Hardness of Non-trivial Generalized Domination Problems Parameterized by Linear Mim-Width

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Abstract

Let σ, ρ be non-empty subsets of $\mathbb{N}$, a (σ, ρ)-dominating set D in a graph G is a subset of $V(G)$ with the following property: $\forall v \in V(G)$ if $v \in D$ then $|N(v) \cap D| \in \sigma$. If $v \notin D$ then $|N(v) \cap D| \in \rho$. Problems on these (σ, ρ)-dominating sets capture many well known problems, such as the Independent Set problem and the Dominating Set problem.

We will show the $W[1]$-hardness of Min-(σ, ρ)-DS param. by l. mim-width + sol. size for all $\sigma, \rho \subseteq \mathbb{N}$, where σ and ρ are both non-empty and $0 \notin \rho$, and that there are no algorithms solving Min-(σ, ρ)-DS param. by l. mim-width + sol. size in $n^{o(w/\log w)}$ time, unless ETH is false, for any graph G with $|V(G)| = n$ and with a linear branch decomposition with mim-width w. Furthermore we will also show the $W[1]$-hardness of Max-(σ, ρ)-DS param. by l. mim-width + sol. size whenever σ and ρ are both finite and $\rho \neq \{0\}$, and that there are no algorithms solving Max-(σ, ρ)-DS param. by l. mim-width + sol. size in $n^{o(w/\log w)}$ time, unless ETH is false.

Moreover we will show that there are no algorithms for Independent Set on H-graphs running in time $n^{o(h/\log h)}$, where $h = |E(H)|$, unless ETH is false.

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1 Introduction

Graphs are an excellent way to abstractly represent binary relations. They are complex enough to have a wide array of applications, while also being simple enough to be able to be adapted to more specific needs. The applications of graphs often require having problems solved on them, and these problems are often of interest outside their application.

Not every problem is easy to solve fast, and often the problems that we do not know how to solve fast are of particular interest. Such problems can sometimes be classified into various groups according to their complexity. However, a problem might only be difficult for general inputs and be easy for others, as if we know more about the problem it might be easier to solve. One way to represent this notion of having information about the input can be represented by (width) parameters. The width parameter of particular interest in this paper is the maximum induced matching width, or mim-width for short, which measures the size of the largest induced matching in cuts induced by recursive decompositions of the vertex set of the input graph.

Say we are given a list of speakers who would like to present their results, however each speaker is only available in a certain time frame, and we only have one room. We would then like to find a set of speakers such that no two speakers present at the same time. We clearly would like to have as many people as possible present, as we have rented the room for the entire day. The problem of finding such a set can be modelled using interval graphs, where the intervals are the time frames the speakers are available, and two speakers have an edge between them if their available time frames overlap. Then the problem reduces to finding the maximum independent set in the interval graph.

Interval graphs can easily be shown to have mim-width of at most 1, and the problem of finding a maximum independent set on any graph with bounded mim-width can be done in polynomial time. We can therefore solve our problem concerning speakers in polynomial time using mim-width. Interval graphs can in one sense be generalized to graphs of bounded mim-width, and independent sets can be generalized to (σ, ρ)-dominating sets, which will be explained later.

Interval graphs are one example of a graph class, where a graph class is a family of graphs having some common attribute(s). In general there are many graph classes which have bounded mim-width, and there are many problems solvable in polynomial time given a graph of bounded mim-width. Therefore algorithms for graphs with bounded mim-width often give algorithmic results on several graph classes, and unifies the study of algorithms on certain graph classes through the study of graph algorithms parameterized by mim-width.

In problems which are not parameterized there two are large groups of problems of particular interest: problems solvable in polynomial time P and problems solvable in non-deterministic polynomial time NP. A third group related to NP is the class of NP-hard problems. NP-hard problems are problems which can solve any problem in NP, we then think of all NP-hard problems which also are in NP as the most difficult problems in NP. Because of how powerful the NP-hard problems are there are many people think that there are no NP-hard problems which also are in P. Analogous to this we have the class of FPT and $W[1]$-hard, where many think that $W[1]$-hard problems are not in FPT.

However unlike P and NP, where for problems in P we have fast algorithms and for problems in NP-hard we do not have fast algorithms – and might not expect that fast algorithms exist – there are two classes which have polynomial-time algorithms solving them in parameterized complexity: FPT and XP. In both cases the speed of the algorithm depends on a parameter, which is a natural number describing something either about the desired solution, or something about the complexity of the input. Problems in XP are polynomial-time solvable where the degree of the polynomial depends on a parameter, and problems in FPT are polynomial-time solvable where the degree of the polynomial does not depend on the parameter. An example of FPT-time is $2^k n^2$, and an example of XP-time is n^{2k+2}, it can then be showed for a large enough value of n that $2^k n^2 < n^{2k+2}$, and in general XP-time is much slower, for $n = 4$, n^{2k+2}

is 2^{3k} times larger than $2^k n^2$.

In analogy to the theory of NP-hardness where under the assumption that $P \neq NP$ we rule out the existence of a polynomial time algorithms for NP-hard problems, we also rule out FPT algorithms for $W[1]$-hard problems. Showing that a problem is $W[1]$-hard can be done by a parameterized reduction from problem A to problem B, taking an instance of problem A which is $W[1]$-hard and we therefore think is probably not FPT, and using a FPT-time algorithm to turn it into an equivalent instance of the problem B we would like to show is probably not FPT. Then if B is FPT then so must A be as FPT-time + FPT-time is still FPT-time. We do not think A is FPT, therefore we can be as sure that B is not FPT.

Moreover we would often like to know how fast a problem theoretically can be solved in terms of concrete running times. For instance if there exists an algorithm for a problem running in n^k time, we would like to know if the problem can be solved in $n^{o(k)}$ time, or if the algorithm is optimal. A common hypothesis which is often used to this end is the exponential time hypothesis, or ETH for short. ETH essentially states that no algorithms can solve the 3-SAT problem in $2^{o(n)}$ time, where n is the number of variables of an instance of 3-SAT. Where 3-SAT, the boolean satisfiability problem where each clause has at most three variables, was one of the first problems proved to be NP-hard. Then if we have a reduction from 3-SAT to another problem B, then algorithms solving the problem B inherits a time upper bound depending on the specifics of the reduction.

We would like to know the complexity class of any problem, but this might not always be easy. If no FPT algorithms can be found for a problem we would then like to have some reason for why it is difficult to find. If we then show the problem is $W[1]$-hard we then get strong evidence that it is indeed difficult – possibly impossible – to find such an algorithm. Grouping many similar problems into one more general problem then allows one to make statements about many problems which all have proven to be difficult to find FPT algorithms for. This then allows us to make statements about a wide array of problems using (hopefully) fewer arguments. One such generalisation, and the generalisation of interest here, are the (σ, ρ)-dominating set problems.

The dominating set problem asks given a graph G and an integer k, if G has a set of a size at most k such that every vertex not in the set is adjacent to at least one vertex in the set. There also exist similar problems asking if there exists some set in G of size at most k such that every vertex not in the set is adjacent to a given amount of vertices in the set. One such example is the perfect dominating set problem where every vertex not in the set is adjacent to exactly one vertex in the set.

Other examples of dominating set-like problems have additional requirements on how the vertices in the dominating set interact. For instance the dominating induced matching problem asks if there exists a dominating set in G of size at most k, which is also an induced matching. That is every vertex in the set has exactly one neighbour also in the set.

Finally there are dominating sets which generalizes the dominating set by requiring vertices to be dominated by a certain amount of vertices, and have additional requirements on how to vertices in the dominating set interact. One example of such a problem is the perfect code problem, which asks given a graph G if there is a perfect dominating set in G which is also an independent set in G.

The (σ, ρ)-dominating set is a type of set which can capture all the generalisations of the dominating set described above, where σ and ρ are sets of natural numbers. σ describes how vertices in the dominating set interact with the other vertices in the dominating set, and ρ describes by how many vertices a vertex outside the dominating set should be dominated by. In particular given a graph G a (σ, ρ)-dominating set in G is a set $D \subseteq V(G)$, such that for all vertices v in D, $|N(v) \cap D| \in \sigma$, and for all vertices v not in D, $|N(v) \cap D| \in \rho$. The (σ, ρ)-dominating set problems we will consider are minimisation and maximisation (σ, ρ)-dominating set problems asking if there is a (σ, ρ)-dominating set in G of size at most k or at least k respectively, for some $k \in \mathbb{N}$.

The goal of this thesis is to show that all non-trivial minimisation and many non-trivial maximisation (σ, ρ)-dominating set problems parameterized by linear mim-width and solution size are $W[1]$-hard, and therefore probably not FPT. And in addition for the same non-trivial minimisation and maximisation (σ, ρ)-dominating set problems we show $W[1]$-hardness for, we show they cannot be solved in $n^{o(w/\log w)}$ time, where w is the mim-width of a linear branch decomposition given with the graph G and $n = |V(G)|$, unless ETH is false.

1.1 Related Work

Many problems which can be formulated using (σ, ρ)-dominating sets are at least $W[1]$-hard Downey and Fellows (1992, 1999); Bodlaender and Kratsch (2001); Downey and Fellows (1995); Cesati (2002); Moser and Thilikos (2009); Golovach et al. (2012). In addition the NP-hardness of several families of (σ, ρ)-dominating set problems have been studied in Telle (1994).

Various width measures have been studied in relation to (σ, ρ)-dominating set problems. (σ, ρ)-dominating set problems are FPT when parameterized by tree-width Telle and Proskurowski (1997), and FPT when parameterized by clique-width Bui-Xuan et al. (2010). In addition (σ, ρ)-dominating set problems are XP when parameterized by mim-width Bui-Xuan et al. (2013).

The $W[1]$-hardness of the Dominating Set and Independent Set problems parameterized by both mim-width and solution size, has been studied in Fomin et al. (2020). Furthermore some (σ, ρ)-domination problems for some pairs of σ and ρ have been further studied in Jaffke et al. (2019).

2 Preliminaries

2.1 Your Sets are My Sets

We use the following notation: $\mathbb{N} = \{0, 1, 2, 3, ...\}$, $[n] = \{1, 2, ..., n\}$, $[n]_0 = \{0, 1, 2, ..., n\}$.

2.2 Graph Theory

Basic familiarity with graph theory is assumed, we refer to the book Diestel (2012) for terms and concepts.

A graph G is a pair of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$ connecting the vertices. We say $u \in V(G)$ is adjacent to $v \in V(G)$ if $uv \in E(G)$.

All graphs we will consider are finite, simple, and have no loops. That is if G is a graph $|V(G)|$ is finite, there are no identical edges in $E(G)$, and for all $v \in V(G)$ the edge $vv \notin E(G)$.

Let the neighbourhood of any vertex $u \in V(G)$ be $N(u) = \{v \mid uv \in E(G)\}$. Furthermore the closed neighbour of u is $N[u] = N(u) \cup \{u\}$. The neighbourhood of a set $A \subseteq V(G)$ is $N(A) = \bigcup_{v \in A} N(v) \setminus A$, and the closed neighbourhood of set A is $N[A] = N(A) \cup A$.

We say that u is a false twin of v if $N(u) = N(v)$, and u is a true twin of v if $N[u] = N[v]$.

The degree of a vertex $u \in V(G)$ is $d(u) = |N(u)|$

A graph G is connected if for all $u, v \in V(G)$ there is a walk from u to v in G, where a walk from u to v is a sequence of edges, $(v_1v_2, v_2v_3, ..., v_{k-1}v_k)$ in G where for all $i \in [k-1]_0$ the edge $v_iv_{i+1} \in E(G)$, and $v_1 = u$ and $v_k = v$.

A cycle is set a of vertices $v_1, v_2, ..., v_k$ for which $(v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kv_1)$ is a walk from v_1 to v_1.

A subgraph H of a graph G has $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The induced subgraph $G[A]$, where $A \subseteq V(G)$, is the graph $G[A] = (A, E(G) \cap (A \times A))$. Furthermore the induced bipartite graph $G[A, B]$ is the graph $G[A, B] = (V(G), E(G) \cap (A \times B))$.

A cycle in a graph G is a set of vertices $\{v_1, ..., v_k\}$ such that $v_iv_{i+1} \in E(G)$ for all $i \in [k-1]$ and $v_1v_k \in E(G)$.

A tree is a graph T which does not have any cycles as a subgraph and is connected. The vertex $\ell \in V(T)$ is a leaf if the degree of ℓ is 1, and we denote the set of leaves of T by $L(T) = \{\ell \in V(T) \mid d(\ell) = 1\}$.

A path is a graph P is connected graph such that exactly two vertices have degree 1 and the rest of the vertices have degree 2. In addition a single vertex is a path. A path with n vertices is P_n.

For a graph G and an edge $e \in E(G)$ we denote by $G - e$ the graph $G' = (V(G'), E(G'))$, where $V(G)' = V(G)$ and $E(G') = E(G) \setminus \{e\}$.

A partition of a set A is some sets $A_1, ..., A_k$ where $A_i \cap A_j = \emptyset$, for all distinct $i, j \in [k]$, and $\bigcup_{i \in [k]} A_i = A$.

Let G be a graph. A matching M is a set of edges such that if $ab \in M$ then $ac \notin M$ and $db \notin M$ for all $c, d \in V(G)$ such that $c \neq b$ and $d \neq a$. An induced matching M is a matching such that if $ab \in M$ then $ac, db \notin E(G)$.

Let G be a graph. An independent set in G is a set I where for all $a, b \in I$ the edge $ab \notin E(G)$. A clique in G is a set C where for all distinct $a, b \in C$ the edge $ab \in E(G)$.

A subdivision of an edge $uv \in E(G)$ in a graph G is the operation adding the vertex u_1 to $V(G)$ with the edges uu_1 and u_1v, and removing uv from $E(G)$.

A subdivision of a graph H is a graph H' obtained from H by subdividing any edges in H any number of times.

An H-graph G is a graph where each vertex v in $V(G)$ has a model $M_v \subseteq V(H')$ such that $H[M_v]$ is a connected subgraph of H', for a subdivision H' of the graph H. Two vertices $u, v \in V(G)$ have an edge in between them in G if and only if $M_u \cap M_v \neq \emptyset$.

Let G be a graph and let $X = \{X_1, ..., X_p\}$ be a partition of $V(G)$. Then the quotient graph G/X is a graph where $V(G/X) = X$ and $X_iX_j \in E(G/X)$ for all distinct $i, j \in [p]$ if there is an edge $x_ix_j \in E(G)$ for some $x_i \in X_i$ and $x_j \in X_j$.

We say that the graph G is isomorphic to H if there exists a function $f : V(G) \to V(H)$ such that whenever $ab \in E(G)$ then $f(a)f(b) \in E(H)$.

2.3 Exponential Time Hypothesis (ETH)

ETH is a hypothesis roughly conjecturing that there are no algorithms solving 3-SAT in $2^{o(n)}$ time, where n is the number of variables of the instance.

More precisely let δ_3 be the infimum – the greatest lower bound – of the set

$$\{c \mid \text{there exists an algorithm solving 3-SAT in } O(2^{c \cdot n} \cdot n^{O(1)}) \text{ time}\}$$

Then ETH conjectures that $\delta_3 > 0$.

ETH is often used to argue lower bounds on the running time of algorithms solving certain problems. This is done through reductions from ETH (or any problem with an established lower bound) to the problem we would like to argue the lower bound on. In particular we will use that unless ETH is false, the Partitioned Subgraph Isomorphism problem cannot be solved in $n^{o(|E(K)|/\log|E(K)|)}$ time Marx (2010).

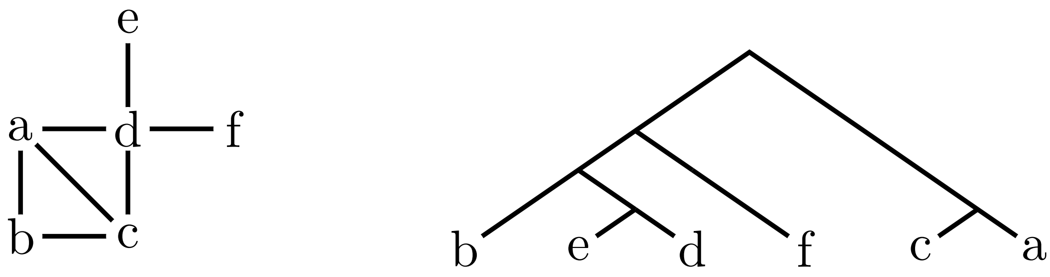
2.4 Parameterized Complexity

For terms and concepts in parameterized complexity we refer to Downey and Fellows (2013); Cygan et al. (2015).

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. For any instance (A, k) of a parameterized problem we call k the parameter.

The set FPT consists of all parameterized problems L which for any input (A, k) it can be determined in in $f(k) \cdot n^{O(1)}$ time, where k is a parameter and $n = |A|$, whether $(A, k) \in L$ or not.

Figure 1: A graph along with one possible branch decomposition of that graph with $\mathcal{L}$ given implicitly.



A parameterized reduction from a parameterized problem X to a parameterized problem Y is an algorithm that given an instance (A, k) of X computes an instance (B, l) of Y in $f(k) \cdot n^{O(1)}$ time where $n = |V(A)|$ and $f(.)$ is some computable function, such that

1. (A, k) is a yes-instance of X if and only if (B, l) is a yes-instance of Y
2. $l \leq g(k)$, for some computable function $g(.)$

We omit the definition of $W[1]$-hardness, but we say a problem is $W[1]$-hard if under a complexity theoretic assumption it is not FPT. Furthermore the property of a problem being $W[1]$-hard transfers to another problem using parameterized reductions in the same way the property of a problem being NP-hard transfers to another problem using polynomial-time reductions.

In particular for two parameterized problems X and Y, if X is $W[1]$-hard and there exists a parameterized reduction from X to Y, then X can be solved in FPT time whenever Y can. This is as any instance of X can be transformed into an equivalent instance of Y in FPT time, and the parameter of the instance of Y remains bounded. Therefore the assumption that X is not FPT transfers to problem Y, and therefore Y is also $W[1]$-hard.

2.5 The Mim-Width Parameter

The mim-width parameter was introduced by Vatshelle (2012).

A branch decomposition of a graph G is a pair $(T, \mathcal{L})$, where T is a tree where all of its vertices have degree at most 3, and $\mathcal{L}$ a bijection mapping the vertices of the graph $V(G)$ to the leaves of the tree T. See Figure 1.

For a graph G and $A, B \subseteq V(G)$ with $A \cap B = \emptyset$ we define

$$cutmim_G(A, B) = \max\{|M| \mid M \text{ is an induced matching in } G[A, B]\}$$

Then we define the mim-width of $(T, \mathcal{L})$, in symbols $\text{mimw}_G(T, \mathcal{L})$ or simply $\text{mimw}(T, \mathcal{L})$ if the graph is clear from context, to be

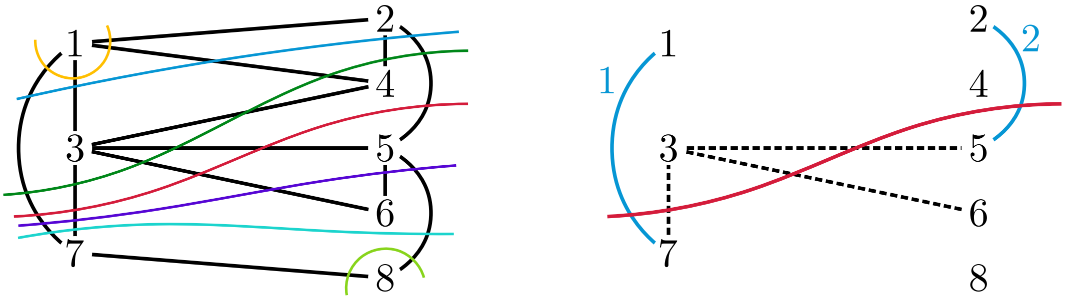
$$\max_{e \in E(T), \{T_1, T_2\} = cc(T - e)} cutmim_G(\mathcal{L}^{-1}(L(T_1)), \mathcal{L}^{-1}(L(T_2)))$$

where cc maps a graph to the set of its connected components.

Furthermore, we define the mim-width of a graph G, $\text{mimw}(G)$, to be the minimum mim-width over all branch decompositions of G.

A type of branch decomposition of particular interest is when T is a caterpillar graph, we then call the branch decomposition for a linear branch decomposition. T is a caterpillar when it has a path P as a subgraph and each vertex of $V(T) \setminus V(P)$ is only adjacent to a single vertex in P. The mim-width of a linear branch decomposition is called linear mim-width, and this linear branch decomposition is in fact equivalent with a linear ordering on the vertices of the graph.

Figure 2: A graph ordered by the numbers in the vertices, where the coloured lines indicate all possible (A, B) cuts induced by the ordering. And an illustration of $cutmim_G(A, B)$ where $A = \{v \mid v \leq 4\}$ and $B = V(G) \setminus A$.



Graph Class	Vertex Model	Mim-Width Bound
Circular Arc	arc on a circle	2
Circular Permutation	line between two concentric circles	2
H-Graphs	$S \subseteq V(H')$ s.t. $H'[S]$ is connected	$2 \cdot \lvert E(H)\rvert$
Interval	intervals on the real line	1
k-Polygon	lines in a regular k-polygon	$2k$
Permutation	line between two lines	2

Table 1: H' is the graph obtained from H by subdividing any amount of edges any amount of times. All the mim-width bounds come from Belmonte and Vatshelle (2013), except for the H-graph mim-width bound which comes from Fomin et al. (2020).

Therefore for linear mim-width instead of considering branch decompositions of a graph we can consider linear orderings of the vertex set of the graph. We denote the linear mim-width of a graph G by linmimw(G). And for some linear ordering Λ of G we say the mim-width of Λ is mimw$_G(\Lambda)$ or mimw(Λ) if the graph is clear from context.

For an example of a graph with a linear ordering see Figure 2.

There are many graph classes which have bounded mim-width. Among them are many intersection graphs, whose edges are defined by intersections between models of their vertices.

Table 2.5 show many intersection graph classes which have bounded mim-width. This implies polynomial time algorithms for problems in XP or FPT when parameterized by mim-width, on graphs where the graph is in one of these graph classes and for which either the vertex models are given or can be computed in polynomial time. In particular this implies polynomial time algorithms for all (σ, ρ)-dominating set problems when the vertex model is given or can be computed in polynomial time, as (σ, ρ)-dominating set problems parameterized by mim-width is XP by Bui-Xuan et al. (2013).

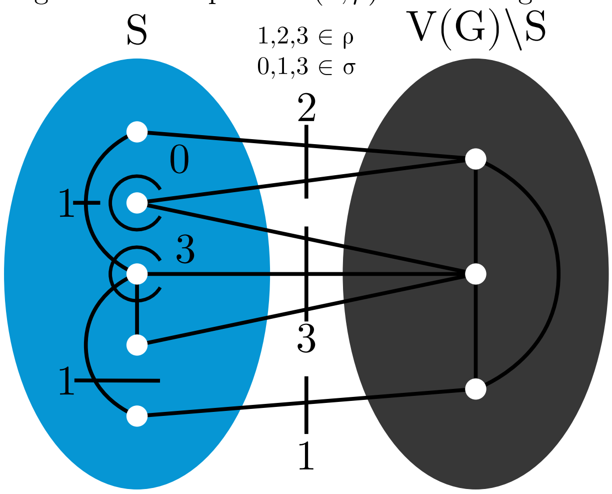
2.6 The (σ, ρ)-Dominating Set

(σ, ρ)-dominating sets were introduced by Telle and Proskurowski (1997), and generalize both dominating set problems and the independent set problem.

Let σ, ρ be non-empty subsets of $\mathbb{N}$, a (σ, ρ)-dominating set D in a graph G is a subset of $V(G)$ with the following property: $\forall v \in V(G)$ if $v \in D$ then $|N(v) \cap D| \in \sigma$. If $v \notin D$ then $|N(v) \cap D| \in \rho$.

Many maximisation and minimisation problems formulated in this manner are not trivial, in the sense that they are NP-hard and $W[1]$-hard with solution size as a parameter, as discussed in Section 1.1. Some of the exceptions where the min. / max. (σ, ρ)-dominating set problems are polynomial-time solvable for certain σ and ρ will be discussed in the next subsection 2.7.

Figure 3: Example of a (σ, ρ)-dominating set S.



For an example of a (σ, ρ)-dominating set see Figure 3.

2.7 Trivial (σ, ρ)-Dominating Set Problems

As discussed many problems which can be formulated with (σ, ρ)-dominating sets are difficult (NP-hard and $W[1]$-hard). However, certain problems are trivial, in the sense that they are polynomial time solvable. We will now discuss some of these trivial problems.

2.7.1 Minimisation

Whenever $0 \in \rho$ notice the empty set will always be a solution of the Min-(σ, ρ)-DS problem, and will be the smallest solution as the empty set has size 0. Clearly this case is then trivial as any algorithm could simply always conclude that the empty set is a valid solution and output accordingly.

Note that these are the only trivial cases for minimisation, as in particular the rest are $W[1]$-hard by this thesis.

2.7.2 Maximisation

If $\sigma = \mathbb{N}$ then notice the entire graph will always be a solution of the Max. (σ, ρ)-Dominating Set problem. Furthermore it is not possible to obtain a larger solution, therefore any algorithm can always conclude that the entire vertex set of the input graph is a valid solution and output accordingly.

If $\rho = \{0\}$ then notice any solution would have to consist of connected components of the input graph. Then for two connected components C_1 and C_2 of the input graph, whether or not C_1 is a $(\sigma, \{0\})$-dominating set is independent of whether or not C_2 is a $(\sigma, \{0\})$-dominating set. Furthermore for any connected component C of the input graph we can verify in polynomial time whether or not C is a $(\sigma, \{0\})$-dominating set.

Therefore we can use a greedy algorithm to solve the problem, by first identifying all connected components of the input graph followed by greedily including them if they are $(\sigma, \{0\})$-dominating sets. This greedy algorithm then clearly runs in polynomial time and indeed finds the maximum size of a $(\sigma, \{0\})$-dominating set. Therefore we do not expect that the maximisation of $(\sigma, \{0\})$-dominating sets is NP-hard nor $W[1]$-hard when parameterized by solution size.

If $\sigma = \mathbb{N}^+$ then notice every isolated vertex in G cannot be in any (σ, ρ)-dominating set. Furthermore let $I = \{v \in V(G) \mid deg(v) = 0\}$, then $V(G) \backslash I$ is a (σ, ρ)-dominating set if and only if $0 \in \rho$. However if $0 \notin \rho$ and $|I| \geq 1$ then there no (σ, ρ)-dominating sets in G. Furthermore if $0 \in \rho$ then $V(G) \setminus I$ is the largest (σ, ρ)-dominating set in G as no vertex in I can be in D. Therefore any algorithm can simply determine that there is a (σ, ρ)-dominating set of size at least $|V(G)| - |I|$ if and only if $0 \in \rho$ or $|I| = 0$.

2.8 Problem Definitions

MIN-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE

Input: (G, k, Λ), where Λ is a linear ordering of $V(G)$
Parameter: $w + k$, where w is the mim-width of Λ
Question: Is there a (σ, ρ)-dominating set in G of size at most k?

MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE

Input: (G, k, Λ), where Λ is a linear ordering of $V(G)$
Parameter: $w + k$, where w is the mim-width of Λ
Question: Is there a (σ, ρ)-dominating set in G of size at least k?

Recall that a linear ordering Λ of G is equivalent with a linear branch decomposition of G.

INDEPENDENT SET on H-graphs

Input: (G, H, k), where G is a H-graph
Parameter: $|E(H)|$
Question: Is there an independent set in G of size k.

PARTITIONED SUBGRAPH ISOMORPHISM

Input: (K, G, ϕ), where $\phi : V(G) \to [V(K)]$ is a colouring function
Parameter: $|E(K)|$
Question: Does there exists an injective function $f : V(K) \to V(G)$ such that $ab \in E(K) \Rightarrow f(a)f(b) \in E(G)$ and $\phi(f(a)) = \phi(a)$ for all $a \in V(K)$

We say a function $f : V(K) \to V(G)$ preserves neighbour if $ab \in E(K) \Rightarrow f(a)f(b) \in E(G)$, and $f(.)$ preserves colours (relative to $\phi : V(G) \to [V(K)]$) if $\phi(f(a)) = \phi(a)$ for all $a \in V(K)$.

We will label the vertices of K by letting $V(K) = [|V(K)|]$.

Note that we will only use results relating to the PARTITIONED SUBGRAPH ISOMORPHISM problem from Marx (2010), in which the results are shown for a K which can be assumed to be connected. We will therefore henceforth assume that K is connected.

3 Lemmas for Linear Mim-Width

If for a partition of a graph the linear mim-width of each part of the partition is bounded and the maximum induced matching between any two parts is bounded, then the linear mim-width of the graph is bounded.

Lemma 1 (Cf. Lemma 7 in Brettell et al. (2022)). *Let G be a graph, let $X = (X_1, ..., X_p)$ be a partition of $V(G)$ such that $\text{cutmim}_G(X_i, X_j) \leq c$ for all distinct $i, j \in [p]$, and let G/X be the quotient graph of X. Then*

$$\text{linmimw}(G) \leq |E(G/X)| \cdot c + \max_{i \in [p]} \text{linmimw}(G[X_i])$$

Moreover, if for all $i \in [p]$, Λ_i is a linear order of X_i, then one can in polynomial time construct a linear order Λ of G with

$$\text{mimw}(\Lambda) \leq |E(G/X)| \cdot c + \max_{i \in [p]} \text{mimw}(\Lambda_i)$$

Proof. Let $\Lambda = \Lambda_1 < ... < \Lambda_p$, with the notation that we let $\Lambda' < \Lambda'' = a_1 < a_2 < ... < a_{\ell'} < b_1 < b_2 < ... < b_{\ell''}$, for two orderings Λ' and Λ'' where $\Lambda' = a_1 < a_2 < ... < a_{\ell'}$ and $\Lambda'' = b_1 < b_2 < ... < b_{\ell''}$ such that Λ' and Λ'' have no vertices in common.

Then let (A, B) be any cut in G induced by Λ, then either $A = X_1 \cup X_2 \cup \cdots \cup X_k$ and $B = X_{k+1} \cup \cdots \cup X_p$, or (A, B) cuts some set X_k in two parts: $X_k \cap A$ and $X_k \cap B$, and $A = X_1 \cup X_2 \cup \cdots \cup X_k \cap A$ and $B = X_k \cap B \cup X_{k+1} \cup \cdots \cup X_p$. Therefore there are at most $\max_{i\in[p]} \operatorname{mimw}(\Lambda_i)$ edges in any induced matching in $G[X_k, X_k]$, as Λ_k cuts X_k in two parts (A', B') in the same way as Λ does (i.e. for any cut (A, B) of G induced by Λ there exists a cut (A', B') of $G[X_k]$ induced by Λ_k where $A' = A \cap X_k$ and $B' = B \cap X_k$).

If $X_iX_j \notin E(G/X)$ then $cutmim_G(X_i, X_j) = 0$, as $E(G[X_i, X_j]) = \emptyset$. There are only $|E(G/X)|$ pairs (i, j) such that $X_iX_j \in E(G/X)$. Therefore there can be at most $|E(G/X)| \cdot c$ edges in any induced matching with one end point in X_i and the other in X_j for all $i \le k$ and $j \ge k$ such that $i \ne j$, as $cutmim_G(X_i, X_j) \le c$.

Therefore we have that $\operatorname{mimw}(\Lambda) = |E(G/X)| \cdot c + \max_{i\in[p]} \operatorname{mimw}(\Lambda_i)$, and that Λ is computable in polynomial time.

□

There can at most be one edge with both end points in a clique in any induced matching.

Lemma 2. *For any induced matching M of $G[A, B]$ and any graph G and bipartition (A, B) of G. If $\mathcal{C}$ is a clique, then $|M \cap \{c'c'' \mid c', c'' \in \mathcal{C}\}| \le 1$.*

Proof. $\mathcal{C}$ is a clique in G and therefore is split in two smaller cliques by the cut (A, B). Therefore, only one edge with endpoints in the two cliques can be in M, as M is an induced matching in G. □

Let G be a graph, and let $\{x_1, ..., x_\ell\} \subseteq V(G)$. Let G' be the graph obtained from G by adding the sets of twins $Y_1, .., Y_\ell$ to G, where for $i \in [\ell]$, Y_i are either all true twins or all false twins, and adding the edge yy' to G' if $x_ix_j \in E(G)$ for all $y \in Y_i$, and for all $y' \in Y_j$ for all distinct $i, j \in [\ell]$. We call this operation adding identical twins of $\{x_1, ..., x_\ell\}$ in G.

Furthermore we call the operation above adding λ identical twins of $\{x_1, ..., x_\ell\}$ in G if $|Y_i| = \lambda$ for all $i \in [\ell]$. We call the modified operation where all twins added are either all false twins or all true twins adding identical false twins or identical true twins respectively. And we say Y_i is the twins of x_i for all $i \in [\ell]$. For an illustration of the operation see Figure 4.

We will show that adding identical twins of $\{x_1, ..., x_\ell\}$ in G changes the linear mim-width of G by at most one.

Lemma 3. *Let G be a graph, and let Λ be a linear ordering of G. Then G' obtained from G by adding identical twins of $\{x_1, ..., x_\ell\}$ in G has a linear ordering Λ' computable in polynomial time from Λ such that* $\operatorname{mimw}(\Lambda') \le \operatorname{mimw}(\Lambda) + 1$.

Proof. We construct a linear ordering Λ' of G' by placing the twins of x_i before x_i in Λ for all $i \in [\ell]$. That is for all $y \in Y_i$ if $a < x_i < b$ then $a < y < x_i < b$. Furthermore let $X_i = \{x_i\} \cup Y_i$ for all $i \in [\ell]$.

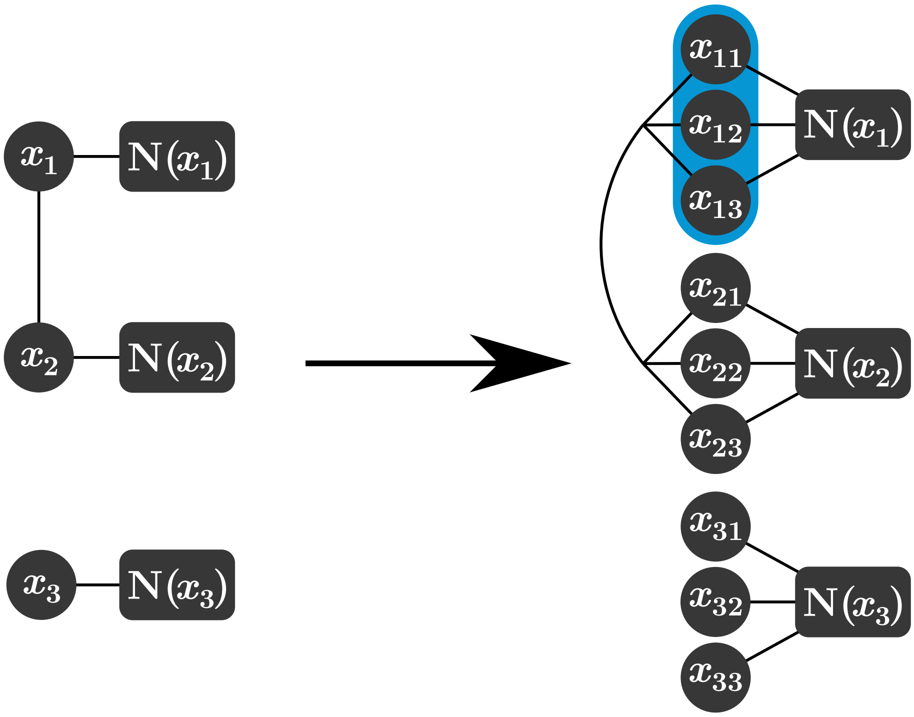
Then let (A, B) be some cut of G' induced by Λ'. Let M' be an induced matching in $G'[A, B]$, we construct an induced matching M of $G[A \cap V(G), B \cap V(G)]$ from M' such that $|M| \ge |M'| - 1$. Notice that $A \supseteq X_1 \cup X_2 \cup \cdots \cup X_i \cap A$ and $B \subseteq X_i \cap B \cup X_{i+1} \cup \cdots \cup X_k$ for some $i \in [\ell]$, where $X_i \cap A$ can be but will not always be equal to A.

Let $f : V(G') \to V(G)$ be the function where $f(a) = \begin{cases} x_i, & \text{if } a \in X_i, \text{ for some } i \in [\ell] \\ a, & \text{otherwise} \end{cases}$.

Then for all $ab \in M'$ we add the edge $f(a)f(b)$ to M if $f(a) \in A \Leftrightarrow f(b) \in B$.

We will first argue that M is then an induced matching. This follows from the fact that if any edge $f(a)f(b) \in M$ "conflicts" with some other edge $f(a')f(b') \in M$ then $ab \in M'$ would

Figure 4: Adding 2 identical twins of $\{x_1, x_2, x_3\}$, where x_{il}, are the twins of x_i for $i, l \in [3]$. The twins of x_1 are true twins, and the twins of x_2 and x_3 are false twins. The blue region is a clique.



also "conflict" with the edge $a'b' \in M'$. Where we say one edge conflicts with another edge if they are both in M, then M cannot be an induced matching.

In particular if $f(a)f(b)$ conflicts with $f(a')f(b')$ for $f(a), f(a') \in A$ and $f(b), f(b') \in B$ then either $f(a) = f(a')$, or $f(b) = f(b')$; or $f(a)f(b') \in E(G[A \cap V(G), B \cap V(G)])$, or $f(a')f(b) \in E(G[A \cap V(G), B \cap V(G)])$. Assume that $f(a) = f(a')$ then notice that $a, a' \in X_j$, for some $j \in [\ell]$. But then a and a' are twins, therefore ab' and $a'b$ are both edges in $E(G')$. The only way for ab' and $a'b$ not be edges in $E(G'[A, B])$ is if $a, b' \in A \Leftrightarrow a', b \in B$. Assume without loss of generality that $a, b' \in A$. Then notice that $f(a') < a'$ since $f(a') \in A$ and $a' \in B$, but $f(a')$ is either equal to a' or $f(a')$ is greater than a' in the ordering. Therefore ab' and $a'b$ are edges in $E(G'[A, B])$, and M is not an induced matching as ab conflicts with $a'b'$. The argument for why $f(b)$ cannot be equal to $f(b')$ is similar.

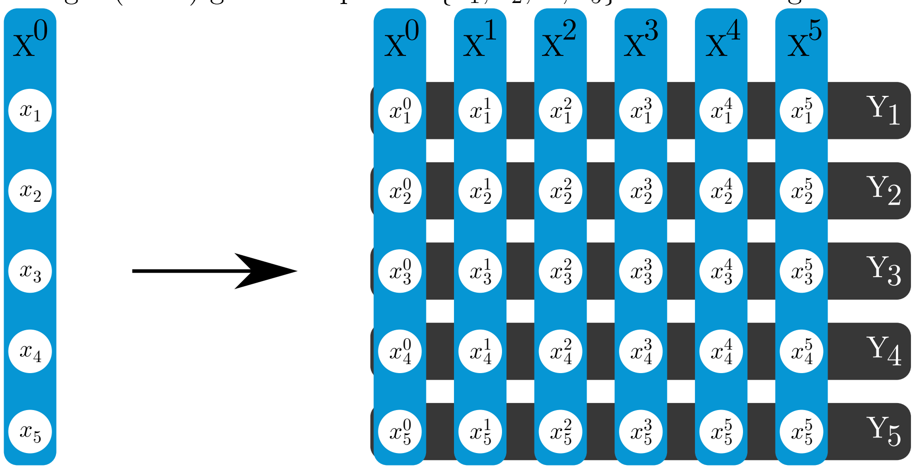
If there is an edge $f(a)f(b') \in E(G[A \cap V(G), B \cap V(G)])$ then $f(a)f(b)$ and $f(a')f(b')$ are both edges in M and the edges ab and $a'b'$ must be in M'. Moreover $ab' \in E(G)$, as not then $f(a), f(b') \in X_i$ but then $f(a), f(b') \in B$ contradicting $f(a) \in A$. Furthermore $ab' \in E(G'[A, B]))$ as if not then either $a, b' \in A$ or $a, b' \in B$. In either case as we have shown above $f(a) > a$ or $f(a') > a'$ which is never true. The argument for when $f(a')f(b) \in E(G[A \cap V(G), B \cap V(G)])$ is similar.

Therefore M is an induced matching. Furthermore for each $xy \in M'$ such that $x, y \notin X_i$ we add exactly one edge to M. This is obvious whenever $x, y \in V(G)$.

If $x \in X_j$ for some $j \neq i$ and $y \notin X_i$ then if there was another edge $x'y' \in M'$ with $x' \in X_j$ and $y' \notin X_i$ then $xy' \in E(G')$ as x and x' are twins. Furthermore $x'y \in E(G'[A, B])$ as for all $i' \neq i$ either $X_{i'} \subseteq A$ or $X_{i'} \subseteq B$. Therefore only one such edge can be in M' for each $j \neq i$. Therefore for each such xy we add one $f(x)f(y)$ to M as $f(x) \in A$ and $f(y) \in B$ whenever $x \in A$ and $y \in B$ respectively.

Finally there can be at most two edges in M' with at least one end point in X_i. Suppose there are three edges $xy, x'y', x''y'' \in M$ such that $x, x', x'' \in X_i$ then notice that the edge $xy' \in E(G')$, but xy' cannot be in $E(G'[A, B])$. Therefore $x, y' \in A$ or $x, y' \in B$. But the edges

Figure 5: Adding a (5×5)-grid of cliques on $\{x_1, x_2, \ldots, x_5\}$. Coloured regions indicate cliques.



$x''y'$ and $x''y$ are also in $E(G')$ but $y' \in A \Leftrightarrow y \in B$ as $x \in A \Leftrightarrow y \in B$. Therefore either $x''y'$ or $x''y$ is in $E(G'[A, B])$ and M' cannot be an induced matching.

Notice that we only add $f(x)f(y)$ to M if $f(x) \in A \Leftrightarrow f(y) \in B$. If both xy and $x'y'$ are in M' with $x, x' \in X_i$ and $y, y' \notin X_i$, then only one of $f(x)f(y)$ and $f(x')f(y')$ is added to M as $f(x) = f(x') = x_i \in B$ and either $y \in A$ or $y' \in A$ but not both by the argument above. Therefore the only edge which can be added to M is $f(x)f(y)$ if $y \in A$ or $f(x')f(y')$ if $y' \in A$. If there is only one edge xy with $x \in X_i$ in M' then at most one edge is added to M depending on whether $x \in B$ or not. Further notice that if $xy \in M'$ with both $x, y \in X_i$, then no more edges $x'y'$ with $x' \in X_i$ can be in M' as xy' would be an edge in $G'[A, B]$. For this case no edges are added to M.

Therefore $|M| \geq |M'| - 1$, and therefore $\text{linmimw}(G') \leq \text{mimw}(\Lambda') \leq \text{mimw}(\Lambda) + 1$, and Λ' is clearly computable in polynomial time. $\square$

Let G be a graph, $\ell \in \mathbb{N}$, and let $X = \{x_1, \ldots, x_k\}$ be a clique in G. The operation of adding a $(k \times \ell)$-grid of cliques on X takes G, and for all $i \in [\ell]$ and for all $j \in [k]$ adds the clique $X^i = \{x^i, \ldots, x^i_k\}$, where we let $x_j = x^0_j$. Furthermore the operation also makes $Y_j = \{x_j, x^1_j, \ldots, x^\ell_j\}$ into a clique.

Furthermore we call X^i the ith column and Y_j the jth row of the grid. For an illustration of the operation see Figure 5.

We will show that adding a $(k \times \ell)$-grid of cliques on X in G increases the linear mim-width of G by at most ℓ.

Lemma 4. *Let G be a graph, let Λ be a linear ordering of G, and let $X = \{x_1, \ldots, x_k\}$ be a clique in G. Then for the graph G' obtained by adding a $(k \times \ell)$-grid of cliques on X in G, there exists a linear ordering Λ' of G' computable in polynomial time from Λ, where* $\text{mimw}(\Lambda') \leq \text{mimw}(\Lambda) + \ell$.

Proof. We can without loss of generality assume that $x_1 < x_2 < \ldots < x_k$ in Λ. For all $j \in [k]$ we place all the vertices in Y_j in Λ such that $(x_{j-1} <) x_j < x^1_j < x^2_j < \ldots < x^\ell_j (< x_{j+1})$, where the vertex in parenthesis may or may not exist depending on whether $j = k$ or $j = 1$, or not. This new ordering we call Λ'.

Let (A, B) be any cut induced by the ordering Λ', and let M' be an induced matching in $G'[A, B]$.

There is some $j \in [k]$ such that $A \supset Y_1 \cup \ldots \cup (Y_j \cap A)$ and $B \supset (Y_j \cap B) \cup Y_{j+1} \cup \ldots \cup Y_k$. Therefore for any induced matching M' in G', $M' \setminus E(G'')$ is an induced matching in G, for

$$G'' = G'\left[\left(\left(\bigcup_{j'<j} Y_{j'}\right) \cup (Y_j \cap A)\right) \setminus X^0, \left((Y_j \cap B) \cup \bigcup_{j''>j} Y_{j''}\right) \setminus X^0\right]$$

First if there are any edges in M' with both end points in some $Y_{j'}$ then $j' = j$ as $Y_{j''} \subseteq A$ or $Y_{j''} \subseteq B$ for all $j'' \neq j$. Furthermore Y_j is a clique and therefore by Lemma 2, M' has at most one edge with both end points in Y_j.

Suppose that $M' \cap E(G'') = M''$ has more than ℓ edges. Then there must be some $i \in [\ell]$ such that there are two edges with both end points in column X_i in M''. Let these two edges be $x^i_j x^i_{j'}$, $x^i_h x^i_{h'}$ for distinct $j, j', h, h' \in [k]$, but X^i is a clique and therefore by Lemma 2, M'' can only have one of these edges. Therefore M'' has at most ℓ edges.

Let M be the induced matching of $G[A \cap V(G), B \cap V(G)]$ such that $M = M' \setminus E(G'')$, it then follows that $\text{mimw}(\Lambda') \leq \text{mimw}(\Lambda) + \ell$. □

If we add a constant amount of vertices to a graph and connect them to the graph in any way, then the mim-width of the new graph can increase by at most a constant relative to the old graph.

Lemma 5. *Let G be a graph, let Λ be a linear order of G, and let G' obtained from G by adding X and connecting X to $V(G)$ in any way. Then $\text{mimw}(\Lambda') \leq \text{mimw}(\Lambda) + |X|$, where Λ' is any linear order of G where the vertices in X are placed anywhere in Λ.*

Proof. Let M' be some maximum induced matching in $G'[A, B]$ where (A, B) is any cut induced by Λ', then M' may or may not have up to $|X|$ edges with one end point in X. Therefore any induced matching M in $G[A \setminus X, B \setminus X]$, where $(A \setminus X, B \setminus X)$ must be some cut induced by Λ, will have the property that $|M| + |X| \geq |M'|$ which in particular implies that $\text{mimw}(\Lambda') \leq \text{mimw}(\Lambda) + |X|$. □

4 $W[1]$-Hardness of (σ, ρ)-Dominating Set Problems

In this section we will prove the main result, which is that all non-trivial minimisation and many non-trivial maximisation (σ, ρ)-dominating set problems parameterized by both the linear mim-width of the input graph and solution size are $W[1]$-hard, and cannot be solved in $n^{o(w/\log(w))}$ time for graphs with linear mim-width w, unless ETH is false.

Theorem 1. *Let σ, ρ be two subsets of $\mathbb{N}$ where $0 \notin \rho$. Then the* Min-(σ,ρ)-DS param. by l. mim-width + sol. size *problem is $W[1]$-hard. Moreover, unless ETH is false, if G is the input graph with $|V(G)| = n$ and with a given linear ordering of G with mim-width w, then there are no algorithms solving* Min-(σ,ρ)-DS param. by l. mim-width + sol. size *in $n^{o(w/\log(w))}$ time.*

Furthermore, if σ and ρ are both finite subsets of $\mathbb{N}$ and $\rho \neq \{0\}$, then the Max-(σ,ρ)-DS param. by l. mim-width + sol. size *problem is $W[1]$-hard, and it cannot be solved in $n^{o(w/\log(w))}$ time, unless ETH is false.*

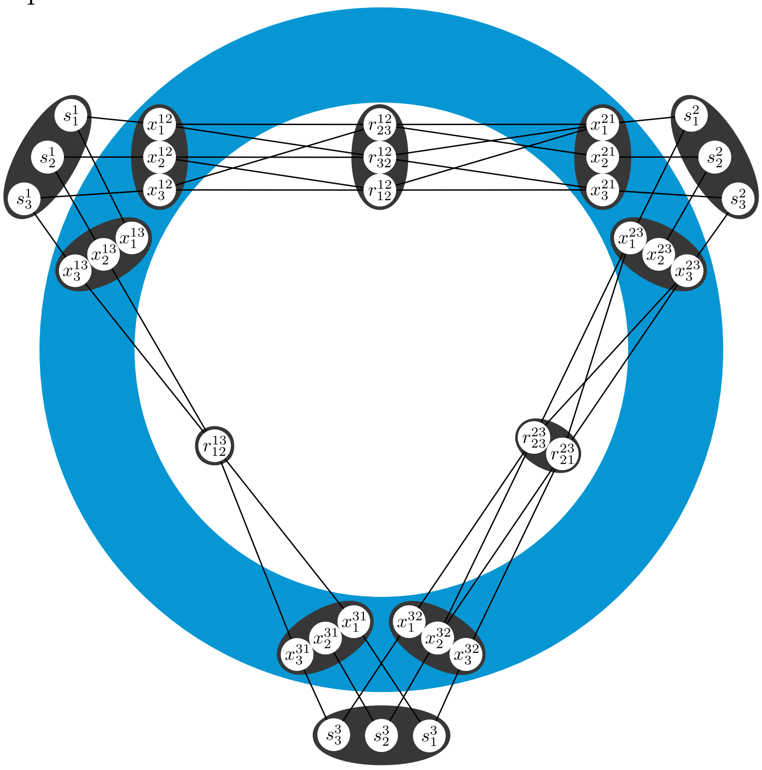
The proof is by a reduction from the $W[1]$-hard problem Partitioned Subgraph Isomorphism, where first a core graph $\mathcal{H}$ is constructed. Afterwards the graph is modified to obtain either H_0, H_1, H_2, or H_3 depending on σ and ρ in manner such that all possible (finite for maximisation) non-trivial σ and ρ are described.

Recall the trivial cases discussed in section 2.7, further note the assumptions on σ and ρ in Theorem 1 imply the (σ, ρ)-dominating set problems are not trivial.

The general idea of the modifications are that we let $\varsigma = \min(\sigma)$ and $\varrho = \min(\rho)$ for minimisation problems and we let $\varsigma = \max(\sigma)$ and $\varrho = \min(\rho)$ for maximisation problems. Then we add some number depending on ς and ϱ vertices such that solving the (σ, ρ)-dominating set problem on that instance becomes similar to solving the $(\{0\}, \{1\})$-domination problem on it, which is also known as the perfect code problem.

We show the linear mim-width of these graphs by first arguing that $\mathcal{H}$ has the linear mim-width of $O(|E(K)|)$ by giving a partition Γ of $V(\mathcal{H})$. Followed by showing that for each part Γ'

Figure 6: Example of $\mathcal{H}$ for $p = 3$, and $k = 3$, and K is the complete graph with three vertices. Note the resemblance to the graph obtained by subdividing every edge in K once. Coloured regions indicate cliques.



of Γ, $\mathcal{H}[\Gamma']$ has bounded linear mim-width and that $cutmim_{\mathcal{H}}(\Gamma', \Gamma'')$ is bounded for all other $\Gamma'' \neq \Gamma'$. This gives the graph a linear mim-width of $O(|E(K)|)$ by Lemma 1.

Then for each modified graph we argue using the lemmas in Section 3, that the linear mim-width increases by at most $|E(K)|$ multiplied some number depending on ς and ϱ. Therefore the linear mim-width of the modified graphs is still $O(|E(K)|)$.

4.1 Construction

We start with an instance of the Partitioned Subgraph Isomorphism problem (K, G, ϕ), and construct an instance of the Min-(σ, ρ)-DS param. by l. mim-width + sol. size or Max-(σ, ρ)-DS param. by l. mim-width + sol. size. This instance depends on whether we are considering the minimisation or maximisation variant, and on σ and ρ. The graph of the constructed instance will be a supergraph of a core graph $\mathcal{H}$ with additional parts depending on σ and ρ.

We can assume that $|V_i| = p$, for all $i \in [k]$, where $p = max\{|V_i| \mid i \in [k]\}$. If this is not the case then we can simply add isolated vertices to the sets whose cardinality is less than p. Isolated vertices clearly do not affect the Partitioned Subgraph Isomorphism instance, as K has no isolated vertices as we assumed it was connected.

For all $i \in [k]$ we name the vertices in V_i: $\{v^i_1, ..., v^i_p\}$.

The core graph $\mathcal{H}$ is constructed as follows:

We add the vertex x^{ij}_a to $V(\mathcal{H})$, for all $i, j \in [k]$ such that $ij \in E(K)$ and for all $a \in [p]$.

Secondly, for all $i, j \in [k]$ such that $ij \in E(K)$ and for all $a, b \in [p]$ such that $v^i_a v^j_b \in E(G)$, we add the vertex $r^{ij}_{ab} = r^{ji}_{ba}$ to $V(\mathcal{H})$. We connect r^{ij}_{ab} to all the vertices in $\{x^{ij}_{a'} \mid a' \neq a, a' \in [p]\}$, and all the vertices in $\{x^{ji}_{b'} \mid b' \neq b, b' \in [p]\}$.

Lastly, for all $i \in [k]$ and for all $a \in [p]$ we add the vertex s^i_a. Furthermore for all $j \in [k]$ such that $ij \in E(K)$, and all $a \in [p]$ we connect s^i_a to x^{ij}_a. Note that there exists at least one pair of i and j such that $ij \in E(K)$ as K is connected.

For all $i, j \in [k]$ such that $ij \in E(K)$ we let $S^i = \{s_a^i \mid a \in [p]\}$, $R^{ij} = \{r_{ab}^{ij} \mid a, b \in [p]\} = R^{ji}$, $X^{ij} = \{x_a^{ij} \mid a \in [p]\}$, $X^{ji} = \{x_a^{ji} \mid a \in [p]\}$, and $X = \bigcup_{i \neq j, i, j \in [k]} X^{ij}$, and make all of these sets into cliques. See Figure 6 for an illustration.

Notice that $R^{ij} = R^{ji}$ but $X^{ij} \neq X^{ji}$ for all $i, j \in [k]$ such that $ij \in E(K)$.

As S^i and R^{ij} have many similar properties we let Z^α be either R^{ij} if $\alpha = ij$ or S^i if $\alpha = i$ for all distinct $i, j \in [k]$ such that $ij \in E(K)$. Furthermore let $\mathcal{I} = [k] \cup \{ij \mid ij \in E(K)\}$ and let $\mathcal{J} = [p] \cup [p] \times [p]$.

We let ς and ϱ be $m(\sigma)$ and $m(\rho)$ respectively, where we let $m(.) = \min(.)$ if we are showing hardness for a Min-(σ, ρ)-DS param. by l. mim-width + sol. size problem and $m(.) = \max(.)$ if we are showing hardness of a Max-(σ, ρ)-DS param. by l. mim-width + sol. size problem.

The core graph $\mathcal{H}$ will be transformed into the graph and solution size pair (H_0, k_0), (H_1, k_1), (H_2, k_2), or (H_3, k_3) by a procedure $\mathcal{A}$ depending on the values of ς and ϱ, and if we are showing hardness of the maximisation or the minimisation variant of the problem.

4.2 Miscellaneous Claims

For arguments later it is useful to be able to assume the size of each S^i or R^{ij} is larger than some constant.

Claim 1. *For all $\alpha \in \mathcal{I}$ and any $n = O(1)$, we can assume that $|Z^\alpha| \geq n$.*

Proof. Z^α can either be some S^i or some R^{ij}, for $i \in [k]$ or $ij \in E(K)$.

If before applying the reduction on the Partitioned Subgraph Isomorphism instance (K, G, ϕ) to create $\mathcal{H}$ we add n isolated vertices to each of $V_1, ..., V_k$, then this does not affect whether or not (K, G, ϕ) is a yes-instance of Partitioned Subgraph Isomorphism or not as K is connected and we can assume that $K \neq P_1$. Similarly, adding n "isolated edges" to R^{ij} by adding one vertex to each of V_i and V_j and connecting them for all $ij \in E(K)$, does not affect whether or not (K, G, ϕ) is a yes-instance of Partitioned Subgraph Isomorphism or not as we can assume that K is connected and is neither P_2 nor P_1.

Adding the isolated vertices increases the size of S^i by n and adding the isolated edges increases the size of R^{ij} by n. Therefore for all $\alpha \in \mathcal{I}$, $|Z^\alpha| \geq n$. ☐

To avoid unnecessary repetition we identify a common scenario which happens for all the instances, which imply that (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem. In particular we will argue that in every case the (σ, ρ)-dominating set of a certain size will always have a subset $D_{\mathcal{H}}$ containing certain vertices.

Claim 2. *Let $D_{\mathcal{H}}$ be a subset of $V(\mathcal{H})$, such that for all $i \in [k]$ there exists a $c_i \in [p]$ such that $s_{c_i}^i \in S^i \cap D_{\mathcal{H}}$ and for all $ij \in E(K)$ the vertex $r_{c_i c_j}^{ij} \in R^{ij} \cap D_{\mathcal{H}}$, then (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem.*

Proof. Let $\{s_{c_i}^i \mid i \in [k]\} \subseteq D_{\mathcal{H}}$, for some $c_1, ..., c_k \in [p]$, then $\{r_{c_i c_j}^{ij} \mid ij \in E(K)\} \subseteq D_{\mathcal{H}}$ by assumption. This implies the edges $\{v_{c_i}^i v_{c_j}^j \mid i, j \in [k]\}$ exist in G. Then the function $f : V(K) \to V(G)$ where $f(i) = v_{c_i}^i$, is a function preserving neighbours and colours as if $ij \in E(K)$ then $r_{c_i c_j}^{ij} \in V(\mathcal{H})$ implying $v_{c_i}^i v_{c_j}^j \in E(G)$. ☐

4.2.1 The Linear Mim-Width of $\mathcal{H}$

For all $i \in [k]$, let $\Gamma_i = S^i \cup \bigcup_{j \in [k], ij \in E(K)} X^{ij}$ and for all $ij \in E(K)$, let $\Gamma_{ij} = R^{ij}$. Then $\Gamma = \{\Gamma_\alpha \mid \alpha \in \mathcal{I}\}$ is a partition of $\mathcal{H}$.

We give the vertices of $\mathcal{H}[\Gamma_i]$ the lexicographic ordering Λ_i: $s_1^i < x_1^{i1} < x_1^{i2} < \cdots < x_1^{ik} < s_2^i < x_2^{i1} < \cdots < x_2^{ik} < \cdots < s_p^i < x_p^{i1} < \cdots < x_p^{ik}$. And we give the vertices of $\mathcal{H}[\Gamma_{ij}]$ any linear ordering Λ_{ij}.

Claim 3. *For all $\alpha \in \mathcal{I}$,* $\text{linmimw}(\mathcal{H}[\Gamma_\alpha]) \leq \text{mimw}(\Lambda_\alpha) \leq 3$.

Proof. Suppose $\alpha = i$ for some $i \in [k]$ and let (A, B) be a cut of $\mathcal{H}[\Gamma_i]$ induced by Λ_i, and let M be an induced matching in $\mathcal{H}[\Gamma_i][A, B]$.

The greatest element in A with respect to the ordering is either s_a^i or x_a^{ij}, for some $a \in [p]$ and $j \in [k]$ such that $ij \in E(K)$.

Firstly suppose that s_a^i is the greatest element in A. Then

$$E(\mathcal{H}[\Gamma_\alpha][A, B]) = \Big((X^{ij} \cap A) \times (X^{ij} \cap B)\Big) \cup \Big((S^i \cap A) \times (S^i \cap B)\Big) \cup \Big\{s_a^i x_a^{ij} \mid j \in [k]\Big\}$$

Note that S^i is a clique, therefore S^i can only contribute one edge to M by Lemma 2. The same goes for X^{ij}.

If $s_a^i x_a^{ij} \in M$ for some $j \in [k]$ such that $ij \in E(K)$, then for all $j' \in [k]$ such that $ij' \in E(K)$, $s_a^i x_a^{ij'}$ cannot be in M as M is a matching. We then conclude that $|M| \leq 3$.

On the other hand suppose that x_a^{ij} is the greatest element of A. Then

$$E(\mathcal{H}[\Gamma_\alpha][A, B]) = \Big((X^{ij} \cap A) \times (X^{ij} \cap B)\Big) \cup \Big((S^i \cap A) \times (S^i \cap B)\Big) \cup \Big\{s_a^i x_a^{ij'} \mid j' > j, j' \in [k]\Big\}$$

But notice then that the same arguments as for when s_a^i was the greatest element in A still works, therefore $|M| \leq 3$.

Therefore $\text{linmimw}(\mathcal{H}[\Gamma_i]) \leq \text{mimw}(\Lambda_i) \leq 3$.

Now suppose that $\alpha = ij$ for $ij \in E(K)$ then let (A, B) be a cut of $\mathcal{H}[\Gamma_{ij}]$ induced by Λ_{ij}, and let M an induced matching in $\mathcal{H}[\Gamma_{ij}][A, B]$. Then Γ_{ij} is a clique, and it therefore only contributes at most one edge to M by Lemma 2.

Therefore in any case $\text{linmimw}(\mathcal{H}[\Gamma_\alpha]) \leq \text{mimw}(\Lambda_\alpha) \leq 3$ for any $\alpha \in \mathcal{I}$, as we have gone through all possibilities of α. ◻

To simplify the arguments for showing that $\mathcal{H}$ has bounded linear mim-width we will argue that a subgraph $\mathcal{H}'$ of $\mathcal{H}$ has bounded linear mim-width. Followed by arguing that the linear mim-width of $\mathcal{H}'$ and $\mathcal{H}$ differ by at most 1. We let $\mathcal{H}'$ be the graph obtained from $\mathcal{H}$ by removing all the edges $x_a^{ij} x_b^{i'j'}$ for all distinct pairs of $ij, i'j' \in E(K)$ such that $i \neq i'$ and for all $a, b \in [p]$.

So $\mathcal{H}'$ is $\mathcal{H}$ but X is not a clique anymore, but for all $i \in [k]$ the vertex set $\bigcup_{j \in [k] \mid ij \in E(K)} X^{ij}$ is a clique still. Then note that since we removed no vertices, only edges, Γ is still a partition of $\mathcal{H}'$. Furthermore for all $\alpha \in \mathcal{I}$, Λ_α is still an ordering of $\mathcal{H}'[\Gamma_\alpha]$, and $\mathcal{H}'[\Gamma_\alpha] = \mathcal{H}[\Gamma_\alpha]$.

Claim 4. *$\text{cutmim}_{\mathcal{H}'}(\Gamma_{\alpha'}, \Gamma_{\alpha''}) \leq 2$, for any distinct $\alpha', \alpha'' \in \mathcal{I}$.*

Proof. Let M be an induced matching in $\mathcal{H}'[\Gamma_{\alpha'}, \Gamma_{\alpha''}]$.

If $(\alpha', \alpha'') = (i, ij)$ for some $ij \in E(K)$, then $E(\mathcal{H}'[\Gamma_{\alpha'}, \Gamma_{\alpha''}]) = \{x_a^{ij} r_{bb'}^{ij} \in E(\mathcal{H}') \mid a \neq b, a, b, b' \in [p]\}$.

Suppose for the sake of contradiction that 3 of these edges are in M. This implies that three vertices from X^{ij}, and three vertices from R^{ij} are endpoints of edges in M.

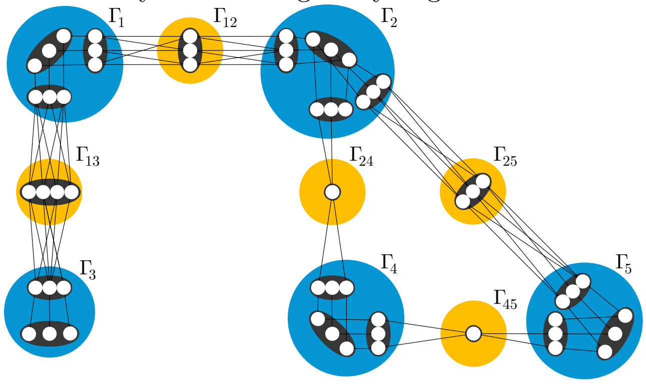
Then suppose the three distinct edges $x_a^{ij} r_{bc}^{ij}, x_{a'}^{ij} r_{b'c'}^{ij}$, and $x_{a''}^{ij} r_{b''c''}^{ij}$ are in M. x_a^{ij} cannot be connected to $r_{b'c'}^{ij}$ nor $r_{b''c''}^{ij}$, as M is an induced matching. Therefore $a = b' = b''$. Similarly $x_{a'}^{ij}$ cannot be connected to $r_{b''c''}^{ij}$ either, and therefore $a' = b''$. So we have $a = b' = b'' = a'$ and therefore $a = a'$, but then M is not a matching.

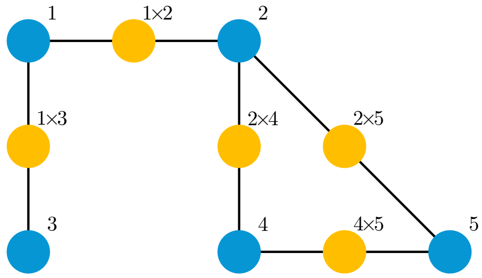
Therefore $|M| \leq 2$. The same argument also works for when $(\alpha', \alpha'') = (j, ij)$ for all $j \in [k]$ and all $i \in [k]$ such that $ij \in E(K)$.

If $(\alpha', \alpha'') = (i, i'j)$, $(\alpha', \alpha'') = (j, ij')$, $(\alpha', \alpha'') = (i, j)$, or $(\alpha', \alpha'') = (i''j'', i'''j''')$, for all $i' \neq i$ such that $i'j \in E(K)$, for all $j' \neq j$ such that $ij' \in E(K)$, for all $i \neq j$, and for all $i''j'', i'''j''' \in E(K)$. Then $E(\mathcal{H}'[\Gamma_{\alpha'}, \Gamma_{\alpha''}]) = \emptyset$ and therefore $M = \emptyset$.

That concludes all the possibilities for α' and α'', and we can conclude that $|M \cap E(\mathcal{H}'[\Gamma_{\alpha'}, \Gamma_{\alpha''}])| \leq 2$, for any distinct pair of $\alpha', \alpha'' \in \mathcal{I}$ ◻

Figure 7: On the left side we see $\mathcal{H}'/\Gamma$ superimposed on $\mathcal{H}'$. On the right side we see K', obtained by subdividing every edge in K once.





Claim 5. linmimw$(\mathcal{H}) \leq$ mimw$(\Lambda) \leq 4|E(K)|+4$, *for an ordering* Λ *computable in polynomial time.*

Proof. We use the partition Γ described above, and use Λ_α for the ordering of $\mathcal{H}'[\Gamma_\alpha]$.

Let $\mathcal{H}'/\Gamma$ be the quotient graph of Γ. Note then that for all distinct $\alpha, \alpha' \in \mathcal{I}$, $\Gamma_\alpha\Gamma_{\alpha'} \in E(\mathcal{H}'/\Gamma)$ if and only if there exists an edge $\gamma_\alpha, \gamma_{\alpha'} \in E(\mathcal{H}')$ for $\gamma_\alpha \in \Gamma_\alpha$, $\gamma_{\alpha'} \in \Gamma_{\alpha'}$.

Further note that $\gamma_\alpha, \gamma_{\alpha'} \in E(\mathcal{H}')$ if and only if $(\alpha, \alpha') = (i, ij)$ or $(\alpha, \alpha') = (j, ij)$ for some $i \in [k]$ and some $j \in [k]$ such that $ij \in E(K)$ or for some $j \in [k]$ and some $i \in [k]$ such that $ij \in E(K)$. Therefore $\mathcal{H}'/\Gamma$ is isomorphic to K', where K' is the graph obtained from K by subdividing every edge in K once. If we let $i \times j$ be the vertex obtained by subdividing the the edge ij once, for all $ij \in E(K)$. Then for all $i \in [k]$, the vertex Γ_i is the vertex mapped to i in the isomorphism, and for all $ij \in E(K)$, the vertex Γ_{ij} is the vertex mapped to $i \times j$. Therefore $|E(\mathcal{H}'/\Gamma)| = 2 \cdot |E(K)|$. See Figure 7.

Therefore by Lemma 1 there exists a linear ordering Λ of $\mathcal{H}'$ computable in polynomial time from Λ_α for all $\alpha \in \mathcal{I}$, such that linmimw$(\mathcal{H}') \leq \text{mimw}_{\mathcal{H}'}(\Lambda) \leq c \cdot |E(\mathcal{H}/\Gamma)| + \max_{\alpha \in \mathcal{I}} \text{mimw}_{\mathcal{H}'}(\Lambda_\alpha)$, where c is $\max_{\Gamma' \neq \Gamma', \Gamma', \Gamma'' \in \Gamma} \mathit{cutmim}_{\mathcal{H}'}(\Gamma', \Gamma'')$.

We have that $c \leq 2$ by Claim 4, and that $\max_{\alpha \in \mathcal{I}} \text{mimw}_{\mathcal{H}'}(\Lambda_\alpha) \leq 3$ by Claim 3. Furthermore Λ_α is clearly computable in polynomial time. Therefore there exists a linear ordering Λ of $\mathcal{H}'$ computable in polynomial time, where $\text{mimw}_{\mathcal{H}'}(\Lambda) \leq 4 \cdot |E(K)| + 3$

Furthermore since no vertices, only edges, are added to $\mathcal{H}'$ to obtain $\mathcal{H}$ we use the same ordering Λ for $\mathcal{H}$. Then X is now a clique and therefore by Lemma 2 any induced matching in a cut (A, B) induced by Λ can have at most one edge with both end points in X. Therefore linmimw$(\mathcal{H}) \leq \text{mimw}_{\mathcal{H}}(\Lambda) \leq \text{mimw}_{\mathcal{H}'}(\Lambda) + 1 \leq 4 \cdot |E(K)| + 4$. □

4.3 Minimisation problems

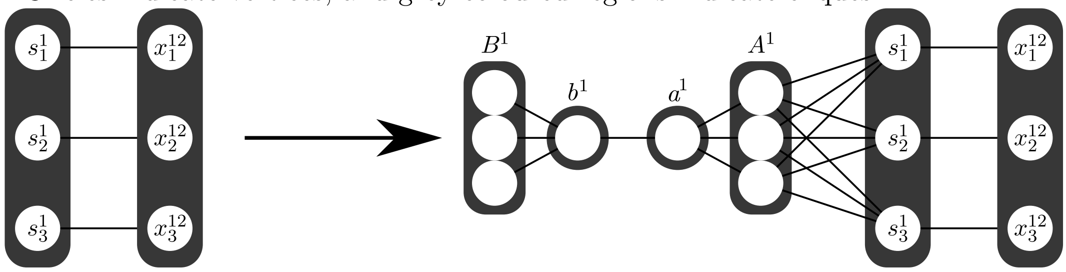
For the minimisation problems we let $\varsigma = \min(\sigma)$ and $\varrho = \min(\rho)$, note that $\varsigma + \varrho = O(1)$.

If $\varsigma = 0$ then $\mathcal{A}$ constructs the graph solution size pair described in Subsection 4.3.4, otherwise $\varsigma \geq 1$. Then if $\varrho = \varsigma + 1$ then $\mathcal{A}$ constructs the graph solution size pair in Subsection 4.3.1, if $\varrho > \varsigma + 1$ the graph solution size pair in Subsection 4.3.2, and if $\varrho < \varsigma + 1$ the graph solution size pair in Subsection 4.3.3. Note that one of these cases must be true, and that if $\varrho < \varsigma + 1$ then $\varsigma \neq 0$ as then $\varrho < 0$ which it cannot be.

4.3.1 When $\varrho = \varsigma + 1$ and $\varsigma \geq 1$

$\mathcal{A}$ transforms $\mathcal{H}$ into the graph solution size pair: (H_0, k_0), where $k_0 = (2\varsigma+2)(k+|E(K)|)+(\varsigma+1)$ and H_0 is constructed as follows:

Figure 8: Example of what $\mathcal{A}$ does to $S^1 \cup X^{12}$ for minimisation problems when $p = 3$, $\varsigma = 3$, $\varrho = 4$. Circles indicate vertices, and grey coloured regions indicate cliques.



For all $\alpha \in \mathcal{I}$, $\mathcal{A}$ creates two cliques A^α and B^α which both have size ς, where A^α is adjacent to all of Z^α. Additionally two vertices are also created a^α and b^α, a^α is adjacent to all vertices in A^α, and b^α is adjacent to all vertices in B^α. Furthermore a^α and b^α are adjacent. Let $Z^\alpha \cup A^\alpha \cup \{a^\alpha\} \cup B^\alpha \cup \{b^\alpha\} = \mathcal{Z}^\alpha$. For an illustration see Figure 8.

Finally $\mathcal{A}$ adds a clique $\mathcal{X}$ of size $\varsigma + 1$, this clique is partitioned in two parts: $\mathcal{X}_1$ and $\mathcal{X}_2$, where $|\mathcal{X}_2| = 1$. Every vertex in $\mathcal{X}_1$ is adjacent to all the vertices in X, and every vertex in $\mathcal{X}_2$ is only adjacent to the vertices in $\mathcal{X}$.

Claim 6. *The graph H_0 has* $\operatorname{linmimw}(H_0) \leq \operatorname{mimw}(\Lambda_0) = O(|E(K)|)$, *for some linear ordering Λ_0 of H_0 computable in polynomial time.*

Proof. The procedure $\mathcal{A}$ transforms $\mathcal{H}$ into H_0 by adding $O(|E(K)|)$ vertex sets of $O(\varsigma + 1)$ size. Then by (the repeated use of) Lemma 5 there exists a linear ordering Λ_0 of H_0 where $\operatorname{linmimw}(H_0) \leq \operatorname{mimw}(\Lambda_0) \leq \operatorname{mimw}(\Lambda) + O(|E(K)|) \cdot O(\varsigma)$, where Λ_0 can be constructed in polynomial time from the linear ordering Λ of $\mathcal{H}$.

Moreover there exists an ordering Λ for which $\operatorname{mimw}(\Lambda) = O(|E(K)|)$ and for which Λ is computable in polynomial time by Claim 5.

Therefore Λ_0 can be constructed in polynomial time and $\operatorname{linmimw}(H_0) \leq \operatorname{mimw}(\Lambda_0) + O(|E(K)|) \cdot O(\varsigma) = O(|E(K)|)$. $\square$

Claim 7. *If (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_0 in H_0*

Proof. Let $f : V(K) \to V(G)$ be the function preserving neighbours and colours. Let $f(i) = v^i_{c_i}$ for all $i \in [k]$ and for some $c_1, ..., c_k \in [p]$. Note that $ij \in E(K)$ implies that $v^i_{c_i} v^j_{c_j} \in E(G)$ further implying that $r^{ij}_{c_i c_j} \in V(\mathcal{H})$ and therefore also $r^{ij}_{c_i c_j}$ is also in H_0.

Then $D = \bigcup_{ij \in E(K)} \{r^{ij}_{c_i c_j}\} \cup \bigcup_{i \in [k]} \{s^i_{c_i}\} \cup \bigcup_{\alpha \in \mathcal{I}} A^\alpha \cup B^\alpha \cup \{b^\alpha\} \cup \mathcal{X}$ is a (σ, ρ)-dominating set of size k_0 in H_0.

$B^\alpha \cup \{b^\alpha\}$ is a clique of size $\varsigma + 1$ and therefore its vertices (all which are in D) are adjacent to ς vertices in D. Furthermore $N(B^\alpha \cup \{b^\alpha\}) = \{a^\alpha\}$ and $a^\alpha \notin D$. So the vertices are adjacent to exactly $\varsigma \in \sigma$ vertices in D. Similarly both $A^\alpha \cup \{z^\alpha_{c_\alpha}\}$, and $\mathcal{X}$ are a cliques of size $\varsigma + 1$ with neighbourhoods not in D.

$N(a^\alpha) \subseteq D$ and $|N(a^\alpha) \cap D| = \varsigma + 1$, so a^α is dominated by $\varsigma + 1 \in \rho$ vertices.

The vertex $s^i_{c'_i}$ for all $i \in [k]$ and all $c'_i \neq c_i$ and the vertex $r^{ij}_{c'_i c'_j}$ for all $ij \in E(K)$ and $c'_i c'_j \neq c_i c_j$ is dominated by all the vertices in $A^i \cup \{s^i_{c_i}\} \subseteq D$ or $A^{ij} \cup \{r^{ij}_{c_i c_j}\} \subseteq D$ respectively. This adds up to $\varsigma + 1 = \varrho \in \rho$ vertices. Furthermore these are all the vertices in $N(s^i_{c'_i}) \cap D$ and $N(r^{ij}_{c_i c_j}) \cap D$ respectively.

Note that $A^\alpha \cup \{a^\alpha\} \cup B^\alpha \cup \{b^\alpha\} \cup Z^\alpha$ only exists if $\alpha \in \mathcal{I}$, and therefore only need to be considered when $\alpha \in \mathcal{I}$, which it is by assumption.

Next for all $ij \in E(K)$ and for all $a \in [p]$ the vertex x^{ij}_a is dominated by the ς vertices $\mathcal{X}_1$ and by either $r^{ij}_{c_i c_j}$ or $s^i_{c_i}$ depending on whether $a \neq c_i$ or $a = c_i$, and as $N(x^{ij}_a) \setminus X =$

$\{s^i_a\} \cup R^{ij} \setminus \{r^{ij}_{c_i d_j} \mid d_j \in [p]\} \cup \mathcal{X}_1$ no other vertices in D. This adds up to $\varsigma + 1 = \varrho \in \rho$ vertices. $\square$

Claim 8. *If there exists a (σ, ρ)-dominating set of size at most k_0 in H_0, then (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem.*

Proof. Let D be a (σ, ρ)-dominating set of size at most k_0, and let $\alpha \in \mathcal{I}$.

Note that for all $b \in B^\alpha$, $deg(b) = \varsigma$, therefore b cannot be dominated by $\varrho = \varsigma + 1$ vertices and therefore $b \in D$. As $b \in D$ the vertex b has to be adjacent to ς vertices also in D, but again $deg(b) = \varsigma$ therefore $N[b] = B^\alpha \cup \{b^\alpha\} \subseteq D$.

Similarly, the single vertex in $\mathcal{X}_2$ has ς neighbours, therefore it cannot possibly be dominated by ς+1 vertices. Therefore $\mathcal{X}_2 \subseteq D$, but then it needs ς neighbours in D. Therefore $\mathcal{X} = N[\mathcal{X}_2] \subseteq D$.

Any vertex $a \in A^\alpha$ must either be in D and have ς neighbours who also are in D, or be dominated by at least $\varsigma + 1$ vertices in D. In either case $|N[A^\alpha] \cap D| \geq \varsigma + 1$. In addition $B^\alpha \cup \{b^\alpha\} \subseteq D$ and $(B^\alpha \cup \{b^\alpha\}) \cap N[A^\alpha] = \emptyset$. Therefore $|\mathcal{Z}^\alpha \cap D| \geq 2\varsigma + 2$ for all $\alpha \in \mathcal{I}$, and it follows that $|V(H_0) \cap D| \geq (|E(K)| + k)(2\varsigma + 2) + \varsigma + 1$. But we know that $|V(H_0) \cap D| = k_0 = (|E(K)| + k)(2\varsigma + 2) + \varsigma + 1$. And consequently $|\mathcal{Z}^\alpha \cap D| = 2\varsigma + 2$ and $X \cap D = \emptyset$.

The vertex a^α must either be in D or not. Suppose for the sake of contradiction that $a^\alpha \in D$. Then $|N(a^\alpha) \cap D| \geq \varsigma$ and note that $|A^\alpha \cap D| \geq \varsigma - 1$ since $b^\alpha \in N(a^\alpha)$, $b^\alpha \in D$, and $N(a^\alpha) \setminus \{b^\alpha\} = A^\alpha$, . Therefore $\varsigma \geq |A^\alpha \cap D| \geq \varsigma - 1$.

First consider when $|A^\alpha \cap D| = \varsigma$ then we have accounted for all $2\varsigma + 2$ vertices in $\mathcal{Z}^\alpha \cap D$ as $B^\alpha \cup \{b^\alpha\} \cup \{a^\alpha\} \subseteq D$. Therefore $Z^\alpha \cap D = \emptyset$. But the for all $c \in \mathcal{J}$ the vertex $z^\alpha_c \notin D$ cannot be dominated by more than the $\varsigma \notin \rho$ vertices in $A^\alpha \subseteq D$ as $Z^\alpha \cap D = \emptyset$ and $X \cap D = \emptyset$. Therefore $|A^\alpha \cap D|$ cannot be ς.

On the other hand if $|A^\alpha \cap D| = \varsigma - 1$, then there is some $a \in A^\alpha$ such that $a \notin D$. But then a must be dominated by at least $\varrho = \varsigma + 1$ vertices in D, but $N(a) = Z^\alpha \cup \{a^\alpha\} \cup A^\alpha$ and $|A^\alpha \cup \{a^\alpha\} \cap D| = \varsigma$. Therefore there exists some $z^\alpha_c \in Z^\alpha \cap D$ for some $c \in \mathcal{J}$. But note that $N(z^\alpha_c) \cap D = (A^\alpha \cup Z^\alpha \setminus \{z^\alpha_c\}) \cap D$ since $X \cap D = \emptyset$, but $|(A^\alpha \cup Z^\alpha \setminus \{z^\alpha_c\}) \cap D| = \varsigma - 1$. Therefore $z^\alpha_c \in D$, but it only has $\varsigma - 1 \notin \sigma$ neighbours in D, which contradicts that D is a (σ, ρ)-dominating set, and therefore $a^\alpha \notin D$.

As $a^\alpha \notin D$ it needs to be dominated by at least $\varrho = \varsigma + 1$ vertices in D, but $deg(a^\alpha) = \varsigma + 1$, therefore $N(a^\alpha) \subseteq D$. Finally z^α_c, for all $c \in \mathcal{J}$, is only adjacent to $\varsigma < \varrho$ vertices in $A^\alpha \cap D$ and zero in $X \cap D$, therefore some $z^\alpha_{c'}$ for some $c' \in \mathcal{J}$ needs to be in D. We have then accounted for $2\varsigma + 2$ vertices in $\mathcal{Z}^\alpha \cap D$, and therefore there cannot be any more vertices in $\mathcal{Z}^\alpha \cap D$.

Therefore we have that $|Z^\alpha \cap D| = 1$ and $X \cap D = \emptyset$. Suppose for all $i \in [k]$ the vertex $s^i_{c_i} \in D$, then for all $j \in [k]$ such that $ij \in E(K)$ and for all $c'_i \neq c_i$ the vertex $x^{ij}_{c'_i}$ is not dominated by $s^i_{c_i}$. As $|S^i \cap D| = 1$ the vertex $x^{ij}_{c'_i}$ can only be dominated by some $r^{ij}_{c_i d_j}$, for some $d_j \in [p]$. The same argument applies for $s^j_{c_j}$ and $r^{ij}_{d_i c_j}$, for some $d_i \in [p]$. It then follows that $r^{ij}_{c_i c_j} \in D$.

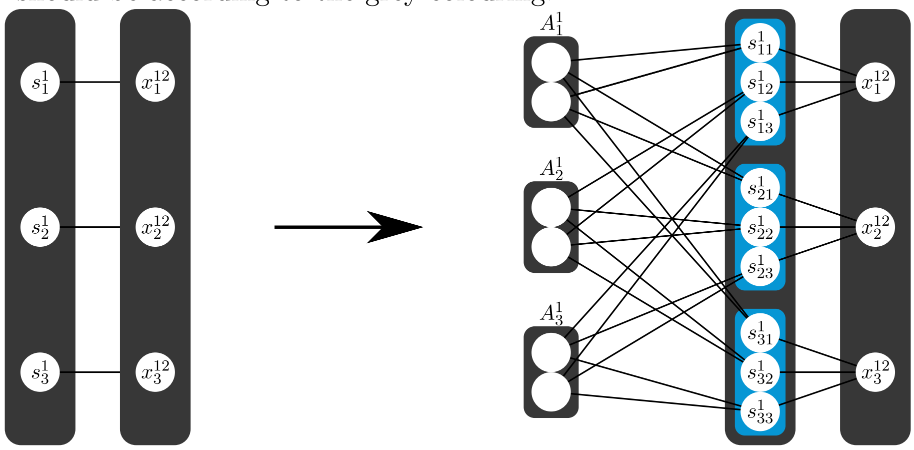
Therefore $D_{\mathcal{H}} = D \cap V(\mathcal{H})$ is a subset of $V(H_0)$ such that for all $i \in [k]$ there exists a $c_i \in [p]$ such that $s^i_{c_i} \in D_{\mathcal{H}}$ and for all $ij \in E(K)$, $r^{ij}_{c_i c_j} \in D_{\mathcal{H}}$, therefore by Claim 2, (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem. $\square$

4.3.2 When $\varrho > \varsigma + 1$ and $\varsigma \geq 1$

Let $\varrho' = \varrho - \varsigma$. The procedure $\mathcal{A}$ then turns $\mathcal{H}$ into the graph and solution size pair: (H_1, k_1), where H_1 is a supergraph of $\mathcal{H}$.

The graph H_1 is generated from $\mathcal{H}$ by for each $\alpha \in \mathcal{I}$ adding $\varrho' - 1$ identical false twins of $\{z^\alpha_\beta \in V(\mathcal{H}) \mid \alpha \in \mathcal{I}, \beta \in \mathcal{J}\}$ in $\mathcal{H}$. Let $\beta \in \mathcal{J}$ be such that $z^\alpha_\beta \in V(\mathcal{H})$, then we call the twins of z^α_β: $z^\alpha_{\beta 2}, ..., z^\alpha_{\beta \varrho'}$, and we let $z^\alpha_\beta = z^\alpha_{\beta 1}$. Followed by adding a clique A^α_ℓ of size ς for all $\ell \in [\varrho']$, where every vertex in A^α_ℓ is adjacent to every vertex in $Z^\alpha_{*\ell} = \{z^\alpha_{\beta\ell} \mid \beta \in \mathcal{J} \text{ s.t. } z^\alpha_\beta \in V(\mathcal{H})\}$. We

Figure 9: Example of what $\mathcal{A}$ does to $S^1 \cup X^{12}$ for $p = 3$, $\varsigma = 2$, $\varrho = 5$. Circles indicate vertices, and grey coloured regions indicate cliques. The blue regions indicate vertices not adjacent even though they should be according to the grey colouring.



call the set containing z_β^α with its $\varrho' - 1$ twins $Z_{\beta *}^\alpha = \{z_{\beta\ell}^\alpha \mid \ell \in [\rho']\}$, and we let $Z_{**}^\alpha = \bigcup_{\ell \in [\varrho']} Z_{*\ell}^\alpha$. We also let $A^\alpha = \bigcup_{\ell \in [\varrho']} A_\ell^\alpha$.

We will also let $R_{\beta *}^{ij} = Z_{\beta *}^\alpha$, $R_{***}^{ij} = Z_{**}^\alpha$, and $R_{*\ell}^{ij} = Z_{*\ell}^\alpha$ whenever $\alpha = ij \in E(K)$, and we let $S_{\beta *}^i = Z_{\beta *}^i$, $S_{**}^i = Z_{**}^\alpha$, and $S_{*\ell}^i = Z_{*\ell}^i$ whenever $\alpha = i$.

Note that the vertices in $Z_{\beta *}^\alpha$ are not adjacent to any other vertices in $Z_{\beta *}^\alpha$, however they are all adjacent to $Z_{\beta' *}^\alpha$ for all $\beta' \neq \beta$. See Figure 9 for an illustration of what $\mathcal{A}$ does to $S^1 \cup X^{12}$.

Furthermore $\mathcal{A}$ adds a clique $\mathcal{X}$ of size $\varsigma + 1$ to H_1, this clique is partitioned into two parts $\mathcal{X}_1$ and $\mathcal{X}_2$, where $|\mathcal{X}_2| = 1$. Every vertex in $\mathcal{X}_1$ is adjacent to all vertices in X, and the vertex in $\mathcal{X}_2$ is only adjacent to $\mathcal{X}$.

Finally we let $k_1 = (\varrho'\varsigma + \varrho')(k + |E(K)|) + (\varsigma + 1)$.

Claim 9. *The graph H_1 has* $\operatorname{linmimw}(H_1) \leq \operatorname{mimw}(\Lambda_1) = O(|E(K)|)$*, for some linear ordering Λ_1 of H_1 computable in polynomial time.*

Proof. $\mathcal{A}$ transforms $\mathcal{H}$ into H_1 by adding $|E(K)| + k + 1 = O(|E(K)|)$ vertex sets A^α and $\mathcal{X}$, of size $O(\varsigma + \varrho)$. Followed by adding false identical twins of $\{z_\beta^\alpha \in V(\mathcal{H}) \mid \alpha \in \mathcal{I}, \beta \in \mathcal{J}\}$ in $\mathcal{H}$.

Suppose without loss of generality that $\mathcal{A}$ first transforms $\mathcal{H}$ into an intermediary graph $\mathcal{H}'$ by adding A^α and $\mathcal{X}$, and finally into the graph H_1 by adding the false twins.

By Lemma 5 there exists a linear ordering Λ' of $\mathcal{H}'$, constructed in polynomial time from any linear ordering Λ of $\mathcal{H}$. Furthermore $\operatorname{mimw}(\Lambda') \leq \operatorname{mimw}(\Lambda) + O(|E(K)|) \cdot O(\varsigma + \varrho)$.

Then by Lemma 3, Λ' can be further transformed in polynomial time into a linear ordering Λ_1 of H_1, where $\operatorname{mimw}(\Lambda_1) \leq \operatorname{mimw}(\Lambda') + 1$.

Finally Λ, an ordering of $\mathcal{H}$, can be computed in polynomial time where $\operatorname{mimw}(\Lambda) = O(|E(K)|)$ by Claim 5. Therefore $\operatorname{linmimw}(H_1) \leq \operatorname{mimw}(\Lambda_1) \leq \operatorname{mimw}(\Lambda') + 1 \leq \operatorname{mimw}(\Lambda) + O(|E(K)|) \cdot O(\varsigma + \varrho) + 1 = O(|E(K)|)$, and Λ_1 is computable in polynomial time as Λ is. □

Claim 10. *If (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_1 in H_1*

Proof. Let $f : V(K) \to V(G)$ be the function preserving neighbours and colours. Let $f(i) = v_{c_i}^i$ for all $i \in [k]$ and for some $c_1, \ldots, c_k \in [p]$. Note that $ij \in E(K)$ implies that $v_{c_i}^i v_{c_j}^j \in E(G)$ further implying that $r_{c_i c_j}^{ij} \in V(\mathcal{H})$ and therefore also in $V(H_1)$.

Then $D = \bigcup_{i \in [k]} S_{c_i *}^i \cup \bigcup_{ij \in E(K)} R_{c_i c_j *}^{ij} \cup \bigcup_{\alpha \in \mathcal{I}} A^\alpha \cup \mathcal{X}$ is a (σ, ρ)-dominating set of size k_1 in H_2.

Note for all $ij \notin \mathcal{I}$ the vertex sets X^{ij} and R^{ij} do not exist and therefore they have no vertices which need to be dominated.

For all $\ell \in [\varrho']$ and for all $ij \in E(K)$ the vertex set $\{r^{ij}_{c_ic_j\ell}\} \cup A^{ij}_\ell$ is a clique contained in D of size $\varsigma + 1$, therefore all its vertices are adjacent to ς vertices in D. Furthermore these are all the vertices in $\{r^{ij}_{c_ic_j\ell}\} \cup A^{ij}_\ell \cap D$. Therefore the vertices in $\{r^{ij}_{c_ic_j\ell}\} \cup A^{ij}_\ell$ are adjacent to exactly $\varsigma \in \sigma$ other vertices in D. The same is true for the vertex set $\{s^i_{c_i\ell}\} \cup A^i_\ell$ for all $i \in [k]$.

The clique $\mathcal{X}$ of size $\varsigma + 1$ is contained in D and therefore all their vertices are adjacent to ς vertices in D. Additionally the vertices in $\mathcal{X}$ are adjacent to exactly $\varsigma \in \sigma$ vertices, as $N(\mathcal{X}) = X$ and $X \cap D = \emptyset$.

For all $ij \in E(K)$ and $a \in [p]$ the vertex x^{ij}_a is being dominated by $R^{ij}_{c_ic_j*}$ if $a \neq c_i$, or by $S^i_{c_i*}$ if $a = c_i$, in addition it is dominated by the ς vertices in $\mathcal{X}_1$. This adds up to $\varrho \in \rho$, furthermore these vertices are all the vertices in $N(x^{ij}_a) \cap D$.

Similarly for all $ij \in E(K)$ the vertex $r^{ij}_{c'_ic'_j\ell}$ for all c'_i, c'_j such that $c'_i \neq c_i \vee c'_j \neq c_j$ is being dominated by the ϱ' vertices in $R^{ij}_{c_ic_j*} \subset D$ in addition to the ς vertices in $A^\alpha_\ell \subset D$, and $N(r^{ij}_{c'_ic'_j\ell}) \cap D$ is exactly these vertices, this adds up to $\varrho \in \rho$ neighbours in D. Similarly $s^i_{c''_i\ell}$ is dominated by the ϱ vertices $S^i_{c_i*} \cup A^\alpha_\ell \subset D$ for all $i \in [k]$ and for all $c''_i \neq c_i$. □

Claim 11. *If there exists a (σ, ρ)-dominating set of size at most k_1 in H_1, then (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem.*

Proof. Let D be the (σ, ρ)-dominating set of size $\leq k_1$ in H_1, and let $\alpha \in \mathcal{I}$.

We will first show that $|Z^\alpha_{**} \cap D| \geq \varrho'$.

Claim 12. *For all $\ell \in [\varrho']$, $A^\alpha_\ell \subseteq D$ or $|N[A^\alpha_\ell] \cap D| \geq \varrho$.*

Proof. For all $a^\alpha \in A^\alpha_\ell$ either $a^\alpha \in D$ or $|N(a^\alpha) \cap D| \geq \varrho$, therefore unless $A^\alpha_\ell \subseteq D$, $|N[A^\alpha_\ell] \cap D| \geq \varrho$. □

By Claim 12 one of two things can be the case, either for some $\ell \in [\varrho']$, $|N[A^\alpha_\ell] \cap D| \geq \varrho$, or for all $\ell \in [\varrho']$, $A^\alpha_\ell \subseteq D$ and therefore $A^\alpha \subseteq D$. Let $a = |\{A^\alpha_\ell \not\subseteq D \mid \ell \in [\varrho']\}|$, then $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq (\varrho' - a)\varsigma + a\varrho = \varrho'\varsigma + a\varrho'$ as $N[A^\alpha_\ell] \cap N[A^\alpha_{\ell'}] = \emptyset$ for all distinct $\ell', \ell \in [\varrho']$.

If $a > 0$ then clearly $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varrho'\varsigma + \varrho'$. But even if $a = 0$, $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varrho'\varsigma + \varrho'$. This is as if $a = 0$ then for all $\ell \in [\varrho']$, $A^\alpha_\ell \subseteq D$ by assumption. But $|A^\alpha_\ell| = \varsigma$ and every vertex in A^α_ℓ has to be adjacent to at least ς vertices in D. Furthermore $N[A^\alpha_\ell] \cap N[A^\alpha_{\ell'}] = \emptyset$ for all distinct $\ell', \ell \in [\varrho']$, therefore for all $\ell \in [\varrho']$ there must be at least one vertex in $N(A^\alpha_\ell) \cap D \subseteq Z^\alpha_{**} \cap D$. It then follows that $|Z^\alpha_{**} \cap D| \geq \varrho'$, and that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varrho'\varsigma + \varrho'$.

$|\mathcal{X} \cap D| \geq \varsigma$, as there is a vertex in $\mathcal{X}$ which is only connected to $\varsigma < \varrho$ vertices. In fact $\mathcal{X} \subseteq D$.

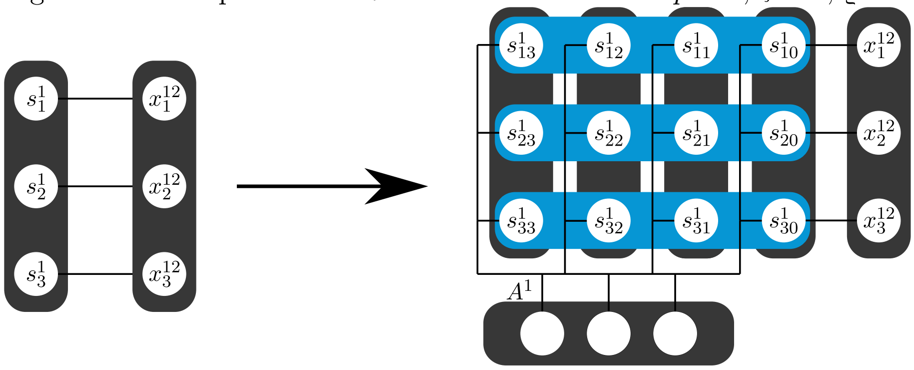
$Z^\alpha_{**} \cup A^\alpha, X \cup \mathcal{X}$, for all $\alpha \in \mathcal{I}$, make a partition of $V(H_1)$. Furthermore, adding up all the size bounds of the sets in the partition yields $|V(H_1) \cap D| \geq k_1$. But $|D| \leq k_1$, therefore $|D| = k_1$. And in particular, $X \cap D = \emptyset$ and $|(Z^\alpha_{**} \cup A^\alpha) \cap D| = \varrho'\varsigma + \varrho'$.

Notice then that $a \leq 1$ as if $a > 1$ then $|(Z^\alpha_{**} \cup A^\alpha) \cap D| > \varrho'\varsigma + \varrho'$ which cannot be the case.

Assume then that $a = 1$ and let A^α_ℓ be the set such that $A^\alpha_\ell \not\subseteq D$. Note that for all $\ell' \neq \ell$ the vertices of $A^\alpha_{\ell'}$ are in D, and therefore have to be adjacent to at least ς vertices in D. But $|A^\alpha_{\ell'}| = \varsigma$, therefore there is at least one vertex in $N(A^\alpha_{\ell'}) \cap D = Z^\alpha_{*\ell'} \cap D$.

Currently we have accounted for $(\varrho' - 1)(\varsigma + 1) = \varrho'\varsigma + \varrho' - \varsigma - 1$ vertices in $(Z^\alpha_{**} \cup A^\alpha) \cap D$. By Claim 12, $|N[A^\alpha_\ell] \cap D| \geq \varrho$, but the currently accounted for vertices are not in $N[A^\alpha_\ell] \cap D$ as $N[A^\alpha_\ell] \cap N[A^\alpha_{\ell'}] = \emptyset$. Therefore, counting these ϱ vertices in addition to the other accounted vertices adds up to $\varrho'\varsigma + 2\varrho' - 1$ accounted vertices in $(Z^\alpha_{**} \cup A^\alpha) \cap D$. However we know that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| = \varrho'\varsigma + \varrho'$, therefore $\varrho' = 1$ but we know this is not the case as $\varrho' > 1$. Therefore $a = 0$.

Figure 10: Example of what $\mathcal{A}$ does to $S^1 \cup X^{12}$ for $p = 3$, $\varsigma = 6$, $\varrho = 3$



For all $\ell \in [\varrho']$, $A^\alpha_\ell \subseteq D$ and therefore $|N[A^\alpha_\ell] \cap D| \geq \varsigma + 1$. Furthermore $|N[A^\alpha_\ell] \cap D| = \varsigma + 1$ since $N[A^\alpha_\ell] \cap N[A^\alpha_{\ell'}] = \emptyset$ for all $\ell' \neq \ell$, $\bigcup_{\ell \in [\varrho']} N[A^\alpha_\ell] = Z^\alpha_{**} \cup A^\alpha$, and $|Z^\alpha_{**} \cup A^\alpha| = \varrho'\varsigma + \varrho'$. Therefore since $|A^\alpha_\ell| = \varsigma$ there must exist exactly one $b \in \mathcal{J}$ for all $\ell \in [\varrho']$ such that $z^\alpha_{b\ell} \in D$. It then follows that $|Z^\alpha_{**} \cap D| = \varrho'$.

Suppose that there does not exist any $c \in \mathcal{J}$ such that $Z^\alpha_{c*} \subseteq D$. Then as $|Z^\alpha_{**} \cap D| = \varrho'$ there must exist some $b \in \mathcal{J}$ such that $|Z^\alpha_{b*} \cap D| \geq 1$. Then $|Z^\alpha_{b*} \cap D| < \varrho'$ by assumption, let $z^\alpha_{b\ell}$ be the vertex which is not in D for some $\ell \in [\varrho']$. Then $N(z^\alpha_{b\ell}) \cap D \subseteq \bigcup_{b' \neq b \ | \ z^\alpha_{b'} \in V(\mathcal{H})} Z^\alpha_{b'*} \cup A^\alpha_\ell \cup X$, as $N(z^\alpha_{b\ell}) \cap Z^\alpha_{b*} = \emptyset$. But $|\bigcup_{b' \neq b \ | \ z^\alpha_{b'} \in V(\mathcal{H})} Z^\alpha_{b'*} \cap D| \leq \varrho' - |Z^\alpha_{b*} \cap D| \leq \varrho' - 1$, $|X \cap D| = 0$, and $|A^\alpha_\ell| = \varsigma$. Therefore $z^\alpha_{b\ell}$ is dominated by at most $\varrho - 1 \notin \rho$ vertices, contradicting that D is a (σ, ρ)-dominating set. Therefore there must exist some $c \in \mathcal{J}$ such that $Z^\alpha_{c*} \subseteq D$.

Then for all $i \in [k]$ there exists a $c_i \in [p]$ such that $S^i_{c_i*} \subseteq D$, and for all $ij \in E(K)$ there exists $d_i, d_j \in [p]$ such that $R^{ij}_{d_i d_j *} \subseteq D$. Suppose that $c_i \neq d_i$ then notice the vertex $x^{ij}_{d_i}$ is only being dominated by the $\varsigma < \varrho$ vertices in $\mathcal{X} \cap D$, but $\varsigma \notin \rho$. Therefore $c_i = d_i$, and by a similar argument $c_j = d_j$. Therefore by Claim 2, (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem. $\square$

4.3.3 When $\varrho < \varsigma + 1$

Let $\varsigma' = \varsigma - \varrho + 1$. The procedure $\mathcal{A}$ turns $\mathcal{H}$ into the graph solution size pair: (H_2, k_2), where $k_2 = (\varsigma + 1) \cdot (|E(K)| + k) + \varsigma + 1$. Let $\alpha \in \mathcal{I}$, and for each α we associate a value $p_\alpha = |Z^\alpha|$. H_2 is constructed from $\mathcal{H}$ as follows:

For all $\beta \in \mathcal{J}$ such that $z^\alpha_\beta \in V(\mathcal{H})$, let $z^\alpha_\beta = z^\alpha_{\beta 0}$.

$\mathcal{A}$ adds a $(p_\alpha \times \varsigma')$-grid of cliques on Z^α in $\mathcal{H}$. We call the ℓth column $Z^\alpha_{*\ell}$, for all $\ell \in [\varsigma']_0$, and the βth row $Z^\alpha_{\beta*}$, for all β such that $z^\alpha_\beta \in Z^\alpha$. Let $Z^\alpha_{**} = \bigcup_{\ell \in [\varsigma']_0} Z^\alpha_{*\ell}$.

Similarly as when $\varrho > \varsigma + 1$ and $\varsigma \geq 1$ we let $R^{ij}_{\beta*} = Z^\alpha_{\beta*}$, $R^{ij}_{***} = Z^\alpha_{**}$, and $R^{ij}_{*\ell} = Z^\alpha_{*\ell}$ whenever $\alpha = ij$ and $S^i_{\beta*} = Z^i_{\beta*}$, $S^i_{**} = Z^\alpha_{**}$, and $S^i_{*\ell} = Z^i_{*\ell}$ whenever $\alpha = i$.

If $\varrho > 1$, then $\mathcal{A}$ adds a clique A^α of size $\varrho - 1$, where the vertices in A^α are adjacent to all vertices in Z^α_{**}. If $\varrho = 1$ then $A^\alpha = \emptyset$.

Finally $\mathcal{A}$ also add a clique $\mathcal{X}$ of size $\varsigma + 1$ to H_2. This clique is partitioned into two parts $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_1$ has size $\varrho - 1$ and all its vertices are adjacent to all vertices in X. The vertices in $\mathcal{X}_2$ are only adjacent to all all vertices in $\mathcal{X}$ and $\mathcal{X}_2$ has size $\varsigma' + 1$.[1]

Claim 13. *The graph H_2 has* $\operatorname{linmimw}(H_2) \leq \operatorname{mimw}(\Lambda_2) = O(|E(K)|)$*, for some linear ordering Λ_2 of H_2 computable in polynomial time.*

Proof. We use the linear ordering Λ of $\mathcal{H}$ constructed in Claim 5, which is computable in polynomial time, and for which $\operatorname{mimw}(\Lambda) = O(|E(K)|)$.

[1]Note that $\mathcal{X}$ is not needed for correctness when $\varsigma = 0$, but for simplicity we include it anyway.

We can assume without loss of generality that first for all $\alpha \in \mathcal{I}$, $(p_\alpha \times \varsigma')$-grids of cliques is added on Z^α in $\mathcal{H}$. By Lemma 4 there exists a linear ordering Λ' of this graph with the grid of cliques added, which can be constructed in polynomial time from Λ, where $\mathrm{mimw}(\Lambda') \leq \mathrm{mimw}(\Lambda) + (|E(K)| + k) \cdot \varsigma' = O(|E(K)|)$.

Finally in order to obtain H_2 the cliques $\mathcal{X}$ and A^α, for all $\alpha \in \mathcal{I}$, are added to the intermediary graph. By Lemma 5 there exists a linear ordering Λ_2 of H_2 constructible from Λ' in polynomial time, where $\mathrm{mimw}(\Lambda_2) \leq (|E(K)| + k) \cdot O(\varsigma + \varrho) + \mathrm{mimw}(\Lambda') = O(|E(K)|)$. □

Claim 14. *If (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_2 in H_2*

Proof. Let $f : V(K) \to V(G)$ be the function preserving neighbours and colours. Let $f(i) = v^i_{c_i}$ for all $i \in [k]$ and for some $c_1, ..., c_k \in [p]$. Note that $ij \in E(K)$ implies that $v^i_{c_i} v^j_{c_j} \in E(G)$ further implying that $r^{ij}_{c_i c_j} \in V(\mathcal{H})$ and therefore also in H_2.

Then $D = \bigcup_{ij \in E(K)} R^{ij}_{c_i c_j *} \cup \bigcup_{i \in [k]} S^i_{c_i *} \cup \bigcup_{\alpha \in \mathcal{I}} A^\alpha \cup \mathcal{X}$ is a (σ, ρ)-dominating set of size k_2 in H_2.

For all $ij \in E(K)$ the vertex sets $R^{ij}_{c_i c_j *} \cup A^{ij}_\ell$, $S^i_{c_i *} \cup A^i_\ell$, and $\mathcal{X}$ are all cliques contained in D of size $\varsigma + 1$. Therefore all the vertices in the cliques are adjacent to at least ς vertices in D. In addition, for all of these cliques there are no vertices in their neighbourhood which are in D. Therefore all their vertices are indeed adjacent to exactly $\varsigma \in \sigma$ vertices in D.

For all $ij \in E(K)$ and for all $a \in [p]$ the vertex x^{ij}_a is dominated by the $\varrho - 1$ vertices in $\mathcal{X}_2$ [2]. Furthermore x^{ij}_a is also dominated by $r^{ij}_{c_i c_j}$ if and only if $a \neq c_i$, and x^{ij}_a is also dominated by $s^i_{c_i}$ if and only if $a = c_i$. These are all the vertices in $N(x^{ij}_a) \cap D$, therefore x^{ij}_a is dominated by exactly $\varrho \in \rho$ vertices in D.

For all $ij \in E(K)$ and for all $c'_i, c'_j \in [p]$ such that $c'_i \neq c_i \vee c'_j \neq c_j$, the vertices in $R^{ij}_{c'_i c'_j *}$ are dominated by the exactly $\varrho \in \rho$ vertices in $R^{ij}_{c_i c_j *} \cup A^{ij} \subset D$. The same is true for all $i \in [k]$, where all the vertices in $S^i_{c''_i *}$, for all $c''_i \neq c_i$, are dominated by the $\varrho \in \rho$ vertices in $S^i_{c_i *} \cup A^i$. □

Claim 15. *If there exists a (σ, ρ)-dominating set of size at most k_2 in H_2, then (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem.*

Proof. Let D be the (σ, ρ)-dominating set in H_2 of size at most k_2, and let $\alpha \in \mathcal{I}$.

We will now show that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varsigma + 1$.

Claim 16. *If $(Z^\alpha_{**} \setminus Z^\alpha) \cap D \neq \emptyset$ then $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varsigma + 1$.*

Proof. Let $z \in Z^\alpha_{**} \setminus Z^\alpha$. If z is in D then it needs ς neighbours in D, these ς vertices have to be in Z^α_{**} as $N(z) \subseteq Z^\alpha_{**}$. Therefore if $z \in D$ then $|Z^\alpha_{**} \cap D| \geq \varsigma + 1$. □

Suppose that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| < \varsigma + 1$ then by Claim 16, $(Z^\alpha_{**} \setminus Z^\alpha) \cap D = \emptyset$. Therefore for all $a \in \mathcal{J}$ and for all $\ell \in [\varsigma']$, the vertex $z^\alpha_{a\ell} \in Z^\alpha_{**} \setminus Z^\alpha$ has to be dominated by some vertex in Z^α as $0 \notin \rho$ by assumption. Note that $N(z^\alpha_{a\ell}) \cap Z^\alpha = \{z^\alpha_a\}$, and therefore $z^\alpha_a \in D$.

Then for all $a \in \mathcal{J}$, $z^\alpha_a \in D$ and therefore $Z^\alpha \subseteq D$. We can by Claim 1 assume that $|Z^\alpha| \geq \varsigma + 1$, therefore $|Z^\alpha \cap D| \geq \varsigma + 1$ contradicting that $|(Z^\alpha_{**} \cup A^\alpha) \cap D| < \varsigma + 1$. We can therefore conclude that in any case $|(Z^\alpha_{**} \cup A^\alpha) \cap D| \geq \varsigma + 1$

There either exists some $x \in \mathcal{X}_2 \cap D$ or $\mathcal{X}_2 \cap D = \emptyset$. For the first case the vertex $x \in D$ needs to be adjacent to at least ς other vertices in D, but $N(x) = \mathcal{X} \setminus \{x\}$ which has size ς, therefore $\mathcal{X} \subseteq D$. For the second case let $x' \in \mathcal{X}_2$, x' then needs to be dominated by at least ϱ vertices in D. Furthermore these vertices have to be in $\mathcal{X}_1$ as $\mathcal{X}_2 \cap D = \emptyset$ by assumption, and $N(x') \subseteq \mathcal{X}$. But $|\mathcal{X}_1| = \varrho - 1$, therefore x' cannot be dominated by at least ϱ vertices which contradicts that

[2] $\mathcal{X}_2$ is empty if $\varrho = 1$, but the arguments still hold.

D is a (σ, ρ)-dominating set, and therefore there must be some $x \in \mathcal{X}_2 \cap D$ which in turn implied that $\mathcal{X} \subseteq D$. And therefore clearly $|\mathcal{X} \cap D| \geq \varsigma + 1$. [3]

The vertex sets $Z^{\alpha}_{**} \cup A^{\alpha}, X \cup \mathcal{X}$, for all $\alpha \in \mathcal{I}$, make a partition of $V(H_2)$. Furthermore, adding up all the size bounds of the sets in the partition yields $|V(H_2) \cap D| \geq k_2$. But $|D| \leq k_2$, therefore $|D| = k_2$, $|(Z^{\alpha} \cup A^{\alpha}) \cap D| = \varsigma + 1$, and $X \cap D = \emptyset$.

Note that because $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| = \varsigma + 1$, the only subset of Z^{α}_{**} which can also be a subset of D must be a clique of size $\varsigma + 1$. If not then there is at least two vertices in D which are adjacent to less than ς vertices which cannot be the case. The only cliques of size at least $\varsigma + 1$ in Z^{α}_{**} are $Z^{\alpha}_{*\ell} \cup A^{\alpha}$ for all $\ell \in [\varsigma']_0$, and $Z^{\alpha}_{b*} \cup A^{\alpha}$ for all $b \in \mathcal{J}$, and possibly their subsets.

Suppose that $Z^{\alpha}_{*\ell} \cup A^{\alpha}$, for some $\ell \in [\varsigma']_0$, has a subset of size $\varsigma + 1$ in D. Then note that because of Claim 1 we can assume that $|Z^{\alpha}_{*\ell''}| > \varsigma + 1$ for all $\ell'' \in [\varsigma']_0$. In particular, there exists $c \in \mathcal{J}$ such that $z^{\alpha}_{c\ell} \notin D$. Furthermore for all $\ell' \neq \ell$ the vertex $z^{\alpha}_{c\ell'}$ is not in D either. But notice that even if $A^{\alpha} \subseteq D$ the vertex $z^{\alpha}_{c\ell'}$ is dominated by less than ϱ vertices in D as $|A^{\alpha}| = \varrho - 1$, $X \cap D = \emptyset$, and $N(z^{\alpha}_{c\ell'}) \cap Z^{\alpha}_{*\ell} = \{z^{\alpha}_{c\ell}\} \nsubseteq D$ and we assumed that $Z^{\alpha}_{**} \cap D = Z^{\alpha}_{*\ell}$.

Therefore, only some subset of $Z^{\alpha}_{b*} \cup A^{\alpha}$, for some $b \in \mathcal{J}$, of size $\varsigma + 1$ can be in D. But $|Z^{\alpha}_{b*} \cup A^{\alpha}| = \varsigma + 1$. Therefore $Z^{\alpha}_{b*} \cup A^{\alpha} \subseteq D$.

Then for all $ij \in E(K)$ there exists $d_i, d_j \in [p]$ such that $R^{ij}_{d_i d_j *} \cup A^{ij} \subseteq D$. Similarly for all $i \in [k]$ there exists $c_i \in [p]$ such that $S^{i}_{c_i *} \cup A^{i} \subseteq D$. Assume that $c_i \neq d_i$ then $x^{ij}_{d_i}$ is adjacent to all of $\mathcal{X}_1$ but not to $R^{ij}_{d_i d_j *}$ nor to $S^{i}_{c_i *}$. But $|\mathcal{X}_1| = \varrho - 1$ and $X \cap D = \emptyset$. Therefore in order for $x^{ij}_{d_i}$ to be dominated by ϱ vertices d_i must be equal to c_i. By a similar argument $d_j = c_j$. Therefore by Claim 2, (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem. $\square$

4.3.4 When $\varrho \geq 1$ and $\varsigma = 0$

$\mathcal{A}$ transforms $\mathcal{H}$ into the graph solution size pair: (H_3, k_3), where $k_3 = (\varrho)(k + |E(K)|)$ and H_3 is constructed as follows:

The graph H_3 is generated from $\mathcal{H}$ by for each $\alpha \in \mathcal{I}$ adding $\varrho - 1$ [4] identical false twins of $\{z^{\alpha}_{\beta} \in V(\mathcal{H}) \mid \alpha \in \mathcal{I}, \beta \in \mathcal{J}\}$ in $\mathcal{H}$. For all $\beta \in \mathcal{J}$ such that $z^{\alpha}_{\beta} \in V(\mathcal{H})$, we call the twins of z^{α}_{β}: $z^{\alpha}_{\beta 2}, \ldots, z^{\alpha}_{\beta \ell}$, and we let $z^{\alpha}_{\beta} = z^{\alpha}_{\beta 1}$.

Similarly as when $\varrho > \varsigma + 1$ and $\varsigma \geq 1$, we let $Z^{\alpha}_{\beta *} = \{z^{\alpha}_{\beta \ell} \mid \ell \in [\rho]\}$, $Z^{\alpha}_{*\ell} = \{z^{\alpha}_{\beta \ell} \mid \beta \in \mathcal{J} \text{ s.t. } z^{\alpha}_{\beta} \in V(\mathcal{H})\}$, and we let $Z^{\alpha}_{**} = Z^{\alpha} \cup \bigcup_{\ell \in [\varrho']} Z^{\alpha}_{*\ell}$.

We will also let $R^{ij}_{\beta *} = Z^{\alpha}_{\beta *}$, $R^{ij}_{***} = Z^{\alpha}_{**}$, and $R^{ij}_{*\ell} = Z^{\alpha}_{*\ell}$ whenever $\alpha = ij \in E(K)$, and we let $S^{i}_{\beta *} = Z^{i}_{\beta *}$, $S^{i}_{**} = Z^{\alpha}_{**}$, and $S^{i}_{*\ell} = Z^{i}_{*\ell}$ whenever $\alpha = i$.

In addition we add a clique A^{α} of size ϱ, and we connect all of its vertices to to all the vertices in Z^{α}_{**}.

Claim 17. *The graph H_3 has* $\operatorname{linmimw}(H_3) \leq \operatorname{mimw}(\Lambda_3) = O(|E(K)|)$*, for some linear ordering Λ_3 of H_3 computable in polynomial time.*

Proof. The proof is very similar to the proof of Claim 9. Essentially by Claim 5, $\mathcal{H}$ has linear mim-width $O(|E(K)|)$. Then false twins are added to $\mathcal{H}$ (possibly 0 false twins in which case use the same ordering as $\mathcal{H}$), for which there is an linear ordering with linear mim-width $O(|E(K)| + 1)$ by Lemma 3. Finally $|E(K)| + k$ cliques of size ϱ are added on top of that to obtain H_3 which has a linear ordering Λ_3 such that $\operatorname{mimw}(\Lambda_3) = O(|E(K)|) + (|E(K)| + k) \cdot \varrho = O(|E(K)|)$ by Lemma 5. Furthermore all of these orderings are computable in polynomial time. $\square$

Claim 18. *If (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem, then there exists a $(\{\varsigma\}, \{\varrho\})$-dominating set of size k_3 in H_3*

[3]The argument still holds even if $\varrho = 1$ and therefore $\mathcal{X}_1 = \emptyset$, as the vertices in $\mathcal{X}$ still need to be dominated by at least one vertex and $|\mathcal{X}| = |\mathcal{X}_2| = \varsigma + 1$.

[4]If $\varrho = 1$ then no identical twins are added.

Proof. Let $f : V(K) \to V(G)$ be the function preserving neighbours and colours. Let $f(i) = v^i_{c_i}$ for all $i \in [k]$ and for some $c_1, ..., c_k \in [p]$. Note that $ij \in E(K)$ implies that $v^i_{c_i} v^j_{c_j} \in E(G)$ further implying that $r^{ij}_{c_i c_j} \in V(\mathcal{H})$ and therefore also $r^{ij}_{c_i c_j}$ is also in H_3.

Then $D = \bigcup_{i\in[k]} S^i_{c_i*} \cup \bigcup_{ij\in E(K)} R^{ij}_{c_i c_j *}$ is a (σ, ρ)-dominating set.

Notice that every vertex in $S^i_{**} \setminus S^i_{c_i*}$ is dominated by $S^i_{c_i*} \subset D$ and only $S^i_{c_i*} \subset D$ which has $\varrho \in \rho$ vertices. The same is true for $R^{ij}_{***} \setminus R^{ij}_{c_i c_j *}$ and $R^{ij}_{c_i c_j *} \subset D$. Furthermore $S^i_{c_i*}$ and $R^{ij}_{c_i c_j *}$ are independent sets as they only consist of false twins of $s^i_{c_i}$ and $r^{ij}_{c_i c_j}$ respectively.

Finally x^{ij}_a is dominated by the ϱ vertices in $R^{ij}_{c_i c_j} \subset D$ if and only if $a \neq c_i$ and to the ϱ vertices in $S^i_{c_i*} \subset D$ if and only if $a = c_i$. And there are no other vertices in $N(x^{ij}_a) \cap D$, therefore in either case x^{ij}_a is dominated by $\varrho \in \rho$ vertices in D. □

Claim 19. *If there exists a (σ, ρ)-dominating set of size at most k_3 in H_3, then (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem.*

Proof. Let D be a (σ, ρ)-dominating set of size at most k_3, and let $\alpha \in \mathcal{I}$.

First note that $|N[A^\alpha] \cap D| \geq \varrho$, as every vertex in A^α either needs to be in D and $|A^\alpha| = \varrho$, or there is a vertex in A^α that is not in D that needs be dominated by at least ϱ vertices in its neighbourhood.

Then $N[A^\alpha]$ for all $\alpha \in \mathcal{I}$, and X make a partition of H_3. Adding up the lower bounds of $|N[A^\alpha] \cap D|$ then yields $|D| \geq k_3$, but $|D| \leq k_3$. Therefore $|N[A^\alpha] \cap D| = \varrho$ and $X \cap D = \emptyset$.

Suppose that there is some $z^\alpha_{a\ell} \in D$ for some $a \in [p]$ and $\ell \in [\varrho]$, then $Z^\alpha_{a*} \subseteq D$. This is trivially true if $\varrho = 1$ and otherwise it is true by the following argument. If $Z^\alpha_{a*} \nsubseteq D$ then there exists some $\ell' \in [\varrho]$ such that $z^\alpha_{a\ell'} \notin D$. But notice then that $z^\alpha_{a\ell'}$ cannot be adjacent to ϱ vertices in D as it is not adjacent to $z^\alpha_{a\ell} \in D$, and $|Z^\alpha_{**} \cap D| \leq \varrho$ as $Z^\alpha_{**} \subseteq N[A^\alpha]$. Furthermore, $X \cap D = \emptyset$. Therefore $|N(z^\alpha_{a\ell'}) \cap D| \leq \varrho - 1 \notin \rho$ but this cannot be the case therefore $Z^\alpha_{a*} \subseteq D$.

For all $ij \in E(K)$ and for all $a \in [p]$, the vertex x^{ij}_a needs to be dominated by ϱ vertices in D. Note that since $X \cap D = \emptyset$, x^{ij}_a needs to be dominated by a subset of S^i_{a*} or a subset of $\bigcup_{a',b\in[p]} R^{ij}_{a'b*}$, or a combination of both subsets. Suppose x^{ij}_a is dominated by a vertex in $\bigcup_{a'\neq a, a',b\in[p]} R^{ij}_{a'b*}$, then notice by the argument above there must be some $a' \neq a$ and some $b \in [p]$ such that $R^{ij}_{a'b*} \subseteq D$. But notice then that $x^{ij}_{a'}$ needs to be dominated by ϱ vertices in D, but this can only be done by $S^i_{a'*}$ as $|R^{ij}_{a'b*}| = \varrho$ and $R^{ij} \cap D = \varrho$, therefore $S^i_{a'*} \subseteq D$. Similarly if x^{ij}_a is being dominated by a vertex in S^i_{a*} then $S^i_{a*} \subseteq D$. And $x^{ij}_{a'}$, for all $a' \neq a$, has to be dominated by ϱ vertices in D. This can only be done by $\bigcup_{b\in[p]} R^{ij}_{ab*}$ but again by the argument above this leads to $R^{ij}_{ab*} \subseteq D$ for some $b \in [p]$.

Therefore for all $i \in [k]$ there exists some $c_i \in [p]$ such that $S^i_{c_i}$ and for all $ij \in E(K)$, $R^{ij}_{c_i c_j *} \subseteq D$. Therefore by Claim 2, (K, G, ϕ) is a yes-instance of Partitioned Subgraph Isomorphism. □

4.4 Maximisation problems

For the maximisation problems we let $\varsigma = \max(\sigma)$ and $\varrho = \max(\rho)$. Here we see why we require σ and ρ to be finite, we can only find a maximum of a subset of the natural numbers if the subset is finite. Recall that $\rho \neq \{0\}$ by assumption, and therefore $\varrho \neq 0$.

4.4.1 When $\varsigma < \varrho$

We use the same procedure $\mathcal{A}$ as for when $\min(\sigma) < \min(\rho) - 1$, except using our new values of ϱ and ς, turning $\mathcal{H}$ into the graph solution size pair (H_1, k_1).

The graph is the same as for the minimisation problem, except new but still constant values of ϱ and ς, and therefore there also exists a linear ordering Λ_1 of H_1 computable in polynomial time that has linear mim-width $O(|E(K)|)$ by Claim 9 as $\varsigma + \varrho$ is still $O(1)$. Additionally, Claim 10 still holds for maximisation.

We still let $\varrho' = \varrho - \varsigma$ and $k_1 = (\varrho'\varsigma + \varrho')(|E(K)| + 1) + (\varsigma + 1)$. Note that ϱ' can be 0, in which case we do not add any identical twins of $\{z^\alpha_\beta \in V(\mathcal{H}) \mid \alpha \in \mathcal{I}, \beta \in \mathcal{J}\}$, and in general we will evaluate $\varrho' - 1$ as 0 if $\varrho' = 0$, but all arguments still hold.

Claim 20. *If there exists a (σ, ρ)-dominating set of size at least k_1 in H_1, then (K, G, ϕ) is a yes-instance of the* Partitioned Subgraph Isomorphism *problem.*

Proof. Let D be a (σ, ρ)-dominating set in H_1 of size at least k_1.

For all $\ell \in [\varrho']$, $A^\alpha_\ell \cup Z^\alpha_{*\ell}$[5] is a clique, therefore any vertex in $(A^\alpha_\ell \cup Z^\alpha_{*\ell}) \cap D$ can at most be adjacent to ς other vertices in D. Therefore $|(A^\alpha_\ell \cup Z^\alpha_{*\ell}) \cap D| \leq \varsigma + 1$. This is true for all $\ell \in [\varrho']$ therefore $|(A^\alpha \cup Z^\alpha_{**}) \cap D| \leq \varrho'\varsigma + \varrho'$.

Furthermore $|(X \cup \mathcal{X}_1) \cap D| \leq \varsigma + 1$ as $\mathcal{X}_1 \cup X$ is a clique. Because $|\mathcal{X}| = |\mathcal{X}_1| + 1$ we then have that $|(X \cup \mathcal{X}) \cap D| \leq \varsigma + 2$.

The vertex sets $A^\alpha \cup Z^\alpha_{**}, X \cup \mathcal{X}$, for all $\alpha \in \mathcal{I}$, make a partition of $V(H_1)$. Therefore, adding up all the size bounds of the sets in the partition yields $|V(H_1) \cap D| \leq k_1 + 1$. But $|D| \geq k_1$, therefore $|D| = k_1$ or $|D| = k_1 + 1$.

First we will show that $|(X \cup \mathcal{X}) \cap D| = \varsigma + 1$. Assume for the sake of contradiction that $|(X \cup \mathcal{X}) \cap D| = \varsigma + 2$. $\mathcal{X}_1 \cup X$ is a clique, therefore at most $\varsigma + 1$ vertices of the clique can be in D. Therefore the last vertex has to be in $\mathcal{X}_2$ and therefore $\mathcal{X}_2 \subseteq D$. Furthermore $X \cap D \neq \emptyset$ as $|\mathcal{X}| = \varsigma + 1$, every vertex in $X \cap D$ is adjacent to ς vertices in D, and $\varrho'\varsigma + \varrho' - 1 \leq |(Z^\alpha_{**} \cup A^\alpha) \cap D| \leq \varrho'\varsigma + \varrho'$. As a consequence $\varrho' - 1 \leq |Z^\alpha_{**} \cap D| \leq \varrho'$ since $|A^\alpha| = \varrho'\varsigma$.

Let x^{ij}_a be a vertex in $X \cap D$ for some $ij \in E(K)$ and $a \in [p]$, then $(N(x^{ij}_a) \setminus (X \cup \mathcal{X})) \cap D$ must be empty as x^{ij}_a has ς neighbours in $(X \cup \mathcal{X}) \cap D$ by assumption and therefore can have no more neighbours in D. Therefore $R^{ij}_{***} \cap D$ must be a subset of $\{r^{ij}_{ab\ell} \in V(H_1) \mid b \in [p], \ell \in [\varrho']\}$, and $S^i_{**} \cap D$ must be a subset of $S^i_{**} \setminus S^i_{a*}$. Note that $R^{ij}_{***} \cap D$ is then adjacent to all the vertices in $X^{ij} \setminus \{x^{ij}_a\}$.

Furthermore there must exist some $b \neq a$ such that $S^i_{b*} \cap D \neq \emptyset$ as $|Z^\alpha_{**} \cap D| \geq \varrho' - 1$ for all $\alpha \in \mathcal{I}$. But then x^{ij}_b cannot be in D as it would be adjacent by at least one vertex in $S^i_{b*} \cap D$, at least $\varrho' - 1$ vertices in $R^{ij}_{***} \cap D$, and ς vertices in $(X \cup \mathcal{X}_1) \cap D$, which adds up to at least $1 + \varrho' - 1 + \varsigma = \varrho > \varsigma$ vertices in $N(x^{ij}_b) \cap D$ which cannot be the case. But notice that if x^{ij}_b is not in D, then it is adjacent to $\varsigma + 1$ vertices in $(X \cup \mathcal{X}_1) \cap D)$ in addition to at least $\varrho' - 1 + 1$ vertices in $(S^i_{b*} \cup R^{ij}_{***}) \cap D$. This adds up to $\varrho + 1 > \varrho$ vertices in D dominating x^{ij}_b which cannot be the case. Therefore $(X \cup \mathcal{X}) \cap D = \varsigma + 1$ and furthermore for all $\alpha \in \mathcal{I}$, $|(Z^\alpha_{**} \cup A^\alpha) \cap D| = \varrho'\varsigma + \varrho'$.

Claim 21. *If for all $\ell \in [\varrho']$, $|(A^\alpha_\ell \cup Z^\alpha_{*\ell}) \cap D| = \varsigma + 1$, then there exists a $c \in \mathcal{J}$ such that $Z^\alpha_{c*} \subseteq D$. Furthermore for all $c' \neq c$, $Z^\alpha_{c'*} \cap D = \emptyset$.*

Proof. First note that for all $\ell \in [\varrho']$, $|(A^\alpha_\ell \cup Z^\alpha_{*\ell}) \cap D| = \varsigma + 1$ and $|A^\alpha_\ell| = \varsigma$. Therefore for all $\ell \in [\varrho']$ there exists a $z^\alpha_{a\ell} \in D$ for some $a \in \mathcal{J}$.

Suppose for the sake of contradiction that there is no $c \in \mathcal{J}$ such that $Z^\alpha_{c*} \subseteq D$. By the argument above there then must exist two distinct $a, b \in \mathcal{J}$ such that $z^\alpha_{a\ell}, z^\alpha_{b\ell'} \in D$, for two distinct $\ell, \ell' \in [\varrho']$. If there are no such distinct a and b then clearly there exists a $c \in \mathcal{J}$ such that $Z^\alpha_{c*} \subseteq D$. But notice that $z^\alpha_{a\ell}$ and $z^\alpha_{b\ell'}$ are adjacent, therefore $z^\alpha_{a\ell}$ is adjacent to the $\varsigma + 1$ vertices in $(Z^\alpha_\ell \cup Z^\alpha_{*\ell} \cup \{z^\alpha_{b\ell'}\}) \cap D$. But this cannot be the case as $\varsigma + 1 \notin \sigma$.

Therefore there exists a $c \in \mathcal{J}$ such that $Z^\alpha_{c*} \subseteq D$. Suppose for the sake of contradiction that there exists $d \in \mathcal{J}$ and $\ell \in [\varrho']$ such that $z^\alpha_{d\ell} \in D$. Then notice that for all $\ell' \neq \ell$ the vertex $z^\alpha_{c\ell'}$ and $z^\alpha_{d\ell}$ are adjacent but $z^\alpha_{c\ell'}$ is adjacent to ς vertices in $(Z^\alpha_{\ell'} \cup Z^\alpha_{*\ell'}) \cap D$ by assumption, and

[5] Note that $\varsigma = 0$ in which case $A^\alpha_\ell = \emptyset$, however all such arguments, and similar arguments when a set is the empty set, still hold.

$z^{\alpha}_{d\ell} \notin Z^{\alpha}_{\ell'} \cup Z^{\alpha}_{*\ell'}$. Therefore $z^{\alpha}_{c\ell'} \in D$ is adjacent to $\varsigma + 1 \notin \sigma$ vertices in D. Therefore no such d and ℓ can exist. $\square$

Claim 22. *If $S^i_{a*} \subseteq D$ and $R^{ij}_{bb'*} \subseteq D$ then $a = b$. Similarly if $S^j_{a*} \subseteq D$ and $R^{ij}_{bb'*} \subseteq D$ then $a = b'$*

Proof. Assume that $a \neq b$. The vertex x^{ij}_a is then adjacent to all vertices in S^i_{a*} and $R^{ij}_{bb'*}$ both of which are contained in D. Therefore x^{ij}_a has $2\varrho'$ neighbours in $S^i_{a*} \cup R^{ij}_{bb'*} \subseteq D$, in addition x^{ij}_a has ς neighbours in $(X \cup \mathcal{X}) \cap D$ as $|(X \cup \mathcal{X}_1) \cap D| = \varsigma$ since $|\mathcal{X}_2| = 1$. This adds up to $\varrho + \varrho' > \varrho > \varsigma$ neighbours in D. Therefore x^{ij}_a can neither be in D nor not be in D which is a contradiction. Therefore $a = b$.

The symmetric case for $S^j_{a*} \subseteq D$ and $R^{ij}_{bb'*} \subseteq D$ has a similar proof. $\square$

Notice that $|(Z^{\alpha}_{**} \cup A^{\alpha}_{*}) \cap D| = \varrho'\varsigma + \varrho'$ and $|(Z^{\alpha}_{*\ell} \cup A^{\alpha}_{\ell}) \cap D| \leq \varsigma + 1$ for all $\ell \in [\varrho']$, therefore $|(Z^{\alpha}_{*\ell} \cup A^{\alpha}_{\ell}) \cap D| = \varsigma + 1$ for all $\ell \in [\varrho']$. Therefore by the Claims 21 and 22, for all $i \in [k]$ there exists $c_i \in [p]$ such that $S^i_{c_i*} \subseteq D$. Furthermore if $ij \in E(K)$ then $R^{ij}_{c_ic_j*} \subseteq D$ and therefore by Claim 2, (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem. $\square$

4.4.2 When $\varsigma \geq \varrho$

We still use the procedure $\mathcal{A}$ to construct (H_2, k_2), except using our new values of ϱ and ς. The graph still has a linear ordering computable in polynomial time with bounded linear mim-width by Claim 13, and Claim 14 still holds for maximisation.

Claim 23. *If there exists a (σ, ρ)-dominating set of size at least k_2 in H_2, then (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem.*

Proof. Let D be a (σ, ρ)-dominating set in H_2 of size at least k_2, and let $\alpha \in \mathcal{I}$.

We will first show that $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| \leq \varsigma + 1$.

If $\varrho > 1$ let $a \in A^{\alpha}$. Note that a is adjacent to all vertices in $Z^{\alpha}_{**} \cup A^{\alpha}$. Therefore if $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| > \varrho$ then $a \in D$ as otherwise it would be dominated too many times. This applies for all vertices in A^{α}, therefore if $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| > \varrho$ then $A^{\alpha} \subseteq D$.

Therefore assume that $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| > \varrho$ and suppose that both $z^{\alpha}_{b\ell}$ and $z^{\alpha}_{b'\ell'}$ are in D, for some $\ell, \ell' \in [\varsigma']_0$ and for two distinct $b, b' \in \mathcal{J}$. If $\ell = \ell'$ then note that $z^{\alpha}_{b''\ell}$, for some $b'' \neq b$ and $b'' \neq b'$, is being dominated by $(\varrho - 1)$ vertices in $A^{\alpha} \subseteq D$ [6] and the vertices $z^{\alpha}_{b\ell}, z^{\alpha}_{b'\ell'} \in D$, which adds up to $\varrho + 1 > \varrho$ vertices in D and therefore $z^{\alpha}_{b''\ell}$ must be in D. This argument can be repeated until all of $Z^{\alpha}_{*\ell}$ is in D. But because by Claim 1 we can assume that $|Z^{\alpha}_{*\ell}| > \varsigma + 1$, therefore all vertices in $Z^{\alpha}_{*\ell}$, which are also in D, are adjacent to more than ς other vertices in D. Therefore $\ell \neq \ell'$.

Because $\ell \neq \ell'$ the vertices $z^{\alpha}_{b\ell'}$ and $z^{\alpha}_{b'\ell}$ are adjacent to both $z^{\alpha}_{b\ell}$ and $z^{\alpha}_{b'\ell'}$. Therefore by a similar argument as above $z^{\alpha}_{b\ell'}$ and $z^{\alpha}_{b'\ell}$ are adjacent to $(\varrho - 1) + 2 > \varrho$ vertices, and hence both $z^{\alpha}_{b\ell'}$ and $z^{\alpha}_{b'\ell}$ must be in D. But then notice that we are in the same situation as above, where all of $Z^{\alpha}_{*\ell}$ (and $Z^{\alpha}_{*\ell'}$) must be in D. But this led to a contradiction, therefore either $b = b'$ or $|Z^{\alpha}_{**} \cap D| = 1$. Furthermore if $b = b'$ all the vertices in Z^{α}_{b*} are being dominated by $\varrho - 1 + 2 > \varrho$ vertices, and therefore must be in D.

We can therefore conclude with that either $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| \leq \varrho$ or $Z^{\alpha}_{**} \cap D = Z^{\alpha}_{c*}$ for some $c \in \mathcal{J}$. In particular $|(Z^{\alpha}_{**} \cup A^{\alpha}) \cap D| \leq \varsigma + 1$ in either case.

We will now show that $|(X \cup \mathcal{X}) \cap D| \leq \varsigma + 1$. First note that $|(X \cup \mathcal{X}_1) \cap D| \leq \varrho$, as if $|(X \cup \mathcal{X}_1) \cap D| > \varrho$ then all vertices in X must be in D, but $|X|$ can be assumed to be bigger than ς, as for all $\alpha' = i \in [k]$, $|X| \geq |X^{\alpha'}| = |Z^{\alpha'}|$ and $Z^{\alpha'}$ can be assumed to be larger than ς by Claim 1.

[6] Note that $\varrho \neq 0$, and even if $\varrho = 1$ the argument, and other arguments later in this proof, still hold.

Therefore if $|(X \cup \mathcal{X}) \cap D| > \varsigma + 1$ then at most ϱ vertices can be in $|(X \cup \mathcal{X}_1) \cap D|$. The rest has to be in $\mathcal{X}_2$ which has size $\varsigma - \varrho + 2$ and therefore $\mathcal{X}_2 \subseteq D$ and $|(X \cup \mathcal{X}) \cap D| = \varsigma + 2$. But then the vertices in $\mathcal{X}_1 \cap D$ are adjacent to $\varsigma - \varrho + 2$ vertices in $\mathcal{X}_2 \cap D$ and at least $\varrho - 1$ vertices in $(X \cup \mathcal{X}_1) \cap D$ which adds up to $\varsigma + 1$ vertices in D. Therefore the vertices in $\mathcal{X}_1$ can neither be in D nor not be in D, which is a contradiction, therefore $|(X \cup \mathcal{X}) \cap D| \leq \varsigma + 1$.

$Z^\alpha_{**} \cup A^\alpha$ and $X, \mathcal{X}$, for all $\alpha \in \mathcal{I}$, make a partition of $V(H_2)$. Therefore, adding up all the size bounds of the sets in the partition yields $|V(H_2) \cap D| \leq k_2$. But $|D| \geq k_2$, therefore $|D| = k_2$, and all the size bounds are exact.

As a consequence $|(A^\alpha \cup Z^\alpha_{**}) \cap D| = \varsigma + 1$ and as $\varsigma + 1 > \varrho$ we have that $Z^\alpha_{**} \cap D = Z^\alpha_{c*}$ for some $c \in \mathcal{J}$.

Note that $|(X \cup \mathcal{X}) \cap D| = \varsigma + 1$. Suppose that $|(X \cup \mathcal{X}_1) \cap D| = \varrho$, then $Z^\alpha_{*0} \cap D = \emptyset$ but by the argument above $z^\alpha_c \in Z^\alpha_{*0} \cap D$. Therefore $|(X \cup \mathcal{X}_1) \cap D| < \varrho$ and in particular $|(X \cup \mathcal{X}_1) \cap D| = \varrho - 1$ and $|\mathcal{X}_2 \cap D| = \varsigma - \varrho + 2$, as $|\mathcal{X}_2| = \varsigma - \varrho + 2$. Furthermore $\mathcal{X}_1 \subseteq D$ as if there is some vertex in $\mathcal{X}_1$ but not in D, then it would be adjacent to $\varsigma + 1 > \varrho$ vertices in D contradicting that $|(X \cup \mathcal{X}_1) \cap D| = \varrho - 1$, as the vertex would have to be in D.

Finally we have shown there exists some $s^i_{c_i} \in D$, implying that $r^{ij}_{c'_i d_j}$ cannot be in D, for some $c'_i \neq c_i$ and all $j \in [k]$. As if both $s^i_{c_i}$ and $r^{ij}_{c'_i d_j}$ are in D then the vertex $x^{ij}_{c_i}$ is being dominated by $\varrho + 1$ vertices, which cannot be the case. Therefore, $x^{ij}_{c_i}$ would have to be in D, but this cannot be the case either as $X \cap D = \emptyset$ by the argument above. Therefore if $s^i_{c_i} \in D$ then $r^{ij}_{c_i d_j} \in D$. And by a similar argument if $s^j_{c_j} \in D$ then $r^{ij}_{d_i c_j} \in D$.

Therefore by Claim 2, (K, G, ϕ) is a yes-instance of the Partitioned Subgraph Isomorphism problem. $\square$

4.5 Proof of Theorem 1

Proof of Theorem 1. Let σ and ρ be two subsets of $\mathbb{N}$ where $0 \notin \rho$.

The $W[1]$-hardness of both Min-(σ,ρ)-DS param. by l. mim-width + sol. size and Max-(σ,ρ)-DS param. by l. mim-width + sol. size will be shown by a reduction from the $W[1]$-hard problem Partitioned Subgraph Isomorphism Fellows et al. (2009); Pietrzak (2003) [7], taking the instance (K, G, ϕ) of the Partitioned Subgraph Isomorphism problem and constructing the graph $\mathcal{H}$. The graph $\mathcal{H}$ is further transformed into the graph solution size pair (H_0, k_0), (H_1, k_1), (H_2, k_2), or (H_3, k_3). The construction of $\mathcal{H}$ can clearly be done in polynomial (polynomial in $k + |V(G)|$) time, and all the transformations on $\mathcal{H}$ can also clearly be done in polynomial time.

For the Min-(σ,ρ)-DS param. by l. mim-width + sol. size problem we let $\varsigma = \min(\sigma)$ and $\varrho = \min(\rho)$. Then if $\varsigma = 0$, $\mathcal{H}$ is transformed into (H_3, k_3), otherwise $\varsigma \geq 1$. If $\varrho = \varsigma + 1$ then $\mathcal{H}$ is transformed into (H_0, k_0), if $\varrho > \varsigma + 1$ into (H_1, k_1), and if $\varrho < \varsigma + 1$ into (H_2, k_2). Note that one of these cases must be true, and that if $\varrho < \varsigma + 1$ then $\varsigma \neq 0$ as then $\varrho < 0$ which it cannot be.

For $a \in [3]_0$ and for some linear ordering Λ_a of H_a, we say (H_a, k_a, Λ_a) retains correctness for a problem $\mathcal{P}$ if (H_a, k_a, Λ_a) is yes-instance of $\mathcal{P}$ is equivalent with (K, G, ϕ) being a yes-instance of the Partitioned Subgraph Isomorphism problem. We first let $\mathcal{P}$ be the Min-(σ,ρ)-DS param. by l. mim-width + sol. size problem.

Then (H_0, k_0, Λ_0) retains correctness for $\mathcal{P}$ by Claim 7 [8] and Claim 8, (H_1, k_1, Λ_1) retains correctness for $\mathcal{P}$ by Claim 10 and Claim 11, (H_2, k_2, Λ_2) retains correctness for $\mathcal{P}$ by Claim 14 and Claim 15, and (H_3, k_3, Λ_3) retains correctness for $\mathcal{P}$ by Claim 18 and Claim 19.

[7] The $W[1]$-hardness follows from the fact that Multicoloured Clique is a special case of Partitioned Subgraph Isomorphism where K is a complete graph, and Multicoloured Clique is $W[1]$-hard Fellows et al. (2009); Pietrzak (2003)

[8] Note that every $(\{\varsigma\}, \{\varrho\})$-dominating set is also a (σ,ρ)-dominating set as $\varsigma \in \sigma$ and $\varrho \in \rho$.

Further note that there exists linear orderings Λ_0, Λ_1, Λ_2, and Λ_3 of the graphs H_0, H_1, H_2, and H_3 computable in polynomial time by the Claims 6, 9, 13, and 17 respectively. Where $\text{linmimw}(H_a) \leq \text{mimw}(\Lambda_a) = O(|E(K)|)$ for all $a \in [3]_0$, as ϱ and ς are both $O(1)$. Therefore the MIN-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problem for the given σ and ρ is $W[1]$-hard.

Now let $\mathcal{P}$ be the MAX-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problem, then by assumption we let σ and ρ be finite subsets of $\mathbb{N}$ and $\rho \neq \{0\}$. We can therefore find maximum elements ς and ϱ of σ and ρ respectively.

The graph $\mathcal{H}$ is then transformed into the graph solution size pair (H_1, k_1) if $\varrho > \varsigma$ and into the pair (H_2, k_2) if $\varrho \leq \varsigma$. Clearly either $\varrho > \varsigma$ or $\varrho \leq \varsigma$.

Then (H_1, k_1, Λ_1) retains correctness for $\mathcal{P}$ by Claim 10 and Claim 20, and (H_2, k_2, Λ_2) retains correctness for $\mathcal{P}$ by Claim 14 and Claim 23.

Furthermore as shown above there exists linear orderings for both H_1 and H_2 computable in polynomial time with linear mim-width $O(|E(K)|)$, as ς and ϱ are still $O(1)$. Therefore the MAX-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problem for the given σ and ρ is $W[1]$-hard.

Moreover, unless ETH is false, there are no algorithms solving the PARTITIONED SUBGRAPH ISOMORPHISM problem in $n^{o(|E(K)|/\log|E(K)|)}$ time, where $n = |V(G)|$ by Corollary 6.3 in Marx (2010). Assume that there exists some algorithm solving MIN-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE or MAX-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE in time $n^{o(w/\log w)}$, for some input graph G with $n = |V(G)|$ and with a linear graph decomposition of G with the mim-width w. We then create an algorithm for PARTITIONED SUBGRAPH ISOMORPHISM determining if the instance (K, G, ϕ) is a yes-instance of PARTITIONED SUBGRAPH ISOMORPHISM.

First we apply the reductions above, which takes polynomial (polynomial in n and $|E(K)|$) time. We then get the instance (H_a, k_a, Λ_a) for some $a \in [3]_0$ where $\text{mimw}(\Lambda_a) \leq b \cdot |E(K)|$ of the problem MIN-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE or of the problem MAX-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE. This instance can then by assumption be solved in time

$$n^{o(w/\log w)} = n^{o(b\cdot|E(K)|/\log(b\cdot|E(K)|))} = n^{o(|E(K)|/\log|E(K)|}$$

But notice that we then have an algorithm solving PARTITIONED SUBGRAPH ISOMORPHISM in $(n + |E(K)|)^{O(1)} + n^{o(|E(K)|/\log|E(K)|)} = n^{o(|E(K)|/\log|E(K)|)}$ time, contradicting that there are no such algorithms.

We only have a reduction from PARTITIONED SUBGRAPH ISOMORPHISM to MIN-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE when $0 \notin \rho$ and from PARTITIONED SUBGRAPH ISOMORPHISM to MAX-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE when $\rho \neq \{0\}$ and σ and ρ are both finite subsets of $\mathbb{N}$.

Therefore there are no algorithms solving MIN-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE when $0 \notin \rho$ in time $n^{o(w/\log w)}$. And there are no algorithms solving MAX-(σ,ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE, when σ and ρ are finite sets and $\rho \neq \{0\}$, in time $n^{o(w/\log w)}$. Where n is the size of the input graph and w is the linear mim-width of a branch decomposition given with the input graph. □

5 ETH Based Lower Bound for the Independent Set Problem on H-graphs

We will show a ETH based lower bound on the running time of any algorithm solving the INDEPENDENT SET problem on H-graphs, using a reduction from PARTITIONED SUBGRAPH ISOMORPHISM to INDEPENDENT SET on H-graphs. The reduction is a modified reduction from Fomin et al. (2020), modified so that it starts from PARTITIONED SUBGRAPH ISOMORPHISM instead of the MULTICOLOURED CLIQUE problem.

Theorem 2. *The* Independent Set *problem on H-graphs cannot be solved in time $n^{o(h/\log h)}$, where n is the number of vertices in the input graph and $h = |E(H)|$, unless ETH is false.*

Note that Theorem 2 does not follow from Theorem 1 as the Independent Set problem is the Max-($\{0\}, \mathbb{N}$)-DS problem, but for maximisation problem we require both σ and ρ to be finite sets, and $\mathbb{N}$ is not finite. The $W[1]$-hardness of Independent Set on H-graphs is already known, therefore we focus on the ETH based time bound.

Given an instance of the Partitioned Subgraph Isomorphism problem (K, G, ϕ), we give a reduction from the Partitioned Subgraph Isomorphism problem to the Independent Set problem on H-graphs.

We will then construct an instance of the Independent Set problem on H-graphs $(G', K', |V(K)|+|E(K)|)$, where G' is a K'-graph, and G' will have an independent set of size $|V(K)| + |E(K)|$ if and only if there is a function mapping $V(K)$ to $V(G)$ which preserves neighbours and colours.

Proof of Theorem 2. Firstly, we partition the vertices of $V(G)$ into k groups such that $V_i = \{v \mid \phi(v) = i\}$. Notice that adding isolated vertices to $V(G)$ does not change whether such a function exists or not as K is connected and therefore has no isolated vertices.

Label each V_i with $\{v_1^i, ..., v_p^i\}$, and recall that we let $V(K) = \{1, ..., k\}$.

Add k vertices, $u_1, ..., u_k$, to K'. For all $1 \le i < j \le k$ if $ij \in E(K)$ add w_{ij} and two pairs of the edges $u_i w_{ij}$ and $w_{ij} u_j$.

Then subdivide the edges p times and label the resulting vertices from one of the two edges between u_i and w_{ij} : $x_1^{ij}, ..., x_p^{ij}$, and label the vertices from the other edge between u_i and w_{ij} : $y_1^{ij}, ..., y_p^{ij}$. Similarly label the vertices from the one of the edges between w_{ij} and u_j : $x_1^{ji}, ..., x_p^{ji}$, and the other edge between w_{ij} and u_j : $y_1^{ji}, ..., y_p^{ji}$.

Furthermore let $x_0^{ij} = u_i = y_0^{ij}$, $x_0^{ji} = u_j = y_0^{ji}$, $x_{p+1}^{ij} = y_{p+1}^{ij} = x_{p+1}^{ji} = y_{p+1}^{ji} = w_{ij}$

For all $i \in [k]$, $s \in [p]$ add the vertex z_i^s to G' with the model

$$M_{z_s^i} = \bigcup_{j \in [k], j \neq i} \{x_0^{ij}, ..., x_{s-1}^{ij}, y_0^{ij}, ..., y_{p-s}^{ij}\}$$

If $v_s^i v_t^j \in E(G)$ and $ij \in E(K)$, for $s, t \in [p]$ and $1 \le i < j \le k$, then add r_{st}^{ij} to G' with the model

$$M_{r_{st}^{ij}} = \{x_s^{ij}, ..., x_{p+1}^{ij}, y_{p-s+1}^{ij}, ..., y_{p+1}^{ij}, x_t^{ji}, ..., x_{p+1}^{ji}, y_{p-t+1}^{ji}, ..., y_{p+1}^{ji}\}$$

This construction is the same as for the Multicoloured Clique problem to the Independent Set problem in Theorem 8 in Fomin et al. (2020), except we only add w_{ij}, and the vertices that depend on w_{ij}, if $ij \in E(K)$.

We will now give some small Claims which all follow from the construction of G'.

Claim 24. $\forall a, b, c, d \in [p], \forall i, j, i', j' \in [k]$:

$$r_{ab}^{ij} r_{cd}^{i'j'} \in E(G') \iff i = i' \wedge j = j'$$

Claim 25. $\forall a, b \in [p], \forall i, j \in [k]$:

$$z_a^i z_b^j \in E(G') \iff i = j$$

Claim 26. $\forall a, b, c \in [p], \forall i, j, i' \in [k]$:

$$z_a^i r_{bc}^{ji'} \in E(G') \iff [i = j \wedge a \neq b] \vee [i = i' \wedge a \neq c]$$

And now we will argue that (K, G, ϕ) is a yes-instance of Partitioned Subgraph Isomorphism if and only if G' has an independent set of size $|E(K)| + |V(K)|$.

Claim 27. *If (K, G, ϕ) is yes-instance of* Partitioned Subgraph Isomorphism*, then G' has an independent set of size $|E(K)| + |V(K)|$.*

Proof. Suppose there exists an injective mapping, $f : V(K) \to V(G)$, which preserves neighbours and colours. Then $f(V(K)) = \{v^1_{c_1}, ..., v^k_{c_k}\}$, for some c_1 to c_k. These vertices correspond to $\{z^1_{c_1}, ..., z^k_{c_k}\} \subseteq V(G')$. By Claim 25 these vertices are pairwise non-adjacent, since they have different "colour" indices. Hence we will add all of them to I.

Secondly $\forall i, j \in [k]$ $i \neq j$, $ij \in E(K) \Rightarrow b^i_{c_i} b^j_{c_j} \in E(G) \Rightarrow r^{ij}_{c_i c_j} \in V(G')$. Add $r^{ij}_{c_i c_j}$ to I.

The vertex $r^{ij}_{c_i c_j}$ is non-adjacent to all other vertices in I, as $r^{ij}_{c_i c_j}$ can only be connected to r^{ij}_{ab} $\forall a, b$ *s.t.* $c_j \neq a \neq b \neq c_i$, by Claim 24. Or it can be connected to z^i_a with $a \neq c_i$, or z^j_b with $b \neq c_j$, by Claim 26. However, r^{ij}_{ab} was never added, as if it were it would imply f maps one vertex in K to multiple in G. Nor can it be connected to z^i_a, nor z^j_b as only one $z^i_{a'}$ and $z^j_{b'}$ was added with $a' = c_i$ and $b' = c_j$ different to a and b. Hence $r^{ij}_{c_i c_j}$ is not connected to any other vertex in I.

Hence I is an independent set of size $|E(K)| + |V(K)|$. □

Claim 28. *If G' has an independent set of size $|E(K)| + |V(K)|$, then (K, G, ϕ) is yes-instance of* Partitioned Subgraph Isomorphism*.*

Proof. Suppose there exists an independent set I of size $|E(K)| + |V(K)|$.

There can at most be $|E(K)|$ vertices from

$$R = \{r^{ij}_{ab} \in V(G') \mid a, b \in [p], i, j \in [k]\}$$

as $r^{ij}_{ab} \in V(G')$ only if $ij \in E(K)$. And if there were multiple vertices from R with same i and j value, say r^{ij}_{ab} and $r^{ij}_{a'b'}$, for some $a \neq a', b \neq b'$ then the set would not be independent as $r^{ij}_{ab} r^{ij}_{a'b'} \in E(G)$ by Claim 24.

Similarly there can be at most $|V(K)|$ vertices from

$$Z = \{z^i_a \in V(G') \mid a \in [p], i \in [k]\}$$

as otherwise, two vertices from Z of the same colour would be in I, say z^i_a and z^i_b. However, these would be connected by Claim 25, and hence the set would not be independent.

The vertex r^{ij}_{ab} is adjacent to all vertices: $z^i_{a'}$, $z^i_{b'}$ with $a' \neq a, b' \neq b$ by Claim 26. Therefore if the vertices: $\{z^1_{c_1}, ..., z^k_{c_k}\}$ are in I then $\{r^{ij}_{c_i c_j} \mid i, j \in [k], i \neq j\}$ also have to be in I, as they are the only vertices in R which are not adjacent to any of the $z^i_{c_i} \in I$.

Assume $\{z^1_{c_1}, ..., z^k_{c_k}\} \subseteq I$ then $\{r^{ij}_{c_i c_j} \mid i, j \in [k], i \neq j\} \subseteq I$ which means that $I = \{z^1_{c_1}, ..., z^k_{c_k}\} \cup \{r^{ij}_{c_i c_j} \mid i, j \in [k], i \neq j\}$.

The vertices of the set $\{z^1_{c_1}, ..., z^k_{c_k}\}$ corresponds to the vertices $\{v^1_{c_1}, ..., v^k_{c_k}\} \subseteq V(G)$. Therefore let the mapping $f : V(K) \to V(G)$. Where $f(i) = v^i_{c_i}$. Clearly this mapping preserves colours as $v^i_{c_i} \in V_i$.

The vertex set $\{r^{ij}_{c_i c_j} \mid i, j \in [k], i \neq j\} \subseteq I$. Therefore $\forall ij \in E(K)$, $r^{ij}_{c_i c_j} \in I \Rightarrow v^i_{c_i} v^j_{c_j} \in E(G)$. Hence the mapping $f(.)$ preserves neighbours and colours, and the Partitioned Subgraph Isomorphism instance is therefore a yes-instance. □

The construction of G' and K' can trivially be done in polynomial in $|V(G)|$ time, as $|V(K)| \leq |V(G)|$.

Furthermore $|E(K')|$ is linearly bounded by $|E(K)|$, as for each edge $ij \in E(K)$ the graph K' has two copies of the edges $u_i w_{ij}$ and $w_{ij} u_j$. Furthermore these are the only edges in K'. Therefore there are 4 edges added per edge in K. Hence $|E(K')| = 4 \cdot |E(K)|$.

Therefore by the Claims 27 and 28 and the same arguments as in Theorem 1, an algorithm solving the Independent Set problem on H-graphs in $n^{o(h/\log h)}$ time, where $h = |E(H)|$

and n is the size of the H-graph, would imply an algorithm solving the PARTITIONED SUBGRAPH ISOMORPHISM problem in $n^{o(|E(K)|/\log|E(K)|)}$ time, contradicting Marx (2010). We can then conclude that unless ETH is false there is no algorithm running in $n^{o(h/\log h)}$ time solving INDEPENDENT SET on H-graphs. $\square$

In particular notice that the linear mim-width on H-graphs is at most $2\cdot|E(H)|$ by Theorem 1 in Fomin et al. (2020), therefore there are no algorithms solving the INDEPENDENT SET problem on graphs with a given linear ordering of the graph with mim-width w in $n^{o(w/\log w)}$ time.

Furthermore, as mentioned the reduction in Theorem 2 is a modification of a reduction from the MULTICOLOURED CLIQUE problem to INDEPENDENT SET problem in Fomin et al. (2020). Another similar reduction in Fomin et al. (2020) is from the MULTICOLOURED CLIQUE problem to the DOMINATING SET problem. The only difference is a vertex which is only connected to $Z^i = \{z^i_a \mid a \in [p]\}$ and a vertex which is only connected to $R^{ij} = \{r^{ij}_{ab} \in V(G') \mid a, b \in [p]\} = R^{ji}$ are added for all distinct $i, j \in [k]$, and this new graph is also a H-graph with the same H as for the INDEPENDENT SET reduction. We claim that the same modifications work for the DOMINATING SET problem, therefore also getting the result that there are no algorithms solving the dominating set problem on H-graphs in $n^{o(h/\log h)}$ time.

There are also other similar reductions to other (σ, ρ)-dominating set problems which probably can be adapted in the same way.

6 Conclusion

We proved that the MIN-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problem, for all non-trivial pairs of $\sigma, \rho \subseteq \mathbb{N}$, is $W[1]$-hard. Furthermore we proved that for all finite pairs $\sigma, \rho \subseteq \mathbb{N}$ such that $\rho \neq \{0\}$, the MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problem is $W[1]$-hard.

Note since a reduction to MIN-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE is also a reduction to MIN-(σ, ρ)-DS PARAM. BY SOLUTION SIZE, we have strengthened Theorem 1 in Golovach et al. (2012), as that showed only $W[1]$-hardness for MIN-(σ, ρ)-DS where σ and ρ are finite subsets of $\mathbb{N}$.

Moreover we proved that unless ETH is false there can be no $n^{o(w/\log w)}$ time algorithms, where $n = |V(G)|$ for some input graph G and w is the linear mim-width of some linear ordering given along with G, solving MIN-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE nor MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE for the same respective sets σ and ρ that we showed $W[1]$-hardness for.

We also proved that unless ETH is false there are no $n^{o(h/\log h)}$ time algorithms solving INDEPENDENT SET on H-graphs, where $n = |V(G)|$ and $h = |E(H)|$ for some input graph G. Which in particular implies that there are no $n^{o(w/\log w)}$ algorithms solving INDEPENDENT SET on graphs with a linear ordering with mim-width w.

Table 6 shows the $W[1]$-hardness of some MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problems. Note that all the problems in the table with $\rho = \mathbb{N}$ are trivial for minimisation, but for the rest they are $W[1]$-hard for minimisation. Furthermore as discussed in Section 2.7, we do not expect $W[1]$-hardness for the MAXIMUM TOTAL DOMINATING SET problem as it is trivial for maximisation since $\sigma = \mathbb{N}^+$.

The reductions of the problems in Table 6, which are shown in Jaffke et al. (2019), are adaptations of the INDEPENDENT SET reduction in Fomin et al. (2020), and therefore these reductions can probably be further adapted to reduce from PARTITIONED SUBGRAPH ISOMORPHISM instead of from MULTICOLOURED CLIQUE. Thereby obtaining a lower bound of $n^{o(|E(H)|/\log|E(H)|)}$, unless ETH is false, for algorithms solving MAX-(σ, ρ)-DS ON H-GRAPHS.

A natural question is then: Can we adapt the reduction in Theorem 2 to work for all non-trivial (σ, ρ)-domination problems? Or more generally are there no algorithms solving Min. or Max. (σ, ρ)-DS ON H-GRAPHS in $n^{o(|E(H)|/\log|E(H)|)}$ time, under ETH?

σ	ρ	Standard Name	$W[1]$-Hardness known for Max.
$\{0\}$	$\mathbb{N}^+$	Maximal Dominating Set	No
$\mathbb{N}^+$	$\mathbb{N}^+$	Total Dominating Set	No
$\{1\}$	$\mathbb{N}^+$	Dominating Induced Matching	Yes
$\{1\}$	$\mathbb{N}$	Induced Matching	Yes
$\{d\}$	$\mathbb{N}$	Induced d-Regular Subgraph	Yes
$\mathbb{N} \setminus [d-1]_0$	$\mathbb{N}$	Subgraph of Min Degree $\geq d$	No
$[d]$	$\mathbb{N}$	Induced Subgraph of Max Degree $\leq d$	Yes

Table 2: $W[1]$-Hardness of MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE, the $W[1]$-hardness is shown in Jaffke et al. (2019).

Open Problem 1. *Under ETH can no non-trivial Min. or Max. (σ, ρ)-domination problem on H-graphs, be solved in $n^{o(|E(H)|/\log|E(H)|)}$ time?*

We only showed $W[1]$-hardness for MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE when σ and ρ are finite subsets of $\mathbb{N}$, and $\rho \neq \{0\}$. Furthermore in Section 2.7 we showed that for $\sigma = \mathbb{N}^+$ and $\sigma = \mathbb{N}$ the MAX-(σ, ρ)-DS problem is trivial. This leads to the question:

Open Problem 2. *For which $\sigma, \rho \subseteq \mathbb{N}$ is the MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE problem $W[1]$-hard?*

Furthermore, we showed that there are no algorithms solving MIN-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE or MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE for certain σ and ρ in $n^{o(w/\log w)}$ time, where for a graph G given with the input $n = |V(G)|$ and w the mim-width of a linear ordering given with G. But are there actually any algorithms solving these problems in $n^{O(w/\log w)}$ time?

Open Problem 3. *Can MIN-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE and MAX-(σ, ρ)-DS PARAM. BY L. MIM-WIDTH + SOL. SIZE for σ and ρ such that the problems are not trivial, be solved in $n^{O(w/\log w)}$ time?*

Note that no such algorithm is currently known for any non-trivial (σ, ρ)-dominating set problem parameterized by mim-width, and the existence of such an algorithm would be surprising. As if there is a reduction from PARTITIONED SUBGRAPH ISOMORPHISM to the (σ, ρ)-dominating set such that the linear mim-width of the constructed graph is $O(|E(H)|)$, the reduction would imply the lower bound in Marx (2010) is tight as the reduction would imply the existence of a $n^{O(|E(H)|/\log|E(H)|)}$ algorithm for the PARTITIONED SUBGRAPH ISOMORPHISM problem. In particular we have these reductions for all σ and ρ such that the minimisation problem is not trivial, and for all finite σ and ρ such that the maximisation problem is not trivial.

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