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On the Sombor characteristic polynomial and Sombor energy of a graph

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Abstract
Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Sombor matrix of G, denoted by $A_{SO}(G)$, is defined as the $n \times n$ matrix whose (i, j)-entry is $\sqrt{d_i^2 + d_j^2}$ if v_i and v_j are adjacent and 0 for another cases. Let the eigenvalues of the Sombor matrix $A_{SO}(G)$ be $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ which are the roots of the Sombor characteristic polynomial $\prod_{i=1}^{n}(\rho - \rho_i)$. The Sombor energy E_{SO} of G is the sum of absolute values of the eigenvalues of $A_{SO}(G)$. In this paper, we compute the Sombor characteristic polynomial and the Sombor energy for some graph classes, define Sombor energy unique and propose a conjecture on Sombor energy.

Keywords Sombor matrix · Sombor energy · Sombor characteristic polynomial · Regular graphs · Eigenvalues

Mathematics Subject Classification 05C12 · 05C50

1 Introduction

In this paper, we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G = (V, E)$ be such a graph, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices v_i and v_j of G are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of the vertices adjacent to v_i.

Let $A(G)$ be adjacency matrix of G and $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues. These are said to be the eigenvalues of the graph G and to form its spectrum (Cvetković et al. 1980). The

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energy $E(G)$ of the graph G is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Details and more information on graph energy can be found in Gutman (2001), Gutman (2005), Gutman et al. (2009), Majstorović et al. (2009). There are many kinds of graph energies, such as Randić energy (Alikhani and Ghanbari 2015; Bozkurt and Bozkurt 2013; Bozkurt et al. 2010; Das and Sorgun 2014; Gutman et al. 2014), distance energy (Stevanović et al. 2013), incidence energy (Bozkurt and Gutman 2013), matching energy (Chen and Shi 2015; Ji et al. 2013) and Laplacian energy (Das et al. 2013).

Sombor index is defined as $SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}$ (see Gutman 2021a). More details on Sombor index can be found in Alikhani and Ghanbari (2021), Chen et al. (2022), Cruz et al. (2021), Das et al. (2021), Deng et al. (xx), Ghanbari and Alikhani (2021), Li et al. (2022), Redžepović (2021), Wang et al. (xx). Recently, in Gutman (2021b), Gutman introduced Sombor matrix of a graph G as $A_{SO}(G) = (r_{ij})_{n\times n}$, and

$$r_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A_{SO}(G)$ are denoted by $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$, and are said to form the Sombor spectrum of the graph G. In Gutman (2021b), Gutman introduced Sombor characteristic polynomial $\phi_{SO}(G, \lambda)$ as

$$\phi_{SO}(G, \lambda) = det(\lambda I - A_{SO}(G)) = \prod_{i=1}^{n} (\lambda - \rho_i),$$

and Sombor energy $E_{SO}(G)$ as

$$E_{SO}(G) = \sum_{i=1}^{n} |\rho_i|.$$

We refer the reader to Gowtham and Swamy (2021); Gutman and Redžepović (2022); Jayanna and Gutman (2021) for more details on Sombor energy.

Two graphs G and H are said to be *Sombor energy equivalent*, or simply $\mathcal{E}_{SO}$-equivalent, written $G \sim H$, if $E_{SO}(G) = E_{SO}(H)$. It is evident that the relation $\sim$ of being $\mathcal{E}_{SO}$-equivalence is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the *$\mathcal{E}_{SO}$-equivalent*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$

We call $[G]$ the equivalence class determined by G. A graph G is said to be *Sombor energy unique*, or simply *$\mathcal{E}_{SO}$-unique*, if $[G] = \{G\}$.

A graph G is called *k-regular* if all vertices have the same degree k. One of the famous graphs is the Petersen graph which is a symmetric non-planar 3-regular graph of order 10. There are exactly twenty one 3-regular graphs of order 10 Khosrovshahi and Maysoori (2001). In the study of Sombor energy, it is interesting to investigate the Sombor characteristic polynomial and Sombor energy of this graph and see if it can be recognised by its Sombor energy and Sombor characteristic polynomial among other 3-regular graphs with the same order. We denote the Petersen graph by P.

In this paper, we consider the Sombor characteristic polynomial and Sombor energy of graphs. In Sect. 2, we bring some known results about Sombor characteristic polynomial and Sombor energy. In addition, the Sombor characteristic polynomial and Sombor energy of some special kind of graphs are computed. In Section 3, we consider to regular graphs, especially cubic graphs of order 10 and state a conjecture. In Section 4, we state some open problems for future direction of this research .

2 Sombor energy of specific graphs

In this section, we study the Sombor characteristic polynomial and the Sombor energy for certain graphs. The following result gives McClelland-type bound for the Sombor energy.

Theorem 2.1 *Gutman (2021b) If G is a graph on n vertices, and $F(G)$ is its forgotten topological index, then*

$$E_{SO}(G) \leq \sqrt{2nF(G)}.$$

The following result gives Koolen–Moulton-type bound for the Sombor energy.

Theorem 2.2 *Gutman (2021b) Let G be a graph on n vertices, with Sombor and forgotten topological indices $SO(G)$ and $F(G)$, respectively. Then,*

$$E_{SO}(G) \leq \frac{2SO(G)}{n} + \sqrt{(n-1)\left(2F(G) - \left(\frac{2SO(G)}{n}\right)^2\right)}.$$

Here, we shall compute the Sombor characteristic polynomial of paths and cycles.

Theorem 2.3 *For every $n \geq 5$, the Sombor characteristic polynomial of the path graph P_n satisfy*

$$\phi_{SO}(P_n, \lambda) = \lambda^2 \Lambda_{n-2} - 10\lambda \Lambda_{n-3} + 25\Lambda_{n-4},$$

where for every $k \geq 3$, $\Lambda_k = \lambda \Lambda_{k-1} - 8\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - 8$. In addition, the characteristic polynomial of P_2, P_3 and P_4 are $\lambda^2 - 2$, $\lambda^3 - 10\lambda$ and $\lambda^4 - 18\lambda^2 + 25$, respectively.

Proof It is easy to see that the characteristic polynomial of P_2 is $\lambda^2 - 2$. In addition, for P_3 is $\lambda^3 - 10\lambda$ and for P_4 is $\lambda^4 - 18\lambda^2 + 25$. Now, for every $k \geq 3$ consider

$$M_k := \begin{pmatrix} \lambda & -\sqrt{8} & 0 & \dots & 0 & 0 \\ -\sqrt{8} & \lambda & -\sqrt{8} & \dots & 0 & 0 \\ 0 & -\sqrt{8} & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -\sqrt{8} \\ 0 & 0 & 0 & \dots & -\sqrt{8} & \lambda \end{pmatrix}_{k\times k},$$

and let $\Lambda_k = det(M_k)$. One can easily check that $\Lambda_k = \lambda\Lambda_{k-1} - 8\Lambda_{k-2}$. Now, consider the path graph P_n. Suppose that $\phi_{SO}(P_n, \lambda) = det(\lambda I - A_{SO}(P_n))$. We have

$$\phi_{SO}(P_n, \lambda) = det \left(\begin{array}{c|ccccc|c} \lambda & -\sqrt{5} & 0\ 0 & \dots & 0 & 0 & 0 \\ \hline -\sqrt{5} & & & & & & 0 \\ 0 & & & & & & 0 \\ 0 & & & & & & 0 \\ \vdots & & & M_{n-2} & & & \vdots \\ 0 & & & & & & 0 \\ 0 & & & & & & -\sqrt{5} \\ \hline 0 & 0 & 0\ 0 & \dots & 0 & -\sqrt{5} & \lambda \end{array} \right)_{n \times n} .$$

Therefore,

$$\phi_{SO}(P_n, \lambda) = \lambda det \left(\begin{array}{cccc|c} & & & & 0 \\ & & M_{n-2} & & \vdots \\ & & & & 0 \\ & & & & -\sqrt{5} \\ \hline 0 & \dots & 0 & -\sqrt{5} & \lambda \end{array} \right) + \sqrt{5} det \left(\begin{array}{c|cccc|c} -\sqrt{5} & -\sqrt{8} & \dots & & 0 & 0 \\ \hline 0 & & & & & 0 \\ \vdots & & & M_{n-3} & & \vdots \\ 0 & & & & & -\sqrt{5} \\ \hline 0 & 0 & & \dots & -\sqrt{5} & \lambda \end{array} \right) .$$

Therefore,

$$\phi_{SO}(P_n, \lambda) = \lambda \left(\lambda\Lambda_{n-2} + \sqrt{5} det \left(\begin{array}{cccc|c} & & & & 0 \\ & & M_{n-3} & & \vdots \\ & & & & 0 \\ & & & & 0 \\ \hline 0 & \dots & 0 & -\sqrt{8} & -\sqrt{5} \end{array} \right) \right) - 5 det \left(\begin{array}{cccc|c} & & & & 0 \\ & & M_{n-3} & & \vdots \\ & & & & 0 \\ & & & & -\sqrt{5} \\ \hline 0 & \dots & 0 & -\sqrt{5} & \lambda \end{array} \right)$$

Hence,

$$\phi_{SO}(P_n, \lambda) = \lambda\left(\lambda\Lambda_{n-2} - 5\Lambda_{n-3}\right)$$

$$
-5\left(\lambda\Lambda_{n-3}+\sqrt{5}det\left(\begin{array}{ccc|c} & M_{n-4} & & \begin{matrix}0\\ \vdots\\ 0\\ 0\end{matrix}\\ \hline 0\ \dots & 0 & -\sqrt{8} & -\sqrt{5}\end{array}\right)\right)
$$

$$
=\lambda\left(\lambda\Lambda_{n-2}-5\Lambda_{n-3}\right)-5\left(\lambda\Lambda_{n-3}-5\Lambda_{n-4}\right),
$$

and, therefore, we have the result. □

Theorem 2.4 *For every $n \geq 3$, the Sombor characteristic polynomial of the cycle graph C_n satisfy*

$$
\phi_{SO}(C_n,\lambda)=\lambda\Lambda_{n-1}-16\Lambda_{n-2}-2(\sqrt{8})^n,
$$

where for every $k \geq 3$, $\Lambda_k=\lambda\Lambda_{k-1}-8\Lambda_{k-2}$ with $\Lambda_1=\lambda$ and $\Lambda_2=\lambda^2-8$.

Proof Similar to the proof of Theorem 2.3, for every $k \geq 3$, we consider

$$
M_k:=\begin{pmatrix}
\lambda & -\sqrt{8} & 0 & 0 & \dots & 0 & 0 & 0\\
-\sqrt{8} & \lambda & -\sqrt{8} & 0 & \dots & 0 & 0 & 0\\
0 & -\sqrt{8} & \lambda & -\sqrt{8} & \dots & 0 & 0 & 0\\
0 & 0 & -\sqrt{8} & \lambda & \dots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \dots & \lambda & -\sqrt{8} & 0\\
0 & 0 & 0 & 0 & \dots & -\sqrt{8} & \lambda & -\sqrt{8}\\
0 & 0 & 0 & 0 & \dots & 0 & -\sqrt{8} & \lambda
\end{pmatrix}_{k\times k},
$$

and let $\Lambda_k=det(M_k)$. We have $\Lambda_k=\lambda\Lambda_{k-1}-8\Lambda_{k-2}$. Suppose that $\phi_{SO}(C_n,\lambda)=det(\lambda I-A_{SO}(C_n))$. We have

$$
\phi_{SO}(C_n,\lambda)=det\left(\begin{array}{c|c}
\lambda & -\sqrt{8}\ 0\ 0\ \dots\ 0\ 0\ -\sqrt{8}\\ \hline
\begin{matrix}-\sqrt{8}\\ 0\\ 0\\ \vdots\\ 0\\ 0\\ -\sqrt{8}\end{matrix} & M_{n-1}
\end{array}\right)_{n\times n}.
$$

Therefore,

$$
\phi_{SO}(P_n,\lambda)=\lambda\Lambda_{n-1}+\sqrt{8}det\left(\begin{array}{c|c}
-\sqrt{8} & -\sqrt{8}\quad 0\quad \dots\ 0\\ \hline
\begin{matrix}0\\ \vdots\\ 0\\ -\sqrt{8}\end{matrix} & M_{n-2}
\end{array}\right)
$$

$$
+(-1)^{n+1}(-\sqrt{8})det\left(\begin{array}{c|ccc} -\sqrt{8} & & & \\ 0 & & M_{n-2} & \\ \vdots & & & \\ 0 & & & \\ \hline -\sqrt{8} & 0\ \dots & 0 & -\sqrt{8} \end{array}\right).
$$

Hence,

$$
\begin{aligned}
\phi_{SO}(P_n,\lambda) &= \lambda\Lambda_{n-1} + \sqrt{8}\left(-\sqrt{8}\Lambda_{n-2} + (-1)^n(-\sqrt{8})^{n-1}\right) \\
&\quad + (-1)^{n+1}(-\sqrt{8})\left((-\sqrt{8})^{n-1} + (-1)^n(-\sqrt{8})\Lambda_{n-2}\right),
\end{aligned}
$$

and, therefore, we have the result. □

Now, we consider to star graph S_n and find its Sombor characteristic polynomial and Sombor energy. We need the following Lemma.

Lemma 2.5 Cvetković et al. (1980) If M is a nonsingular square matrix, then

$$
det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = det(M)det(Q - PM^{-1}N).
$$

Theorem 2.6 *For $n \geq 2$,*

(i) The Sombor characteristic polynomial of the star graph $S_n = K_{1,n-1}$ is

$$
\phi_{SO}(S_n,\lambda)) = \lambda^{n-2}\left(\lambda^2 - (n-1)(n^2-2n+2)\right).
$$

(ii) The Sombor energy of S_n is

$$
E_{SO}(S_n) = 2\sqrt{(n-1)(n^2-2n+2)}.
$$

Proof (i) One can easily check that the Sombor matrix of $K_{1,n-1}$ is

$$
A_{SO}(S_n) = \sqrt{n^2-2n+2}\begin{pmatrix} 0_{1\times 1} & J_{1\times n-1} \\ J_{n-1\times 1} & 0_{n-1\times n-1} \end{pmatrix}.
$$

We have

$$
det(\lambda I - A_{SO}(S_n)) = det\begin{pmatrix} \lambda & -\sqrt{n^2-2n+2}J_{1\times(n-1)} \\ -\sqrt{n^2-2n+2}J_{(n-1)\times 1} & \lambda I_{n-1} \end{pmatrix}.
$$

Using Lemma 2.5,

$$
det(\lambda I - A_{SO}(S_n)) = \lambda det(\lambda I_{n-1} - \sqrt{n^2-2n+2}J_{(n-1)\times 1}\frac{1}{\lambda}\sqrt{n^2-2n+2}J_{1\times(n-1)}).
$$

Since $J_{(n-1)\times 1}J_{1\times(n-1)} = J_{n-1}$, therefore,

$$
\begin{aligned}
det(\lambda I - A_{SO}(S_n)) &= \lambda det(\lambda I_{n-1} - \frac{1}{\lambda}(n^2-2n+2)J_{n-1}) \\
&= \lambda^{2-n}det(\lambda^2 I_{n-1} - (n^2-2n+2)J_{n-1}).
\end{aligned}
$$

On the other hand, the eigenvalues of J_{n-1} are $n-1$ (once) and 0 ($n-2$ times), the eigenvalues of $(n^2-2n+2)J_{n-1}$ are $(n-1)(n^2-2n+2)$ (once) and 0 ($n-2$ times). Hence,

$$
\phi_{SO}(S_n,\lambda)) = \lambda^{n-2}\left(\lambda^2 - (n-1)(n^2-2n+2)\right).
$$

(ii) It follows from Part (i).

□

Here, we shall investigate the Sombor energy of complete graphs.

Theorem 2.7 *For $n \geq 2$,*

(i) The Sombor characteristic polynomial of complete graph K_n is

$$\phi_{SO}(K_n, \lambda) = (\lambda - (n-1)^2\sqrt{2})(\lambda + (n-1)\sqrt{2})^{n-1}.$$

(ii) The Sombor energy of K_n is

$$E_{SO}(K_n) = 2(n-1)^2\sqrt{2}.$$

Proof (i) The Sombor matrix of K_n is $(n-1)\sqrt{2}(J - I)$. Therefore,

$$\begin{aligned}\phi_{SO}(K_n, \lambda) &= det(\lambda I - (n-1)\sqrt{2}J + (n-1)\sqrt{2}I)\\ &= det((\lambda + (n-1)\sqrt{2})I - (n-1)\sqrt{2}J).\end{aligned}$$

Since the eigenvalues of J_n are n (once) and 0 ($n-1$ times), the eigenvalues of $(n-1)\sqrt{2}J_n$ are $n(n-1)\sqrt{2}$ (once) and 0 ($n-1$ times). Therefore,

$$\phi_{SO}(K_n, \lambda) = (\lambda - (n-1)^2\sqrt{2})(\lambda + (n-1)\sqrt{2})^{n-1}.$$

(ii) It follows from Part (i).

□

We end this section by finding Sombor characteristic polynomial of complete bipartite graphs and their Sombor energy.

Theorem 2.8 *For natural number $m, n \neq 1$,*

(i) The Sombor characteristic polynomial of complete bipartite graph $K_{m,n}$ is

$$\phi_{SO}(K_{m,n}, \lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^2 + n^2)).$$

(ii) The Sombor energy of $K_{m,n}$ is $2\sqrt{mn(m^2+n^2)}$.

Proof (i) It is easy to see that the Sombor matrix of $K_{m,n}$ is $\sqrt{m^2+n^2}\begin{pmatrix} 0_{m\times m} & J_{m\times n} \\ J_{n\times m} & 0_{n\times n}\end{pmatrix}$.

Using Lemma 2.5, we have

$$det(\lambda I - A_{SO}(K_{m,n})) = det\begin{pmatrix} \lambda I_m & -\sqrt{m^2+n^2}J_{m\times n} \\ -\sqrt{m^2+n^2}J_{n\times m} & \lambda I_n\end{pmatrix}.$$

Therefore,

$$det(\lambda I - A_{SO}(K_{m,n})) = det(\lambda I_m)det(\lambda I_n - \sqrt{m^2+n^2}J_{n\times m}\frac{1}{\lambda}I_m\sqrt{m^2+n^2}J_{m\times n}).$$

We know that $J_{n\times m}J_{m\times n} = mJ_n$. Therefore,

$$\begin{aligned}det(\lambda I - A_{SO}(K_{m,n})) &= \lambda^m det(\lambda I_n - \frac{1}{\lambda}m(m^2+n^2)J_n)\\ &= \lambda^{m-n}det(\lambda^2 I_n - m(m^2+n^2)J_n).\end{aligned}$$

The eigenvalues of J_n are n (once) and 0 ($n-1$ times). Therefore, the eigenvalues of $m(m^2+n^2)J_n$ are $mn(m^2+n^2)$ (once) and 0 ($n-1$ times). Hence,

$$\phi_{SO}(K_{m,n}, \lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^2 + n^2)).$$

(ii) It follows from Part (i). □

3 Sombor energy of 2-regular and 3-regular graphs

In this section, we consider 2-regular and 3-regular graphs. As a beginning of this section, we have the following easy lemma:

Lemma 3.1 *Let $G = G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n$. Then*

(i) $\phi_{SO}(G) = \prod_{i=1}^{n} \phi_{SO}(G_i)$.
(ii) $E_{SO}(G) = \sum_{i=1}^{n} E_{SO}(G_i)$.

As an immediate result of Lemma 3.1, we have the following results:

Proposition 3.2 *(i) If $e = v_r v_{r+1} \in E(P_n)$, then $E_{SO}(P_n - e) = E_{SO}(P_r) + E_{SO}(P_s)$, where $r + s = n$.*
(ii) If $e \in E(C_n)$, ($n \geq 3$), then $E_{SO}(C_n - e) = E_{SO}(P_n)$.
(iii) Let S_n be the star on n vertices and $e \in E(S_n)$. Then, for any $n \geq 3$,

$$E_{SO}(S_n - e) = E_{SO}(S_{n-1}).$$

Now, consider to the 2-regulars. Every 2-regular graph is a disjoint union of cycles. By Theorem 2.4, we can find all the eigenvalues of Sombor matrix of cycle graphs. Therefore, by Lemma 3.1, we can find Sombor characteristic polynomial and Sombor energy of 2-regular graphs. Now, we consider to the characteristic polynomial of 3-regular graphs of order 10. In addition, we shall compute Sombor energy of this class of graphs. There are exactly 21 cubic graphs of order 10 given in Fig. 1 (see Khosrovshahi and Maysoori (2001)).

Using Maple, we computed the Sombor characteristic polynomials of 3-regular graphs of order 10 in Table 1. By finding the roots of Sombor characteristic polynomial of cubic graphs of order 10, we can have the Sombor energy of these graphs. We compute them to three decimal places. Therefore, we have them in Table 2.

Proposition 3.3 *Six cubic graphs of order 10 are not $\mathcal{E}_{SO}$-unique.*

Proof By observing Table 2, we see that $[G_1] = \{G_1, G_8\}$, $[G_{12}] = \{G_{12}, G_{17}\}$ and $[G_{16}] = \{G_{16}, G_{20}\}$. Therefore, we have 15 cubic graphs of order 10 which are $\mathcal{E}_{SO}$-unique. □

As an immediate result of Proposition 3.3, we have

Corollary 3.4 *In general, two k-regular graphs of the same order may not have same Sombor energy.*

Theorem 3.5 *Let $\mathcal{G}$ be the family of 3-regular graphs of order 10. For the Petersen graph P (Fig. 2), we have the following properties:*

(i) *P is not $\mathcal{E}_{SO}$-unique in $\mathcal{G}$.*
(ii) *P has the maximum Sombor energy in $\mathcal{G}$.*

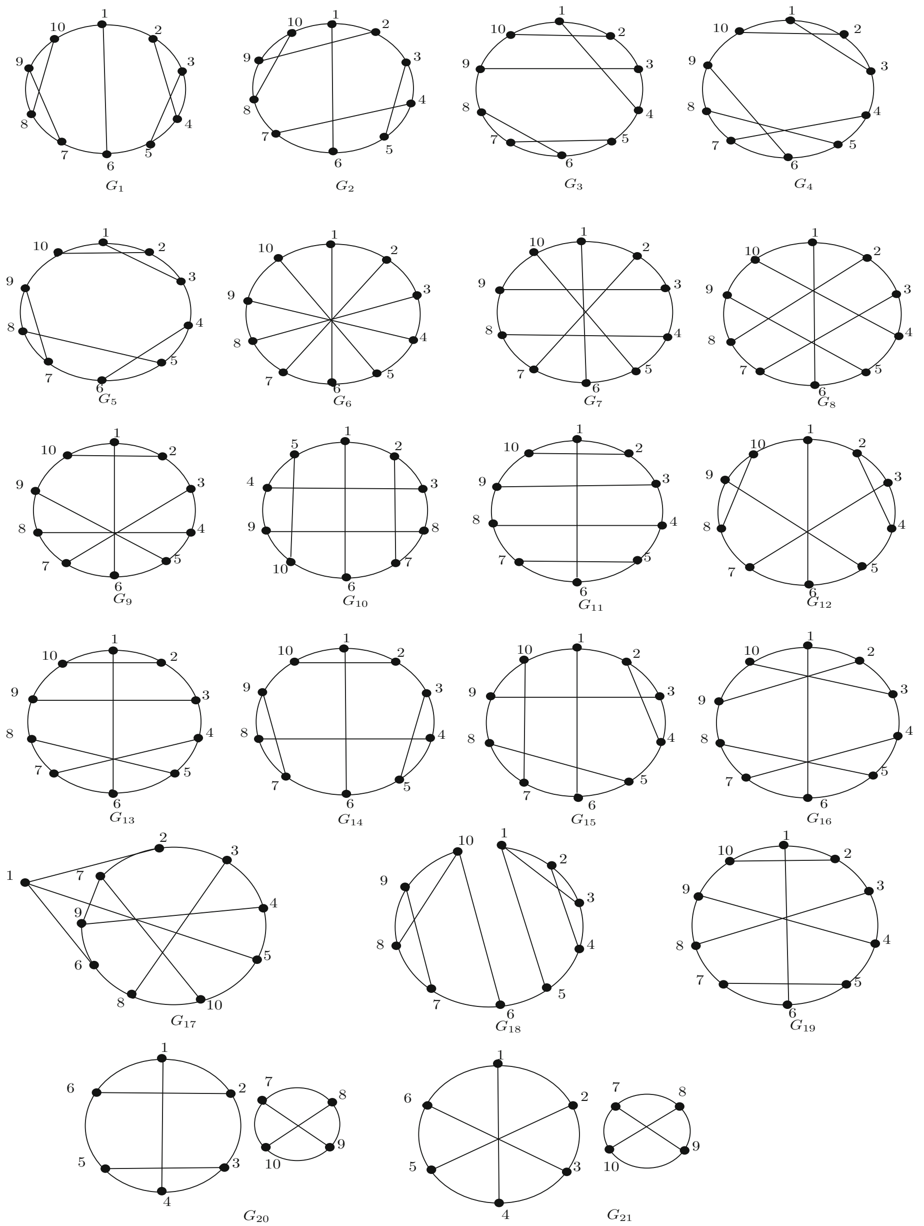

Fig. 1 Cubic graphs of order 10

Proof (i) The Sombor matrix of P is

$$A_{SO}(P)=\begin{pmatrix} 0 & 3\sqrt{2} & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0 & 0\\ 3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0\\ 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 & 0\\ 0 & 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0\\ 3\sqrt{2} & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2}\\ 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} & 0\\ 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2}\\ 0 & 0 & 3\sqrt{2} & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2}\\ 0 & 0 & 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\phi_{SO}(P,\lambda)=det(\lambda I-A_{SO}(P))=(\lambda-9\sqrt{2})(\lambda+6\sqrt{2})^4(\lambda-3\sqrt{2})^5.$$

Therefore, we have

$$\lambda_1=9\sqrt{2},\ \ \lambda_2=\lambda_3=\lambda_4=\lambda_5=-6\sqrt{2},\ \ \lambda_6=\lambda_7=\lambda_8=\lambda_9=\lambda_{10}=3\sqrt{2},$$

and so we have $E_{SO}(P)=48\sqrt{2}$. By Table 2, we have $P\in\{G_{12},G_{17}\}$. Hence, P is not $\mathcal{E}_{\mathcal{SO}}$-unique in $\mathcal{G}$.

(ii) It follows from Part (i) and Table 2.

□

Now, we check that is there any relationship between Sombor energy and permanent of adjacency matrix of two connected k-regular graphs of the same order?

Observation 3.6 *If two connected k-regular graphs have the same Sombor energy, then their adjacency matrices may have or have not the same permanent.*

Proof We consider to the cubic graphs of order 10. By Table 2, $E_{SO}(G_1)=E_{SO}(G_8)$ and $E_{SO}(G_{16})=E_{SO}(G_{20})$. Now, we find $per(A(G_1))$, $per(A(G_8))$, $per(A(G_{16}))$ and $per(A(G_{20}))$. For graph G_1, we have

$$A(G_1)=\begin{pmatrix} 0&1&0&0&0&1&0&0&0&1\\ 1&0&1&1&0&0&0&0&0&0\\ 0&1&0&1&1&0&0&0&0&0\\ 0&1&1&0&1&0&0&0&0&0\\ 0&0&1&1&0&1&0&0&0&0\\ 1&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&1&0&1&1&0\\ 0&0&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&1&1&0&1\\ 1&0&0&0&0&0&0&1&1&0 \end{pmatrix}.$$

By Ryser's method, we have $per(A(G_1))=72$. Similarly, we have

$$per(A(G_8))=72,\ \ per(A(G_{16}))=144,\ \ per(A(G_{20}))=180.$$

Therefore, we have the result. □

Table 1 Sombor characteristic polynomial $P(G_i, \lambda)$, for $1 \le i \le 21$

G_i	$P(G_i, \lambda)$
G_1	$\lambda^{10} - 270\lambda^8 - 432\sqrt{2}\lambda^7 + 23004\lambda^6 + 62208\sqrt{2}\lambda^5 - 589032\lambda^4 - 1819584\sqrt{2}\lambda^3 + 4618944\lambda^2$ $+15116544\sqrt{2}\lambda$
G_2	$\lambda^{10} - 270\lambda^8 - 216\sqrt{2}\lambda^7 + 23004\lambda^6 + 27216\sqrt{2}\lambda^5 - 705672\lambda^4 - 839808\sqrt{2}\lambda^3 + 6718464\lambda^2$ $+7558272\sqrt{2}\lambda$
G_3	$\lambda^{10} - 270\lambda^8 - 324\sqrt{2}\lambda^7 + 22356\lambda^6 + 46656\sqrt{2}\lambda^5 - 559872\lambda^4 - 1329696\sqrt{2}\lambda^3 + 3149280\lambda^2$ $+8188128\sqrt{2}\lambda + 5668704$
G_4	$\lambda^{10} - 270\lambda^8 - 216\sqrt{2}\lambda^7 + 20412\lambda^6 + 34992\sqrt{2}\lambda^5 - 355752\lambda^4 - 979776\sqrt{2}\lambda^3 - 1259712\lambda^2$
G_5	$\lambda^{10} - 270\lambda^8 - 432\sqrt{2}\lambda^7 + 23004\lambda^6 + 66096\sqrt{2}\lambda^5 - 542376\lambda^4 - 2309472\sqrt{2}\lambda^3 - 3779136\lambda^2$
G_6	$\lambda^{10} - 270\lambda^8 + 21060\lambda^6 - 612360\lambda^4 + 5773680\lambda^2 - 17006112$
G_7	$\lambda^{10} - 270\lambda^8 + 22356\lambda^6 - 11664\sqrt{2}\lambda^5 - 682344\lambda^4 + 629856\sqrt{2}\lambda^3 + 6193584\lambda^2 - 3779136\sqrt{2}\lambda$ -17006112
G_8	$\lambda^{10} - 270\lambda^8 + 23004\lambda^6 - 15552\sqrt{2}\lambda^5 - 775656\lambda^4 + 1119744\sqrt{2}\lambda^3 + 7978176\lambda^2 - 15116544\sqrt{2}\lambda$
G_9	$\lambda^{10} - 270\lambda^8 - 108\sqrt{2}\lambda^7 + 23004\lambda^6 + 7776\sqrt{2}\lambda^5 - 769824\lambda^4 - 34992\sqrt{2}\lambda^3 + 9552816\lambda^2$ $-2519424\sqrt{2}\lambda - 22674816$
G_{10}	$\lambda^{10} - 270\lambda^8 + 21060\lambda^6 - 3888\sqrt{2}\lambda^5 - 495720\lambda^4 - 349920\sqrt{2}\lambda^3 + 3674160\lambda^2 + 6298560\sqrt{2}\lambda$ $+5668704$
G_{11}	$\lambda^{10} - 270\lambda^8 - 216\sqrt{2}\lambda^7 + 22356\lambda^6 + 31104\sqrt{2}\lambda^5 - 612360\lambda^4 - 1119744\sqrt{2}\lambda^3 + 2414448\lambda^2$ $+6298560\sqrt{2}\lambda + 5668704$
G_{12}	$\lambda^{10} - 270\lambda^8 - 216\sqrt{2}\lambda^7 + 24300\lambda^6 + 23328\sqrt{2}\lambda^5 - 915624\lambda^4 - 629856\sqrt{2}\lambda^3 + 15116544\lambda^2$ $+5038848\sqrt{2}\lambda - 90699264$
G_{13}	$\lambda^{10} - 270\lambda^8 - 108\sqrt{2}\lambda^7 + 21708\lambda^6 + 11664\sqrt{2}\lambda^5 - 559872\lambda^4 - 384912\sqrt{2}\lambda^3 + 3674160\lambda^2$ $+3779136\sqrt{2}\lambda$
G_{14}	$\lambda^{10} - 270\lambda^8 - 324\sqrt{2}\lambda^7 + 24300\lambda^6 + 46656\sqrt{2}\lambda^5 - 839808\lambda^4 - 1994544\sqrt{2}\lambda^3 + 7873200\lambda^2$ $+21415104\sqrt{2}\lambda + 22674816$
G_{15}	$\lambda^{10} - 270\lambda^8 - 108\sqrt{2}\lambda^7 + 22356\lambda^6 + 11664\sqrt{2}\lambda^5 - 676512\lambda^4 - 419904\sqrt{2}\lambda^3 + 5668704\lambda^2$ $+8188128\sqrt{2}\lambda + 5668704$
G_{16}	$\lambda^{10} - 270\lambda^8 + 20412\lambda^6 - 495720\lambda^4 + 3779136\lambda^2$
G_{17}	$\lambda^{10} - 270\lambda^8 + 24300\lambda^6 - 23328\sqrt{2}\lambda^5 - 962280\lambda^4 + 2099520\sqrt{2}\lambda^3 + 12597120\lambda^2 - 50388480\sqrt{2}\lambda$ $+9069926$
G_{18}	$\lambda^{10} - 270\lambda^8 - 432\sqrt{2}\lambda^7 + 20412\lambda^6 + 62208\sqrt{2}\lambda^5 - 215784\lambda^4 - 979776\sqrt{2}\lambda^3 - 1259712\lambda^2$
G_{19}	$\lambda^{10} - 270\lambda^8 - 216\sqrt{2}\lambda^7 + 23652\lambda^6 + 27216\sqrt{2}\lambda^5 - 822312\lambda^4 - 909792\sqrt{2}\lambda^3 + 10392624\lambda^2$ $+5038848\sqrt{2}\lambda - 39680928$
G_{20}	$\lambda^{10} - 270\lambda^8 - 648\sqrt{2}\lambda^7 + 20412\lambda^6 + 93312\sqrt{2}\lambda^5 - 75816\lambda^4 - 1469664\sqrt{2}\lambda^3 - 3779136\lambda^2$
G_{21}	$\lambda^{10} - 270\lambda^8 - 432\sqrt{2}\lambda^7 + 16524\lambda^6 + 69984\sqrt{2}\lambda^5 + 157464\lambda^4$

By Observation 3.6, we know that if two connected k-regular graphs have the same Sombor energy, we can say nothing about the permanent of their adjacency matrices. Now by the following Remark, we show that if two graphs have the same permanent, then we cannot conclude that they have same Sombor energy. Therefore, in general, there is no relation between Sombor energy and permanent of adjacency matrices of k-regular graphs with the same order.

Remark 3.7 In the class of cubic graphs of order 10, we have $per(A(G_7)) = per(A(G_{11})) = 85$, but as we see in Table 2, $E_{SO}(G_7) \neq E_{SO}(G_{11})$.

As an observation, we see that every graph does not have integer-valued Sombor energy. We end this section with the following conjecture:

Conjecture 3.8 *There is no graph with integer-valued Sombor energy.*

4 Conclusions

In this paper, we obtained the Sombor characteristic polynomial and Sombor energy of specific graphs such as paths, cycles, stars, complete bipartite graphs and complete graphs. In addition, we studied Sombor energy of 2-regular and 3-regular graphs.

Future topics of interest for future research include the following suggestions:

- Proving Conjecture 3.8 or giving a graph with integer-valued Sombor energy.

Table 2 Sombor energy of cubic graphs of order 10

G_i	$E_{SO}(G_i)$	G_i	$E_{SO}(G_i)$	G_i	$E_{SO}(G_i)$
G_1	64.161	G_8	64.161	G_{15}	62.767
G_2	63.043	G_9	64.981	G_{16}	59.396
G_3	62.880	G_{10}	61.399	G_{17}	67.882
G_4	57.336	G_{11}	62.375	G_{18}	57.517
G_5	60.638	G_{12}	67.882	G_{19}	66.096
G_6	63.403	G_{13}	61.000	G_{20}	59.396
G_7	63.969	G_{14}	65.835	G_{21}	50.911

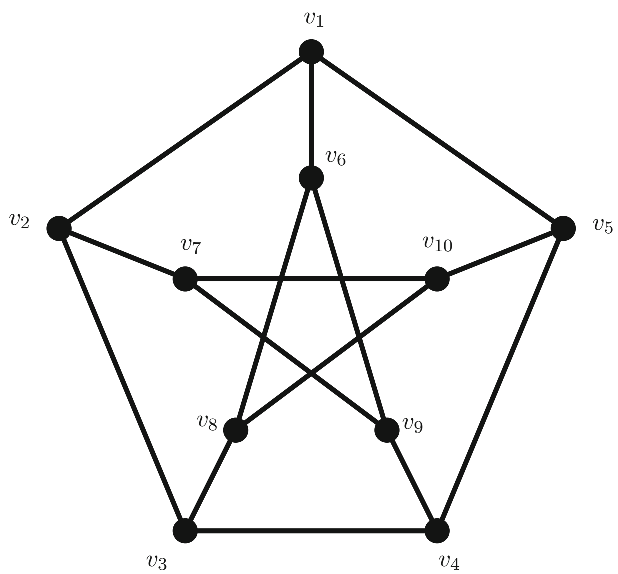

Fig. 2 Petersen graph

- What is the relationship between $E_{SO}(G)$ and $E_{SO}(G-e)$ where $e \in E(G)$?
- What can we say about $E_{SO}(G)$ and $E_{SO}(G-v)$ where $v \in V(G)$?
- What is the Sombor energy of $G * H$ where $*$ is some kind of operation on two graph?
- If two graphs of the same order be $\mathcal{E}_{SO}$-*equivalent*, do they have any properties in common?

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