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Towards Stronger Lagrangean Bounds for Stable Spanning Trees

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ABSTRACT

Given a graph $G = (V, E)$ and a set C of unordered pairs of edges regarded as being in conflict, a stable spanning tree in G is a set of edges T inducing a spanning tree in G, such that for each $\{e_i, e_j\} \in C$, at most one of the edges e_i and e_j is in T. The existing work on Lagrangean algorithms to the NP-hard problem of finding minimum weight stable spanning trees is limited to relaxations with the integrality property. We have recently initiated the combinatorial and polyhedral study of fixed cardinality stable sets [17], which motivates a new formulation for stable spanning trees based on Lagrangean Decomposition. By optimizing over the spanning tree polytope of G and the fixed cardinality stable set polytope of the conflict graph $\hat{G} = (E, C)$ in the subproblems, we are able to determine stronger Lagrangean bounds (equivalent to dualizing exponentially-many subtour elimination constraints), while limiting the number of multipliers in the dual problem to $|E|$. This naturally asks for more sophisticated dual algorithms, requiring the fewest iterations possible, and we derive a collection of Lagrangean dual ascent directions to this end.

KEYWORDS

Stable spanning trees, conflict-free spanning trees, Lagrangean decomposition, dual ascent, fixed cardinality stable sets.

1 INTRODUCTION

Given an undirected graph $G = (V, E)$, with edge weights $w : E \rightarrow \mathbb{Q}$, and a family C of unordered pairs of edges that are regarded as being in conflict, a stable (or conflict-free) spanning tree in G is a set of edges T inducing a spanning tree in G, such that for each $\{e_i, e_j\} \in C$, at most one of the edges e_i and e_j is in T. The minimum spanning tree under conflict constraints (MSTCC) problem is to determine a stable spanning tree of least weight, or decide that none exists. It was introduced by [8, 9], who also prove its NP-hardness.

Different combinatorial and algorithmic results about stable spanning trees explore the associated conflict graph $\hat{G} = (E, C)$, which has a vertex corresponding to each edge in the original graph G, and where we represent each conflict constraint by an edge connecting the corresponding vertices in $\hat{G}$. Note that each conflict-free spanning tree in G is a subset of E which corresponds both to a spanning tree in G and to a stable set (or independent set, or co-clique: a subset of pairwise non-adjacent vertices) in $\hat{G}$. Therefore, one can equivalently search for stable sets in $\hat{G}$ of cardinality exactly $|V| - 1$ which do not induce cycles in the original graph G.

We have recently initiated the combinatorial study of stable sets of cardinality exactly k in a graph [17], where k is a positive integer given as part of the input. There are appealing research directions around algorithms, combinatorics and optimization for problems defined over fixed cardinality stable sets. Also from an applications perspective, conflict constraints arise naturally in operations research and management science. Stable spanning trees, in particular, model real-world settings such as communication networks with different link technologies (which might be mutually exclusive in some cases), and utilities distribution networks. In fact, the latter is a standard application of the quadratic minimum spanning tree problem [1], which generalizes the MSTCC one.

Exact algorithms to find stable spanning trees have been investigated for a decade now, building on branch-and-cut [6, 18], or Lagrangean relaxation [7, 20] strategies. Consider the natural integer programming (IP) formulation for the MSTCC problem:

$$\min \sum_{e \in E} w_e x_e \tag{1}$$

$$\text{s.t.} \sum_{e \in E(S)} x_e \leq |S| - 1, \quad \text{for each } S \subsetneq V, S \neq \emptyset, \tag{2}$$

$$\sum_{e \in E} x_e = |V| - 1, \tag{3}$$

$$x_{e_i} + x_{e_j} \leq 1, \quad \text{for each } \{e_i, e_j\} \in C, \tag{4}$$

$$x_e \in \{0, 1\}, \quad \text{for each } e \in E. \tag{5}$$

While a considerable effort in the development of branch-and-cut algorithms led to more sophisticated formulations and contributed to a better understanding of our capacity to solve MSTCC instances by judicious use of valid inequalities, the existing Lagrangean algorithms are limited to the most elementary approach. Namely, a relaxation scheme dualizing conflict constraints (4), which thus has the integrality property. We review other aspects of the corresponding references in Section 2.

The present paper takes the standpoint that the development of a full-fledged Lagrangean strategy to find stable spanning trees is an unsolved problem. While we recognize different merits of previous work, we argue that it is worth investigating stronger Lagrangean bounds for the MSTCC structure: exploring more creative relaxation schemes, designing improved dual methods, all the while harnessing the progress in IP computation.

The main idea of this paper is to offer an alternative starting point for this problem. In Section 3, we call attention to a stronger relaxation scheme, based on Lagrangean *Decomposition*. We explain how classical results from the literature guarantee the superiority of such a reformulation: both with respect to the quality of dual bounds, when compared to the straightforward relaxation, and with regard to the number of multipliers, when compared to an alternative framework to determine the same bounds (relax-and-cut dualizing violated subtour elimination constraints (2) dynamically). The decomposition naturally leads to the dual ascent paradigm to solve the Lagrangean dual problem, and Section 4 is devoted to presenting maximal ascent directions. These are fundamental ingredients in tailored methods guaranteeing monotone bound improvement when optimizing the Lagrangean dual.

We see the opportunity for renewed interest in Lagrangean Decomposition in light of the progress in mixed-integer linear programming (MILP) computation. Given the impressive speedup of MILP solvers over the past two decades, Dimitris Bertsimas and Jack Dunn are among a group of distinguished researchers who make a case for (exact) optimization over integers as the natural, correct model for several tasks within machine learning and towards interpretable artificial intelligence. This is the theme of their recent book [2]; see also [3, 4]. We are interested in exploring whether this philosophy (challenging assumptions previously deemed computationally intractable) should also imply less hesitation towards designing Lagrangean algorithms that exploit subproblems for which, albeit strongly NP-hard, specialized solvers attain good performance. In the particular case of the MSTCC problem, one could leverage state-of-the-art branch-and-cut algorithms for stable sets (in particular, of fixed cardinality) to find stable spanning trees more efficiently by means of Lagrangean Decomposition, such as we outline in this paper.

In summary, our contributions are the following.

(1) We bring attention to the quality of different Lagrangean bounds for the MSTCC problem as an inviting margin for designing improved algorithms, and we discuss the advantages, in theory, of a reformulation based on Lagrangean Decomposition.
(2) We determine a collection of Lagrangean dual ascent directions for optimizing the Lagrangean dual problem corresponding to the new MSTCC reformulation, hence contributing towards a new family of algorithms and dual bounds for the problem.

2 DRAWBACKS OF EXISTING LAGRANGEAN APPROACHES FOR MSTCC

The work of [20] contributes in many research directions about stable spanning trees, including particular cases which are polynomially solvable, feasibility tests, several heuristics, and two exact algorithms based on Lagrangean relaxation. The first formulation is the straightforward one we mentioned, dualizing all conflict constraints (4); they denote the corresponding dual bound L^*. The second approach relaxes a subset of inequalities (4): using an approximation to the maximum edge clique partitioning problem [10], this scheme dualizes a subset of conflict constraints such that the remaining conflict graph is a collection of disjoint cliques; the resulting dual bound is denoted ℓ^*. The authors argue that the latter reformulation is stronger than the former, and present extensive computational results justifying their claims.

Unfortunately, the Lagrangean dual bounds L^* and ℓ^* in [20] are in fact identical, as we show next. The first relaxation clearly has the integrality property, as the remaining constraints correspond to a description of the spanning tree polytope or, equivalently, to bases of the graphic matroid of G. The second relaxation scheme is designed so that the conflict constraints which remain in the subproblem of relaxation ℓ^* induce a collection of disjoint cliques in $\hat{G}$. The subproblem thus corresponds to the intersection of two matroids: the graphic matroid of G and the partition matroid of subsets of E that intersect the enumerated cliques in $\hat{G}$ at most once. It follows that the second relaxation also has the integrality property [15, Theorem III.3.5.9], and consequently, L^* and ℓ^* both equal the optimal objective function value in the continuous relaxation of (1)–(5) [15, Corollary II.3.6.6]. In this perspective, the computational results in Tables 2–4 of [20] diverge from what Lagrangean duality theory prescribes.

Recently, [7] presented thorough computational experiments of a new Lagrangean algorithm for the MSTCC problem. They use the same relaxation scheme dualizing all conflict constraints, and focus on a combination of dual ascent and the subgradient method to compute the Lagrangean bound, namely, L^* in [20], equal to the LP-relaxation of (1) – (5). In Table 1 of [7], the performance of the new algorithm is compared to the results published in [20]. That is, the issue we analyse above regarding the computational results of [20] is repeated as a baseline of the new numerical evaluation.

Another drawback of the new algorithm is that dual ascent steps are intertwined with subgradient optimization. While *not* incorrect, this choice undermines the advantages of a strategy to solve the dual problem in fewer iterations. A passage from a classical work of Guignard and Rosenwein [14] is conclusive: *"An ascent procedure may also serve to initialize multipliers in a subgradient procedure. This scheme is particularly useful at the root node of an enumeration tree. However, an ascent method cannot guarantee improved bounds over bounds obtained by solving the Lagrangean dual with a subgradient procedure."*

Moreover, the ascent steps rely on a greedy heuristic, and not on *maximal ascent directions*, *i.e.* optimal step size in a direction of bound increase; see Definition 4.1. In the algorithm of [7], if a conflicting pair of edges exists in a Lagrangean solution, the multiplier adjustment is derived from the observation that the dual bound shall improve by at least the increased cost of replacing one of the edges by its cheapest successor (in a list of edges ordered by current costs). The authors remedy the resulting low adjustment values by alternating subgradient optimization iterations and the ascent procedure.

We stress again that references [7] and [20] have many virtues and present concrete contributions to the MSTCC literature. Our only remark is that the first Lagrangean strategy designed to improve upon the LP-relaxation bound is matter-of-factly yet to be introduced. In the next sections, we offer an interesting approach to tackle this challenge.

3 LAGRANGEAN DECOMPOSITION

Renaming the variables in (4) as y, and introducing linking constraints $x_e = y_e$ for each $e \in E$, we have the same formulation. Now, dualizing the linking constraints with Lagrangean multipliers $\lambda \in \mathbb{Q}^{|E|}$, we arrive at the Lagrangean Decomposition formulation:

$$z(\lambda) \overset{\text{def}}{=} \min_{x \in \mathcal{F}_{\text{sp.tree}}(G)} (w - \lambda)^\intercal x \quad + \min_{y \in \mathcal{F}_{\text{kstab}}(\hat{G}, |V|-1)} \lambda^\intercal y \qquad (6)$$

where $\mathcal{F}_{\text{sp.tree}}(G)$ is given by

$$\sum_{e \in E(S)} x_e \leq |S| - 1, \quad \text{for each } S \subsetneq V, S \neq \emptyset, \tag{7}$$

$$\sum_{e \in E} x_e = |V| - 1, \tag{8}$$

$$x_e \in \{0, 1\}, \qquad \text{for each } e \in E, \tag{9}$$

and $\mathcal{F}_{\text{kstab}}(\hat{G}, |V| - 1)$ is given by

$$\sum_{e \in E} y_e = |V| - 1, \tag{10}$$

$$y_{e_i} + y_{e_j} \leq 1, \quad \text{for each } \{e_i, e_j\} \in C, \tag{11}$$

$$y_e \in \{0, 1\}, \quad \text{for each } e \in E. \tag{12}$$

The Lagrangean dual problem is to determine the tightest such bound:

$$\zeta \overset{\text{def}}{=} \max_{\lambda \in \mathbb{Q}^{|E|}} \{z(\lambda)\}. \tag{13}$$

The first systematic study of Lagrangean Decomposition as a general purpose reformulation technique was presented by Guignard and Kim [13]. They indicate earlier applications of variable splitting/layering, especially [19] and [16]. See also the outstanding presentation in [12, Section 7].

One of the main virtues of the decomposition principle over traditional Lagrangean relaxation schemes is that the bound from the Lagrangean decomposition dual is equal to the optimum of the primal objective function over the intersection of the convex hulls of both constraint sets [13, Corollary 3.4]. The decomposition bound is thus equal to the strongest of the two Lagrangean relaxation schemes corresponding to dualizing either of the constraint sets.

In our application to the MSTCC problem, we recognize the integrality of the spanning tree formulation described by (7) – (8) over $x \in \mathbb{Q}^{|E|}$. Hence the decomposition bound matches that of the stronger scheme where constraints (10) – (11) enforcing fixed cardinality stable sets are kept in the subproblem (which is thus *convexified*), and all subtour elimination constraints (7) are dualized. This means that we can compute stronger Lagrangean bounds, while limiting the number of multipliers in the dual problem to $|E|$, instead of dealing with exponentially-many multipliers *e.g.* in a relax-and-cut approach.

We defend the advantages of breaking the original problem into two parts, exploiting their rich combinatorial and polyhedral structures, so as to derive stronger dual bounds. The price of this strategy is to solve a strongly NP-hard subproblem, which naturally leads to more sophisticated dual algorithms, requiring the fewest iterations possible. Customized dual ascent is a technique that integrates naturally with Lagrangean decomposition [13], and may be the key ingredient towards effective computation of such stronger bounds.

4 LAGRANGEAN DUAL ASCENT

In this section we present the main contributions of the paper. In what follows, let $\mathfrak{e}_i \in \mathbb{R}^m$ denote the standard unit vector in the i-th direction, let **conv** S denote the convex hull of a set S, and let **ext** Q denote the set of extreme points of a given polyhedron Q. We let

$$\mathcal{P}_{\text{sp.tree}}(G) \overset{\text{def}}{=} \textbf{conv}\ \mathcal{F}_{\text{sp.tree}}(G)$$

denote the spanning tree polytope of a graph G, and let

$$\mathfrak{C}(G, k) \overset{\text{def}}{=} \textbf{conv}\ \mathcal{F}_{\text{kstab}}(G, k)$$

denote the polytope of stable sets of cardinality k in G. Note that $\mathcal{P}_{\text{sp.tree}}$ and $\mathfrak{C}$ are bounded (polytopes contained in the 0,1 hypercube), and do not contain extreme rays.

The Lagrangean dual function $z : \mathbb{Q}^{|E|} \to \mathbb{Q}$ is an implicit function of λ. It is determined by the the lower envelope of

$$\Big\{(w - \lambda)^\top x^r + \lambda^\top y^s \ : \ x^r \in \textbf{ext}\ \mathcal{P}_{\text{sp.tree}}(G), \\ y^s \in \textbf{ext}\ \mathfrak{C}(\hat{G}, |V| - 1)\Big\}.$$

Hence, it is piecewise linear concave, and differentiable almost everywhere, with breakpoints at λ' such that the optimal solution of $z(\lambda')$ is not unique.

Such breakpoints are the key ingredient in the dual ascent paradigm to solve a Lagrangean dual problem. In particular, the following kind of point deserves special attention to guide progress in this framework.

Definition 4.1. A **maximal ascent direction** of the Lagrangean dual function $z : \mathbb{Q}^m \to \mathbb{Q}$ at λ^r is a vector $u \in \mathbb{Q}^m$ in a direction of increase from $z(\lambda^r)$, *i.e.*

$$z(\lambda^r + u) > z(\lambda^r),$$

such that $\lambda + u$ is a breakpoint of z, *i.e.*

$$z(\lambda^r + u) \geq z(\lambda^r + \alpha u), \text{ for all } \alpha \in \mathbb{Q}.$$

A maximal ascent direction determines an optimal multiplier adjustment in a given direction of increase of the Lagrangean dual function. It need not correspond to a *steepest* ascent direction from $z(\lambda^r)$, in general.

The technique of optimizing the Lagrangean dual function by means of ascent directions uses the formulation structure to determine monotone bound improving sequences of multipliers. It was pioneered by [5] and [11] in the context of the facility location problem. An actual algorithm of this kind thus relies on analysing the specific problem and the information available from subproblem solutions. Nevertheless, we found it instructive to summarize and systematically review the following instructions as a *guiding principle* of Lagrangean decomposition based dual ascent:

i. Update multiplier λ_e corresponding to a violation $x_e \neq y_e$
ii. in the sense of improving the Lagrangean dual bound,
iii. analysing the implications of changes in λ_e *alone*,
iv. so as to induce alternative subproblem solutions
v. while avoiding bound-decreasing effects.

Although one cannot hope for a pragmatic, problem-independent algorithm, this principle is the intuitive foundation of the arguments that follow.

So as not to overload notation, we omit the transposition symbol in the remainder of the text, whenever it is clear from the context *e.g.* in vector products like $(w - \lambda^r)^\top x^r$.

Theorem 4.2. *Let (x^r, y^r) be an optimal solution of subproblem $z(\lambda^r)$, such that $x^r_e = 0 < 1 = y^r_e$. Define the non-negative quantities*

$$\Delta^r_e \stackrel{def}{=} \min\left\{\lambda^r y : y \in \mathcal{F}_{kstab}(\hat{G}, |V|-1), y_e = 0\right\} - \lambda^r y^r, \tag{14}$$

$$\partial^r_e \stackrel{def}{=} \min\left\{(w-\lambda^r)x : x \in \mathcal{F}_{sp.tree}(G), x_e = 1\right\} - \left(w - \lambda^r\right) x^r. \tag{15}$$

If $\min\left\{\Delta^r_e, \partial^r_e\right\} \neq 0$*, then* $\min\left\{\Delta^r_e, \partial^r_e\right\} \cdot \mathfrak{e}_e$ *is a maximal ascent direction of z at λ^r.*

Proof.

(i.) Given that $x^r_e = 0$ and $y^r_e = 1$, increasing λ^r_e corresponds to increasing the dual bound, until alternative optimal solutions where that hypothesis fails are induced. Specifically,

$$z(\lambda^r + \epsilon\mathfrak{e}_e) > z(\lambda^r) \tag{16}$$

for all $\epsilon > 0$ such that

$$x^r \in \arg\min\left\{\Big(w - \left(\lambda^r + \epsilon\mathfrak{e}_e\right)\Big)x : x \in \mathcal{F}_{\text{sp.tree}}(G)\right\}, \tag{17}$$

$$y^r \in \arg\min\left\{\left(\lambda^r + \epsilon\mathfrak{e}_e\right) y : y \in \mathcal{F}_{\text{kstab}}(\hat{G}, |V|-1)\right\}. \tag{18}$$

As long as ϵ can be made positive, $\epsilon\mathfrak{e}_e$ is a direction of increase from $z(\lambda^r)$. The necessity of conditions (17) and (18) follows from noting that the contribution of the e-th variables x_e and y_e to z,

$$\left(w_e - \left(\lambda^r_e + \epsilon\mathfrak{e}_e\right)\right) x_e + \left(\lambda^r_e + \epsilon\mathfrak{e}_e\right) y_e,$$

remains constant as we increase ϵ after x_e joins, or y_e leaves, an optimal solution. For, if $x_e = y_e = 1$, meaning that the coefficient of edge e is attractive enough in (17), any further increase $+\epsilon y_e$ is cancelled by $-\epsilon x_e$. Moreover, if $x_e = y_e = 0$, which means that the coefficient of vertex e is no longer attractive enough in (18), further increasing ϵ in $\left(\lambda^r_e + \epsilon\mathfrak{e}_e\right) y_e = 0$ has no effect.

(ii.) To determine ϵ such that we find a breakpoint of z, we use the limiting conditions (17), (18).
For x^r to no longer be the unique optimum in (17), the cost of edge e decreases so much that an alternative solution $\tilde{x} \in \mathcal{F}_{\text{sp.tree}}(G)$ which includes e is determined. Note that $\tilde{x}$ is well-defined, as the choice of edges in a minimum spanning tree where e is fixed *a priori* does not depend on the cost of e (all other costs are kept unchanged). Also note that, since the existing solution is such that $x^r_e = 0$, the cost of $\tilde{x}$ is no less than that of x^r. The difference is precisely ∂^r_e in (15).
If $\partial^r_e = 0$, the bound cannot be improved by adjusting λ^r_e, as an alternative minimum spanning tree including e is readily available; equivalently, we should have $\epsilon = 0$ in part (i). If $\partial^r_e > 0$, it is the maximum increase in λ^r_e (*i.e.* decrease in the cost of edge e in the x subproblem) before $\tilde{x}$ becomes optimal and z starts to decrease. That is, enforcing (17) yields

$$\epsilon \leq \partial^r_e\ . \tag{19}$$

(iii.) For y^r to no longer be the unique optimum in (18), the cost of vertex e increases so much that an alternative fixed cardinality stable set $\tilde{y} \in \mathcal{F}_{\text{kstab}}(\hat{G}, |V|-1)$ which does not include e is determined.
Analogous to the situation in part (ii), $\tilde{y}$ is well-defined because the multipliers corresponding to all other vertices are kept constant: choosing $\tilde{y}$ amounts to finding a minimum cost fixed cardinality stable set in $\hat{G} - e$. Also, its cost is no less than that of y^r, the existing optimal solution to the y subproblem. The difference is exactly Δ^r_e in (14).
If $\Delta^r_e = 0$, no bound improvement by changing λ^r_e is possible, as an alternative fixed cardinality stable set of least cost not including e is readily available; *i.e.* we should have $\epsilon = 0$ in part (i). On the other hand, if $\Delta^r_e > 0$, it is the maximum increase in λ^r_e before $\tilde{y}$ becomes optimal and z stops increasing. That is, enforcing (18) yields

$$\epsilon \leq \Delta^r_e\ . \tag{20}$$

(iv.) In conclusion, if $\min\left\{\Delta^r_e, \partial^r_e\right\} = 0$, then $\epsilon = 0$ and $\epsilon\mathfrak{e}_e$ fails to be a direction of increase from $z(\lambda^r)$. Otherwise, we combine bounds (19) and (20) into (16):

$$\forall \epsilon > 0,\ z(\lambda^r + \min\left\{\Delta^r_e, \partial^r_e\right\} \cdot \mathfrak{e}_e) \geq z(\lambda^r + \epsilon\mathfrak{e}_e),$$

showing that $\lambda^r + \min\left\{\Delta^r_e, \partial^r_e\right\} \cdot \mathfrak{e}_e$ is a breakpoint of z, and $\min\left\{\Delta^r_e, \partial^r_e\right\} \cdot \mathfrak{e}_e$ is a maximal ascent direction.

□

To determine a minimum spanning tree with edge $e = \{i, j\}$ fixed *a priori* in part (ii), we may *contract* that edge in G. If the contraction operator is defined so as to allow parallel edges between the new vertex ij and $k \in N(i) \cap N(j)$, where $N(u) \subset V$ denotes the neighbourhood of vertex u, we must ensure that not more than one edge between two vertices is chosen (*e.g.* in Kruskal's algorithm; this is not an issue in Prim's method). Now, if the contraction operator forbids parallel edges, we make an unambiguous choice in the original graph G by recognizing the proper edge ($\{i, k\}$ or $\{j, k\}$) yielding the correct spanning tree.

A maximal ascent direction from Lagrangean solutions where $x^r_e = 1$ but $y^r_e = 0$ is derived by an argument analogous to that of Theorem 4.2. The next proof if thus significantly streamlined.

Theorem 4.3. *Let (x^r, y^r) be an optimal solution of subproblem $z(\lambda^r)$, such that $x^r_e = 1 > 0 = y^r_e$. Define the non-negative quantities*

$$\Delta^r_e \stackrel{def}{=} \min\left\{\lambda^r y : y \in \mathcal{F}_{kstab}(\hat{G}, |V|-1), y_e = 1\right\} - \lambda^r y^r, \tag{21}$$

$$\partial^r_e \stackrel{def}{=} \min\left\{(w-\lambda^r)x : x \in \mathcal{F}_{sp.tree}(G), x_e = 0\right\} - \left(w - \lambda^r\right) x^r. \tag{22}$$

If $\min\left\{\Delta^r_e, \partial^r_e\right\} \neq 0$*, then* $\min\left\{\Delta^r_e, \partial^r_e\right\} \cdot (-\mathfrak{e}_e)$ *is a maximal ascent direction of z at λ^r.*

Proof. Decreasing λ^r_e corresponds to increasing the dual bound, in this case. Hence, $\epsilon(-\mathfrak{e}_e)$ is a direction of increase from $z(\lambda^r)$, as long as ϵ can be made positive in

$$z\big(\lambda^r + \epsilon(-\mathfrak{e}_e)\big) > z\big(\lambda^r\big), \tag{23}$$

where

$$x^r \in \arg\min\left\{\left[w - \big(\lambda^r + \epsilon(-\mathfrak{e}_e)\big)\right]x : x \in \mathcal{F}_{\text{sp.tree}}(G)\right\}, \tag{24}$$

$$y^r \in \arg\min\left\{\left[\lambda^r + \epsilon(-\mathfrak{e}_e)\right]y : y \in \mathcal{F}_{\text{kstab}}(\hat{G}, |V|-1)\right\}. \tag{25}$$

For y^r to no longer be the unique optimum in (25), the cost of vertex e decreases enough for an alternative solution including e to be determined. Since all other multipliers are kept constant, such point $\tilde{y} \in \mathcal{F}_{\text{kstab}}(\hat{G}, |V|-1)$ corresponds to a minimum cost stable set of cardinality $|V| - 2$ in $\hat{G} - N[e]$, that is, the conflict graph where the closed neighbourhood of vertex e is removed. As the existing solution is such that $y_e^r = 0$, the cost of $\tilde{y}$ is no less than that of y^r. The difference is precisely Δ_e^r in (21).

Now, for x^r to no longer be the unique optimum in (24), the cost of edge e increases as far as determining an alternative minimum spanning tree not including e. Let $\tilde{x} \in \mathcal{F}_{\text{sp.tree}}(G)$ denote that point, which corresponds to a minimum spanning tree in $G - e$, since all other multipliers are held constant. The cost of $\tilde{x}$ is no less than that of x^r, the existing optimal solution to the x subproblem. The difference is exactly ∂_e^r in (22).

If $\min\left\{\Delta_e^r, \partial_e^r\right\} = 0$, then $\epsilon = 0$, and $\epsilon(-\mathfrak{e}_e)$ fails to be a direction of increase from $z(\lambda^r)$. Otherwise, we have

$$\forall \epsilon > 0,\ z\big(\lambda^r + \min\left\{\Delta_e^r, \partial_e^r\right\} \cdot (-\mathfrak{e}_e)\big) \geq z\big(\lambda^r + \epsilon(-\mathfrak{e}_e)\big),$$

showing that $\lambda^r + \min\left\{\Delta_e^r, \partial_e^r\right\} \cdot (-\mathfrak{e}_e)$ is a breakpoint of z, and $\min\left\{\Delta_e^r, \partial_e^r\right\} \cdot (-\mathfrak{e}_e)$ is a maximal ascent direction. □

5 CONCLUDING REMARKS

We bring attention to a research question that we consider both attractive and promising. Stable spanning trees comprise appealing combinatorial and polyhedral structures, and designing a Lagrangean algorithm that may yield stronger dual bounds to optimal stable trees is an open problem. This paper presents the first steps in a sensible direction: Lagrangean Decomposition inducing a non-integral relaxation, coupled with carefully designed dual ascent.

Our development relies on the solid foundation that the pioneers of Lagrangean duality in IP have laid, through which we are able to justify the shortcomings of existing approaches and the virtues of the one we propose. We also make an effort for our exposition of the design principle of Lagrangean dual ascent to be fairly tutorial, and for the main proof of the maximal ascent direction to be instructive.

The definitive proof of concept should be actually computing the stronger bounds and finding optimal stable spanning trees in computationally challenging benchmark instances more efficiently. We are currently crafting an implementation of the method outlined in this paper. Regardless of the success of our current efforts and one particular algorithm, we stand in the position put forth at the end of the Introduction. In light of the progress in MILP computation, it seems worthwhile to further investigate the strategy of Lagrangean Decomposition based on harder subproblems, possibly replacing the common sense boundary of weakly NP-hard choices by the weaker requirement that our choice be *computationally tractable*.

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