Mathematical Programming (2021) 189:339–377
https://doi.org/10.1007/s10107-020-01553-7

FULL LENGTH PAPER

Series B

Strongly stable C-stationary points for mathematical programs with complementarity constraints

Daniel Hernández Escobar[1] · Jan-J. Rückmann[1]

Received: 7 November 2019 / Accepted: 11 August 2020 / Published online: 19 August 2020

Abstract
In this paper we consider the class of mathematical programs with complementarity constraints (MPCC). Under an appropriate constraint qualification of Mangasarian–Fromovitz type we present a topological and an equivalent algebraic characterization of a strongly stable C-stationary point for MPCC. Strong stability refers to the local uniqueness, existence and continuous dependence of a solution for each sufficiently small perturbed problem where perturbations up to second order are allowed. This concept of strong stability was originally introduced by Kojima for standard nonlinear optimization; here, its generalization to MPCC demands a sophisticated technique which takes the disjunctive properties of the solution set of MPCC into account.

Keywords MPCC · Strong stability · C-stationary point · Parametric optimization · Algebraic characterization · C-Mangasarian–Fromovitz constraint qualification · Basic Lagrange vector

Mathematics Subject Classification. 90C33 · 90C31 · 49K40 · 65K10

1 Introduction

We consider the class of mathematical programs with complementarity constraints (MPCC) given as

$$\mathcal{P}^{cc}(f, r, s): \min_{x \in M[r,s]} f(x) \tag{1.1}$$

with

$$M[r, s] = \left\{x \in \mathbb{R}^n : \min\{r_m(x), s_m(x)\} = 0, \ m \in L\right\}$$

✉ Jan-J. Rückmann
jru061@uib.no

1 Department of Informatics, University of Bergen, Postbox 7803, 5020 Bergen, Norway

where $L = \{1, \ldots, l\}$, $l \in \mathbb{N}$ and the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $r_m, s_m : \mathbb{R}^n \to \mathbb{R}$, $m \in L$, are assumed to be twice continuously differentiable.

The new results in this paper are mainly related to complementarity constraints and can be easily extended to programs with additional finitely many equality or inequality constraints. Note that there is a huge variety of applications for MPCC, see e.g. [24,34].

The goal of this paper is to present necessary and sufficient conditions for the strong stability of a C-stationary point for MPCC. The concept of strong stability was introduced by Kojima [22] for standard nonlinear optimization programs and it refers to the local existence and uniqueness of a stationary point for each sufficiently small perturbed problem. There, the values of a perturbation and its derivatives up to second order are taken into consideration, but do not necessarily depend on real parameters. In particular, results on strong stability can be immediately applied whenever only sufficiently small linear and quadratic perturbations are allowed. Several results related to strong stability have been established, we refer e.g. to [5,9,15,21,31,32]. First results for a generalization to MPCC are given in [33] and [18] where the latter presents a characterization of strong stability of a C-stationary point under MPCC-LICQ.

There are several stationary concepts for MPCC and many related references, e.g. [6,7,19,28,35,36]. Note that e.g. M- and S-stationarity are stronger concepts concerning optimality conditions; in particular, C-stationarity does not exclude trivial first-order descend directions.

However, C-stationarity is related to certain geometric properties which are described by the so-called Morse-relations [13] and which refer to the topological changes of the feasible level set when the level varies. For sensitivity analysis and solution (homotopy) methods [30], it is important to know where and whether topological changes may arise. Such changes could be that a new connected component is born, or two connected components merge or, in general, the geometric shape (sphere, torus...) of the feasible level set changes. This geometric shape is in particular relevant for the possible number of different local minimizers. Moreover, one is interested in conditions under which these topological changes remain unchanged after small perturbations (stability). The classical (unconstrained) Morse Theory [25] and its generalizations to standard nonlinear optimization [13] show that such changes happen if and only if a level containing a stationary point is passed. Otherwise the feasible level sets remain homeomorphic (topologically identical). A corresponding result for MPCC was presented in [16]: here, topological changes happen exactly at levels that contain a *C-stationary point*. Therefore, strong stability of a C-stationary point refer to stability of geometric properties of MPCC which are important for sensitivity analysis and design of solutions methods. As a consequence, we might miss some of this topological changes if we consider strong stability (only) for M- or S-stationary points. On the other hand, when concerning optimality conditions, it remains an open and interesting question how to establish strong stability for M- or S-stationarity.

The adaptation to MPCC of Kojima's topological definition of strong stability is straightforward; the challenge is to find an algebraic characterization which is equivalent to this topological definition. Thus, the goal of this paper is to present such an algebraic characterization of a strongly stable C-stationary point $\bar{x}$ of MPCC where we assume that:

- MPCC-LICQ does not hold at $\bar{x}$.
- A constraint qualification of Mangasarian–Fromovitz type holds at $\bar{x}$.

As we will see, the disjunctive structure of MPCC implies the use of algebraic techniques which are different to those used in the standard nonlinear case. Moreover, we refer to our previous paper [11] where we characterized strong stability of C-stationary points for the particular case with $n+1$ active constraints; some of the results from that paper will be used here. We also refer to some related papers. Other stability results are established [2,4] (Lipschitz properties) and in [26,29] (Tilt stability); solutions methods are discussed e.g. in [8,12,20,23,30].

This paper is organized as follows. Section 2 contains some auxiliary results and notations. Section 3 summarizes some known results from standard nonlinear optimization and MPCC which are needed later. In Sect. 4 we introduce the crucial notation of a *basic Lagrange vector*. In Sect. 5 a necessary second order condition (Condition C^*) for the strong stability of a C-stationary point for MPCC is shown; moreover, several properties are proved in a series of preliminary lemmas. Section 6 contains the main results. Under two appropriate assumptions (A1 and A2), equivalent algebraic characterizations for the strong stability of a C-stationary point are presented. Finally, Sect. 7 delivers some final remarks.

2 Preliminary notations and results

In this section we describe some basic notations which will be used later. Main parts of this description are taken from our previous paper [11, Sect. 2]. For $p \in \mathbb{N}$ and $w \in \mathbb{R}^p$ define

$$I^0(w) = \{i \in \{1, \dots, p\} : w_i = 0\},$$
$$I^*(w) = \{i \in \{1, \dots, p\} : w_i \neq 0\}.$$

If $E \subset \mathbb{R}^n$ is a linear subspace and A is an $n \times n$ symmetric matrix, then A is called *positive definite on* E if

$$v^T A v > 0$$

for all $v \in E \backslash \{0\}$, which is denoted by $A|_E \succ 0$. When $E = \mathbb{R}^n$, we simply write $A \succ 0$.

Let $\bar{x} \in \mathbb{R}^n$ and $\delta > 0$. The Euclidean norm of $\bar{x}$ will be denoted by $\|\bar{x}\|$, the closed Euclidean ball centered at $\bar{x}$ with radius δ by $B(\bar{x}, \delta)$ and the Euclidean sphere centered at $\bar{x}$ with radius δ by $S(\bar{x}, \delta)$. We abbreviate the sentence "V is a neighborhood of $\bar{x}$" by letting $\mathcal{V}(\bar{x})$ to be the *set of all neighborhoods* of $\bar{x}$. This allows us to write the aforementioned sentence as "$V \in \mathcal{V}(\bar{x})$".

Let $\mathcal{C}^k(A^1, A^2)$ be the space of $k-$times continuously differentiable mappings with domain $A^1 \subset \mathbb{R}^n$ and codomain $A^2 \subset \mathbb{R}^m$. Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $\bar{x} \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, with $\|v\| = 1$. As usual, $\frac{\partial f(\bar{x})}{\partial x_i}$ and $\frac{\partial f(\bar{x})}{\partial v}$, denote the partial derivative with respect to x_i and the directional derivative with respect to v, respectively, of the function f at $\bar{x}$.

In addition, $D_x f(\bar{x})$ stands for its gradient taken as a row vector and $D_x^2 f(\bar{x})$ for its Hessian.

Moreover, for $\Omega \subset \mathbb{R}^n$ let bd Ω denote the boundary of Ω and

$$\Omega^{\perp} = \{z \in \mathbb{R}^n : z^T x = 0, x \in \Omega\}$$

its orthogonal subspace. Furthermore, if Ω is convex, let

$$\text{ext}\,\Omega = \{x \in \Omega : x \notin \text{conv}[\Omega \backslash \{x\}]\}$$

denote the set of its extreme points.

For defining strong stability we need a seminorm for functions. Let $V \in \mathcal{V}(\bar{x})$ and $\bar{F} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$. Following [22], let

$$\|\bar{F}\|^V = \max\left\{\sup_{x\in V}\max_i\left\{|\bar{F}_i(x)|\right\}, \sup_{x\in V}\max_{i,j}\left\{\left|\frac{\partial \bar{F}_i(x)}{\partial x_j}\right|\right\}, \sup_{x\in V}\max_{i,j,k}\left\{\left|\frac{\partial^2 \bar{F}_i(x)}{\partial x_j \partial x_k}\right|\right\}\right\}, \tag{2.1}$$

where the indices i and j, k are varying in the sets $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. The sets of all neighborhoods of $\bar{F}$, with respect to this seminorm, is denoted by $\mathcal{U}^V(\bar{F})$.

Let $\bar{F}: \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function. Let $\partial_x^C \bar{F}(\bar{x})$ denote the Clarke subdifferential of $\bar{F}$ at $\bar{x}$, see [3, Definition 2.6.1] (there, it is actually called *generalized Jacobian*). If $\bar{F}(\bar{x}) = 0$ and all elements of $\partial_x^C \bar{F}(\bar{x})$ are nonsingular, then we have the following result for the zeros of sufficiently small perturbations of $\bar{F}$.

Theorem 2.1 *Assume that $\bar{F}(\bar{x}) = 0$ and that $\partial_x^C \bar{F}(\bar{x})$ is nonsingular. Then, the following condition hold:*

(1) *There exist $V \in \mathcal{V}(\bar{x})$ and $U \in \mathcal{U}^V(\bar{F})$ such that for all $F \in U$ the set V contains exactly one solution to the equation $F(x) = 0$, which we denote by $\check{x}(F)$.*
(2) *The mapping $\check{x}: U \to V$, $F \mapsto \check{x}(F)$ is continuous.*

Proof It is a straightforward adaptation of [14, Theorem 2.1 (Implicit Function Theorem)]. □

The previous theorem is similar to upcoming Theorems 3.1 and 3.2 which present characterizations for strong stability of stationary and C-stationary points, respectively. We end this section by presenting a property of the Clarke subdifferential of min-type functions.

Lemma 2.1 *Let $\bar{F}^1, \bar{F}^2: \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable functions with $\bar{F}^1(\bar{x}) = \bar{F}^2(\bar{x}) = 0$. If*

$$\bar{F}(x) = \left(\min\{\bar{F}_i^1(x), \bar{F}_i^2(x)\}\right)_{i=1,\ldots,n},$$

then

$$\partial_x^C \bar{F}(\bar{x}) \subset \bigtimes_{i=1}^{n} \mathrm{conv}\{D\bar{F}_i^1(\bar{x}), D\bar{F}_i^2(\bar{x})\}$$

where the latter denotes the set of all (n, n)*-matrices whose ith row belongs to the set* $\mathrm{conv}\{D\bar{F}_i^1(\bar{x}), D\bar{F}_i^2(\bar{x})\}$.

Proof It is a straightforward consequence of Propositions 2.3.1, 2.3.12 and 2.6.2 in [3]. □

3 Auxiliary results for standard nonlinear programs and for MPCC

In this section we present some auxiliary results and definitions that are mainly taken from [11, Sects. 3 and 4]. The exception is the forthcoming Lemma 3.2 which, to our knowledge, is new although its proof is essentially an adaptation of the proof of [22, Theorem 7.2]. Let $P = \mathcal{P}^{\mathrm{sn}}(f, h, g)$ denote the standard nonlinear program

$$\begin{aligned} &\min\ f(x) \\ &\text{s. t. } x \in M^{\mathrm{sn}}[h, g] = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} h_i(x) = 0,\ i \in I, \\ g_j(x) \geq 0,\ j \in J \end{array} \right\} \end{aligned}$$

where the index sets I and J are finite, $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $h_i \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $i \in I$ and $g_j \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $j \in J$. We say that two nonlinear programs P^1 and P^2 are equal ($P^1 = P^2$) if they are defined by the same functions $f, h_i, i \in I, g_j, j \in J$. For $\bar{x} \in M^{\mathrm{sn}}[h, g]$ define

$$J_g^0(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\}.$$

A point $\bar{x} \in \mathbb{R}^n$ is called a *stationary point* for $P = \mathcal{P}^{\mathrm{sn}}(f, h, g)$ if there exist $\lambda_i \in \mathbb{R}$, $i \in I$ and $\mu_j \in \mathbb{R}$, $j \in J$ such that

$$D_x \mathbf{L}^{\mathrm{sn}}(\bar{x}, \lambda, \mu) = 0,\ \ h_i(\bar{x}) = 0,\ \ i \in I,\ \ \min\{\mu_j, g_j(\bar{x})\} = 0,\ \ j \in J \tag{3.1}$$

where

$$\mathbf{L}^{\mathrm{sn}}(x, \lambda, \mu) = f(x) - \sum_{i \in I} \lambda_i h_i(x) - \sum_{j \in J} \mu_j g_j(x)$$

is the *Lagrange function* for P. The set of stationary points for P is denoted by $\Sigma(P)$. The set of (λ, μ) such that (3.1) holds is denoted by $\mathcal{L}(P, \bar{x})$.

It is well-known that the following constraint qualifications relate local minimizers to stationary points:

- The *Linear Independence constraint qualification (LICQ)* holds at $\bar{x} \in M^{\mathrm{sn}}[h, g]$ if the vectors

$$D_x h_i(\bar{x}),\ i \in I,\ D_x g_j(\bar{x}),\ j \in J_g^0(\bar{x}),$$

  are linearly independent.
- The *Mangasarian–Fromovitz constraint qualification (MFCQ)* holds at $\bar{x} \in M^{\mathrm{sn}}[h, g]$ if the vectors

$$D_x h_i(\bar{x}),\ i \in I$$

  are linearly independent and there exists $v \in \mathbb{R}^n$ such that

$$D_x h_i(\bar{x})v = 0,\ i \in I,\ D_x g_j(\bar{x})v > 0,\ j \in J_g^0(\bar{x}).$$

It is well known that LICQ implies MFCQ and that if MFCQ holds at a local minimizer $\bar{x}$ for P, then $\bar{x}$ is a stationary point for P.

Since we deal with stationary points under sufficiently small perturbations, we recall the concept of a strongly stable stationary point introduced by Kojima in [22]. For this we need a seminorm for functions. Given $V \in \mathcal{V}(\bar{x})$ and $P = \mathcal{P}^{\mathrm{sn}}(f, h, g)$, we define

$$\|P\|^V = \|(f, h, g)\|^V,$$

where $\|(f, h, g)\|^V$ is obtained by taking $\bar{F} = (f, h, g)$ in (2.1). Let $\bar{P} = \mathcal{P}^{\mathrm{sn}}(\bar{f}, \bar{h}, \bar{g})$ and $\delta > 0$ be fixed and

$$B^V(\bar{P}, \delta) = \{P : \|P - \bar{P}\|^V \leq \delta\}$$

where P and $\bar{P}$ have the same number of equality and inequality constraints; the set of all neighborhoods of $\bar{P}$ is denoted by $\mathcal{W}^V(\bar{P})$. Now, we recall Kojima's [22] definition of a strongly stable stationary point and a convenient characterization of it.

Definition 3.1 [22] Let $\bar{P} = \mathcal{P}^{\mathrm{sn}}(\bar{f}, \bar{h}, \bar{g})$. A point $\bar{x} \in \Sigma(\bar{P})$ is called *strongly stable* if there exists $\bar{\delta} > 0$ such that for all $\delta \in (0, \bar{\delta}]$ there exists $\varepsilon > 0$ such that for every $P \in B^{B(\bar{x}, \bar{\delta})}(\bar{P}, \varepsilon)$ it holds that

$$|\Sigma(P) \cap B(\bar{x}, \bar{\delta})| = |\Sigma(P) \cap B(\bar{x}, \delta)| = 1,$$

The set of strongly stable stationary points for $\bar{P}$ is denoted by $\Sigma^S(\bar{P})$.

Theorem 3.1 [9] *Let $\bar{P} = \mathcal{P}^{\mathrm{sn}}(\bar{f}, \bar{h}, \bar{g})$. The point $\bar{x} \in \Sigma(\bar{P})$ is strongly stable if and only if the following conditions hold:*

(1) *There exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that for all $P \in W$ the set $\Sigma(P) \cap V$ contains exactly one element, which we denote by $\hat{x}(P)$.*
(2) *The mapping $\hat{x} \colon W \to V,\ P \mapsto \hat{x}(P)$ is continuous.*

Under MFCQ, the set $\mathcal{L}(\bar{P}, \bar{x})$ remains contained in a certain compact set after any sufficiently small perturbations (see e.g. [22, Lemma 7.4]). Moreover, it holds that $(\bar{\lambda}, \bar{\mu}) \in \operatorname{ext} \mathcal{L}(\bar{P}, \bar{x})$ if and only if the gradients

$$D_x \bar{h}_i(\bar{x}),\ i \in I,\ D_x \bar{g}_j(\bar{x}),\ j \in I^*(\bar{\mu})$$

are linearly independent. By the latter fact together with a continuity argument, the next result readily follows.

Lemma 3.1 *Assume that MFCQ holds at $\bar{x}$. Then, there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that for all $P \in W$, $x \in V \cap \Sigma(P)$ and all $(\lambda, \mu) \in \operatorname{ext} \mathcal{L}(P, x)$ there exists $(\bar{\lambda}, \bar{\mu}) \in \operatorname{ext} \mathcal{L}(\bar{P}, \bar{x})$ such that*

$$I^*(\bar{\lambda}) \subset I^*(\lambda),\ I^*(\bar{\mu}) \subset I^*(\mu).$$

In addition, for (λ, μ) and $(\bar{\lambda}, \bar{\mu})$ it holds that $\lambda_i \cdot \bar{\lambda}_i > 0, i \in I^*(\bar{\lambda})$.

In the remainder of this section, we assume that the vectors $D_x \bar{h}_i(\bar{x})$, $i \in I$ are linearly independent. By Carathéodory's theorem, the latter ensures that $\operatorname{ext} \mathcal{L}(\bar{P}, \bar{x}) \neq \emptyset$ whenever $\bar{x} \in \Sigma(\bar{P})$. For $\bar{x} \in \Sigma(\bar{P})$ and $(\bar{\lambda}, \bar{\mu}) \in \mathcal{L}(\bar{P}, \bar{x})$ let

$$T_{\bar{x}}(\bar{h}, \bar{g}, \bar{\lambda}, \bar{\mu}) = \{v \in \mathbb{R}^n : D_x \bar{h}_i(\bar{x})v = 0, i \in I, D_x \bar{g}_j(\bar{x})v = 0, j \in I^*(\bar{\mu})\}.$$

The next lemma relates a second order condition to the existence of two stationary points near $\bar{x}$ after a sufficiently small perturbation of $\bar{P}$.

Lemma 3.2 *Assume that LICQ does not hold at $\bar{x} \in \Sigma(\bar{P})$. If for some $(\bar{\lambda}, \bar{\mu}) \in \operatorname{ext} \mathcal{L}(\bar{P}, \bar{x})$ the condition*

$$D_x^2 \bar{\mathbf{L}}^{\text{sn}}(\bar{x}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}}(\bar{h}, \bar{g}, \bar{\lambda}, \bar{\mu})} \succ 0 \tag{3.2}$$

does not hold, then there exist sequences $P^k \to P$, $x^{1,k}, x^{2,k} \to \bar{x}$ with $x^{1,k} \neq x^{2,k}$ and $x^{1,k}, x^{2,k} \in \Sigma(P^k)$ such that LICQ holds at $x^{1,k}, x^{2,k}$ and that if

$$\mathcal{L}(P^k, x^{1,k}) = \{(\lambda^{1,k}, \mu^{1,k})\},\ \mathcal{L}(P^k, x^{2,k}) = \{(\lambda^{2,k}, \mu^{2,k})\},$$

then

$$(\lambda^{1,k}, \mu^{1,k}) \to (\bar{\lambda}, \bar{\mu}),\ (\lambda^{2,k}, \mu^{2,k}) \to (\bar{\lambda}, \bar{\mu}).$$

Proof The main idea of the proof is given in the "only if" part of [22, Theorem 7.2]. There, the condition MFCQ is only needed to ensure that $\mathcal{L}(\bar{P}, \bar{x})$ is bounded and to express its elements as a convex combination of its extreme points. Afterwards, a vector $(\bar{\lambda}, \bar{\mu}) \in \operatorname{ext} \mathcal{L}(\bar{P}, \bar{x})$ is fixed and $\bar{P}$ perturbed sufficiently small in such ways that LICQ holds at $\bar{x} \in \Sigma(\bar{P}) \backslash \Sigma^S(\bar{P})$ and $\mathcal{L}(\bar{P}, \bar{x}) = \{(\bar{\lambda}, \bar{\mu})\}$, whenever (3.2) does not hold. Thus, by applying [22, Theorem 4.2], the desired result follows. □

The novelty of the latter result consists in its independence from the condition MFCQ. As we already mentioned, this condition is necessary for strong stability. However, in our MPCC setting it is worth studying auxiliary standard nonlinear programs whose stationary points do not fulfill MFCQ.

In the remainder of this section we turn our attention to MPCC and recall now that $P = \mathcal{P}^{\mathrm{cc}}(f, r, s)$ is a problem with the objective function f and the feasible set $M[r, s]$ as given in (1.1) where $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $r_m, s_m \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $m \in L$. Analogously to the standard nonlinear program, we say that two MPCCs are equal if they are defined by the same functions (f, r, s). Moreover, for $\bar{x} \in M[r, s]$ we define the *active index sets*:

$$\begin{aligned} I_r(\bar{x}) &= \{i \in L : r_i(\bar{x}) = 0, s_i(\bar{x}) > 0\}, \\ I_s(\bar{x}) &= \left\{j \in L : r_j(\bar{x}) > 0, s_j(\bar{x}) = 0\right\}, \\ I_{rs}(\bar{x}) &= \{m \in L : r_m(\bar{x}) = 0, s_m(\bar{x}) = 0\}, \\ \bar{I}_r(\bar{x}) &= \{i \in L : r_i(\bar{x}) = 0\}, \\ \bar{I}_s(\bar{x}) &= \left\{j \in L : s_j(\bar{x}) = 0\right\}. \end{aligned}$$

Concerning the number of active constraints for P at $\bar{x} \in M[r, s]$ define

$$N^0(P, \bar{x}) = |\bar{I}_r(\bar{x})| + |\bar{I}_s(\bar{x})|$$

and

$$\hat{N}(P, \bar{x}) = N^0(P, \bar{x}) - \dim \operatorname{span}\left\{Dr_i(\bar{x}), i \in \bar{I}_r(\bar{x}), Ds_j(\bar{x}), j \in \bar{I}_s(\bar{x})\right\}.$$

Remark 3.1 To simplify notation, we use the same letters that were used for defining sets for standard nonlinear programs, now for defining analogous sets for MPCCs. From now on, we assume that $P = \mathcal{P}^{\mathrm{cc}}(f, r, s)$ and $\bar{P} = \mathcal{P}^{\mathrm{cc}}(\bar{f}, \bar{r}, \bar{s})$ are two MPCCs with the same number of complementarity constraint. In addition, we use auxiliary standard nonlinear programs that we denote by the superscript "aux", for instance P^{aux}, $\bar{P}^{\mathrm{aux}}$, $P^{\mathrm{aux},1}$, etc.

Definition 3.2 A point $\bar{x} \in M[r, s]$ is called an *FJC point* for P if there exist $\mu_0 \in \mathbb{R}$, $\rho \in \mathbb{R}^l$ and $\sigma \in \mathbb{R}^l$, not all of them being zero, such that

$$D_x \mathbf{L}^{\mathrm{cc}}(\bar{x}, \mu_0, \rho, \sigma) = 0, \tag{3.3}$$

$$\rho_m \cdot r_m(\bar{x}) = \sigma_m \cdot s_m(\bar{x}) = 0, \;\; \rho_m \cdot \sigma_m \geq 0, \;\; m \in L \tag{3.4}$$

where

$$\mathbf{L}^{\mathrm{cc}}(x, \mu_0, \rho, \sigma) = \mu_0 f(x) - \sum_{m \in L} [\rho_m r_m(x) + \sigma_m s_m(x)]$$

is the *MPCC-Lagrange function* for P. The set of FJC points for P is denoted by $\Sigma^F(P)$.

The motivation for defining FJC points comes from the fact that for a local minimizer $\bar{x}$ for P it holds that $\bar{x} \in \Sigma^F(P)$, see [33, Lemma 1]. Now, we recall the definitions of C-MFCQ and C-stationarity. Note that C-MFCQ is called MFC in [10,11,17,18,34].

Definition 3.3 We say that *C-MFCQ* holds at $\bar{x} \in M[r, s]$ if the vectors

$$D_x r_i(\bar{x}), i \in I_r(\bar{x}), D_x s_j(\bar{x}), j \in I_s(\bar{x}), \lambda_m D_x r_m(\bar{x}) + (1 - \lambda_m) D_x s_m(\bar{x}), m \in I_{rs}(\bar{x}),$$

are linearly independent for any choice of $\lambda_m \in [0, 1], m \in I_{rs}(\bar{x})$.

Definition 3.4 The set of all $(\rho, \sigma) \in \mathbb{R}^{2l}$ with (3.3), (3.4) and $\mu_0 = 1$ is denoted by $\mathcal{L}(P, \bar{x})$ and is called the *set of Lagrange vectors* for P at $\bar{x}$. The point $\bar{x} \in \Sigma^F(P)$ is called a *C-stationary point* for P if $\mathcal{L}(P, \bar{x}) \neq \emptyset$. The set of C-stationary points for P is denoted by $\Sigma^C(P)$.

For sake of simplicity, we write $\mathbf{L}^{cc}(x, \rho, \sigma)$ when $\mu_0 = 1$. If C-MFCQ holds at a local minimizer $\bar{x}$, then $\bar{x} \in \Sigma^C(P)$, see [17, Proposition 2.1]. The abbreviation FJC refers to Fritz John and C-stationarity. Furthermore, we recall that the *Linear Independence constraint qualification for MPCC (MPCC-LICQ)* holds at $\bar{x} \in M[r, s]$ if the vectors

$$D_x r_i(\bar{x}), i \in \bar{I}_r(\bar{x}), D_x s_j(\bar{x}), j \in \bar{I}_s(\bar{x}),$$

are linearly independent. Obviously, MPCC-LICQ implies C-MFCQ.

In order to present the concept of a strongly stable C-stationary point for MPCC we introduce a seminorm analogously as above. Given $V \in \mathcal{V}(\bar{x})$ and $P = \mathcal{P}^{cc}(f, r, s)$, we define

$$\|P\|^V = \|(f, r, s)\|^V, \tag{3.5}$$

where $\|(f, r, s)\|^V$ is obtained by taking $\bar{F} = (f, r, s)$ in (2.1). For $\delta > 0$ define

$$B^V(\bar{P}, \delta) = \{P : \|P - \bar{P}\|^V \leq \delta\}.$$

The set of all neighborhoods of $\bar{P}$ is denoted by $\mathcal{W}^V(\bar{P})$ and the set of neighborhoods of $(\bar{r}, \bar{s})$ by $\mathcal{U}^V(\bar{r}, \bar{s})$. Now, we present the definition of a strongly stable C-stationary point.

Definition 3.5 [18] A point $\bar{x} \in \Sigma^C(\bar{P})$ is called *strongly stable* if there exists $\bar{\delta} > 0$ such that for all $\delta \in (0, \bar{\delta}]$ there exists $\varepsilon > 0$ such that for every $P \in B^{B(\bar{x}, \bar{\delta})}(\bar{P}, \varepsilon)$ it holds that

$$|\Sigma^C(P) \cap B(\bar{x}, \bar{\delta})| = |\Sigma^C(P) \cap B(\bar{x}, \delta)| = 1,$$

The set of strongly stable C-stationary points for $\bar{P}$ is denoted by $\Sigma^S(\bar{P})$.

Furthermore, we have the following characterizations of the strong stability of a C-stationary point.

Theorem 3.2 [10, Lemma 2.5 and Theorem 4.5] *The point* $\bar{x} \in \Sigma^C(\bar{P})$ *is strongly stable if and only if the following condition hold:*

(1) *There exist* $V \in \mathcal{V}(\bar{x})$ *and* $W \in \mathcal{W}^V(\bar{P})$ *such that for all* $P \in W$ *the set* $\Sigma^C(P) \cap V$ *contains exactly one element, say* $\hat{x}(P)$.
(2) *The mapping* $\hat{x}\colon W \to V,\ P \mapsto \hat{x}(P)$ *is continuous.*

Corollary 3.1 [10, Corollary 4.6] *The point* $\bar{x} \in \Sigma^C(\bar{P})$ *is strongly stable if and only if there exist* $V \in \mathcal{V}(\bar{x})$ *and* $W \in \mathcal{W}^V(\bar{P})$ *such that*

$$\Sigma^F(P) \cap V = \Sigma^C(P) \cap V = \Sigma^S(P) \cap V = \{\hat{x}(P)\},$$

for all $P \in W$.

We terminate this section by presenting a brief discussion about the relationship between MFCQ, MPCC-MFCQ and C-MFCQ. Note that C-MFCQ appeared (probably) first in [17] in the context of topological stability of the feasible set of MPCC.

Lemma 3.3 [10, Lemmas 3.1 and 3.3] *Assume that* $\bar{x} \in M[\bar{r}, \bar{s}]$. *The following conditions are equivalent:*

(i) *C-MFCQ holds at* $\bar{x}$.
(ii) *There does not exist* $(\bar{\alpha}, \bar{\beta}) \in S(0, 1) \subset \mathbb{R}^{2l}$ *such that*

$$\sum_{m \in L} \left[\bar{\alpha}_m D_x \bar{r}_m(\bar{x}) + \bar{\beta}_m D_x \bar{s}_m(\bar{x})\right] = 0,$$
$$\bar{\alpha}_m \cdot \bar{r}_m(\bar{x}) = \bar{\beta}_m \cdot \bar{s}_m(\bar{x}) = 0,\ \bar{\alpha}_m \cdot \bar{\beta}_m \geq 0,\ m \in L.$$

(iii) *There exist* $V \in \mathcal{V}(\bar{x})$, $W \in \mathcal{W}^V(\bar{P})$ *and a compact set* $K^1 \subset \mathbb{R}^{2l}$ *such that*

$$\mathcal{L}(P, x) \subset K^1$$

for all $P \in W$ *and all* $x \in V \cap \Sigma^C(P)$.

Remark 3.2 Consider for a moment a standard nonlinear program as defined in the beginning of this section and a feasible point $\bar{x} \in M^{\mathrm{sn}}[\bar{h}, \bar{g}]$. Then, the following conditions are equivalent:

(a) MFCQ holds at $\bar{x}$.
(b) There does not exist $(\bar{\alpha}, \bar{\beta}) \in S(0, 1) \subset \mathbb{R}^{|I|+|J|}$ such that

$$\sum_{i \in I} \bar{\alpha}_i D_x \bar{h}_i(\bar{x}) + \sum_{j \in J} \bar{\beta}_j D_x \bar{g}_j(\bar{x}) = 0$$
$$\min\{\bar{\beta}_j, \bar{g}_j(\bar{x})\} = 0,\ j \in J.$$

(c) There exists a compact set $K^2 \subset \mathbb{R}^{|I|+|J|}$ which contains the set of Lagrange vectors for any sufficiently small perturbed problem and x near $\bar{x}$ [22, Lemma 7.4].

Note that (ii) in Lemma 3.3 and (b) in Remark 3.2 are dual formulations of C-MFCQ and MFCQ, respectively. Moreover, the properties (ii) and (iii) in Lemma 3.3 are analogous to (b) and (c) in Remark 3.2, respectively. That is the reason why C-MFCQ is called a constraint qualification of Mangasarian–Fromovitz-type.

Remark 3.3 Now, we consider an MPCC, which might have standard constraints, and MPCC-MFCQ [33]. If the problem under consideration has no standard constraints, then MPCC-LICQ and MPCC-MFCQ are equivalent. Analogously to (ii) in Lemma 3.3, one obtains the dual formulation of MPCC-LICQ which is obviously related to the so-called weak stationarity [33]. Roughly speaking, MPCC-MFCQ relates to weak stationarity in the same way as C-MFCQ relates to C-stationarity. Moreover, MPCC-MFCQ implies C-MFCQ. Since we deal with C-stationarity, C-MFCQ is the appropriate constraint qualification in the context of this paper.

4 Basic Lagrange vectors

In [22, Theorem 7.2], the concept of extreme points of a convex set plays an essential role. However, in our MPCC setting the set $\mathcal{L}(\bar{P}, \bar{x})$ is, in general, not convex and, therefore, this concept cannot be applied. In the following, we consider instead the concept of a *basic Lagrange vector* which is crucial for necessary and sufficient conditions for strong stability. Throughout this section, we do not always assume that C-MFCQ holds at $\bar{x}$.

Definition 4.1 We say that $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$ is a *basic Lagrange vector* if there does not exist $(\bar{\rho}^0, \bar{\sigma}^0) \in \mathcal{L}(\bar{P}, \bar{x})$ with $(\bar{\rho}^0, \bar{\sigma}^0) \neq (\bar{\rho}, \bar{\sigma})$ and

$$I^*(\bar{\rho}^0) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\bar{\rho}), \;\; I^*(\bar{\sigma}^0) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\bar{\sigma}).$$

The set of basic Lagrange vectors is denoted by $\mathcal{L}^0(\bar{P}, \bar{x})$.

Lemma 4.1 *If $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$, then: $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ if and only if the vectors*

$$D_x\bar{r}_i(\bar{x}), i \in I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho}), \;\; D_x\bar{s}_j(\bar{x}), j \in I_{\bar{s}}(\bar{x}) \cup I^*(\bar{\sigma}) \tag{4.1}$$

are linearly independent.

Proof Obviously, the linear independence of the vectors in (4.1) implies $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. Now, assume $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ and suppose contrarily that for some nontrivial $\bar{\alpha}_i, i \in I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho})$ and $\bar{\beta}_j, j \in I_{\bar{s}}(\bar{x}) \cup I^*(\bar{\sigma})$ it holds that

$$0 = \sum_{i \in I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho})} \bar{\alpha}_i D_x\bar{r}_i(\bar{x}) + \sum_{j \in I_{\bar{s}}(\bar{x}) \cup I^*(\bar{\sigma})} \bar{\beta}_j D_x\bar{s}_j(\bar{x}).$$

After defining $\bar{\alpha}_i = 0,\ i \in L\backslash[I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho})]$ and $\bar{\beta}_j = 0,\ j \in L\backslash[I_{\bar{s}}(\bar{x}) \cup I^*(\bar{\sigma})]$, we have for $\varepsilon \in \mathbb{R}$ sufficiently small that $I^*(\bar{\rho} + \varepsilon\bar{\alpha}) = I^*(\bar{\rho})$, $I^*(\bar{\sigma} + \varepsilon\bar{\beta}) = I^*(\bar{\sigma})$ and that $(\bar{\rho}, \bar{\sigma}) + \varepsilon(\bar{\alpha}, \bar{\beta}) \in \mathcal{L}(\bar{P}, \bar{x})$ which contradicts $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. □

Note that a basic Lagrange vector is an extreme point (vertex) in case that $\mathcal{L}(\bar{P}, \bar{x})$ is a convex polyhedron. Furthermore, we refer again to [11] where we considered the particular case with $n + 1$ active constraints. There, the definition of a basic Lagrange vector becomes much simpler [11, Definition 5.4]. The latter is equivalent to Definition 4.1 whenever the assumptions in [11] are fulfilled. The next result states that $\mathcal{L}(\bar{P}, \bar{x})$ is the union of certain polyhedrons whose extreme points belong to $\mathcal{L}^0(\bar{P}, \bar{x})$.

Lemma 4.2 *For any $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ define the polyhedron*

$$\mathcal{L}(\bar{P}, \bar{x}, I) = \left\{ (\rho, \sigma) \in \mathbb{R}^{2l} \;\middle|\; \begin{array}{r} D_x \bar{\mathbf{L}}^{cc}(\bar{x}, \rho, \sigma) = 0, \\ \rho_m \cdot \bar{r}_m(\bar{x}) = \sigma_m \cdot \bar{s}_m(\bar{x}) = 0,\ m \in L, \\ \rho_m \geq 0, \sigma_m \geq 0,\ m \in I, \\ \rho_m \leq 0, \sigma_m \leq 0,\ m \in I_{\bar{r}\bar{s}}(\bar{x})\backslash I, \end{array} \right\}.$$

Then, the following holds

$$\mathcal{L}(\bar{P}, \bar{x}) = \bigcup_{I \subset I_{\bar{r}\bar{s}}(\bar{x})} \mathcal{L}(\bar{P}, \bar{x}, I),\ \ \mathcal{L}^0(\bar{P}, \bar{x}) = \bigcup_{I \subset I_{\bar{r}\bar{s}}(\bar{x})} \operatorname{ext} \mathcal{L}(\bar{P}, \bar{x}, I).$$

Proof The first equality follows from the definition of $\mathcal{L}(\bar{P}, \bar{x})$; the second one from the first one and by Lemma 4.1. □

Now, we provide a characterization of the existence of basic Lagrange vectors and a necessary condition for the strong stability of a C-stationary point.

Lemma 4.3 *If $\bar{x} \in \Sigma^C(\bar{P})$, then: $\mathcal{L}^0(\bar{P}, \bar{x}) \neq \emptyset$ if and only if the vectors*

$$D_x\bar{r}_i(\bar{x}), i \in I_{\bar{r}}(\bar{x}),\ \ D_x\bar{s}_j(\bar{x}), j \in I_{\bar{s}}(\bar{x}) \tag{4.2}$$

are linearly independent.

Proof By Lemma 4.1, if $\mathcal{L}^0(\bar{P}, \bar{x}) \neq \emptyset$, then the vectors in (4.2) are linearly independent. Now, assume the latter condition. By Lemma 4.2, for some $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ it holds that $\mathcal{L}(\bar{P}, \bar{x}, I) \neq \emptyset$. Since the vectors in (4.2) are linearly independent, application of [1, Proposition 3.3.1] to $\mathcal{L}(\bar{P}, \bar{x}, I)$ yields $\operatorname{ext} \mathcal{L}(\bar{P}, \bar{x}, I) \neq \emptyset$. By using Lemma 4.2 again, we obtain $\mathcal{L}^0(\bar{P}, \bar{x}) \neq \emptyset$. □

Corollary 4.1 *If $\bar{x} \in \Sigma^S(\bar{P})$, then $\mathcal{L}^0(\bar{P}, \bar{x}) \neq \emptyset$.*

Proof It is a straightforward consequence of [10, Theorem 5.5] and Lemma 4.3. □

In the following theorem C-MFCQ is assumed and Lemma 4.2 is strengthened. Moreover, a result analogous to Lemma 3.1 follows immediately.

Theorem 4.1 *Assume that C-MFCQ holds at $\bar{x} \in \Sigma^C(\bar{P})$. Then*

$$\mathcal{L}(\bar{P}, \bar{x}) = \bigcup_{I \subset I_{\bar{r}\bar{s}}(\bar{x})} \text{conv}\left[\mathcal{L}^0(\bar{P}, \bar{x}) \cap \mathcal{L}(\bar{P}, \bar{x}, I)\right].$$

Proof Since C-MFCQ holds at $\bar{x}$, the set $\mathcal{L}(\bar{P}, \bar{x})$ is bounded [10, Lemma 3.3]. Hence, by Lemma 4.2, for $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ the set $\mathcal{L}(\bar{P}, \bar{x}, I)$ is compact. By Krein-Milman theorem (see e.g. [1, p. 181]), we get

$$\mathcal{L}(\bar{P}, \bar{x}, I) = \text{conv}[\text{ext}\, \mathcal{L}(\bar{P}, \bar{x}, I)]. \tag{4.3}$$

Moreover, by Lemma 4.1, we obtain

$$\text{ext}\, \mathcal{L}(\bar{P}, \bar{x}, I) = \mathcal{L}^0(\bar{P}, \bar{x}) \cap \mathcal{L}(\bar{P}, \bar{x}, I). \tag{4.4}$$

By (4.3), (4.4) and Lemma 4.2 the desired result follows. □

Lemma 4.4 *Assume that C-MFCQ holds at $\bar{x} \in \Sigma^C(\bar{P})$. Then, there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that for all $P \in W$, $x \in V \cap \Sigma^C(P)$ and all $(\rho, \sigma) \in \mathcal{L}^0(P, x)$ there exists $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ such that*

$$I^*(\bar{\rho}) \subset I^*(\rho), \;\; I^*(\bar{\sigma}) \subset I^*(\sigma).$$

In addition, for (ρ, σ) and $(\bar{\rho}, \bar{\sigma})$ it holds that

$$\rho_i \cdot \bar{\rho}_i > 0, i \in I^*(\bar{\rho}), \;\; \sigma_j \cdot \bar{\sigma}_j > 0, j \in I^*(\bar{\sigma}).$$

5 A necessary condition for strong stability

In the remainder of this paper let $\bar{x} \in \Sigma^C(\bar{P})$ be our point under consideration and assume that MPCC-LICQ does *not* hold at $\bar{x} \in M[r, s]$. As already mentioned in Sect. 1, strong stability under MPCC-LICQ is completely described in [18]. In this section we present a necessary second order condition (Condition C^*) for the strong stability of a C-stationary point. We define for $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$ the sets

$$I^{\bar{\rho}} = I^0(\bar{\rho}) \cap I_{\bar{r}\bar{s}}(\bar{x}),$$
$$I^{\bar{\sigma}} = I^0(\bar{\sigma}) \cap I_{\bar{r}\bar{s}}(\bar{x}).$$

Definition 5.1 Let $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ and

$$T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma}) = \{v \in \mathbb{R}^n : D_x\bar{r}_i(\bar{x})v = 0, i \in I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho}), D_x\bar{s}_j(\bar{x})v = 0, \\ j \in I_{\bar{s}}(\bar{x}) \cup I^*(\bar{\sigma})\}.$$

We say that $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ fulfills *Condition* C^* if

$$D_x^2 \bar{\mathbf{L}}^{\mathrm{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma})\bar{\sigma}_i|_{T_{\bar{x}}(\bar{r},\bar{s},\bar{\rho},\bar{\sigma})} \succ 0, i \in I^{\bar{\rho}}$$

and

$$D_x^2 \bar{\mathbf{L}}^{\mathrm{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma})\bar{\rho}_j|_{T_{\bar{x}}(\bar{r},\bar{s},\bar{\rho},\bar{\sigma})} \succ 0, j \in I^{\bar{\sigma}}.$$

Note that the set $T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma})$ is a so-called *tangent space*, see e.g. [34]. The next result is obvious and therefore its proof is skipped.

Lemma 5.1 *Assume that for some $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$ and some sets $I, J \subset L$ it holds that $I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x}) \subset I$, $I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x}) \subset J$ and that the vectors $D\bar{r}_i(\bar{x}),\ i \in I,\ D\bar{s}_j(\bar{x}),\ j \in J$ are linearly independent. Let the vectors $\xi^q \in \mathbb{R}^n, q \in Q$ form an orthonormal basis of the subspace*

$$\left\{D\bar{r}_i(\bar{x}), i \in I, D\bar{s}_j(\bar{x}), j \in J\right\}^{\perp}$$

where Q is an appropriate index set. Then, there exist $V \in \mathcal{V}(\bar{x})$ and functions

$$\hat{\rho}_i \in \mathcal{C}^1(V, \mathbb{R}),\ i \in I,\ \hat{\sigma}_j \in \mathcal{C}^1(V, \mathbb{R}),\ j \in J,\ \hat{\mu}_q \in \mathcal{C}^1(V, \mathbb{R}),\ q \in Q$$

such that

$$D\bar{f}(x) = \sum_{i\in I} \hat{\rho}_i(x) D\bar{r}_i(x) + \sum_{j\in J} \hat{\sigma}_j(x) D\bar{s}_j(x) + \sum_{q\in Q} \hat{\mu}_q(x)[\xi^q]^T$$

for $x \in V$. Moreover, it holds that

$$v^T D_x^2 \bar{\mathbf{L}}^{\mathrm{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma})w = \sum_{i\in I} \frac{\partial \hat{\rho}_i(\bar{x})}{\partial v} \cdot \frac{\partial \bar{r}_i(\bar{x})}{\partial w} + \sum_{j\in J} \frac{\partial \hat{\sigma}_j(\bar{x})}{\partial v} \cdot \frac{\partial \bar{s}_j(\bar{x})}{\partial w} + \sum_{q\in Q} \frac{\partial \hat{\mu}_q(\bar{x})}{\partial v} \cdot [\xi^q]^T w$$

for $v, w \in S(0, 1)$.

Corollary 5.1 *Let $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ and ξ be the matrix whose columns are $\xi^q, q \in Q$. Assume that $V \in \mathcal{V}(\bar{x})$ and $\hat{\mu}_q \in \mathcal{C}^1(V, \mathbb{R}), q \in Q$, are given as in Lemma 5.1 with $I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x}) = I$ and $I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x}) = J$. If the matrix $\xi^T D_x^2 \bar{\mathbf{L}}^{\mathrm{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma})\xi$ is nonsingular, then for any $v \in S(0, 1) \cap T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma})$ there exists $q \in Q$ such that*

$$\frac{\partial \hat{\mu}_q(\bar{x})}{\partial v} \neq 0. \tag{5.1}$$

Proof Suppose contrarily that

$$\frac{\partial \hat{\mu}_q(\bar{x})}{\partial \bar{v}} = 0, q \in Q$$

for some $\bar{v} \in S(0,1) \cap T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma})$. Let $v^{\xi} \in \mathbb{R}^{|Q|}\backslash\{0\}$ be such that $\bar{v} = \xi v^{\xi}$. By Lemma 5.1, we get

$$0 = \frac{\partial \hat{\mu}_q(\bar{x})}{\partial \bar{v}} = \bar{v}^T D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma})\xi^q = [\xi v^{\xi}]^T D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma})\xi^q, q \in Q.$$

Hence

$$[v^{\xi}]^T [\xi^T D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma})\xi] = 0,$$

which contradicts the nonsingularity of $\xi^T D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma})\xi$. □

The next lemma presents a first necessary condition for strong stability.

Lemma 5.2 *If*

$$|I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x})| + |I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x})| + |I^{\bar{\rho}} \cap I^{\bar{\sigma}}| + 1 \leq n \tag{5.2}$$

and $I^{\bar{\rho}} \cap I^{\bar{\sigma}} \neq \emptyset$ *for some* $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$, *then* $\bar{x} \notin \Sigma^S(\bar{P})$.

Proof Suppose contrarily that $\bar{x} \in \Sigma^S(\bar{P})$ and that (5.2) holds for some $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with $I^{\bar{\rho}} \cap I^{\bar{\sigma}} \neq \emptyset$. After possibly interchanging constraints $\bar{r}_m, \bar{s}_m, m \in I_{\bar{r}\bar{s}}(\bar{x})$ and perturbing $\bar{f}$ sufficiently small, assume without loss of generality that

$$I_{\bar{r}}(\bar{x}) \subset I^*(\bar{\rho}),\ I_{\bar{s}}(\bar{x}) \subset I^*(\bar{\sigma}) \text{ and } I^0(\bar{\sigma}) \subset I^0(\bar{\rho}) \subset I_{\bar{r}\bar{s}}(\bar{x}).$$

Moreover, by (5.2) we obtain

$$|I^*(\bar{\rho})| + |\bar{I}_{\bar{s}}(\bar{x})| + 1 \leq n. \tag{5.3}$$

Fix $m^0 \in I^{\bar{\rho}} \cap I^{\bar{\sigma}}$ and by (5.3) let $e^{i,r} \in \mathbb{R}^n$, $i \in I^*(\bar{\rho}) \cup \{m^0\}$, $e^{j,s} \in \mathbb{R}^n$, $j \in \bar{I}_{\bar{s}}(\bar{x})$ be pairwise distinct unit vectors.

For $\varepsilon > 0$ sufficiently small define

$$\begin{aligned}
r_i^{\varepsilon}(x) &= \bar{r}_i(x) + \varepsilon,\ i \in I^0(\bar{\rho})\backslash\{m^0\},\\
r_i^{\varepsilon}(x) &= \bar{r}_i(x) + \varepsilon e^{i,r}(x - \bar{x}),\ i \in I^*(\bar{\rho}) \cup \{m^0\},\\
s_j^{\varepsilon}(x) &= \bar{s}_j(x) + \varepsilon e^{j,s}(x - \bar{x}),\ j \in \bar{I}_{\bar{s}}(\bar{x}),\\
f^{\varepsilon}(x) &= \bar{f}(x) + \sum_{i \in I^*(\bar{\rho}) \cup \{m^0\}} \varepsilon \bar{\rho}_i e^{i,r}(x - \bar{x}) + \sum_{j \in \bar{I}_{\bar{s}}(\bar{x})} \varepsilon \bar{\sigma}_j e^{j,s}(x - \bar{x})
\end{aligned}$$

By construction, MPCC-LICQ holds at $\bar{x} \in \Sigma^C(P^\varepsilon)$ and $\mathcal{L}(P^\varepsilon, \bar{x}) = \{(\bar{\rho}, \bar{\sigma})\}$. Since $\bar{\rho}_{m^0} = \bar{\sigma}_{m^0} = 0$, by [18, Theorem 3.1], we get $\bar{x} \notin \Sigma^S(P^\varepsilon)$ which contradicts $\bar{x} \in \Sigma^S(\bar{P})$. □

Corollary 5.2 *If $\bar{x} \in \Sigma^S(\bar{P})$ and $I^{\bar{\rho}} \cap I^{\bar{\sigma}} \neq \emptyset$ for some $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$, then $N^0(\bar{P}, \bar{x}) \geq n + 1$.*

Proof Suppose contrarily that $N^0(\bar{P}, \bar{x}) \leq n$. A contradiction easily follows by noting that $N^0(\bar{P}, \bar{x})$ is an upper bound for the left hand side of (5.2). □

Since MPCC-LICQ does not hold at $\bar{x}$, there exists $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l} \setminus \{0\}$ such that

$$\sum_{m \in L} \left[\bar{\alpha}_m D_x \bar{r}_m(\bar{x}) + \bar{\beta}_m D_x \bar{s}_m(\bar{x})\right] = 0, \tag{5.4}$$

$$\bar{\alpha}_m \cdot \bar{r}_m(\bar{x}) = \bar{\beta}_m \cdot \bar{s}_m(\bar{x}) = 0, \ m \in L. \tag{5.5}$$

In the following lemmas we assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. As mentioned in [11], this assumption implies the following characterization of the set $\mathcal{L}(\bar{P}, \bar{x})$.

Lemma 5.3 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. Then, there exists $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l} \setminus \{0\}$ uniquely determined, up to a common multiple, such that (5.4) and (5.5) hold. In addition, if $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$, then $\mathcal{L}(\bar{P}, \bar{x}) = \{(\bar{\rho}, \bar{\sigma}) + (\bar{\alpha}, \bar{\beta})t : t \in T\}$ where $T = \{t \in \mathbb{R} : (\bar{\rho}_m + \bar{\alpha}_m t)(\bar{\sigma}_m + \bar{\beta}_m t) \geq 0, \ m \in I_{\bar{r}\bar{s}}(\bar{x})\}$.*

If $\hat{N}(\bar{P}, \bar{x}) = 1$, then we define the sets $I^{\bar{\alpha}}$ and $I^{\bar{\beta}}$ analogously as $I^{\bar{\rho}}$ and $I^{\bar{\sigma}}$.

Lemma 5.4 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. If $\bar{x} \in \Sigma^S(\bar{P})$ and $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$, then $I^{\bar{\rho}^1} \cap I^{\bar{\sigma}^1} = \emptyset$ and $I^{\bar{\rho}^2} \cap I^{\bar{\sigma}^2} = \emptyset$.*

Proof Suppose contrarily that $I^{\bar{\rho}^1} \cap I^{\bar{\sigma}^1} \neq \emptyset$. By Lemma 5.2, we obtain $N^0(\bar{P}, \bar{x}) = n + 1$ and $I^{\bar{\rho}^1} \cap I^{\bar{\sigma}^1} = \{m^0\}$ for some $m^0 \in I_{\bar{r}\bar{s}}(\bar{x})$. Furthermore, Lemma 4.1 implies $m^0 \notin I^{\bar{\alpha}} \cap I^{\bar{\beta}}$. Assume without loss of generality that $\bar{\alpha}_{m^0} > 0$. By Lemma 5.6, we get $(\bar{\rho}^2, \bar{\sigma}^2) = \bar{t}(\bar{\alpha}, \bar{\beta})$ for some $\bar{t} \in \mathbb{R} \setminus \{0\}$ and, thus, $\bar{\beta}_{m^0} \geq 0$. For $\varepsilon > 0$ perturb

$$r^\varepsilon_{m^0}(x) = \bar{r}_{m^0}(x) - \varepsilon \sum_{i \in I^{\bar{\alpha}}} D\bar{r}_i(\bar{x})(x - \bar{x}) - \varepsilon \sum_{j \in I^{\bar{\beta}}} D\bar{s}_j(\bar{x})(x - \bar{x}). \tag{5.6}$$

For simplicity of notation denote r^ε and $(\alpha^\varepsilon, \beta^\varepsilon)$ again by $\bar{r}$ and $(\bar{\alpha}, \bar{\beta})$, respectively. The latter perturbation ensures that $\bar{\alpha}_m \cdot \bar{\beta}_m \neq 0$, $m \in I_{\bar{r}\bar{s}}(\bar{x})$ and, in particular, $\bar{\alpha}_{m^0} \cdot \bar{\beta}_{m^0} > 0$. Hence, [11, Lemma 5.10] yields $\bar{x} \notin \Sigma^S(\bar{P})$ which is a contradiction. □

The next two lemmas relate the strong stability of $\bar{x}$ to $\mathcal{L}^0(\bar{P}, \bar{x})$ and the signs of some components of $(\bar{\alpha}, \bar{\beta})$.

Lemma 5.5 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. If there exist $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$, $i^1 \in I^{\bar{\rho}^1}$ and $j^2 \in I^{\bar{\sigma}^2}$ such that $\bar{\alpha}_{i^1} \cdot \bar{\beta}_{j^2} > 0$, then $\bar{x} \notin \Sigma^S(\bar{P})$.*

Proof Suppose contrarily that $\bar{x} \in \Sigma^S(\bar{P})$. From Lemma 5.4, it follows that $I^{\bar{\rho}^1} \cap I^{\bar{\sigma}^1} = \emptyset$ and $I^{\bar{\rho}^2} \cap I^{\bar{\sigma}^2} = \emptyset$. Hence, $\bar{\sigma}^1_{i^1} \neq 0$ and $\bar{\rho}^2_{j^2} \neq 0$. We will perturb $\bar{P}$ in two steps such that

$$I^{\bar{\rho}^1} = \{i^1\},\ I^{\bar{\sigma}^1} = \emptyset,\ I^{\bar{\rho}^2} = \emptyset,\ I^{\bar{\sigma}^2} = \{j^2\}. \tag{5.7}$$

Step 1 Fix $\varepsilon > 0$ sufficiently small and let r^ε and s^ε be given as follows

$$\begin{aligned} r_i^\varepsilon(x) &= \bar{r}_i(x) + \varepsilon,\ i \in I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2}, \qquad & r_i^\varepsilon(x) &= \bar{r}_i(x),\ i \in L\backslash[I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2}],\\ s_j^\varepsilon(x) &= \bar{s}_j(x) + \varepsilon,\ j \in I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2}, \qquad & s_j^\varepsilon(x) &= \bar{s}_j(x),\ j \in L\backslash[I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2}]. \end{aligned}$$

Let $P^\varepsilon = \mathcal{P}^{\mathrm{cc}}(\bar{f}, r^\varepsilon, s^\varepsilon)$. By Lemma 5.3, it is easy to see that

$$\bar{\alpha}_i = 0,\ i \in I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2},\ \bar{\beta}_j = 0,\ j \in I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2}.$$

Hence, the vector $(\bar{\alpha}, \bar{\beta})$ is the same for P^ε at $\bar{x}$ and $\hat{N}(P^\varepsilon, \bar{x}) = 1$. For simplicity of notation denote P^ε, r^ε and s^ε again by $\bar{P}$, $\bar{r}$ and $\bar{s}$, respectively. After this step we get $I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2} = \emptyset$ and $I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2} = \emptyset$.

Step 2 By $I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2} = I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2} = I^{\bar{\rho}^1} \cap I^{\bar{\sigma}^1} = I^{\bar{\rho}^2} \cap I^{\bar{\sigma}^2} = \emptyset$, perhaps after interchanging constraints assume that $I^{\bar{\sigma}^1} = \emptyset$ and $I^{\bar{\rho}^2} = \emptyset$. Fix $\varepsilon > 0$ sufficiently small and let

$$f^\varepsilon(x) = \bar{f}(x) + \varepsilon \sum_{i \in I^{\bar{\rho}^1}\backslash\{i^1\}} \bar{\sigma}_i^1 \bar{r}_i(x) + \varepsilon \sum_{j \in I^{\bar{\sigma}^2}\backslash\{j^2\}} \bar{\rho}_j^2 \bar{s}_j(x).$$

For simplicity of notation we denote f^ε and the corresponding basic Lagrange vectors again by $\bar{f}$, $(\bar{\rho}^1, \bar{\sigma}^1)$ and $(\bar{\rho}^2, \bar{\sigma}^2)$, respectively. Now, we are in the situation as described in (5.7).

Next, we will show that $(\bar{\rho}^1, \bar{\sigma}^1)$ fulfills Condition C^*. Suppose the contrary and consider the following auxiliary standard nonlinear program

$$\begin{aligned} &\bar{P}^{\mathrm{aux}}: \quad \min\ \bar{\sigma}^1_{i^1} \cdot \bar{f}(x)\\ &\text{s. t.}\quad \bar{r}_{i^1}(x) \geq 0,\ \bar{r}_i(x) = 0,\ i \in \bar{I}_{\bar{r}}(\bar{x})\backslash\{i^1\},\ \bar{s}_j(x) = 0,\ j \in \bar{I}_{\bar{s}}(\bar{x}). \end{aligned}$$

After rearranging constraints, we have $(\bar{\lambda}, \bar{\mu}) = \bar{\sigma}^1_{i^1} \cdot (\bar{\rho}^1, \bar{\sigma}^1) \in \operatorname{ext} \mathcal{L}(\bar{P}^{\mathrm{aux}}, \bar{x})$. Since $(\bar{\rho}^1, \bar{\sigma}^1)$ does not fulfill Condition C^*, the property

$$D_x^2 \bar{\mathbf{L}}^{\mathrm{sn}}(\bar{x}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}}(\bar{h}, \bar{g}, \bar{\lambda}, \bar{\mu})} \succ 0$$

is not fulfilled for $\bar{P}^{\mathrm{aux}}$. By Lemma 3.2, there exist sequences $P^{k,\mathrm{aux}} \to \bar{P}^{\mathrm{aux}}$, $x^{1,k}, x^{2,k} \to \bar{x}$ with $x^{1,k} \neq x^{2,k}$ and $x^{1,k}, x^{2,k} \in \Sigma(P^{k,\mathrm{aux}})$ such that LICQ holds at

$x^{1,k}, x^{2,k}$ and that if

$$\mathcal{L}\big(P^{k,\text{aux}}, x^{1,k}\big) = \big\{(\lambda^{1,k}, \mu^{1,k})\big\},\ \mathcal{L}\big(P^{k,\text{aux}}, x^{2,k}\big) = \big\{(\lambda^{2,k}, \mu^{2,k})\big\},$$

then

$$\begin{aligned}\big(\lambda^{1,k}, \mu^{1,k}\big) &\to \big(\bar\lambda, \bar\mu\big),\\ \big(\lambda^{2,k}, \mu^{2,k}\big) &\to \big(\bar\lambda, \bar\mu\big).\end{aligned} \tag{5.8}$$

By (5.8) and $(\bar\lambda, \bar\mu) = \bar\sigma^1_{i^1} \cdot (\bar\rho^1, \bar\sigma^1)$, we obtain $x^{1,k}, x^{2,k} \in \Sigma^C(P^k)$ which contradicts $\bar x \in \Sigma^S(\bar P)$. Consequently $(\bar\rho^1, \bar\sigma^1)$ fulfills Condition C^*.

By using the terminology from Lemma 5.1, we obtain

$$D\bar f(x) = \sum_{i\in \bar I_{\bar r}(\bar x)\setminus\{i^1\}} \hat\rho_i(x) D\bar r_i(x) + \sum_{j\in \bar I_{\bar s}(\bar x)} \hat\sigma_j(x) D\bar s_j(x) + \sum_{q\in Q} \hat\mu_q(x)[\xi^q]^T$$

for x near $\bar x$. For $\varepsilon \geq 0$ define the mapping

$$F^\varepsilon(x) = \begin{pmatrix} \min\{\bar r_{i^1}(x), -\varepsilon - \bar s_{j^2}(x)\} \\ \bar r_i(x), i \in \bar I_{\bar r}(\bar x)\setminus\{i^1\} \\ \bar s_j(x), j \in \bar I_{\bar s}(\bar x)\setminus\{j^2\} \\ \hat\mu_q(x), q \in Q \end{pmatrix}.$$

We will verify that $\partial^C_x F^0(\bar x)$ is nonsingular. By Lemma 2.1, suppose contrarily that for some $\bar v \in S(0, 1)$ and some $\lambda \in [0, 1]$ it holds that

$$\lambda \frac{\partial \bar r_{i^1}(\bar x)}{\partial \bar v} - (1-\lambda)\frac{\partial \bar s_{j^2}(\bar x)}{\partial \bar v} = 0, \tag{5.9}$$

$$\frac{\partial \bar r_i(\bar x)}{\partial \bar v} = 0,\ i \in \bar I_r(\bar x)\setminus\{i^1\},\ \frac{\bar s_j(\bar x)}{\partial \bar v} = 0,\ j \in \bar I_s(\bar x)\setminus\{j^2\}, \tag{5.10}$$

$$\frac{\partial \hat\mu_q(\bar x)}{\partial \bar v} = 0, q \in Q. \tag{5.11}$$

By $I^{\bar\sigma^1} \cap I^{\bar\sigma^2} = \emptyset$, assume that $\bar\beta_{j^2} = 1$ and, by (5.10) and Lemma 5.3, we get

$$\frac{\partial \bar s_{j^2}(\bar x)}{\partial \bar v} = -\alpha_{i^1} \frac{\partial \bar r_{i^1}(\bar x)}{\partial \bar v}.$$

Substituting the latter in (5.9) yields $\bar v \in T_{\bar x}(\bar r, \bar s, \bar\rho^1, \bar\sigma^1)$. Recall that $(\bar\rho^1, \bar\sigma^1)$ fulfills Condition C^* which, in particular, implies that the matrix $\xi^T D^2_x \bar{\mathbf{L}}^{\text{cc}}(\bar x, \bar\rho, \bar\sigma)\xi$ given in Corollary 5.1 is nonsingular for $(\bar\rho, \bar\sigma) = (\bar\rho^1, \bar\sigma^1)$. Consequently, by (5.11) and Corollary 5.1, specifically by (5.1), we get a contradiction to $\bar v \in S(0, 1)$.

Finally, since $\partial^C_x F^0(\bar x)$ is nonsingular, according to Theorem 2.1, for $\varepsilon > 0$ sufficiently small there exists $x^\varepsilon \in \mathbb{R}^n$ with $F^\varepsilon(x^\varepsilon) = 0$ and $x^\varepsilon \to \bar x$ as $\varepsilon \to 0$. Fix $\varepsilon > 0$

sufficiently small and define

$$s^{\varepsilon}_{j^2}(x) = \bar{s}_{j^2}(x) - \bar{s}_{j^2}(x^{\varepsilon}),\ s^{\varepsilon}_{j}(x) = \bar{s}_{j}(x),\ j \in \bar{I}_s(\bar{x})\backslash\{j^2\}.$$

Let $P^{\varepsilon} = \mathcal{P}(\bar{f}, \bar{r}, s^{\varepsilon})$. Note that $\bar{x}, x^{\varepsilon} \in \Sigma^C(P^{\varepsilon})$ which contradicts $\bar{x} \in \Sigma^S(\bar{P})$. □

Corollary 5.3 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. If $\bar{x} \in \Sigma^S(\bar{P})$, then $|\mathcal{L}^0(\bar{P}, \bar{x})| \leq 2$.*

Proof Suppose contrarily that there are pairwise distinct vectors $(\bar{\rho}^1, \bar{\sigma}^1)$, $(\bar{\rho}^2, \bar{\sigma}^2)$, $(\bar{\rho}^3, \bar{\sigma}^3) \in \mathcal{L}^0(\bar{P}, \bar{x})$. By Lemma 5.3, we get

$$I^0(\bar{\rho}^1) \cap I^0(\bar{\alpha}) = I^0(\bar{\rho}^2) \cap I^0(\bar{\alpha}) = I^0(\bar{\rho}^3) \cap I^0(\bar{\alpha}), \tag{5.12}$$
$$I^0(\bar{\sigma}^1) \cap I^0(\bar{\beta}) = I^0(\bar{\sigma}^2) \cap I^0(\bar{\beta}) = I^0(\bar{\sigma}^3) \cap I^0(\bar{\beta}). \tag{5.13}$$

Fix $\varepsilon > 0$ and let r^{ε} and s^{ε} be given as follows

$$r^{\varepsilon}_i(x) = \bar{r}_i(x) + \varepsilon,\ i \in I^{\bar{\rho}^1} \cap I^{\bar{\alpha}}, \qquad r^{\varepsilon}_i(x) = \bar{r}_i(x),\ i \in L\backslash[I^{\bar{\rho}^1} \cap I^{\bar{\alpha}}],$$
$$s^{\varepsilon}_j(x) = \bar{s}_j(x) + \varepsilon,\ j \in I^{\bar{\sigma}^1} \cap I^{\bar{\beta}}, \qquad s^{\varepsilon}_j(x) = \bar{s}_j(x),\ j \in L\backslash[I^{\bar{\sigma}^1} \cap I^{\bar{\beta}}].$$

Let $P^{\varepsilon} = \mathcal{P}^{cc}(\bar{f}, r^{\varepsilon}, s^{\varepsilon})$. The latter perturbation, together with (5.12) and (5.13), yields

$$\bar{\alpha}_i \neq 0,\ i \in [I^0(\bar{\rho}^1) \cup I^0(\bar{\rho}^2) \cup I^0(\bar{\rho}^3)] \cap I_{r^{\varepsilon}s^{\varepsilon}}(\bar{x}),$$
$$\bar{\beta}_j \neq 0,\ j \in [I^0(\bar{\sigma}^1) \cup I^0(\bar{\sigma}^2) \cup I^0(\bar{\sigma}^3)] \cap I_{r^{\varepsilon}s^{\varepsilon}}(\bar{x}).$$

Besides, the vector $(\bar{\alpha}, \bar{\beta})$ is the same for P^{ε} at $\bar{x}$ and $\hat{N}(P^{\varepsilon}, \bar{x}) = 1$. For simplicity of notation denote P^{ε}, r^{ε} and s^{ε} again by $\bar{P}$, $\bar{r}$ and $\bar{s}$, respectively. Now, by inspecting the signs of the components of $(\bar{\alpha}, \bar{\beta})$ and by applying Lemma 5.5 to all choices of a pair from $\mathcal{L}^0(\bar{P}, \bar{x})$, we get a contradiction to $\bar{x} \in \Sigma^S(\bar{P})$. □

Lemma 5.6 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. If for some $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ and some $i^0 \in I^{\bar{\rho}}$, $j^0 \in I^{\bar{\sigma}}$ it holds that $\bar{\alpha}_{i^0} \cdot \bar{\beta}_{j^0} > 0$ and $\bar{\rho}_{j^0} \cdot \bar{\sigma}_{i^0} \leq 0$, then $\bar{x} \notin \Sigma^S(\bar{P})$.*

Proof Assume contrarily that $\bar{x} \in \Sigma^S(\bar{x})$. If $N^0(\bar{P}, \bar{x}) = n + 1$, then, by perturbing analogously to (5.6), it follows that $\bar{\alpha}_m \cdot \bar{\beta}_m \neq 0$, $m \in I_{\bar{r}\bar{s}}(\bar{x})$. An application of [11, Lemma 5.10] yields the desired result. Therefore, in the remainder of this proof we assume $N^0(\bar{P}, \bar{x}) \leq n$. By the latter and Lemma 5.2, we get $i^0 \neq j^0$. Moreover, after perturbing $\bar{f}$ sufficiently small, assume $I^{\bar{\rho}} = \{i^0\}$, $I^{\bar{\sigma}} = \{j^0\}$ and $\bar{\rho}_{j^0} \cdot \bar{\sigma}_{i^0} < 0$. For $\varepsilon > 0$ sufficiently small consider

$$f^{1,\varepsilon}(x) = \bar{f}(x) + \varepsilon\bar{\rho}_{j^0} D_x\bar{s}_{j^0}(\bar{x})(x - \bar{x}) \text{ and } f^{2,\varepsilon}(x) = \bar{f}(x) + \varepsilon\bar{\sigma}_{i^0} D_x\bar{r}_{i^0}(\bar{x})(x - \bar{x}).$$

Let $P^{1,\varepsilon} = \mathcal{P}(f^{1,\varepsilon}, \bar{r}, \bar{s})$ and $P^{2,\varepsilon} = \mathcal{P}(f^{2,\varepsilon}, \bar{r}, \bar{s})$. Obviously, $\bar{x} \in \Sigma^C(P^{1,\varepsilon}) \cap \Sigma^C(P^{2,\varepsilon})$ with corresponding basic Lagrange vectors $(\rho^{1,\varepsilon}, \sigma^{1,\varepsilon})$, $(\rho^{2,\varepsilon}, \sigma^{2,\varepsilon})$ that we obtain from $(\bar{\rho}, \bar{\sigma})$ by substituting $\bar{\sigma}_{j^0} = 0$ by $\varepsilon\bar{\rho}_{j^0}$ and $\bar{\rho}_{i^0} = 0$ by $\varepsilon\bar{\sigma}_{i^0}$, respectively. A moment of reflection shows that

$$D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma}) = D_x^2 \mathbf{L}^{1,\varepsilon}(\bar{x}, \rho^{1,\varepsilon}, \sigma^{1,\varepsilon}) = D_x^2 \mathbf{L}^{2,\varepsilon}(\bar{x}, \rho^{2,\varepsilon}, \sigma^{2,\varepsilon}) \tag{5.14}$$

where $\mathbf{L}^{1,\varepsilon}$ and $\mathbf{L}^{2,\varepsilon}$ denote the MPCC-Lagrange functions for $P^{1,\varepsilon}$ and $P^{2,\varepsilon}$, respectively. Furthermore, by $\bar{\alpha}_{i^0} \cdot \bar{\beta}_{j^0} \neq 0$ and $N^0(\bar{P}, \bar{x}) \leq n$ we get

$$T_{\bar{x}}(\bar{r}, \bar{s}, \rho^{1,\varepsilon}, \sigma^{1,\varepsilon}) = T_{\bar{x}}(\bar{r}, \bar{s}, \rho^{2,\varepsilon}, \sigma^{2,\varepsilon}) \neq \{0\}. \tag{5.15}$$

Now, we define the two auxiliary problems

$$\begin{aligned}
&P^{\text{aux},1,\varepsilon}: \quad \min\ \sigma_{i^0}^{1,\varepsilon} \cdot f^{1,\varepsilon}(x)\\
&\text{s. t.} \quad \bar{r}_{i^0}(x) \geq 0,\ \bar{r}_i(x) = 0,\ i \in \bar{I}_{\bar{r}}(\bar{x})\backslash\{i^0\},\ \bar{s}_j(x) = 0,\ j \in \bar{I}_{\bar{s}}(\bar{x}).\\
&P^{\text{aux},2,\varepsilon}: \quad \min\ \rho_{j^0}^{2,\varepsilon} \cdot f^{2,\varepsilon}(x)\\
&\text{s. t.} \quad \bar{s}_{j^0}(x) \geq 0,\ \bar{r}_i(x) = 0,\ i \in \bar{I}_{\bar{r}}(\bar{x}),\ \bar{s}_j(x) = 0,\ j \in \bar{I}_{\bar{s}}(\bar{x})\backslash\{j^0\}.
\end{aligned}$$

We apply an analogous technique as in the proof of Lemma 5.5 and obtain that $(\rho^{1,\varepsilon}, \sigma^{1,\varepsilon})$ and $(\rho^{2,\varepsilon}, \sigma^{2,\varepsilon})$ fulfill Condition C^*. The latter, (5.14) and (5.15) imply

$$D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma})\bar{\sigma}_{i^0}|_{T_{\bar{x}}(\bar{r},\bar{s},\rho^{1,\varepsilon},\sigma^{1,\varepsilon})} \succ 0 \text{ and } D_x^2 \bar{\mathbf{L}}^{cc}(\bar{x}, \bar{\rho}, \bar{\sigma})\bar{\rho}_{j^0}|_{T_{\bar{x}}(\bar{r},\bar{s},\rho^{2,\varepsilon},\sigma^{2,\varepsilon})} \succ 0 \tag{5.16}$$

However, by $N^0(\bar{P}, \bar{x}) \leq n$ and (5.16) we obtain $\bar{\rho}_{i^0} \cdot \bar{\sigma}_{j^0} > 0$ which contradicts our assumption. Therefore, we have $\bar{x} \notin \Sigma^S(\bar{P})$. □

Remark 5.1 We observe in the previous proof that the condition $\bar{\alpha}_{i^0} \cdot \bar{\beta}_{j^0} > 0$ is needed when $N^0(\bar{P}, \bar{x}) = n+1$. If $N^0(\bar{P}, \bar{x}) \leq n$, then it is sufficient to assume $\bar{\alpha}_{i^0} \cdot \bar{\beta}_{j^0} \neq 0$ in Lemma 5.6.

Next, we consider a case with exactly one basic Lagrange vector.

Lemma 5.7 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$, $\mathcal{L}^0(\bar{P}, \bar{x}) = \{(\bar{\rho}, \bar{\sigma})\}$ and $I^{\bar{\sigma}} = \emptyset$. If C-MFCQ holds at $\bar{x}$ or $\bar{x} \in \Sigma^S(\bar{P})$, then there exist $i^1, i^2 \in I^{\bar{\rho}}$ such that $\bar{\alpha}_{i^1} \cdot \bar{\sigma}_{i^1} \cdot \bar{\alpha}_{i^2} \cdot \bar{\sigma}_{i^2} < 0$.*

Proof We distinguish two cases.

Case 1 C-MFCQ does not hold at $\bar{x}$ and $\bar{x} \in \Sigma^S(\bar{P})$. By [10, Theorem 5.8] and $\hat{N}(\bar{P}, \bar{x}) = 1$ we obtain $N^0(\bar{P}, \bar{x}) = n + 1$. Then, by [11, Lemma 5.2] we get $\bar{\alpha}_m \cdot \bar{\beta}_m \neq 0$, $m \in I_{\bar{r}\bar{s}}(\bar{x})$ and, moreover, [11, Lemma 5.9] implies that C-MFCQ holds at $\bar{x}$. Therefore, Case 1 is not possible.

Case 2 C-MFCQ holds at $\bar{x}$. Suppose contrarily that $\bar{\alpha}_{i^1} \cdot \bar{\sigma}_{i^1} \cdot \bar{\alpha}_{i^2} \cdot \bar{\sigma}_{i^2} \geq 0$, for all $i^1, i^2 \in I^{\bar{\rho}}$. Assume without loss of generality that $\bar{\alpha}_i \cdot \bar{\sigma}_i \geq 0$, $i \in I^{\bar{\rho}}$. By Lemma 5.3, there exists $\bar{t} > 0$ such that

$$\left\{(\bar{\rho}, \bar{\sigma}) + t(\bar{\alpha}, \bar{\beta}), t \in [0, \bar{t}]\right\} \subset \mathcal{L}(\bar{P}, \bar{x}).$$

Since C-MFCQ holds at $\bar{x}$, the set $\mathcal{L}(\bar{P}, \bar{x})$ is bounded [10, Lemma 3.3] and, thus, $\mathcal{L}^0(\bar{P}, \bar{x})$ cannot be a singleton. □

Remark 5.2 According to the proof of Lemma 5.7, we can delete the words "or $\bar{x} \in \Sigma^S(\bar{P})$" in Lemma 5.7 (since Case 1 in this proof is not possible).

The following corollary completes the preparation for the forthcoming Theorem 5.1.

Corollary 5.4 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. If $\bar{x} \in \Sigma^S(\bar{P})$ and $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with $I^{\bar{\sigma}} = \emptyset$, then*

$$\bar{\sigma}_{i^1} \cdot \bar{\sigma}_{i^2} > 0 \textit{ for all } i^1, i^2 \in I^{\bar{\rho}}. \tag{5.17}$$

Proof By $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$, choose $i^0 \in I^{\bar{\rho}}$ in such a way that $\alpha_{i^0} \neq 0$. Perturb

$$r^{\varepsilon}_{i^0}(x) = \bar{r}_{i^0}(x) + \varepsilon \sum_{i \in I^{\bar{\alpha}}} D\bar{r}_i(\bar{x})(x - \bar{x}) + \varepsilon \sum_{j \in I^{\bar{\beta}}} D\bar{s}_j(\bar{x})(x - \bar{x}).$$

After this perturbation we get $\bar{\alpha}_m \cdot \bar{\beta}_m \neq 0, m \in I_{\bar{r}\bar{s}}(\bar{x})$. The latter property, together with Lemma 5.6 and Remark 5.1, implies (5.17) whenever $N^0(\bar{P}, \bar{x}) \leq n$. If $N^0(\bar{P}, \bar{x}) = n + 1$, then (5.17) follows from the proof of [11, Theorem 5.14]. □

Now, we present the main result of this section.

Theorem 5.1 *If $\bar{x} \in \Sigma^S(\bar{P})$, then $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ fulfills Condition C^* whenever*

$$I^{\bar{\rho}} \cap I^{\bar{\sigma}} = \emptyset. \tag{5.18}$$

Proof Suppose contrarily that there exists $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with (5.18) and $(\bar{\rho}, \bar{\sigma})$ does not fulfill Condition C^*. Perhaps after interchanging components of $\bar{r}$ and $\bar{s}$, assume that $I^{\bar{\sigma}} = \emptyset$. Now, we perturb $\bar{P}$ in such a way that $\hat{N}(\bar{P}, \bar{x}) = 1$. Choose $I \subset L$ with $I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x}) \subset I$ such that the vectors

$$D\bar{r}_i(\bar{x}),\ i \in I,\ \ D\bar{s}_j(\bar{x}),\ \ j \in \bar{I}_{\bar{s}}(\bar{x})$$

form a basis of the subspace generated by

$$D\bar{r}_i(\bar{x}),\ i \in \bar{I}_{\bar{r}}(\bar{x}),\ \ D\bar{s}_j(\bar{x}),\ \ j \in \bar{I}_{\bar{s}}(\bar{x}).$$

Since MPCC-LICQ does not hold at $\bar{x}$, it follows that $I^{\bar{\rho}} \backslash I \neq \emptyset$. Fix $\varepsilon > 0$ sufficiently small, choose arbitrarily $i' \in I^{\bar{\rho}} \backslash I$ and let

$$r^{\varepsilon}_i(x) = \bar{r}_i(x) + \varepsilon,\ i \in I^{\bar{\rho}} \backslash [I \cup \{i'\}],\ \ r^{\varepsilon}_i(x) = \bar{r}_i(x),\ i \in L \backslash (I^{\bar{\rho}} \backslash [I \cup \{i'\}]).$$

Therefore, without loss of generality assume now that $\hat{N}(\bar{P},\bar{x}) = 1$. By Corollary 5.4, we obtain (5.17) for all $i^1, i^2 \in I^{\bar{\rho}}$. Now, fix $i^0 \in I^{\bar{\rho}}$ and consider the following auxiliary standard nonlinear program

$$
\begin{aligned}
P^{\text{aux}}:\quad & \min\ \bar{\sigma}_{i^0}\cdot f(x)\\
\text{s. t.}\quad & r_i(x) \geq 0,\, i \in I^{\bar{\rho}},\ \ s_j(x) \geq 0,\, j \in I^{\bar{\rho}},\\
& r_i(x) = 0,\, i \in \bar{I}_{\bar{r}}(\bar{x})\backslash I^{\bar{\rho}},\ \ s_j(x) = 0,\, j \in \bar{I}_{\bar{s}}(\bar{x})\backslash I^{\bar{\rho}}.
\end{aligned}
$$

Analogously to the proof of Lemma 5.5, application of Lemma 3.2 yields a contradiction to $\bar{x} \in \Sigma^S(\bar{P})$.

Therefore, $(\bar{\rho},\bar{\sigma})$ fulfills Condition C^*. □

Remark 5.3 By [11, Lemma 5.2, Theorem 5.14], it follows that the statement of the latter theorem holds independently from C-MFCQ.

6 Main results

In the remainder of this paper we assume that:

A1 C-MFCQ holds at $\bar{x}$.
A2 For all $(\bar{\rho},\bar{\sigma}) \in \mathcal{L}^0(\bar{P},\bar{x})$ it holds that $I^{\bar{\rho}} \cap I^{\bar{\sigma}} = \emptyset$.

Note that by A1, we obtain that $\mathcal{L}(\bar{P},\bar{x})$ is bounded, see e. g. [10, Lemma 3.3]. Let

$$
\mathbf{I}^\rho = \bigcup_{(\bar{\rho},\bar{\sigma})\in\mathcal{L}^0(\bar{P},\bar{x})} I^{\bar{\rho}},
$$
$$
\mathbf{I}^\sigma = \bigcup_{(\bar{\rho},\bar{\sigma})\in\mathcal{L}^0(\bar{P},\bar{x})} I^{\bar{\sigma}},
$$

and for $(\bar{\rho},\bar{\sigma}) \in \mathcal{L}^0(\bar{P},\bar{x})$ and $m^0 \in I^{\bar{\rho}} \cup I^{\bar{\sigma}}$ define

$$
\hat{\mathcal{L}}(\bar{P},\bar{x},\bar{\rho},\bar{\sigma},m^0) = \left\{ (\rho,\sigma) \in \mathbb{R}^{2l} \;\middle|\; \begin{array}{r} D_x\,\bar{\mathbf{L}}^{\text{cc}}(\bar{x},\rho,\sigma) = 0,\\ \rho_m\cdot\bar{r}_m(\bar{x}) = \sigma_m\cdot\bar{s}_m(\bar{x}) = 0,\ m \in L,\\ (\bar{\rho}_{m^0}+\bar{\sigma}_{m^0})\cdot\rho_i \geq 0,\ i \in \mathbf{I}^\rho,\\ (\bar{\rho}_{m^0}+\bar{\sigma}_{m^0})\cdot\sigma_j \geq 0,\ j \in \mathbf{I}^\sigma \end{array} \right\}.
$$

If the set $\hat{\mathcal{L}}(\bar{P},\bar{x},\bar{\rho},\bar{\sigma},m^0)$ is independent from the choices of $(\bar{\rho},\bar{\sigma}) \in \mathcal{L}^0(\bar{P},\bar{x})$ and $m^0 \in I^{\bar{\rho}} \cup I^{\bar{\sigma}}$, then we simply denote it by $\hat{\mathcal{L}}(\bar{P},\bar{x})$.

Let $G(\bar{P},\bar{x})$ denote the graph whose set of vertices is $\mathcal{L}^0(\bar{P},\bar{x})$ and whose set of edges is

$$
E(\bar{P},\bar{x}) = \Big\{\{(\bar{\rho}^1,\bar{\sigma}^1),(\bar{\rho}^2,\bar{\sigma}^2)\} : (\bar{\rho}^1,\bar{\sigma}^1) \neq (\bar{\rho}^2,\bar{\sigma}^2) \text{ and } (I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2}) \cup (I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2}) \neq \emptyset\Big\}.
$$

Furthermore, for $(\bar{\rho}^0, \bar{\sigma}^0) \in \mathcal{L}^0(\bar{P}, \bar{x})$ let $C(\bar{\rho}^0, \bar{\sigma}^0)$ denote the set of vertices of the connected component of $G(\bar{P}, \bar{x})$ which contains $(\bar{\rho}^0, \bar{\sigma}^0)$. For $t \in \mathbb{R}$ define

$$t^- = \min\{t, 0\}, \; t^+ = \max\{t, 0\}.$$

For $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$ define

$$I^+(\bar{\rho}, \bar{\sigma}) = \{m \in I_{\bar{r}\bar{s}}(\bar{x}) | \, \bar{\rho}_m + \bar{\sigma}_m > 0\}\,.$$

The next result is straightforward but crucial for the characterization of strong stability.

Theorem 6.1 *If $\bar{x} \in \Sigma^S(\bar{P})$, then at least one of the following conditions hold:*

(1) *There exists $(\bar{\rho}^0, \bar{\sigma}^0) \in \mathcal{L}^0(\bar{P}, \bar{x})$ such that, after possibly interchanging constraints, it holds that $I^{\bar{\sigma}^0} = \emptyset$ and that $\bar{\sigma}^0_{i^1} \cdot \bar{\sigma}^0_{i^2} > 0$ for all $i^1, i^2 \in I^{\bar{\rho}^0}$.*

(2) *$|I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x})| + |I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x})| = n$ for all $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$.*

Proof It is a direct consequence of Theorem 5.1. □

We will characterize strong stability when (1) or (2) in Theorem 6.1 holds. Next, we present two preliminary results.

Lemma 6.1 *Assume that $\hat{N}(\bar{P}, \bar{x}) = 1$. If $\bar{x} \in \Sigma^S(\bar{P})$, then $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$.*

Proof By Corollaries 4.1 and 5.3, we distinguish two cases.

Case 1 $\mathcal{L}^0(\bar{P}, \bar{x}) = \{(\bar{\rho}, \bar{\sigma})\}$. Perhaps after interchanging constraints assume that $\mathbf{I}^\sigma = I^{\bar{\sigma}} = \emptyset$ and that $\mathbf{I}^\rho = I^{\bar{\rho}}$. By Lemma 5.7 and Corollary 5.4, it follows that $\bar{\sigma}_{i^1} \cdot \bar{\sigma}_{i^2} > 0$ for all $i^1, i^2 \in I^{\bar{\rho}}$ and that $\bar{\alpha}_{i^3} \cdot \bar{\alpha}_{i^4} < 0$ for some $i^3, i^4 \in I^{\bar{\rho}}$. Fix an arbitrary $m^0 \in I^{\bar{\rho}}$. Obviously, the condition

$$(\bar{\alpha}_{i^3} t) \cdot \bar{\sigma}_{m^0} \geq 0, \; (\bar{\alpha}_{i^4} t) \cdot \bar{\sigma}_{m^0} \geq 0$$

holds only when $t = 0$. Then, by Lemma 5.3, we obtain

$$\mathcal{L}(\bar{P}, \bar{x}) = \{(\bar{\rho}, \bar{\sigma})\} = \left\{ (\rho, \sigma) \in \mathbb{R}^{2l} \; \middle| \; \begin{array}{r} D_x \bar{\mathbf{L}}^{cc}(\bar{x}, \rho, \sigma) = 0, \\ \rho_m \cdot \bar{r}_m(\bar{x}) = \sigma_m \cdot \bar{s}_m(\bar{x}) = 0, \; m \in L, \\ \bar{\sigma}_{m^0} \cdot \rho_i \geq 0, \; i \in \mathbf{I}^\rho \end{array} \right\}$$
$$= \hat{\mathcal{L}}(\bar{P}, \bar{x}).$$

Case 2 $\mathcal{L}^0(\bar{P}, \bar{x}) = \{(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2)\}$ with $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$. Fix $\varepsilon > 0$ and let P^ε be given as in the proof of Corollary 5.3. As in the latter, we see that $\bar{\alpha}_i \neq 0, \; i \in \mathbf{I}^\rho, \; \bar{\beta}_j \neq 0, \; j \in \mathbf{I}^\sigma$ and the vector $(\bar{\alpha}, \bar{\beta})$ is the same for P^ε at $\bar{x}$; $\hat{N}(P^\varepsilon, \bar{x}) = 1$, $\mathcal{L}(P^\varepsilon, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x})$ and $\mathcal{L}^0(P^\varepsilon, \bar{x}) = \mathcal{L}^0(\bar{P}, \bar{x})$. For simplicity of notation denote P^ε, r^ε and s^ε again by $\bar{P}$, $\bar{r}$ and $\bar{s}$, respectively. By A2, assume without loss of generality that $I^{\bar{\sigma}^1} = \emptyset$, $I^{\bar{\rho}^2} = \emptyset$ and, by Lemma 5.5 and Corollary 5.4, that $\bar{\alpha}_i \cdot \bar{\sigma}^1_i > 0, \; i \in I^{\bar{\rho}^1}$. Note that $\mathbf{I}^\rho = I^{\bar{\rho}^1}$

and that $\mathbf{I}^\sigma = I^{\bar{\sigma}^2}$. Since, by A1, $\mathcal{L}(\bar{P}, \bar{x})$ is bounded and $|\mathcal{L}^0(\bar{P}, \bar{x})| = 2$, there exists $\bar{t} \neq 0$ such that $(\bar{\rho}^1, \bar{\sigma}^1) + (\bar{\alpha}, \bar{\beta})\bar{t} = (\bar{\rho}^2, \bar{\sigma}^2)$ and

$$\mathcal{L}(\bar{P}, \bar{x}) = \{(\bar{\rho}^1, \bar{\sigma}^1) + (\bar{\alpha}, \bar{\beta})t : t \in [\bar{t}^-, \bar{t}^+]\}.$$

Fix an arbitrary $m^0 \in I^{\bar{\rho}^1}$. Now, by Lemmas 5.3, 5.5 and Corollary 5.4 we obtain

$$\mathcal{L}(\bar{P}, \bar{x}) = \left\{ (\rho, \sigma) \in \mathbb{R}^{2l} \;\middle|\; \begin{array}{r} D_x \bar{\mathbf{L}}^{cc}(\bar{x}, \rho, \sigma) = 0, \\ \rho_m \cdot \bar{r}_m(\bar{x}) = \sigma_m \cdot \bar{s}_m(\bar{x}) = 0,\ m \in L, \\ \bar{\sigma}^1_{m^0} \cdot \rho_i \geq 0,\ i \in \mathbf{I}^\rho, \\ \bar{\sigma}^1_{m^0} \cdot \sigma_j \geq 0,\ j \in \mathbf{I}^\sigma \end{array} \right\} = \hat{\mathcal{L}}(\bar{P}, \bar{x}).$$

□

Corollary 6.1 *Assume that* $\hat{N}(\bar{P}, \bar{x}) = 1$. *If* $\bar{x} \in \Sigma^S(\bar{P})$, *then* $\mathcal{L}(\bar{P}, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x}, I)$ *for some* $I \subset I_{\bar{r}\bar{s}}(\bar{x})$.

Proof By Lemma 6.1, the set

$$I^{++} = I^+(\bar{\rho}, \bar{\sigma}) \backslash [\mathbf{I}^\rho \cup \mathbf{I}^\sigma]$$

is independent from the choice of $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}(\bar{P}, \bar{x})$. Now, choose $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ and $m^0 \in I^{\bar{\rho}} \cup I^{\bar{\sigma}}$. If $\bar{\rho}_{m^0} + \bar{\sigma}_{m^0} > 0$, then put $I = I^+(\bar{\rho}, \bar{\sigma})$, else take $I = I^{++}$. By using Lemma 6.1 again, the desired result follows. □

Assuming (1) in Theorem 6.1, we characterize now the strong stability of $\bar{x} \in \Sigma^C(\bar{P})$.

Theorem 6.2 *Assume that there exists* $(\bar{\rho}^0, \bar{\sigma}^0) \in \mathcal{L}^0(\bar{P}, \bar{x})$ *with* $I^{\bar{\sigma}^0} = \emptyset$ *and* $\bar{\sigma}^0_{i^1} \cdot \bar{\sigma}^0_{i^2} > 0$ *for all* $i^1, i^2 \in I^{\bar{\rho}^0}$. *Then the following conditions are equivalent:*

(i) $\bar{x} \in \Sigma^S(\bar{P})$
(ii) $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$ *and each* $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ *fulfills Condition* C^*.

Proof (ii) ⇒ (i). Take $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ as in Lemma 4.4. Fix $i^0 \in I^{\bar{\rho}^0}$ and for each $P \in W$ consider the following standard nonlinear program

$$P^{\text{aux}} : \quad \min \bar{\sigma}^0_{i^0} \cdot f(x)$$
$$\text{s. t.} \quad r_i(x) \geq 0, i \in \mathbf{I}^\rho,\ s_j(x) \geq 0,\ j \in \mathbf{I}^\sigma,$$
$$r_i(x) = 0, i \in \bar{I}_{\bar{r}}(\bar{x}) \backslash \mathbf{I}^\rho,\ s_j(x) = 0,\ j \in \bar{I}_{\bar{s}}(\bar{x}) \backslash \mathbf{I}^\sigma.$$

After rearranging constraints it holds that

$$\mathcal{L}(\bar{P}^{\text{aux}}, \bar{x}) = \bar{\sigma}^0_{i^0} \cdot \hat{\mathcal{L}}(\bar{P}, \bar{x}) = \bar{\sigma}^0_{i^0} \cdot \mathcal{L}(\bar{P}, \bar{x}).$$

Since C-MFCQ holds at $\bar{x}$ for $\bar{P}$, it follows that $\mathcal{L}(\bar{P}, \bar{x})$ is bounded. Hence $\mathcal{L}(\bar{P}^{\text{aux}}, \bar{x})$ is bounded and consequently MFCQ holds at $\bar{x}$ for $\bar{P}^{\text{aux}}$. Shrink V and W as in Lemma 3.1. Next, we show that

$$\Sigma^C(P) \cap V = \Sigma(P^{\text{aux}}) \cap V \tag{6.1}$$

for all $P \in W$. Let $x \in \Sigma^C(P) \cap V$. Obviously. x is feasible for P^{aux}. Fix $(\rho, \sigma) \in \mathcal{L}^0(P, x)$ and take $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ as in Lemma 4.4. From the inequalities

$$\begin{aligned} \bar{\sigma}^0_{i^0} \cdot \bar{\rho}_i > 0, i \in I^*(\bar{\rho}) \cap \mathbf{I}^\rho,\ \bar{\rho}_i \cdot \rho_i > 0, i \in I^*(\bar{\rho}), \\ \bar{\sigma}^0_{i^0} \cdot \bar{\sigma}_j > 0, j \in I^*(\bar{\sigma}) \cap \mathbf{I}^\sigma,\ \bar{\sigma}_j \cdot \sigma_j > 0, j \in I^*(\bar{\sigma}) \end{aligned}$$

we get

$$\bar{\sigma}^0_{i^0} \cdot \rho_i \geq 0, i \in \mathbf{I}^\rho,\ \bar{\sigma}^0_{i^0} \cdot \sigma_j \geq 0, j \in \mathbf{I}^\sigma.$$

Thus, after rearranging constraints, we obtain $\bar{\sigma}^0_{i^0} \cdot (\rho, \sigma) \in \mathcal{L}(P^{\text{aux}}, x)$ and $x \in \Sigma(P^{\text{aux}})$. Now, let $x \in \Sigma(P^{\text{aux}}) \cap V$ and $(\lambda, \mu) \in \text{ext}\, \mathcal{L}(P^{\text{aux}}, x)$. After rearranging constraints, let $(\rho, \sigma) = [\bar{\sigma}^0_{i^0}]^{-1} \cdot (\lambda, \mu)$. By A2, we have $I^\rho \cap I^\sigma = \emptyset$ and, hence, $x \in M[r, s]$. By using the same argument as before, we get $x \in \Sigma^C(P)$. Now, take an arbitrarily chosen $(\bar{\lambda}, \bar{\mu}) \in \text{ext}\, \mathcal{L}(\bar{P}^{\text{aux}}, \bar{x})$ and $(\bar{\rho}, \bar{\sigma}) = [\bar{\sigma}^0_{i^0}]^{-1} \cdot (\bar{\lambda}, \bar{\mu})$. By Lemma 4.1 we obtain $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. Since MPCC-LICQ does not hold at $\bar{x}$, assume without loss of generality that $\bar{\rho}_{i^1} = 0$ for some $i^1 \in I_{\bar{r}\bar{s}}(\bar{x})$. Since $(\bar{\rho}, \bar{\sigma})$ fulfills Condition C^* and $\bar{\sigma}^0_{i^0} \cdot \bar{\sigma}_{i^1} > 0$, by $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$, it follows that

$$D^2_x \bar{\mathbf{L}}^{\text{aux}}(\bar{x}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\lambda}, \bar{\mu})} = \frac{\bar{\sigma}^0_{i^0}}{\bar{\sigma}_{i^1}} D^2_x \bar{\mathbf{L}}^{\text{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma}) \bar{\sigma}_{i^1}|_{T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma})} \succ 0 \tag{6.2}$$

and, therefore, according to [22, Theorem 7.2], $\bar{x} \in \Sigma^S(\bar{P}^{\text{aux}})$. Thus, by (6.1), we obtain $\bar{x} \in \Sigma^S(\bar{P})$.

(i) $\Rightarrow$ (ii). By Theorem 5.1, it follows that $(\bar{\rho}, \bar{\sigma})$ fulfills Condition C^* for all $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. In order to show $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$ we proceed by induction on $|I_{\bar{r}\bar{s}}(\bar{x})|$. Suppose that $|I_{\bar{r}\bar{s}}(\bar{x})| = 1$. By A2, we have $\hat{N}(\bar{P}, \bar{x}) = 1$ and, by Lemma 6.1, it follows that $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$. Assume now that $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$ holds whenever $|I_{\bar{r}\bar{s}}(\bar{x})| = p \geq 1$ and $\bar{x} \in \Sigma^S(\bar{P})$. For $|I_{\bar{r}\bar{s}}(\bar{x})| = p+1$ assume $\bar{x} \in \Sigma^S(\bar{P})$, $\hat{N}(\bar{P}, \bar{x}) > 1$, fix $i^0 \in I^{\bar{\rho}^0}$ and define

$$\begin{aligned} \hat{\mathbf{I}}^\rho(\bar{\rho}^0, \bar{\sigma}^0) &= \bigcup_{(\bar{\rho}, \bar{\sigma}) \in C(\bar{\rho}^0, \bar{\sigma}^0)} I^{\bar{\rho}}, \\ \hat{\mathbf{I}}^\sigma(\bar{\rho}^0, \bar{\sigma}^0) &= \bigcup_{(\bar{\rho}, \bar{\sigma}) \in C(\bar{\rho}^0, \bar{\sigma}^0)} I^{\bar{\sigma}}. \end{aligned}$$

The remainder of the proof is given in eight steps.

Step 1 Fix an edge $\{(\bar\rho^1,\bar\sigma^1),(\bar\rho^2,\bar\sigma^2)\}\in E(\bar P,\bar x)$ with $(\bar\rho^1,\bar\sigma^1)=(\bar\rho^0,\bar\sigma^0)$. The case where such an egde does not exist runs analogously. After possibly interchanging constraints, choose without loss of generality

$$i^1\in I^{\bar\rho^1}\backslash I^{\bar\rho^2},\ i^2\in I^{\bar\rho^1}\cap I^{\bar\rho^2}.$$

Fix $\varepsilon>0$ sufficiently small and set

$$r^{\varepsilon}_{i^2}(x)=\bar r_{i^2}(x)+\varepsilon,\ r^{\varepsilon}_i(x)=\bar r_i(x),\ i\in L\backslash\{i^2\},\ P^{\varepsilon}=\mathcal{P}(\bar f,r^{\varepsilon},\bar s).\quad (6.3)$$

Note that $(\bar\rho^1,\bar\sigma^1),(\bar\rho^2,\bar\sigma^2)\in\mathcal{L}^0(P^{\varepsilon},\bar x)$ and that $|I_{r^{\varepsilon}\bar s}(\bar x)|=p$. Hence, by induction hypothesis, we get $\mathcal{L}(P^{\varepsilon},\bar x)=\hat{\mathcal{L}}(P^{\varepsilon},\bar x)$. Therefore, the set $\mathcal{L}(P^{\varepsilon},\bar x)$ is convex and

$$\begin{aligned}&\bar\sigma^1_{i^1}\cdot\bar\rho^1_i\geq 0,\ \bar\sigma^1_{i^1}\cdot\bar\rho^2_i\geq 0,\ i\in I^{\bar\rho^1}\cup I^{\bar\rho^2},\\&\bar\sigma^1_{i^1}\cdot\bar\sigma^1_j\geq 0,\ \bar\sigma^1_{i^1}\cdot\bar\sigma^2_j\geq 0,\ j\in I^{\bar\sigma^1}\cup I^{\bar\sigma^2},\end{aligned}$$

Moreover, by A2 and the convexity of $\mathcal{L}(P^{\varepsilon},\bar x)$, we obtain $\bar\sigma^1_{i^1}\cdot\bar\sigma^2_{i^2}>0$ and

$$\bar\rho^1_i\cdot\bar\rho^2_i>0,\ i\in I_{\bar r\bar s}(\bar x)\backslash\big[I^{\bar\rho^1}\cup I^{\bar\rho^2}\big],\ \bar\sigma^1_j\cdot\bar\sigma^2_j>0,\ j\in I_{\bar r\bar s}(\bar x)\backslash\big[I^{\bar\sigma^1}\cup I^{\bar\sigma^2}\big].$$

Step 2 The previous argument can be repeated along $C(\bar\rho^0,\bar\sigma^0)$ by taking adjacent vertices. Thus, for each $(\bar\rho,\bar\sigma)\in C(\bar\rho^0,\bar\sigma^0)$ it holds that

$$\bar\sigma^0_{i^0}\cdot\bar\rho_i\geq 0,\ i\in\hat{\mathbf{I}}^{\rho}\big(\bar\rho^0,\bar\sigma^0\big),\ \bar\sigma^0_{i^0}\cdot\bar\sigma_j\geq 0,\ j\in\hat{\mathbf{I}}^{\sigma}\big(\bar\rho^0,\bar\sigma^0\big)\quad (6.4)$$

and

$$\bar\rho^0_i\cdot\bar\rho_i>0,\ i\in I_{\bar r\bar s}(\bar x)\backslash\hat{\mathbf{I}}^{\rho}\big(\bar\rho^0,\bar\sigma^0\big),\ \bar\sigma^0_j\cdot\bar\sigma_j>0,\ j\in I_{\bar r\bar s}(\bar x)\backslash\hat{\mathbf{I}}^{\sigma}\big(\bar\rho^0,\bar\sigma^0\big).\quad (6.5)$$

Now, choose $(\bar\rho,\bar\sigma)\in C(\bar\rho^0,\bar\sigma^0)$, $m^0\in I^{\bar\rho}\cup I^{\bar\sigma}$ and let

$$\hat{\mathcal{L}}(\bar P,\bar x,C(\bar\rho^0,\bar\sigma^0))=\left\{(\rho,\sigma)\in\mathbb{R}^{2l}\ \middle|\ \begin{array}{c}D_x\,\bar{\mathbf{L}}^{\mathrm{cc}}(\bar x,\rho,\sigma)=0,\\ \rho_m\cdot\bar r_m(\bar x)=\sigma_m\cdot\bar s_m(\bar x)=0,\ m\in L,\\ (\bar\rho_{m^0}+\bar\sigma_{m^0})\cdot\rho_i\geq 0,\ i\in\hat{\mathbf{I}}^{\rho}(\bar\rho,\bar\sigma),\\ (\bar\rho_{m^0}+\bar\sigma_{m^0})\cdot\sigma_j\geq 0,\ j\in\hat{\mathbf{I}}^{\sigma}(\bar\rho,\bar\sigma)\end{array}\right\}.$$

From (6.4), it follows that the convex set $\hat{\mathcal{L}}(\bar P,\bar x,C(\bar\rho^0,\bar\sigma^0))$ is well defined, that is, it is independent from the choice of $(\bar\rho,\bar\sigma)\in C(\bar\rho^0,\bar\sigma^0)$ and $m^0\in I^{\bar\rho}\cup I^{\bar\sigma}$. In addition, it holds

$$C\big(\bar\rho^0,\bar\sigma^0\big)\subset\operatorname{ext}\hat{\mathcal{L}}\big(\bar P,\bar x,C\big(\bar\rho^0,\bar\sigma^0\big)\big).\quad (6.6)$$

Step 3 Next, we show that $\hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$ is bounded. Suppose contrarily that there exists an unbounded sequence $(\rho^k, \sigma^k) \in \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$ with

$$\frac{(\rho^k, \sigma^k)}{\|(\rho^k, \sigma^k)\|} \to (\bar{\alpha}, \bar{\beta})$$

for some $(\bar{\alpha}, \bar{\beta}) \in S(0, 1)$. By taking the limit, we obtain

$$\sum_{m \in L} [\bar{\alpha}_m D_x \bar{r}_m(\bar{x}) + \bar{\beta}_m D_x \bar{s}_m(\bar{x})] = 0,$$
$$\bar{\alpha}_m \cdot \bar{r}_m(\bar{x}) = \bar{\beta}_m \cdot \bar{s}_m(\bar{x}) = 0, m \in L,$$
$$\left(\bar{\rho}^0_{m^0} + \bar{\sigma}^0_{m^0}\right) \cdot \bar{\alpha}_i \geq 0,\ i \in \hat{\mathbf{I}}^\rho\left(\bar{\rho}^0, \bar{\sigma}^0\right),$$
$$\left(\bar{\rho}^0_{m^0} + \bar{\sigma}^0_{m^0}\right) \cdot \bar{\beta}_j \geq 0,\ j \in \hat{\mathbf{I}}^\sigma\left(\bar{\rho}^0, \bar{\sigma}^0\right).$$

By $\hat{N}(\bar{P}, \bar{x}) > 1$, assume $\bar{\alpha}_{i^2} = 0$ for some $i^2 \in \hat{\mathbf{I}}^\rho(\bar{\rho}^0, \bar{\sigma}^0)$ and perturb as in (6.3). Furthermore, since $i^2 \in \hat{\mathbf{I}}^\rho(\bar{\rho}^0, \bar{\sigma}^0)$, we fix $(\bar{\rho}, \bar{\sigma}) \in C(\bar{\rho}^0, \bar{\sigma}^0)$ with $i^2 \in I^{\bar{\rho}}$. By the induction hypothesis, it follows that $\mathcal{L}(P^\varepsilon, \bar{x}) = \hat{\mathcal{L}}(P^\varepsilon, \bar{x})$. Obviously, $(\bar{\rho}, \bar{\sigma}) + (\bar{\alpha}, \bar{\beta})t \in \hat{\mathcal{L}}(P^\varepsilon, \bar{x})$ for $t \geq 0$. However, A1 implies that $\mathcal{L}(P^\varepsilon, \bar{x})$ is compact, which is a contradiction. Thus, the set $\hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$ is bounded.

Step 4 Now, we show that

$$\operatorname{ext} \hat{\mathcal{L}}\left(\bar{P}, \bar{x}, C\left(\bar{\rho}^0, \bar{\sigma}^0\right)\right) \subset \mathcal{L}\left(\bar{P}, \bar{x}\right). \tag{6.7}$$

Assume contrarily that for some $m^3 \in I_{\bar{r}\bar{s}}(\bar{x})$ and some $(\bar{\rho}^3, \bar{\sigma}^3) \in \operatorname{ext} \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$ it holds that $\bar{\rho}^3_{m^3} \cdot \bar{\sigma}^3_{m^3} < 0$. Since $(\bar{\rho}^3, \bar{\sigma}^3) \in \operatorname{ext} \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$, it follows that the vectors

$$\begin{aligned} &D_x \bar{r}_i(\bar{x}), i \in \left[I_{\bar{r}}(\bar{x}) \backslash \hat{\mathbf{I}}^\rho(\bar{\rho}^0, \bar{\sigma}^0)\right] \cup I^*(\bar{\rho}^3),\ D_x \bar{s}_j(\bar{x}), \\ &j \in \left[I_{\bar{s}}(\bar{x}) \backslash \hat{\mathbf{I}}^\sigma(\bar{\rho}^0, \bar{\sigma}^0)\right] \cup I^*(\bar{\sigma}^3) \end{aligned} \tag{6.8}$$

are linearly independent. Hence, since MPCC-LICQ does not hold at $\bar{x}$, assume without loss of generality that there exists $i^3 \in \hat{\mathbf{I}}^\rho(\bar{\rho}^0, \bar{\sigma}^0) \cap I^{\bar{\rho}^3}$. Fix $\varepsilon > 0$ sufficiently small and let

$$r^\varepsilon_{i^3}(x) = \bar{r}_{i^3}(x) + \varepsilon,\ r^\varepsilon_i(x) = \bar{r}_i(x),\ i \in L \backslash \{i^3\},\ P^\varepsilon = \mathcal{P}\left(\bar{f}, r^\varepsilon, \bar{s}\right).$$

Then, we have $|I_{r^\varepsilon \bar{s}}(\bar{x})| = p$ and by induction hypothesis we get

$$\left(\bar{\rho}^3, \bar{\sigma}^3\right) \in \hat{\mathcal{L}}\left(P^\varepsilon, \bar{x}\right) = \mathcal{L}\left(P^\varepsilon, \bar{x}\right)$$

which contradicts $\bar{\rho}^3_{m^3} \cdot \bar{\sigma}^3_{m^3} < 0$. Thus, (6.7) holds.

Step 5 Next, we show that

$$\operatorname{ext} \hat{\mathcal{L}}\big(\bar{P}, \bar{x}, C\big(\bar{\rho}^0, \bar{\sigma}^0\big)\big) = C\big(\bar{\rho}^0, \bar{\sigma}^0\big). \tag{6.9}$$

Let $(\bar{\rho}^3, \bar{\sigma}^3) \in \operatorname{ext} \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$. Following (6.8) the vectors

$$D_x \bar{r}_i(\bar{x}),\ i \in I_{\bar{r}}(\bar{x}) \cup I^*(\bar{\rho}^3),\ \ D_x \bar{s}_j(\bar{x}),\ j \in I_{\bar{s}}\big(\bar{x}\big) \cup I^*(\bar{\sigma}^3\big)$$

are linearly independent. By Lemma 4.1, we get $(\bar{\rho}^3, \bar{\sigma}^3) \in \mathcal{L}^0(\bar{P}, \bar{x})$. Hence, we have

$$\operatorname{ext} \hat{\mathcal{L}}\big(\bar{P}, \bar{x}, C\big(\bar{\rho}^0, \bar{\sigma}^0\big)\big) \subset \mathcal{L}^0\big(\bar{P}, \bar{x}\big). \tag{6.10}$$

Furthermore, by (6.6), (6.10) and since

$$\big[I^{\bar{\rho}} \cap \hat{\mathbf{I}}^{\rho}\big(\bar{\rho}^0, \bar{\sigma}^0\big)\big] \cup \big[I^{\bar{\sigma}} \cap \hat{\mathbf{I}}^{\sigma}\big(\bar{\rho}^0, \bar{\sigma}^0\big)\big] \neq \emptyset,$$

holds for all $(\bar{\rho}, \bar{\sigma}) \in \operatorname{ext} \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$, we obtain (6.9).

Step 6 Now, consider the following standard nonlinear program

$$\begin{aligned} P^{\text{aux}}: \quad & \min\ \bar{\sigma}_{\bar{i}^0} \cdot f(x) \\ \text{s. t.} \quad & r_i(x) \geq 0,\ i \in \hat{\mathbf{I}}^{\rho}(\bar{\rho}^0, \bar{\sigma}^0),\ \ s_j(x) \geq 0,\ j \in \hat{\mathbf{I}}^{\sigma}(\bar{\rho}^0, \bar{\sigma}^0), \\ & r_i(x) = 0,\ i \in \bar{I}_{\bar{r}}(\bar{x}) \backslash \hat{\mathbf{I}}^{\rho}(\bar{\rho}^0, \bar{\sigma}^0),\ \ s_j(x) = 0,\ j \in \bar{I}_{\bar{s}}(\bar{x}) \backslash \hat{\mathbf{I}}^{\sigma}(\bar{\rho}^0, \bar{\sigma}^0). \end{aligned}$$

After rearranging constraints, it holds that

$$\mathcal{L}(\bar{P}^{\text{aux}}, \bar{x}) = \bar{\sigma}_{\bar{i}^0} \cdot \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0)).$$

Hence, the set $\mathcal{L}(\bar{P}^{\text{aux}}, \bar{x})$ is bounded and, consequently, MFCQ holds at $\bar{x}$ for $\bar{P}^{\text{aux}}$. In addition, by (6.9), it follows that

$$\operatorname{ext} \mathcal{L}(\bar{P}^{\text{aux}}, \bar{x}) = \bar{\sigma}_{\bar{i}^0} \cdot \operatorname{ext} \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0)) = \bar{\sigma}_{\bar{i}^0} \cdot C(\bar{\rho}^0, \bar{\sigma}^0). \tag{6.11}$$

Since each $(\bar{\rho}, \bar{\sigma}) \in C(\bar{\rho}^0, \bar{\sigma}^0)$ fulfills Condition C^*, MFCQ holds at $\bar{x}$ for $\bar{P}^{\text{aux}}$ and (6.11), by [22, Theorem 7.2] and an expression analogous to (6.2), it follows that $\bar{x} \in \Sigma^S(\bar{P}^{\text{aux}})$. Moreover, analogously to (6.1), there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that

$$\Sigma(P^{\text{aux}}) \cap V \subset \Sigma^C(P), \tag{6.12}$$

for all $P \in W$.

Step 7 Now, we will show that

$$C(\bar{\rho}^0, \bar{\sigma}^0) = \mathcal{L}^0(\bar{P}, \bar{x}). \tag{6.13}$$

Suppose contrarily that there exists $(\bar{\rho}^4, \bar{\sigma}^4) \in \mathcal{L}^0(\bar{P}, \bar{x}) \backslash C(\bar{\rho}^0, \bar{\sigma}^0)$ and without loss of generality fix $j^4 \in I^{\bar{\sigma}^4} \backslash \hat{\mathbf{I}}^{\sigma}(\bar{\rho}^0, \bar{\sigma}^0)$. For $\varepsilon > 0$ sufficiently small perturb

$$s^{\varepsilon}_{j^4}(x) = \bar{s}_{j^4}(x) + \varepsilon.$$

Since $\bar{x} \in \Sigma^S(\bar{P}^{\text{aux}})$, there exists a solution $x^{\varepsilon} \in \Sigma^S(P^{\text{aux},\varepsilon})$ near $\bar{x}$. From (6.12), we get $x^{\varepsilon} \in \Sigma^C(P^{\varepsilon})$. Moreover, it is easy to see that $\bar{x} \in \Sigma^C(P^{\varepsilon})$ and $\bar{x} \neq x^{\varepsilon}$. Therefore, we get

$$|\Sigma^C(P^{\varepsilon}) \cap V| \geq |\{\bar{x}, x^{\varepsilon}\}| = 2$$

which contradicts $\bar{x} \in \Sigma^S(\bar{P})$. Consequently, we have (6.13).

Step 8 Finally, we will show that $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$. Note that (6.13) implies

$$\mathbf{I}^{\rho} = \hat{\mathbf{I}}^{\rho}(\bar{\rho}^0, \bar{\sigma}^0), \ \mathbf{I}^{\sigma} = \hat{\mathbf{I}}^{\sigma}(\bar{\rho}^0, \bar{\sigma}^0),$$

and $\hat{\mathcal{L}}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0))$. By (6.4) and (6.5) we get $\mathcal{L}^0(\bar{P}, \bar{x}) \subset \mathcal{L}(\bar{P}, \bar{x}, I^+(\bar{\rho}^0, \bar{\sigma}^0))$ and that for $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ with $I \neq I^+(\bar{\rho}^0, \bar{\sigma}^0)$ it holds that $\mathcal{L}^0(\bar{P}, \bar{x}) \cap \mathcal{L}(\bar{P}, \bar{x}, I) = \emptyset$. The latter, Theorem 4.1 and (6.9) give

$$\begin{aligned}\mathcal{L}(\bar{P}, \bar{x}) = \bigcup_{I \subset I_{\bar{r}\bar{s}}(\bar{x})} \operatorname{conv}\left[\mathcal{L}^0(\bar{P}, \bar{x}) \cap \mathcal{L}(\bar{P}, \bar{x}, I)\right] = \operatorname{conv} \mathcal{L}^0(\bar{P}, \bar{x}) = \\ = \operatorname{conv} \operatorname{ext} \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0)) = \hat{\mathcal{L}}(\bar{P}, \bar{x}, C(\bar{\rho}^0, \bar{\sigma}^0)) = \hat{\mathcal{L}}(\bar{P}, \bar{x}).\end{aligned}$$

This completes the proof. □

Next, we provide an example in which the previous theorem is used. Note that in this case the Hessian is not positive definite but negative definite on the corresponding tangent space.

Example 6.1 Let $n = 5$, $\bar{x} = 0$ and consider the problem $\bar{P}$ given by

$$\begin{aligned} \min 4x_1 - 6x_2 - 6x_3 + x_4 - x_5^2 \\ \text{s.t.} \\ \min\{x_1, x_4\} = 0, \\ \min\{x_2, x_1 + 3x_2 + x_3 + x_4\} = 0, \\ \min\{x_3, 3x_1 - x_2 + 2x_3 + x_4\} = 0. \end{aligned}$$

The set of Lagrange vectors at $\bar{x}$ is

$$\mathcal{L}(\bar{P}, \bar{x}) = \left\{ (\bar{\rho}, \bar{\sigma}) \in \mathbb{R}^6 \,\middle|\, \begin{aligned} \bar{\rho}_1 + \bar{\sigma}_2 + 3\bar{\sigma}_3 = 4, \\ \bar{\rho}_2 + 3\bar{\sigma}_2 - \bar{\sigma}_3 = -6, \\ = \bar{\rho}_3 + \bar{\sigma}_2 + 2\bar{\sigma}_3 = -6, \\ \bar{\sigma}_1 + \bar{\sigma}_2 + \bar{\sigma}_3 = 1, \\ \bar{\rho}_2, \bar{\rho}_3, \bar{\sigma}_2, \bar{\sigma}_3 \leq 0 \end{aligned} \right\} = \hat{\mathcal{L}}(\bar{P}, \bar{x})$$

and the elements of $\mathcal{L}^0(\bar{P}, \bar{x})$ are those listed in the following table.

$\bar{\rho}_1$	$\bar{\rho}_2$	$\bar{\rho}_3$	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{\sigma}_3$
4	−6	−6	1	0	0
13	−9	0	4	0	−3
6	0	−4	3	−2	0
11.71	0	0	5.29	−2.57	−1.71

Moreover, for any $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ it holds that

$$D_x^2 \bar{\mathbf{L}}^{\mathrm{cc}}(\bar{x}, \bar{\rho}, \bar{\sigma}) = \mathrm{diag}(0, 0, 0, 0, -2), \ T_{\bar{x}}(\bar{r}, \bar{s}, \bar{\rho}, \bar{\sigma}) = \mathrm{span}\{(0, 0, 0, 0, 1)\}$$

and, therefore, $(\bar{\rho}, \bar{\sigma})$ fulfills Condition C^*. By Theorem 6.2, it follows that $0 \in \Sigma^S(\bar{P})$.

Now, we provide a characterization of strong stability of a C-stationary point when $N^0(\bar{P}, \bar{x}) \leq n$. We point out that in the following corollary both A1 and A2 appear as necessary conditions for strong stability (and not as assumptions throughout this section).

Corollary 6.2 *If $N^0(\bar{P}, \bar{x}) \leq n$, then the following statements are equivalent:*

(i) $\bar{x} \in \Sigma^S(\bar{P})$
(ii) *A1 and A2 hold, $\mathcal{L}(\bar{P}, \bar{x}) = \hat{\mathcal{L}}(\bar{P}, \bar{x})$ and each $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$ fulfills Condition C^*.*

Proof (ii) $\Rightarrow$ (i). It immediately follows from Theorem 6.2.
(i) $\Rightarrow$ (ii). It follows from [10, Theorem 5.8], Corollary 5.2 and Theorems 5.1 and 6.2. □

In the next lemmas we assume (2) in Theorem 6.1.

Lemma 6.2 *Assume that $|I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x})| + |I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x})| = n$ for all $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. Then, the following statements are equivalent:*

(i) *There exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{x})$ with $|\Sigma^C(P) \cap V| \leq 1$ for all $P \in W$.*
(ii) *For $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$ there exists $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l}$ such that (5.4) and (5.5) hold with*

$$\bar{\alpha}_i > 0, \ i \in I^{\bar{\rho}^1} \backslash I^{\bar{\rho}^2}, \ \bar{\alpha}_i = 0, \ i \in I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2}, \ \bar{\alpha}_i < 0, \ i \in I^{\bar{\rho}^2} \backslash I^{\bar{\rho}^1}, \quad (6.14)$$
$$\bar{\beta}_j > 0, \ j \in I^{\bar{\sigma}^1} \backslash I^{\bar{\sigma}^2}, \ \bar{\beta}_j = 0, \ j \in I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2}, \ \bar{\beta}_j < 0, \ j \in I^{\bar{\sigma}^2} \backslash I^{\bar{\sigma}^1}. \quad (6.15)$$

Proof (i) $\Rightarrow$ (ii). Suppose contrarily that there exist $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in \mathcal{L}^0(\bar{P}, \bar{x})$ with $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$ and that there does not exist $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l}$ as described in (ii). Assume, after interchanging constraints, fixing $\varepsilon > 0$ sufficiently small and perturbing

$$r_i^{\varepsilon}(x) = \bar{r}_i(x) + \varepsilon, \ i \in I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2}, \ s_j^{\varepsilon}(x) = \bar{s}_j(x) + \varepsilon, \ j \in I^{\bar{\sigma}^1} \cap I^{\bar{\sigma}^2}, \quad (6.16)$$

that $I^{\bar{\sigma}^1} = I^{\bar{\rho}^2} = \emptyset$. By a dual argument there exist $v \in S(0, 1)$ such that

$$\frac{\partial \bar{r}_i(\bar{x})}{\partial v} \geq 0,\ i \in I^{\bar{\rho}^1},\ \frac{\partial \bar{r}_i(\bar{x})}{\partial v} = 0,\ i \in \bar{I}_{\bar{r}}(\bar{x}) \backslash I^{\bar{\rho}^1},$$
$$\frac{\partial \bar{s}_j(\bar{x})}{\partial v} \leq 0,\ j \in I^{\bar{\sigma}^2},\ \frac{\partial \bar{s}_j(\bar{x})}{\partial v} = 0,\ j \in \bar{I}_{\bar{s}}(\bar{x}) \backslash I^{\bar{\sigma}^2}.$$

Fix $\varepsilon > 0$ sufficiently small and perturb

$$s_j^{\varepsilon}(x) = \bar{s}_j(x) - \varepsilon v^T (x - \bar{x}),\ j \in I^{\bar{\sigma}^2}.$$

For simplicity of notation denote s^{ε} again by $\bar{s}$. After this perturbation, we obtain

$$\frac{\partial \bar{s}_j(\bar{x})}{\partial v} < 0,\ j \in I^{\bar{\sigma}^2}.$$

Now, consider the mapping

$$\Phi(x, t) = \begin{pmatrix} \bar{r}_i(x) - t \dfrac{\partial \bar{r}_i(\bar{x})}{\partial v}, & i \in \bar{I}_{\bar{r}}(\bar{x}) \\ \bar{s}_j(x) - t \dfrac{\partial \bar{s}_j(\bar{x})}{\partial v}, & j \in \bar{I}_{\bar{s}}(\bar{x}) \backslash I^{\bar{\sigma}^2} \end{pmatrix}.$$

Note that $\Phi(\bar{x}, 0) = 0$ and that $D_x \Phi(\bar{x}, 0)$ has full rank. By the implicit function theorem, there exist $\bar{t} > 0$ and $\mathcal{C}^2$ curve $\varphi \colon (-\bar{t}, \bar{t}) \to \mathbb{R}^n$ such that $\Phi(\varphi(t), t) = 0$ for $t \in (-\bar{t}, \bar{t})$. By using the derivative at $t = 0$ of $\Phi(\varphi(t), t) = 0$, we obtain

$$\frac{d\bar{s}_j(\varphi(0))}{dt} = \frac{\partial \bar{s}_j(\bar{x})}{\partial v} < 0,\ \ j \in I^{\bar{\sigma}^2}.$$

Perhaps by shrinking $\bar{t}$ assume that the functions $\bar{s}_j(\varphi(t)),\ j \in I^{\bar{\sigma}^2}$ are strictly decreasing on $(-\bar{t}, \bar{t})$. Now take $\varepsilon > 0$ sufficiently small and perturb

$$s^{\varepsilon}(x) = \bar{s}(x) - \bar{s}(\varphi(\varepsilon)).$$

Let $P^{\varepsilon} = \mathcal{P}(\bar{f}, \bar{r}, s^{\varepsilon})$ and $x^{\varepsilon} = \varphi(\varepsilon)$. It is easy to see that $\bar{x} \in \Sigma^C(P^{\varepsilon})$. Moreover, by applying a continuity argument, using

$$D_x \bar{f}(\bar{x}) = \sum_{i \in \bar{I}_{\bar{r}}(\bar{x}) \backslash I^{\bar{\rho}^1}} \bar{\rho}_i^1 D_x \bar{r}_i(\bar{x}) + \sum_{j \in \bar{I}_{\bar{s}}(\bar{x})} \bar{\sigma}_j^1 D_x \bar{s}_s(\bar{x})$$

and $\bar{r}_i(x^{\varepsilon}) \geq 0, i \in I^{\bar{\rho}^1}$, we get $x^{\varepsilon} \in \Sigma^C(P^{\varepsilon})$. Since the functions $\bar{s}_j(\varphi(t)),\ j \in I^{\bar{\sigma}^2}$ are strictly decreasing on $(-\bar{t}, \bar{t})$, it follows that $\varphi(0) \neq \varphi(\varepsilon)$. Therefore, $\bar{x} \neq x^{\varepsilon}$ and

$$|\Sigma^C(P^{\varepsilon}) \cap V| \geq |\{\bar{x}, x^{\varepsilon}\}| = 2$$

which is a contradiction.

(ii) $\Rightarrow$ (i).

Suppose contrarily that there exist sequences $x^{1,k} \to \bar{x}$, $x^{2,k} \to \bar{x}$, $x^{1,k} \neq x^{2,k}$, $P^k \to \bar{P}$ with $x^{1,k}, x^{2,k} \in \Sigma^C(P^k)$ and

$$\frac{x^{2,k} - x^{1,k}}{\left\| x^{2,k} - x^{1,k} \right\|} \to v \neq 0.$$

Take $(\rho^{1,k}, \sigma^{1,k}) \in \mathcal{L}^0(P^k, x^{1,k})$, $(\rho^{2,k}, \sigma^{2,k}) \in \mathcal{L}^0(P^k, x^{2,k})$ and, by Lemma 4.4, assume that

$$I^*(\bar{\rho}^1) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\rho^{1,k}),\ I^*(\bar{\sigma}^1) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\sigma^{1,k}),$$
$$I^*(\bar{\rho}^2) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\rho^{2,k}),\ I^*(\bar{\sigma}^2) \cap I_{\bar{r}\bar{s}}(\bar{x}) \subset I^*(\sigma^{2,k}).$$

Moreover, after perhaps perturbing $(\bar{r}, \bar{s})$ and (r^k, s^k) as in (6.16) assume that

$$I^{\bar{\sigma}^1} = I^{\bar{\rho}^2} = \emptyset.$$

Note that

$$r_i^k(x^{2,k}) - r_i^k(x^{1,k}) \leq 0,\ i \in I^{\bar{\rho}^1},\ s_j^k(x^{2,k}) - s_j^k(x^{1,k}) \geq 0,\ j \in I^{\bar{\sigma}^2},$$
$$r_i^k(x^{2,k}) - r_i^k(x^{1,k}) = 0,\ \ i \in \bar{I}_{\bar{r}}(\bar{x}) \backslash I^{\bar{\rho}^1},\ s_j^k(x^{2,k}) - s_j^k(x^{1,k}) = 0,\ \ j \in \bar{I}_{\bar{s}}(\bar{x}) \backslash I^{\bar{\sigma}^2}.$$

Letting $k \to +\infty$, we get

$$D_x\bar{r}_i(\bar{x})v \leq 0,\ i \in I^{\bar{\rho}^1},\ D_x\bar{s}_j(\bar{x})v \geq 0,\ j \in I^{\bar{\sigma}^2}, \tag{6.17}$$
$$D_x\bar{r}_i(\bar{x})v = 0,\ \ i \in \bar{I}_{\bar{r}}(\bar{x}) \backslash I^{\bar{\rho}^1},\ D_x\bar{s}_j(\bar{x})v = 0,\ \ j \in \bar{I}_{\bar{s}}(\bar{x}) \backslash I^{\bar{\sigma}^2}. \tag{6.18}$$

By Lemma 4.1, after possibly interchanging constraints, assume that

$$D_x\bar{r}_{i^0}(\bar{x})v < 0 \tag{6.19}$$

for some $i^0 \in I^{\bar{\rho}^1}$. A dual statement to (6.17–6.19) yields a contradiction to the existence of $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l}$ with (5.4), (5.5), (6.14) and (6.15). □

Lemma 6.3 *Assume that* $|I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x})| + |I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x})| = n$ *for all* $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. *If* $\hat{N}(\bar{P}, \bar{x}) \geq 2$, *then for any* $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ *the subgraph of* $G(\bar{P}, \bar{x})$ *whose set of vertices is* ext $\mathcal{L}(\bar{P}, \bar{x}, I)$ *is connected.*

Proof Let $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ and assume that ext $\mathcal{L}(\bar{P}, \bar{x}, I) \neq \emptyset$. We will mainly follow [37, Sect. 3.2]. Fix $c \in \mathbb{R}^{2l}$ in such a way that

$$c^T(\bar{\rho}^1, \bar{\sigma}^1) \neq c^T(\bar{\rho}^2, \bar{\sigma}^2)$$

for any $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in \text{ext}\, \mathcal{L}(\bar{P}, \bar{x}, I)$ with $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$. The latter and the compactness of $\mathcal{L}(\bar{P}, \bar{x}, I)$ imply the existence of a unique solution $(\bar{\rho}^0, \bar{\sigma}^0) \in \text{ext}\, \mathcal{L}(\bar{P}, \bar{x}, I)$ to the linear program

$$\min c^T(\rho, \sigma) \;\; \text{s. t.}\; (\rho, \sigma) \in \mathcal{L}(\bar{P}, \bar{x}, I).$$

Let now $(\bar{\rho}, \bar{\sigma}) \in \text{ext}\, \mathcal{L}(\bar{P}, \bar{x}, I)$ with $(\bar{\rho}, \bar{\sigma}) \neq (\bar{\rho}^0, \bar{\sigma}^0)$. By applying the Simplex method to this linear program with initial feasible solution $(\bar{\rho}, \bar{\sigma})$ and taking into account that $\hat{N}(\bar{P}, \bar{x}) \geq 2$, we find a feasible path connecting $(\bar{\rho}, \bar{\sigma})$ and $(\bar{\rho}^0, \bar{\sigma}^0)$. Since $(\bar{\rho}, \bar{\sigma})$ was arbitrarily chosen, the result follows. □

Now, assuming (2) in Theorem 6.1, we characterize the strong stability of $\bar{x} \in \Sigma^C(\bar{P})$.

Theorem 6.3 *Assume that* $|I^*(\bar{\rho}) \cup I_{\bar{r}}(\bar{x})| + |I^*(\bar{\sigma}) \cup I_{\bar{s}}(\bar{x})| = n$ *for all* $(\bar{\rho}, \bar{\sigma}) \in \mathcal{L}^0(\bar{P}, \bar{x})$. *Then, the following statements are equivalent:*

(i) $\bar{x} \in \Sigma^S(\bar{P})$
(ii) $\mathcal{L}(\bar{P}, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x}, I)$ *for some* $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ *and for* $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in \mathcal{L}^0(\bar{P}, \bar{x})$ *with* $(\bar{\rho}^1, \bar{\sigma}^1) \neq (\bar{\rho}^2, \bar{\sigma}^2)$ *there exists* $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l}$ *with* (5.4)*,* (5.5)*,* (6.14) *and* (6.15)*.*

Proof (ii) ⇒ (i). Assume without loss of generality that $I_{\bar{r}\bar{s}}(\bar{x}) = L$ and for a problem P with the same number of constraints as $\bar{P}$ consider the mapping $F^{P,I} : \mathbb{R}^{n+2l} \to \mathbb{R}^{n+2l}$ given by

$$F^{P,I}(x, \tau, \zeta) = \begin{pmatrix} D_x f(x) - \sum\limits_{m \in I} \left[\tau_m^+ D_x r_m(x) + \zeta_m^+ D_x s_m(x)\right] + \\ + \sum\limits_{m \in L \setminus I} \left[\tau_m^+ D_x r_m(x) + \zeta_m^+ D_x s_m(x)\right] \\ r_m(x) + \tau_m^-,\; m \in L \\ s_m(x) + \zeta_m^-,\; m \in L \end{pmatrix} \tag{6.20}$$

where $(\tau, \zeta) \in \mathbb{R}^{2l}$ and $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ is chosen as in (ii). Consider the sets

$$\mathcal{S}^I(P, x) = \left\{(\tau, \zeta) \in \mathbb{R}^{2l} \,|\, F^{P,I}(x, \tau, \zeta) = 0\right\} \text{ and } \Sigma^I(P) = \{x \in \mathbb{R}^n \,|\, \mathcal{S}^I(P, x) \neq \emptyset\}.$$

Note that

$$\mathcal{L}(\bar{P}, \bar{x}, I) = A\, \mathcal{S}^I(\bar{P}, \bar{x}) \tag{6.21}$$

where A is a diagonal matrix with entry 1 in $|I|$ rows and entry -1 in $|L \setminus I|$ rows. By C-MFCQ, the set $\mathcal{S}^I(\bar{P}, \bar{x})$ is bounded. Hence, analogously to (6.1), there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that

$$\Sigma^I(P) \cap V = \Sigma^C(P) \cap V$$

for $P \in W$. By Lemma 6.2, assume after possibly shrinking V and W that

$$|\Sigma^I(P) \cap \text{cl } V| \leq 1 \tag{6.22}$$

for $P \in W$. Since $\mathcal{S}^I(\bar{P}, \bar{x})$ is bounded, there exists an open and bounded set $\Theta \subset \mathbb{R}^{2l}$ such that, after possibly shrinking V and W, it holds that $\mathcal{S}^I(P, x) \subset \Theta$ for $x \in V$ and $P \in W$. Assume without loss of generality that V is open. Next, we show that

$$0 \notin F^{\bar{P},I}(\text{bd}[V \times \Theta]). \tag{6.23}$$

Since $V \in \mathcal{V}(\bar{x})$ and Θ is open, it follows that

$$\text{bd}[V \times \Theta] \cap [\{\bar{x}\} \times \Theta] = \emptyset. \tag{6.24}$$

Moreover, it holds that

$$\text{bd}[V \times \Theta] \subset \text{cl } V \times \text{cl } \Theta. \tag{6.25}$$

By (6.24) and (6.25) we get

$$F^{\bar{P},I}(\text{bd}[V \times \Theta]) \subset F^{\bar{P},I}([\text{cl } V \times \text{cl } \Theta] \backslash [\{\bar{x}\} \times \Theta]). \tag{6.26}$$

Hence, by (6.22) and (6.26) we obtain (6.23). Consequently, $\deg(F^{\bar{P},I}, V \times \Theta, 0)$ is well defined where deg denotes the *degree of a mapping*, see [27] for more details. By the homotopy invariance theorem [27, Theorem 6.2.2], after perhaps shrinking W, we assume that $\deg(F^{P,I}, V \times \Theta, 0)$ is well defined for $P \in W$. Following [22, pp. 126 – 127] we get

$$\deg(F^{P,I}, V \times \Theta, 0) \in \{-1, 1\}.$$

Thus, by Kronecker's theorem [27, Theorem 6.3.1] it follows that $|\Sigma^I(P) \cap V| \geq 1$. Note that V and W can be further shrunk. Consequently, $\bar{x} \in \Sigma^S(\bar{P})$.

(i) $\Rightarrow$ (ii). By Lemma 6.2, it remains to be shown that $\mathcal{L}(\bar{P}, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x}, I)$ for some $I \subset I_{\bar{r}\bar{s}}(\bar{x})$. We proceed by induction on $|I_{\bar{r}\bar{s}}(\bar{x})|$. Suppose that $|I_{\bar{r}\bar{s}}(\bar{x})| = 1$. Then, $\hat{N}(\bar{P}, \bar{x}) = 1$ and, by Corollary 6.1, it follows that $\mathcal{L}(\bar{P}, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x}, I)$ for some $I \subset I_{\bar{r}\bar{s}}(\bar{x})$. Assume that $\mathcal{L}(\bar{P}, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x}, I)$ for some $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ whenever $|I_{\bar{r}\bar{s}}(\bar{x})| = p \geq 1$. For $|I_{\bar{r}\bar{s}}(\bar{x})| = p + 1$ assume that $\hat{N}(\bar{P}, \bar{x}) > 1$. The remainder of the proof is given in five steps.

Step 1 Fix an edge $\{(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2)\} \in E(\bar{P}, \bar{x})$ with $(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2) \in C(\bar{\rho}^0, \bar{\sigma}^0)$ for some $(\bar{\rho}^0, \bar{\sigma}^0) \in \mathcal{L}^0(\bar{P}, \bar{x})$. The case where such an egde does not exist runs analogously. We show that $I^+(\bar{\rho}^1, \bar{\sigma}^1) = I^+(\bar{\rho}^2, \bar{\sigma}^2)$. Assume without loss of generality that there is an index $i^0 \in I^{\bar{\rho}^1} \cap I^{\bar{\rho}^2}$, fix $\varepsilon > 0$ sufficiently small and let

$$r^{\varepsilon}_{i^0}(x) = \bar{r}_{i^0}(x) + \varepsilon,\ r^{\varepsilon}_i(x) = \bar{r}_i(x),\ i \in L \backslash \{i^0\},\ P^{\varepsilon} = \mathcal{P}(\bar{f}, r^{\varepsilon}, \bar{s}).$$

Note that $|I_{r^\varepsilon \bar{s}}(\bar{x})| = p$. By induction hypothesis, we get $\mathcal{L}(P^\varepsilon, \bar{x}) = \mathcal{L}(P^\varepsilon, \bar{x}, I)$ for some $I \subset I_{\bar{r}\bar{s}}(\bar{x})\backslash\{i^0\}$. Hence, $\mathcal{L}(P^\varepsilon, \bar{x})$ is convex. By the latter and A2, we get

$$I^+(\bar{\rho}^1, \bar{\sigma}^1)\backslash\{i^0\} = I^+(\bar{\rho}^2, \bar{\sigma}^2)\backslash\{i^0\},\ \bar{\sigma}^1_{i^0} \cdot \bar{\sigma}^2_{i^0} > 0.$$

Thus, $I^+(\bar{\rho}^1, \bar{\sigma}^1) = I^+(\bar{\rho}^2, \bar{\sigma}^2)$.

Step 2 The previous argument can be repeated along $C(\bar{\rho}^0, \bar{\sigma}^0)$ by taking adjacent vertices. Therefore, there exists $I \subset I_{\bar{r}\bar{s}}(\bar{x})$ such that for each $(\bar{\rho}, \bar{\sigma}) \in C(\bar{\rho}^0, \bar{\sigma}^0)$ it holds that $I^+(\bar{\rho}, \bar{\sigma}) = I$. By Lemmas 4.1 and 4.2, we get

$$\mathcal{L}(\bar{P}, \bar{x}, I) \subset \mathcal{L}(\bar{P}, \bar{x}),\ C(\bar{\rho}^0, \bar{\sigma}^0) \subset \operatorname{ext} \mathcal{L}(\bar{P}, \bar{x}, I) \subset \mathcal{L}^0(\bar{P}, \bar{x})$$

and, by Lemma 6.3,

$$\operatorname{ext} \mathcal{L}(\bar{P}, \bar{x}, I) = C(\bar{\rho}^0, \bar{\sigma}^0). \tag{6.27}$$

Step 3 Consider the mapping $F^{P,I}(x, \tau, \zeta)$ in (6.20) and the property (6.21). By Lemma 6.2 and the previous argument with the *degree of a mapping*, it follows that there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}(\bar{P})$ such that

$$|\Sigma^I(P) \cap V| = 1,\ \ P \in W \tag{6.28}$$

and

$$\Sigma^I(P) \cap V \to \{\bar{x}\} \text{ as } P \to \bar{P}. \tag{6.29}$$

Furthermore, analogously to (6.1) we obtain

$$\Sigma^I(P) \cap V \subset \Sigma^C(P). \tag{6.30}$$

Step 4 Next, we will show that

$$C(\bar{\rho}^0, \bar{\sigma}^0) = \mathcal{L}^0(\bar{P}, \bar{x}). \tag{6.31}$$

Suppose contrarily that there exists $(\bar{\rho}^3, \bar{\sigma}^3) \in \mathcal{L}^0(\bar{P}, \bar{x})\backslash C(\bar{\rho}^0, \bar{\sigma}^0)$. Let $j^3 \in I^{\bar{\sigma}^3}$ and for $\varepsilon > 0$ sufficiently small perturb $\bar{P}$ by

$$s^\varepsilon_{j^3}(x) = \bar{s}_{j^3}(x) + \varepsilon.$$

By (6.28) and (6.29) there exists $x^\varepsilon \in \Sigma^I(P^\varepsilon)$ near $\bar{x}$. Since $(\bar{\rho}^3, \bar{\sigma}^3) \notin C(\bar{\rho}^0, \bar{\sigma}^0)$, by (6.21) it follows that $\bar{\zeta}_{j^3} \neq 0$ for all $(\bar{\tau}, \bar{\zeta}) \in \mathcal{S}^I(\bar{P}, \bar{x})$. By the latter, a continuity argument yields $s^\varepsilon_{j^3}(x^\varepsilon) = 0$. From (6.30), we get $x^\varepsilon \in \Sigma^C(P^\varepsilon)$. Now, it is easy to verify that $\bar{x} \in \Sigma^C(P^\varepsilon)$ and that $\bar{x} \neq x^\varepsilon$. Therefore, we obtain

$$|\Sigma^C(P^\varepsilon) \cap V| \geq |\{\bar{x}, x^\varepsilon\}| = 2$$

which contradicts $\bar{x} \in \Sigma^S(\bar{P})$. Thus, we get (6.31).

Step 5 Finally, analogously to Step 8 in the proof of Theorem 6.2, by A1, Lemma 4.1 and (6.31), it follows that

$$\mathcal{L}(\bar{P}, \bar{x}) = \bigcup_{I' \subset I_{\bar{r}\bar{s}}(\bar{x})} \text{conv}\left[\mathcal{L}^0(\bar{P}, \bar{x}) \cap \mathcal{L}(\bar{P}, \bar{x}, I')\right] = \text{conv}\, \mathcal{L}^0(\bar{P}, \bar{x}) =$$
$$= \text{conv ext}\, \mathcal{L}(\bar{P}, \bar{x}, I) = \mathcal{L}(\bar{P}, \bar{x}, I).$$

This completes the proof. □

In the following example we apply Theorem 6.3.

Example 6.2 Let $n = 4$, $\bar{x} = 0$ and consider the problem $\bar{P}$ given by

$$\begin{aligned} \min -3x_1 - 3x_2 + x_4 \\ \text{s. t.} \\ \min\{x_1, x_4\} = 0, \\ \min\{x_2, -3x_1 + x_2 + x_3 - 3x_4\} = 0, \\ \min\{x_3, -2x_1 - 3x_2 + x_3 - x_4\} = 0. \end{aligned}$$

The set of Lagrange vectors at $\bar{x}$ is

$$\mathcal{L}(\bar{P}, \bar{x}) = \left\{ (\bar{\rho}, \bar{\sigma}) \in \mathbb{R}^6 \;\middle|\; \begin{aligned} \bar{\rho}_1 - 3\bar{\sigma}_2 - 2\bar{\sigma} &= -3, \\ \bar{\rho}_2 + \bar{\sigma}_2 - 3\bar{\sigma}_3 &= -3, \\ \bar{\rho}_3 + \bar{\sigma}_2 + \bar{\sigma}_3 &= 0, \\ = \bar{\sigma}_1 - 3\bar{\sigma}_2 - \bar{\sigma}_3 &= 1, \\ \bar{\rho}_1, \bar{\sigma}_1 &\leq 0, \\ \bar{\rho}_2, \bar{\sigma}_2 &\leq 0, \\ \bar{\rho}_3, \bar{\sigma}_3 &\geq 0 \end{aligned} \right\} = \mathcal{L}(\bar{P}, \bar{x}, \{3\})$$

which verifies the first part of condition (ii) in Theorem 6.3, that is $\mathcal{L}(\bar{P}, \bar{x}) = \mathcal{L}(\bar{P}, \bar{x}, I)$ for some $I \subset I_{\bar{r}\bar{s}}(\bar{x})$. The four elements of $\mathcal{L}^0(\bar{P}, \bar{x})$ are those listed in the following table.

$\bar{\rho}_1$	$\bar{\rho}_2$	$\bar{\rho}_3$	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{\sigma}_3$
−4	−2.67	0.33	0	−0.33	0
−3.5	−1	0	0	−0.5	0.5
−12	0	3	−8	−3	0
−3.75	0	0	−0.5	−0.75	0.75

To verify the remaining of condition (ii) in Theorem 6.3, for each pair of basic Lagrange vectors we list the corresponding $(\bar{\alpha}, \bar{\beta}) \in \mathbb{R}^{2l}$ in the following table.

	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\alpha}_3$	$\bar{\beta}_1$	$\bar{\beta}_2$	$\bar{\beta}_3$
$(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^2, \bar{\sigma}^2)$	−1	−3.33	0.67	0	0.33	−1
$(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^3, \bar{\sigma}^3)$	−3	1	1	−3	−1	0
$(\bar{\rho}^1, \bar{\sigma}^1), (\bar{\rho}^4, \bar{\sigma}^4)$	−3.5	0.5	0.5	−3.5	−1.5	−0.5
$(\bar{\rho}^2, \bar{\sigma}^2), (\bar{\rho}^3, \bar{\sigma}^3)$	−0.5	−2.83	1.17	0.5	0.83	−0.5
$(\bar{\rho}^2, \bar{\sigma}^2), (\bar{\rho}^4, \bar{\sigma}^4)$	1	−4	0	2	1	−1
$(\bar{\rho}^3, \bar{\sigma}^3), (\bar{\rho}^4, \bar{\sigma}^4)$	−11	0	4	−10	−3	−1

By Theorem 6.3, it follows that $0 \in \Sigma^S(\bar{P})$.

We terminate this section by presenting a result about the convexity of $\mathcal{L}(\bar{P}, \bar{x})$, which easily follows from Theorems 6.2 and 6.3.

Corollary 6.3 *If $\bar{x} \in \Sigma^S(\bar{P})$, then $\mathcal{L}(\bar{P}, \bar{x}) = \text{conv}\, \mathcal{L}^0(\bar{P}, \bar{x})$.*

7 Final remarks

In this paper we considered mathematical programs with complementarity constraints (MPCC) and presented a topological as well as an equivalent algebraic characterization of the strong stability of a C-stationary point of MPCC. Strong stability refers to the local uniqueness, existence and continuous dependence of a solution for each sufficiently small perturbed problem where perturbations up to second order are allowed. Moreover, a second order necessary condition, which we called Condition C^*, was presented.

Since an MPCC has a more combinatorial structure than a standard nonlinear optimization problem, more sophisticated algebraic techniques were necessarily applied to establish our results. For example, we characterized the *set of Lagrange vectors* and defined the set of *basic* Lagrange vectors, which we denoted by $\mathcal{L}(\bar{P}, \bar{x})$ and $\mathcal{L}^0(\bar{P}, \bar{x})$, respectively. As mentioned in the beginning of Sect. 4, in [22, Theorem 7.2], the concept of extreme points plays an essential role. However, in our setting the set $\mathcal{L}(\bar{P}, \bar{x})$ is, in general, not convex and, consequently, this concept cannot be directly employed in the characterization of strongly stable C-stationary points. Our solution to this issue was to define and characterize the set $\mathcal{L}^0(\bar{P}, \bar{x})$, which plays a role in our results similarly to that of the set of extreme points in [22, Theorem 7.2].

Moreover, note that Condition C^* plays a crucial role in Theorems 6.2 and 6.3, but not in [18, Theorem 3.1] where MPCC-LICQ holds. Hence, our results differ from those in [18] not just in the fulfillment of MPCC-LICQ, but in the matter of the second order Condition C^*. Furthermore, the set $\mathcal{L}(\bar{P}, \bar{x})$ is always a singleton in [18], whereas in our context, that need not to be the case, see for instance Examples 6.1 and 6.2. Instead, under Assumptions A1 and A2, we showed that the convexity of $\mathcal{L}(\bar{P}, \bar{x})$ is necessary for the strong stability of C-stationary points.

Finally, we recall that there are several other concepts of stationarity for MPCC such as A-, B-, M- and S-stationarity; we refer e.g. to [34,36] for an overview on relations among them. The characterization of strong stability of these types of stationary points are topic of current research.

Acknowledgements The authors wish to thank both referees for their constructive and deep review of the original manuscript which has definitely improved the quality of this paper.

Funding Open Access funding provided by University of Bergen.

References

1. Bertsekas, D.P., Nedic, A., Ozdaglar, A.E.: Convex Analysis and Optimization. Athena Scientific, Belmont (2003)
2. Cervinka, M., Outrata, J.V., Pistek, M.: On stability of M-stationary points in MPCCs. Set-Valued Var. Anal. **22**, 575–595 (2014)
3. Clarke, F.H.: Optimization and Nonsmooth Analysis. Society for Industrial and Applied Mathematics, Philadelphia (1990)
4. Dontchev, A.L., Tyrrell Rockafellar, R.: Characterizations of Lipschitzian stability in nonlinear programming. In: Mathematical Programming with Data Perturbations, pp. 65–82. Marcel Dekker, Inc (1998)
5. Dorsch, D., Günzel, H., Guerra Vázquez, F., Rückmann, J.-J.: Semi-infinite programming: strong stability implies EMFCQ. J. Convex Anal. **19**(4), 999–1008 (2012)
6. Flegel, M.L., Kanzow, C.: On M-stationary points for mathematical programs with equilibrium constraints. J. Math. Anal. Appl. **310**(1), 286–302 (2005)
7. Flegel, M.L., Kanzow, C., Outrata, J.V.: Optimality conditions for disjunctive programs with application to mathematical programs with equilibrium constraints. Set-Valued Anal. **15**(2), 139–162 (2007)
8. Fletcher, R., Leyffer, S., Ralph, D., Scholtes, S.: Local convergence of SQP methods for mathematical programs with equilibrium constraints. SIAM J. Optim. **17**(1), 259–286 (2006)
9. Günzel, H., Jongen, H.T.: Strong stability implies Mangasarian–Fromovitz Constraint qualification. Optimization **55**(5–6), 605–610 (2006)
10. Hernández Escobar, D., Rückmann, J.-J.: MPCC: on necessary conditions for the strong stability of C-stationary points. Optimization **68**(2–3), 593–613 (2019)
11. Hernández Escobar, D., Rückmann, J.-J.: MPCC: strongly stable C-stationary points when the number of active constraints is n + 1. Optimization **69**(5), 1039–1067 (2020)
12. Hoheisel, T., Kanzow, C., Schwartz, A.: Theoretical and numerical comparison of relaxation methods for mathematical programs with complementarity constraints. Math. Program. **137**, 257–288 (2013)
13. Jongen, H.T., Jonker, P., Twilt, F.: Nonlinear Optimization in Finite Dimensions. Kluwer Academic Publishers, Dordrecht (2000)
14. Jongen, H.T., Klatte, D., Tammer, K.: Implicit functions and sensitivity of stationary points. Math. Program. **49**(1), 123–138 (1990)
15. Jongen, H.T., Möbert, T., Rückmann, J.-J., Tammer, K.: On inertia and Schur complement in optimization. Linear Algebra Appl. **95**, 97–109 (1987)
16. Jongen, H.T., Rückmann, J.-J., Shikhman, V.: MPCC: critical point theory. SIAM J. Optim. **20**(1), 473–484 (2009)
17. Jongen, H.T., Rückmann, J.-J., Shikhman, V.: On stability of the feasible set of a mathematical problem with complementarity problems. SIAM J. Optim. **20**(3), 1171–1184 (2009)
18. Jongen, H.T., Shikhman, V., Steffensen, S.: Characterization of strong stability for C-stationary points in MPCC. Math. Program. **132**(1), 295–308 (2012)

19. Kanzow, C., Schwartz, A.: Mathematical programs with equilibrium constraints: enhanced Fritz John-conditions, new constraint qualifications, and improved exact penalty results. SIAM J. Optim. **20**(5), 2730–2753 (2010)
20. Kanzow, C., Schwartz, A.: A new regularization method for mathematical programs with complementarity constraints with strong convergence properties. SIAM J. Optim. **23**, 770–798 (2016)
21. Klatte, D., Tammer, K.: Strong stability of stationary solutions and Karush–Kuhn–Tucker points in nonlinear optimization. Ann. Oper. Res. **27**(1), 285–308 (1990)
22. Kojima, M.: Strongly stable stationary solutions in nonlinear programs. In: Robinson, S.M. (ed.) Analysis and Computation of Fixed Points, pp. 93–138. Academic Press, New York (1980)
23. Leyffer, S., López-Calva, G., Nocedal, J.: Interior methods for mathematical programs with complementarity constraints. SIAM J. Optim. **17**(1), 52–77 (2006)
24. Luo, Z.-Q., Pang, J.-S., Ralph, D.: Mathematical Programs with Equilibrium Constraints. Cambridge University Press, Cambridge (1996)
25. Milnor, J.: Morse Theory. Annals of Mathematics Studies, vol. 51. Princeton University Press, Princeton (1963)
26. Mordukhovich, B.S., Outrata, J.V.: Tilt stability in nonlinear programming under Mangasarian–Fromovitz constraint qualification. Kybernetika **49**, 446–464 (2013)
27. Ortega, J.M., Rheinboldt, W.C.: Chapter 6—The degree of a mapping. In: Iterative Solution of Nonlinear Equations in Several Variables, pp. 147–177. Academic Press, Cambridge(1970)
28. Outrata, J.V.: Optimality conditions for a class of mathematical programs with equilibrium constraints. Math. Oper. Res. **24**(3), 627–644 (1999)
29. Poliquin, R.A., Tyrrell Rockafellar, R.: Tilt stability of a local minimum. SIAM J. Optim. **8**, 287–299 (1998)
30. Ralph, D., Stein, O.: The c-index: a new stability concept for quadratic programs with complementarity constraints. Math. Oper. Res. **36**(08), 504–526 (2011)
31. Robinson, S.M.: Strongly regular generalized equations. Math. Oper. Res. **5**(1), 43–62 (1980)
32. Rückmann, J.-J.: On existence and uniqueness of stationary points in semi-infinite optimization. Math. Program. **86**(2), 387–415 (1999)
33. Scheel, H., Scholtes, S.: Mathematical programs with complementarity constraints: stationarity, optimality, and sensitivity. Math. Oper. Res. **25**(1), 1–22 (2000)
34. Shikhman, V.: Topological Aspects of Nonsmooth Optimization. Springer, New York (2012)
35. Ye, J.J.: Optimality conditions for optimization problems with complementarity constraints. SIAM J. Optim. **9**(2), 374–387 (1999)
36. Ye, J.J.: Necessary and sufficient optimality conditions for mathematical programs with equilibrium constraints. J. Math. Anal. Appl. **307**(1), 350–369 (2005)
37. Ziegler, G.M.: Lectures on Polytopes. Springer, New York (1995)