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Market equilibria and money

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Abstract

By the first welfare theorem, competitive *market equilibria* belong to the *core* and hence are *Pareto optimal*. Letting *money* be a commodity, this paper turns these two inclusions around. More precisely, by generalizing the second welfare theorem we show that the said solutions may coincide as a common *fixed point* for one and the same system.

Mathematical arguments invoke conjugation, convolution, and generalized gradients. Convexity is merely needed via subdifferentiablity of aggregate "cost", and at one point only.

Economic arguments hinge on idealized market mechanisms. Construed as algorithms, each stops, and a steady state prevails if and only if price-taking markets clear *and* value added is nil.

Keywords: Conjugation; Convexity; Convolution; Fixed point; Generalized gradients; Competitive equilibrium; Core; Money; Pareto optimum

1 Introduction

Consider allocation of perfectly divisible and transferable commodities among diverse parties. To simplify, suppose each party has preferences as to what he gets only. Which sharing of goods and "bads" might then prove reasonably stable?

Various concepts relate to this question. Most notable are those on *competitive equilibrium, core*, and *Pareto optimum*. Each among these indicates some steady state. However, neither mentions any mechanism prone to preserve, select, or underpin such states. This silence motivates two questions:

First, *may the said concepts be brought under one shared umbrella as manifestation of a unifying fixed point*?

Second, *can some idealized mechanisms come to halt at such a common solution*?

To address the first question, in this paper, we invoke a *money commodity* and choose Debreu's *valuation equilibrium* as a focal point [1]. Such an equilibrium presumes price-taking behavior and market clearing; it is *competitive* alright. It differs from the Walrasian version in that each agent's wealth equals the value of his *final* holding, not that of his *initial* counterpart.

Addressing the second question, in this paper, we place diverse market mechanisms within *one* frame, idealized and price-based. *Double auctions* suit best and directly. But the same frame may also fit *direct deals* and *order markets*. Inspired by these three instances, the task—and the contribution here below—is to:

* reduce somewhat the omnipresent role of convexity,
* dispense with nontransferable utility via money,
* measure Pareto improvement by a potential,
* allow that endowments affect behavior,
* accommodate parties having imperfect foresight or knowledge, and finally, to
* open for out-of-equilibrium market mechanisms.

The setting—and the story—is broadly as follows. Each agent holds some money alongside a bundle of other goods. Contemplating various changes in that bundle and valuing each change by money, he comes up with an *indifference criterion* in the form of an extended real-valued function. The latter, being conditioned by his holdings, reflects reservation costs for supplying commodities or threshold payments for purchase of such. "Derivatives" of the said criterion provide linear valuations. These serve as "personal prices", not necessarily unique. *Off* equilibrium, somebody *bids* more for one or more goods than another party *asks*. Hence *bid-ask spreads* drive trade. *In* equilibrium, all such spreads are nil, trade stops, and markets clear. Formally, equilibrium prevails iff *pricing is fixed as a subgradient common to all indifference criteria*.

Arguments for this novelty are organized as follows. In Sect. 2, we clarify the construction of any generic agent's indifference criterion. In Sect. 3, we use that criterion to characterize price-taking behavior. In Sect. 4, we apply extremal convolution of indifference criteria, one for each agent, to capture "diverse" solutions as common fixed points. Modeling of disequilibrium dynamics in markets is called for but falls outside the scope of this paper. In the final Sect. 5, we invite agent-based computations and coordinated or distributed procedures. Central there are various *price-based market mechanisms.* As modeled last, *direct deals, double auctions*, and *order markets* all comprise an idealized mechanism of that special sort. These three "institutions" have steady states, which are simultaneously Pareto optimal, in the core, *and* a valuation equilibrium (Theorem 4.1). This coincidence simplifies the problem of fixed point existence (Theorem 4.2). Brought together, and thus extended, are recent results in [3–5].

This paper touches interfaces of several fields. Hence it addresses diverse readers. Arguments relate to auctions [11], convex and variational analysis [14, 15], cooperative games [16], and markets and money [10, 12]. Familiarity with these fields is not necessary, and no knowledge of economic theory is needed. Focal are just several solution concepts the satisfaction of which "freezes" competitive markets in common fixed points.

Notation and preliminaries. Let $\mathcal{X}$ be a real vector space, possibly of infinite dimension. Topological properties will be added when needed.

Any $\chi \in \mathcal{X}$ is construed as a bundle of assets or goods, financial or real. By assumption these are privately held "commodities", all of homogenous and known qualities. Each is perfectly divisible, marketable, and transferable, without any frictions, fees, or extra costs.[1] No consumption, production, or transfer has external impacts.

The notation $\chi^* \in \mathcal{X}^*$ is shorthand for a *linear functional*

$$\chi \in \mathcal{X} \quad \mapsto \quad \chi^*\chi := \chi^*(\chi) \in R.$$

Any such functional may serve as a *price regime*. By tacit convention, when $\mathcal{X}$ is topological, take each $\chi^* \in \mathcal{X}^*$ to be continuous.

[1] In particular, walking to and from the bank uses up no shoe leather [20].

A pair $x = (r, \chi)$ in the augmented space $X := R \times \mathcal{X}$ reflects a bank roll or reserves $r \in R$ of *money* alongside a holding $\chi \in \mathcal{X}$ of "real goods". Each linear pricing $x^* \in X^*$ takes the form $x^* = (r^*, \chi^*) \in R^* \times \mathcal{X}^*$ and operates by

$$x \in X \quad \mapsto \quad x^* x = (r^*, \chi^*)(r, \chi) = r^* r + \chi^* \chi \in R.$$

Trade proceeds in real goods for money or *quid pro quo*. Money is never exchanged for itself. Being numéraire, a unit of presently available money always commands price 1. So, from now on,

$$\text{whatever linear } x^* = (r^*, \chi^*), \text{ intended or used for valuation, has } r^* = 1. \tag{1}$$

The component $\chi^* \in \mathcal{X}^*$ just prices "real goods". Conversely, any "price vector" $\chi^* \in \mathcal{X}^*$ extends to a unique valuation regime $x^* = (1, \chi^*) \in X^*$. Convention (1) was motivated here by elementary economics and first principles. It is later underpinned by Proposition 3.1.

A (cost) criterion $c : \mathcal{X} \to R \cup \{+\infty\}$ is *proper* iff it has nonempty *effective domain* $\mathit{dom} c := \{\chi \in \mathcal{X} : c(\chi) \in R\}$. Any proper c has a *conjugate function*

$$\chi^* \in \mathcal{X}^* \quad \mapsto \quad c^*(\chi^*) := \sup\{\chi^* \chi - c(\chi) : \chi \in \mathcal{X}\} \in R \cup \{+\infty\}, \tag{2}$$

which reports the *value added* or profit, potentially obtained, under exogenous price χ^*. Call the latter a *subgradient* of c at $\chi \in \mathit{dom} c$, as signalled by writing

$$\chi^* \in \partial c(\chi) \quad \text{iff} \quad \chi \in \arg\max\{\chi^* - c\}. \tag{3}$$

This holds precisely when, by *Fenchel's equality*

$$\chi^* \chi = c^*(\chi^*) + c(\chi), \tag{4}$$

the total revenue $\chi^* \chi$ splits between price-taking profit $c^*(\chi^*)$ (2) and cost $c(\chi)$.

A proper *payoff criterion* $\pi : \mathcal{X} \to R \cup \{-\infty\}$ has a *supgradient*

$$\chi^* \in \hat{\partial} \pi(\chi) := -\partial[-\pi](\chi) \quad \text{iff} \quad \chi \in \arg\max\{\pi - \chi^*\}. \tag{5}$$

2 Preferences, money, and indifference criteria

To prepare the ground, in this section, we consider just *one* generic agent, alongside his *preferences*, affected by *money*, and represented by an *indifference criterion*.

The preferences of the agent at hand are mirrored by a binary relation $\succsim$ over $X = R \times \mathcal{X}$. An outcome $x \in X$ belongs to the *effective domain* of $\succsim$, denoted *dom* $\succsim$ and supposed nonempty, iff the upper level, *preferred* set

$$P(x) := \{\succsim x\} := \{\hat{x} \in X : \hat{x} \succsim x\} \tag{6}$$

contains x but differs from X.[2] Consequently, $\succsim$ is *reflexive* on its proper domain but not necessarily *complete*. *Transitivity* will be invoked only when needed. *Strict preference* $\hat{x} \succ$

[2] Since the typical agent does not care for all goods [10], his *viability set dom* $\succsim$ may well have empty interior or bear little resemblance to nonnegative orthants.

x means that $\hat{x} \succsim x$ but *not* $x \succsim \hat{x}$. Preferences are primitives throughout. None needs a functional representation.

Money [4, 10] is a commodity in itself and *desirable* as such:

$$r > 0 \quad \& \quad x \in dom \succsim \quad \Longrightarrow \quad (r, 0) + x \succ x. \tag{7}$$

Moreover,

$$\hat{x} \succ x \quad \Longrightarrow \quad \hat{x} - (r, 0) \succsim x \quad \text{for small enough } r > 0. \tag{8}$$

Money is the only good for which each agent's preferences must display monotonicity and nonsatiation. Further, *some money is always retained*; it is essential or indispensable in that

$$x \in dom \succsim \quad \Longrightarrow \quad x - (r, 0) \in dom \succsim \quad \text{for small enough } r > 0. \tag{9}$$

Together (7)–(9) allow the agent to balance the amenity of holding money against the benefits procured by suitable bundles of real goods:

Indifference criterion. Suppose the agent holds *endowment* $x \in dom \succsim$. If contemplating to *supply* $\chi \in \mathcal{X}$, he would *ask* for no less money compensation than

$$a(\chi \mid x) := \inf\{r \in R : (r, -\chi) + x \succsim x\}. \tag{10}$$

$a(\chi \mid x)$ is attained if the set mentioned in (10) is bounded below, closed, and nonempty.[3] Then $a(\chi \mid x)$ equals the minimal payment the agent would accept for supplying χ. Similarly, if considering to *buy* $\chi \in \mathcal{X}$, he would *bid* no more money than his resulting *benefit* [12]

$$b(\chi \mid x) := \sup\{r \in R : (-r, \chi) + x \succsim x\}. \tag{11}$$

$b(\chi \mid x)$ is attained if the set in question is bounded above, closed, and nonempty. Then $b(\chi \mid x)$ equals the maximal payment the agent offers for χ. Note that

$$a(\chi \mid x) = -b(-\chi \mid x) =: c(\chi \mid x). \tag{12}$$

So, from here onward, to simplify and synthesize, *supply is negative demand,* and *revenue is negative expense.* Accordingly, by (12), making money a means of all transactions, the *indifference criterion*

$$\chi \in \mathcal{X} \quad \mapsto \quad c(\chi \mid x) \in R \cup \{+\infty\}$$

becomes a unifying object. It reflects the agent's idiosyncratic valuations, reservation costs, or threshold payments, all denominated in money and depending on his actual endowment x.[4]

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[3] The convention $\inf \varnothing = +\infty$ applies.

[4] There is no "*money illusion*" here: If one unit of old currency matches $\rho > 0$ units of a new one, then $\rho c(\cdot \mid x)$ replaces $c(\cdot \mid x)$.

Most likely, the criterion $c(\cdot \mid x)$ is known only to the agent himself, and maybe just locally, near 0. He might communicate it to nobody. Indeed, rather than frankly reporting $c(\cdot \mid x)$, he may, for strategic reasons, use the shaded version

$$\chi \in \mathcal{X} \quad \mapsto \quad C(\chi \mid x) \geq c(\chi \mid x), \tag{13}$$

briefly called his *market curve*. When the latter drives a wedge between acceptable "revenue" r and indifference "cost" $c(\chi \mid x)$, it overstates the agent's demands.

By assumption, $c(\cdot \mid x) > -\infty$. Naturally, $C(0 \mid x) \leq 0$ for any $x \in dom \succsim$. Hence both $C(\cdot \mid x)$ and $c(\cdot \mid x)$ are proper. The latter criterion extends from the real-good subspace $\mathcal{X}$ to $X = R \times \mathcal{X}$ by

$$\hat{x} \in X \quad \mapsto \quad c(\hat{x} \mid x) := \inf\{r \in R : (r, 0) - \hat{x} + x \succsim x\}.$$

However, $\hat{x}$ should best be construed right here as *change* $(0, \chi)$ in the real-good component of endowment x. Anyway, with $\hat{x} = (\hat{r}, \hat{\chi}) \in domc(\cdot \mid x)$, convention (1), just like (10)–(12), implies

$$c(\hat{x} \mid x) = \hat{r} + c(\hat{\chi} \mid x). \tag{14}$$

Relations (7)–(9) and (14) entail *transferable utility* or *quasi-linearity* [18], and thereby, $\frac{\partial}{\partial \hat{r}} c((\hat{r}, \hat{\chi}) \mid x) = 1$, a property shared with many financial measures [7].

Remark (On convexity and closure of preferred sets) If $\{\succsim x\}$, then (6) is *convex*, and so is also the function $\chi \mapsto c(\chi \mid x)$. Similarly, with $\mathcal{X}$ topological and $\{\succsim x\}$ *closed*, $c(\cdot \mid x)$ is also closed (i.e. lower semicontinuous). These observations might motivate a blanket hypothesis that each preferred set $\{\succsim x\}$ comes closed convex. Yet, since Guesnerie [9], many studies have contended with weaker assumptions; see [6, 14], and references therein.

3 Price-taking behavior

How might a generic agent fare in face of a fixed price $x^* \in X^*$? Answers to this question will inform subsequent arguments. Therefore, upon addressing it, this section presumes that the agent behaves as a price-taking optimizer.

Having preference order $\succsim$, endowment $x \in dom \succsim$, and thereby indifference criterion $c(\cdot \mid x)$, any exogenous price regime $x^* \in X^*$ offers him *value added* (2):

$$c^*(x^* \mid x) := \sup\{x^*\tilde{x} - c(\tilde{x} \mid x) : \tilde{x} \in X\}. \tag{15}$$

From $c(0 \mid x) \leq 0$ and (15) it follows that $c^*(\cdot \mid x) \geq 0$. Thus

$$c^*(x^* \mid x) = 0 \quad \Longleftrightarrow \quad \tilde{x} = 0 \quad \text{solves (15) and } c(0 \mid x) = 0. \tag{16}$$

Since annulment of profit serves as common certificate of efficacy, henceforth *suppose* $c(0 \mid x) = 0$ (16) *at each* $x \in dom \succsim$. By (1) and (14) a money unit has unitary price:

Proposition 3.1 (On linear pricing and minimal expenditure [5]) *Value added* (15) *cannot be finite unless* $x^* = (r^*, \chi^*)$ *has money price* $r^* = 1$. *Then it equals*

$$c^*(x^* \mid x) = \sup\{x^*(x - \hat{x}) : \hat{x} \succsim x\}, \tag{17}$$

and $\hat{x}$ *is a best choice in* (17) *iff it solves the problem of* minimal expenditure [12]:

$$\mathcal{E}(x^* \mid x) := \inf\{x^*\hat{x} : \hat{x} \succsim x\} = x^*x - c^*(x^* \mid x). \tag{18}$$

Proof Proof is included for completeness. Observe that

$$\begin{aligned} c^*(x^* \mid x) &= \sup\{x^*\tilde{x} - r \mid \hat{x} := (r, 0) - \tilde{x} + x \succsim x, r \in R, \tilde{x} \in X\} \\ &= \sup\{x^*(x - \hat{x}) + r(r^* - 1) : r \in R \;\&\; \hat{x} \succsim x\} \\ &= \begin{cases} \sup\{x^*(x - \hat{x}) : \hat{x} \succsim x\}, & \text{hence (17) holds iff } r^* = 1, \\ +\infty & \text{otherwise,} \end{cases} \\ &= \begin{cases} x^*x - \mathcal{E}(x^* \mid x) & \text{if } r^* = 1, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

□

Still presuming that the agent faces an exogenous fixed price $x^* \in X^*$, we have:

Proposition 3.2 (On best competitive choice) *Consider a price-taking agent having endowment* $x \in dom \succsim$. *Suppose his indifference criterion* $c(\cdot \mid x)$ *attains each finite value, and recall that* $c(0 \mid x) = 0$. *Upon facing a price regime* $x^* = (1, \chi^*) \in X^*$, *he has a best update*

$$x^{+1} := (r, -\chi) + x \succsim x$$

iff $r = \chi^*\chi$ *and* $\chi^* \in \partial c(\chi \mid x)$. *Thereby he gets value added or profit* $c^*(\chi^* \mid x) = \chi^*\chi - c(\chi \mid x) \geq 0$. *In particular,* $c^*(\chi^* \mid x) = 0$ *if* $(r, \chi) = (0, 0)$, *meaning that he makes no trade*: $x^{+1} = x$.

Proof Any "supply" $\chi \in \mathcal{X}$ generates gross revenue $r = \chi^*\chi$, attained cost $c(\chi \mid x)$, and profit $\chi^*\chi - c(\chi \mid x)$. The latter becomes maximal at χ with value $c^*(\chi^* \mid x)$ iff $\chi^* \in \partial c(\chi \mid x)$. Then, in particular, $c^*(\chi^* \mid x) = 0 \Longleftrightarrow 0 \in \partial c^*(\chi^* \mid x) \Longleftrightarrow \chi^* \in \partial c(0 \mid x) \Longleftrightarrow$ (by (4))

$$0 = \chi^*0 = c^*(\chi^* \mid x) + c(0 \mid x).$$

□

For links to received microeconomic theory on price-taking behavior [12, 18], this section concludes by briefly looking *first*, at a *consumer* constrained by his budget; *second*, at a *producer* constrained by his technology. The impatient reader may skip the rest of this section or return to it later.

Proposition 3.3 (On the price-taking consumer [5]) *If* $x \in dom \succsim$ *and* $c^*(x^* \mid x) > 0$ *with budget* $\beta := x^*x$, *then the affordable, strictly preferred set* $\{\hat{x} \succ x : x^*\hat{x} \leq \beta\}$ *cannot be empty. Conversely, if that set is indeed nonempty, then* $c^*(x^* \mid x) > 0$. *In short,*

$$c^*(x^* \mid x) = 0 \quad \Longleftrightarrow \quad \{\hat{x} \succ x : x^*\hat{x} \leq x^*x\} = \varnothing. \tag{19}$$

Proof Proof included for completeness. Note that $x^* \neq 0$. From (11) it follows that

$$\sup\{b(\check{x} \mid x) - x^*\check{x} : \check{x} \in X\} = c^*(x^* \mid x). \tag{20}$$

So, if $c^*(x^* \mid x) > 0$, some $\check{x} \in X$ satisfies $b(\check{x} \mid x) - x^*\check{x} > 0$, and thereby

$$x^*[\check{x} - b(\check{x} \mid x)(1,0) + x] < \beta = x^*x.$$

Hence the bundle $\hat{x} := \check{x} - [b(\check{x} \mid x) + r, 0] + x$ costs $x^*\hat{x} \leq \beta$ for sufficiently small $r > 0$. At the same time, from (7),

$$\hat{x} \succ \check{x} - b(\check{x} \mid x)(1,0) + x \succsim x.$$

Consequently, $\hat{x}$ is affordable (within budget x^*x) and strictly preferred to x.

For the converse, suppose some $\hat{x}$ is such a bundle. Then, by (8), for sufficiently small $r > 0$, we have $(-r,0) + (\hat{x} - x) + x \succ x$ and thereby $b(\hat{x} - x \mid x) > 0 = b(0 \mid x)$. Then

$$b(\hat{x} - x \mid x) - x^*(\hat{x} - x) > b(0 \mid x) - x^*0 \geq 0.$$

In turn, by (20) this implies $c^*(x^* \mid x) > 0$. □

Remark (On extremality of preference and budget) By (19), $c^*(x^* \mid x) = 0$ implies that the preferred set $\{\succsim x\}$ and the half-space $\{x^* \leq x^*x\}$ are *extremal* at their common point x; see [14], I, Def. 2.1. Indeed, by (7),

$$\{(r,0) + \hat{x} : \hat{x} \succsim x\} \cap \{\hat{x} : x^*\hat{x} \leq x^*x\} = \varnothing \quad \text{for each } r > 0.$$

So, if the space $\mathcal{X}$ is Banach, hence X likewise, and $\{\succsim x\}$ is *locally convex* (or more generally, *normally regular*) at x, then the nonzero price x^* belongs to the (basic, limiting) reversed normal cone $-N(\{\succsim x\}, x)$ of $\{\succsim x\}$ at x [14].

Complementing the consumer is the producer:

Proposition 3.4 (On the price-taking producer) *Suppose a prototypical producer having endowment $x \in X$ operates a closed technology $T \subset X$ containing x. Let*

$$\hat{x} := (\hat{r}, \hat{\chi}) \succsim (r, \chi) =: x \in T \quad \Longleftrightarrow \quad \hat{x} \in T \quad \& \quad \hat{r} \geq r. \tag{21}$$

Then, with price $x^* = (1, \chi^*)$, it obtains

$$c^*(x^* \mid x) = \sup\{x^*(x - \hat{x}) : \hat{x} \succsim x\} \tag{22}$$

with $c^*(x^* \mid x) \geq 0$, and

$$c^*(x^* \mid x) = 0 \quad \Longleftrightarrow \quad x \in \arg\max\{x^*\hat{x} : \hat{x} \in T\}. \tag{23}$$

Thus, at zero profit, we have the customary normal cone condition (or variational inequality) $x^*(\hat{x} - x) \le 0$ for all $\hat{x} \in T$. Then extremality of preference and payoff prevails in that

$$\left\{(r,0) + \hat{x} : \hat{x} \succsim x\right\} \cap \left\{\hat{x} : x^*\hat{x} \ge x^*x\right\} = \varnothing \quad \textit{for each } r > 0.$$

So, with X Banach, if $\{\succsim x\}$ is normally regular at x, then the price x^* belongs to the (basic, limiting) normal cone $N(\{\succsim x\}, x)$.

Proof Verbatim follows the demonstration of Proposition 3.1 until equality (22) comes up. Now invoke (21) to have (23). The inclusion $x^* \in N(\{\succsim x\}, x)$ follows the arguments after Definition 2.5 in [14]. □

Since $c(0 \mid x) \le 0$, a price-taking agent can aim at value added $c^*(\chi^* \mid x) \ge 0$. By (19) and (23) he ought stay put iff $c^*(\chi^* \mid x) = 0$. When nobody sees any value added, Pareto optimality, the core and valuation equilibrium come up, all at once, as is argued next.

4 Pareto optimality, core, and valuation equilibrium

Pareto optimality often serves as a weak welfare criterion [12]. Modulo money as commodity (and currency), this section identifies such optimality with *core* solutions [16] and *valuation equilibria* [1]. What makes these notions coincide is the event of *value added being nil in the large and in the small* (36). Then, as seen below, the three solutions, whence the two welfare theorems [12], are obtained in one shot (Theorem 4.1).

Accommodated henceforth is a fixed finite ensemble I of economic agents. They need not be many but, clearly, at least two. Member $i \in I$ has endowment $x_i \in X$, preference $\succsim_i$, and indifference criterion $c_i(\cdot \mid x_i)$.

For brevity, call any nonempty ensemble $\mathcal{I} \subseteq I$ a *coalition*. Its members hold some prescribed *endowment* $x^0_{\mathcal{I}} \in X$ for shared use. By assumption, if coalitions $\mathcal{I}_1, \mathcal{I}_2$ are disjoint, then $x^0_{\mathcal{I}_1} + x^0_{\mathcal{I}_2} = x^0_{\mathcal{I}_1 \cup \mathcal{I}_2}$. Let

$$\mathbf{X}_{\mathcal{I}} := \left\{\mathbf{x}_{\mathcal{I}} = (x_i) \in X^{\mathcal{I}} : x_i \in dom \succsim_i \forall i \in \mathcal{I} \;\&\; \sum_{i\in\mathcal{I}} x_i = x^0_{\mathcal{I}}\right\} \tag{24}$$

be the set of *feasible allocations* across $\mathcal{I}$. Given any $\chi_{\mathcal{I}} \in \mathcal{X}$ and $\mathbf{x}_{\mathcal{I}} \in \mathbf{X}_{\mathcal{I}}$, the *inf-convolution*

$$c_{\mathcal{I}}(\chi_{\mathcal{I}} \mid \mathbf{x}_{\mathcal{I}}) := \inf\left\{\sum_{i\in\mathcal{I}} c_i(\chi_i \mid x_i) : \sum_{i\in\mathcal{I}} \chi_i = \chi_{\mathcal{I}}\right\} \tag{25}$$

denotes the "minimal cost" incurred by $\mathcal{I}$ at total "supply" χ_I. Accordingly, for fixed $\mathbf{x}_{\mathcal{I}} \in \mathbf{X}_{\mathcal{I}}$, the best *benefit* (11), (12)

$$b_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}}) := \sup\left\{\sum_{i\in\mathcal{I}} b_i(\chi_i \mid x_i) : \sum_{i\in\mathcal{I}} \chi_i = 0\right\} = -c_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}}) \ge 0 \tag{26}$$

presumes efficient *reallocation* of goods and money across $\mathcal{I}$. Entities $b_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}})$ and $c_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}})$, being two sides of the same coin, are later linked to *value added* for coalition $\mathcal{I}$, as seen below.

Definition 4.1 (Three solution concepts) *(Pareto optimality).* $\mathbf{x}_I \in \mathbf{X}_I$ is strongly (weakly) *Pareto optimal* iff the "inequality" system

$$x_i^{+1} = (r_i, -\chi_i) + x_i \succsim_i x_i \quad \forall i \in I \tag{27}$$

has no solution $(x_i^{+1}) =: \mathbf{x}_I^{+1} \in \mathbf{X}_I$ with $x_i^{+1} \succ_i x_i$ for some (all) $i \in I$.

(Core). $\mathbf{x}_I \in \mathbf{X}_I$ belongs to the *core*, as implemented by a monetary benefit profile $(\bar{b}_i) \in R^I$, iff

$$\sum_{i \in \mathcal{I}} \bar{b}_i \geq b_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}}) \quad (26)\ \forall \mathcal{I} \subset I, \text{ with equality for } \mathcal{I} = I. \tag{28}$$

(Valuation equilibrium). A price-cum-allocation pair $(\chi^*, \mathbf{x}_I) \in \mathcal{X}^* \times \mathbf{X}_I$ is declared a competitive *valuation equilibrium* iff χ^* incites no further trade, meaning that

$$\chi^* \in \partial c_i(0 \mid x_i) \quad \forall i \in I. \tag{29}$$

All three concepts revolve around *efficacy*. That property is most conveniently tested or verified in differential but generalized terms:

Proposition 4.0 (On price-based efficacy [5]) *Suppose* (χ_i) *solves* (25) *for* $\mathcal{I} = I$, *some fixed* $\chi_I \in \mathcal{X}$, *and* $(x_i) = \mathbf{x}_I \in \mathbf{X}_I$ (24). *Then*

$$\partial c_I(\chi_I \mid \mathbf{x}_I) \subseteq \cap \partial_{i \in I} c_i(\chi_i \mid x_i). \tag{30}$$

Conversely, if $\sum_{i \in I} \chi_i = \chi_I$, *then the turned-around inclusion also holds*:

$$\partial c_I(\chi_I \mid \mathbf{x}_I) \supseteq \bigcap_{i \in I} \partial c_i(\chi_i \mid x_i). \tag{31}$$

Moreover, if $\bigcap_{i \in I} \partial c_i(\chi_i \mid x_i)$ *is nonempty, then* (χ_i) *solves* (25).

Proof Proof is included for completeness. If $\chi^* \in \partial c_I(\chi_I \mid \mathbf{x}_I)$ and (χ_i) solves (25) with $\mathcal{I} = I$, then $\sum_{i \in I} \hat{\chi}_i = \hat{\chi}_I$ implies

$$\begin{aligned} \sum_{i \in I} c_i(\hat{\chi}_i \mid x_i) &\geq c_I(\hat{\chi}_I \mid \mathbf{x}_I) \geq c_{\mathcal{I}}(\chi_I \mid \mathbf{x}_I) + \chi^*(\hat{\chi}_I - \chi_I) \\ &= \sum_{i \in I} \big[c_i(\chi_i \mid x_i) + \chi^*(\hat{\chi}_i - \chi_i)\big]. \end{aligned} \tag{32}$$

In this string, posit $\hat{\chi}_j = \chi_j$ for each $j \in I \diagdown i$ to get

$$c_i(\hat{\chi}_i \mid x_i) \geq c_i(\chi_i \mid x_i) + \chi^*(\hat{\chi}_i - \chi_i). \tag{33}$$

Since $i \in I$ and $\hat{\chi}_i \in \mathcal{X}$ were arbitrary, $\chi^* \in \partial c_i(\chi_i \mid x_i)$ for all $i \in I$, and inclusion (30) holds.

Conversely, if $\sum_{i \in I} \chi_i = \chi_I$, then for any $i \in I$, $\chi^* \in \cap \partial_{i \in I} c_i(\chi_i \mid x_i)$, and $\hat{\chi}_i \in \mathcal{X}$, inequality (33) holds. Summation of these, subject to $\sum_{i \in I} \hat{\chi}_i =: \hat{\chi}_I$, gives the inequalities in (32), and hence $\chi^* \in \partial c_{\mathcal{I}}(\chi_I \mid \mathbf{x}_I)$, this taking care of (31). Further, letting $\hat{\chi}_I = \chi_I$, we obtain the optimality of (χ_i). □

It merits noticing that Proposition 4.0 needed no hypotheses as to convexity, differentiability, or set separation. Its applicability hinges, however, on $\partial c_I(0 \mid \mathbf{x}_I)$ being nonempty. Then $c_I(\cdot \mid \mathbf{x}_I)$ is convex at 0, as follows forthwith:

Lemma 4.1 (On convexity and closure at *one* point) *If some criterion* $c : \mathcal{X} \to R \cup \{+\infty\}$ *has* $\partial c(\chi)$ *nonempty at* $\chi \in \mathcal{X}$, *then c coincides with its* convex envelope *at* χ. *If moreover,* $\mathcal{X}$ *is topological, then c must be closed at* χ *to the effect that c equals its* closed convex envelope *there. Consequently, if* $\mathcal{X}$ *is topological, then*

$$\chi^* \in \partial c(\chi) \iff \chi \in \partial c^*(\chi^*).$$

What comes to the fore here is the particular instance $c = c_I(\cdot \mid \mathbf{x}_I)$:

Lemma 4.2 (On *aggregate* closure and convexity under reallocation) *With* $\mathcal{X}$ *topological,*

$$\chi^* \in \partial c_I(0 \mid \mathbf{x}_I) \iff 0 \in \partial c_I^*(\chi^* \mid \mathbf{x}_I).$$

Thus, for any shadow price $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$,

$$0 = \chi^* 0 = c_I^*(\chi^* \mid \mathbf{x}_I) + c_I(0 \mid \mathbf{x}_I).$$

Given $\mathbf{x}_I = (x_i)$, *this happens iff*

$$c_i^*(\chi^* \mid x_i) = c_i(0 \mid x_i) = 0 \quad \forall i.$$

Henceforth assume that $\mathcal{X}$ *is topological.*

By Lemma 4.2,

$$\chi^* \in \arg\min c_I^*(\cdot \mid \mathbf{x}_I) \iff 0 \in \partial c_I^*(\chi^* \mid \mathbf{x}_I) \iff \partial c_I(0 \mid \mathbf{x}_I) = \bigcap_{i \in I} \partial c_i(0 \mid x_i) \neq \varnothing.$$

Also, like (16),

$$c_I^*(\chi_I^* \mid \mathbf{x}_I) = 0 \iff 0 \in \arg\max\{\chi_I^* - c_I(\cdot \mid \mathbf{x}_I)\} \quad \& \quad c_I(0 \mid \mathbf{x}_I) = 0.$$

Lemma 4.2 points to the focal role of value added being nil, as is confirmed next.

Proposition 4.1 (On Pareto optimality and value added) *If* $\mathbf{x}_I = (x_i)$ *is strongly Pareto optimal, and* $\chi^* \in \bigcap_{i \in I} \partial c_i(0 \mid x_i)$, *then* $c_I^*(\chi^* \mid \mathbf{x}_I) = 0$.

Conversely, if $c_I^*(\chi^* \mid \mathbf{x}_I) = 0$ *for some* $\chi^* \in \mathcal{X}^*$, *then* $\mathbf{x}_I$ *is strongly Pareto optimal.*

There is no distinction here between strong and weak Pareto optimality; either prevails iff

$$c_i^*(\chi^* \mid x_i) = c_i(0 \mid x_i) = 0 \quad \forall i \in I. \tag{34}$$

Proof When $\mathbf{x}_I$ is strongly Pareto optimal for the grand coalition $\mathcal{I} = I$, each solution $\mathbf{x}_I^{+1} \in \mathbf{X}_I$ to (27) satisfies $r_i = c_i(\chi_i \mid x_i)$ $\forall i$ and $\sum_{i\in I}(r_i, \chi_i) = (0,0)$. Consequently, $c_I(0 \mid \mathbf{x}_I) = 0$. But then the profile $i \in I \mapsto (r_i, \chi_i) = (0,0)$ also yields the invariant solution $\mathbf{x}_I^{+1} = \mathbf{x}_I \in \mathbf{X}_I$. Now, for any $\chi^* \in \bigcap_{i\in I} \partial c_i(0 \mid x_i)$, from Proposition 4.0 it follows that $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$. Hence

$$0 = \chi^* 0 = c_I^*(\chi^* \mid \mathbf{x}_I) + c_I(0 \mid \mathbf{x}_I) = c_I^*(\chi^* \mid \mathbf{x}_I).$$

Conversely, if $\mathbf{x}_I$ is not strongly Pareto optimal, then system (27) is solvable for $\mathcal{I} = I$ with $\mathbf{x}_I^{+1} \in \mathbf{X}_I$ and at least one $x_j^{+1} \succ_j x_j$. Then $\sum_{i\in I}(r_i, \chi_i) = (0,0)$, $r_i \geq c_i(\chi_i \mid x_i)$ $\forall i$, and by (8), $r_j > c_j(\chi_j \mid x_j)$. Consequently,

$$\begin{aligned} &c_I^*(\chi^* \mid \mathbf{x}_I) \\ &\quad= \sup_{\hat{\chi}_I}\{\chi^*\hat{\chi}_I - c_I(\hat{\chi}_I \mid \mathbf{x}_I)\} = \sup_{(\hat{\chi}_i)}\left\{\sum_{i\in I}[\chi^*\hat{\chi}_i - c_i(\hat{\chi}_i \mid x_i)] : \sum_{i\in I}\hat{\chi}_i = \hat{\chi}_I\right\} \\ &\quad\geq \sup_{(\hat{\chi}_i)}\left\{\sum_{i\in I}[\chi^*\hat{\chi}_i - c_i(\hat{\chi}_i \mid x_i)] : \sum_{i\in I}\hat{\chi}_i = 0\right\} \\ &\quad\geq \sum_{i\in I}[\chi^*\chi_i - c_i(\chi_i \mid x_i)] > \sum_{i\in I}[\chi^*\chi_i - r_i] = 0. \end{aligned}$$

Finally, as just argued, if $\mathbf{x}_I$ is not strongly Pareto optimal, then system (27) is solvable for $\mathcal{I} = I$ with $\mathbf{x}_I^{+1} \in \mathbf{X}_I$, $r_i \geq c_i(\chi_i \mid x_i)$ $\forall i$, and at least one $r_j > c_j(\chi_j \mid x_j)$. Subtract some money amount $\rho \in \,]0, r_j - c_j(\chi_j \mid x_j)[$ from that agent j and distribute ρ equally among the others. Compared to $\mathbf{x}_I$, doing so leaves everybody with strict Pareto improvement. Finally, because $c_I^*(\chi^* \mid \mathbf{x}) = \sum_{i\in I} c_i^*(\chi^* \mid x_i)$ for every $\chi^* \in \mathcal{X}^*$, this takes care of (34). □

The concept of core strengthens that of Pareto optimality. Yet, if efficiency is price-supported as in Proposition 4.0, then the two coincide:

Proposition 4.2 (On price-supported core solutions) *Fix any $\mathbf{x}_I \in \mathbf{X}_I$ and suppose $\partial c_I(0 \mid \mathbf{x}_I)$ is nonempty. Consider the cooperative game in which coalition $\mathcal{I} \subseteq I$ can aim at no less benefit than $b_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}}) = -c_{\mathcal{I}}(0 \mid \mathbf{x}_{\mathcal{I}}) \geq 0$ (26).*

Then, for any shadow price *$\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$ (3), by offering agent $i \in I$ value added or benefit $\bar{b}_i := c_i^*(\chi^* \mid x_i)$ (2), the said game generates a core solution (28). If, moreover, (χ_i) solves (25) for $\mathcal{I} = I$ and $\chi = 0$, then $c_i^*(\chi^* \mid x_i) = \chi^*\chi_i - c_i(\chi_i) \geq 0$.*

Since $c_I^*(\cdot \mid \mathbf{x}_I) = \sum_{i\in I} c_i^*(\cdot \mid x_i)$, value added is constructively and explicitly shared. In particular,

$$c_I^*(\chi^* \mid \mathbf{x}_I) = 0 \quad\Longleftrightarrow\quad c_i^*(\chi^* \mid x_i) = c_i(0 \mid x_i) = 0 \quad \forall i.$$

Proof (2) implies, for *any* $\chi^* \in \mathcal{X}^*$, that each nonempty coalition $\mathcal{I} \subseteq I$ would get the aggregate value

$$\sum_{i\in\mathcal{I}} c_i^*(\chi^* \mid \mathbf{x}_I) = c_{\mathcal{I}}^*(\chi^* \mid \mathbf{x}_I) \geq -c_{\mathcal{I}}(0 \mid \mathbf{x}_I) = b_{\mathcal{I}}(0 \mid \mathbf{x}_I) \quad (26). \tag{35}$$

In particular, $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$ iff $c_I^*(\chi^* \mid \mathbf{x}_I) = -c_I(0 \mid \mathbf{x}_I)$. Then, by Proposition 4.0, $\chi^* \in \partial c_i(\chi_i \mid x_i)$, so that $\chi^*\chi_i = c_i^*(\chi^* \mid x_i) + c_i(\chi_i \mid x_i)$ for each i (4). Because $c_i(0 \mid x_i) \leq 0$, we have $c_i^*(\chi^* \mid x_i) \geq 0$, and hence $\chi^* \chi_i \geq c_i(\chi_i \mid x_i)$. The statement on shared value is derived from (35) when $\mathcal{I} = I$. □

Much market theory fits the frames of *primal-dual problems* in optimization, primal on allocation and dual on valuation. The inequalities in (28) and (35), which required virtually nothing, reflect *weak duality*. By contrast, equality for the grand coalition $\mathcal{I} = I$, which just hinges on *aggregate convexity*, amounts to *strong duality*. Apart from the possibility that $|c_I(0|\mathbf{x}_I)| > 0$—and then apart from positive profits or values added—the outcome resembles competitive equilibrium. When, moreover, $c_I^*(\chi^* \mid \mathbf{x}_I) = 0$, such equilibrium comes up:

Proposition 4.3 (On equilibrium and no value added) *Valuation equilibrium* $(\chi^*, \mathbf{x}_I)$ *prevails iff value added is nil, that is,* $c_I^*(\chi^* \mid \mathbf{x}_I) = 0$, *and the price is a shadow, that is,* $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$. *This happens iff* $c_i^*(\chi^* \mid x_i) = c_i(0 \mid x_i) = 0\ \forall i \in I$.

Proof From (29) and Proposition 4.0 it follows that $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$ as well as $c_I(0 \mid \mathbf{x}_I) = \sum_{i \in I} c_i(0 \mid x_i) = 0$ with each $c_i(0 \mid x_i) = 0$. Hence

$$0 = \chi^* 0 = c_I^*\left(\chi^* \mid \mathbf{x}_I\right) + c_I(0 \mid \mathbf{x}_I) = c_I^*\left(\chi^* \mid \mathbf{x}_I\right) = \sum_{i \in I} c_i^*\left(\chi^* \mid x_i\right), \tag{36}$$

and thereby each $c_i^*(\chi^* \mid x_i) = 0$. Tracking these arguments, the opposite direction is straightforward. □

As promised, three solutions coincide. This is forthwith brought out next by collecting arguments from respective Propositions 4.1–4.3:

Theorem 4.1 (On economic welfare) *Suppose that* $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$ *and* $c_I(0 \mid \mathbf{x}_I)$ *is attained. Then the following three statements are equivalent*:

1: $(x_i) = \mathbf{x}_I$ *is Pareto optimal and supported by* χ^*.

2: (x_i) *generates a core solution, implemented by* χ^*.[5]

3: $(\chi^*, \mathbf{x}_I)$ *is a valuation equilibrium.*

Choosing Pareto optimality as focal solution, Theorem 4.1 opens a direct take on when there is at least one fixed point:

Theorem 4.2 (On equilibrium existence) *Let the linear space* $\mathcal{X}$ *of real commodity bundles be Hausdorff topological. Suppose that the set* $\mathbf{X}_I$ *of feasible allocations* (24) *is nonempty compact convex and that all preference orders* $\succsim_i$ *are transitive. If, for each* $i \in I$ *and* $x_i \in \operatorname{dom} \succsim_i$,

$$x_i \notin \operatorname{conv}\{\bar{x}_i : \bar{x}_i \succ_i x_i\}, \quad \textit{and} \quad \{\underline{x}_i : x_i \succ_i \underline{x}_i\} \textit{ is open}, \tag{37}$$

then the set of Pareto optima is nonempty compact. Moreover, if $\partial c_I(0 \mid \mathbf{x}_I)$ *is nonempty at each Pareto optimum* $\mathbf{x}_I \in \mathbf{X}_I$*,then these are all competitive valuation equilibria.*

[5]"Paradoxalement, c'est avec les jeux coalitionnels que la liason équilibre general/théorie des jeux a été la plus féconde" [19].

Proof Define an irreflexive transitive order $\succ$ on $\mathbf{X}_I$ (24) by

$$\hat{\mathbf{x}}_I \succ \mathbf{x}_I \quad \Longleftrightarrow \quad \hat{x}_i \succ_i x_i \quad \forall i \in I.$$

By condition (37) and Proposition 7.4.13 in [8] the set of maximal elements $\mathbf{x}_I \in \mathbf{X}_I$ is nonempty compact. Each such element is a Pareto optimum. Invoking Theorem 4.1 suffices to conclude. □

5 Market mechanisms

Economic theory shows some predilection with equilibria.[6] Constructive approaches are, however, fairly few. Some appear hardly convincing; others are neither factual nor real [13].

This motivates some glimpses on disequilibrium dynamics in markets. Construe transactions as driven or executed there by the participants themselves. Allowing various institutions and mechanisms, each just sketched, it is presumed here that trade always is voluntary, in full, liberal compliance with agents' incentives.

Accordingly, by (13), if agent $i \in I$ holds $x_i \in X$ and "supplies" $\chi_i \in \mathcal{X}$, then he accepts "revenue" r_i iff

$$r_i \geq C_i(\chi_i \mid x_i) \geq c_i(\chi_i \mid x_i). \tag{38}$$

As before, demand is negative supply, and expense is negative revenue. Now *let each preference order* $\succsim_i$ *be transitive and presume that* $\partial c_I(0 \mid \mathbf{x}_I)$ *is nonempty at every* $\mathbf{x}_I \in \mathbf{X}_I$ (24).

Suppose $(x_i) = \mathbf{x}_I \in \mathbf{X}_I$ is the profile actually held. If merely the members of $\mathcal{I} \subseteq I$, $\#\mathcal{I} \geq 2$, conclude deals among themselves, then the overall updated holding $(x_i^{+1}) = \mathbf{x}_I^{+1} \in \mathbf{X}_I$ satisfies

$$x_i^{+1} = (r_i, -\chi_i) + x_i \succsim_i x_i \quad \forall i \in \mathcal{I} \quad \text{and} \quad x_i^{+1} = x_i \quad \forall i \notin \mathcal{I}. \tag{39}$$

A correspondence $M : \mathbf{X}_I \rightrightarrows \mathbf{X}_I$, so defined, is called a *market mechanism.*[7] It is purely redistributive in that $\sum_{i \in \mathcal{I}}(r_i, \chi_i) = (0, 0)$. Moreover, (38) holds for all $i \in \mathcal{I}$. In particular, the mechanism is said to be *price-based* iff $\mathcal{I} = I$ *and*, besides (39),

$$r_i = \chi^* \chi_i \quad \forall i \in I \text{ with } \chi^* \in \bigcap_{i \in I} \partial c_i(\chi_i \mid x_i) \subseteq \bigcap_{i \in I} \partial C_i(\chi_i \mid x_i). \tag{40}$$

Proposition 4.3 entails forthwith:

Proposition 5.1 (On stationarity) *$(\chi^*, \mathbf{x}_I)$ is a valuation equilibrium iff each price-based mechanism, applied in state $\mathbf{x}_I \in \mathbf{X}_I$ with price $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$, has an invariant outcome $\mathbf{x}_I^{+1} = \mathbf{x}_I$.*

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[6] The same observation applies to game theory [16].

[7] Its values need not be closed or convex.

A market session may compose—and comprise—several *mechanisms*, maybe different, operating in finite, possibly random sequence.[8] The components considered below are *double auctions*, *order markets*, and *direct deals*.

Agent $i \in I$ enters the session with endowment $x_i \in dom \succsim_i$ and exits with $x_i^{+1} \in dom \succsim_i$. During the session, he sees a finite improving chain of own holdings:

$$x_i \precsim_i x_i^{(1)} \precsim_i \cdots \precsim_i x^{(k)} \precsim_i x_i^{+} \precsim_i x_i^{+1}, \tag{41}$$

x_i^{+} denoting his next to last endowment. The session is regulated by clock and closure:

Assumption 5.1 (On ultimate updates) Each session closes with a price-based mechanism (40), generating a last update $\mathbf{x}_I^{+} \longrightarrow \mathbf{x}_I^{+1}$ in (41).

Double auctions are market institutions for which each session clears directly and immediately [5], that is, in (41), $x_i = x_i^{+}$ for every agent $i \in I$. Moreover, allocation and pricing is managed by a system operator or *auctioneer* who:

* counteracts strategic behavior by requiring that all *market curves* $C_i(\cdot \mid x_i)$, $i \in I$ (13), (38) be submitted anonymously, "silently", "simultaneously", and to him only,[9]
* ensures Pareto efficiency and shaves off any constant markup $\gamma_i \geq 0$ on $c_i(\cdot \mid x_i)$,
* precludes arbitrage or second-hand deals by linear pricing, and
* induces participation in so far as no party should see any loss of value.

To these many ends, for fixed $\mathbf{x}_I = \mathbf{x}_I^{+} \in \mathbf{X}_I$, the auctioneer solves the *inf-convolution*

$$C_I(0 \mid \mathbf{x}_I) := \inf\left\{\sum_{i \in I} C_i(\chi_i \mid x_i) : \sum_{i \in I} \chi_i = 0\right\} \tag{42}$$

of market curves (13) and (38). By assumption, (42) allows an optimal solution (χ_i). Also by assumption, the mechanism being price-based, a subgradient

$$\chi^* \in \bigcap_{i \in I} \partial c_i(\chi_i \mid x_i) \subseteq \bigcap_{i \in I} \partial C_i(\chi_i \mid x_i) = \partial C_I(0 \mid \mathbf{x}_I)$$

is used by for common pricing; see Proposition 4.0. Thus participant i "supplies" χ_i for which he receives "revenue" $r_i = \chi^* \chi_i$. The latter covers his "cost" $c_i(\chi \mid x_i)$ because by (4) and $c_i^*(\chi^* \mid x_i) \geq 0$,

$$r_i = \chi^* \chi_i = c_i^*\left(\chi^* \mid x_i\right) + c_i(\chi_i \mid x_i) \geq c_i(\chi_i \mid x_i),$$

and hence $x_i^{+1} \succsim_i x_i$. Moreover, $c_i^*(\chi^* \mid x_i) > 0 \Longrightarrow x_i^{+1} \succ_i x_i$; otherwise, agent i remains indifferent.

Order markets offer platforms for anonymous picking or posting of *limit orders.* The functioning of such a market is most easily grasped or implemented if agents deal in just one real good at a time.

[8] Trade is a punctuated process. It proceeds in discrete endogenous time steps and unfolds during repeated but nonoverlapping intervals. Considered here is just *one* such interval, called a *session*.

[9] By assumption, if $C_i(\chi_i \mid x_i) < +\infty$, then agent i can honour his commitment. In particular, he has no concerns with liquidity.

Anyway, if agent $i \in I$ submits a *limit sell/ask order* $A_i(\cdot \mid x_i) := C_i(\cdot \mid x_i) \geq c_i(\cdot \mid x_i)$ (10), (13), then he commits (with no name) to supply any $\chi_i \in domA_i(\cdot \mid x_i)$ for payment $A_i(\chi_i \mid x_i)$ or better.

Alternatively, if having a (maybe hidden, not posted) *market buy/bid order*

$$B_i(\cdot \mid x_i) := -C_i(-\cdot \mid x_i) \leq -c_i(-\cdot \mid x_i), \tag{43}$$

he may pick off—or anonymously be matched with—some worthwhile bundle $\chi_j \in domA_j(\cdot \mid x_j)$, already committed to by another agent j. When

$$C_{ij}(0 \mid \mathbf{x}_{ij}) := \inf\left\{C_i(\chi_i \mid x_i) + C_j(\chi_j \mid x_j) : \chi_i + \chi_j = 0\right\} < 0,$$

there is a nonmarginal, bilateral "*bid-ask spread*"

$$\sup_{\chi \in \mathcal{X}} \left[B_i(-\chi \mid x_i) - A_j(\chi \mid x_j)\right] = -C_{ij}(0 \mid \mathbf{x}_{ij}) > 0.$$

Quite likely, that spread might be realized for pair a $\{i,j\} \subseteq I$, having $|\, C_{ij}(0|\mathbf{x}_{ij})|$ maximal, to the effect that spreads usually vanish swiftly. Anyway, agent i receives some real bundle $\chi \in domB_i(-\cdot \mid x_i) \cap domA_j(\cdot \mid x_j)$ from j for suitable payment r, between $A_j(\chi \mid x_j)$ and $B_i(-\chi \mid x_i)$, going the other way.

Immediately thereafter i and j pull back their orders, update respective endowments to

$$x_i \longleftarrow (-r, \chi) + x_i \succsim_i x_i \quad \& \quad x_j \longleftarrow (r, -\chi) + x_j \succsim_j x_j,$$

and submit new orders, if any. This transaction makes for *one* improvement of agent i in the chain (41), and quite similarly for j.

Any agent can change, convolute, pick off, place or withdraw orders, as many and often as he pleases, yet incurring no fee.

Ultimately, just prior to session closure, a price-based clearing (40) transforms the profile $\mathbf{x}^+$ to $\mathbf{x}^{+1}$. At that moment the attending order executions reflect perfect foresight, on the part of each agent, during the very last minute of opening hours.

Direct deals [2, 3] between members of $\mathcal{I} \subseteq I$, $\#\mathcal{I} \geq 2$, at some market venue, seem tempting whenever marginal valuations differ, meaning that $\bigcap_{i \in \mathcal{I}} \partial C_i(0 \mid x_i) = \varnothing$. The bilateral instance $\#\mathcal{I} = 2$ is classic, frequent, simple, and most time-honored.

Let $\mathcal{X}$ be Euclidean or Hilbert here and fix $(x_i) = \mathbf{x}_{\mathcal{I}} \in \mathbf{X}_{\mathcal{I}}$ (24). Then, with no loss of generality, a bounded *reallocation* $(\chi_i) \in \mathcal{X}^{\mathcal{I}}$ of real goods across $\mathcal{I}$ yields changes $\chi_i^{+1} = \sigma \chi_i$ for some step-size σ with $\sum_{i \in \mathcal{I}} \chi_i = 0$ and $\sum_{i \in \mathcal{I}} \|\chi_i\|^2 \leq 1$. Reflecting on this, use supdifferentials $\hat{\partial}$ (5) of shaded bid functions $B_i(\cdot \mid x_i)$ (11), (43) to define a *marginal bid-ask spread* $\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) :=$

$$\max_{(\chi_i)} \inf_{(\chi_i^*)} \left\{ \sum_{i \in \mathcal{I}} \chi_i^* \chi_i : \chi_i^* \in \hat{\partial} B_i(0 \mid x_i), \sum_{i \in \mathcal{I}} \chi_i = 0, \sum_{i \in \mathcal{I}} \|\chi_i\|^2 \leq 1 \right\}. \tag{44}$$

Clearly, $\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) \geq 0$, and a larger ensemble $\mathcal{I}$ gives no less spread. By Theorem 1.86 in [17] we may replace max inf (44) with inf max. This done, the inner operation $\max_{(\chi_i)}$ is easily performed. Indeed, using shorthand $\bar{\chi}^* := \sum_{i\in\mathcal{I}} \chi_i^*/\#\mathcal{I}$ for the average supgradient (5), we obtain the alternative formula

$$\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) = \min\left\{\left[\sum_{i\in\mathcal{I}}\|\chi_i^* - \bar{\chi}^*\|^2\right]^{-1/2}\sum_{i\in\mathcal{I}}\chi_i^*(\chi_i^* - \bar{\chi}^*) : \chi_i^* \in \hat{\partial}B_i(0 \mid x_i)\right\}.$$

When $\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) > 0$, the members of $\mathcal{I}$ have good reasons to transact among themselves. In fact, a positive spread signals that agents' marginal valuations diverge:

Proposition 5.2 (On marginal bid-ask spreads) *Suppose each supdifferential $\hat{\partial}B_i(0 \mid x_i)$, $i \in \mathcal{I}$, is compact and nonempty* (5). *Then $\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) = 0 \Longleftrightarrow \bigcap_{i\in\mathcal{I}} \hat{\partial}B_i(0 \mid x_i) = \emptyset$.*

Proof If there is some $\chi^* \in \bigcap_{i\in\mathcal{I}} \hat{\partial}B_i(0 \mid x_i)$, then, clearly, $\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) = 0$. Otherwise, if $\bigcap_{i\in\mathcal{I}} \hat{\partial}B_i(0 \mid x_i) = \emptyset$, then use a nonzero vector $(\chi_i) \in \mathcal{X}^{\mathcal{I}}$ to separate the compact convex product set $\prod_{i\in\mathcal{I}} \hat{\partial}B_i(0 \mid x_i) \subset \mathcal{X}^{*\mathcal{I}}$ strictly from the "diagonal" $\{(\chi_i^*) : \text{all } \chi_i^* \text{ are equal}\}$. Then $\sum_{i\in\mathcal{I}} \chi_i = 0$, and we may take $\sum_{i\in\mathcal{I}} \|\chi_i\|^2 = 1$. Finally, by choosing appropriate sign of (χ_i) it follows that $\mathfrak{B}_{\mathcal{I}}(\mathbf{x}_{\mathcal{I}}) > 0$. □

(40) reflects a simple feature of markets for direct deals, namely: At clearing or closing time, presumably known or predicted, *each real commodity trades at **one** price which equals the marginal valuation of every party*. Otherwise, astute traders would make profitable, last-second transactions.

Relation (40) also reflects another, more subtle feature: *the last deals* (*or limit orders*) *are executed to the full.* This feature bears on agents' short-term *foresight*. Through the session, that capacity of his might be imperfect, albeit *not* at the ultimate moment.

As long as $c_I^*(\chi^* \mid \mathbf{x}_I) > 0$, any payment profile

$$i \in I \quad \mapsto \quad r_i \in \left[0, c_i^*(\chi^* \mid x_i)\right],$$

atop costs, generates strict improvement for agent i iff $r_i > 0$. For such agents, another market session appears attractive. Indeed, to conclude, provided that preferences are transitive, *the aggregate value added decreases monotonically*. Thus prospects seem good for convergence of iterated sessions. This is argued as a conclusion:

Theorem 5.1 (Monotone convergence in value added) *Suppose every preference $\succsim_i$ is transitive and that $\partial c_I(0 \mid \mathbf{x}_I)$ is nonempty for each $\mathbf{x}_I \in \mathbf{X}_I$. Let $\mathbf{x}_I^- \in \mathbf{X}_I$ denote the profile just prior to the price-based mechanism that generated the entering profile $\mathbf{x}_I \in \mathbf{X}_I$ in* (41). *Then, with respective clearing prices χ^{*+} and χ^{*-}, value added decreases*:

$$c_I^*(\chi^{*+} \mid \mathbf{x}_I^+) \le c_I^*(\chi^{*-} \mid \mathbf{x}_I^-). \tag{45}$$

Proof The proof invokes an auxiliary observation: *every price-based mechanism, using some shadow price $\chi^* \in \partial c_I(0 \mid \mathbf{x}_I)$, adds the minimal value*:

$$c_I^*(\chi^* \mid \mathbf{x}_I) = \inf_{\hat{\chi}^*} c_I^*(\hat{\chi}^* \mid \mathbf{x}_I) \tag{46}$$

as is derived from $0 \in \partial c_I^*(\chi^* \mid \mathbf{x}_I)$, and hence $c_I^*(\cdot \mid \mathbf{x}_I)$ is minimal at χ^*.

For the main argument, from (41) and transitivity it follows that $x_i^+ \succsim_i x_i^-$ for each $i \in I$. So expenditure $\mathcal{E}_i(\cdot \mid x_i^+) \geq \mathcal{E}_i(\cdot \mid x_i^-)$ for all $i \in I$; see (18). Consequently, writing $x_I := \sum_{i\in I} x_i = x_I^0$ and $x^{*\pm} := (r^*, \chi^*)^\pm := (1, \chi^{*\pm})$ (1), inequality (45) follows from (18) and (46) by

$$c_I^*(\chi^{*+} \mid \mathbf{x}_I^+) = \inf_{\hat{x}^*}\Big\{\hat{x}^* x_I - \sum_{i\in I} \mathcal{E}_i(\hat{x}^* \mid x_i^+)\Big\}$$
$$\leq \inf_{\hat{x}^*}\Big\{\hat{x}^* x_I - \sum_{i\in I} \mathcal{E}_i(\hat{x}^* \mid x_i^-)\Big\} = c_I^*(\chi^{*-} \mid \mathbf{x}_I^-). \qquad \square$$

Remark (On "externalities" and more) If the choice profile $\mathbf{x}_{-i} := (x_j)_{j\in I\setminus i}$ of other participants affects the preferences of agent i, then $\succsim_i \subset X^I \times X$, and

$$c_i(\chi_i \mid \mathbf{x}_I) := \inf\big\{r_i \in R : \big([r_i, -\chi_i] + x_i, \mathbf{x}_{-i}\big) \succsim_i \mathbf{x}_I\big\}.$$

Extensions of this sort are important in reality and theory. Regarding many economic "goods", issues often come up as to its divisibility, exclusivity, external effects, transferability, pricing, or property rights. This paper addressed no such issues.

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