

Modal Logics and Group Polarization*

Mina Young Pedersen¹, Sonja Smets^{2,1}, and Thomas Ågotnes^{3,1}

- ¹ Dept. of Information Science and Media Studies, University of Bergen,
5007 Bergen, Norway
{mina.pedersen,thomas.agotnes}@uib.no
- ² ILLC, University of Amsterdam, 1098 XG Amsterdam, The Netherlands
s.j.l.smets@uva.nl
- ³ ILI, Southwest University, Chongqing, China

Abstract. This paper proposes different ways of modally defining properties related to the concept of balance in signed social networks where relations can be either positive or negative. The motivation is to be able to formally reason about the social phenomenon of group polarization based on balance theory. The starting point is a recently developed basic modal logic that axiomatizes the class of social networks that are balanced up to a certain degree. This property is not modally definable but can be captured using a deduction rule. In this work, we examine different possibilities for extending this basic language to define frame properties such as balance and related properties such as non-overlapping positive and negative relations and collective connectedness as axioms. Furthermore, we define the property of *full* balance rather than balanced-up-to-a-degree. We look into the complexity of the model checking problem and show a non-compactness result of the extended language. Along the way, we provide axioms for weak balance. We also look at a full hybrid extension and reason about network changes with dynamic modalities. Then, to explore measures of how far a network is from polarization, we consider variations of measures in relation to balance.

Keywords: Polarization · Balance · Social network logic · Modal logic · Network theory

1 Introduction

The way in which we receive and exchange information changes rapidly with the advances of new technology in our current world. Simultaneously we are facing local and global issues that are driving our opinions to the extremes of the political landscape. A social phenomenon related to these trends considered to be increasingly dangerous is *group polarization*.

Group polarization, or polarization for short, is not a new concept but has gained increasing relevance in the age of social media [28]. The phenomenon has been

* This is an extended and revised version of [36], first published in the proceedings of *Third International Conference on Logic and Argumentation (CLAR 2020)*.

extensively researched by, among others, Cass Sunstein [39,40]. Polarization describes the tendency for people to develop more extreme views after deliberation within a group. Although issues up for debate often are complex and dependent on a number of factors, an effect of polarization is that fine lines are blurred and that answers to complicated questions are driven into opposing parties of either “for” or “against”. This applies to juries in court rooms and participants in political discussions, but can also find its way into mundane everyday social settings.

Reasons behind polarization include a combination of peer pressure and the way information exchange is carried out within group settings [40]. One important aspect of this process is that individuals with a weak inclination towards one opinion are likely to be confronted with louder voices expressing a radicalized version of the same opinion. As a result of exposure to new arguments and desire to be part of a community, uncertain agents might leave their insecurities behind and adopt a stronger position.

One purely network-theoretical factor related to polarization phenomena is *balance theory*. Balance theory goes back to the foundation of the field of social network analysis [25], and asserts that certain configurations of connections between friends and enemies in a signed network with positive and negative links, such as a triangle of two positive and one negative relation, are unstable and therefore comparatively rarely observed. Key results in balance theory, originating in the works of Frank Harary [25], link this local property of unstable configurations to the global property of a formation of groups of friends who are enemies with everyone else.

Group polarization is captured precisely in this global balance property of networks divided into opposing parties. In this work, we study a polarized network as a balanced graph of clusters of agents positively related within, but negatively related to the other clusters. On this interpretation of group polarization as balance, we can consider the positive and negative relations between agents as agreement or disagreement on a given issue, which is another reading of the friend and enemy-relation in the original theory.

We are interested in formal reasoning about polarization and therefore about balance. First steps in this direction have been made: positive and negative relations logic (**PNL**) [42,43] is a basic modal logic that uses Kripke frames with two distinct relations to model networks where agents can be related positively or negatively, but not both, and is used to axiomatize the class of networks that are balanced (to a certain degree). While **PNL** can be seen as a logical foundation for reasoning about balance, it has two particular downsides. First, balance properties, and related properties such as non-overlapping positive and negative relations and collective connectedness, are not modally definable in the logical language, but can only be captured using a deduction rule. Second, the logic only axiomatizes the class of “almost” balanced networks, networks that are balanced up to a degree set by a fixed parameter, and not the class of all

fully balanced networks. In this paper we study possible extensions of the basic language of **PNL** in order to increase the expressive power in general and in particular to be able to define the mentioned properties, including balance, as axioms. Towards this end, we systematically look at several expressive modalities known from the literature, both static modalities such as the universal and the difference modalities, the intersection modality, and nominals known from hybrid logic, and dynamic global bridge modalities known from sabotage modal logic [7]. We focus on the possibility of modal definability of the mentioned properties, as first steps towards possible axiomatizations. In particular, we provide a logical-dynamic characterization of balance. Along the way we characterize *weak* balance, which has not been logically characterized before. We also show that by extending our framework to a full hybrid logic, we get an axiomatic system that is sound and complete with respect to balanced frames. In this hybrid framework, we add local dynamic modalities to reason about network changes towards balance, which we in turn argue give a dynamic picture of polarization. Additionally, we introduce and evaluate a set of functions on the class of models determining how balanced a network is. This is to develop a tool for measuring how far the network is from being polarized.

The structure of the remainder of the paper is as follows. The following Section 2 consists of preliminaries: we present the social concept of balance and polarization as well as the basic **PNL**. In the next Section 3 we propose additions to **PNL** in order to define a balance axiom, and show that the logic with these extensions is not compact, and that its model checking problem is PSPACE-hard. Then, in Section 4 we look at the in **PNL** modally undefinable frame properties collective connectedness and non-overlapping positive/negative relations. We discuss and compare the inclusion of various known modalities with respect to definability. In Section 5 we extend **PNL** with nominals and hybrid operators and show a full axiomatization. We prove soundness and completeness of this axiomatic system with respect to balanced frames. Section 6 introduces local dynamic modalities to reason about change on hybrid frames. Section 7 is devoted to measures of balance to analyze how close a network is to polarization. We present three metrics and discuss strengths and weaknesses before using an example for comparison. Then we consider a measure of reconciliation to judge how far a network is from a situation where no agents are enemies. In the final Section 8 we conclude the paper and assess some related work on logics for reasoning about social networks. We also discuss future directions of the work.

2 Preliminaries: Balance and PNL

We begin by presenting **PNL** which we extend and use in later sections. We also look at structural balance theory and its relation to polarization while studying how this particular logic captures essential properties of the theory. The section concludes with a discussion of motivations to expand on this framework.

As a well-known concept from the field of social network analysis, balance is defined on *signed* social networks. A signed network is an undirected graph consisting of agents and relations between them, represented as strictly either positive or negative, but not both. In relation to polarization, it is more fitting to think of these relations as agreement or disagreement on a particular topic than friends or enemies in a wider sense, but we keep the terms “friend” and “enemy” for simplification.

Positive and negative relations logic (**PNL**) [42,43] models signed networks as Kripke frames⁴ with a set of possible worlds representing agents and two distinct positive “+” and negative “-” binary relations representing friendships and enmities, respectively. See Fig. 1 for an example.

2.1 Syntax and Semantics of PNL

Definition 1 (Syntax of PNL [42]). *Let At be a countable set of propositional letters. We define the well-formed formulas of the language \mathcal{L}_{PNL} to be generated by the following grammar:*

$$\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \boxplus\phi \mid \boxminus\phi$$

where $p \in \text{At}$. We define propositional connectives like \vee, \rightarrow and the formulas \top, \perp as usual. Further, we define the duals as standard $\boxplus := \neg \boxminus \neg$ and $\boxminus := \neg \boxplus \neg$.

Edges in a signed graph are represented using two relations R^+ and R^- . It is required that both relations are symmetric, as signed graphs are undirected. R^+ is chosen to be reflexive, demanding agents to have a positive relation to themselves. Moreover, relations are *non-overlapping*: no two agents can be related by both a positive and negative relation. Some networks, but not all, are *collectively connected*: all agents in the graph or subgraph that constitutes the network under consideration, are related, either positively or negatively. Formal definitions of non-overlapping and collective connectedness follow.

Definition 2 (Non-overlapping and Collective Connectedness). *Let A be a non-empty set of agents and R^+ and R^- be two binary relations on A . We define the following properties of R^+ and R^- :*

- R^+ and R^- are **non-overlapping** iff $\forall a, b \in A : (a, b) \notin R^+ \text{ or } (a, b) \notin R^-$.
- R^+ and R^- are **collectively connected** iff $\forall a, b \in A : aR^+b \text{ or } aR^-b$.

We can now define signed frames and models, and the semantics of **PNL**.

Definition 3 (Signed Frame and Model [42]). *Let A be a non-empty set of agents and R^+ and R^- be two symmetric and non-overlapping binary relations on A where R^+ is reflexive. Further, let $V : \text{At} \rightarrow \wp(A)$ be a valuation function.*

⁴ We assume some familiarity with Kripke semantics for modal logic; see, e.g., [12].

A **signed model** is a tuple $\mathbb{M} = \langle A, R^+, R^-, V \rangle$. We define a **pointed signed model** (\mathbb{M}, a) where \mathbb{M} is a signed model and $a \in A$ its distinguished point, at which evaluation takes place.

We call a signed model without valuation $\mathbb{F} = \langle A, R^+, R^- \rangle$ a **signed frame**.

Definition 4 (Semantics of PNL [42]). Let \mathbb{M} be a signed model and a an agent in A . We define the truth conditions for **PNL** as follows:

$$\begin{aligned} \mathbb{M}, a \Vdash p & \text{ iff } a \in V(p) \\ \mathbb{M}, a \Vdash \neg\phi & \text{ iff } \mathbb{M}, a \not\Vdash \phi \\ \mathbb{M}, a \Vdash \phi \wedge \psi & \text{ iff } \mathbb{M}, a \Vdash \phi \text{ and } \mathbb{M}, a \Vdash \psi \\ \mathbb{M}, a \Vdash \heartsuit\phi & \text{ iff } \exists b \in A \text{ such that } aR^+b \text{ and } \mathbb{M}, b \Vdash \phi \\ \mathbb{M}, a \Vdash \spadesuit\phi & \text{ iff } \exists b \in A \text{ such that } aR^-b \text{ and } \mathbb{M}, b \Vdash \phi \end{aligned}$$

For a signed frame \mathbb{F} and a formula $\phi \in \mathcal{L}_{\text{PNL}}$, we write $\mathbb{F} \Vdash \phi$ when ϕ is valid in \mathbb{F} : if ϕ is true at every agent in every signed model $\langle \mathbb{F}, V \rangle$ for any valuation V on \mathbb{F} .

Intuitively, we read $\heartsuit\phi$ to hold at an agent if and only if the current agent is positively related to an agent where ϕ is true. Similarly, we read $\spadesuit\phi$ to hold at an agent if and only if the current agent is related negatively to an agent where ϕ is true.

2.2 The Balance Theorem: Polarized Networks

Structural balance, referred to as balance for short, originates from theories in social psychology [27], and also carries empirical support (e.g. [32]). We first define balance on a collectively connected network. A collectively connected network with the balance property consists of triangles with either all positive edges, or two negative edges and one positive edge. These triangles correspond to the socio-psychological motivation that “the enemy of my enemy is my friend” and similarly “the friend of my enemy is my enemy” and “the friend of my friend is my friend”. The last tendency has also been characterized as triadic closure in social networks [24] and has been formalized in a logical framework in [35]. We formally define balance on collectively connected signed frames as *local balance*.

Definition 5 (Local Balance⁵ [43]). A signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ has the **local balance** property iff $\forall a, b, c \in A$:

$$- \text{ if } aR^+b \text{ and } bR^+c, \text{ or } aR^-b \text{ and } bR^-c, \text{ then } aR^+c, \text{ and}$$

⁵ As noted by one of the reviewers, this definition contains redundancies due to the already symmetric signed frames. This redundancy is also found in Definition 6 and suggests equivalent alternatives to formulas **4B** and **4W**.

– if aR^+b and bR^-c , or aR^-b and bR^+c , then aR^-c .

We note that a network can have the local balance property without being collectively connected: it can have single disconnected agents or consist of disconnected subgraphs each of which are collectively connected.

The Balance Theorem, proved by Frank Harary in 1953 [25] shows an equivalence on collectively connected networks between the local property of sets of three agents and a global property of the network in its entirety: that all agents can be divided into two groups where agents within groups are friends and agents across groups are enemies.

The general version of the Balance Theorem defined on signed networks as discussed in [21] states the following equivalence: a signed network can be divided into two opposing groups if and only if it is possible to “fill in the missing edges” to construct a collectively connected signed frame with the local balance property. See the signed frame \mathbb{F} in Fig. 1 for an example. For simplicity, we have omitted positive reflexive relations. Here we can divide agents into the two sets $X = \{a, c\}$ and $Y = \{b, d\}$ where within the sets, agents are friends, and, if related, enemies towards members of the other set. Note that we can “fill in” a negative relation between c and d such that the signed frame has the local balance property.

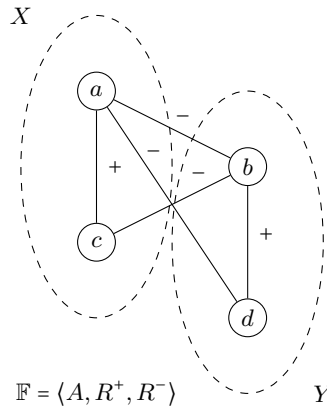


Fig. 1. Division of agents in A into sets X and Y .

The characterization of balance does not end here: a signed frame is balanced if and only if there are no *simple cycles* with an odd number of negative edges [15]. We refer to these cycles as *negative cycles*. A simple cycle, often just called a *cycle*, is defined in graph theory as a path of nodes and at least three edges, in which the first and last nodes are the same and visited exactly twice [21]. Otherwise all nodes are distinct. Examples of cycles in Fig. 1 are $aR^-bR^-cR^+a$

and $aR^-bR^+dR^-a$. We note in agreement with the Balance Theorem that the cycles are not negative. The Balance Theorem is summarized below.

Theorem 1 (The Balance Theorem). *Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame. The three following properties are equivalent:*

1. *There exists a collectively connected signed frame $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$ that has the local balance property;*
2. *There exists a set of agents $S \subseteq A$ such that $\forall a, b \in A$:*
 - *if aR^+b , then $a, b \in S$ or $a, b \in A \setminus S$, and*
 - *if aR^-b , then $a \in S$ and $b \in A \setminus S$, or $a \in A \setminus S$ and $b \in A$;*
3. *There are no negative cycles in \mathbb{F} .*

We say that a signed frame is **balanced**, or has the **balance property**, if it has any, and all, of these properties.

The connection between balance and polarization is captured in the global definition of balance, when the positive and negative relations are thought of as agreement and disagreement on a particular topic, rather than friends and enemies as in the original literature that introduced balance theory. A balanced signed frame can be divided into two opposing groups, just as in a polarized social setting. A balanced network is a polarized one. However, it is important to note that this interpretation of polarization is not a general definition; we are looking specifically at signed networks of positive and negative relations constructed on certain properties like symmetry and non-overlapping. Still, to speak of polarization one must assume some positive and negative attitudes. Signed networks turn out to provide a useful foundation for analyzing a simplification of personal and collective opinion, and of polarization in particular. Analogies between balance and polarization is not novel in this paper, and can be found in literature such as [13,18,41]. We will therefore use the terms balance and polarization interchangeably in what follows.

2.3 Weak Balance: More Than Two Extremes

Before we turn to examine the axiomatization of **PNL**, we briefly introduce the notion of *weak balance*, first proposed by James A. Davis in 1967 [18]. Weakly locally balanced frames are supersets of locally balanced frames that disallow only one type of triangle: those with two positive edges and one negative edge.

Definition 6 (Weak Local Balance). *A signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ has the **weak local balance property** iff $\forall a, b, c \in A$:*

- *if aR^+b and bR^+c , then aR^+c , and*
- *if aR^+b and bR^-c , or aR^-b and bR^+c , then aR^-c .*

Davis [18] proved a similar Balance Theorem for weak balance, although in this case the global property of weak balance characterizes the possibility of dividing agents into not just two, but any number of sets of “friends”. Weakly balanced signed frames are polarized with respect to a collection of groups, where relations within each group are positive and all relations between agents in different groups are negative. Analogous to (strong) balance, a weakly balanced signed frame also has a cycle property: it cannot contain a simple cycle with only *one* negative edge. We state the Weak Balance Theorem as follows.

Theorem 2 (Weak Balance Theorem). *Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame. The following three properties are equivalent:*

1. *There exists a collectively connected signed frame $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$ that has the weak local balance property;*
2. *There exists a partition S_1, \dots, S_n of A for $n \in \mathbb{N}$ such that $\forall a, b \in A$:*
 - *if aR^+b , then $a, b \in S_m$ for some m , and*
 - *if aR^-b , then $a \in S_s$ and $b \in S_t$ for some $s \neq t$;*
3. *There are no cycles with exactly one negative edge in \mathbb{F} .*

We say that a signed frame is **weakly balanced**, or has the **weak balance property**, if it has any, and all, of these properties.

Studies, such as [30], have found strong balance to be too restrictive as a common property of real-world social networks and propose weak balance as a more realistic alternative. In the literature on **PNL**, weak balance is only mentioned in passing and not included in the formalization. We keep both definitions as they serve different purposes. A network of football fans might converge to a weakly balanced graph structure where supporters of the same team agree and disagree with supporters from other teams in plural. In the context of particular political issues, like Brexit or anti-vaccination, the same network could converge to a strongly polarized network in camps of “yes” and “no”. Depending on the social context and research goal, both balance definitions are valuable in their own respect.

2.4 Axiomatization

The original literature on **PNL** [43] presents a complete axiomatization of balanced signed frames. In this axiomatization, balance and non-overlapping is characterized in a derivation rule, not an axiom. This is because balance, non-overlapping and collective connectedness in fact are modally undefinable in **PNL**. This means that there cannot be a formula ϕ of $\mathcal{L}_{\mathbf{PNL}}$ such that ϕ is valid on a signed frame if and only if the frame has any of these properties. Moreover, balance is here characterized up to a degree n , which will be explained further in this section. There is no axiomatization for balance regardless

of a given degree. In later sections we discuss possible extensions to the language of **PNL** to define these properties with axioms.

As shown in [43], local balance is definable by the following axiom **4B**.

$$((\diamond \diamond p \vee \diamond \diamond p) \rightarrow \diamond p) \wedge ((\diamond \diamond p \vee \diamond \diamond p) \rightarrow \diamond p) \quad (4B)$$

Lemma 1 ([43]). *For any signed frame \mathbb{F} , $\mathbb{F} \models 4B$ iff \mathbb{F} has the local balance property.*

Recall that local balance is the balance property relevant for collectively connected signed frames, where we assume that all agents are related to one another. Since signed frames are not restricted with this assumption, the **4B**-axiom is not included in the axiomatic system for **PNL**.

By modifying the **4B**-axiom to adapt to the local weak balance conditions, we get the **4W**-axiom with the corresponding lemma, novel in this paper.

$$(\diamond \diamond p \rightarrow \diamond p) \wedge ((\diamond \diamond p \vee \diamond \diamond p) \rightarrow \diamond p) \quad (4W)$$

Lemma 2. *For any signed frame \mathbb{F} , $\mathbb{F} \models 4W$ iff \mathbb{F} has the weak local balance property.*

Proof. (\Rightarrow) Proof by contraposition. Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame without the weak local balance property. Then, without loss of generality $\exists a, b, c \in A$ such that aR^+b , bR^+c and aR^-c . Now, let V be a valuation on \mathbb{F} such that $V(p) = \{c\}$. It follows that $\langle \mathbb{F}, V \rangle, a \models \diamond \diamond p$. However, by the non-overlapping property, we have that $(a, c) \notin R^+$. Thus $\langle \mathbb{F}, V \rangle, a \not\models \diamond p$. We have that $\langle \mathbb{F}, V \rangle, a \not\models \diamond \diamond p \rightarrow \diamond p$ and hence $\mathbb{F} \not\models 4W$.

(\Leftarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame with the weak local balance property and fix an arbitrary valuation V and $a \in A$. Assume that $\langle \mathbb{F}, V \rangle, a \models \diamond \diamond p$. Then $\exists b \in A$ such that aR^+b and $\langle \mathbb{F}, V \rangle, b \models \diamond p$. Thus it follows that $\exists c \in A$ such that bR^+c and $\langle \mathbb{F}, V \rangle, c \models p$. By the weak local balance property aR^+c and hence $\langle \mathbb{F}, V \rangle, a \models \diamond \diamond p \rightarrow \diamond p$. Now assume that $\langle \mathbb{F}, V \rangle, a \models \diamond \diamond p$. Then $\exists b, c \in A$ such that aR^+b and bR^-c where $\langle \mathbb{F}, V \rangle, c \models p$. The weak local balance property of \mathbb{F} demand aR^-c and therefore $\langle \mathbb{F}, V \rangle, a \models \diamond p$. By similar reasoning $\langle \mathbb{F}, V \rangle, a \models \diamond p$ if we assume $\langle \mathbb{F}, V \rangle, a \models \diamond \diamond p$. Hence it follows that $\langle \mathbb{F}, V \rangle, a \models (\diamond \diamond p \rightarrow \diamond p) \wedge ((\diamond \diamond p \vee \diamond \diamond p) \rightarrow \diamond p)$ and as we fixed an arbitrary V and $a \in A$ we conclude that $\mathbb{F} \models 4W$. \square

In [43], the axiomatic system called **pnl_n** over the language $\mathcal{L}_{\mathbf{PNL}}$ is introduced for each $n \in \mathbb{N}^+$.⁶ The number represents balance up to the *degree* n : that there are no negative cycles of length less than or equal to n . [43] proves that **pnl_n**

⁶ We denote $\mathbb{N} \setminus \{0\}$ as \mathbb{N}^+ .

is sound and weakly complete with respect to the class of n -balanced models. This gives us an axiomatization of n -balance, not of balance in the general sense. Included in the axiomatization as the only component dependent on n is an inference rule called \mathbf{Nb}_n . As this rule requires an extensive presentation of concepts with details outside the scope of this paper, we refer to the original literature in [43] or to the presentation of \mathbf{PNL} in [34]. As a final note, we mention that both balance and weak balance are invariant under bisimulation⁷.

We conclude this preliminary section with a short recap of the essential properties of \mathbf{PNL} and the motivations leading us to expand on this framework. \mathbf{PNL} is a modal logic developed to analyze the concept of balance in social networks. A full axiomatic system is given, but there is no formula in $\mathcal{L}_{\mathbf{PNL}}$ that characterizes the frame property of being balanced for all signed frames; balance is captured by a rule, not an axiom. Additionally, this rule only characterizes n -balance, not balance. Furthermore, the non-overlapping property, i.e., that no two agents can be both positively and negatively related, is similarly not modally definable in \mathbf{PNL} and also captured by the \mathbf{Nb}_n -rule. Local balance is the balance property on signed frames that are collectively connected: where all agents have a relation to each other. Signed frames with single disconnected points or a set of collectively connected frames disconnected to one another can also have the local balance property. Local balance is characterized with the $\mathbf{4B}$ -axiom, whereas collective connectedness is modally undefinable. We extended the formal landscape by taking weak balance into account and defined the $\mathbf{4W}$ -axiom. Motivated by the undefinability of the properties mentioned above, in the next sections we explore additions to the language of \mathbf{PNL} to get definability.

3 Speaking of Balance

In this section we introduce the universal operator and dynamic modalities for global link addition to define a dynamic characterization of balance. We also show that this extension of \mathbf{PNL} has a PSPACE-hard model checking problem, inherited from sabotage modal logic, and that it is non-compact.

3.1 A Balance Axiom

Recall that the only formula we have in \mathbf{PNL} to define balance on a signed frame is axiom $\mathbf{4B}$ defining the local balance property. To begin to resolve the issue of defining balance, we introduce the standard global modality $[A]$ and global link-adding modalities $\langle \mathbb{A}+ \rangle$ and $\langle \mathbb{A}- \rangle$. The global link-adding modalities take inspiration from sabotage modal logic [7] and bridge operators found in literature such as [5]. The semantics of these modalities is presented below.

⁷ This follows directly from the standard definition of bisimulation. We leave out the technical details as they are not important in the following.

Definition 7 (Semantics of Universal Operator and Global Addition Modalities). Let $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ be a signed model and $a \in A$. We define truth conditions for the universal operator and global addition modalities as follows:

$$\begin{aligned} \mathbb{M}, a \Vdash [A]\phi &\text{ iff } \forall b \in A : \mathbb{M}, b \Vdash \phi \\ \mathbb{M}, a \Vdash \langle \mathbb{M}+ \rangle \phi &\text{ iff } \exists b, c \in A \text{ such that } (b, c) \notin R^- \text{ and} \\ &\quad \langle A, R^+ \cup \{(b, c), (c, b)\}, R^-, V \rangle, a \Vdash \phi \\ \mathbb{M}, a \Vdash \langle \mathbb{M}- \rangle \phi &\text{ iff } \exists b, c \in A \text{ such that } (b, c) \notin R^+ \text{ and} \\ &\quad \langle A, R^+, R^- \cup \{(b, c), (c, b)\}, V \rangle, a \Vdash \phi \end{aligned}$$

Intuitively, the formula $[A]\phi$ states that ϕ is true at all agents in the network. $\langle \mathbb{M}+ \rangle \phi$ is forced at an agent if and only if ϕ is true at the current agent after adding a positive link somewhere in the network. Similarly in the negative sense for $\langle \mathbb{M}- \rangle \phi$.

We also include *choice* and *iteration* modalities inspired by known dynamic logics such as Propositional Dynamic Logic (PDL) [12]. We accommodate them to the global link-adding modalities of our language and define them accordingly.

Definition 8 (Semantics of Choice and Iteration Modalities). Let $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ be a signed model and $a \in A$. We define truth conditions for the global addition choice and iteration modalities as follows:

$$\begin{aligned} \mathbb{M}, a \Vdash \langle \mathbb{M} + \cup \mathbb{M} - \rangle \phi &\text{ iff } \langle \mathbb{M}+ \rangle \phi \text{ or } \langle \mathbb{M}- \rangle \phi \\ \mathbb{M}, a \Vdash \langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle \phi &\text{ iff } \exists n \geq 0 \text{ such that} \\ \mathbb{M}, a \Vdash \underbrace{\langle \mathbb{M} + \cup \mathbb{M} - \rangle \dots \langle \mathbb{M} + \cup \mathbb{M} - \rangle}_n \phi & \end{aligned}$$

$\langle \mathbb{M} + \cup \mathbb{M} - \rangle \phi$ is true at an agent if and only if ϕ is true at the current agent after adding a positive or negative link somewhere in the network. We read the iterated modality as $\langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle \phi$ true at an agent if and only if ϕ holds at the current agent after adding a finite number of positive or negative edges to the signed frame. With the newly defined modalities we present the axiom \mathbf{B}_G and the subsequent lemma.

$$\langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle [A] \mathbf{4B} \quad (\mathbf{B}_G)$$

Lemma 3. For any finite signed frame \mathbb{F} , $\mathbb{F} \Vdash \mathbf{B}_G$ iff \mathbb{F} has the balance property.

Proof. (\Rightarrow) Proof by contraposition. We show that for any finite signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$: if \mathbb{F} is unbalanced then $\langle \mathbb{F}, V \rangle, x \not\Vdash \mathbf{B}_G$ for any V such that $V(p) = \{c\}$ where c is any member of the shortest odd negative cycle in \mathbb{F} and

any $x \in A$. We prove this by induction on the length of the shortest odd negative cycle in \mathbb{F} .

Base case: the shortest negative cycle in \mathbb{F} is a triangle. This triangle is either of one negative and two positive edges, or of all negative edges. Assume the case where the cycle is $aR^+bR^+cR^-a$ for $a \neq b \neq c \in A$ and let $V(p) = \{c\}$. Here, $\langle \mathbb{F}, V \rangle, a \Vdash \diamond \diamond p \wedge \neg \diamond p$. Thus, $\langle \mathbb{F}, V \rangle, a \not\Vdash \mathbf{4B}$. Therefore, for any $x \in A$: $\langle \mathbb{F}, V \rangle, x \not\Vdash [A]\mathbf{4B}$. No extension of additional relations to \mathbb{F} could force $\mathbf{4B}$ at a . This would require aR^+c , but as already aR^-c and the frame satisfies non-overlapping, this is not possible. Hence $\langle \mathbb{F}, V \rangle, x \not\Vdash \langle (\mathbb{M} + \cup \mathbb{M} -) \rangle [A]\mathbf{4B}$. The other cases are similar.

Inductive case: assume that the property holds for length less than or equal to m . Let the shortest odd negative cycle in \mathbb{F} be of length $m + 1$, and let $V(p) = \{c\}$ where c is a member of that cycle. Let $b, a, b \neq c, a \neq c, b \neq a$ be members of the cycle such that aR^+b or aR^-b and bR^+c or bR^-c . Assume the former cases, that aR^+b and bR^+c . Now assume, towards a contradiction, that $\langle \mathbb{F}, V \rangle, x \Vdash \mathbf{B}_G$. That means that there is a \mathbb{F}' such that $\mathbb{F} \subseteq \mathbb{F}'$ and $\langle \mathbb{F}', V \rangle, x \Vdash [A]\mathbf{4B}$. Since $\langle \mathbb{F}', V \rangle, a \Vdash \mathbf{4B}$ we must have that aR^+c in \mathbb{F}' . That means that \mathbb{F}' has an odd cycle with length m or less, that contains c . By the induction hypothesis, $\langle \mathbb{F}', V \rangle, x \not\Vdash \mathbf{B}_G$. But that contradicts the fact that $\langle \mathbb{F}', V \rangle, x \Vdash [A]\mathbf{4B}$. The other cases are similar.

(\Leftarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a finite signed frame with the balance property. Then by the Balance Theorem (Theorem 1) there exists a collectively connected frame $\mathbb{F}' = \langle A', R^{+'}, R^{-'} \rangle$ such that $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$ that has the local balance property. It follows from Lemma 1 that $\mathbb{F}' \Vdash \mathbf{4B}$. Fix an arbitrary valuation V and an arbitrary $a \in A$. It follows that $\langle \mathbb{F}', V \rangle, a \Vdash [A]\mathbf{4B}$. Since $A = A'$, $R^+ \subseteq R^{+'}$ and $R^- \subseteq R^{-'}$, it follows directly that $\langle \mathbb{F}, V \rangle, a \Vdash \langle (\mathbb{M} + \cup \mathbb{M} -) \rangle [A]\mathbf{4B}$. As we chose an arbitrary V and $a \in A$, we conclude that $\mathbb{F} \Vdash \mathbf{B}_G$. \square

The \mathbf{B}_G formula holds at any agent in the network if and only if axiom $\mathbf{4B}$ will be forced at all agents after adding a finite number of positive and negative edges anywhere in the signed frame. This is essentially characterizing the Balance Theorem: the formula holds at an agent in a signed frame \mathbb{F} if and only if there exists a superframe of \mathbb{F} where the local balance property holds.

It follows directly that we have the analogous \mathbf{B}_W -axiom relative to weak balance with a lemma proved similarly as in the case of Lemma 3.

$$\langle (\mathbb{M} + \cup \mathbb{M} -) \rangle [A]\mathbf{4W} \quad (\mathbf{B}_W)$$

Lemma 4. *For any finite signed frame \mathbb{F} , $\mathbb{F} \Vdash \mathbf{B}_W$ iff \mathbb{F} has the weak balance property.*

3.2 Model Checking

As mentioned the global addition modalities introduced above are very similar to the (global) *bridge* modality \diamond_{gbr} [5]. Intuitively, $\diamond_{gbr}\phi$ is true if and only if

ϕ is true after adding a new link somewhere (“globally”) in the model. There are some subtle differences though:

- the global bridge operator always adds a *new* link, while our operators do not;
- our operators check that a corresponding link does not exist for the “opposite” (positive or negative) relation;
- our modalities add a link in both directions to preserve symmetry.

The first of these differences, in particular, precludes a direct translation of the model checking problem for the basic bridge modal logic to our extended language. We do, however, get the following.

Theorem 3. *The model checking problem for the extended language of \mathcal{L}_{PNL} with global addition modalities is PSPACE-hard.*

Proof. The proof is a straightforward adaption of the hardness proof for bridge modal logic in [5], reducing the satisfiability problem for Quantified Boolean Formulas to the model checking problem for the extended logic with global addition modalities.

3.3 Non-compactness

Logics with iteration modalities such as common knowledge [23] or iteration in PDL [12], are often not *compact*. Non-compactness is a consequence of the interaction of the iteration modalities and the other modalities, and it is not completely obvious that the extended logic with the iteration modality introduced above is not compact. However, we show that it is not by presenting a counterexample. Let \mathcal{S}^* denote the set of finite strings of symbols from a set \mathcal{S} . Then $\{\langle \mathbb{M}+ \rangle, \langle \mathbb{M}- \rangle\}^*$ below denotes the set of all sequences of the $\langle \mathbb{M}+ \rangle$ and $\langle \mathbb{M}- \rangle$ modalities, e.g., $\langle \mathbb{M}+ \rangle \langle \mathbb{M}- \rangle \langle \mathbb{M}- \rangle$. The counterexample is then the following theory:

$$\Gamma = \{ \langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle \boxminus \diamond p, \neg X \boxminus \diamond p : X \in \{ \langle \mathbb{M}+ \rangle, \langle \mathbb{M}- \rangle \}^* \}$$

Γ is not satisfiable (there is no pointed signed model that satisfies all formulas in Γ): if $\mathbb{M}, a \Vdash \langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle \boxminus \diamond p$ then there is some n such that $\mathbb{M}, a \Vdash \langle \mathbb{M} + \cup \mathbb{M} - \rangle^n \boxminus \diamond p$, which by the semantics means that $\mathbb{M}, a \Vdash X \boxminus \diamond p$ for some $X \in \{ \langle \mathbb{M}+ \rangle, \langle \mathbb{M}- \rangle \}$ with $|X| = n$, which is a contradiction.

However, every finite subset Γ' of Γ is satisfiable. First, observe that if Γ' does not contain $\langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle \boxminus \diamond p$ it is trivially satisfiable (say, by two connected points where p is false in both). Second, assume that $\langle (\mathbb{M} + \cup \mathbb{M} -)^* \rangle \boxminus \diamond p \in \Gamma'$ and let $m \geq 0$ be the largest m such that there is a $X \in \{ \langle \mathbb{M}+ \rangle, \langle \mathbb{M}- \rangle \}^*$ with $|X| = m$ and $\neg X \boxminus \diamond p \in \Gamma'$. Let \mathbb{M} be the model in Fig. 2. We see that $\mathbb{M}, a \Vdash \neg Y \boxminus \diamond p$ for any $Y \in \{ \langle \mathbb{M}+ \rangle, \langle \mathbb{M}- \rangle \}^*$ with $|Y| \leq m$; all $m+1$ dotted edges need to be added

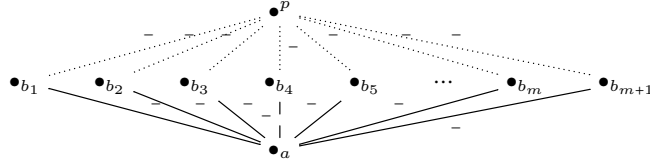


Fig. 2. Model \mathbb{M} , consisting of all points and the solid edges. The dotted edges are the potential edges that if added make $\boxplus \diamond p$ true in a . All edges are symmetric. Positive reflexive loops omitted.

for $\boxplus \diamond p$ to be true in a – which is exactly why $\mathbb{M}, a \Vdash \langle (\boxplus + \cup \boxplus -) * \rangle \boxplus \diamond p$ as well.

Consequently, the logic with the iteration modality is not strongly axiomatizable.

4 Collective Connectedness and Non-overlapping

We proceed with a discussion of additions to \mathcal{L}_{PNL} to define collective connectedness and non-overlapping. We show that collective connectedness can be defined by the inclusion of the standard universal modality introduced in the previous section. We discuss nominals, the intersection modality and the difference operator as possible candidates for extensions guaranteeing an axiom for non-overlapping, where the latter also gives us a collective connectedness axiom without the universal modality.

4.1 Universal Modality

Recall that the collective connectedness property is modally undefinable in **PNL**. By adding the global modality $[A]$ we get the axiom **C** for collective connectedness.

$$(\boxplus p \rightarrow [A]p) \vee (\boxminus p \rightarrow [A]p)^8 \quad (\mathbf{C})$$

Lemma 5. *For any signed frame \mathbb{F} , $\mathbb{F} \Vdash \mathbf{C}$ iff \mathbb{F} satisfies collective connectedness.*

Proof. (\Rightarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame and $\mathbb{F} \Vdash (\boxplus p \rightarrow [A]p) \vee (\boxminus p \rightarrow [A]p)$. Fix $a \in A$ arbitrarily. For any $V : \langle \mathbb{F}, V \rangle, a \Vdash (\boxplus p \rightarrow [A]p) \vee (\boxminus p \rightarrow [A]p)$. Then $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus p \rightarrow [A]p$ or $\langle \mathbb{F}, V \rangle, a \Vdash \boxminus p \rightarrow [A]p$. Let $V(p) = \{b \mid aR^+b \text{ or } aR^-b\}$. Fix $c \in A$ arbitrarily. We want to prove that $\langle \mathbb{F}, V \rangle, c \Vdash p$. Assume that $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus p \rightarrow [A]p$. By V , we have $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus p$ and thus

⁸ As noted by one of the reviewers, axiom **C** is equivalent to $\neg[A]\neg p \rightarrow (\boxplus p \vee \boxminus p)$ where the connection between formula and property is arguably more explicit.

$\langle \mathbb{F}, V \rangle, a \Vdash [A]p$. Therefore $\langle \mathbb{F}, V \rangle, c \Vdash p$. Similarly for the case where $\langle \mathbb{F}, V \rangle, a \Vdash \exists p \rightarrow [A]p$. Since we fixed $a, c \in A$ arbitrarily, we conclude that \mathbb{F} is collectively connected.

(\Leftarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame with the collective connectedness property. Then $\forall a, b \in A : aR^+b$ or aR^-b . Suppose for reductio that $\exists a \in A$ and V such that $\langle \mathbb{F}, V \rangle, a \not\Vdash (\boxplus p \rightarrow [A]p) \vee (\exists p \rightarrow [A]p)$. Then $\langle \mathbb{F}, V \rangle, a \Vdash \neg(\boxplus p \rightarrow [A]p) \wedge \neg(\exists p \rightarrow [A]p)$. Thus $\langle \mathbb{F}, V \rangle, a \Vdash (\boxplus p \wedge \exists p) \wedge \neg[A]p$. Then $\exists b \in A$ such that $\langle \mathbb{F}, V \rangle, b \not\Vdash p$. As $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus p \wedge \exists p$ and aR^+b or aR^-b , this is a contradiction. Hence $\mathbb{F} \Vdash (\boxplus p \rightarrow [A]p) \vee (\exists p \rightarrow [A]p)$. \square

The following corollaries follow directly from this lemma.

Corollary 1. *For any signed frame \mathbb{F} , $\mathbb{F} \Vdash \mathbf{4B} \wedge \mathbf{C}$ iff \mathbb{F} is locally balanced and has the collective connectedness property.*

Corollary 2. *For any finite signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$, $\mathbb{F} \Vdash \mathbf{B}_G$ iff $\exists \mathbb{F}' = \langle A', R^+, R^- \rangle$ such that $A = A'$, $R^+ \subseteq R'^+$, $R^- \subseteq R'^-$ and $\mathbb{F}' \Vdash \mathbf{4B} \wedge \mathbf{C}$.*

4.2 Nominals

One option of extending the expressivity of $\mathcal{L}_{\mathbf{PNL}}$ to be able to characterize non-overlapping is to add nominals in the hybrid tradition. We keep the formal discussion of nominals rather brief for now as a full hybrid extension of \mathbf{PNL} will be presented in the next Section 5. Nominals are a set of propositional variables where output of the valuation function is a singleton: any nominal can only be true at exactly one world. In practice, this lets us assign a *name* to individual agents in the network.

We extend the set of propositional variables to be the union of two sets \mathbf{At} and \mathbf{Nom} with an empty intersection. \mathbf{At} is the set of propositional atoms, whereas \mathbf{Nom} is the set of nominals. We also modify our valuation function and call it V_H such that $V_H : \mathbf{At} \cup \mathbf{Nom} \rightarrow \wp(A)$ satisfies the property: for all $i \in \mathbf{Nom}$, $|V_H(i)| = 1$. We denote members of $\mathbf{At} = \{p, q, r, \dots\}$ and $\mathbf{Nom} = \{i, j, k, \dots\}$. Satisfaction of nominals in a signed model with nominals $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ and $a \in A$ is defined as with propositional variables:

$$\mathbb{M}, a \Vdash i \text{ iff } a \in V_H(i)$$

We call the language of \mathbf{PNL} including nominals $\mathcal{L}_{\mathbf{PNL}_i}$ and present the nominal axiom for non-overlapping \mathbf{N}_H .

$$i \rightarrow \boxplus(\boxplus i \rightarrow i) \quad (\mathbf{N}_H)$$

Since we defined signed frames as already having the non-overlapping property, we define *general frames* and *general models* to be signed frames and models without any restrictions on the binary relations R^+ and R^- .

Definition 9 (General Frame and Model). Let A be a non-empty set of agents and R^+ and R^- be two binary relations on A . Further, let $V : \text{At} \rightarrow \wp(A)$ be a valuation function. A **general model** is a tuple $\mathbb{M} = \langle A, R^+, R^-, V \rangle$.

We call a general model without valuation $\mathbb{F} = \langle A, R^+, R^- \rangle$ a **general frame**.

Lemma 6. For any symmetric general frame $\mathbb{F} = \langle A, R^+, R^- \rangle$ of \mathcal{L}_{PNL_i} , $\mathbb{F} \models \mathbf{N}_H$ iff \mathbb{F} has the non-overlapping property.

Proof. (\Rightarrow) Proof by contraposition. Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric general frame without the non-overlapping property. Then $\exists a, b \in A$ such that aR^+b and aR^-b . Let $V_H(i) = \{a\}$ be a valuation on \mathbb{F} . It follows that $\langle \mathbb{F}, V_H \rangle, b \not\models i$. By symmetry bR^+a and thus $\langle \mathbb{F}, V_H \rangle, b \models \Diamond i$. Therefore we have $\langle \mathbb{F}, V_H \rangle, b \models \neg(\Diamond i \rightarrow i)$ and as aR^-b it follows that $\langle \mathbb{F}, V_H \rangle, a \not\models \Box(\Diamond i \rightarrow i)$. Hence $\mathbb{F} \not\models i \rightarrow \Box(\Diamond i \rightarrow i)$.

(\Leftarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric general frame with the non-overlapping property. Suppose for reductio that there exists $a \in A$ and a valuation V_H on \mathbb{F} such that $\langle \mathbb{F}, V_H \rangle, a \not\models i \rightarrow \Box(\Diamond i \rightarrow i)$. Then $\langle \mathbb{F}, V_H \rangle, a \models i$ and $\langle \mathbb{F}, V_H \rangle, a \models \Box(\Diamond i \wedge \neg i)$. It follows that $\exists b \in A$ such that aR^-b and $\langle \mathbb{F}, V_H \rangle, b \models \Diamond i \wedge \neg i$. Since $|V_H(i)| = 1$, it must be the case that bR^+a . By symmetry bR^-a and thus we have reached a contradiction by non-overlapping. We conclude that $\mathbb{F} \models i \rightarrow \Box(\Diamond i \rightarrow i)$. \square

Nominals greatly extend the expressivity of a logic. A common motivation for including nominals in a language is to express otherwise undefinable properties like irreflexivity, asymmetry, antisymmetry and intransitivity, to mention some. However, when modeling agent based networks with a logic like **PNL**, we also have an incentive to add nominals to make it clear *who* we are modeling. This is not a novel approach to social network logics, see e.g., [17,35,38].

4.3 Intersection

Another possible option for extension to gain a non-overlapping axiom is to introduce the intersection modality, perhaps most commonly used as a distributed knowledge operator known in the literature of epistemic logic, such as [19,23,37]. We modify it to our purpose.

Definition 10 (Semantics of Intersection Modality). Let $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ be a general model and let $a \in A$. We define the semantics of the intersection modality $\langle + \cap - \rangle$ as follows:

$$\mathbb{M}, a \models \langle + \cap - \rangle \phi \text{ iff } \exists b \in A \text{ such that } aR^+b, aR^-b \text{ and } \mathbb{M}, b \models \phi$$

By including this operator, the axiom for non-overlapping \mathbf{N}_I would simply be:

$$\langle + \cap - \rangle \perp \quad (\mathbf{N}_I)$$

Lemma 7. *For any general frame $\mathbb{F} = \langle A, R^+, R^- \rangle$, $\mathbb{F} \Vdash \mathbf{N}_I$ iff \mathbb{F} has the non-overlapping property.*

We read $\langle +\cap-\rangle\phi$ to hold at an agent if and only if there exists another agent that is both a friend and an enemy of the current agent where ϕ is true. That two agents *cannot* be both friends and enemies is a property assumed in the original work on signed graphs, and it is therefore difficult to see how the intersection operator would have any application outside axiomatizing the non-overlapping property.

4.4 Difference

A third possible solution is to introduce the difference operator $\langle D \rangle$.

Definition 11 (Semantics of Difference Operator [12]). *Let $\mathbb{M} = \langle A, R^+, R^-, V \rangle$ be a general model and let $a \in A$. The semantics of $\langle D \rangle$ is defined as follows:*

$$\mathbb{M}, a \Vdash \langle D \rangle\phi \text{ iff } \exists b \in A \text{ such that } b \neq a \text{ and } \mathbb{M}, b \Vdash \phi.$$

With this definition, we introduce the axiom \mathbf{N}_D for the non-overlapping property:

$$(p \wedge \neg\langle D \rangle p) \rightarrow (\boxplus(\diamond p \rightarrow p) \wedge \boxminus(\diamond p \rightarrow p)) \quad (\mathbf{N}_D)$$

Inclusion of the $\langle D \rangle$ modality is not hard to motivate. $\langle D \rangle\phi$ holds at an agent if and only if there is another agent in the network where ϕ is true. We show the following lemma.

Lemma 8. *For any symmetric general frame $\mathbb{F} = \langle A, R^+, R^- \rangle$, $\mathbb{F} \Vdash \mathbf{N}_D$ iff \mathbb{F} has the non-overlapping property.*

Proof. (\Rightarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric general frame such that $\mathbb{F} \Vdash \mathbf{N}_D$. Let $a, b \in A$ and without loss of generality assume that aR^+b . We want to prove that $(a, b) \notin R^-$. Let V be a valuation on \mathbb{F} such that $V(p) = \{a\}$. It follows that $\langle \mathbb{F}, V \rangle, a \Vdash p \wedge \neg\langle D \rangle p$. Since $\mathbb{F} \Vdash (p \wedge \neg\langle D \rangle p) \rightarrow (\boxplus(\diamond p \rightarrow p) \wedge \boxminus(\diamond p \rightarrow p))$, we have that $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus(\diamond p \rightarrow p) \wedge \boxminus(\diamond p \rightarrow p)$. As aR^+b , then $\langle \mathbb{F}, V \rangle, b \Vdash \diamond p \rightarrow p$. We know that $\langle \mathbb{F}, V \rangle, b \not\Vdash p$, thus $\langle \mathbb{F}, V \rangle, b \not\Vdash \diamond p$. Hence, $(b, a) \notin R^-$ and by symmetry $(a, b) \notin R^-$.

(\Leftarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a symmetric general frame with the non-overlapping property. Fix an arbitrary valuation V on \mathbb{F} and $a \in A$. Assume that $\langle \mathbb{F}, V \rangle, a \Vdash p \wedge \neg\langle D \rangle p$. Then $\neg\exists b \in A$ such that $b \neq a$ and $\langle \mathbb{F}, V \rangle, b \Vdash p$. It follows that $V(p) = \{a\}$. Let $c \in A$ such that aR^+c . By symmetry and non-overlapping $(c, a) \notin R^-$. Thus $\langle \mathbb{F}, V \rangle, c \not\Vdash \diamond p$ and hence $\langle \mathbb{F}, V \rangle, c \Vdash \diamond p \rightarrow p$. Then $\langle \mathbb{F}, V \rangle, a \Vdash \boxplus(\diamond p \rightarrow p)$. Now, let $d \in A$ such that aR^-d . By similar reasoning $\langle \mathbb{F}, V \rangle, d \Vdash \diamond p \rightarrow p$ and thus $\langle \mathbb{F}, V \rangle, a \Vdash \boxminus(\diamond p \rightarrow p)$. It follows that $\langle \mathbb{F}, V \rangle, a \Vdash (p \wedge \neg\langle D \rangle p) \rightarrow (\boxplus(\diamond p \rightarrow p) \wedge \boxminus(\diamond p \rightarrow p))$ and as we chose an arbitrary V and $a \in A$, we conclude that $\mathbb{F} \Vdash \mathbf{N}_D$. \square

We show that we can also define collective connectedness with this operator:

$$(p \vee \langle D \rangle p) \rightarrow (\Diamond p \vee \Diamond p) \quad (\mathbf{C}_D)$$

Lemma 9. *For any signed frame $\mathbb{F} = \langle A, R^+, R^- \rangle$, $\mathbb{F} \Vdash \mathbf{C}_D$ iff \mathbb{F} has the collective connectedness property.*

Proof. (\Rightarrow) Proof by contraposition. Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame without the collective connectedness property. Then $\exists a, b \in A$ such that $(a, b) \notin R^+$ and $(a, b) \notin R^-$. Now let $V(p) = \{b\}$ be a valuation on \mathbb{F} . Thus, $\langle \mathbb{F}, V \rangle, a \Vdash \langle D \rangle p$. Yet we have that $\langle \mathbb{F}, V \rangle, a \Vdash \neg \Diamond p \wedge \neg \Diamond p$ as a and b are neither positively nor negatively related. It follows that $\langle \mathbb{F}, V \rangle, a \not\Vdash (p \vee \langle D \rangle p) \rightarrow (\Diamond p \vee \Diamond p)$ and hence $\mathbb{F} \not\Vdash (p \vee \langle D \rangle p) \rightarrow (\Diamond p \vee \Diamond p)$.

(\Leftarrow) Let $\mathbb{F} = \langle A, R^+, R^- \rangle$ be a signed frame with the collective connectedness property. Let $a \in A$ be an arbitrary agent and V an arbitrary valuation on \mathbb{F} . Now assume that $\langle \mathbb{F}, V \rangle, a \Vdash p \vee \langle D \rangle p$. Then $\exists b \in A$ such that $\langle \mathbb{F}, V \rangle, b \Vdash p$. By the collective connectedness property aR^+b or aR^-b and thus $\langle \mathbb{F}, V \rangle, a \Vdash \Diamond p \vee \Diamond p$. Since we chose a and V arbitrarily we conclude that $\mathbb{F} \Vdash (p \vee \langle D \rangle p) \rightarrow (\Diamond p \vee \Diamond p)$. \square

5 Hybrid Extension

In Section 4.2, we showed that we get an axiom for non-overlapping by including nominals in the language of **PNL**. In this section, we properly extend **PNL** to the hybrid logic **PNL_i**. We show a full axiomatization and prove completeness with respect to the class of balanced signed frames and weakly balanced signed frames, respectively.

5.1 Syntax and Semantics of **PNL_i**

The language of **PNL_i** does not only extend **PNL** with nominals, but also with the hybrid operators $\@$ and \downarrow . Intuitively, $\@_i$ shifts the evaluation to the specific agent named i , whereas $\downarrow x$ ties the name “ x ” to the current agent. In addition to the set of nominals $\mathbf{Nom} = \{i, j, k, \dots\}$, we also need a set of nominal variables denoted $\mathbf{Var} = \{x, y, z, \dots\}$ to speak generally about names in our language. The standard propositional atoms are still denoted $\mathbf{At} = \{p, q, r, \dots\}$. We define the syntax of **PNL_i** formally as follows.

Definition 12 (Syntax of **PNL_i).** *Let \mathbf{At} be a set of propositional atoms, \mathbf{Nom} be a set of nominals and \mathbf{Var} be a set of agent variables, all countable and pairwise disjoint. We define the well-formed formulas of the language $\mathcal{L}_{\mathbf{PNL}_i}$ to be generated by the following grammar:*

$$\phi ::= p \mid s \mid \neg\phi \mid (\phi \wedge \phi) \mid \Diamond\phi \mid \Diamond\phi \mid \@_s\phi \mid \downarrow x.\phi$$

where $p \in \text{At}$, $s \in \text{Nom} \cup \text{Var}$ and $x \in \text{Var}$. We define propositional connectives like \vee, \rightarrow and the formulas \top, \perp as usual. Further, we define the duals as standard $\boxplus := \neg \boxtimes \neg$ and $\boxminus := \neg \boxtimes \neg$.

Signed hybrid frames are defined similarly to signed frames. Signed hybrid models include a valuation function that takes both propositional atoms and nominals.

Definition 13 (Signed Hybrid Frames and Models). *Let A be a non-empty set of agents and R^+ and R^- be two symmetric and non-overlapping binary relations on A where R^+ is reflexive. Further, let $V_H : \text{At} \cup \text{Nom} \rightarrow \wp(A)$ be a valuation function such that $\forall i \in \text{Nom} : |V_H(i)| = 1$. A **signed hybrid model** is a tuple $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$.*

*We call a signed hybrid model without valuation $\mathbb{F} = \langle A, R^+, R^- \rangle$ a **signed hybrid frame** or just a **signed frame**.*

To define the semantics of PNL_i , we need an assignment function for the nominal variables $g : \text{Var} \rightarrow A$. Define an x -variant of g as $g_a^x(x) = a$ and $g_a^x(y) = g(y)$ for all $y \neq x$. Further, for $i \in \text{Nom}$, $[i]^{\mathbb{M}, g}$ is the state $a \in A$ called “ i ”, i.e. the unique a such that $a \in V(i)$. For $x \in \text{Var}$, $[x]^{\mathbb{M}, g} = g(x)$.

Definition 14 (Semantics of PNL_i). *Let $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ be a signed hybrid model, a an agent in A and $g : \text{Var} \rightarrow A$ an assignment function. We define the truth conditions for PNL_i as follows:*

$$\begin{aligned}
 \mathbb{M}, g, a \Vdash p & \text{ iff } a \in V(p) \text{ for } p \in \text{At} \\
 \mathbb{M}, g, a \Vdash s & \text{ iff } a = [s]^{\mathbb{M}, g} \text{ for } s \in \text{Nom} \cup \text{Var} \\
 \mathbb{M}, g, a \Vdash \neg \phi & \text{ iff } \mathbb{M}, g, a \not\Vdash \phi \\
 \mathbb{M}, g, a \Vdash \phi \wedge \psi & \text{ iff } \mathbb{M}, g, a \Vdash \phi \text{ and } \mathbb{M}, g, a \Vdash \psi \\
 \mathbb{M}, g, a \Vdash \boxplus \phi & \text{ iff } \exists b \in A \text{ such that } aR^+b \text{ and } \mathbb{M}, g, b \Vdash \phi \\
 \mathbb{M}, g, a \Vdash \boxminus \phi & \text{ iff } \exists b \in A \text{ such that } aR^-b \text{ and } \mathbb{M}, g, b \Vdash \phi \\
 \mathbb{M}, g, a \Vdash @_s \phi & \text{ iff } \mathbb{M}, g, [s]^{\mathbb{M}, g} \Vdash \phi \text{ for } s \in \text{Nom} \cup \text{Var} \\
 \mathbb{M}, g, a \Vdash \downarrow x. \phi & \text{ iff } \mathbb{M}, g_a^x, a \Vdash \phi
 \end{aligned}$$

5.2 Axiomatization, Completeness and Undecidability

As shown in Section 4.2, the axiom \mathbf{N}_H characterizes non-overlapping on any symmetric general frame.

$$i \rightarrow \boxminus (\boxplus i \rightarrow i) \quad (\mathbf{N}_H)$$

Symmetry of both R^+ and R^- is characterized by axiom \mathbf{B}^{+-} .

$$i \rightarrow (\boxplus \boxplus i \wedge \boxminus \boxminus i) \quad (\mathbf{B}^{+-})$$

Positive reflexivity is characterized by axiom \mathbf{T}^+ .

$$i \rightarrow \diamond i \quad (\mathbf{T}^+)$$

To axiomatize balance on signed hybrid frames, we introduce the axiom schema \mathbf{B}_i . For convenience we use the following definition from [43].

Definition 15 ((\boxplus ; \boxminus) [43]). *Let $m, n \in \mathbb{N}$ and $\phi \in \mathcal{L}_{PNLi}$.*

$(\boxplus; \boxminus)_\phi^{m,n}$ is the set of all formulas obtained by prefixing ϕ with a sequence of m positive (\boxplus) and n negative (\boxminus) box modalities in some order.

$(\boxplus; \boxminus)_\phi^{m,n}$ gives us a set of formulas with all permutations of a given number of positive and negative boxes ending with a given ϕ . As an example $(\boxplus; \boxminus)_p^{1,2} = \{\boxplus\boxminus\boxminus p, \boxminus\boxplus\boxminus p, \boxminus\boxminus\boxplus p\}$. Using this notation, we present the axiom schema \mathbf{B}_i .

$$\downarrow x.\psi, \quad \psi \in (\boxplus; \boxminus)_{-x}^{m,2n+1} \quad (\mathbf{B}_i)$$

for $m, n \in \mathbb{N}$.

$(\boxplus; \boxminus)_{-x}^{m,2n+1}$ is a countable set of formulas on the form $\square(\dots)\square\neg x$ for all combinations of \boxplus and \boxminus with an odd number of \boxminus -modalities. \mathbf{B}_i therefore gives us a set of formulas that represents all cycles with an odd number of negative edges – and forbids them. These are formulas like $\downarrow x\boxminus\boxminus\boxminus\neg x$, $\downarrow x\boxplus\boxplus\boxplus\boxminus\neg x$, etc. The binder $\downarrow x$ lets us “lock down” the variable x and disallow a return to it with $\neg x$. \mathbf{B}_i thus characterizes the cycle-property of the Balance Theorem.

We present the axiomatic system \mathbf{pnl}_i in Table 1, which is the standard normal hybrid logic $\mathbf{K}_{\mathcal{H}(\boxplus, \boxminus)}$ with the axioms mentioned in this section. Including the set of formulas in \mathbf{B}_i in the axiomatic system is not problematic as $(\boxplus; \boxminus)_{-x}^{m,2n+1}$ is recursively enumerable. It can be compared to the countable set of tautologies that are also in the axiomatic system, in Table 1 called CT .

Theorem 4. *\mathbf{pnl}_i is sound and complete with respect to the class of balanced signed hybrid frames.*

Proof. (Soundness). Let \mathcal{F} be the class of balanced signed hybrid frames. $\mathbf{K}_{\mathcal{H}(\boxplus, \boxminus)}$ is sound with respect to the class of standard hybrid frames [11], thus it follows that $\mathcal{F} \models \mathbf{K}_{\mathcal{H}(\boxplus, \boxminus)}$. We therefore only need to show validity of the axioms \mathbf{N}_H , \mathbf{B}^{+-} , \mathbf{T}^+ and all axioms in the schema \mathbf{B}_i . Validity of \mathbf{N}_H is shown in Lemma 6. For \mathbf{B}^{+-} and \mathbf{T}^+ , validity follows straightforwardly. For any valuation V_H on a frame $\mathbb{F} \in \mathcal{F}$, each individual formula in \mathbf{B}_i represents a negative cycle that is not allowed in the model (\mathbb{F}, V_H) . Since \mathbb{F} is balanced, by the Balance Theorem, there are no negatives cycles in \mathbb{F} . It follows that all formulas in \mathbf{B}_i are valid and thus \mathbf{pnl}_i is sound with respect to the class of balanced signed hybrid frames.

Table 1. Axiomatic system \mathbf{pnl}_i , where $\diamond \in \{\Phi, \Downarrow\}$, $\square \in \{\boxplus, \boxminus\}$ and $m, n \in \mathbb{N}$.

(CT)	All classical tautologies
(K_\square)	$\vdash \square(\phi \rightarrow \psi) \rightarrow \square\phi \rightarrow \square\psi$
($K_\@$)	$\vdash \@_i(\phi \rightarrow \psi) \rightarrow \@_i\phi \rightarrow \@_i\psi$
(Selfdual)	$\vdash \@_i\phi \leftrightarrow \neg\@_i\neg\phi$
(Ref $_@$)	$\vdash \@_i i$
(Agree)	$\vdash \@_i\@_j\phi \leftrightarrow \@_j\phi$
(Intro)	$\vdash i \rightarrow (\phi \leftrightarrow \@_i\phi)$
(Back)	$\vdash \diamond\@_i\phi \rightarrow \@_i\phi$
(DA)	$\vdash \@_i(\downarrow x.\phi \leftrightarrow \phi[x/i])$
(\mathbf{N}_H)	$\vdash i \rightarrow \boxplus(\Phi i \rightarrow i)$
(\mathbf{B}^{+-})	$\vdash i \rightarrow (\boxplus \Phi i \wedge \boxminus \Phi i)$
(\mathbf{T}^+)	$\vdash i \rightarrow \Phi i$
(\mathbf{B}_i)	$\vdash \downarrow x.\psi, \quad \psi \in (\boxplus; \boxminus)_{-x}^{m, 2n+1}$
(MP)	If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$ then $\vdash \psi$
(Subst)	If $\vdash \phi$ then $\vdash \phi^\sigma$, for σ a substitution
(Gen $_@$)	If $\vdash \phi$ then $\vdash \@_i\phi$
(Gen $_\square$)	If $\vdash \phi$ then $\vdash \square\phi$
(Name)	If $\vdash \@_i\phi$ and i does not occur in ϕ , then $\vdash \phi$
(BG)	If $\vdash \@_i \diamond j \rightarrow \@_j\phi, j \neq i$ and j does not occur in ϕ , then $\vdash \@_i \square \phi$

(Completeness). Recall that $\mathbf{pnl}_i = \mathbf{K}_{\mathcal{H}(\@, \downarrow)} + \{\mathbf{N}_H, \mathbf{B}^{+-}, \mathbf{T}^+, \mathbf{B}_i\}$. By a theorem proved in [11], if Σ is a set of pure formulas, $\mathbf{K}_{\mathcal{H}(\@, \downarrow)} + \Sigma$ is complete with respect to the class of frames defined by Σ . None of the formulas in $\{\mathbf{N}_H, \mathbf{B}^{+-}, \mathbf{T}^+, \mathbf{B}_i\}$ contains any propositional letters and they can all be formulated in the language $\mathcal{H}(\@, \downarrow)$ with two box modalities. Hence, the formulas are pure. \mathbf{N}_H characterizes non-overlapping, \mathbf{B}^{+-} characterizes positive and negative symmetry and \mathbf{T}^+ characterizes positive reflexivity. Each formula in \mathbf{B}_i represents the disallowance of a unique negative cycle, for all combinations up to infinity. All formulas of \mathbf{B}_i are therefore valid on a signed hybrid frame \mathbb{F} if and only if there are no negative cycles in \mathbb{F} and thus, by the Balance Theorem, \mathbf{B}_i characterizes the balance property. We conclude that \mathbf{pnl}_i is complete with respect to the class of balanced signed hybrid frames. \square

It is natural to ask whether there are any known decidability results from the hybrid logic literature that can help us determine whether this logic has a decidable satisfiability problem. The answer is yes, but the result is negative. $\mathcal{L}_{\mathbf{PNL}_i}$ is the hybrid language $\mathcal{H}(\@, \downarrow)$ [3] with two relational operators instead of the standard modal diamond. The following result therefore follows from the proof of undecidability of $\mathcal{H}(\@, \downarrow)$ in [4].

Lemma 10. *The satisfiability problem for \mathbf{PNL}_i is undecidable.*

5.3 Weak Balance

Perhaps not surprising, we can also get an axiom schema on the same form as \mathbf{B}_i that characterizes weak balance. Recall that a weakly balanced network only disallows cycles with a single negative edge. The schema for weak balance \mathbf{B}_{W_i} therefore uses $(\boxplus; \boxminus)_{-x}^{m,1}$ in place of $(\boxplus; \boxminus)_{-x}^{m,2n+1}$ in \mathbf{B}_i . Note that $(\boxplus; \boxminus)_{-x}^{m,1} \subset (\boxplus; \boxminus)_{-x}^{m,2n+1}$ for $m, n \in \mathbb{N}$. We define \mathbf{B}_{W_i} accordingly.

$$\downarrow x.\psi, \quad \psi \in (\boxplus; \boxminus)_{-x}^{m,1} \quad (\mathbf{B}_{W_i})$$

for $m \in \mathbb{N}$.

Let \mathbf{pnl}_{W_i} be the axiomatic system that differs from \mathbf{pnl}_i only in the inclusion of axiom \mathbf{B}_{W_i} instead of \mathbf{B}_i . The following result can be proved in a similar fashion as Theorem 4 in the last section.

Corollary 3. *\mathbf{pnl}_{W_i} is sound and complete with respect to the class of weakly balanced signed hybrid frames.*

6 Reasoning on Connected Frames

We have shown that by using a hybrid framework, we get an axiom schema that characterizes balance on signed hybrid frames. In this section, we show that balance on collectively connected signed hybrid frames can be defined with a single axiom. We use this axiom together with the universal modality and local link-changing modalities to analyze change in social networks towards balance.

6.1 Balance and $\mathbf{PNL}_i[A]$

On the class of collectively connected signed hybrid frames, balance is characterized by the axiom \mathbf{B}_H .

$$\downarrow x.(\boxminus \boxminus \boxminus \neg x) \wedge (\boxminus \boxplus \boxminus \neg x) \quad (\mathbf{B}_H)$$

Lemma 11. *For any collectively connected signed hybrid frame \mathbb{F} , $\mathbb{F} \models \mathbf{B}_H$ iff \mathbb{F} has the balance property.*

Proof of this lemma follows easily when we observe that \mathbf{B}_H characterizes the local property of balance on a network where all agents are either friends or enemies: the network is balanced if and only if every triangle is of either all positive edges, or two negative edges and one positive edge. In this axiom, we force this property by disallowing triangles of all negative edges, and of two positive and one negative edge. With a similar reasoning while only disallowing triangles of one single negative edge, we can define the axiom \mathbf{B}_{WH} for weak balance on collectively connected signed hybrid frames.

$$\downarrow x.(\boxminus \boxplus \boxplus \neg x) \quad (\mathbf{B}_{WH})$$

Lemma 12. *For any collectively connected signed hybrid frame \mathbb{F} , $\mathbb{F} \Vdash \mathbf{B}_{WH}$ iff \mathbb{F} has the weak balance property.*

Recall that we already in [42] had an axiom for balance on collectively connected signed frames, namely the axiom **4B**. However, interestingly, axiom **B_H** and **B_{WH}** have an applicable attribute that axiom **4B** does not have. By including the universal modality $[A]$ in our language, this modality with the hybrid binder \downarrow enables us to speak of balance in the network at individual agents. We can have a formula $[A]\mathbf{B}_H$ that is forced at an agent in a network if and only if the network is balanced. We show how this can be useful when analyzing dynamics in the next Section 6.2.

Extending **PNL_i** with the universal modality $[A]$ gives us **PNL_i[A]** where the following semantic clause is added to the semantics of **PNL_i**. Let \mathbb{M} be a signed hybrid model, $a \in A$ an agent and $g : \text{Var} \rightarrow A$ an assignment function.

$$\mathbb{M}, g, a \Vdash [A]\phi \text{ iff } \forall b \in A : \mathbb{M}, g, b \Vdash \phi$$

It is no surprise that hybrid languages are expressive, however the language of **PNL_i[A]**, $\mathcal{L}_{\text{PNL}_i[A]}$, is even strictly more expressive than $\mathcal{L}_{\text{PNL}_i}$. So expressive, in fact, that we can define the strong Priorean quantifiers \exists and \forall [11] in terms of $\mathcal{L}_{\text{PNL}_i[A]}$. For $x, y \in \text{Var}$ and $\phi \in \mathcal{L}_{\text{PNL}_i[A]}$, $\forall s. \phi$ is equivalent to $\downarrow y.[A] \downarrow x. @_y \phi$ where y does not occur in ϕ [11]. Additionally, with the universal modality $[A]$ in the language, we can also define $@_s \phi$ as $[A](s \rightarrow \phi)$ for $s \in \text{Nom} \cup \text{Var}$.

6.2 Dynamics

To analyze change in collectively connected signed hybrid frames, we include the following local link change modalities inspired, as our previous global bridge modalities in Section 3.1, by sabotage modal logic, most notably [5, 7].

Definition 16 (Semantics of Local Dynamic Modalities). *Let $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ be a signed hybrid model and $a \in A$. We define truth conditions for the local link change modalities as follows:*

$$\begin{aligned} \mathbb{M}, g, a \Vdash \langle \oplus \rangle \phi & \text{ iff } \exists b \in A \text{ such that } (a, b) \in R^- \text{ and} \\ & \langle A, R^+ \cup \{(a, b), (b, a)\}, R^- \setminus \{(a, b), (b, a)\}, V_H \rangle, a \Vdash \phi \\ \mathbb{M}, g, a \Vdash \langle \ominus \rangle \phi & \text{ iff } \exists b \in A \text{ such that } (a, b) \in R^+, a \neq b \text{ and} \\ & \langle A, R^+ \setminus \{(a, b), (b, a)\}, R^- \cup \{(a, b), (b, a)\}, V_H \rangle, a \Vdash \phi \end{aligned}$$

We read $\langle \oplus \rangle \phi$ to be true at an agent if and only if ϕ holds at the current agent after changing one edge connected to this agent from *negative to positive*. Similarly, we read $\langle \ominus \rangle \phi$ to be true at an agent if and only if ϕ holds at the current agent after changing one edge connected to this agent from *positive to negative*. These modalities enable stepwise analyses of the network dynamics from the perspective of single agents. We show the use of them with an example.

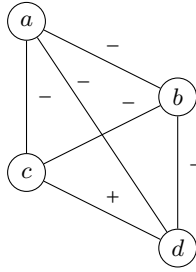


Fig. 3. A collectively connected signed hybrid model \mathbb{M} .

Consider the signed hybrid model $\mathbb{M} = \langle A, R^+, R^-, V_H \rangle$ in Fig. 3 with $A = \{a, b, c, d\}$. Positive reflexive edges have been omitted for simplicity. Let $a, b, c, d \in \text{Nom}$ be nominals that hold at their respective agents: for instance $\mathbb{M}, g, a \Vdash a$, $\mathbb{M}, g, b \Vdash b$, and so on. \mathbb{M} is not balanced, as it has the two negative triangles abc and abd . \mathbb{M} is weakly balanced, as there are no triangles with a single negative edge. This can be formulated by the formula $\neg[A]\mathbf{B}_H \wedge [A]\mathbf{B}_{WH}$ which is indeed valid on the underlying frame of \mathbb{M} . We now observe some, of many, satisfiable formulas on \mathbb{M} .

$$- \mathbb{M}, g, a \Vdash \langle \oplus \rangle [A]\mathbf{B}_H$$

“After changing an enmity to a friendship at agent a , the network can become balanced.”

Specifically, this is if agent a and agent b becomes friends.

$$- \mathbb{M}, g, c \Vdash \neg \langle \oplus \rangle [A]\mathbf{B}_H \wedge \langle \ominus \rangle \neg [A]\mathbf{B}_{WH}$$

“The network cannot become balanced after agent c reconciles with an enemy, but it is possible that after agent c becomes enemies with a current friend, the network will stop being weakly balanced.”

Agent c cannot with a single relationship change resolve both negative triangles to make a balanced network. If agent c 's goal was to disrupt the system, we observe that by becoming enemies with agent b , the network will not anymore be weakly balanced. It will then have the triangle bcd with a single negative edge.

$$- \mathbb{M}, g, d \Vdash \langle \ominus \rangle (\langle \ominus \rangle c \wedge @_a \neg \langle \oplus \rangle [A]\mathbf{B}_H)$$

“After agent d makes an enemy with a current friend, agent c is an enemy of agent d and then it cannot be the case that agent a can reconcile with another agent to make the network balanced.”

If agent d and agent c would become enemies, this would make the triangle bcd of all negative edges. This would mean that no relations to agent a could be changed and result in a balanced network.

These local dynamic modalities have similarities and differences with *local* bridge operators like the global addition modalities introduced in Section 3.2 have with *global* bridge operators. We get the following in the same way as for the global bridge operators.

Theorem 5. *The model checking problem for $\mathcal{L}_{PNL_i[A]}$ extended with local dynamic modalities is PSPACE-hard.*

As mentioned in Section 4.2, there are other known hybrid logics used to reason about social networks, and in particular how they change over time. In Section 8, we discuss some of these logics in relation to $\mathbf{PNL}_i[A]$.

7 Measuring Polarization

The aim of this section is to investigate networks changing from imbalanced to balanced, and in particular to analyze how far a network is from being polarized. We begin by assessing different properties that such a measure might have. Then we introduce several measures of balance found in the literature, but accommodated to a logical framework. We contrast advantages and disadvantages of each metric and evaluate them in an example. We also briefly consider a measure of reconciliation, towards a network where no agents disagree.

7.1 Measure Properties

In literature such as [1,6,14,31], the distance between two standard Kripke models is defined as a mapping from an ordered pair of two models to a real number. This mapping usually has to satisfy certain properties. The core feature of what we call a *balance measure* is to judge how far a signed frame is from being balanced. Therefore, we define a balance measure as a mapping from *one* signed frame to a real number. We only consider the balance measure on finite signed frames. This is not an unrealistic restriction on a social network, as analyses of social networks are often done on a given finite set of agents and relations between them.

Definition 17 (Balance Measure). *Let \mathcal{F} be the class of finite signed frames. A **balance measure** is a mapping $d : \mathcal{F} \rightarrow \mathbb{R}$ which satisfies the following properties:*

$$[\textit{nonnegativity}] \quad d(\mathbb{F}) \geq 0,$$

$$[\textit{balance indistinguishability}] \quad d(\mathbb{F}) = 0 \textit{ iff } \mathbb{F} \textit{ is balanced.}$$

There are other restrictions we can impose on a balance measure depending on motivation and purpose. One candidate is *long cycle discrimination*. Studies show that longer cycles have less effect on people's tension than shorter cycles [22]. Moreover, the number of cycles in a network of a given length generally

increases with length [30]. A count of cycles would therefore be dominated by long cycles. This might motivate the need for a metric that downplays the role of longer cycles in the calculation.

By simply counting the number of negative cycles, we do not distinguish between cases where the cycles overlap and cases where they do not. Consider the example in Fig. 4, of two networks both having the ratio $\frac{2}{3}$ of negative cycles to all cycles. Positive reflexive relations are again omitted for simplicity. In the network to the left, two of a total of three cycles in the network are negative. There is only need of a single link change, of seven total, namely between c and d , for the network to become balanced, because the negative cycles overlap on the link that needs to be changed. In the network to the right, we require two links of seven to change to make all negative triangles balanced. However, counting the ratio $\frac{4}{6}$ of negative cycles determines the same balance measure between these two networks. This observation might provoke the need for an *overlapping cycle discrimination*.

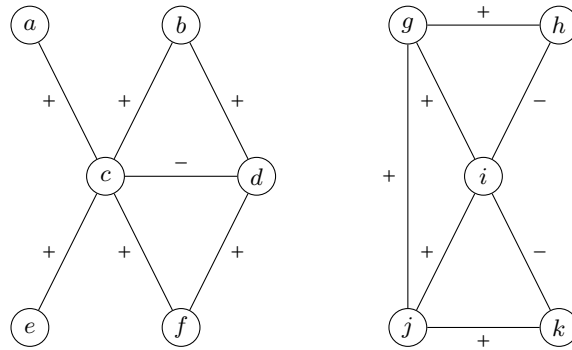


Fig. 4. Two networks that are equally polarized by one measure, but not by another.

It is difficult to give a general formal definition of long cycle discrimination and overlapping cycle discrimination. Intuitively, they say that “all things being equal” short cycles should count more than long cycles and non-overlapping cycles should count more than overlapping ones, respectively, but what “all things being equal” means is highly dependant on the concrete measure being used. In the context of the concrete measures we discuss in the following, the meaning of these informal definitions will be intuitively clear.

For every balance measure there is a corresponding weakly balanced version. As balance always entails weak balance, balance and weakly balance measures might, but not necessarily, output the same number. With all properties listed here in mind, we turn to examine some options for a concrete notion of a balance measure.

7.2 Counting Cycles

By the Balance Theorem, imbalance is directly related to negative cycles. This observation was applied to measuring balance already in a paper by Cartwright and Harary in 1956 [15] and realized as *degree of balance*. Degree of balance in its original form is the number of non-negative cycles, divided by the total number of cycles. To ensure output 0 when the model is balanced, we appropriately rename our variation *degree of imbalance* and divide the number of negative cycles by the number of cycles.

Denote $c^-(\mathbb{F})$ as the number of negative cycles in signed frame \mathbb{F} , and $c(\mathbb{F})$ as the total number of cycles in \mathbb{F} . Let $c^{-W}(\mathbb{F})$ be the number of cycles in \mathbb{F} that have exactly one single negative edge. Note that $c^{-W}(\mathbb{F}) \subseteq c^-(\mathbb{F}) \subseteq c(\mathbb{F})$. We define degree of imbalance and weak imbalance in the following definition.

Definition 18 (Degree of Imbalance). *Let \mathcal{F} be the class of finite signed frames. The **degree of imbalance** is the map $d_{DB} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{DB}(\mathbb{F}) = \frac{c^-(\mathbb{F})}{c(\mathbb{F})}$. The **degree of weak imbalance** is the map $d_{DBW} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{DBW}(\mathbb{F}) = \frac{c^{-W}(\mathbb{F})}{c(\mathbb{F})}$.*

We observe that although this simple measure of distance is a balance measure by Definition 17, it does not satisfy neither the long cycle nor the overlapping cycle discrimination property. [30] defines another cycle counting measure of balance motivated by long cycle discrimination, called *level of imbalance*.

Definition 19 (Level of Imbalance [30]). *Let \mathcal{F} be the class of finite signed frames. The **level of imbalance** is the map $d_{Bz} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{Bz}(\mathbb{F}) = \sum_{k=1}^{\infty} \frac{c_k^-(\mathbb{F})}{z^k}$ where $c_k^-(\mathbb{F})$ is the number of negative cycles of length k and $z > 1$ is a free parameter. The **level of weak imbalance** is the map $d_{BzW} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{BzW}(\mathbb{F}) = \sum_{k=1}^{\infty} \frac{c_k^{-W}(\mathbb{F})}{z^k}$ where $c_k^{-W}(\mathbb{F})$ is the number of cycles with a single negative edge of length k .*

The level of imbalance satisfies the long cycle discrimination property in addition to being a balance measure. The measure divides the number of negative cycles by a free parameter that increases by the negative cycle's length. Like the degree of imbalance, this metric does not satisfy the overlapping cycle discrimination property.

7.3 Line Index of Imbalance

Line index of imbalance was proposed by Harary in 1959 [26] and follows a simple idea: it measures the minimal number of edges to be deleted for the network to be balanced. The measure has also been implemented in terms of weak balance in [20].

Transition from a signed frame to a subframe of fewer edges can seem unintuitive when we imagine links between agents to represent positive and negative relations. Where it is easy to imagine relations in a network to be created, it might be slightly harder to think of situations where agents completely lose touch. We can still of course regard line index of imbalance as a fruitful measurement, although we also remark that the minimal number of edges deleted is the same number as the smallest number of edges changing signs in order to make the network balanced. The reasoning is as follows. By the Balance Theorem, a network is balanced if and only if it has the potential to have the local structural balance property for each set of three agents. That is, as long as it is possible to fill in missing edges to create a collectively connected frame where all triangles have either three positive signs or one positive and two negative, the signed frame is balanced. Thus, changing signs in an imbalanced network have the same purpose as deleting edges in terms of balance: each edge needed to change signs could be deleted and now have the potential of the desired sign.

We present a definition of line index of imbalance for signed frames.

Definition 20 (Line Index of Imbalance). *Let \mathcal{F} be the class of finite signed frames. The **line index of imbalance** is the map $d_{LI} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{LI}(\mathbb{F}) = \min\{\sum_{i \in \{+, -\}} \left| \frac{|R^i| - |R^j|}{2(|R^+ \cup R^-| - |A|)} \right| : \mathbb{F}' = \langle A, R^+, R^- \rangle \text{ where } \mathbb{F}' \text{ is balanced}\}$ where $\mathbb{F} = \langle A, R^+, R^- \rangle$. The **line index of weak imbalance** is the map $d_{LIW} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{LIW}(\mathbb{F}) = \min\{\sum_{i \in \{+, -\}} \left| \frac{|R^i| - |R^j|}{2(|R^+ \cup R^-| - |A|)} \right| : \mathbb{F}' = \langle A, R^+, R^- \rangle \text{ where } \mathbb{F}' \text{ is weakly balanced}\}$ where $\mathbb{F} = \langle A, R^+, R^- \rangle$.*

Line index of imbalance satisfies the properties of a balance measure. It does not discriminate long cycles: in a network with both shorter and longer negative cycles, line index of imbalance will output a number independent on the ratio between short and long negative cycles. As mentioned, line index of imbalance satisfies the overlapping cycle property: in networks where cycles overlap, this metric will not count twice any edges needed to be changed for the purpose of balance. This is exactly what is pictured in Fig. 4. The left network is closer to polarization by line index of imbalance: d_{LI} outputs $\frac{1}{7}$ on the left versus $\frac{2}{7}$ on the right. Simultaneously, the degree of imbalance of both networks are the same: d_{DB} is $\frac{2}{3}$ for both networks.

7.4 Example: How Far From Polarization?

We now look at an example to visually view the different measures we have considered in this section. How far the network is from being polarized or weakly polarized is decided with respect to the measure one chooses to adopt. The measures are ordinal, rather than cardinal values. That is, values for the same measure are comparable, but not across measures. We therefore present two networks and compare the measures individually, on both networks. Consider the two networks \mathbb{F} and \mathbb{F}' in Fig. 5. Positive reflexive edges are omitted for simplicity. We calculate and judge the distance from polarization in Table 2.

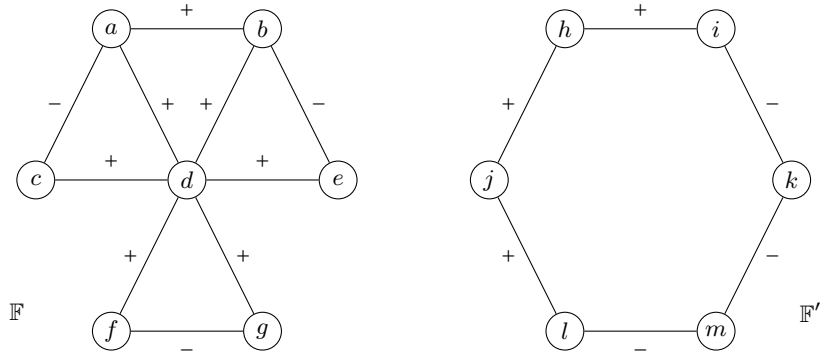


Fig. 5. Two networks that are not yet strongly polarized.

Table 2. How far are \mathbb{F} and \mathbb{F}' in Fig. 5 from being polarized?

\mathbb{F}	\mathbb{F}'
Strong Polarization	
Degree of Imbalance	
$d_{DB}(\mathbb{F}) = \frac{c^-(\mathbb{F})}{c(\mathbb{F})} = \frac{5}{7}$	$d_{DB}(\mathbb{F}') = \frac{c^-(\mathbb{F}')}{c(\mathbb{F}')} = 1$
Level of Imbalance ($z = 2$)	
$d_{Bz}(\mathbb{F}) = \sum_{k=1}^{\infty} \frac{c_k^-(\mathbb{F})}{z^k} = \frac{3}{2^3} + \frac{2}{2^4} = \frac{1}{2}$	$d_{Bz}(\mathbb{F}') = \sum_{k=1}^{\infty} \frac{c_k^-(\mathbb{F}')}{z^k} = \frac{1}{2^6} = \frac{1}{64}$
Line Index of Imbalance	
$d_{LI}(\mathbb{F}) = \frac{3}{10}$	$d_{LI}(\mathbb{F}') = \frac{1}{6}$
Weak Polarization	
Degree of Weak Imbalance	
$d_{DBW}(\mathbb{F}) = \frac{c^{-W}(\mathbb{F})}{c(\mathbb{F})} = \frac{5}{7}$	$d_{DBW}(\mathbb{F}') = \frac{c^{-W}(\mathbb{F}')}{c(\mathbb{F}')} = 0$
Level of Weak Imbalance ($z = 2$)	
$d_{BzW}(\mathbb{F}) = \sum_{k=1}^{\infty} \frac{c_k^{-W}(\mathbb{F})}{z^k} = \frac{3}{2^3} + \frac{2}{2^4} = \frac{1}{2}$	$d_{BzW}(\mathbb{F}') = \sum_{k=1}^{\infty} \frac{c_k^{-W}(\mathbb{F}')}{z^k} = 0$
Line Index of Weak Imbalance	
$d_{LIW}(\mathbb{F}) = \frac{3}{10}$	$d_{LIW}(\mathbb{F}') = 0$

We make some observations on Fig. 5 and Table 2. The networks depict quite different social configurations. \mathbb{F} , on the left, is a relatively highly connected network, where agent d is in a particular position as the single common friend of all agents. \mathbb{F}' , on the right, is a network with considerably lower connectivity than \mathbb{F} and consists of a single negative cycle. From the table, we first consider the measures of strong polarization. We observe that whereas degree of imbalance is higher for \mathbb{F}' than for \mathbb{F} , level of imbalance is higher for \mathbb{F} than for \mathbb{F}' . In other words, according to degree of imbalance, \mathbb{F}' is closer to strong polarization than \mathbb{F} , but according to level of imbalance, \mathbb{F} is the network closest of the two to being strongly polarized. The reason for this is long cycle discrimination. \mathbb{F} has several shorter negative cycles, where \mathbb{F}' only consists of one (relative to the network) long negative cycle. The single cycle in \mathbb{F}' is negative, thus according to degree of imbalance, the network is as strongly polarized as possible. However, the long cycle discrimination gives the cycle in \mathbb{F}' little impact in the level of imbalance.

Regarding weak polarization, we see that all measures for weak polarization of \mathbb{F}' outputs 0. Thus, the network is already weakly polarized: it can be divided into groups of agents where there is friendship within, but hostility towards other groups. According to the definition, these groups can consist of single agents, and in the case of \mathbb{F}' , the groups are $\{h, i, j, l\}$, $\{k\}$ and $\{m\}$. We also observe that across all measures of weak polarization for \mathbb{F} , the value is equal to the measures of strong polarization. This entails that the all the negative cycles in \mathbb{F} have a single negative edge. Would there have been any cycles with an odd number of negative edges higher than one, they would not be counted in degree or level of weak imbalance.

7.5 Reconciliation

In the context of polarization, there is one subclass of balanced frames that are particularly interesting: the networks where no agents disagree. These frames have all positive relations, and we name them *reconciled frames*. In a reconciled network, one of the polarized sets is empty, and we can group the agents into a single set of friends. Although this network will be balanced, it socially differs from other balanced networks.

An axiom for reconciliation for all signed frames is $\boxplus \perp$. While this is a trivial observation, it is more interesting to note that using the universal modality we can write $[A]_{\boxplus \perp}$ which holds at any agent in a network if and only if the network is reconciled. It follows that $[A]_{\boxplus \perp} \rightarrow [A]_{\mathbf{B}_H}$ is valid on all collectively connected signed hybrid frames.

We also introduce *line index of reconciliation* as a measure of reconciliation based on line index of imbalance in Section 7.3.

Definition 21 (Line Index of Reconciliation). *Let \mathcal{F} be the class of finite signed frames. The **line index of reconciliation** is the map $d_{LI_R} : \mathcal{F} \rightarrow \mathbb{R}$ such that $d_{LI_R}(\mathbb{F}) = \frac{|R^-|}{|R^+ \cup R^-| - |A|}$ where $\mathbb{F} = \langle A, R^+, R^- \rangle$.*

The intuitive idea behind line index of reconciliation is to measure how far a network is from reconciliation by counting how many edges need to change for the network to be reconciled, divided by the total number of edges. We subtract the number of agents from the denominator since we do not want to count the positive reflexive relations. Line index of reconciliation is based on line index of imbalance and therefore like the latter discriminates overlapping cycles, but not long cycles. In the earlier Fig. 5 we have $d_{LI_R}(\mathbb{F}) = \frac{3}{10}$ and $d_{LI}(\mathbb{F}) = \frac{3}{6} = \frac{1}{2}$. Comparing line index of reconciliation to line index of imbalance and weak imbalance, we see that $d_{LI_R}(\mathbb{F}) = d_{LI}(\mathbb{F}) = d_{LIW}(\mathbb{F})$, whereas $d_{LI_R}(\mathbb{F}') > d_{LI}(\mathbb{F}') > d_{LIW}(\mathbb{F}')$. The closest strongly, and weakly, polarized network to \mathbb{F} in terms of changing edges is exactly a reconciled network. For \mathbb{F}' , there is a strongly polarized network closer to \mathbb{F} than to a fully reconciled network in terms of changing edges, namely any network obtained by changing any one single edge in the cycle.

8 Discussion, Conclusions and Future Work

After introducing structural balance and group polarization through positive and negative relations logic known from the literature, we set out to expand this logical framework with several intentions in mind. We presented a number of additions to **PNL** to be able to define previously undefinable frame properties. By extending the language with the universal operator and dynamic modalities we introduced a dynamic characterization of the balance property. We also showed that an axiom for collective connectedness is secured by the universal modality or the difference operator, while non-overlapping can be characterized with an axiom by inclusion of nominals, the intersection modality or the difference operator. Then, we extended **PNL** into a hybrid framework **PNL_i**. We proved soundness and completeness of an axiomatic system over the hybrid language with respect to balanced signed hybrid frames. Furthermore, we added some local dynamic modalities to reason about change in collectively connected networks. Finally, we considered a variation of distances in relation to balance to explore measures of how far a network is from polarization, and from reconciliation exclusively.

Of other known hybrid logics used to reason about social networks, the most well-known is perhaps the “Facebook” logic, or Dynamic Epistemic Friendship Logic (DEFL) [38]. As in **PNL_{i[A]}**, formulas in DEFL are evaluated at agents in a social network modeled as a Kripke frame. In contrast to the logics presented in this paper, DEFL is epistemic, and the dynamic operators are inspired by General Dynamic Dynamic Logic. This dynamic framework is quite different in nature to our approach, and requires terms from Propositional Dynamic

Logic in its language. Another hybrid social network logic is the dynamic logic for diffusion in social networks found in [17], also here with Kripke models for social networks. In this framework, the dynamic transformations changes the valuation at specific agents, whereas the network structure itself remains static. In our case, it is the structure of the social network that is subject to change. The most similar hybrid logic for social networks to $\mathbf{PNL}_{i[A]}$ is probably Tied Epistemic Logic (TEL) introduced in [35]. This logic also uses dynamic operators from sabotage modal logic, but is in this context used on social networks of strong and weak ties, not related to balance or group polarization. Furthermore, the models of TEL are threshold models, in which relations between agents are specified by a quantitative similarity measure between agents. The most important difference of the extended \mathbf{PNL} these other logics of social networks is that \mathbf{PNL} is specifically constructed to reason about social networks of positive and negative relations. The only other known logic to analyze social balance theory is the Logic of Allies and Enemies (LAE) presented in [29]. LAE is not a hybrid logic, but expressive enough to refer directly to specific relations in the network. Additionally, the framework is temporal, and the dynamics of relation change is modeled with temporal operators. As in the extended \mathbf{PNL} , LAE is expressive enough to express that a network is balanced, with the use of a universal quantifier. However, there is no known axiomatization of LAE and the framework generally differs quite a lot from \mathbf{PNL} .

Regarding expressivity of the extended \mathbf{PNL} , we note that both the local and the global addition modalities come at a considerable computational cost, rendering the model checking problem PSPACE-hard. This comes at no surprise; we have a similar situation in other dynamic logics with quantification. For example, both Arbitrary Public Announcement Logic (APAL) [8] and Group Announcement Logic [2] are PSPACE-complete as shown in [2]. To characterize balance using these modalities we also needed the iteration modalities and the universal modality.

One obvious direction for future work is to prove completeness for the different logics with the axioms we have identified. This is not trivial. First, as far as we know there is no complete axiomatization of sabotage modal logic with the bridge modality. Second, as we already pointed out, if we include the global iteration modality together with the global addition modalities, the logic becomes non-compact. As a consequence, the standard canonical model method cannot be used (the standard truth lemma does not hold), but finitary methods, such as using an appropriate notion of closure like in completeness proofs for PDL, can possibly be used.

For the hybrid $\mathbf{PNL}_{i[A]}$, completeness with respect to the class of collectively connected balanced signed hybrid frames is not trivial either. In the standard \mathbf{PNL} , collective connectedness has been proven modally undefinable [43]. However, we have shown that with the universal modality, the axiom \mathbf{C} characterizes collective connectedness. There is therefore reason to believe that we could axiomatize the subclass of collectively connected signed frames in a language like

$\mathcal{L}_{PNL_{i[A]}}$ that includes the universal modality. But, as proved in [16], connectedness cannot be defined with a pure formula, and thus we cannot use a reasoning relying on the results in [11]. And so it remains to see whether we can get a completeness result for this subclass of frames.

Yet another direction for future work is to look at possibilities for reasoning about the balance measures in the object language. This would be challenging, among other things because the measures are not invariant under bisimulation (the latter point could be alleviated by requiring all agents to be named by some nominal⁹).

On a final note, in light of logical analyses of social concepts it could be interesting to give certain attributes to agents in the network. One alternative is knowledge, as in social epistemic frameworks such as [9,10,33,38]: could a polarized setting change depending on what agents know about their social situations? Another is communication: by implementing an information flow in the network we could analyze which agents are likely to become friends based on information and information access, e.g., by adopting a dynamic epistemic logic approach including public and private announcements [19].

Acknowledgments

The authors would like to thank the anonymous reviewers for their sharp eyes and helpful suggestions. This includes the reviewers of Third International Conference on Logic and Argumentation (CLAR 2020), who gave useful feedback on the conference proceedings paper that this extended version is based upon. Thank you also to the participants of the Logic and Interactive Rationality seminar hosted at the ILLC, University of Amsterdam in March 2020 for an engaging discussion that led to improvements of this work.

References

1. Ågotnes, T., van der Hoek, W., Wooldridge, M.: Conservative Social Laws. In: Proceedings of the 20th European Conference on Artificial Intelligence, pp.49–54. IOS Press (2012)
2. Ågotnes, T., Balbiani, P., van Ditmarsch, H., Seban, P.: Group announcement logic. *Journal of Applied Logic* **8**(1), 62–81 (2010)
3. Areces, C., ten Cate, B.: Hybrid Logics. In: van Benthem, J., Blackburn, P. and Wolter, F. (eds.): *Handbook of Modal Logic*, pp.821–868. Elsevier, Amsterdam (2006)
4. Areces, C., Blackburn, P., Marx, M.: Hybrid logics: Characterization, interpolation and complexity. *Journal of Symbolic Logic* **66**(3), 977–1010 (2001)
5. Areces, C., Fervari, R., Hoffmann, G.: Relation-changing modal operators. *Logic Journal of the IGPL* **23**(4), 601–627 (2015)
6. Aucher, G.: Generalizing AGM to a multi-agent setting. *Logic Journal of the IGPL* **18**(4), 530–558 (2010)

⁹ As pointed out by one of the reviewers.

7. Aucher, G., van Benthem, J., Grossi, D.: Modal Logics of Sabotage Revisited. *Journal of Logic and Computation* **28**(2), 269–303 (2017)
8. Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., De Lima, T.: ‘Knowable’ as ‘known after an announcement’. *Review of Symbolic Logic* **1**(3), 305–334 (2008)
9. Baltag, A., Christoff, Z., Hansen, J.U., Smets, S.: Logical Models of Informational Cascades. *Studies in Logic* 47, 405–432 (2013)
10. Baltag, A., Christoff, Z., Rendsvig, R.K., Smets, S.: Dynamic epistemic logics of diffusion and prediction in social networks. *Studia Logica* **107**(3), 1–43 (2018)
11. Blackburn, P., ten Cate, B.: Pure extensions, proof rules, and hybrid axiomatics. *Studia Logica* **84**(2), 277–322 (2006)
12. Blackburn, P., de Rijke, M., Venema, Y.: *Modal Logic*. Cambridge University Pr. (2001)
13. Bramson, A. et al.: Understanding polarization: Meanings, measures, and model evaluation. *Philosophy of Science* **84**(1), 115–159 (2017)
14. Caridroit, T., Konieczny, S., de Lima, T., Marquis, P.: On Distances Between KD45n Kripke Models and Their Use for Belief Revision. In: *Proc. of ECAI*, pp.1053–1061. IOS Press (2016)
15. Cartwright, D., Harary, F.: Structural Balance: A Generalization of Heider’s Theory. *Psychological Review* **63**(5), 277–293 (1956)
16. ten Cate, B., Litak, T.: Topological Perspective on the Hybrid Proof Rules. *Electronic Notes in Theoretical Computer Science* **174**(6), 79–94 (2007)
17. Christoff, Z., Hansen, J. U.: A logic for diffusion in social networks. *Journal of Applied Logic*, **13**(1), 48–77 (2015)
18. Davis, J.A.: Clustering and Structural Balance in Graphs. *Human relations* **20**(2), 181–187 (1967)
19. van Ditmarsch, H., van der Hoek, W., Kooi, B.: *Dynamic Epistemic Logic*, Springer, (2007)
20. Doreian, P., Mrvar, A.: A Partitioning Approach to Structural Balance. *Social networks* **18**(2), 149–168 (1996)
21. Easley, D., Kleinberg, J.: *Networks, Crowds and Markets*. Cambridge University Pr. (2010)
22. Estrada, E., Benzi, M.: Walk-based measure of balance in signed networks: Detecting lack of balance in social networks. *Physical Review E* **90**(4), 042802. (2014)
23. Fagin, R., Halpern, J. Y., Moses, Y., Vardi, M. Y.: *Reasoning About Knowledge*. The MIT Press (1995)
24. Granovetter, M.S.: The Strength of Weak Ties. *Social Networks*. Academic Press, 347–367 (1977)
25. Harary, F.: On the Notion of Balance of a Signed Graph. *Michigan Mathematical Journal* **2**(2), 143–146 (1953)
26. Harary, F.: On the Measurement of Structural Balance. *Behavioral Sc.* **4**(4), 316–323 (1959)
27. Heider, F.: Attitudes and Cognitive Organization. *Journal of Psychology* 21, 107–112 (1946)
28. Hendricks, V. F., Hansen, P. G.: *Infostorms*. Springer, Switzerland (2016)
29. van der Hoek, W., Kuijjer, L., Wang Y.: Logics of Allies and Enemies: A Formal Approach to the Dynamics of Social Balance Theory. In: *Proc. of IJCAI 2020*, vol. 2021, pp.210–216 (2020)
30. Kirkley, A., Cantwell, G.T., Newman, M.E.J.: Balance in signed networks. *Physical Review E* **99**(1), 012320 (2019)

31. Klein, D., Rendsvig, R. K.: Convergence, continuity, recurrence and Turing completeness in dynamic epistemic logic. *Journal of Logic and Computation* **30**(6), 1213–1238 (2020)
32. Leskovec, J., Huttenlocker, D., Kleinberg, J.: Signed Networks in Social Media. In: Proc. of the SIGCHI conference on human factors in computing systems, pp. 1361–1370 (2010)
33. Liberman, A. O., Rendsvig, R. K.: Dynamic Term-Modal Logic for Epistemic Social Network Dynamics. In: Proc. of LORI 2019. LNCS, vol. 11813, pp.168–182. Springer (2019)
34. Pedersen, M.Y.: Polarization and Echo Chambers: A logical analysis of balance and triadic closure in social networks. Master of Logic Thesis Series, MoL-2019-10, <https://eprints.illc.uva.nl/1700/> (2019)
35. Pedersen, M.Y., Smets, S., Ågotnes, T.: Analyzing Echo Chambers: A logic of strong and weak ties. In: Proc. of LORI 2019. LNCS, vol. 11813, pp.183–198. Springer (2019)
36. Pedersen M.Y., Smets S., Ågotnes T.: Further Steps Towards a Logic of Polarization in Social Networks. In: Dastani M., Dong H., van der Torre L. (eds) Logic and Argumentation. CLAR 2020. Lecture Notes in Computer Science, vol. 12061, pp.324–345. Springer, Cham (2020)
37. Roelofsen, F.: Distributed Knowledge. *J. of Applied Non-Classical Logics* **17**(2), 255–273 (2007)
38. Seligman, J., Liu, F., Girard, P.: Facebook and the epistemic logic of friendship. In Proc. of the conference on Theoretical Aspects of Rationality and Knowledge, pp.229–238 (2013)
39. Sunstein, C.R.: The Law of Group Polarization. *The J. of Pol. Phil.* **10**(2), 175–195 (2002)
40. Sunstein, C.R.: Group Polarization and 12 Angry Men. *Negotiation J.* **23**(4), 443–447 (2007)
41. Xia, W., Cao, M., Johansson, K.H.: Structural balance and opinion separation in trust–mistrust social networks. *IEEE Trans. on Control of Network Systems* **3**(1), 46–56 (2015)
42. Xiong, Z.: On the Logic of Multicast Messaging and Balance in Social Networks. Doctoral dissertation, University of Bergen (2017)
43. Xiong, Z., Ågotnes, T.: On the Logic of Balance in Social Networks. *Journal of Logic, Language and Information*, 1–23 (2019)