∞-categorical Thom spectra

Harald Ivarsen

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Master's Programme in Mathematics
Matematisk institutt
Universitetet i Bergen

Contents

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Introduction

This paper is meant as an introduction to an ∞-categorical theory of Thom spectra. We assume a basic knowledge of algebraic topology and simplicial homotopy theory. We begin by constructing the simplicial category of orthogonal spectra, $\mathrm{Sp}^{\mathbb{O}}$, a category of spectra with a monoidal structure. In particular a tensor product, the smash product. This smash product enables us to define orthogonal ring spectra, i.e an orthogonal spectrum R, equipped with a map $R \wedge R \to R$, satisfying certain coherence diagrams. In other words a monoid in $\mathrm{Sp}^{\mathbb{O}}$. To each such orthogonal ring spectrum R, we have the simplicial category of R-modules, $R-$mod. We continue by defining a model structure, called the stable model structure, showing that it is compatible with the simplicial structure. We can for each simplicial category $\mathcal{S}$, associate a simplicial set $N_\Delta(\mathcal{S})$, an enhanced version of the usual nerve functor $N : \mathrm{Cat} \to Set_\Delta$, capturing more of the homotopical information. If the mapping spaces of the simplicial category $\mathcal{S}$ are Kan complexes, then $N_\Delta(\mathcal{S})$ is a particularly nice simplicial set, by fulfilling a certain horn-filler lifting condition, and we call these simplicial sets ∞-categories. In ∞-categories we can define notions from ordinary category theory, such as composition, colimits, slice categories and so on, although often only up to homotopy equivalence. Let $R-$line be an ∞-category of R-modules weakly equivalent to R. We then define the Thom spectra of a simplicial map

$$f : K \to R\text{−line} \hookrightarrow N_\Delta(R\text{−mod}_{cf})$$

to be the ∞-categorical colimit of f. We finally go through some examples, and show that $R-$line is equivalent as a ∞-category to the classifying space of the ∞-category $\mathrm{Aut}(R)$ of automorphisms of R as an R-module, which helps us analyse the Thom spectra of a morphism $\widehat{f} : \Sigma K \to R-\mathrm{line} \hookrightarrow N_\Delta(R-\mathrm{mod}_{cf})$, when $\widehat{f}$ is adjoint to a map $f : K \to \mathrm{Aut}(R)$.

1 Orthogonal Spectra

1.1 Orthogonal Spectra

Definition 1.1.1. An *orthogonal spectrum* X, is a sequence of pointed topological spaces $\{X_n \,|\, n \in \mathbb{N}\}$, with X_n equipped with an $O(n)$-action, and structure maps

$$\sigma_n : X_n \wedge S^1 \to X_{n+1}$$

such that any composition

$$\begin{gathered}\sigma^k : X_n \wedge S^k \cong X_n \wedge S^1 \wedge S^{k-1} \xrightarrow{\sigma_n \wedge 1} X_{n+1} \wedge S^{k-1} \cong \\ X_{n+1} \wedge S^1 \wedge S^{k-2} \xrightarrow{\sigma_{n+1} \wedge 1} \cdots \xrightarrow{\sigma_{n+k-2} \wedge 1} M_{n+k-1} \wedge S^1 \xrightarrow{\sigma_{n+k-1}} X_{n+k}\end{gathered}$$

is $O(n) \times O(k)$-equivariant. Note that by *pointed topological spaces* one often means a restriction to an appropriate category of pointed topological spaces, in our case we will use compactly generated Hausdorff spaces. We will continue to refer to this category as Top_*. A *map of orthogonal spectra* $f : X \to Y$, is a collection of maps $\{f_n : X_n \to Y_n \,|\, n \in \mathbb{N}\}$, that commute with the structure maps

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "\sigma_n"] \arrow[d, "f_n \wedge 1"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ Y_n \wedge S^1 \arrow[r, "\sigma_n"] & Y_{n+1} \end{tikzcd}$$

Definition 1.1.2. The *sphere spectrum*, $\mathbb{S}$, is the spectrum with $\mathbb{S}_n = S^n$ for every $n \geq 0$, and structure maps given by isomorphisms $S_n \wedge S^1 \cong S^{n+1}$.

Theorem 1.1.3. [Rog21, Theorem 9.7.1] *The category of orthogonal spectra, denoted $Sp^{\mathbb{O}}$, is symmetric monoidal, with unit object given by the sphere spectrum $\mathbb{S}$, and with the monoidal paring $X \wedge Y$ given levelwise by coequalizer*

$$\begin{tikzcd} \bigvee_{a+b+c=n} O(n)_+ \wedge_{O(a)\times O(b)\times O(c)} (X_a \wedge S^b \wedge Y_c) \arrow[d, shift left] \arrow[d, shift right] \\ \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} (X_i \wedge Y_j) \arrow[d] \\ (X \wedge Y)_n \end{tikzcd}$$

where the left map is given by

$$O(n)_+ \wedge X_a \wedge S^b \wedge Y_c \xrightarrow{\sigma^b \wedge 1} O(n)_+ \wedge X_{i=a+b} \wedge Y_c$$

and the right map is given on elements by

$$\begin{array}{c} A \wedge x \wedge s \wedge y \mapsto (A \cdot I_a \times \chi_{c,b}) \wedge x \wedge y \wedge s \\ \downarrow \\ (A \cdot I_a \times \chi_{c,b}) \wedge x \wedge \sigma^b(y \wedge s) \in O(n)_+ \wedge X_b \wedge Y_{j=b+c} \end{array}$$

Here $\chi_{c,b}$ is the $(c+b \times c+b)$-matrix permutating the first b coordinates with the c last ones. The twist map is induced by the mappings

$$\begin{array}{c} A \wedge x \wedge y \in \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} (X_i \wedge Y_j) \\ \downarrow \\ A \cdot \chi_{j,i} \wedge x \wedge y \in \bigvee_{i+j=n} O(n)_+ \wedge_{O(i)\times O(j)} (X_i \wedge Y_j) \end{array}$$

on the spaces defining the coequalizer, applying it multiple times on the first space.

1.2 Model structure on $\mathbf{Sp}^{\mathbb{O}}$

Definition 1.2.1. Let $\alpha_n : \pi_n(M_k) \to \pi_{n+1}(M_{k+1})$ be the map given by

$$(\gamma : S^n \to M_k) \mapsto (\sigma \circ (\gamma \wedge 1) : S^n \wedge S^1 \to M_k \wedge S^1 \to M_{k+1})$$

the *graded homotopy groups* $\pi_*(X)$ of an orthogonal spectra X, is given in degree n by the colimit of the sequence

$$\cdots \to \pi_{n+k}(M_k) \xrightarrow{\alpha_{n+k}} \pi_{n+k+1}(M_{k+1}) \to \cdots$$

for $n + k \geq 2$. A spectra map $f : X \to Y$ gives morphisms

$$\pi_{n+k}(f) : \pi_{n+k}(X_k) \to \pi_{n+k}(Y_k)$$

which are compatible with the colimit. Therefore we have a functor $\pi_* : \mathrm{Sp}^{\mathbb{O}} \to \mathrm{Ab}^{\mathbb{Z}}$.

Definition 1.2.2. A *stable equivalence* is a spectra map $f : X \to Y$, which induces a isomorphism $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$.

Definition 1.2.3. For every $l \geq 0$, let the degree-l *evaluation functor*, denoted $Ev_l : Sp^{\mathbb{O}} \to \mathrm{Top}$, be $Ev_l(X) = X_l$. Ev_l has a left adjoint, the *degree-l free functor*, $F_l : \mathrm{Top} \to Sp^{\mathbb{O}}$. We see that

$$(F_l(A))_n = \begin{cases} A \wedge S^{n-l} & \text{for} n \geq l \\ * & \text{otherwise.} \end{cases}$$

Definition 1.2.4. Let $I = \{i : S^{n-1}_+ \to D^n_+ \,|\, n \geq 0\}$. Let $FI = \{F_l i : F_l S^{n-1}_+ :\to D^n_+ \,|\, l \geq 0, \, i \in I\}$. A spectra map $i : X \to Y$ is called a *relative cell spectrum* if i is the colimit of a sequence

$$X = Y(0) \to Y(1) \to ... \to Y(n) \to Y(n+1) \to ... \to Y$$

where each map $Y(n) \to Y(n+1)$ given by a pushout

$$\begin{tikzcd}
\bigvee_\alpha S(\alpha) \arrow[r, "\alpha"] \arrow[d, "\phi"] & \bigvee_\alpha D(\alpha) \arrow[d] \\
Y(n) \arrow[r] & Y(n+1)
\end{tikzcd}$$

where $\alpha \in FI$.

Definition 1.2.5. A spectra map $i : X \to Y$ is called a *Quillen cofibration* if it is a retract of a relative cell spectrum, meaning that there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i"] & X' \arrow[r] \arrow[d, "i'"] & X \arrow[d, "i"] \\
Y \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

where $i : X' \to Y'$ is a relative cell spectrum.

Remark 1.2.6. Notice that since colimits are taken levelwise in $Sp^{\mathbb{O}}$, a Quillen-cofibration is levelwise a cofibration in Top_*. But note that if a spectra map is levelwise a cofibration in Top_*, it is not necessarily a cofibration in $Sp^{\mathbb{O}}$.

Definition 1.2.7. Let $\bar{\sigma} : X_n \to \Omega X_{n+1}$, be the adjoint map of $\sigma : X_n \wedge S^1 \to X_{n+1}$ in the loop-suspension adjunction. A map $p : X \to Y$ is called a *stable fibration* if $p_n : X_n \to Y_n$ is a Serre fibration for every $n \geq 0$, and the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\bar{\sigma}"] \arrow[d, "p_n"] & \Omega X_{n+1} \arrow[d, "\Omega p_{n+1}"] \\
Y_n \arrow[r, "\bar{\sigma}"] & \Omega Y_{n+1}
\end{tikzcd}$$

is a weak homotopy pullback for every $n \geq 0$, meaning that the induced map $X_n \to Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$, is weak homotopy equivalence.

Theorem 1.2.8. [BR20, Theorem 5.2.16] *$Sp^{\mathbb{O}}$ has a model structure with the stable fibrations, Quillen cofibrations, and stable equivalences being the fibrations, cofibrations, and weak equivalences respectively. We call this the stable model structure on $Sp^{\mathbb{O}}$.*

Theorem 1.2.9. [BR20, p. 185] *$Sp^{\mathbb{O}}$ has a model structure with the levelwise fibrations and levelwise topological weak equivalences being the fibrations, and weak equivalences respectively. We call this the levelwise model structure on $Sp^{\mathbb{O}}$.*

Theorem 1.2.10. [BR20, Theorem 2.3.12] *A spectral map $f : X \to Y$ is trivial stable fibration if and only if it is a levelwise trivial fibration. Thus the levelwise model structure and the stable model structure on $Sp^{\mathbb{O}}$ share trivial fibrations and cofibrations.*

Proof. Consider a trivial stable fibration $f : X \to Y$, we know that it is a levelwise fibration, so it remains to show that it is a levelwise weak homotopy equivalence. Since f is a levelwise fibration, the fiber is given levelwise by the homotopy fiber $Ff_n = f_n^{-1}$. Looking at the pullback diagram

$$\begin{tikzcd}
Ff \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
* \arrow[r] & Y
\end{tikzcd}$$

we have that $Ff \to *$ is a trivial stable fibration. This have two consequences. First we a have a weak homotopy equivalence

$$Ff_n \cong \Omega Ff_{n+1} \times_{\Omega *_{n+1}} *_n \cong \Omega Ff_{n+1}.$$

Second, we have $\pi_k(Ff) = 0$ for every k, but from the first consequence, and the definition of homotopy groups of spectra we have $\pi_k(Ff_n) \cong \pi_k(Ff) = 0$ for every k and $n \geq 0$. From the long exact sequence of homotopy groups we then have

$$\pi_k(X_n) \cong \pi_k(Y_n)$$

for every $n \geq 0$ and $k > 0$. But for loop-spaces we have a weak homotopy equivalence

$$\Omega f_n : \Omega X_n \to \Omega Y_n$$

for every $n \geq 0$. Consider the diagram

$$\begin{tikzcd}
X_n \arrow[dr] \arrow[ddr, bend right, "f_n"'] & & \\
& Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1} \arrow[r] \arrow[d] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \\
& Y_n \arrow[r] & \Omega Y_{n+1}
\end{tikzcd} \tag{1}$$

we see that we can write f_n as the composite $X_n \to Y_n \times_{\Omega Y_n} \Omega X_n \to Y_n$. The first map is a weak homotopy equivalence since f is stable fibration. The second map is a weak homotopy equivalence since Ωf_n is a trivial fibration, thus f_n is a weak homotopy equivalence for every n.

Now consider diagram (1) when f is a levelwise trivial fibration. We know Ωf_{n+1} is a trivial fibration, and since trivial fibrations are preserved by pullbacks $\times_{\Omega Y_n} \Omega X_n \to Y_n$ is on as well. Since $f_n : X_n \to Y_n$ is a weak homotopy equivalence, $X_n \to Y_n \times_{\Omega Y_n} \Omega X_n$ is a weak equivalence as well by the 2-out-of-3 property.

□

2 Simplicial Structures

2.1 The simplicial model structure on $\mathbf{Sp}^{\mathbb{O}}$

Lemma 2.1.1. [GJ09, Lemma II.2.4] *Let $\mathcal{C}$ be a category equipped with a functor*

$$- \otimes - : \mathcal{C} \times Set_\Delta \to \mathcal{C}.$$

Suppose the following conditions hold

1. *$- \otimes K : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\mathbf{hom}_{\mathcal{C}}(K, -)$ for fixed $K \in Set_\Delta$.*
2. *$X \otimes - : Set_\Delta \to \mathcal{C}$ commutes with colimits, and $X \otimes * \cong X$, for fixed $X \in \mathcal{C}$.*
3. *For $X \in \mathcal{C}$, and $K, T \in Set_\Delta$, there is a natural isomorphism $(X \otimes K) \otimes T \cong X \otimes (K \times T)$.*

Then $\mathcal{C}$ is a simplicial category with simplicial mapping space given by $\mathbf{Hom}_{\mathcal{C}}(X, Y)_n = \hom_{\mathcal{C}}(X \otimes \Delta^n, Y)$.

Definition 2.1.2. Let $\mathcal{C}$ be a category that is both a model category and a simplicial category. If for every cofibration $i : A \to B$ and fibration $p : Y \to Y$, the induced map

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration, which is acyclic if either i or p is, then $\mathcal{C}$ is called a *simplicial model category*, and the above condition is called the *simplicial model axiom.*

Proposition 2.1.3. [GJ09, Proposition 2.3.11, Corollary 2.3.12, Proposition 2.3.13]

Let $\mathcal{C}$ be a simplicial category, and a model category. Then the following are equivalent

1. *The simplicial model axiom holds.*
2. *that for any cofibration $i : X \to Y$ in $\mathcal{C}$, and $n \geq 0$ the induced map*

$$\begin{tikzcd}
X \otimes \partial\Delta^n \arrow[r] \arrow[d] & Y \otimes \partial\Delta^n \arrow[d] \arrow[ddr, bend left] & \\
X \otimes \Delta^n \arrow[r] \arrow[drr, bend right] & X \otimes \Delta^n \cup_{X \otimes \partial\Delta^n} Y \otimes \partial\Delta^n \arrow[dr, dashed] & \\
 & & Y \otimes \Delta^n
\end{tikzcd}$$

is a cofibration, which is trivial if i is, and for $e = 1$ or 0

$$X \otimes \Delta^1 \cup_{X \otimes \{e\}} Y \otimes \{e\} \to Y \otimes \Delta^1$$

is a trivial cofibration.

3. *For any cofibration $i : X \to Y$ in $\mathcal{C}$, and cofibration $j : A \to B$, the induced map*

$$\begin{tikzcd}
X \otimes A \arrow[r] \arrow[d] & Y \otimes A \arrow[d] \arrow[ddr, bend left] & \\
X \otimes B \arrow[r] \arrow[drr, bend right] & X \otimes B \cup_{X \otimes A} Y \otimes A \arrow[dr, dashed] & \\
 & & Y \otimes B
\end{tikzcd}$$

is a cofibration, which is trivial if either i or j is.

4. *For a cofibration $i : K \to L$ in Set_Δ and fibration $p : X \to Y$ in $\mathcal{C}$ the induced map*

$$
\begin{tikzcd}
\mathbf{hom}_{\mathcal{C}}(L,X) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathbf{hom}_{\mathcal{C}}(K,X) \times_{\mathbf{hom}_{\mathcal{C}}(K,Y)} \mathbf{hom}_{\mathcal{C}}(L,Y) \arrow[r] \arrow[d] & \mathbf{hom}_{\mathcal{C}}(L,Y) \arrow[d] \\
& \mathbf{hom}_{\mathcal{C}}(K,X) \arrow[r] & \mathbf{hom}_{\mathcal{C}}(K,Y)
\end{tikzcd}
$$

is a fibration which is trivial if either i or p is.

Example 2.1.4. The category of pointed compactly generated Hausdorff spaces, denoted Top_* is a simplicial model category, with

$$X \otimes K = X \times |K|_+.$$

Theorem 2.1.5. *$Sp^{\mathbb{O}}$ is a simplicial model category.*

Proof. First we show that $Sp^{\mathbb{O}}$ is a simplicial category. Define the functor $- \otimes - : Sp^{\mathbb{O}} \times Set_\Delta \to Sp^{\mathbb{O}}$, for $X \in Sp^{\mathbb{O}}$ and $K \in Set_\Delta$

$$X \otimes K = X \wedge |K|_+ = \{X_n \wedge |K|_+ \mid n \geq 0\}$$

with structure maps

$$X_n \wedge |K|_+ \wedge S^1 \cong X_n \wedge S^1 \wedge |K|_+ \xrightarrow{\sigma \wedge 1} X_{n+1} \wedge |K|_+ \ .$$

Where σ is the structure map for X. Now we have to verify that this functor satisfies the three conditions of a simplicial category.

1. Let $\mathbf{hom}_{Sp^{\mathbb{O}}}(K, -) : Sp^{\mathbb{O}} \to Sp^{\mathbb{O}}$ be defined by

   $$\mathbf{hom}_{Sp^{\mathbb{O}}}(K, X)_n = \mathrm{Map}_{\mathrm{Top}_*}(|K|, X_n)$$

   with structure maps

   $$\mathrm{Map}_{\mathrm{Top}_*}(|K|, X_n) \wedge S^1 \to \mathrm{Map}_{\mathrm{Top}_*}(|K|, X_n \wedge S^1) \xrightarrow{\sigma_*} \mathrm{Map}_{\mathrm{Top}_*}(|K|, X_{n+1}).$$

2. Fix an $X \in Sp^{\mathbb{O}}$, and let $K : J \to Set_\Delta$, be a functor. We want show that there is a natural isomorphism

   $$\mathrm{colim}((X \otimes -) \circ K) \cong X \otimes (\mathrm{colim}\, K).$$

   Since colimits are taken degreewise in $Sp^{\mathbb{O}}$ this becomes

$$\operatorname{colim}((X_n \wedge -) \circ |-|_+ \circ K) \cong X_n \wedge |(\operatorname{colim} K)|_+.$$

We can break this into three parts. First we know that $|-| : Set_\Delta \to \mathrm{Top}$ preserves colimits since it is left adjoint to the singular complex functor. Second, the functor $(-)_* : \mathrm{Top} \to \mathrm{Top}_*$ adding a disjoint basepoint is adjoint to the forgetful functor functor $U : \mathrm{Top}_* \to \mathrm{Top}$, so it preserves colimits. Third, $- \wedge - : \mathrm{Top}_* \times \mathrm{Top}_* \to \mathrm{Top}_*$ is a left adjoint since Top_* is a closed category, and $- \wedge -$ is precisely the product in Top_*

3. By inspection we can see that for $X \in Sp^{\mathbb{O}}$, $K, L \in Set_\Delta$

$$X \otimes (K \times L) \cong (X \otimes K) \otimes L$$

since

$$X_n \wedge |K \times L|_+ \cong (X \wedge |K|_+) \wedge |L|_+$$

since $|K \times L| = |K| \times |L|$, and in general for $X, Y, Z \in \mathrm{Top}$ $X \wedge (Z \times Y)_+ \cong X \wedge Z_+ \wedge Y_+$

Now it remains to show that $Sp^{\mathbb{O}}$ fulfills the simplicial model axiom. We will show that the second condition in proposition 2.1.3 holds. Consider the levelwise model structure on $Sp^{\mathbb{O}}$, where the fibrations are the levelwise fibrations, and the weak equivalences are the levelwise weak equivalences. If we consider the third version of the simplicial model axiom given in definition 2.1.2, it is clear that he levelwise model structure on $Sp^{\mathbb{O}}$ is a simplicial model structure. This also means that for any cofibration $i : X \to Y$ in $Sp^{\mathbb{O}}$, the induced map

$$\begin{tikzcd}
X \otimes \partial\Delta^n \arrow[r] \arrow[d] & Y \otimes \partial\Delta^n \arrow[d] \arrow[ddr, bend left] & \\
X \otimes \Delta^n \arrow[r] \arrow[drr, bend right] & X \otimes \Delta^n \cup_{X \otimes \partial\Delta^n} Y \otimes \partial\Delta^n \arrow[dr, dashed] & \\
 & & Y \otimes \Delta^n
\end{tikzcd}$$

is a cofibration. Since the levelwise model structure, and the stable model structure share cofibrations this is also true for the stable model structure.

Now assume i is a trivial cofibration. First we show that

$$i \otimes 1 : X \otimes \partial\Delta^n \to Y \otimes \partial\Delta^n$$

is a cofibration. We can show this directly from Proposition 2.1.3, setting $A = \emptyset$ and $B = \partial\Delta^n$, but we will here provide an alternate proof. Let $p : A \to B$ be a trivial fibration, and consider the diagram

$$\begin{tikzcd}
X \otimes \partial\Delta^n \arrow[r] \arrow[d] & A \arrow[d] \\
Y \otimes \partial\Delta^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

By the definition of $- \otimes -$, this diagram is equal to

$$\begin{tikzcd}
X \otimes \partial\Delta^n \cong X \otimes \partial\Delta^n \coprod_{X \otimes \emptyset} Y \otimes \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\
Y \otimes \partial\Delta^n \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

By adjunction of $X \otimes -$ and $\mathbf{hom}(X, -)$, for a fixed X, and by the universal properties of pushback and pullout, finding a lift in the diagram above is equivalent to finding a lift in the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbf{hom}(\partial\Delta^n, A) \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed] & \mathbf{hom}(\emptyset, A) \times_{\mathbf{hom}(\emptyset, B)} \mathbf{hom}(\partial\Delta^n, B) \cong \mathbf{hom}(\partial\Delta^n, B)
\end{tikzcd}$$

We know that the levelwise model structure is simplicial. We also know that the levelwise model structure share trivial fibrations with the stable structure. Since $p : A \to B$ was assumed to be a trivial fibration, we know that $\mathbf{hom}(\partial\Delta^n, A) \to \mathbf{hom}(\partial\Delta^n, B)$ is a trivial fibration in the levelwise model structure by proposition 2.1.3(4), and hence in the stable model structure. Then since $i : X \to Y$ was assumed to be a cofibration, we can find the desired lift.

Second, we show that $i \otimes 1 : X \otimes \partial\Delta^n \to Y \otimes \partial\Delta^n$ is a weak equivalence. Both X and Y define homology theories with coefficient groups $\pi_*(X) = \pi_*(Y)$ [Rog21, Proposition 9.4.2]. $i : X \to Y$ induces a isomorphism on coefficient groups, and therefore by [Rog21, Corollary 3.3.11] for any CW complex Z a isomorphism $X_*(Z) \cong Y_*(Z)$. Putting $Z = \partial\Delta^n_+$ and expanding the definition of X_* and Y_* we get.

$$\begin{aligned}
\pi_*(X \otimes \partial\Delta^n) = \pi_*(X \wedge \partial\Delta^n_+) = X_*(\partial\Delta^n_+) \\
\cong Y_*(\partial\Delta^n_+) = \pi_*(Y \wedge \partial\Delta^n_+) = \pi_*(Y \otimes \partial\Delta^n)
\end{aligned}$$

So $i \otimes 1 : X \otimes \partial\Delta^n \to Y \otimes \partial\Delta^n$ is a weak equivalence. Since $X \otimes \Delta^n \to Y \otimes \Delta^n$ is obviously levelwise weakly equivalent to $i : X \to Y$, and is therefore a weak equivalence. Then because pushout preserves trivial cofibrations, and the 2-out-of-3 property, the dashed map is a weak equivalence, in the stable model structure.

The last condition is fulfilled since $\{e\} \simeq \Delta^1$, and therefore $Y \otimes \{e\} \simeq Y \otimes \Delta^1$

$$\begin{tikzcd}
X \otimes \{e\} \arrow[rr, "\simeq"] \arrow[d] & & X \otimes \Delta^1 \arrow[d] \\
Y \otimes \{e\} \arrow[r] \arrow[drr, bend right, "\simeq"] & Y \otimes \{e\} \cup_{X \otimes \{e\}} X \otimes \Delta^1 \arrow[dr, dashed] & \\
& & Y \otimes \Delta^1
\end{tikzcd}$$

Note that $Y \otimes \{e\} \to Y \otimes \{e\} \cup_{X \otimes \{e\}} X \otimes \Delta^1$ is equivalent to $Y \to Y \cup_X X \otimes \Delta^1 = Mi$, where $(Mi)_n = M(i_n)$, which is levelwise homotopy equivalent to Y, and the lower horizontal map is therefore a stable equivalence, then by the 2-out-of-3 property the dashed map is an stable equivalence.

$\square$

Definition 2.1.6. Let R be an orthogonal spectra equipped with maps

$$\mu : R \wedge R \to R \text{ and } \eta : \mathbb{S} \to R$$

where $\mathbb{S}$ is the sphere spectrum, and the maps satisfy diagrams

$$\begin{tikzcd}
\mathbb{S} \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & R \wedge R \arrow[d, "\mu"] & R \wedge \mathbb{S} \arrow[l, "1\wedge\eta"'] \arrow[dl, "\cong"] \\
& R &
\end{tikzcd}$$

$$\begin{tikzcd}
R \wedge R \wedge R \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"] & R \wedge R \arrow[d, "\mu"] \\
R \wedge R \arrow[r, "\mu"] & R
\end{tikzcd}$$

expressing unitality and associativity, respectively. We call such an R an *orthogonal ring spectra*. The ring spectra is said to be commutative if

$$\begin{tikzcd} R \wedge R \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & R \wedge R \arrow[dl, "\mu"] \\ & R & \end{tikzcd}$$

We can characterize an (commutative) orthogonal ring spectra as a (commutative) monoid in $Sp^{\mathbb{O}}$.

Definition 2.1.7. A *(left) R-Module* is an orthogonal spectra M, equipped with a map

$$\lambda : R \wedge M \to M$$

satisfying the following diagrams

$$\begin{tikzcd} R \wedge R \wedge M \arrow[r, "1\wedge\lambda"] \arrow[d, "\mu\wedge 1"] & R \wedge M \arrow[d, "\lambda"] \\ R \wedge M \arrow[r, "\lambda"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \wedge M \arrow[r, "\eta\wedge 1"] \arrow[dr, "\cong"'] & R \wedge M \arrow[d, "\lambda"] \\ & M \end{tikzcd}$$

2.2 The simplicial nerve

Definition 2.2.1. For an ordinary category $\mathcal{C}$, the *nerve*, denoted $N(\mathcal{C})$ is a simplicial set characterized by

$$N(\mathcal{C})_n = \hom_{Cat}([n], \mathcal{C})$$

where $[n]$ is the linearly ordered set $\{0, 1, ..., n\}$ regarded as a category. This construction gives us a functor

$$N : Cat \to Set_{\Delta}$$

Remark 2.2.2. We can extend this nerve functor to simplicial categories by applying it to the underlying ordinary category. But by doing this we will in general lose homotopical information. Consider the simplicial category with objects $\{0, 1, 2\}$, four non-identity morphisms, $\{0, 1\} : 0 \to 1$, $\{0, 2\} : 0 \to 2$, $\{1, 2\} : 1 \to 2$, and $\{1, 2\} \circ \{0, 1\} : 0 \to 2$, and a single homotopy, the one between $\{0, 2\}$ and $\{1, 2\} \circ \{0, 1\}$.

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=4, "\Uparrow"] & & 2 \end{tikzcd}$$

This is easily seen to be a simplicial category. Now consider the same category without the homotopy between $\{0, 2\}$ and $\{1, 2\} \circ \{0, 1\}$.

$$
\begin{tikzcd}
 & 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] & & 2
\end{tikzcd}
$$

This is also a simplicial category, although a discrete one. Since these have isomorphic underlying categories, they have isomorphic nerve, even though they are fundamentally different as simplicial categories. To amend this we replace $[n]$ with another simplicial category which can capture the homotopical information better, which we will denote $C[\Delta^n]$.

Definition 2.2.3. The objects of $C[\Delta^n]$ are the numbers $0, 1, ..., n$. The mapping spaces are defined by the vertices it goes through, more precisely, let $P_{i,j}$ be the set of subsets

$$P_{i,j} = \{I \subseteq [i, j] \,|\, i, j \in I\}$$

If we order this set by inclusions we get a partially ordered set, which can be regarded as a category. We can then take the nerve of this category to obtain a simplicial set, which we define to be the mapping space between i and j. We define composition to be induced by the union. To summarize $C[\Delta^n]$

- **Objects:** the numbers $\{i \in \mathbb{N} \,|\, 0 \leq i \leq n\}$
- **Mapping space:** $\mathrm{Map}_{C[\Delta^n]}(i, j) = N(P_{i,j})$
- **Composition:** $P_{j,k} \times P_{i,j} \to P_{i,k}$, is given by $(I_1, I_2) \mapsto I_1 \cup I_2$
- **Identities:** $\mathrm{id}_i = N(P_{i,i}) = \{i\}$

Lemma 2.2.4. [Lan21, Lemma 1.2.7] *For $i \leq j$, $N(P_{i,j}) \cong (\Delta^1)^{j-i-1}$. In particular* $\mathrm{Map}_{C[\Delta^n]}(0, n) = N(P_0, n) = (\Delta^1)^{n-1}$

Example 2.2.5. Here is $\mathrm{Map}_{C[\Delta^4]}(0, 4)$ with only some of the 1-homotopies, or 1-simplices drawn.

$$
\begin{tikzcd}
 & \{0,1,3,4\} \arrow[rr] & & \{0,1,2,3,4\} \\
\{0,1,4\} \arrow[ur] \arrow[rr] & & \{0,1,2,4\} \arrow[ur] & \\
 & \{0,3,4\} \arrow[uu] \arrow[rr] & & \{0,2,3,4\} \arrow[uu] \\
\{0,4\} \arrow[uu] \arrow[ur] \arrow[rr] & & \{0,2,4\} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

Here is $\mathrm{Map}_{C[\Delta^3]}(0, 3)$ drawn with all homotopies

$$\begin{tikzcd}
 & \{0,1,3\} \arrow[dr] & \\
\{0,3\} \arrow[ur] \arrow[rr] \arrow[dr] & \arrow[u, Rightarrow] \arrow[d, Rightarrow] & \{0,1,2,3\} \\
 & \{0,2,3\} \arrow[ur] &
\end{tikzcd}$$

Both examples express the idea that although the mapping spaces are non-trivial, the compositions are all homotopic to each other in a coherent way, and the mapping space is contractible.

Definition 2.2.6. The *simplicial nerve* or the *homotopy coherent nerve* of an simplicial category $\mathcal{S}$, denoted $N_\Delta(\mathcal{S})$, is characterized by

$$N_\Delta(\mathcal{S})_n = \hom_{Cat_\Delta}(C[\Delta^n], \mathcal{S}).$$

Theorem 2.2.7. [Lan21, Lemma 1.2.67] *There exists a unique colimit preserving functor*

$$C[-] : Set_\Delta \to Cat_\Delta$$

which is adjoint to

$$N_\Delta(-) : Cat_\Delta \to Set_\Delta$$

2.3 ∞-categories

Definition 2.3.1. An ∞-*category* $\mathcal{C}$, is a simplicial set satisfying an inner horn-filling condition

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

for $0 < i < n$. This property is supposed to enable a form of composition which we will define later. A functor $F : \mathcal{C} \to \mathcal{D}$ is a simplicial map. Recall that if a simplicial set satisfies lifting property for all $0 \leq i \leq n$, we call it a Kan complex.

Definition 2.3.2. An 1-simplex $f : \Delta^1 \to \mathcal{C}$ is called an *equivalence* if there exists 2-simplices $\alpha_l, \alpha_r : \Delta^2 \to \mathcal{C}$, such that $\alpha_l|\Lambda^2_0 = (f, 1)$ and $\alpha_r|\Lambda^2_2 = (1, f)$

$$\begin{tikzcd}
 & y \arrow[dr] \arrow[d, Rightarrow] & \\
x \arrow[ur, "f"] \arrow[rr, "1"'] & \, & x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & x \arrow[dr, "f"] \arrow[d, Rightarrow] & \\
y \arrow[ur] \arrow[rr, "1"'] & \, & y
\end{tikzcd}$$

Usually a ∞-groupoid is defined as a ∞-category where each 1-simplex, (morphism) is an equivalence. It is easily seen that a Kan complex is an ∞-groupoid, and it can be shown, although it is non-trivial, that a ∞-groupoid is a Kan complex, so the two notions coincide.

Theorem 2.3.3. [Lan21, p. 1.2.70] *The simplicial nerve of a simplicial category $\mathcal{S}$ is a ∞-category if all simplicial mapping spaces are Kan complexes.*

Remark 2.3.4. In particular for a simplicial model category $\mathcal{S}$, the mapping spaces between fibrant-cofibrant objects are Kan complexes. So the simplicial nerve of the restriction to fibrant-cofibrant objects $\mathcal{S}_{cf}$ is a ∞-category

Definition 2.3.5. The ∞-category of ∞-categories, Cat_∞, is the simplicial nerve of the simplicial category with objects ∞-categories and mapping spaces between ∞-categories $\mathcal{C}$ and $\mathcal{D}$ given by the maximal ∞-groupoid of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. It can be shown that passing to maximal ∞-groupoid is a right adjoint and therefore preserves products, Therefore this defines a fibrant simplicial category, and so Cat_∞ is in fact a ∞-category

Definition 2.3.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is *Joyal equivalence* if it is an equivalence in Cat_∞. By [Lan21, Observation 2.2.11] Joyal equivalences between ∞-groupoids are exactly homotopy equivalences.

Definition 2.3.7. The composite of two morphisms f, g in an ∞-category $\mathcal{C}$, i.e two 1-simplicies $f, g : \Delta^1 \to \mathcal{C}$, such that $d^1 f = d^0 g$ is an element of the pullback in Set_Δ

$$\begin{tikzcd}
\mathrm{Comp}_{\mathcal{C}}(f,g) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d, "i^*"] \\
* \arrow[r, "(f,g)"] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C})
\end{tikzcd}$$

By [Lan21, Corollary 1.3.44] $\mathrm{Comp}_{\mathcal{C}}(f,g) \to *$ is a trivial fibration, and therefore is $\mathrm{Comp}_{\mathcal{C}}(f,g)$ contractible.

Definition 2.3.8. A simplicial map between a simplicial set K and a ∞-category $\mathcal{C}$, $f : K \to \mathcal{C}$ defines two simplicial sets $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ defined by the levelwise by

$$n \mapsto \mathrm{Hom}_{(Set_\Delta)_{K/}}(K \to K \star \Delta^n, f : K \to \mathcal{C})$$

and

$$n \mapsto \mathrm{Hom}_{(Set_\Delta)_{K/}}(K \to \Delta^n \star K, f : K \to \mathcal{C})$$

respectively. In other words we consider objects of $\mathcal{C}$ "under" and "over" f. $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ are ∞-categories by [Lan21, Corollary 1.4.24]

Definition 2.3.9. Let x, y be two objects in an ∞-category $\mathcal{C}$. The *mapping space* of x and y, $\mathrm{map}_{\mathcal{C}}(x, y)$ is defined by the following pullback in Set_Δ

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1,\mathcal{C}) \arrow[d, "i^*"] \\
* \arrow[r, "{(x,y)}"] & \mathrm{Fun}(\partial\Delta^1,\mathcal{C})
\end{tikzcd}$$

By [Lan21, Proposition 1.3.48], $\mathrm{map}_{\mathcal{C}}(x,y)$ is an ∞-groupoid.

Definition 2.3.10. A functor between ∞-categories $F : \mathcal{C} \to \mathcal{D}$, is *fully faithful* if the induced map $\mathrm{map}_{\mathcal{C}}(x,y) \to \mathrm{map}_{\mathcal{D}}(Fx,Fy)$ is an homotopy equivalence for every $x,y \in \mathcal{C}$.

Definition 2.3.11. If a functor between ∞-categories $F : \mathcal{C} \to \mathcal{D}$, induces a essentially surjective functor $hF : h\mathcal{C} \to h\mathcal{D}$ of ordinary categories, it is called *essentially surjective.*

Definition 2.3.12. An object x in a ∞-category $\mathcal{C}$ is *initial* if for every $y \in \mathcal{C}$, $\mathrm{map}_{\mathcal{C}}(x,y)$ is contractible. We see that initial objects are unique up to homotopy.

Definition 2.3.13. The colimit of a simplicial map $f : K \to \mathcal{C}$ is an initial object in the category $\mathcal{C}_{f/}$.

Definition 2.3.14. $R-$Mod is the simplicial nerve of $R-\mathrm{mod}_{cf}$, the full subcategory of $R-$mod of fibrant-cofibrant objects. Note that $R-$Mod is an ∞-category

Definition 2.3.15. $R-$line is the maximal connected Kan complex of $R-$Mod containing R. In other words it is the restriction of $R-$Mod to R-modules stably equivalent to R

Definition 2.3.16. The *Thom spectra* of a morphism

$$f : X \to R\text{−line}$$

is the colimit of

$$f : X \to R\text{−line} \hookrightarrow R\text{−Mod}.$$

We want to introduce an alternate but equivalent notion of colimits, that will be useful later.

Definition 2.3.17. For a functor $F : K \to \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(F,x)$ is defined by the pullback diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(F,x) \arrow[r] \arrow[d] & \mathrm{Fun}(K \star \Delta^0,\mathcal{C}) \arrow[d] \\
* \arrow[r, "{(F,x)}"] & \mathrm{Fun}(K,\mathcal{C}) \times \mathcal{C}
\end{tikzcd}$$

In essence, this means we look at cones $X \star \Delta^0 \to \mathcal{C}$, but with base given by $F : K \to \mathcal{C}$, and with cone point given by x.

Theorem 2.3.18. *$\overline{F} : K \star \Delta^0 \to \mathcal{C}$ is a colimit cone of $F : K \to \mathcal{C}$ if and only if the map*

$$\mathrm{Map}_{\mathcal{C}}(\overline{F}, x) \to \mathrm{Map}_{\mathcal{C}}(F, x) \tag{2}$$

induced by

$$\begin{tikzcd}
\mathrm{Fun}(K \star \Delta^0, \mathcal{C}) \arrow[r] & \mathrm{Fun}(K, \mathcal{C}) \times \mathcal{C} & \ast \arrow[l, "{(F,x)}"'] \\
\mathrm{Fun}(K \star \Delta^0 \star \Delta^0, \mathcal{C}) \arrow[u] \arrow[r] & \mathrm{Fun}(K \star \Delta^0, \mathcal{C}) \times \mathcal{C} \arrow[u] & \ast \arrow[l, "{(\overline{F},x)}"] \arrow[u]
\end{tikzcd}$$

is a homotopy equivalence. The maps in the diagram are induced by $K \to K \star \Delta^0$

Proof. This is [Lan21, Definition 4.3.4] and [Lan21, Theorem 4.3.11]. □

Lemma 2.3.19. *The induced map*

$$\mathrm{Map}_{\mathcal{C}}(\overline{F}, x) \to \mathrm{Map}_{\mathcal{C}}(\overline{F}(\ast), x) \tag{3}$$

where $\overline{F}(\ast)$ is regarded as a functor $\overline{F}|\Delta^0 : \Delta^0 \to \mathcal{C}$, is a homotopy equivalence. Note that $\mathrm{Map}_{\mathcal{C}}(\overline{F}(\ast), x) = \mathrm{map}_{\mathcal{C}}(\overline{F}(\ast), x)$.

Proof. This is [Lan21, Remark 4.3.5] and [Lan21, Lemma 4.3.2] □

Lemma 2.3.20. [Lan21, Corollary 4.3.20] *Every functor $F : \Delta^n \to \mathcal{C}$ admits limits and colimits, with the limit and colimit object given $F(0)$ and $F(n)$, respectively.*

Remark 2.3.21. A colimit cone in the above lemma is given by a initial object in $\mathcal{C}_{F/}$. A obvious example of such an initial object is

$$F \circ s_n : \Delta^n \star \Delta^0 \simeq \Delta^{n+1} \to \Delta^n \to \mathcal{C}. \tag{4}$$

We have can generalize this result for arbitrary $F : X \star \Delta^0 \to \mathcal{C}$. We have in general for any cone $F : X \star \Delta^0 \to \mathcal{C}$ a homotopy equivalence

$$\mathrm{Map}_{\mathcal{C}}(F, x) \cong \mathrm{Map}_{\mathcal{C}}(F(\ast), x) = \mathrm{map}_{\mathcal{C}}(F(\ast), x)$$

where $F(\ast)$ is the cone point. Now consider the extension

$$\overline{F} : X \star \Delta^0 \star \Delta^0 \cong X \star \Delta^1 \xrightarrow{1 \star s_0} X \star \Delta^0 \xrightarrow{F} \mathcal{C}$$

again we have a general homotopy equivalence

$$\begin{aligned} \mathrm{Map}_{\mathcal{C}}(\overline{F}, x) \cong \mathrm{Map}_{\mathcal{C}}(\overline{F}(\{1\}), x) = \mathrm{Map}_{\mathcal{C}}(\overline{F}(\{0\}), x) \\ = \mathrm{map}_{\mathcal{C}}(F(\{0\}), x) \cong \mathrm{Map}_{\mathcal{C}}(F, x) \end{aligned}$$

so together we have a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(\overline{F}, x) \cong \mathrm{Map}_{\mathcal{C}}(F, x)$, so $\overline{F}$ is a colimit cone.

3 Examples

3.1 Simplicial circle

Proposition 3.1.1. [Lan21, Corollary 4.3.26] *Let the simplicial set K be given by a pushout*

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "i"'] & Z \arrow[d, "p'"] \\ Y \arrow[r, "p"'] & K \end{tikzcd}$$

where $i : Y \to X$. Suppose we are given an ∞-category $\mathcal{C}$, and a functor $F : K \to \mathcal{C}$. If the restrictions $F \circ p \circ i, F \circ p$ and $F \circ p'$ has colimit objects x, y and z, and $\mathcal{C}$ has pushouts, then F has a colimit object $y \coprod_x z$.

Remark 3.1.2. The maps in the pushout $y \coprod_x z$ is given accordingly. Let $\overline{F \circ p \circ i} : X \star \Delta^0 \to \mathcal{C}$ be the colimit cone of $F \circ p \circ i$, and similar for $\overline{F \circ p}$. Then

$$\overline{F \circ p} \circ (i \star 1) : X \star \Delta^0 \to Y \star \Delta^0 \to \mathcal{C} \tag{5}$$

restricts to $F \circ p \circ i$ on X, and is therefore an object in $\mathcal{C}_{F \circ p \circ i/}$. We know $\overline{F \circ p \circ i}$ is initial in $\mathcal{C}_{F \circ p \circ i/}$ since it is a colimit cone of $F \circ p \circ i$. We then know that the mapping space between $\overline{F \circ p \circ i}$ and $\overline{F \circ p} \circ (i \star 1)$ is contractible, which we see parameterize a contractible choice of maps between the colimit objects x and y, by the commutative diagram

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}_{F \circ p \circ i/}}(\overline{x}, \overline{y} \circ (i \star 1)) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}_{F \circ p \circ i/}) \arrow[r, "pr"] \arrow[d, "i^*"] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "i^*"] \\ * \arrow[r, "(\overline{x}, \overline{y} \circ (i \star 1))"'] & \mathrm{Fun}(\partial\Delta^1, C_{F \circ p \circ i/}) \arrow[r, "pr"'] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \end{tikzcd}$$

where $\overline{x}$ and $\overline{y}$ are short for $\overline{F \circ p \circ i}$ and $\overline{F \circ p}$. Similar for z

Theorem 3.1.3. [Lur09, Theorem 4.2.4.1] *Let J and $\mathcal{C}$ be fibrant simplicial categories, and $F : J \to \mathcal{C}$ a simplicial functor. Suppose we are given $c \in \mathcal{C}$ and an extension of F, $\bar{F} : J \star \Delta^0 \to \mathcal{C}$, with $\bar{F} : |\Delta^0 = c$. Then the following are equivalent.*

1. *$\bar{F}$ is a homotopy colimit of F*

2. *$N_\Delta(\bar{F})$ is a colimit of $N_\Delta(F) : N_\Delta(J) \to N_\Delta(\mathcal{C})$*

Example 3.1.4. We can compute an analog of the Thom spectra for a discrete ring R. Let $R\text{−mod}_\Delta = \mathrm{Fun}(\Delta^{op}, R\text{−mod})$ be the category of simplicial R-modules. This category has a simplicial model structure by [GJ09, Example II.6.2]. Let $R\text{−Mod}_\Delta$ be the coherent nerve of the full subcategory of fibrant-cofibrant objects in $R\text{−mod}_\Delta$. By [Lan21, Lemma 1.2.70] this is an ∞-category. Let $R\text{−line}_\Delta$ be the maximal connected Kan-complex containing R, the functor from Δ^{op} constant on R. The objects in R-line are the simplicial R-modules which are weakly equivalent to R. By [GJ09, Corollary III.2.5] this amounts to there being a quasi-isomorphism between the chain complexes which for each $A \in R\text{−mod}_\Delta$ is given degreewise by A_n and boundary maps

$$\sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1}$$

where the d_i are the face maps of A. Consider the case when $R = \mathbb{Z}$, and let S^1_Δ be the pushout diagram in Set_Δ

$$\begin{tikzcd}
\{0,1\} \arrow[r, "i"] \arrow[d, "i"'] & \Delta^1 \arrow[d, "p"] \\
\Delta^1 \arrow[r, "p'"'] & S^1_\Delta
\end{tikzcd}$$

and let $F : S^1_\Delta \to \mathbb{Z}\text{−line}_\Delta$ be $\mathbb{Z}$ (the discrete simplicial abelian group) on objects and on one of the Δ^1 components, $1 : \mathbb{Z} \to \mathbb{Z}$ and $-1 : \mathbb{Z} \to \mathbb{Z}$ on the other one. Lemma 2.3.20 gives us that the colimit of the restriction to the components of the pushout should be $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}$ and $\mathbb{Z}$. In the first case regarding $1 : \mathbb{Z} \to \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ the colimit cones are simple

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "i_1"'] \\
\mathbb{Z} \oplus \mathbb{Z} & \\
 & \mathbb{Z} \arrow[ul, "i_0"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[dr, "1"] & \\
 & \mathbb{Z} \\
\mathbb{Z} \arrow[uu, "1"] \arrow[ur, "1"'] &
\end{tikzcd}$$

Following the remark after 3.1.1, we restrict the base of the right diagram. We can regard it as a element in the category $R\text{−mod}_{p \circ i/}$, where the left diagram is an initial element, giving us an contractible space of maps, similar for $-1 : \mathbb{Z} \to \mathbb{Z}$. Since the diagrams are so simple we easily find explicit maps

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "i_1"'] \arrow[dr, "1"] & \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[rr, dashed, "x+y"] & & \mathbb{Z} \\
 & \mathbb{Z} \arrow[ul, "i_0"] \arrow[ur, "1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, "i_1"'] \arrow[dr, "1"] & \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[rr, dashed, "y-x"] & & \mathbb{Z} \\
 & \mathbb{Z} \arrow[ul, "i_0"] \arrow[ur, "-1"] &
\end{tikzcd}$$

This results in a new pushout diagram

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "y-x"] \arrow[d, "x+y"'] & \mathbb{Z} \arrow[d] \\
\mathbb{Z} \arrow[r] & \operatorname{colim} F
\end{tikzcd}$$

Now consider the composite double pushout diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "{(x,-x)}"] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "x-y"] \arrow[d, "x+y"] & \mathbb{Z} \arrow[d] \\
* \arrow[r] & \mathbb{Z} \arrow[r] & \operatorname{colim} F
\end{tikzcd}$$

By [Lur09, Lemma 4.4.2.1] the given that the left square is an ∞-categorical pushout diagram, the whole square is also an ∞-categorical pushout if, and only if the right square is one. This gives us a cofiber sequence

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \operatorname{colim} F$$

and thus a long exact sequence of homotopy groups

$$\cdots \to \pi_n(\mathbb{Z}) \xrightarrow{2} \pi(\mathbb{Z}) \to \pi_n(\operatorname{colim} F) \to \pi_{n-1}(\mathbb{Z}) \to \cdots$$

$\mathbb{Z}$ has only one non-trivial homotopy group, $\pi_0(\mathbb{Z})$.

Example 3.1.5. Let S^1_Δ be as previously, and let $F : S^1_\Delta \to \mathbb{S}\text{−line} \to \mathbb{S}\text{−mod}$ be the functor sending one of the Δ^1 factors to $1 : \mathbb{S} \to \mathbb{S}$, and the other to $-1 : \mathbb{S} \to \mathbb{S}$. We want to compute the colimit of F. By lemma 2.3.20 and Proposition 3.1.1 we can reduce this to a diagram

$$\begin{tikzcd}
\mathbb{S} \vee \mathbb{S} \arrow[r] \arrow[d] & \mathbb{S} \arrow[d] \\
\mathbb{S} \arrow[r] & \operatorname{colim} F
\end{tikzcd}$$

but we first need to analyse the restricted colimits to understand what the maps in the pushout are supposed to be. Using Remark 3.1.2 and Lemma 2.3.20 we get diagrams

$$\begin{tikzcd}
 & \mathbb{S} \arrow[dl, "i_1"'] \arrow[dr, "1"] & \\
\mathbb{S} \vee \mathbb{S} \arrow[rr, dashed, "1 \vee 1"] & & \mathbb{S} \\
 & \mathbb{S} \arrow[ul, "i_0"] \arrow[ur, "1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{S} \arrow[dl, "i_1"'] \arrow[dr, "1"] & \\
\mathbb{S} \vee \mathbb{S} \arrow[rr, dashed, "-1 \vee 1"] & & \mathbb{S} \\
 & \mathbb{S} \arrow[ul, "i_0"] \arrow[ur, "-1"] &
\end{tikzcd}$$

So the complete pushout diagram we need to compute is

$$
\begin{tikzcd}
\mathbb{S} \vee \mathbb{S} \arrow[r, "-1\vee 1"] \arrow[d, "1\vee 1"'] & \mathbb{S} \arrow[d] \\
\mathbb{S} \arrow[r] & \operatorname{colim} F
\end{tikzcd}
\tag{6}
$$

Now consider the diagram, where $\{-1,1\} : \mathbb{S} \xrightarrow{-1\times 1} \mathbb{S} \times \mathbb{S} \cong \mathbb{S} \vee \mathbb{S}$

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\{-1,1\}"] \arrow[d] & \mathbb{S} \vee \mathbb{S} \arrow[r, "-1\vee 1"] \arrow[d, "1\vee 1"'] & \mathbb{S} \arrow[d] \\
* \arrow[r] & \mathbb{S} \arrow[r] & \operatorname{colim} F
\end{tikzcd}
$$

We can show that the left square is a pushout, i.e a that $\mathbb{S} \xrightarrow{\{-1,1\}} \mathbb{S} \vee \mathbb{S} \xrightarrow{1\vee 1} \mathbb{S}$ is a cofiber sequence. Since the right square is pushout by definition, we have that $\mathbb{S} \xrightarrow{(-1\vee 1)\circ\{-1,1\}} \mathbb{S} \to \operatorname{colim} F$ is a cofiber sequence. The first composite is equal to $\mathbb{S} \xrightarrow{2} \mathbb{S}$., and we have a long exact sequence of homotopy groups

$$
\cdots \to \pi_n(\mathbb{S}) \xrightarrow{2} \pi_n(\mathbb{S}) \to \pi_n(\operatorname{colim} F) \xrightarrow{\partial} \pi_{n-1}(\mathbb{S}) \to \cdots \tag{7}
$$

Alternatively we can try to compute it more directly. By construction we can easily lift diagram (6). Now we need to compute the homotopy colimit $R\text{−mod}_{cf}$ To do this we need a cofibrant replacement for $\pm 1 \vee 1 : \mathbb{S} \vee \mathbb{S} \to \mathbb{S}$. Consider the diagram

$$
\begin{tikzcd}
\mathbb{S} \vee \mathbb{S} \arrow[r, "1\vee\pm 1"] \arrow[d, "i_0\vee i_1"'] & \mathbb{S} \vee \mathbb{S} \arrow[d] \arrow[ddr, bend left, "\pm 1\vee 1"] & \\
\mathbb{S} \wedge I_+ \arrow[r] \arrow[drr, bend right, "\pm 1\circ pr"'] & M_{\pm 1} \arrow[dr, dashed] & \\
 & & \mathbb{S}
\end{tikzcd}
$$

We know $i_0 \vee i_1 : \mathbb{S} \vee \mathbb{S} \to \mathbb{S} \wedge I_+$ is a cofibration, hence $\mathbb{S} \vee \mathbb{S} \to M_{\pm 1}$ is a cofibration. We know that $\pm 1 \vee 1 : \mathbb{S} \vee \mathbb{S} \to \mathbb{S}$ is a weak equivalence, and since $R\text{−mod}_{cf}$ is left proper by [Hir03, Corollary 13.1.3], $\mathbb{S} \wedge I_+ \to M_{\pm 1}$ is a weak equivalence. Since both ± 1 and the projection $\mathbb{S} \wedge I_+ \to \mathbb{S}$ are weak equivalences, the composite are by the 2-out-of-3-rule, and by the same rule the dashed map $M_{\pm 1} \to \mathbb{S}$ is a weak equivalence. Note that this holds for any left proper category and weak equivalence $f : B \to B$ for a cofibrant object B. Then the new pushout diagram is

$$\begin{tikzcd}
\mathbb{S} \vee \mathbb{S} \arrow[r] \arrow[d] & M_{-1} \arrow[d] \\
M_1 \arrow[r] & \operatorname{colim} F
\end{tikzcd}$$

The resulting object can be described as a mapping torus with a collapsed inner circle.

3.2 Aut(R)

Definition 3.2.1. Recall that for two objects x, y in an ∞-category $\mathcal{C}$, $\mathrm{map}_{\mathcal{C}}(x, y)$ is defined by the pullback in Set_Δ

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "i^*"] \\
* \arrow[r, "(x,y)"] & \mathrm{Fun}(\partial\Delta^1, \mathcal{C})
\end{tikzcd}$$

We define the endomorphism space of R in R–Mod as $\mathrm{End}_R(R) = \mathrm{map}_{R-\mathrm{Mod}}(R, R)$.

Theorem 3.2.2. [Lan21, Proposition 2.5.35] *Let $\mathrm{Map}_{\mathcal{C}}(x, y)$ be the simplicial mapping space of a fibrant simplicial category $\mathcal{C}$, then*

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{map}_{N_\Delta(\mathcal{C})}(x, y) \tag{8}$$

the homotopy class of this map is natural in x and y. In particular composition, which is only defined up to homotopy in $N_\Delta(\mathcal{C})$, commutes up to homotopy.

Remark 3.2.3. Notice that $\mathrm{End}_R(R) = \mathrm{map}_{R-\mathrm{Mod}}(R, R) \simeq \mathrm{Map}_{R-\mathrm{mod}}(R, R)$.

Remark 3.2.4. Note that an orthogonal ring spectrum R induces a monoid structure on R_0, since in degree 0 we have $R_0 \wedge R_0 \to R_{0+0} = R_0$. This is associative and unital, by the definition of the ring structure.

Theorem 3.2.5. *Let $\mathrm{Sing}(-)$ be the regular singular functor and R_0 the 0-th space of an orthogonal ring R, then*

$$\mathrm{End}_R(R) \simeq \mathrm{Sing}(R_0).$$

Proof. There is an adjunction

$$R \wedge (-)_+ : \mathrm{Top} \rightleftarrows R\mathrm{-mod} : \mathrm{Ev}_0$$

which gives us

$$\mathrm{Map}_{R-\mathrm{mod}}(R, R)_k := R\mathrm{-mod}(R \wedge |\Delta^k|_+, R) \simeq \mathrm{Map}_{\mathrm{Top}}(|\Delta^k|, R_0) = \mathrm{Sing}(R_0)_k$$

Applying π_0 to this equivalence gives us

$$\begin{aligned}\mathrm{Ho}(R\text{−Mod})(R,R) = \pi_0\mathrm{End}_R(R) = \pi_0\mathrm{map}_{R-\mathrm{Mod}}(R,R) = \pi_0\mathrm{Map}_{R-\mathrm{mod}}(R,R) = \\ \mathrm{Ho}(R\text{−mod})(R,R) = \pi_0\mathrm{Sing}(R_0) = \pi_0 R\end{aligned}$$

The last equality follows since R is an Ω-spectrum. ☐

Definition 3.2.6. Let $\mathrm{Aut}_R(R) \subseteq \mathrm{End}_R(R)$ be the union of connected components corresponding to stable homotopy equivalences $R \to R$.

Definition 3.2.7. Let $GL_1(R) \subseteq R$ be the union of connected components corresponding to units in $\pi_0 R_0$.

We see that we have an induced equivalence $\mathrm{Aut}_R(R) \simeq GL_1(R)$, since both correspond to the units in equivalent monoids.

Definition 3.2.8. Let $B\mathrm{Aut}_R(R) \subset R$−line be the full subgroupoid with R as the single object.

Proposition 3.2.9. *$B\mathrm{Aut}_R(R) \simeq R$−line.*

Proof. The inclusion of $B\mathrm{Aut}_R(R)$ is by definition fully faithful, and since R−line is an ∞-groupoid, it is also essentially surjective. By [Lan21, Theorem 2.3.20] the inclusion is then a Joyal equivalence. ☐

Lemma 3.2.10. *$\mathrm{Aut}_R(R) = \mathrm{map}_{R-\mathrm{line}}(R,R)$.*

Proof. By [Lan21, Lemma 2.3.8] the map

$$\mathrm{map}_{\mathcal{C}^{\simeq}}(x,y) \to \mathrm{map}_{\mathcal{C}}(x,y)$$

for an ∞-category $\mathcal{C}$ and objects $x, y \in \mathcal{C}$ is the inclusion of the path components which have equivalences as points. In the case where $\mathcal{C} = R\text{−Mod}'$, the connected component of R−Mod containing R, $x = y = R$, we have by definition $\mathcal{C}^{\simeq} = R$−line and we see that $\mathrm{Aut}_R(R) \subseteq \mathrm{End}_R(R)$ coincides with $\mathrm{map}_{R-\mathrm{line}}(R,R) \subseteq \mathrm{map}_{R-\mathrm{Mod}'}(R,R)$. ☐

Theorem 3.2.11. *$\mathrm{Aut}_R(R) \simeq \Omega B\mathrm{Aut}_R(R)$. So $B\mathrm{Aut}_R(R)$ is in fact a classifying space of $\mathrm{Aut}_R(R)$.*

Proof. $\mathrm{Aut}_R(R) = \mathrm{map}_{R-\mathrm{line}}(R,R)$ is defined by the pullback

$$\begin{tikzcd}
\mathrm{map}_{R-\mathrm{line}}(R,R) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, R\text{−line}) \arrow[d] \\
* \arrow[r, "{(R,R)}"] & \mathrm{Fun}(\partial\Delta^1, R\text{−line})
\end{tikzcd}$$

but this precisely defines the loop space of R-line based at R. Since $\partial\Delta^1 \to \Delta^1$ is a cofibration, the induced map $\mathrm{Fun}(\Delta^1, R\text{–line}) \to \mathrm{Fun}(\partial\Delta^1, R\text{–line})$ is a fibration, and since R–line is a ∞-groupoid, i.e a Kan complex, $\mathrm{Fun}(\partial\Delta^1, R\text{–line})$ and $\mathrm{Fun}(\partial\Delta^1, R\text{–line})$ are fibrant. Therefore the pullback is a homotopy pullback, and from proposition 0.0.10, we have an equivalence $\mathrm{Aut}_R(R) = \Omega R\text{–line} \simeq \Omega B\mathrm{Aut}_R(R)$. $\square$

3.3 Suspensions

Suppose we have a finite simplicial set X, and we want to analyse mappings

$$\Sigma X \to B\mathrm{Aut}_R(R) \simeq R\text{–line} \hookrightarrow R\text{–Mod}$$

and their colimits, the Thom spectra. We can decompose ΣX as a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

Then a map $\widehat{f} : \Sigma X \to B\mathrm{Aut}_R(R)$ is adjoint to a map $f : X \to \Omega B\mathrm{Aut}_R(R) \simeq \mathrm{Aut}_R(R)$.

Remark 3.3.1. Note that by the particular choice of construction of the suspension ΣX, we do not require in this case that either $\widehat{f}$ of f to be based preserving. Note that $B\mathrm{Aut}(R)$ only has one object, so $\widehat{f}$ is in a sense automatically based. If we wanted a such an adjoint for a space, Y, with more than one 0-simplex, we need to restrict to maps $\Sigma X \to Y$ that is constant on the 0-simplicies of ΣX to the basepoint used in the construction of ΩY. For alternate suspensions see [GJ09, Chapter III.5].

Lets consider such a map $f : X \to \mathrm{Aut}_R(R)$, and the adjoint map $\widehat{f} : \Sigma X \to B\mathrm{Aut}_R(R) \simeq R\text{–line}$. Now consider a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & CX \arrow[d, "p"] \arrow[ddr, bend left] & \\ * \arrow[r] \arrow[drr, bend right] & \Sigma X \arrow[dr, "\widehat{f}"] & \\ & & B\mathrm{Aut}_R(R) \end{tikzcd}$$

from [Lan21, Proposition 4.3.26], we have a homotopy pushout

$$
\begin{tikzcd}
T(\widehat{f}pi) \arrow[r] \arrow[d] & T(\widehat{f}p) \arrow[d] \\
T(*) \arrow[r] & T(\widehat{f})
\end{tikzcd}
\tag{9}
$$

where $T(-)$ denotes the Thom spectra, i.e the homotopy colimit of the functor composed with inclusion to $R-$Mod

Lemma 3.3.2. *The colimit cone of $\widehat{f}p$, which we also denote $T(\widehat{f}p)$, is*

$$T(\widehat{f}p) : X \star \Delta^0 \star \Delta^0 \cong X \star \Delta^1 \xrightarrow{1 \star s_0} X \star \Delta^0 \xrightarrow{\widehat{f}p} R-\text{Mod}. \tag{10}$$

*In particular $T(\widehat{f}p) \circ (i \star i) : X \star \Delta^0 \to R-$Mod *equals* $\widehat{f}p$.*

Proof. This is simply remark 2.3.21. □

Remark 3.3.3. For an arbitrary simplicial set X, the R-module $R \wedge |X|_+$ is cofibrant, but generally not fibrant. To simplify notion, we will still denote the fibrant replacement of $R \wedge |X|_+$ by $R \wedge |X|_+$.

Lemma 3.3.4. *For a finite simplicial set X, the constant map to R*

$$c_X : X \to R-\text{mod}$$

has colimit $R \wedge |X|_+$. In particular $T(\widehat{f}pi) \simeq R \wedge |X|_+$.

Proof. Notice that $\widehat{f}pi$ is constant since it factors through $*$. We continue by induction on the dimension of X, the highest degree of non-degenerate simplices. Assume that the inclusion $(n-1)$-skeleton, $X^{n-1} \hookrightarrow X$ has colimit object $R \wedge X_+^{n-1}$. Assume that the colimit cone can be described by as follows. For notational simplicity will describe it for a general X. Let $\overline{c_X}$ denote the colimit cone of c_X, the constant map on R, $X \to R-$Mod. This is a map

$$\overline{c_X} : X \star \Delta^0 \to R-\text{Mod}$$

which corresponds to a map

$$\overline{c_X} : X \to R-\text{Mod}_{/R \wedge X_+}.$$

We define how this maps simplices $\gamma : \Delta^m \to X$. Consider a 0-simplex $x : \Delta^0 \to X$. This corresponds to a map $\Delta^0 \star \Delta^0 \xrightarrow{x \star 1} X \star \Delta^0 \xrightarrow{\overline{c_X}} R-$Mod. By definition of $R-$Mod as the simplicial nerve of the full subcategory of fibrant-cofibrant objects of $R-$mod, this is equivalent by adjointness to a map

$$\widehat{x} : C[\Delta^0 \star \Delta^0] \cong C[\Delta^1] \to R-\text{mod}_{cf}.$$

In particular we have a map

$$\widehat{x}_* : \mathrm{Map}_{C[\Delta^1]}(0,1) \cong (\Delta^1)^{1-1} \cong \Delta^0 \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X|_+).$$

We define this map by

$$\Delta^0 \mapsto (1 \wedge x : R \cong R \wedge |\Delta^0|_+ \to R \wedge |X|_+)$$

where $1 \wedge x : R \wedge |\Delta^0|_+ \to R \wedge |X|_+$ is the map $(r, *) \mapsto (r, x)$. Now consider a 1-simplex $\alpha : x \to y$ in X, or equivalently $\alpha : \Delta^1 \to X$. Now we define the map

$$\Delta^2 \cong \Delta^1 \star \Delta^0 \xrightarrow{\alpha \wedge 1} X^\star \Delta^0 \xrightarrow{\overline{c_X}} R\mathrm{-Mod}$$

As above this is equivalent to a map

$$\widehat{\alpha} : C[\Delta^2] \to R\mathrm{-mod}_{cf}.$$

In particular we have an induced map on the mapping spaces

$$\widehat{\alpha}_* : \mathrm{Map}_{C[\Delta^2]}(0,2) \cong (\Delta^1)^{2-1} \cong \Delta^1 \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X|_+)$$

We define this map by

$$(1 \wedge x : R \wedge |\Delta^0|_+ \to R \wedge |X|_+) \xrightarrow{\widehat{\alpha \wedge 1}} (1 \wedge y : R \wedge |\Delta^0|_+ \to R \wedge |X|_+)$$

where $\widehat{\alpha \wedge 1}$ is adjoint to the map $1 \wedge \alpha : R \wedge |\Delta^1|_+ \to R \wedge |X|_+$, mapping $(r, i) \mapsto (r, \alpha(i))$. Now consider a 2-simplex $\psi : \Delta^2 \to X$

$$\begin{tikzcd} & y \arrow[dr, "\beta"] \arrow[d, phantom, "\Uparrow \psi"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & {} & z . \end{tikzcd}$$

By the same argument as above this corresponds to a map

$$\widehat{\psi} : C[\Delta^3] \to R\mathrm{-mod}_{cf}$$

and in particular a map

$$\widehat{\psi}_* : \mathrm{Map}_{C[\Delta^3]}(0,3) \cong (\Delta^1)^{3-1} \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X|_+)$$

where $(\Delta^1)^3 - 1 = (\Delta^1)^2$ is identified with the diagram

$$\begin{tikzcd} \{0,3\} \arrow[r, "\subset"] \arrow[d, "\cap"'] \arrow[dr] & \{0,1,3\} \arrow[d, "\cap"] \\ \{0,2,3\} \arrow[r, "\subset"'] & \{0,1,2,3\} \end{tikzcd}$$

which we map to

$$\begin{tikzcd}
1 \wedge x \arrow[rr, "\widehat{1\wedge\alpha}"] \arrow[d, "\widehat{1\wedge\gamma}"'] \arrow[drr, "\widehat{1\wedge\gamma}" description] \arrow[drr, phantom, bend left=40, "\widehat{1 \wedge \psi}"] \arrow[drr, phantom, bend right=40, "s_1(\widehat{1 \wedge \gamma})"] & & 1 \wedge y \arrow[d, "\widehat{1\wedge\beta}"] \\
1 \wedge z \arrow[rr, "\widehat{1\wedge 1}"'] & & 1 \wedge z
\end{tikzcd}$$

Now we want to define a similar map for a arbitrary $\theta : \Delta^n \to X$. Same as before we define a map $\widehat{\theta} : C[\Delta^{n+1}] \to R-\mathrm{mod}_{cf}$, such that $0, 1, \dots n \mapsto R$, and $n+1 \mapsto R \wedge |X|_+$. This map needs to be compatible with maps $\Delta^m \to \Delta^n \xrightarrow{\theta} X$. We need to ensure that for $0 \leq i \leq k < n$

$$\mathrm{Map}_{C[\Delta^{n+1}]}(i, k) \to \mathrm{Map}_{R-\mathrm{mod}}(R, R)$$

is constant. So consider $\{0, ..., k, n+1\} \in \mathrm{Map}_{C[\Delta^{n+1}]}(0, n+1)$. By definition $\{0, ..., k, n+1\} = \{0, ..., k\} \cup \{k, n+1\} = \{k, n+1\} \circ \{0, ..., k\}$. This is then mapped to $1 \wedge \theta(k) \circ 1_R = 1 \wedge \theta(k)$. This holds in general for inclusions coming from

$$\mathrm{Map}_{C[\Delta^{n+1}]}(k, n+1) \times \mathrm{Map}_{C[\Delta^{n+1}]}(i, k) \to \mathrm{Map}_{C[\Delta^{n+1}]}(0, n+1)$$

for $0 \leq i \leq k \leq n$. So the only information not determined by the inductive process is how the sequence of inclusion

$$i_n : \{0, , n+1\} \subset \{0, 1, n+1\} \subset \cdots \subset \{0, 1, \dots, n, n+1\}$$

is mapped. This amounts to a map

$$\widehat{\theta}_*(i_n) : \Delta^{n+1} \to \mathrm{Map}_{R-\mathrm{mod}}(R, R \wedge |X|_+)$$

adjoint to $1 \wedge \theta \in \mathrm{Map}_{R-\mathrm{mod}}(R \wedge \Delta^{n+1}, R \wedge |X|_+)$, mapping $(r, i) \mapsto (r, \theta(i))$. So now we have a collection of maps $\widehat{\theta} : C[\Delta^n \star \Delta^0] \to R-\mathrm{mod}$ compatible with diagrams

$$\begin{tikzcd}
\Delta^n \arrow[dr, "\theta"] \arrow[dd, "\eta"'] & \\
& X^{n-1} \\
\Delta^m \arrow[ur, "\theta'"'] &
\end{tikzcd}$$

for maps $[n] \to [m]$ in Δ. Using the canonical isomorphism $C[X \star \Delta^0] \cong \operatorname*{colim}_{\theta:\Delta^n \to X} C[\Delta^n \star \Delta^0]$, and maps $\widehat{\theta} : C[\Delta^n \star \Delta^0] \to R-$mod, the universal property of colimits gives a map $C[X \star \Delta^0] \to R-$mod. This is equivalent by adjointness to a map $X \star \Delta^0 \to R-$Mod. Now we prove the base case. First we want to show that the colimit object of the constant map $c : \coprod_{\alpha \in I} \Delta^0 \to R-$Mod equals $R \wedge |\coprod_{\alpha \in I} \Delta^0|_+$. Considering the definition of the smash product, we see that

$$R \wedge |\coprod_{\alpha \in I} \Delta^0|_+ = \bigvee_{\alpha \in I} R$$

which is exactly the coproduct in $R-$Mod, which is defined as the colimit of $c : \coprod_{\alpha \in I} \Delta^0 \to R-$Mod. The cone as a map

$$\coprod_{\alpha \in I} \Delta^0 \to R-\mathrm{Mod}_{/\bigvee_{\alpha \in I} R}$$

obviously maps a 0-simplex to $\Delta^0_\alpha \hookrightarrow \coprod_{\alpha \in I} \Delta^0$ to the coproduct inclusion. The n-skeleton of X, X^n can be written as the pushout

$$\begin{tikzcd}
\coprod_{x \in NX^n} \partial\Delta^n \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\
\coprod_{x \in NX^n} \Delta^n \arrow[r] & X^n
\end{tikzcd}$$

where NX^n are the non-degenerate simplices of X_n. Again by 3.1.1 theorem and 2.3.20 we can then express the colimit as the homotopy pushout.

$$\begin{tikzcd}
\coprod_{x \in NX^n} R \wedge |\partial\Delta^n|_+ \arrow[r, "\Phi"] \arrow[d] & R \wedge |X^{n-1}|_+ \arrow[d] \\
\coprod_{x \in NX^n} R \arrow[r] & T(X^n)
\end{tikzcd}$$

We want to establish that

$$\Phi : \coprod_{x \in NX^n} R \wedge |\partial\Delta^n|_+ \to R \wedge |X^{n-1}|_+$$

is natural, i.e that it is the inclusion $1_R \wedge \alpha : R \wedge |\partial\Delta^n|_+ \to R \wedge |X^{n-1}|_+$ for each $\alpha : \partial\Delta^n \to X^{n-1}$. To do this we need an analyse how this map is induced, which is restricting the cone $\overline{c_X}_{n+1}$to $\overline{c_{X_{n+1}}} \circ (\alpha \star 1)$. We see that this is an element of $R\text{--Mod}_{c_{X_{n+1}}\circ\alpha/}$. Recall that by definition the mapping space $\mathrm{map}_{R\text{--Mod}_{c_{X_{n+1}}\circ\alpha/}}(\overline{c_{X_{n+1}}\circ\alpha}, \overline{c_{X_{n+1}}}\circ(\alpha\star 1))$ is contractible so we only need to show that there exist a map $\overline{\Phi}_\alpha : \partial\Delta^n \star \Delta^1 \to R\text{--Mod}$, such that

$$\overline{\Phi}_\alpha|\partial\Delta^n \star \{0\} = \overline{c_{X_{n+1}}\circ\alpha}$$

$$\overline{\Phi}_\alpha|\partial\Delta^n \star \{1\} = \overline{c_{X_{n+1}}}\circ(\alpha\star 1)$$

and

$$\overline{\Phi}|\Delta^1 = 1_R \wedge \alpha : R\wedge|\partial\Delta^n|_+ \to R\wedge|X^{n-1}|_+$$

for each $\alpha : \partial\Delta^n \to X^{n-1}$. Consider a map $\theta : \Delta^m \to \partial\Delta^n$, $m < n$. This induces a map

$$\Delta^{m+2} \cong \Delta^m \star \Delta^1 \xrightarrow{\theta\star 1} \partial\Delta^n \star \Delta^1 \xrightarrow{\overline{\Phi}_\alpha} R\text{--Mod}$$

and we want to define this. This corresponds to a map

$$\widehat{\overline{\Phi}_\alpha \circ (\theta\star 1)} : C[\Delta^{m+2}] \to R\text{--mod}_{cf}.$$

The restriction on $\overline{\Phi}_\alpha|\partial\Delta^n\star\{0\}$ forces the induced map on $\mathrm{Map}_{C[\Delta^{m+2}]}(0, m+1)$ to be

$$\widehat{\theta}_* : \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+1) \cong (\Delta^1)^m \to \mathrm{Map}_{R\text{--mod}_{cf}}(R, R\wedge|\partial\Delta^n|_+).$$

Let $\mathrm{Map}_{C[\Delta^2]}(0, \widehat{m+1}, m+2) \subset \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+2)$ be the inclusion

$$N(\{I \subseteq [0, m+2]\,|\, 0, m+2 \in I,\ m+1 \notin I\}) \subset N(\{I \subseteq [0, m+2]\,|\, 0, m+2 \in I, \}).$$

Recall that the last set is $\mathrm{Map}_{C[\Delta^{m+2}]}(0, m+2)$ by definition. Then the restriction on $\overline{\Phi}|X\star\{1\}$ forces the induced map on $\mathrm{Map}_{C[\Delta^2]}(0, \widehat{m+1}, m+2)$ to be

$$\begin{aligned}\widehat{\alpha\circ\theta}_* : \mathrm{Map}_{C[\Delta^{m+2}]}(0, \widehat{m+1}, m+2) &\cong (\Delta^1)^m \\ &\to \mathrm{Map}_{R\text{--mod}_{cf}}(R, R\wedge|X^{n-1}|_+).\end{aligned}$$

The map such that

$$\widehat{\alpha\circ\theta}_*(i_m) : \Delta^m \to \mathrm{Map}_{R\text{--mod}_{cf}}(R, R\wedge|X^{n-1}|_+)$$

is adjoint to

$$1\wedge(\alpha\circ\theta) = (1\wedge\alpha)\circ(1\wedge\theta) : \Delta^m \to R\wedge|\partial\Delta^n|_+ \to R\wedge|X^{n-1}|_+.$$

Now consider the diagram

$$\begin{tikzcd} \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+1) \arrow[r, "\widehat{\theta}_*"] \arrow[d, "(\{m+1,m+2\})_*"'] & \mathrm{Map}_{R\text{--mod}_{cf}}(R, R\wedge|\partial\Delta^n|_+) \arrow[d, "(1\wedge\alpha)_*"] \\ \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+2) \arrow[r, dashed, "\widehat{\overline{\Phi}_\alpha\circ(\theta\star 1)}"] & \mathrm{Map}_{R\text{--mod}_{cf}}(R, R\wedge|X^{n-1}|_+) \end{tikzcd} \tag{11}$$

Let $\mathrm{Map}_{C[\Delta^{m+2}]}(0, m+1, m+2) \subset \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+2)$ be the inclusion

$N(\{I \subseteq [0, m+2] \,|\, 0, m+1, m+2 \in I, \}) \subset N(\{I \subseteq [0, m+2] \,|\, 0, m+2 \in I, \})$.

From the diagram (15) we see that the map induced by $\Phi_\alpha \widehat{\circ (\theta \star 1)}$ on

$$\begin{aligned}(\Delta^1)^m \cong \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+1, m+2) \subset \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+2)& \\ \cong (\Delta^1)^{m+1} \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X^{n+1}|_+)&\end{aligned}$$

is equal to $(1 \wedge \alpha)_* \circ \widehat{\theta}_*$. Notice that $(1 \wedge \alpha)_* \circ \widehat{\theta}(i_n)$ is adjoint also adjoint to

$$(1 \wedge \alpha) \circ (1 \wedge \theta) : R \wedge |\Delta^m|_+ \to R \wedge |\partial \Delta^n|_+ \to R \wedge |X^{n-1}|_+.$$

So extending what we are given

$$\begin{aligned}(\Delta^1)^m \coprod (\Delta^1)^m \cong \mathrm{Map}_{C[\Delta^{m+2}]}(0, m{+}1, m{+}2) \coprod \mathrm{Map}_{C[\Delta^{m+2}]}(0, \widehat{m+1}, m{+}2)& \\ \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X^{n+1}|_+)&\end{aligned}$$

to $(\Delta^1)^m \times \Delta^1 = (\Delta^1)^{m+1} \cong \mathrm{Map}_{C[\Delta^{m+2}]}(0, m+2) \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X^{n-1}|_+)$ is easily and naturally done, since we can just take the identity homotopy. Now we make a cofibrant replacement of the left vertical map, and the obvious choice is

$$\coprod_{x \in NX^n} R \wedge |\partial \Delta^n|_+ \hookrightarrow \coprod_{x \in NX^n} R \wedge |\Delta^n|_+ \xrightarrow{\sim} \coprod_{x \in NX^n} R. \tag{12}$$

This gives us the pushout

$$\begin{array}{ccc} \coprod_{x \in NX^n} R \wedge |\partial \Delta^n|_+ & \xrightarrow{\coprod_{x \in NX^n} 1 \wedge x} & R \wedge |X^{n-1}|_+ \\ \downarrow & & \downarrow \\ \coprod_{x \in NX^n} R \wedge |\Delta^n|_+ & \longrightarrow & T(X^n)\,. \end{array}$$

Since $R \wedge (-)_+$ commutes with colimits, we have $T(X^n) = R \wedge |X^n|_+$. $\square$

Now diagram 9 takes the form

$$\begin{array}{ccc} R \wedge X_+ & \xrightarrow{\Phi} & R \\ \downarrow & & \downarrow \\ R \wedge |\Delta^0|_+ & \longrightarrow & T(\widehat{f}) \end{array} \tag{13}$$

Proposition 3.3.5. Φ *is given up to homotopy by the composition*

$$R \wedge |X|_+ \xrightarrow{1\wedge f} R \wedge |\mathrm{Aut}_R(R)|_+ \to R \wedge \mathrm{map}_{R-\mathrm{Mod}}(R,R) \to R \tag{14}$$

Proof. The goal is to show the existence of a

$$\overline{\Phi} : X \star \Delta^1 \to R\text{−Mod}$$

such that $\overline{\Phi}|X \star (\{0\} \coprod \{1\}) = \overline{c_X} \coprod \widehat{f}p$, and $\overline{\Phi}|\Delta^1 = ev \circ (1 \wedge f)$. Consider a map $\theta : \Delta^n \to X$. This induces a map

$$\Delta^{n+2} \cong \Delta^n \star \Delta^1 \xrightarrow{\theta\star 1} X \star \Delta^1 \xrightarrow{\overline{\Phi}} R\text{−Mod}$$

and we want to define this. This corresponds to a map

$$\widehat{\overline{\Phi} \circ (\theta \star 1)} : C[\Delta^{n+2}] \to R\text{−mod}_{cf}.$$

The restriction on $\overline{\Phi}|X \star \{0\}$ forces the induced map on $\mathrm{Map}_{C[\Delta^{n+2}]}(0, n+1)$ to be

$$\widehat{\theta}_* : \mathrm{Map}_{C[\Delta^2]}(0, n+1) \cong (\Delta^1)^n \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R \wedge |X|_+)$$

Now consider the diagram

$$\begin{tikzcd}
\mathrm{Map}_{C[\Delta^{n+2}]}(0,n+1) \arrow[r, "\widehat{\theta}_*"] \arrow[d, "(\{n+1,n+2\})_*"'] & \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R\wedge|X|_+) \arrow[d, "(ev\circ(1\wedge f))_*"] \\
\mathrm{Map}_{C[\Delta^2]}(0,n+2) \arrow[r, dashed, "\widehat{\overline{\Phi}\circ(\theta\star 1)}_*"] & \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R,R)
\end{tikzcd} \tag{15}$$

From the diagram (15) we see that the map induced by $\Phi \circ \widehat{(\theta \star 1)}$ on

$$\begin{aligned}(\Delta^1)^n \cong \mathrm{Map}_{C[\Delta^{m+2}]}(0, n+1, n+2) \subset \mathrm{Map}_{C[\Delta^{m+2}]}(0, n+2)&\\ \cong (\Delta^1)^{n+1} \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R,R)&\end{aligned}$$

is equal to $(ev \circ (1 \wedge f))_* \circ \widehat{\theta}_*$. Notice that the adjoint of $(ev \circ (1 \wedge f))_* \circ \widehat{\theta}_*(i_n)$ is

$$ev\circ(1\wedge f)\circ(1\wedge\theta) : R\wedge|\Delta^n|_+ \to R\wedge|X|_+ \to R\wedge|\mathrm{Aut}(R)|_+ \hookrightarrow R\wedge|\mathrm{End}(R)|_+ \to R$$

defined on elements as

$$(r, i) \mapsto (r, \theta(i)) \mapsto (r, f(\theta(i))) \mapsto f(\theta(i))(r)$$

The restriction on $\overline{\Phi}|X \star \{1\}$ forces the induced map on

$$\widehat{fp_\theta} : C[\Delta^{n+1}] \cong C[d_{n+1}\Delta^{n+2}] \subset C[\Delta^{n+2}] \to R\text{−mod}$$

to be the adjoint of

$$\widehat{f}p \circ (1 \star \theta) : \Delta^{n+1} \cong \Delta^n \star \Delta^0 \to X \star \Delta^0 \to R\text{−Mod}\,.$$

We then have an induced map

$$\widehat{fp_{\theta *}} : (\Delta^1)^n \cong \mathrm{Map}_{C[\Delta^{n+2}]}(0, \widehat{n+1}, n+2) \subset \mathrm{Map}_{C[\Delta^{n+2}]}(0, n+2) \\ \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R)$$

and in particular a map

$$\widehat{fp_{\theta *}}(i_n) : \Delta^n \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R)$$

which is exactly the map

$$\Delta^n \xrightarrow{\theta} X \xrightarrow{f} \mathrm{End}(R) \cong \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R)$$

which is adjoint $R \wedge |\Delta^n|_+ \to R$ defined on elements by $(r, i) \mapsto f \circ \theta(i)(r) = f(\theta(i))(r)$. Hence $(ev \circ (1 \wedge f))_* \circ \widehat{\theta}_* = \widehat{fp_{\theta *}}$. So extending what we are given

$$(\Delta^1)^n \coprod (\Delta^1)^n \cong \mathrm{Map}_{C[\Delta^{n+2}]}(0, n+1, n+2) \coprod \mathrm{Map}_{C[\Delta^{n+2}]}(0, \widehat{n+1}, n+2) \\ \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R)$$

to $(\Delta^1)^n \times \Delta^1 = (\Delta^1)^{n+1} \cong \mathrm{Map}_{C[\Delta^{n+2}]}(0, n+2) \to \mathrm{Map}_{R-\mathrm{mod}_{cf}}(R, R)$ is easily and naturally done, since we can just take the identity homotopy.

□

Example 3.3.6. Consider an Ω ring spectrum R, such that $\pi_n(R) = \pi_n(\Omega^\infty R)$, for $n \geq 0$. Recall that $\pi_0(R)$ is a discrete ring, with addition given by canonical group structure as a stable homotopy group, and multiplication given by the multiplication $\lambda : R \wedge R \to R$, and passing to $\pi_0(R)$, notice that there is an multiplicative identity. S^n is connected for $n > 0$, so a based map $S^n \to \Omega^\infty R$ is contained in the connected component containing the basepoint, denoted $\Omega^\infty_0(R)$. Notice that $\Omega^\infty_0(R)$ corresponds to the 0 in the ring $\pi_0(R) \cong \pi_0(\Omega^\infty R)$. Let $\Omega^\infty_1(R)$ corresponds to 1 in $\pi_0(R)$. $\Omega^\infty R$ has a H-space structure so we have a homotopy equivalence

$$\Omega^\infty_0(R) \xrightarrow{+1} \Omega^\infty_1(R).$$

Recall that $\mathrm{End}_R(R) \simeq \mathrm{Sing}(R_0) = \Omega^\infty R$. Under this equivalence $\Omega^\infty_1(R)$ corresponds to the connected component of $\mathrm{End}_R(R)$ of homotopy equivalences homotopic to the identity, hence it lies in $\mathrm{Aut}_R(R)$. Therefore given an element $[f : S^n \to \Omega^\infty(R)] \in \pi_n(R)$ we get a map $(f+1) : S^n \to \mathrm{Aut}_R(R)$, by composing with $+1$. Now we can ask what the Thom spectra is for $\widehat{(f+1)}$: $\Sigma S^n \to B\mathrm{Aut}_R(R)$. We know we have a homotopy pushout.

$$\begin{tikzcd}
R \wedge S^n_+ \arrow[r] \arrow[d] & R \arrow[d] \\
R \arrow[r] & T(\widehat{(f+1)})
\end{tikzcd}$$

We have a stable equivalence

$$R \wedge S^n_+ \simeq R \wedge S^n \times R \simeq R \wedge S^n \vee R$$

such that $R \wedge S^n \hookrightarrow R \wedge R \wedge S^n_+ \simeq R \wedge S^n \vee R$ corresponds to the inclusion. So now we have the double homotopy pushout

$$\begin{tikzcd}
R \wedge S^n \arrow[r] \arrow[d] & R \wedge S^n_+ \arrow[r] \arrow[d] & R \arrow[d] \\
* \arrow[r] & R \arrow[r] & T(\widehat{(f+1)})
\end{tikzcd}$$

where by proposition (3.3.5) the composite $R \wedge S^n \to R \wedge S^n_+ \to R$ is given by

$$R \wedge S^n \xrightarrow{1 \wedge f} R \wedge (R_0^\times)_+ \to R \wedge R_0 \xrightarrow{\lambda} R.$$

Considering the outer pushout square, we get a cofibration sequence

$$R \wedge S^n \to R \to T(\widehat{(f+1)}).$$

On homotopy groups this means

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_i(R \wedge S^n) \arrow[r] \arrow[d, Rightarrow] & \pi_i(R) \arrow[r] \arrow[d, Rightarrow] & \pi_i(T(\widehat{f+1})) \arrow[r] \arrow[d, Rightarrow] & \pi_{i-1}(R \wedge S^n) \arrow[r] \arrow[d, Rightarrow] & \cdots \\
\cdots \arrow[r] & \pi_{i-n}(R) \arrow[r, "\cdot[f]"] & \pi_i(R) \arrow[r] & \pi_i(T(\widehat{(f+1)})) \arrow[r] & \pi_{i-1-n}(R \wedge S^n) \arrow[r] & \cdots
\end{tikzcd}$$

where $\cdot[f]$ is multiplication in the graded ring $\pi_*(R)$ induced by $\lambda : R \wedge R \to R$.

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