Total dominator chromatic number of k-subdivision of graphs

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Abstract

Let G be a simple graph. A total dominator coloring of G, is a proper coloring of the vertices of G in which each vertex of the graph is adjacent to every vertex of some color class. The total dominator chromatic (TDC) number $\chi_d^t(G)$ of G, is the minimum number of colors among all total dominator coloring of G. For any $k \in \mathbb{N}$, the k-subdivision of G is a simple graph $G^{\frac{1}{k}}$ which is constructed by replacing each edge of G with a path of length k. In this paper, we study the total dominator chromatic number of k-subdivision of G.

Keywords: Total dominator chromatic number, k-subdivision.

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1 Introduction

In this paper, we are concerned with simple finite graphs. Let $G = (V, E)$ be such a graph and $\lambda \in \mathbb{N}$. A mapping $f \colon V(G) \longrightarrow \{1, 2, ..., \lambda\}$ is called a λ-*proper coloring* of G, if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G. A *color class* of this coloring, is a set consisting of all those vertices assigned the same color. If f is a proper coloring of G with the coloring classes $V_1, V_2, ..., V_\lambda$ such that every vertex in V_i has color i, then

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sometimes write simply $f = (V_1, V_2, ..., V_\lambda)$. The *chromatic number* $\chi(G)$ of G is the minimum number of colors needed in a proper coloring of a graph. The concept of a graph coloring and chromatic number is very well-studied in graph theory. The total dominator coloring, abbreviated TD-coloring studied in [4, 5, 6, 7]. Let G be a graph with no isolated vertex, the *total dominator coloring* is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some (other) color class. The *total dominator chromatic number*, abbreviated TDC-number, $\chi_d^t(G)$ of G is the minimum number of color classes in a TD-coloring of G. A set $D \subset V$ is a total dominating set if every vertex of V is adjacent to some vertices of D. The total dominating number $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G.

Computation of the TDC-number is NP-complete ([5]). The TDC-number of some graphs, such as paths, cycles, wheels and the complement of paths and cycles has computed in [5]. Also Henning in [2] established the lower and the upper bounds on the TDC-number of a graph in terms of its total domination number $\gamma_t(G)$. He has shown that, every graph G with no isolated vertex satisfies $\gamma_t(G) \leq \chi_d^t(G) \leq \gamma_t(G) + \chi(G)$. The properties of TD-colorings in trees has studied in [2, 5]. Trees T with $\gamma_t(T) = \chi_d^t(T)$ has characterized in [2]. We have examined the effects on $\chi_d^t(G)$ when G is modified by operations on vertex and edge of G in [1].

The *k-subdivision* of G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge $v_i v_j$ of G with a path of length k., say $P_{\{v_i,v_j\}}$. These k-paths are called *superedges*, any new vertex is an internal vertex, and is denoted by $x_l^{\{v_i,v_j\}}$ if it belongs to the superedge $P_{\{v_i,v_j\}}$, $i < j$ with distance l from the vertex v_i, where $l \in \{1, 2, \dots, k-1\}$. Note that for $k = 1$, we have $G^{1/1} = G^1 = G$, and if the graph G has v vertices and e edges, then the graph $G^{\frac{1}{k}}$ has $v + (k-1)e$ vertices and ke edges.

In this paper, we study the TDC-number of the k-subdivision of a graph.

2   Main results

We start by proposing upper and lower bounds for the TDC-number of the k-subdivision of a graph. First we need the TDC-number of path graph. Note that the value of TDC-number of paths and cycles which have computed in [5] are lower and upper bounds for $\chi_d^t(P_n)$, $\chi_d^t(C_n)$ and are not the exact value. For example by formula in [5], $\chi_d^t(P_{60}) = 40$ which is not true and the correct value is 32 which can obtain by the following theorem.

Theorem 2.1. *If P_n is the path graph of order $n \geq 8$, then*

$$\chi_d^t(P_n) = \begin{cases} 2k + 2 & \text{if } n = 4k, \\ 2k + 3 & \text{if } n = 4k + 1, \\ 2k + 4 & \text{if } n = 4k + 2, \, n = 4k + 3. \end{cases}$$

Also $\chi_d^t(P_3) = 2$, $\chi_d^t(P_4) = 3$, $\chi_d^t(P_5) = \chi_d^t(P_6) = 4$ and $\chi_d^t(P_7) = 5$.

Proof. It is easy to show that $\chi_d^t(P_3) = 2$, $\chi_d^t(P_4) = \chi_d^t(P_5) = 3$, $\chi_d^t(P_6) = 4$ and $\chi_d^t(P_7) = 5$. Now let $n \geq 8$. First we show that for each four consecutive vertices we have to use at least two new colors. Consider Figure 1. We have two cases. If we give an old color to v_{i+1}, then we need to give a new color to v_{i+2} and v_{i+3} to have a TD-coloring. Also if we give a new color to v_{i+1}, then we have to give a new color to v_{i+2} or v_i to have a TD-coloring. So we need at least two new colors in every four consecutive vertices.

$$v_i \qquad v_{i+1} \qquad v_{i+2} \qquad v_{i+3}$$
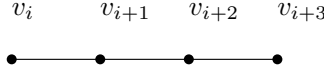
Figure 1: Four consecutive vertices of the Path graph P_n.

Suppose that $n = 4k$, for some $k \in \mathbb{N}$. We give a TD-coloring for the path P_{4k} which use only two new colors in every four consecutive vertices. Define a function f_0 on the vertices of P_{4k}, i.e., $V(P_{4k})$ such that for any vertex v_i,

$$f_0(v_i) = \begin{cases} 1 & \text{if } i = 1 + 4k, \\ 2 & \text{if } i = 4k, \end{cases}$$

and for any v_i, $i \neq 4k$ and $i \neq 4k + 1$, $f_0(v_i)$ is a new number. Then f_0 is a TD-coloring of P_{4k} with the minimum number $2k + 2$.

If $n = 4k + 1$, for some $k \in \mathbb{N}$, then first color the $4k - 4$ vertices using f_0. Now for the rest of vertices define $f_1(v_{4k-3}) = 1$, $f_1(v_{4k-2}) = 2k + 1$, $f_1(v_{4k-1}) = 2k + 2$, $f_1(v_{4k}) = 2k + 3$ and $f_1(v_{4k+1}) = 2$. Since for every five consecutive vertices we have to use at least three new colors, so f_1 is a TD-coloring of P_{4k+1} with the minimum number $2k + 3$.

If $n = 4k + 2$, for some $k \in \mathbb{N}$, then first color the $4k - 4$ vertices using f_0. Now for the rest of vertices define $f_2(v_{4k-3}) = 1$, $f_2(v_{4k-2}) = 2k + 1$, $f_2(v_{4k-1}) = 2k + 2$, $f_2(v_{4k}) = 2k+3$, $f_2(v_{4k+1}) = 2k+4$ and $f_2(v_{4k+2}) = 2$. Since for every six consecutive vertices we have to use at least four new colors, so f_2 is a TD-coloring of P_{4k+2} with the minimum number $2k + 4$.

If $n = 4k + 3$, for some $k \in \mathbb{N}$, then first color the $4k - 4$ vertices using f_0. Now for the rest of vertices define $f_3(v_{4k-3}) = 1$, $f_3(v_{4k-2}) = 2k + 1$, $f_3(v_{4k-1}) = 2k + 2$, $f_3(v_{4k}) = 2$, $f_3(v_{4k+1}) = 2k + 3$, $f_3(v_{4k+2}) = 2k + 4$ and $f_3(v_{4k+2}) = 2$. Then f_3 is a TD-coloring of P_{4k+2} with the minimum number $2k+4$. Therefore we have the result. $\qquad\square$

Theorem 2.2. *If G is a connected graph with m edges and $k \geq 2$, then*

$$\chi_d^t(P_{k+1}) \leq \chi_d^t(G^{\frac{1}{k}}) \leq (m-1)\chi_d^t(P_k) + \chi_d^t(P_{k+1}).$$

Proof. For the right inequality, let $e = uu_1$ be an arbitrary edge of G. This edge is replaced with the superedge $P^{\{u,u_1\}}$ in $G^{\frac{1}{k}}$, with vertices $\{u, x_1^{\{u,u_1\}}, \ldots, x_{k-1}^{\{u,u_1\}}, u_1\}$. We color this superedge with $\chi_d^t(P_{k+1})$ colors as a total dominator coloring of P_{k+1} (Theorem 2.1). If $N_G(u) = \{u_1, \ldots, u_s\}$ then we color the vertices of paths $P^{\{u,u_i\}}$ such that

1. The color of u in $P^{\{u,u_i\}}$, for any $2 \leq i \leq s$, is the same as color u in total dominator coloring of $P^{\{u,u_1\}}$.

2. The superedges $P^{\{u,u_i\}}$, for any $1 \leq i \leq s$, have been colored as a total dominator coloring of P_{k+1} such that for any $i, i' \in \{1, \ldots, s\}$, (note that $c(y)$ is the color of vertex y in our coloring)

$$\left(\bigcup_{j=1}^{k-1} c(x_j^{\{u,u_i\}}) \cup c(u_i) \right) \cap \left(\bigcup_{j=1}^{k-1} c(x_j^{\{u,u_{i'}\}}) \cup c(u_{i'}) \right) = \emptyset, \text{ where } i \neq i'.$$

Thus we need at most $(s-1)\chi_d^t(P_k) + \chi_d^t(P_{k+1})$ colors for such coloring of vertices of superedges $P^{\{u,u_i\}}$, $1 \le i \le s$. Note that we need at most $\chi_d^t(P_k)$ new colors for a TD-coloring of $P^{\{u,u_i\}}$, since the vertex u has been colored in all superedges $P^{\{u,u_i\}}$, $1 \le i \le s$. We do not use the colors used for superedges $P^{\{u,u_i\}}$, $1 \le i \le s$, any more. In the next step, we consider that superedges in $G^{\frac{1}{k}}$ which are replaced instead of incident edges to u_i's in G, and have not been colored in the prior step. Now we color the vertices of these superedges as a TD-coloring of P_k, such that the vertices $u_2, \ldots, u_s$ have been colored in prior step, and the pairwise intersection of the set of colors used for coloring of vertices of these superedges is the empty set. We continue this process to color all vertices of $G^{\frac{1}{k}}$. This coloring is a TD-coloring of $G^{\frac{1}{k}}$, because every superedge have been colored with distinct color set, except the end vertices of the superedges, possibly. Finally, since we used at most $(m-1)\chi_d^t(P_k) + \chi_d^t(P_{k+1})$ colors, the right inequality follows.

For the left inequality, if G is a path then the result is true. So we suppose that G is a connected graph which is not a path. Let $P^{\{v,w\}}$ be an arbitrary superedge of $G^{\frac{1}{k}}$ with vertex set $\{v, x_1^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}, w\}$. Since G is not a path, so at least one of v and w is adjacent to some vertices of $G^{\frac{1}{k}}$ except $x_1^{\{v,w\}}$ and $x_{k-1}^{\{v,w\}}$, respectively. Let c be a total dominator coloring of $G^{\frac{1}{k}}$. The two following cases can be occured: either the restriction of c to vertices of $P^{\{v,w\}}$ is a total dominator coloring and so we have the result, or not. If the restriction of c to vertices of $P^{\{v,w\}}$ is not a total dominator coloring then since c is a total dominator coloring of $G^{\frac{1}{k}}$ we conclude that at least one of vertices v and w, as the vertices of the induced subgraph $P^{\{v,w\}}$, are not adjacent to every vertex of some color class. Without loss of generality we assume that the vertex v, as the vertex of the induced subgraph $P^{\{v,w\}}$, is not adjacent to every vertex of some color class. But c is a total dominator coloring of $G^{\frac{1}{k}}$ so the vertex v is adjacent to every vertex of some color class, as the vertex of $G^{\frac{1}{k}}$. Hence there is a new color for an adjacent vertex of v, except the vertex $x_1^{\{v,w\}}$. Thus if we use this new color for the vertex $x_1^{\{v,w\}}$ and consider the restriction of c for the remaining vertices of superedge $P^{\{v,w\}}$, then $P^{\{v,w\}}$ has a total dominator coloring. Therefore the total coloring c has at least $\chi_d^t(P_{k+1})$ colors. $\square$

By the following Proposition we show that the upper bound of $\chi_d^t(G^{\frac{1}{k}})$ in Theorem 2.2 is sharp for $G = K_{1,n}$ and $k = 3$.

Proposition 2.3. *For every $n \ge 3$, $\chi_d^t(K_{1,n}^{\frac{1}{3}}) = 2n + 1$.*

Proof. Let $p_1, \ldots, p_n$ be the pendant vertices of $K_{1,n}^{\frac{1}{3}}$. The adjacent vertex to p_i is denoted by q_i, and the adjacent vertex to q_i of degree 2 is denoted by w_i for any $1 \le i \le n$. The center of $K_{1,n}^{\frac{1}{4}}$ is denoted by v. Since the vertex q_i is the only vertex adjacent to p_i, so the color of q_i should not be used for any other vertices of graph, where $1 \le i \le n$. Thus we color the vertices $q_1, \ldots, q_n$ with colors $1, \ldots, n$, respectively, and do not use these colors any more. For every $1 \le i \le n$, the vertex q_i is adjacent to p_i and w_i, thus we need a new color for at least one of p_i and w_i. Here we consider the three following cases:

Case 1: The vertices $p_1, \ldots, p_n$ have been colored with colors $n+1, n+2, \ldots, 2n$, respectively, and these colors do not use any more. In this case we must color the vertices $w_1, \ldots, w_n$ with a new color, say color $2n+1$. Since we need a proper coloring, we color the vertex v with color $2n+2$. So it can be seen that we have a TD-coloring of $K_{1,n}^{\frac{1}{3}}$ with $2n+2$ colors.

Case 2: The vertices $w_1, \ldots, w_n$ have been colored with colors $n+1, n+2, \ldots, 2n$, respectively, and these colors do not use any more. Since we need a proper coloring, we must color the vertex v with a new color. If we color all vertices $v, p_1, \ldots, p_n$ with color $2n+1$ then we have a TD-coloring of $K_{1,n}^{\frac{1}{3}}$ with $2n+1$ colors.

Case 3: The vertices w_i, p_i have been colored with color $n+i$, for any i $1 \leq i \leq n$ and these colors do not use any more. Thus we need a new color for the vertex v, say color $2n+1$. Now we have a TD-coloring of $K_{1,n}^{\frac{1}{3}}$ with $2n+1$ colors.

Therefore $\chi_d^t(K_{1,n}^{\frac{1}{3}}) = 2n+1$, by Cases 1, 2 and 3.

Here we improve the lower bound of Theorem 2.2 for $k \geq 9$.

Theorem 2.4. *If G is a connected graph with m edges and $k \geq 9$, then*

$$m(\chi_d^t(P_{k-1}) - 2) + 2 \leq \chi_d^t(G^{\frac{1}{k}}).$$

Proof. Let $e = vw$ be an edge of G. We consider the superedge $P^{\{v,w\}}$ with vertex set $\{v, x_1^{\{v,w\}}, \ldots, x_{k-1}^{\{v,w\}}, w\}$. It is clear that $P^{\{v,w\}} \setminus \{v, w\}$ is the path graph P_{k-1}. Since we use repetitious colors for the vertices $x_1^{\{v,w\}}$ and $x_{k-1}^{\{v,w\}}$ in the TD-coloring of paths, so we need at least $\chi_d^t(P_{k-1}) - 2$ colors for each superedges, and hence the result follows. □

Theorem 2.5. *If G is a connected graph with m edges and $k \geq 9$, then*

$$\chi_d^t(G^{\frac{1}{k}}) \geq \begin{cases} \frac{mk}{2} + 2 & k \equiv 0 \ (mod \ 4) \\ m(\frac{k-1}{2}) + 2 & k \equiv 1 \ (mod \ 4) \\ m(\frac{k-2}{2} + 1) + 2 & k \equiv 2 \ (mod \ 4) \\ m(\frac{k-3}{2} + 2) + 2 & k \equiv 3 \ (mod \ 4). \end{cases}$$

Proof. It follows by Theorems 2.1 and 2.4. □

Theorem 2.6. *If G is a connected graph with m edges and $k \geq 7$, then*

$$\chi_d^t(G^{\frac{1}{k}}) \leq m(\chi_d^t(P_{k+1}) - 2) + 2.$$

Proof. As we have seen in TD-coloring of paths, we can use the same color for the pendant vertices. So we give the color 1 or 2 to all the vertices belong to G and we color other vertices of any superedges with $\chi_d^t(P_{k+1}) - 2$ colors. This is a TD-coloring for $G^{\frac{1}{k}}$ and hence the result follows. □

Theorem 2.7. *If G is a connected graph with m edges and $k \geq 9$, then*

$$\chi_d^t(G^{\frac{1}{k}}) \leq \begin{cases} m(\frac{k}{2} + 1) + 2 & k \equiv 0 \ (mod \ 4) \\ m(\frac{k-1}{2} + 2) + 2 & k \equiv 1 \ (mod \ 4) \\ m(\frac{k-2}{2} + 2) + 2 & k \equiv 2 \ (mod \ 4) \\ m(\frac{k-3}{2} + 2) + 2 & k \equiv 3 \ (mod \ 4). \end{cases}$$

Proof. It follows by Theorems 2.1 and 2.6. □

Iradmusa [3], showed that if G is a connected graph and k is a positive integer greater than one, then at most three colors are enough to achieve a proper coloring of $G^{\frac{1}{k}}$.

Theorem 2.8. *If G is a connected graph and k be a positive integer number greater than one, then $\gamma_t(G^{\frac{1}{k}}) \leq \chi_d^t(G^{\frac{1}{k}}) \leq \gamma_t(G^{\frac{1}{k}}) + 2$.*

Proof. If G is a connected graph and k is a positive integer greater than one, then at most three colors are enough to achieve a proper coloring of $G^{\frac{1}{k}}$. In continue we want to show that there is no graph with $\chi_d^t(G^{\frac{1}{k}}) = \gamma_t(G^{\frac{1}{k}}) + 3$. Let $\gamma_t(G^{\frac{1}{k}}) = s$ and $\Gamma = \{y_1, \ldots, y_s\}$ be the total dominating set of $G^{\frac{1}{k}}$. We color the vertices $y_1, \ldots, y_s$ with colors $1, \ldots, s$, respectively, and do not use these colors any more. Let x be an arbitrary uncolored vertices of $G^{\frac{1}{k}}$ which is adjacent to one of $y_1, \ldots, y_s$. Since $k \geq 2$, so the set of all uncolored adjacent vertices to x, denote by $N_{un-col}(x)$, make an independent set of vertices, since otherwise we find a triangle, i.e., K_3, in $G^{\frac{1}{k}}$, which is a contradiction (note that if G is a graph with girth g, then $G^{\frac{1}{k}}$ is graph with girth gk). If $N_{un-col}(x) = \emptyset$, then we color x with color $s + 1$. If $N_{un-col}(x) \neq \emptyset$, then we suppose that $N_{un-col}(x) = \{x_1, \ldots, x_t\}$ for some $t \geq 1$. Since $k \geq 2$ and $\{x_1, \ldots, x_t\} \cap \{y_1, \ldots, y_s\} = \emptyset$, so each of vertices x_i has no uncolored adjacent vertex in $G^{\frac{1}{k}}$, except x, because otherwise $\{y_1, \ldots, y_s\}$ is not a total dominating set. In this case we color the vertex x with color $s + 1$, and all vertices $x_1, \ldots, x_t$ with color $s + 2$. It can be seen that we have a TD-coloring of $G^{\frac{1}{k}}$. By the above argument, we conclude that $\chi_d^t(G^{\frac{1}{k}}) \leq \gamma_t(G^{\frac{1}{k}}) + 2$, for any $k \geq 2$. $\square$

The following proposition gives the exact value TDC-number of 4-subdivision of stars graph.

Proposition 2.9. *For every $n \geq 3$, $\chi_d^t(K_{1,n}^{\frac{1}{4}}) = 2n + 2$.*

Proof. Let $p_1, \ldots, p_n$ be the pendant vertices of $K_{1,n}^{\frac{1}{4}}$. The adjacent vertex to p_i is denoted by q_i, and the adjacent vertex to q_i of degree 2 is denoted by w_i for any $1 \leq i \leq n$. If the center of $K_{1,n}^{\frac{1}{4}}$ is denoted by v then the adjacent vertices of v are denoted by $z_1, \ldots, z_n$ where z_i and w_i are adjacent for any $1 \leq i \leq n$. Since p_i is a vertex of degree one, for any $1 \leq i \leq n$, and p_i must be adjacent to all vertices of a color class, so we must color the vertices $q_1, \ldots, q_n$ with different colors, say $1, \ldots, n$, respectively, and we should not use these colors any more. Now we consider the vertices $q_1, \ldots, q_n$. These vertices have exactly two adjacent vertices, w_i's and p_i's. To have a TD-coloring, at least one of these two vertices, w_i and p_i, must be colored with a new color, for every $1 \leq i \leq n$, such that these new colors do not use any more. Hence the three following cases can be occured:

Case 1: The vertices $w_1, \ldots, w_n$ have been colored with colors $n + 1, n + 2, \ldots, 2n$, respectively, and these colors do not use any more. It is clear that we need at least two colors for coloring of vertices $v, z_1, \ldots, z_n$, since the vertex v is adjacent to vertices $z_1, \ldots, z_n$. If we label all vertices $z_1, \ldots, z_n$ with color $2n + 1$, and the vertices $v, p_1, \ldots, p_n$ with color $2n + 2$, then it can be seen that we have a TD-coloring of $K_{1,n}^{\frac{1}{4}}$ with $2n + 2$ colors.

Case 2: The vertices $p_1, \ldots, p_n$ have been colored with colors $n + 1, n + 2, \ldots, 2n$, respectively, and these colors do not use any more. In this case we consider the vertices $z_1, \ldots, z_n$. Each of z_i, $1 \leq i \leq n$, has two adjacent vertices, v and w_i. Since z_i must be adjacent to all vertices of some color class, so at least one of w_i

and v must be colored with a new color such that this color do not use any more. Since v is the common neighbors of all vertices $z_1, \ldots, z_n$, and we want to use the minimum number of colors, so we color the vertex v with color $2n+1$, and do not use this color any more. Since our coloring must be proper, we need at least two new colors for coloring of the remaining uncolored vertices: one color, say $2n+2$, for vertices $z_1, \ldots, z_n$, and another color for coloring of all vertices $w_1, \ldots, w_n$, say color $2n+3$. Now we have a TD-coloring of $K_{1,n}^{\frac{1}{4}}$ with $2n+3$ colors.

Case 3: The vertices w_i, p_i have been colored with color $n+i$, for any i ($1 \leq i \leq n$) and these colors do not use any more. Since our coloring is proper so we need at least two new colors for coloring of vertices v and $z_1, \ldots, z_n$. If we color the vertex v with color $2n+1$, and all vertices $z_1, \ldots, z_n$ with color $2n+2$, then it can be seen that we have a TD-coloring of $K_{1,n}^{\frac{1}{4}}$ with $2n+2$ colors.

Therefore $\chi_d^t(K_{1,n}^{\frac{1}{4}}) = 2n+2$, by Cases 1, 2 and 3. $\qquad\qquad\square$

Theorem 2.10. *For any $k \geq 3$, $\chi_d^t(G^{\frac{1}{k}}) \leq \chi_d^t(G^{\frac{1}{k+1}})$.*

Proof. First we give a TD-coloring to the vertices of $G^{\frac{1}{k+1}}$. Let $P^{\{v,w\}}$ be an arbitrary superedge of $G^{\frac{1}{k+1}}$ with vertex set $\{v, x_1^{\{v,w\}}, \ldots, x_k^{\{v,w\}}, w\}$. We have the following cases:

Case 1: There exists a vertex $u \in \{x_1^{\{v,w\}}, \ldots, x_k^{\{v,w\}}\}$ such that other vertices of graph are not adjacent to all vertices with color class of vertex u. Consider the graph in Figure 2. Suppose that the vertex u has the color i and the vertex n has color α. The vertex m is adjacent to all vertices with color class j and $j \neq i$ and the vertex n is adjacent to all vertices with color class k and $k \neq i$. Since $k \geq 3$, without loss of generality, suppose that $m \neq v$. We have two cases:

  Case i: The color of the vertex m is not α. We omit the vertex u and put an edge between n and m. So without adding a new color we have a TD-coloring for this new graph.

  Case ii: The color of the vertex m is α. Since the vertex u is adjacent to color class α, so any other vertices does not have color α. In this case, by removing the vertex m and putting an edge between u and t, we have a TD-coloring for this new graph. Because the vertex t is adjacent to color class which is not α, the color of t is not i (because of our assumptions), the vertex n is adjacent to all vertices with color class k and the vertex u is adjacent to all vertices with color class α.

Case 2: For every vertex $u \in \{x_1^{\{v,w\}}, \ldots, x_k^{\{v,w\}}\}$, there exists a vertex such that is adjacent to all vertices with color of vertex u. Consider the graph in Figure 2. Suppose that the vertex u has the color i and the vertex p has color j and the vertex p is adjacent to all vertices with color i. we have two cases:

  Case i: The color of vertex q is not i. We omit the vertex r and put an edge between u and s. So without adding a new color we have a TD-coloring for this new graph, since there is no other vertex with color i.

Case ii: The color of the vertex q is i. In this case the vertex r is adjacent to color class of vertex s and the color of vertex s does not use for other vertices. Now we omit the vertex u and put an edge between r and p. If the color of r is j, then we change it to i and obviously since the vertex s was adjacent to a color class except j, so we have a TD-coloring. If the color of the vertex r is not j, then we do not change its color and so we have a TD-coloring again.

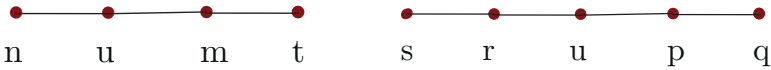Now we give the same TD-coloring for all superedges. So we have the result. □

Figure 2: A part of a superedge in the proof of Theorem 2.10, respectively.

Here we prove that Theorem 2.10 is true for $k = 2$, too.

Theorem 2.11. *For any graph G, $\chi_d^t(G^{\frac{1}{2}}) \leq \chi_d^t(G^{\frac{1}{3}})$.*

Proof. First we give a TD-coloring to the vertices of $G^{\frac{1}{3}}$. Let $P^{\{s,w\}}$ be an arbitrary superedge of $G^{\frac{1}{3}}$ with vertex set $\{s, x_1^{\{s,w\}} = v, x_2^{\{s,w\}} = u, w\}$ (see Figure 3) and suppose that the vertex v has the color α. We have the following cases:

Case 1: The vertices u and s are adjacent with a vertex with a color class which is not α. we have two cases:

Case i: The color of vertices u and s are different. In this case, by removing vertex v and put an edge between u and s, we have a TD-coloring for this new graph. Because two vertices u and s are adjacent with a vertex with color class which is not α.

Case ii: The color of vertices u and s are the same. Suppose that u and s have color β. In this case β does not use for any other vertices. So w is adjacent with a vertex with color class except β. Now we remove vertex u and put an edge between v and w. So we have a TD-coloring for this new graph.

Case 2: The vertex u is adjacent to all vertices with color class α. we have two cases:

Case i: The color of the vertex w is not α. Suppose that the vertex u has color γ. If the vertex v is adjacent with all vertices with color γ, and if the color of s is γ, we remove the vertex u and put an edge between v and w. But if the color of vertex s is not γ, then we remove the vertex u, put an edge between v and w and give the color γ to the vertex w. So we have a TD-coloring for this new graph. If the vertex v is adjacent to all vertices

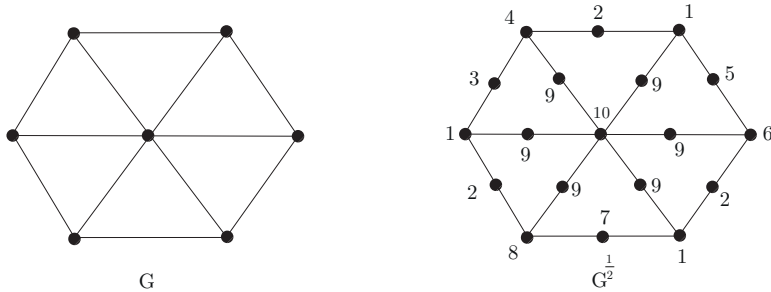Figure 3: A superedge in $G^{\frac{1}{3}}$.

Figure 4: An example which shows that Theorem 2.12 is not true for $k = 2$.

whose have a color except γ (vertex s), then we remove the vertex u and put an edge between v and w. So we have a TD-coloring for this new graph.

Case ii: The color of the vertex w is α. We have two new cases. First, the vertex v is adjacent to a vertex with color class γ. Any adjacent vertex with w is not adjacent to vertex with color class α (except u). So we remove the vertex u, put an edge between v and w and give the color γ to w. This is a TD-coloring for this new graph. Second, v is not adjacent with color class γ. So the color of the vertex s does not use any more. Also the vertex s is not adjacent to vertex with color class α. So we remove vertex v and put an edge between s and u. This is a TD-coloring for this new graph.

Case 3: The vertex s is adjacent to all vertices with color class α. We have two cases:

Case i: If v is the only vertex which has color α, then we remove the vertex u and put an edge between v and w. This is a TD-coloring for this new graph.

Case ii: If there exist some vertices with color α, then the vertex u is adjacent with color class except α. So we remove v and put an edge between s and u. This is a TD-coloring for this new graph.

We apply this TD-coloring for all superedges. So we obtain a TD-coloring for $G^{\frac{1}{2}}$. Therefore we have $\chi_d^t(G^{\frac{1}{2}}) \leq \chi_d^t(G^{\frac{1}{3}})$. $\square$

We end the paper with the following theorem:

Theorem 2.12. *If G is a graph with m edges, then $\chi_d^t(G^{\frac{1}{k}}) \geq m$, for $k \geq 4$.*

Proof. For $k = 4$, in any superedge $P^{\{v,w\}}$ such as $\{v, x_1^{\{v,w\}}, x_2^{\{v,w\}}, x_3^{\{v,w\}}, w\}$ without considering vertices v and w, there is a path graph of length two. The vertex $x_2^{\{v,w\}}$ need to use a new color in at least one of its adjacent vertices, and we cannot use this color in any other superedges. So we have the result. $\square$

Note that Theorem 2.12 is not true for $k = 2$. As an example, for the graph G of size 12 in Figure 4 ,we have $\chi_d^t(G^{\frac{1}{2}}) \not\geq 12$. Also consider the graph G of size 18 in Figure 5. For this graph we have $\chi_d^t(G^{\frac{1}{3}}) \not\geq 18$ which show that Theorem 2.12 is not true for $k = 3$.
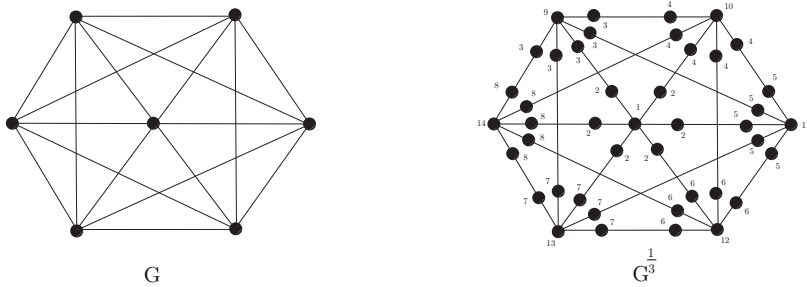
Figure 5: An example which show that Theorem 2.12 is not true for $k = 3$.

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