A_{\mathbb{H}} = \operatorname{diag}_p \mathbf{j},$$

we conclude that $\operatorname{Aut}^0(\mathfrak{n}_{1,6}(U)) \cong \mathrm{O}^*(2p)$.

$V^{5,2}_{min}$					dim = 16
$E^{\pm}_{P_1}$	+		-		dim = 8
$E^{\pm}_{P_2}$	+ $E^*_{5,2}$	-	+	-	dim = 4
Basis for $E^*_{5,2}$	$x_1 = v$	$\dots$	$\dots$	$\dots$	$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_6J_7$ $P_3 = J_1J_2J_5$
	$x_2 = \mathbf{I}v$	$\dots$	$\dots$	$\dots$	$\mathbf{I} = J_2J_3J_5J_6$
	$x_3 = \mathbf{J}v$	$\dots$	$\dots$	$\dots$	$\mathbf{J} = J_1J_2$
	$x_4 = \mathbf{K}v$	$\dots$	$\dots$	$\dots$	$\mathbf{K} = J_1J_3J_5J_6$ $\mathbf{\Pi} = J_5$
G_I		J_7	J_3	J_1	

Observe that $P_3 = -\operatorname{Id}$ according to $E^*_{5,2} \subset V^{5,2}_{min;+}$ with $\Omega^{5,2} = \operatorname{Id}$ on $V^{5,2;N}_{min;+}$. The calculation, similar to the case of $\mathfrak{n}_{1,6}(U)$, shows that $\operatorname{Aut}^0(\mathfrak{n}_{5,2}(U)) \cong \mathrm{O}^*(2p)$.

$V^{6,1}_{min}$					dim = 16
$E^{\pm}_{P_1}$	+		-		dim = 8
$E^{\pm}_{P_2}$	+ $E^*_{6,1}$	-	+	-	dim = 4
Basis for $E^*_{6,1}$	$x_1 = v$	$\dots$	$\dots$	$\dots$	$P_1 = J_1J_2J_3J_4$ $P_2 = J_1J_2J_5J_6$ $P_3 = J_1J_3J_5$
	$x_2 = \mathbf{I}v$	$\dots$	$\dots$	$\dots$	$\mathbf{I} = J_1J_2$
	$x_3 = \mathbf{J}v$	$\dots$	$\dots$	$\dots$	$\mathbf{J} = J_1J_3J_5J_7$
	$x_4 = \mathbf{K}v$	$\dots$	$\dots$	$\dots$	$\mathbf{K} = J_2J_3J_5J_7$ $\mathbf{\Pi} = J_7$
G_I		J_5	J_3	J_1	

Observe that $E^*_{6,1} = E^+_{P_3} \oplus E^-_{P_3}$, with $E^+_{P_3} = \operatorname{span}\{v, \mathbf{K}v\}$ and $E^-_{P_3} = \operatorname{span}\{\mathbf{I}v, \mathbf{J}v\}$. We obtain $\eta\,\mathbf{\Pi} = \operatorname{diag}_p \mathbf{j}$ for $U = \oplus^p V^{6,1;N}_{min}$. Thus, $\operatorname{Aut}^0(\mathfrak{n}_{6,1}(U)) \cong \mathrm{O}^*(2p)$.

The calculations and the table for $\mathfrak{n}_{2,5}(U)$ are similar to $\mathfrak{n}_{6,1}(U)$ and we conclude $\operatorname{Aut}^0(\mathfrak{n}_{2,5}(U)) \cong \mathrm{O}^*(2p)$.

$V^{5,0}_{min}$			dim = 8
$E^{\pm}_{P_1}$	+ $E^*_{5,0}$	-	dim = 4
Basis for $E^*_{5,0}$	$x_1 = v$	$\dots$	$P_1 = J_2J_3J_4J_5$ $P_2 = J_1J_2J_3$
	$x_2 = \mathbf{I}v$	$\dots$	$\mathbf{I} = J_3J_4$
	$x_3 = \mathbf{J}v$	$\dots$	$\mathbf{J} = J_3J_2$
	$x_4 = \mathbf{K}v$	$\dots$	$\mathbf{K} = J_4J_2$ $\mathbf{\Pi} = J_1$
G_I		J_5	

Note that $E^*_{5,0} = E^+_{P_2} \oplus E^-_{P_2}$ with $E^+_{P_2} = \mathrm{span}\{v, \mathbf{J}v\}$ and $E^-_{P_2} = \mathrm{span}\{\mathbf{I}v, \mathbf{K}v\}$. Thus $\mathbf{\Pi} = \mathrm{diag}_p\, \mathbf{j}$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{5,0}(U)) \cong \mathrm{O}^*(2p)$ for $U = \oplus^p V^{5,0;+}_{min}$.

For the case $\mathfrak{n}_{1,4}(U)$ we use the quaternion structure $\mathbf{I} = J_3J_4$, $\mathbf{J} = J_3J_2$, $\mathbf{K} = J_2J_4$. The rest of calculations are similar to $\mathfrak{n}_{5,0}(U)$ and we obtain $\mathrm{Aut}^0(\mathfrak{n}_{1,4}(U)) \cong \mathrm{O}^*(2p)$.

$V^{3,0}_{min}$		dim=4
Basis	$x_1 = v$	$P_1 = J_1J_2J_3$
	$x_2 = \mathbf{I}v$	$\mathbf{I} = J_1J_2$
	$x_3 = \mathbf{J}v$	$\mathbf{J} = J_2J_3$
	$x_4 = \mathbf{K}v$	$\mathbf{K} = J_3J_1$ $\mathbf{\Pi} = J_1J_2J_3$

Observe that $\mathbf{\Pi} = \Omega^{3,0} = \mathrm{Id}$. We obtain that $A^T\mathbf{\Pi}A = \mathbf{\Pi}$ is equivalent to $\bar{A}^T_{\mathbb{H}}\,\mathrm{Id}_{p,q}\,A_{\mathbb{H}} = \mathrm{Id}_{p,q}$. Thus $\mathrm{Aut}^0(\mathfrak{n}_{3,0}(U)) = \mathrm{Sp}(p,q)$ for $U = \big(\overset{p}{\oplus} V^{3,0;+}_{\mathrm{min};+}\big) \oplus \big(\overset{q}{\oplus} V^{3,0;-}_{\mathrm{min};+}\big)$.

$V^{3,6}_{min}$									dim = 32
$E^{\pm}_{P_1}$	+				-				dim = 16
$E^{\pm}_{P_2}$	+		-		+		-		dim = 8
$E^{\pm}_{P_3}$	+ $E^*_{3,6}$	-	+	-	+	-	+	-	dim = 4
Basis for $E^*_{3,6}$	$x_1 = v$	…	…	…	…	…	…	…	$P_1 = J_1J_2J_8J_9$ $P_2 = J_4J_5J_8J_9$
	$x_2 = \mathbf{I}v$	…	…	…	…	…	…	…	$P_3 = J_6J_7J_8J_9$ $P_4 = J_3J_8J_9$
	$x_3 = \mathbf{J}v$	…	…	…	…	…	…	…	$\mathbf{I} = J_8J_9$ $\mathbf{J} = J_1J_4J_7J_8$
	$x_4 = \mathbf{K}v$	…	…	…	…	…	…	…	$\mathbf{K} = -J_1J_4J_7J_9$ $\mathbf{\Pi} = J_3J_8J_9$
G_I		J_7	J_4	J_5	J_1	J_1J_6	J_7J_8	J_9	

We have $E^*_{3,6} = E^+_{P_4} \oplus E^-_{P_4}$, with $E^+_{P_4} = \mathrm{span}\{v, \mathbf{I}v\}$, $E^-_{P_4} = \mathrm{span}\{\mathbf{J}v, \mathbf{K}v\}$. Since $\eta\,\mathbf{\Pi} = \mathrm{Id}_{p,q}$, we obtain

$$A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi} \quad \Longleftrightarrow \quad \bar{A}_{\mathbb{H}}\,\mathrm{Id}_{p,q}\,A_{\mathbb{H}} = \mathrm{Id}_{p,q}\,.$$

So $\mathrm{Aut}^0(\mathfrak{n}_{3,6}(U)) = \mathrm{Sp}(p,q)$ for $U = \big(\overset{p}{\oplus} V^{3,6;+}_{\mathrm{min}}\big) \oplus \big(\overset{q}{\oplus} V^{3,6;-}_{\mathrm{min}}\big)$.

The calculation and the table for $\mathfrak{n}_{7,2}(U)$ are similar to the case $\mathfrak{n}_{3,6}(U)$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{7,2}(U)) = \mathrm{Sp}(p,q)$.

4.5.2. $\dim_{\mathbb{H}}(E^*_{r,s}) = 2$*: cases* $\mathfrak{n}_{0,3}(U)$, $\mathfrak{n}_{6,3}(U)$, $\mathfrak{n}_{2,7}(U)$, $\mathfrak{n}_{0,5}(U)$, $\mathfrak{n}_{4,1}(U)$

$V^{0,3}_{min}$		dim = 8
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}v$	$\mathbf{I} = J_2J_1$
	$x_3 = \mathbf{J}v$	$\mathbf{J} = J_3J_2$
	$x_4 = \mathbf{K}v$	$\mathbf{K} = J_1J_3$
	$x_5 = J_1J_2J_3v$	
	$x_6 = \mathbf{I}x_5$	
	$x_7 = \mathbf{J}x_5$	$\mathbf{\Pi} = J_1J_2J_3$
	$x_8 = \mathbf{K}x_5$	

We make calculations on $V^{0,3}_{min}$ and note that $\eta\,\mathbf{\Pi} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In the basis

$$\begin{aligned} &y_1 = x_1 + x_5, \quad && y_2 = \mathbf{I}y_1, \quad && y_3 = \mathbf{J}y_1, \quad && y_4 = \mathbf{K}y_1, \\ &y_5 = x_1 - x_5, \quad && y_6 = \mathbf{I}y_5, \quad && y_7 = \mathbf{J}y_5, \quad && y_8 = \mathbf{K}y_5, \end{aligned} \tag{4.6}$$

the operator $\eta\,\mathbf{\Pi}$ takes the form $\mathrm{Id}_{1,1}$. Thus

$$A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi} \quad \Longleftrightarrow \bar{A}^T_{\mathbb{H}}\,\mathrm{Id}_{1,1}\,A_{\mathbb{H}} = \mathrm{Id}_{1,1} \quad \Longrightarrow A \in \mathrm{Sp}(1,1).$$

We conclude that $\mathrm{Aut}^0(\mathfrak{n}_{0,3}(U)) \cong \mathrm{Sp}(p,p)$ for $U = \overset{p}{\oplus}\, V^{0,3;N}_{min}$.

$V^{6,3}_{min}$									dim = 64
$E^{\pm}_{P_1}$	+				-				dim = 32
$E^{\pm}_{P_2}$	+		-		+		-		dim = 16
$E^{\pm}_{P_3}$	+ $E^*_{6,3}$	-	+	-	+	-	+	-	dim = 8
Basis for $E^*_{6,3}$	$x_1 = v$	...	...	...	...	...	...	...	$P_1 = J_1J_2J_3J_4$
	$x_2 = \mathbf{I}v$	...	...	...	...	...	...	...	$P_2 = J_1J_2J_5J_6$
	$x_3 = \mathbf{J}v$	...	...	...	...	...	...	...	$P_3 = J_1J_2J_7J_8$
	$x_4 = \mathbf{K}v$	...	...	...	...	...	...	...	$\mathbf{I} = J_1J_3J_6J_8$
	$x_5 = J_2J_1J_9v$	...	...	...	...	...	...	...	$\mathbf{J} = J_2J_1$
	$x_6 = \mathbf{I}J_2J_1J_9v$	...	...	...	...	...	...	...	$\mathbf{K} = J_2J_3J_6J_8$
	$x_7 = \mathbf{J}J_2J_1J_9v$	...	...	...	...	...	...	...	$\mathbf{\Pi} = J_2J_1J_9$
	$x_8 = \mathbf{K}J_2J_1J_9v$	...	...	...	...	...	...	...	
G_I		J_7	J_5	J_1J_3	J_3	J_1J_5	J_1J_7	J_1	

We have that $\eta\,\mathbf{\Pi} = \mathrm{Id}_{1,1}$ in the basis (4.6). It leads to $\mathrm{Aut}^0(\mathfrak{n}_{6,3}(U)) \cong \mathrm{Sp}(p,p)$.

The calculations for $\mathfrak{n}_{2,7}(U)$ are similar to $\mathfrak{n}_{6,3}(U)$ and $\mathrm{Aut}^0(\mathfrak{n}_{2,7}(U)) \cong \mathrm{Sp}(p,p)$.

$V^{0,5}_{min}$			dim = 16
$E^{\pm}_{P_1}$	+ $E^*_{0,5}$	-	dim = 8
Basis for $E^*_{0,5}$	$x_1 = v$	...	$P_1 = J_1J_2J_3J_4$
	$x_2 = \mathbf{I}v$	...	
	$x_3 = \mathbf{J}v$	...	$\mathbf{I} = J_1J_2$
	$x_4 = \mathbf{K}v$	...	$\mathbf{J} = J_1J_3$
	$x_5 = J_5v$	...	$\mathbf{K} = J_3J_2$
	$x_6 = \mathbf{I}J_5v$	...	
	$x_7 = \mathbf{J}J_5v$	...	$\mathbf{\Pi} = J_5$
	$x_8 = \mathbf{K}J_5v$	...	
G_I		J_1	

We have $\eta\,\mathbf{\Pi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $V^{0,5}_{min}$, and $\eta\,\mathbf{\Pi} = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}$ in the basis

$$\begin{aligned} &y_1 = x_1 + x_3 - x_5 + x_7, \quad && y_2 = \mathbf{I}y_1, \quad && y_3 = \mathbf{J}y_1, \quad && y_4 = \mathbf{K}y_1, \\ &y_5 = x_2 + x_4 + x_6 - x_8, \quad && y_6 = \mathbf{I}y_5, \quad && y_7 = \mathbf{J}y_5, \quad && y_8 = \mathbf{K}y_5. \end{aligned} \tag{4.7}$$

It leads to $A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi}$ that is equivalent to $\bar{A}^T_{\mathbb{H}}\,\mathrm{diag}_p\begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix} A_{\mathbb{H}} = \mathrm{diag}_p\begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}$. Thus we showed that $\mathrm{Aut}^0(\mathfrak{n}_{0,5}(U)) \cong \mathrm{O}^*(4p)$.

In the case $\mathfrak{n}_{4,1}(U)$ we use the quaternion structure $\mathbf{I} = J_1J_2$, $\mathbf{J} = J_4J_2$, $\mathbf{K} = J_1J_4$. The rest of calculations are similar to $\mathfrak{n}_{0,5}(U)$. Thus $\mathrm{Aut}^0(\mathfrak{n}_{4,1}(U)) \cong \mathrm{O}^*(4p)$.

4.6. Modules over $\mathbb{R}$ caring a negative involution

4.6.1. Cases $\mathfrak{n}_{1,1}(U)$, $\mathfrak{n}_{3,3}(U)$

In these cases there are no complex or quaternion structures, but only a negative involution $\mathbf{Q}$ leaving the space $E^*_{r,s}$ invariant. The involution $\mathbf{Q}$ commutes with involutions of type (1)-(3) and therefore decomposes the space $E^*_{r,s}$ into its eigenspaces: $E^*_{r,s} = N_+ \oplus N_-$. The admissible scalar product is degenerate on both $N_\pm$, but the decomposition still orthogonal with respect to the admissible product. In these cases the determination of $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ reduces to the calculations on $N_\pm$.

$V^{1,1}_{min}$		dim = 4
Basis	$x_1 = v$	
	$x_2 = \mathbf{Q}\,v$	$\mathbf{Q} = J_1J_2$
	$x_3 = J_1v$	
	$x_4 = \mathbf{Q}\,J_1v$	$\mathbf{\Pi} = J_1$

We have $V^{1,1;N}_{min} = N_+ \oplus N_-$ with the bases

$$N_+ = \mathrm{span}\Big\{y_1 = \frac{x_1+x_2}{2},\ y_2 = \frac{x_3+x_4}{2}\Big\},\ N_- = \mathrm{span}\Big\{y_3 = \frac{x_1-x_2}{2},\ y_4 = \frac{x_3-x_4}{2}\Big\}$$

Since $A\,\mathbf{Q} = \mathbf{Q}\,A$ we can decompose $A = A_+ \oplus A_-$ such that $A_\pm \colon N_\pm \to N_\pm$. We have $\eta\,\mathbf{\Pi}\,\mathbf{Q} = \mathbf{Q}\,\eta\,\mathbf{\Pi}$ and $\eta\,\mathbf{\Pi} = \mathrm{diag}_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ in the basis $\{y_k\}_{k=1}^4$. Thus the condition $A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi}$ is equivalent to two independent conditions

$$A^T_\pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}A_\pm = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We conclude that $\mathrm{Aut}^0(\mathfrak{n}_{1,1}(V^{1,1;N}_{min})) \cong \mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$. We obtain $\mathrm{Aut}^0(\mathfrak{n}_{1,1}(U)) \cong \mathrm{Sp}(2p,\mathbb{R}) \times \mathrm{Sp}(2p,\mathbb{R})$ for a general admissible module $U = \overset{p}{\oplus} V^{1,1;N}_{min}$.

$V^{3,3}_{min}$					dim = 8
$E^{\pm}_{P_1}$	+		-		dim = 4
$E^{\pm}_{P_2}$	+ $E^*_{3,3}$	-	+	-	dim = 2
Basis for $E^*_{3,3}$	$x_1 = v$	...	...	...	$P_1 = J_1J_2J_5J_6$ $P_2 = J_1J_3J_4J_6$ $P_3 = J_1J_2J_3$
	$x_2 = \mathbf{Q}\,v$	...	...	...	$\mathbf{Q} = J_1J_6$ $\mathbf{\Pi} = P_3$
G_I		J_3	J_2	J_6	

We have $E^*_{3,3} = E^+_{\mathbf{\Pi}} \oplus E^-_{\mathbf{\Pi}}$, with $E^+_{\mathbf{\Pi}} = \mathrm{span}\{v\}$, $E^-_{\mathbf{\Pi}} = \mathrm{span}\{\mathbf{Q}\,v\}$. Thus

$$E^*_{3,3} = N_+ \oplus N_-,\quad N_+ = \mathrm{span}\Big\{y_1 = \frac{x_1+x_2}{2}\Big\},\quad N_- = \mathrm{span}\Big\{y_2 = \frac{x_1-x_2}{2}\Big\}.$$

We write $A_\pm\colon N_\pm \to N_\pm$. We have $\eta\,\mathbf{\Pi}\,\mathbf{Q} = \mathbf{Q}\,\eta\,\mathbf{\Pi}$ and $\eta\,\mathbf{\Pi} = \mathrm{Id}_2$ in the basis $\{y_k\}$, $k = 1, 2$. The condition $A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi}$ is equivalent to two independent conditions $A_\pm^T A_\pm = \mathrm{Id}$. We conclude $\mathrm{Aut}^0(\mathfrak{n}_{3,3}(V^{3,3;+}_{min})) \cong \mathrm{O}(1)\times\mathrm{O}(1)$. We obtain $\mathrm{Aut}^0(\mathfrak{n}_{3,3}(U)) \cong \mathrm{O}(p, q) \times \mathrm{O}(p, q)$ for $U = (\overset{p}{\oplus}\, V^{3,3;+}_{min}) \oplus (\overset{q}{\oplus}\, V^{3,3;-}_{min})$.

4.7. Modules over $\mathbb{C}$, caring a negative involution

In these cases we continue to consider eigenspaces of the negative involution $\mathbf{Q}$. The complex structure can preserve eigenspaces of $\mathbf{Q}$ or reverse them. It leads to the different results.

4.7.1. Cases $\mathfrak{n}_{2,2}(U)$, $\mathfrak{n}_{3,2}(U)$, $\mathfrak{n}_{2,3}(U)$, $\mathfrak{n}_{1,2}(U)$

$V^{2,2}_{min}$			$\dim = 8$
$E^{\pm}_{P_1}$	$+\ E^*_{2,2}$	-	$\dim = 4$
Basis for $E^*_{2,2}$	$x_1 = v$	$\dots$	
	$x_2 = \mathbf{I}v$	$\dots$	$P_1 = J_1J_2J_3J_4$
	$x_3 = J_2J_3v$	$\dots$	$\mathbf{I} = J_1J_2$
	$x_4 = \mathbf{I}J_2J_3v$	$\dots$	$\mathbf{Q} = J_2J_3$
G_I		J_3	

We have the decomposition $E^*_{2,2} = N_+ \oplus N_-$ with the bases:

$$N_+ = \mathrm{span}\Big\{y_1 = \frac{x_1 + x_3}{2},\ y_2 = \frac{x_2 - x_4}{2}\Big\}, \qquad (4.8)$$
$$N_- = \mathrm{span}\Big\{y_3 = \frac{x_1 - x_3}{2},\ y_4 = \frac{x_2 + x_4}{2}\Big\}.$$

We write $A = A_+ \oplus A_-$, where $A_+ \in \mathrm{GL}(2, \mathbb{R})$, $A_+\colon N_+ \to N_+$. The map $A_-\colon N_- \to N_-$ can be found from the relation $A_- = J_1J_2A_+J_2J_1$. We conclude that for minimal admissible module $A \in \mathrm{GL}(2; \mathbb{R})$. In general $\mathrm{Aut}^0(\mathfrak{n}_{2,2}(U)) = \mathrm{GL}(2p, \mathbb{R})$ for $U = \overset{p}{\oplus} V^{2,2;N}_{min}$.

$V^{1,2}_{min}$		$\dim = 4$
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}v$	$\mathbf{I} = J_2J_3$
	$x_3 = J_1J_2v$	$\mathbf{Q} = J_1J_2$
	$x_4 = \mathbf{I}J_1J_2v$	$\mathbf{\Pi} = J_1J_2J_3$

In this case there are two minimal admissible modules but they are metrically isotypic and we set $\mathbf{\Pi}\,v = v$. We start from a minimal admissible module and write $V^{1,2;N}_{min} = N_+ \oplus N_-$. We also write $A = A_+ \oplus A_-$, where $A_+ \in \mathrm{GL}(2; \mathbb{R})$ and $A_- = J_2J_3A_+J_3J_2$. We obtain $\eta\,\mathbf{\Pi} = \begin{pmatrix} 0 & \mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix}$ in the basis (4.8). The condition

$$A^T\eta\,\mathbf{\Pi}\,A = \eta\,\mathbf{\Pi} \iff A_+^T \oplus A_-^T \begin{pmatrix} 0 & \mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix} A_+ \oplus A_- = \begin{pmatrix} 0 & \mathrm{Id}_2 \\ \mathrm{Id}_2 & 0 \end{pmatrix}$$

is equivalent to

$$A_+^T A_- = \mathrm{Id}_2 \iff A_+^T J_2 J_3 A_+ J_3 J_2 = \mathrm{Id} \iff A_+^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{1,2}(V^{1,2;N}_{min})) = \mathrm{Sp}(2,\mathbb{R})$. For a general admissible module $U = \overset{p}{\oplus} V^{1,2;N}_{min}$ we obtain $\mathrm{Aut}^0(\mathfrak{n}_{1,2}(U)) = \mathrm{Sp}(2p,\mathbb{R})$.

$V^{3,2}_{min}$			$\dim = 8$
$E^{\pm}_{P_1}$	$+\ E^*_{3,2}$	-	$\dim = 4$
Basis for $E^*_{3,2}$	$x_1 = v$	$\dots$	$P_1 = J_1J_2J_4J_5$ $P_2 = J_3J_4J_5$
	$x_2 = \mathbf{I}v$	$\dots$	$\mathbf{I} = J_4J_5$
	$x_3 = \mathbf{Q}\,v$	$\dots$	$\mathbf{Q} = J_1J_4$
	$x_4 = \mathbf{I}\,\mathbf{Q}\,v$	$\dots$	$\mathbf{\Pi} = J_3J_4J_5$
G_I		J_1	

We have $E^*_{3,2} = E^+_{\mathbf{\Pi}} \oplus E^-_{\mathbf{\Pi}}$, with $E^+_{\mathbf{\Pi}} = \mathrm{span}\{v, \mathbf{I}v\}$ and $E^-_{\mathbf{\Pi}} = \mathrm{span}\{\mathbf{Q}\,v, \mathbf{Q}\,\mathbf{I}v\}$, and $\eta\,\mathbf{\Pi} = \mathrm{Id}$ in the basis (4.8). As before we decompose $A = A_+ \oplus A_-$ on $E^*_{3,2}$ with $A_+ \in \mathrm{GL}(2p+2q;\mathbb{R})$ and $A_- = -\mathbf{I}A_+\mathbf{I}$. The condition

$$A_+^T\, \mathrm{Id}_{2p,2q}\, A_+ = \mathrm{Id}_{2p,2q} \quad \text{on} \quad U = \big(\overset{p}{\oplus} V^{3,2;+}_{min}\big) \oplus \big(\overset{q}{\oplus} V^{3,2;-}_{min}\big)$$

leads to the conclusion that $\mathrm{Aut}^0(\mathfrak{n}_{3,2}(U)) = \mathrm{O}(2p,2q)$.

$V^{2,3}_{min}$			$\dim = 8$
$E^{\pm}_{P_1}$	$+\ E^*_{2,3}$	-	$\dim = 4$
Basis for $E^*_{2,3}$	$x_1 = v$	$\dots$	$P_1 = J_1J_2J_4J_5$ $P_2 = J_1J_3J_5$
	$x_2 = \mathbf{I}v$	$\dots$	$\mathbf{I} = J_4J_5$
	$x_3 = \mathbf{Q}\,v$	$\dots$	$\mathbf{Q} = J_1J_4$
	$x_4 = \mathbf{I}\,\mathbf{Q}\,v$	$\dots$	$\mathbf{\Pi} = J_1J_3J_5$
G_I		J_1	

Arguing as in the case $V^{3,2}_{min}$ and by making use the basis (4.8), we come to condition

$$A_+^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{for} \quad V^{2,3;N}_{min}.$$

For a general module $U = \overset{p}{\oplus} V^{2,3;N}_{min}$ we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{2,3}(U)) = \mathrm{O}(p,p)$.

$V^{2,1}_{min}$		$\dim = 8$
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}v$	
	$x_3 = \mathbf{Q}\,v$	
	$x_4 = \mathbf{I}\,\mathbf{Q}\,v$	$\mathbf{I} = J_1J_2$
	$x_5 = J_1J_2J_3v$	$\mathbf{Q} = J_2J_3$
	$x_6 = \mathbf{I}J_1J_2J_3v$	$\mathbf{\Pi} = J_1J_2J_3$
	$x_7 = \mathbf{Q}\,J_1J_2J_3v$	
	$x_8 = \mathbf{I}\,\mathbf{Q}\,J_1J_2J_3v$	

We use the basis

$$\begin{array}{llll} y_1 = \frac{x_1+x_3}{2}, & y_2 = \frac{x_2-x_4}{2}, & y_3 = \frac{x_5+x_7}{2}, & y_4 = \frac{x_6-x_8}{2}, \\ y_5 = \frac{x_1-x_3}{2}, & y_6 = \frac{x_2+x_4}{2}, & y_7 = \frac{x_5-x_7}{2}, & y_8 = \frac{x_6+x_8}{2}, \end{array} \tag{4.9}$$

for $V^{2,1;N}_{min} = N_+ \oplus N_-$ with $N_+ = \text{span}\{y_1, y_2, y_3, y_4\}$, $N_- = \text{span}\{y_5, y_6, y_7, y_8\}$. We write $A = A_+ \oplus A_-$ with $A_+ \in \text{GL}(4;\mathbb{R})$ and $A_- = J_1J_2A_+J_2J_1$ in the basis (4.9). Then

$$\eta\,\mathbf{\Pi} = \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \quad \text{with} \quad S = -\begin{pmatrix} 0 & \text{Id}_2 \\ \text{Id}_2 & 0 \end{pmatrix}.$$

Thus we need to check the condition

$$A_+^T S A_- = S \quad \Longleftrightarrow \quad A_+^T S J_1 J_2 A_+ = S J_1 J_2 \quad \text{with} \quad S J_1 J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, we conclude that $\text{Aut}^0(\mathfrak{n}_{2,1}(U)) \cong \text{Sp}(4p, \mathbb{R})$.

4.8. Modules over $\mathbb{H}$ caring a negative involution

4.8.1. Cases $\mathfrak{n}_{1,3}(U)$, $\mathfrak{n}_{3,1}(U)$, $\mathfrak{n}_{1,5}(U)$, $\mathfrak{n}_{5,1}(U)$, $\mathfrak{n}_{7,3}(U)$, $\mathfrak{n}_{3,7}(U)$,

$V^{1,3}_{min}$		$\dim = 8$
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}v$	$P = J_1J_2J_3$
	$x_3 = \mathbf{J}v$	
	$x_4 = \mathbf{K}v$	$\mathbf{I} = J_2J_3$
	$x_5 = J_4v$	$\mathbf{J} = J_3J_4$
	$x_6 = \mathbf{I}J_4v$	$\mathbf{K} = J_2J_4$
	$x_7 = \mathbf{J}J_4v$	$\mathbf{Q} = J_1J_2$
	$x_8 = \mathbf{K}J_4v$	$\mathbf{\Pi} = J_1$

We choose $P_1v = v$ and the basis for $V^{1,3;N}_{min} = N_+ \oplus N_-$:

$$\begin{aligned} N_+ &= \text{span}\left\{y_1 = \frac{x_1+x_7}{2},\ y_2 = \frac{x_3-x_5}{2},\ y_3 = \frac{x_4+x_6}{2},\ y_4 = \frac{x_2-x_8}{2}\right\}, \\ N_- &= \text{span}\left\{y_5 = \frac{x_1-x_7}{2},\ y_6 = \frac{x_3+x_5}{2},\ y_7 = \frac{x_4-x_6}{2},\ y_8 = \frac{x_2+x_8}{2}\right\}. \end{aligned} \tag{4.10}$$

Since $A\,\mathbf{Q} = \mathbf{Q}\,A$ and $\mathbf{Q}\,\mathbf{I} = -\mathbf{I}\,\mathbf{Q}$ we write $A = A_+ \oplus A_-$, where $A_+ \in \text{GL}(4;\mathbb{R})$ and $A_- = J_2J_3A_+J_3J_2$. Since $\mathbf{Q}\,\mathbf{J} = \mathbf{J}\,\mathbf{Q}$ we deduce that $A_+ \in \text{GL}(2;\mathbb{C})$. Moreover

$$N_+ = \text{span}\{y_1, \quad y_2 = \mathbf{J}y_1, \quad y_3 = y_3, \quad y_4 = \mathbf{J}y_3\}. \tag{4.11}$$

We also have $\eta\,\mathbf{\Pi}\,\mathbf{Q} = \mathbf{Q}\,\eta\,\mathbf{\Pi}$, $\eta\,\mathbf{\Pi}\,\mathbf{J} = \mathbf{J}\eta\,\mathbf{\Pi}$ with the matrix $\eta\,\mathbf{\Pi}\,|_{N_+} = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$. It leads to

$$A_+^T \eta\,\mathbf{\Pi}\,A_+ = \eta\,\mathbf{\Pi} \quad\Longleftrightarrow\quad \overline{(A_+)_{\mathbb{C}}^T}\begin{pmatrix}0&1\\1&0\end{pmatrix}(A_+)_{\mathbb{C}} = \begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The matrix $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is Hermitian of index $(1,1)$. We conclude that $\mathrm{Aut}^0(\mathfrak{n}_{1,3}(V^{1,3;N}_{min})) \cong \mathrm{U}(1,1)$ and $\mathrm{Aut}^0(\mathfrak{n}_{1,3}(U)) \cong \mathrm{U}(p,p)$ for $U = \overset{p}{\oplus} V^{1,3;N}_{min}$.

$\mathfrak{n}_{3,1}$		$\dim = 8$
Basis	$x_1 = v$	
	$x_2 = \mathbf{I}v$	$P_1 = J_1J_2J_3$
	$x_3 = \mathbf{J}v$	
	$x_4 = \mathbf{K}v$	$\mathbf{I} = J_2J_3$
	$x_5 = J_4v$	$\mathbf{J} = J_1J_2$
	$x_6 = \mathbf{I}J_4v$	$\mathbf{K} = J_1J_3$
	$x_7 = \mathbf{J}J_4v$	$\mathbf{Q} = J_3J_4$
	$x_8 = \mathbf{K}J_4v$	$\mathbf{\Pi} = J_4$

We have $V^{3,1;+}_{min} = E^+_{P_1} \oplus E^-_{P_1}$, with

$$E^+_{P_1} = \mathrm{span}\{v, \mathbf{I}v, \mathbf{J}v, \mathbf{K}v\}, \quad E^-_{P_1} = \mathrm{span}\{J_4v, \mathbf{I}J_4v, \mathbf{J}J_4v, \mathbf{K}J_4v\}.$$

The negative involution $\mathbf{Q}$ decomposes $V^{3,1;+}_{min} = N_+ \oplus N_-$ with the basis given by (4.10). Since $A\,\mathbf{Q} = \mathbf{Q}\,A$ and $\mathbf{Q}\,\mathbf{I} = -\mathbf{I}\,\mathbf{Q}$ we write $A = A_+ \oplus A_-$, where $A_+ \in \mathrm{GL}(4;\mathbb{R})$ and $A_- = -\mathbf{I}A_+\mathbf{I}$. The condition $\mathbf{Q}\,\mathbf{J} = \mathbf{J}\,\mathbf{Q}$ implies $A_+ \in \mathrm{GL}(2;\mathbb{C})$. We also have $\eta\,\mathbf{\Pi}\,\mathbf{Q} = \mathbf{Q}\,\eta\,\mathbf{\Pi}$ and $\eta\,\mathbf{\Pi}\,\mathbf{J} = \mathbf{J}\eta\,\mathbf{\Pi}$ with $\eta\,\mathbf{\Pi}\,|_{N_+} = \mathrm{diag}_2\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ in the basis (4.11). It leads to

$$A_+^T \eta\,\mathbf{\Pi}\,A_+ = \eta\,\mathbf{\Pi} \quad\Longleftrightarrow\quad \overline{(A_+)_{\mathbb{C}}^T}(A_+)_{\mathbb{C}} = \mathrm{Id}_2\,.$$

The conclusion is that $\mathrm{Aut}^0(\mathfrak{n}_{3,1}(V^{3,1;+}_{min})) \cong \mathrm{U}(2)$ and $\mathrm{Aut}^0(\mathfrak{n}_{3,1}(U)) \cong \mathrm{U}(2p,2q)$ for $U = (\overset{p}{\oplus} V^{3,1;+}_{min}) \oplus (\overset{q}{\oplus} V^{3,1;-}_{min})$.

$V^{5,1}_{min}$			$\dim = 16$
$E^{\pm}_{P_1}$	$+\ E^*_{5,1}$	-	$\dim = 8$
Basis for $E^*_{5,1}$	$x_1 = v$	...	$P_1 = J_1J_2J_3J_4$
	$x_2 = \mathbf{I}v$	...	$P_2 = J_1J_2J_5$
	$x_3 = \mathbf{J}v$	...	
	$x_4 = \mathbf{K}v$	...	$\mathbf{I} = J_1J_3$
	$x_5 = J_6v$	...	$\mathbf{J} = J_1J_2$
	$x_6 = \mathbf{I}J_6v$	...	$\mathbf{K} = J_3J_2$
	$x_7 = \mathbf{J}J_6v$	...	$\mathbf{Q} = J_5J_6$
	$x_8 = \mathbf{K}J_6v$	...	$\mathbf{\Pi} = J_6$
G_I		J_1	

We have $E^*_{5,1} = E^+_{P_2} \oplus E^-_{P_2}$, with

$$E^+_{P_2} = \mathrm{span}\{v, \mathbf{J}v, \mathbf{I}J_6v, \mathbf{K}J_6v\}, \quad E^-_{P_2} = \mathrm{span}\{J_6v, \mathbf{I}v, \mathbf{J}J_6v, \mathbf{K}v, \}.$$

The negative involution $\mathbf{Q}$ decomposes $E^*_{5,1}$ into two eigenspaces $E^*_{5,1} = N_+ \oplus N_-$ with the bases

$$\begin{aligned} N_+ &= \operatorname{span}\Big\{y_1 = \frac{x_1+x_7}{2},\ y_2 = \frac{x_2+x_8}{2},\ y_3 = \frac{x_3-x_5}{2},\ y_4 = \frac{x_4-x_6}{2}\Big\},\\ N_- &= \operatorname{span}\Big\{y_5 = \frac{x_1-x_7}{2},\ y_6 = \frac{x_2-x_8}{2},\ y_7 = \frac{x_3-x_5}{2},\ y_8 = \frac{x_4+x_6}{2}\Big\}. \end{aligned} \tag{4.12}$$

Since $A\mathbf{Q} = \mathbf{Q}A$ and $\mathbf{Q}\mathbf{I} = \mathbf{I}\mathbf{Q}$, $\mathbf{Q}\mathbf{J} = \mathbf{J}\mathbf{Q}$ we write $A = A_+ \oplus A_-$, where $A_\pm \in \mathrm{GL}(1;\mathbb{H})$. Moreover

$$N_+ = \operatorname{span}\{y_1,\quad y_2 = \mathbf{I}y_1,\quad y_3 = y_3,\quad y_4 = \mathbf{I}y_3\}. \tag{4.13}$$

We also have $\eta\,\mathbf{\Pi}\,\mathbf{Q} = \mathbf{Q}\,\eta\,\mathbf{\Pi}$, $\eta\,\mathbf{\Pi}\,\mathbf{I} = \mathbf{I}\eta\,\mathbf{\Pi}$, $\eta\,\mathbf{\Pi}\,\mathbf{J} = \mathbf{J}\eta\,\mathbf{\Pi}$. Thus $\eta\,\mathbf{\Pi}$ is quaternion linear and $\eta\,\mathbf{\Pi}\,|_{N_\pm} = \mathbf{j}$, written in the basis (4.13). It leads to

$$A_\pm^T \eta\,\mathbf{\Pi}\,A_\pm = \eta\,\mathbf{\Pi}\,|_{N_+} \quad\Longleftrightarrow\quad \overline{(A_\pm)^T_{\mathbb{H}}}\mathbf{j}(A_\pm)_{\mathbb{H}} = \mathbf{j}.$$

The conclusion is that $\mathrm{Aut}^0(\mathfrak{n}_{5,1}(V^{5,1;N}_{min})) \cong \mathrm{O}^*(2)\times\mathrm{O}^*(2)$ and $\mathrm{Aut}^0(\mathfrak{n}_{5,1}(U)) \cong \mathrm{O}^*(2p)\times\mathrm{O}^*(2p)$ for $U = \overset{p}{\oplus} V^{5,1;N}_{min}$.

$V^{1,5}_{min}$			dim = 16
$E^{\pm}_{P_1}$	$+\ E^*_{1,5}$	-	dim = 8
Basis for $E^*_{1,5}$	$x_1 = v$	$\dots$	$P_1 = J_2J_3J_4J_5$
	$x_2 = \mathbf{I}v$	$\dots$	$P_2 = J_1J_2J_3$
	$x_3 = \mathbf{J}v$	$\dots$	
	$x_4 = \mathbf{K}v$	$\dots$	$\mathbf{I} = J_3J_4$
	$x_5 = J_6v$	$\dots$	$\mathbf{J} = J_2J_3$
	$x_6 = \mathbf{I}J_6v$	$\dots$	$\mathbf{K} = J_4J_2$
	$x_7 = \mathbf{J}J_6v$	$\dots$	$\mathbf{Q} = J_1J_6$
	$x_8 = \mathbf{K}J_6v$	$\dots$	$\mathbf{\Pi} = J_6$
G_I		J_5	

With the chosen operators $\mathbf{I}, \mathbf{J}, \mathbf{Q}, \mathbf{\Pi}$ the calculations are identical to the case of $\mathfrak{n}_{5,1}$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{1,5}(U)) \cong \mathrm{O}^*(2p)\times\mathrm{O}^*(2p)$ for $U = \overset{p}{\oplus} V^{1,5;N}_{min}$.

$V^{7,3}_{min}$									dim = 64
$E^{\pm}_{P_1}$	+				-				dim = 32
$E^{\pm}_{P_2}$	+		-		+		-		dim = 16
$E^{\pm}_{P_3}$	$+\ E^*_{7,3}$	-	+	-	+	-	+	-	dim = 8
Basis for $E^*_{7,3}$	$x_1 = v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$P_1 = J_1J_2J_4J_5$
	$x_2 = \mathbf{I}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$P_2 = J_1J_2J_6J_7$
	$x_3 = \mathbf{J}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$P_3 = J_1J_2J_8J_9$
	$x_4 = \mathbf{K}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$P_4 = J_1J_2J_3$
	$x_5 = J_{10}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{I} = J_1J_2$
	$x_6 = \mathbf{I}J_{10}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{J} = J_1J_4J_6J_8$
	$x_7 = \mathbf{J}J_{10}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{K} = J_2J_4J_6J_8$
	$x_8 = \mathbf{K}J_{10}v$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\mathbf{Q} = J_3J_{10}$ $\mathbf{\Pi} = J_{10}$
G_I		J_8	J_6	J_1J_4	J_4	J_1J_6	J_1J_8	J_1	

Observe that $E^*_{7,3} = E^+_{P_4} \oplus E^-_{P_4}$, with

$$E^+_{P_4} = \mathrm{span}\{v, \mathbf{I}v, \mathbf{J}J_{10}v, \mathbf{K}J_{10}v\}, \quad E^-_{P_4} = \mathrm{span}\{J_{10}v, \mathbf{I}J_{10}v, \mathbf{J}v, \mathbf{K}v\}.$$

We start from the minimal admissible module. The negative involution $\mathbf{Q}$ decomposes $E^*_{7,3}$ into two eigenspaces $E^*_{7,3} = N_+ \oplus N_-$ with the bases

$$\begin{aligned} N_+ &= \mathrm{span}\Big\{y_1 = \frac{x_1+x_6}{2},\ y_2 = \frac{x_2-x_5}{2},\ y_3 = \frac{x_4+x_7}{2},\ y_4 = \frac{x_8-x_3}{2}\Big\}, \\ N_- &= \mathrm{span}\Big\{y_5 = \frac{x_1-x_6}{2},\ y_6 = \frac{x_2+x_5}{2},\ y_7 = \frac{x_7-x_4}{2},\ y_8 = \frac{x_8+x_3}{2}\Big\}. \end{aligned} \tag{4.14}$$

Since $A\,\mathbf{Q} = \mathbf{Q}\,A$ and $\mathbf{Q}\,\mathbf{I} = \mathbf{I}\,\mathbf{Q}$, $\mathbf{Q}\,\mathbf{J} = \mathbf{J}\,\mathbf{Q}$ we write $A = A_+ \oplus A_-$, where $A_\pm \in \mathrm{GL}(1;\mathbb{H})$. Moreover

$$N_+ = \mathrm{span}\{y_1, \quad y_2 = \mathbf{I}y_1, \quad y_3 = y_3, \quad y_4 = \mathbf{I}y_3\}. \tag{4.15}$$

We also have $\eta\,\mathbf{\Pi}\,\mathbf{Q} = \mathbf{Q}\,\eta\,\mathbf{\Pi}$, $\eta\,\mathbf{\Pi}\,\mathbf{I} = \mathbf{I}\eta\,\mathbf{\Pi}$, $\eta\,\mathbf{\Pi}\,\mathbf{J} = -\mathbf{J}\eta\,\mathbf{\Pi}$ with $\big(\eta\,\mathbf{\Pi}\,|_{N_\pm}\big)_{\mathbb{C}} = \mathbf{i}\,\mathrm{Id}_2$, written in the basis (4.15). It leads to

$$A^T_\pm \eta\,\mathbf{\Pi}\,A_\pm = \eta\,\mathbf{\Pi}\,|_{N_+} \quad \Longleftrightarrow \quad \overline{(A_\pm)^T_{\mathbb{C}}}\,\mathrm{Id}_2 (A_\pm)_{\mathbb{C}} = \mathrm{Id}_2\,.$$

Thus we conclude $\mathrm{Aut}^0(\mathfrak{n}_{7,3}(V^{7,3;+}_{min})) \cong \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ for a minimal admissible module. If $U = (\overset{p}{\oplus}\, V^{7,3;+}_{min}) \oplus (\overset{q}{\oplus}\, V^{7,3;-}_{min})$, then $\mathrm{Aut}^0(\mathfrak{n}_{7,3}(U)) \cong \mathrm{Sp}(p,q) \times \mathrm{Sp}(p,q)$.

The calculations and the table for $\mathfrak{n}_{3,7}(U)$ are identical to $\mathfrak{n}_{7,3}(U)$ and we conclude that $\mathrm{Aut}^0(\mathfrak{n}_{3,7}(U)) \cong \mathrm{Sp}(p,q) \times \mathrm{Sp}(p,q)$ for $U = (\overset{p}{\oplus}\, V^{3,7;+}_{min}) \oplus (\overset{q}{\oplus}\, V^{3,7;-}_{min})$.

5. Appendix

5.1. *Comparison of* $\mathrm{Aut}^0(\mathfrak{n}_{r,s}(U))$ *for isomorphic algebras*

Cases $\mathfrak{n}_{1,0}(U)$, $\mathfrak{n}_{0,1}(V)$; $\mathfrak{n}_{2,0}(U)$, $\mathfrak{n}_{0,2}(V)$; $\mathfrak{n}_{5,1}(U)$, $\mathfrak{n}_{1,5}(V)$.

$$\mathfrak{n}_{1,0}(\overset{p}{\oplus}\, V^{1,0;+}_{min}) \cong \mathfrak{n}_{0,1}(\overset{p}{\oplus}\, V^{0,1;N}_{min}), \quad \mathfrak{n}_{2,0}(\overset{p}{\oplus}\, V^{2,0;+}_{min}) \cong \mathfrak{n}_{0,2}(\overset{p}{\oplus}\, V^{0,2;N}_{min})$$
$$\mathfrak{n}_{5,1}(\overset{p}{\oplus}\, V^{5,1;N}_{min}) \cong \mathfrak{n}_{1,5}(\overset{p}{\oplus}\, V^{1,5;N}_{min})$$
$$\mathrm{Aut}^0(\mathfrak{n}_{1,0}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{0,1}(V)) \cong \mathrm{Sp}(2p,\mathbb{R}),$$
$$\mathrm{Aut}^0(\mathfrak{n}_{2,0}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{0,2}(V)) = \mathrm{Sp}(2p,\mathbb{C}),$$
$$\mathrm{Aut}^0(\mathfrak{n}_{1,5}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{5,1}(V)) = \mathrm{O}^*(2p) \times \mathrm{O}^*(2p).$$

Cases $\mathfrak{n}_{4,0}(V)$, $\mathfrak{n}_{0,4}(U)$; $\mathfrak{n}_{2,6}(U)$, $\mathfrak{n}_{6,2}(V)$; $\mathfrak{n}_{8,0}(U)$, $\mathfrak{n}_{0,8}(V)$, $\mathfrak{n}_{4,4}(W)$; $\mathfrak{n}_{1,6}(U)$, $\mathfrak{n}_{6,1}(U)$; $\mathfrak{n}_{2,5}(U)$, $\mathfrak{n}_{5,2}(V)$.

$$\mathfrak{n}_{4,0}(\overset{p}{\oplus} V^{4,0;+}_{min}) \cong \mathfrak{n}_{0,4}(\overset{p}{\oplus} V^{0,4;N}_{min}), \qquad \mathfrak{n}_{2,6}(\overset{p}{\oplus} V^{2,6;N}_{min}) \cong \mathfrak{n}_{6,2}(\overset{p}{\oplus} V^{6,2;N}_{min})$$
$$\mathfrak{n}_{1,6}(\overset{p}{\oplus} V^{1,6;N}_{min}) \cong \mathfrak{n}_{6,1}(\overset{p}{\oplus} V^{6,1;N}_{min}), \qquad \mathfrak{n}_{2,5}(\overset{p}{\oplus} V^{2,5;N}_{min}) \cong \mathfrak{n}_{5,2}(\overset{p}{\oplus} V^{5,2;N}_{min})$$
$$\mathfrak{n}_{8,0}(\overset{p}{\oplus} V^{8,0;+}_{min}) \cong \mathfrak{n}_{0,8}(\overset{p}{\oplus} V^{0,8;+}_{min}) \ncong \mathfrak{n}_{4,4}(\overset{p}{\oplus} V^{4,4;+}_{min})$$
$$\mathrm{Aut}^0(\mathfrak{n}_{4,0}(V)) = \mathrm{Aut}^0(\mathfrak{n}_{0,4}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{2,6}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{6,2}(V) = \mathrm{GL}(p,\mathbb{H});$$
$$\mathrm{Aut}^0(\mathfrak{n}_{1,6}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{6,1}(V)) = \mathrm{Aut}^0(\mathfrak{n}_{2,5}(U)) = \mathrm{Aut}^0(\mathfrak{n}_{5,2}(V)) = \mathrm{O}^*(2p).$$
$$\mathrm{Aut}^0\left(\mathfrak{n}_{8,0}(U)\right) = \mathrm{Aut}^0\left(\mathfrak{n}_{0,8}(V)\right) = \mathrm{Aut}^0\left(\mathfrak{n}_{4,4}(W)\right) = \mathrm{GL}(p,\mathbb{R}).$$

Cases $\mathfrak{n}_{5,0}(U)$, $\mathfrak{n}_{0,5}(V)$; $\mathfrak{n}_{1,4}(U)$, $\mathfrak{n}_{4,1}(V)$; $\mathfrak{n}_{6,0}(U)$, $\mathfrak{n}_{0,6}(U)$; $\mathfrak{n}_{2,4}(U)$, $\mathfrak{n}_{4,2}(U)$; $\mathfrak{n}_{1,2}(U)$, $\mathfrak{n}_{2,1}(U)$ Here we have that

$$\mathfrak{n}_{5,0}(\overset{2p}{\oplus} V^{5,0;+}_{min}) \cong \mathfrak{n}_{0,5}(\overset{p}{\oplus} V^{0,5;N}_{min}), \quad \mathfrak{n}_{1,4}(\overset{2p}{\oplus} V^{1,4;+}_{min}) \cong \mathfrak{n}_{4,1}(\overset{p}{\oplus} V^{4,1;N}_{min})$$
$$\mathfrak{n}_{6,0}(\overset{2p}{\oplus} V^{6,0;+}_{min}) \cong \mathfrak{n}_{0,6}(\overset{p}{\oplus} V^{0,6;N}_{min}), \quad \mathfrak{n}_{2,4}(\overset{2p}{\oplus} V^{2,4;+}_{min}) \cong \mathfrak{n}_{4,2}(\overset{p}{\oplus} V^{4,2;N}_{min}),$$
$$\mathfrak{n}_{1,2}(\overset{2p}{\oplus} V^{1,2;N}_{min}) \cong \mathfrak{n}_{2,1}(\overset{p}{\oplus} V^{2,1;N}_{min}).$$

We also showed

$$\mathrm{Aut}^0\left(\mathfrak{n}_{5,0}(\overset{p}{\oplus} V^{5,0;+}_{min})\right) \cong \mathrm{O}^*(2p) \quad \text{and} \quad \mathrm{Aut}^0(\mathfrak{n}_{0,5}(\overset{p}{\oplus} V^{0,5;N}_{min})) \cong \mathrm{O}^*(4p).$$
$$\mathrm{Aut}^0\left(\mathfrak{n}_{1,4}(\overset{p}{\oplus} V^{1,4;+}_{min})\right) \cong \mathrm{O}^*(2p) \quad \text{and} \quad \mathrm{Aut}^0(\mathfrak{n}_{4,1}(\overset{p}{\oplus} V^{4,1;N}_{min})) \cong \mathrm{O}^*(4p).$$
$$\mathrm{Aut}^0\left(\mathfrak{n}_{6,0}(\overset{p}{\oplus} V^{6,0;+}_{min})\right) \cong \mathrm{O}(p,\mathbb{C}) \quad \text{and} \quad \mathrm{Aut}^0\left(\mathfrak{n}_{0,6}(\overset{p}{\oplus} V^{0,6;N}_{min})\right) \cong \mathrm{O}(2p,\mathbb{C}).$$
$$\mathrm{Aut}^0\left(\mathfrak{n}_{2,4}(\overset{p}{\oplus} V^{2,4;+}_{min})\right) \cong \mathrm{O}(p,\mathbb{C}) \quad \text{and} \quad \mathrm{Aut}^0\left(\mathfrak{n}_{4,2}(\overset{p}{\oplus} V^{4,2;N}_{min})\right) \cong \mathrm{O}(2p,\mathbb{C}).$$
$$\mathrm{Aut}^0(\mathfrak{n}_{1,2}(\overset{p}{\oplus} V^{1,2;N}_{min})) \cong \mathrm{Sp}(2p,\mathbb{C}) \quad \text{and} \quad \mathrm{Aut}^0\left(\mathfrak{n}_{2,1}(\overset{p}{\oplus} V^{2,1;N}_{min})\right) \cong \mathrm{Sp}(4p,\mathbb{R}).$$

CASES $\mathfrak{n}_{3,0}(V)$, $\mathfrak{n}_{0,3}(U)$; $\mathfrak{n}_{7,0}(U)$, $\mathfrak{n}_{0,7}(V)$; $\mathfrak{n}_{3,4}(U)$, $\mathfrak{n}_{4,3}(V)$.

$$\mathfrak{n}_{0,3}(V^{0,3;N}_{min}) \cong \mathfrak{n}_{3,0}\Big(\big(\overset{p}{\oplus} V^{3,0;+}_{min;+}\big) \oplus \big(\overset{p}{\oplus} V^{3,0;-}_{min;+}\big)\Big),$$
$$\mathfrak{n}_{0,7}(\overset{p}{\oplus} V^{0,7;N}_{min}) \cong \mathfrak{n}_{7,0}\Big(\big(\overset{p}{\oplus} V^{7,0;+}_{min;+}\big) \oplus \big(\overset{p}{\oplus} V^{7,0;-}_{min;+}\big)\Big),$$
$$\mathfrak{n}_{4,3}(\overset{p}{\oplus} V^{4,3;N}_{min}) \cong \mathfrak{n}_{3,4}\Big(\big(\overset{p}{\oplus} V^{3,4;+}_{min;+}\big) \oplus \big(\overset{p}{\oplus} V^{3,4;-}_{min;+}\big)\Big),$$
$$\mathrm{Aut}^0(\mathfrak{n}_{0,3}\big(\overset{p}{\oplus} V^{0,3;N}_{min}\big) = \mathrm{Sp}(p,p), \quad \mathrm{Aut}^0(\mathfrak{n}_{3,0}\Big(\big(\overset{p}{\oplus} V^{3,0;+}_{min;+}\big) \oplus \big(\overset{q}{\oplus} V^{3,0;-}_{min;+}\big)\Big) = \mathrm{Sp}(p,q),$$
$$\mathrm{Aut}^0(\mathfrak{n}_{0,7}\big(\overset{p}{\oplus} V^{0,7;N}_{min}\big) \cong \mathrm{Aut}^0(\mathfrak{n}_{4,3}\big(\overset{p}{\oplus} V^{4,3;N}_{min}\big) \cong \mathrm{O}(p,p),$$
$$\mathrm{Aut}^0(\mathfrak{n}_{7,0}\Big(\big(\overset{p}{\oplus} V^{7,0;+}_{min;+}\big) \oplus \big(\overset{q}{\oplus} V^{7,0;-}_{min;+}\big)\Big) \cong \mathrm{Aut}^0(\mathfrak{n}_{3,4}\Big(\big(\overset{p}{\oplus} V^{3,4;+}_{min;+}\big) \oplus \big(\overset{q}{\oplus} V^{3,4;-}_{min;+}\big)\Big)$$
$$\cong \mathrm{O}(p,q).$$

CASES $\mathfrak{n}_{1,7}(U)$, $\mathfrak{n}_{7,1}(V)$; $\mathfrak{n}_{5,3}(U)$, $\mathfrak{n}_{3,5}(V)$; $\mathfrak{n}_{2,7}(U)$, $\mathfrak{n}_{7,2}(V)$; $\mathfrak{n}_{6,3}(U)$, $\mathfrak{n}_{3,6}(V)$.

$$\mathfrak{n}_{1,7}(\overset{p}{\oplus} V^{1,7;N}_{min}) \cong \mathfrak{n}_{7,1}\big((\overset{p}{\oplus} V^{7,1;+}_{\min}) \oplus (\overset{p}{\oplus} V^{7,1;-}_{\min})\big).$$
$$\mathfrak{n}_{5,3}(\overset{p}{\oplus} V^{5,3;N}_{min}) \cong \mathfrak{n}_{3,5}\big((\overset{p}{\oplus} V^{3,5;+}_{\min}) \oplus (\overset{p}{\oplus} V^{3,5;-}_{\min})\big).$$
$$\mathfrak{n}_{2,7}(\overset{p}{\oplus} V^{2,7;N}_{min}) \cong \mathfrak{n}_{7,2}\big((\overset{p}{\oplus} V^{7,2;+}_{\min}) \oplus (\overset{p}{\oplus} V^{7,2;-}_{\min})\big),$$
$$\mathfrak{n}_{6,3}(\overset{p}{\oplus} V^{6,3;N}_{min}) \cong \mathfrak{n}_{3,6}\big((\overset{p}{\oplus} V^{3,6;+}_{\min}) \oplus (\overset{p}{\oplus} V^{3,6;-}_{\min})\big),$$
$$\mathrm{Aut}^0\left(\mathfrak{n}_{5,3}(\overset{p}{\oplus} V^{5,3;N}_{min})\right) \cong \mathrm{Aut}^0\left(\mathfrak{n}_{1,7}(\overset{p}{\oplus} V^{1,7;N}_{min})\right) \cong \mathrm{U}(p,p)$$
$$\mathrm{Aut}^0(\mathfrak{n}_{7,1}\big(\overset{p}{\oplus} V^{7,1;+}_{\min}) \oplus (\overset{q}{\oplus} V^{7,1;-}_{\min})\big) \cong \mathrm{Aut}^0(\mathfrak{n}_{3,5}\big(\overset{p}{\oplus} V^{3,5;+}_{\min}) \oplus (\overset{q}{\oplus} V^{3,5;-}_{\min})\big) \cong \mathrm{U}(p,q).$$
$$\mathrm{Aut}^0\left(\mathfrak{n}_{2,7}(\overset{p}{\oplus} V^{2,7;N}_{min})\right) \cong \mathrm{Aut}^0\left(\mathfrak{n}_{6,3}(\overset{p}{\oplus} V^{6,3;N}_{min})\right) \cong \mathrm{Sp}(p,p),$$
$$\mathrm{Aut}^0(\mathfrak{n}_{7,2}\big(\overset{p}{\oplus} V^{7,2;+}_{\min}) \oplus (\overset{q}{\oplus} V^{7,2;-}_{\min})\big) \cong \mathrm{Aut}^0(\mathfrak{n}_{3,6}\big(\overset{p}{\oplus} V^{3,6;+}_{\min}) \oplus (\overset{q}{\oplus} V^{3,6;-}_{\min})\big) \cong \mathrm{Sp}(p,q).$$

Cases $\mathfrak{n}_{1,3}(U)$, $\mathfrak{n}_{3,1}(V)$; $\mathfrak{n}_{2,3}(U)$, $\mathfrak{n}_{3,2}(V)$; $\mathfrak{n}_{3,7}(U)$, $\mathfrak{n}_{7,3}(V)$. In all these cases the pairs of the Lie algebras are not isomorphic for any choice of admissible modules. We have

$$\mathrm{Aut}^0(\mathfrak{n}_{1,3}(U)) = \mathrm{U}(p,p), \quad \mathrm{Aut}^0(\mathfrak{n}_{3,1}(V)) = \mathrm{U}(2p,2q);$$
$$\mathrm{Aut}^0(\mathfrak{n}_{2,3}(U)) \cong \mathrm{O}(p,p), \quad \mathrm{Aut}^0(\mathfrak{n}_{3,2}(V)) = \mathrm{O}(2p,2q);$$
$$\mathrm{Aut}^0(\mathfrak{n}_{3,7}(U)) \cong \mathrm{Aut}^0(\mathfrak{n}_{7,3}(V)) \cong \mathrm{Sp}(p,q) \times \mathrm{Sp}(p,q).$$

5.2. Some isomorphisms

In the work [24, Theorem 11] it was shown the existence of an isomorphism $\mathfrak{n}_{1,7}(U^{1,7;N}_{\min}) \cong \mathfrak{n}_{7,1}(V^{7,1;\pm}_{min} \oplus V^{7,1;\pm}_{min})$. The proof was not constructive and did not show how the metric changes under the isomorphism. Therefore we propose here the constructive proof of $\mathfrak{n}_{1,7}(U^{1,7;N}_{\min}) \cong \mathfrak{n}_{7,1}(V^{7,1;+}_{min} \oplus V^{7,1;-}_{min})$. We will construct the isomorphism only for minimal dimensional module. Thus we choose the basis $(z_1, \ldots, z_8)$

$$\langle z_k, z_k \rangle_{1,7} = -1, \quad k = 1, \ldots, 7, \quad \langle z_8, z_8 \rangle_{1,7} = 1 \quad \text{for} \quad \mathbb{R}^{1,7}.$$
$$y_1 = u, \quad y_2 = \mathbf{I}y_{l1}, \quad y_3 = J_{z_1}J_{z_2}J_{z_7}y_1, \quad y_4 = \mathbf{I}y_3 = J_{z_8}y_1 \quad \text{for} \quad E^{1,7} \subset U^{1,7;N}_{\min}$$

with $\langle u, u \rangle_{E^{1,7}} = 1$ and the complex structure $\mathbf{I} = J_{z_1}J_{z_2}J_{z_7}J_{z_8}$. We also choose the basis $(w_1, \ldots, w_8)$

$$\langle w_k, w_k \rangle_{7,1} = 1, \quad k = 1, \ldots, 7, \quad \langle w_8, w_8 \rangle_{7,1} = -1 \quad \text{for} \quad \mathbb{R}^{7,1}.$$
$$x_1 = v_1, \quad x_2 = \tilde{\mathbf{I}}x_1 = -J_{w_8}v_1, \quad x_3 = J_{w_1}J_{w_2}J_{w_7}v_2, \quad x_4 = \tilde{\mathbf{I}}x_3 = -J_{w_8}v_2$$

for $E^{7,1;+} \oplus E^{7,1;-} \subset V^{7,1;+}_{min} \oplus V^{7,1;-}_{min}$ with $\langle v_1, v_1 \rangle_{E^{7,1;+1}_{J_{w_1}J_{w_2}J_{w_7}}} = -\langle v_2, v_2 \rangle_{E^{7,1;+1}_{J_{w_1}J_{w_2}J_{w_7}}} = 1$ and the complex structure $\tilde{\mathbf{I}} = J_{w_1}J_{w_2}J_{w_7}J_{w_8}$. According to [24, Corollary 5, Theorem 3] we define $C\colon \mathbb{R}^{1,7} \to \mathbb{R}^{7,1}$ by $C(z_k) = w_k$ and $C^\tau(w_k) = -z_k$, $k = 1, \ldots, 8$. The complex structure $\mathbf{I}$ will correspond the complex structure $\tilde{\mathbf{I}}$.

We define $A\colon E^{1,7;N} \to E^{7,1;+} \oplus E^{7,1;-}$ by setting

$$Ay_1 = \sum_{m=1}^{4} a_m x_m, \quad Ay_3 = \sum_{m=1}^{4} b_m x_m.$$

Using the properties $A\mathbf{I} = \tilde{\mathbf{I}}A$ we deduce that

$$A_{\mathbb{C}} = \begin{pmatrix} \bar{\lambda}_1 & \bar{\mu}_1 \\ \bar{\lambda}_2 & \bar{\mu}_2 \end{pmatrix}, \quad \eta^{1,7} J_{z_8} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \eta^{7,1} J_{w_8} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

We need to check the condition

$$A^T \eta^{7,1} J_{w_8} A = -\eta^{1,7} J_{z_8} \iff \bar{A}_{\mathbb{C}}^T \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A_{\mathbb{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It leads to finding the solution of the system

$$\begin{cases} -|\lambda_1|^2 + |\lambda_2|^2 = 0 \\ -|\mu_1|^2 + |\mu_2|^2 = 0 \\ -\lambda_1 \bar{\mu}_1 + \lambda_2 \bar{\mu}_2 = 1. \end{cases} \Longrightarrow \begin{cases} -\lambda_1 = \lambda_2 = \frac{1}{2}, \\ \mu_1 = \mu_2 = 1. \end{cases}$$

As we see the Lie algebras $\mathfrak{n}_{1,7}(V_{\min}^{1,7;N})$ and $\mathfrak{n}_{7,1}(V_{min}^{7,1;+} \oplus V_{min}^{7,1;-})$ are isomorphic.

The isomorphism is extended to any modules and the algebras $\mathfrak{n}_{1,7}(\overset{p}{\oplus} V_{min}^{1,7;N})$ and $\mathfrak{n}_{7,1}\big((\overset{p}{\oplus} V_{\min}^{7,1;+}) \oplus (\overset{p}{\oplus} V_{\min}^{7,1;-})\big)$. Analogously we can show

$$\mathfrak{n}_{2,7}(\overset{p}{\oplus} V_{min}^{2,7;N}) \cong \mathfrak{n}_{7,2}\big((\overset{p}{\oplus} V_{\min}^{7,2;+}) \oplus (\overset{p}{\oplus} V_{\min}^{7,2;-})\big),$$
$$\mathfrak{n}_{l,3}(\overset{p}{\oplus} V_{min}^{l,3;N}) \cong \mathfrak{n}_{3,l}\big((\overset{p}{\oplus} V_{\min}^{3,l;+}) \oplus (\overset{p}{\oplus} V_{\min}^{3,l;-})\big), \quad l = 5, 6,$$
$$\mathfrak{n}_{0,l}(\overset{p}{\oplus} V_{min}^{0,l;N}) \cong \mathfrak{n}_{l,0}\Big(\big(\overset{p}{\oplus} V_{min;+}^{l,0;+}\big) \oplus \big(\overset{p}{\oplus} V_{min;+}^{,0;-}\big)\Big), \quad l = 3, 7,$$
$$\mathfrak{n}_{4,3}(\overset{p}{\oplus} V_{min}^{4,3;N}) \cong \mathfrak{n}_{3,4}\Big(\big(\overset{p}{\oplus} V_{min;+}^{3,4;+}\big) \oplus \big(\overset{p}{\oplus} V_{min;+}^{3,4;-}\big)\Big).$$

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