Classes of Intersection Digraphs with Good Algorithmic Properties

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Abstract

While intersection graphs play a central role in the algorithmic analysis of hard problems on undirected graphs, the role of intersection *digraphs* in algorithms is much less understood. We present several contributions towards a better understanding of the algorithmic treatment of intersection digraphs. First, we introduce natural classes of intersection digraphs that generalize several classes studied in the literature. Second, we define the directed locally checkable vertex (DLCV) problems, which capture many well-studied problems on digraphs such as (Independent) Dominating Set, Kernel, and H-Homomorphism. Third, we give a new width measure of digraphs, *bi-mim-width*, and show that the DLCV problems are polynomial-time solvable when we are provided a decomposition of small bi-mim-width. Fourth, we show that several classes of intersection digraphs have bounded bi-mim-width, implying that we can solve all DLCV problems on these classes in polynomial time given an intersection representation of the input digraph. We identify reflexivity as a useful condition to obtain intersection digraph classes of bounded bi-mim-width, and therefore to obtain positive algorithmic results.

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1 Introduction

The computational intractability of graph problems is often dealt with by restricting the input graph to be a member of some graph class and exploit the structural properties of this class to design efficient algorithms. Intersection graph classes are an extensively studied family of classes of undirected graphs where vertices are represented by sets with two vertices being adjacent if and only if their corresponding sets intersect. For instance, a graph is an *interval graph* if it is an intersection graph of intervals on a line. The literature on algorithmic aspects of classes of intersection graphs is vast, and we refer to [13] for an overview. Even though the concept of intersection *digraphs* has already been introduced in the early 1980s [9], these classes of directed graphs have not received nearly as much attention in the algorithmic literature as their undirected counterparts. That is not to say that they have not been considered before; for instance, interval digraphs [38], circular-arc digraphs [39], and permutation digraphs [33] have been introduced quite early on.

Formally, a digraph G is an *intersection digraph* if there exists a family $\{(S_v, T_v) : v \in V(G)\}$ of ordered pairs of sets such that there is an edge from v to w in G if and only if S_v intersects T_w. Note that we add a loop on a vertex v if S_v and T_v intersect. Even for interval digraphs, a natural starting point for the investigation of algorithmic properties of intersection digraphs, no algorithmic applications are known besides a polynomial-time recognition algorithm of the class [33]. One possible explanation for this is that the class of interval digraphs appears to be much richer than their undirected counterparts. We observe that interval digraphs contain, for each integer n, some orientation of the $(n \times n)$-grid (see Proposition 15); in contrast, interval graphs do not contain an induced subgraph isomorphic to the 1-subdivision of the claw. This shows that the underlying undirected graphs of interval digraphs are very different from interval graphs.

The case of interval digraphs suggests that further structural restrictions are necessary to make classes of intersection digraphs amenable for algorithmic treatment. In this vein, restrictions of interval digraphs have been considered in the literature [20, 35] with applications to digraph problems such as Independent Dominating Set, Kernel, and List Homomorphism. A common feature of the restrictions considered in [20, 35] is that the digraphs are *reflexive*, meaning that each vertex has a loop. Note that for a class of intersection digraphs, reflexivity gives much more additional structure than just added loops.

In this work, we give a host of algorithmic applications of intersection digraph classes, in the following manner:

- We give new and more general classes of intersection digraphs, namely H-digraphs, rooted directed path digraphs, and H-convex digraphs. (See the discussion below Theorem 3 for definitions.)
- We introduce directed analogues of the locally checkable vertex problems [43], which include many well-studied digraph problems such as (Independent) Dominating Set, Kernel, H-Homomorphism, and Oriented k-Coloring, see Tables 1 and 2.
- We define a new width measure of digraphs, called *bi-mim-width*, and prove that the directed locally checkable vertex problems can be solved in polynomial time when a decomposition of bounded bi-mim-width of the input graph is given.
- We prove that fairly general subclasses of these intersection digraph classes have bounded bi-mim-width, see Figure 1.

Note in particular that the last item implies that given a representation of the input digraph, all directed locally checkable problems are solvable in polynomial time on the classes of intersection digraphs in question. For H-digraphs, we identify reflexivity as the additional restriction that gives bounded bi-mim-width, and therefore algorithmic applications, while we prove that the bi-mim-width is unbounded when we drop this requirement. Recently, Francis, Hell, and Jacob [22] obtained polynomial-time algorithms for Kernel, Dominating Set, and Absorbing Set on reflexive interval digraphs. Our results are more general in two ways: we give algorithms for more problems, including the aforementioned ones (see Tables 1 and 2), and on much broader digraph classes (see Figure 1). Naturally, the specific algorithms presented in [22] are more efficient than the algorithm following from our general framework. In the following, we discuss the above items in more detail.

Bi-mim-width. We introduce a new digraph width parameter, called *bi-mim-width*, which is a directed analogue of the mim-width of an undirected graph introduced by Vatshelle [44]. Roughly speaking, the bi-mim-width of a digraph G is defined as a branch-width with a cut

Table 1 Examples of $(\sigma^+, \sigma^-, \rho^+, \rho^-)$-sets, represented by finite or co-finite sets. For any row there is an associated NP-complete problem, usually maximizing or minimizing the cardinality of a set with the property. Some properties are known under different names; e.g. Efficient Total Dominating sets are also called Efficient Open Dominating sets, and here even the existence of such a set in a digraph G is NP-complete, as it corresponds to deciding if $V(G)$ can be partitioned by the open out-neighborhoods of some $S \subseteq V(G)$. If rows A and B have their in-restrictions and out-restrictions swapped for both σ and ρ (i.e. σ^+ of row A equals σ^- of row B and vice-versa, and same for ρ^+ and ρ^-), then a row-A set in G is always a row-B set in the digraph with all arcs of G reversed; this is the case for Dominating set vs in-Dominating set and for Kernel vs Independent Dominating set.

σ^+	σ^-	ρ^+	ρ^-	Standard name
$\{0\}$	$\{0\}$	$\mathbb{N} \setminus \{0\}$	$\mathbb{N}$	Kernel [45]
$\{0, ..., k-1\}$	$\{0\}$	$\{i : i \geq l\}$	$\mathbb{N}$	(k,l)-out Kernel [36]
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	Dominating set [24]
$\{0\}$	$\{0\}$	$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	Independent Dominating set [16]
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	$\mathbb{N}$	In-Dominating set/Absorbing set [23]
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	$\mathbb{N} \setminus \{0\}$	Twin Dominating set [17]
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$	$\{i : i \geq k\}$	k-Dominating set [34]
$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	$\mathbb{N}$	$\mathbb{N} \setminus \{0\}$	Total Dominating set [2]
$\{0\}$	$\{0\}$	$\mathbb{N}$	$\{1\}$	Efficient (Closed) Dominating set [8]
$\mathbb{N}$	$\{1\}$	$\mathbb{N}$	$\{1\}$	Efficient Total Dominating set [37]
$\{k\}$	$\{k\}$	$\mathbb{N}$	$\mathbb{N}$	k-Regular Induced Subdigraph [15]

function that measures for a vertex partition (A, B) of G, the sum of the sizes of maximum induced matchings in two bipartite digraphs, one induced by edges from A to B, and the other induced by edges from B to A. This is similar to how rank-width is generalized to bi-rank-width for digraphs [29, 30]. We formally define bi-mim-width and linear bi-mim-width in Section 3. We compare bi-mim-width and other known width parameters. The mim-width of an undirected graph is exactly the half of the bi-mim-width of the digraph obtained by replacing each edge with bi-directed edges, and this observation can be used to argue that a bound on the bi-mim-width of a class of digraphs implies a bound on the mim-width of a certain class of undirected graphs.

Directed Locally Checkable Vertex (DLCV) Problems. We introduce directed locally checkable vertex subset (DLCVS) and partitioning (DLCVP) problems, in analogy with [43]. We abbreviate the union of these two families of problems to "DLCV problems". A DLCVS problem is represented as a $(\sigma^+, \sigma^-, \rho^+, \rho^-)$-problem for some $\sigma^+, \sigma^-, \rho^+, \rho^- \subseteq \mathbb{N}$, and it asks to find a maximum or minimum vertex set S in a digraph G such that for every vertex v in S, the numbers of out/in-neighbors in S are contained in σ^+ and σ^-, respectively, and for every vertex v in $V(G) \setminus S$, the numbers of out/in-neighbors in S are contained in ρ^+ and ρ^-, respectively. If each $\mu \in \{\sigma^+, \sigma^-, \rho^+, \rho^-\}$ is either finite or co-finite (i.e., $\mathbb{N} \setminus \mu$ is finite), then we say that the problem is *represented by finite or co-finite sets*. See Table 1 for several examples of DLCVS problems that appear in the literature and note that they are all represented by finite or co-finite sets. In particular, it includes the KERNEL problem, which was introduced by von Neumann and Morgenstern [45].

A DLCVP problem is represented by a $(q \times q)$-matrix D for some positive integer q, where for all $i, j \in \{1, \ldots, q\}$, $D[i, j] = (\mu^+_{i,j}, \mu^-_{i,j})$ for some $\mu^+_{i,j}, \mu^-_{i,j} \subseteq \mathbb{N}$. The problem asks to find a vertex partition of a given digraph into $X_1, X_2, \ldots, X_q$ such that for all $i, j \in [q]$,

Table 2 Examples of directed LCVP problems that are represented by finite or co-finite sets. For every row there are choices of values for which the problems are NP-complete. For Directed H-Homomorphism let $V(H) = \{1, \ldots, |V(H)|\}$ and denote by $H\colon \overrightarrow{K_k}$ that H is an orientation of a complete graph on k vertices, and by $H\colon \overrightarrow{K_k^\circ}$ that H is an orientation of a complete graph on k vertices, with loops. (*) For Simple k-Coloring, we require two nonempty color classes to avoid trivial solutions. The general algorithm can easily be modified to take this into account.

Problem name	q	DLCVP $(q \times q)$-matrix D
Directed H-Homomorphism [25]	$\|V(H)\|$	$\forall(i,j) \in E(H)\colon D[i,j] = (\mathbb{N},\mathbb{N})$ $\forall(i,j) \notin E(H)\colon D[i,j] = (\{0\},\{0\})$
Oriented k-Coloring [18, 42]	k	$\bigvee_{H\colon \overrightarrow{K_k}}$ Directed H-Homomorphism
Simple k-Coloring (*) [40]	k	$\bigvee_{H\colon \overrightarrow{K_k^\circ}}$ Directed H-Homomorphism
$\exists$ $(\sigma^+,\sigma^-,\rho^+,\rho^-)$-set [This paper]	2	$\begin{pmatrix} (\sigma^+,\sigma^-) & (\mathbb{N},\mathbb{N}) \\ (\rho^+,\rho^-) & (\mathbb{N},\mathbb{N}) \end{pmatrix}$
$(\delta^+ \geq k_1, \delta^- \geq k_2)$-Partition [6]	2	$\begin{pmatrix} (\{j\colon j \geq k_1\},\mathbb{N}) & (\mathbb{N},\mathbb{N}) \\ (\mathbb{N},\mathbb{N}) & (\mathbb{N},\{j\colon j \geq k_2\}) \end{pmatrix}$
$(\delta^+ \geq k_1, \delta^+ \geq k_2)$-Partition [5]	2	$\begin{pmatrix} (\{j\colon j \geq k_1\},\mathbb{N}) & (\mathbb{N},\mathbb{N}) \\ (\mathbb{N},\mathbb{N}) & (\{j\colon j \geq k_2\},\mathbb{N}) \end{pmatrix}$
$(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-Partition [3]	2	$\begin{pmatrix} (\{j\colon j \leq k_1\},\mathbb{N}) & (\mathbb{N},\mathbb{N}) \\ (\mathbb{N},\mathbb{N}) & (\{j\colon j \leq k_2\},\mathbb{N}) \end{pmatrix}$
$(\delta^+ \geq k_1, \delta^- \geq k_2)$-Bipartite-Partition [4]	2	$\begin{pmatrix} (\mathbb{N},\mathbb{N}) & (\{j\colon j \geq k_1\},\mathbb{N}) \\ (\mathbb{N},\{j\colon j \geq k_2\}) & (\mathbb{N},\mathbb{N}) \end{pmatrix}$
$(\delta^+ \geq k_1, \delta^+ \geq k_2)$-Bipartite-Partition [4]	2	$\begin{pmatrix} (\mathbb{N},\mathbb{N}) & (\{j\colon j \geq k_1\},\mathbb{N}) \\ (\{j\colon j \geq k_2\},\mathbb{N}) & (\mathbb{N},\mathbb{N}) \end{pmatrix}$
2-Out-Coloring [1]	2	$\begin{pmatrix} (\mathbb{N}\setminus\{0\},\mathbb{N}) & (\mathbb{N}\setminus\{0\},\mathbb{N}) \\ (\mathbb{N}\setminus\{0\},\mathbb{N}) & (\mathbb{N}\setminus\{0\},\mathbb{N}) \end{pmatrix}$

the numbers of out/in-neighbors of a vertex of X_i in X_j are contained in $\mu^+_{i,j}$ and $\mu^-_{i,j}$, respectively. In analogy with subset problems, we say that the problem is *represented by finite or co-finite sets* if each set appearing in a pair that is an entry of D is either finite or co-finite. Directed H-Homomorphism is a directed LCVP problem represented by finite or co-finite sets: For a digraph H on vertices $\{1, \ldots, q\}$, we can view a homomorphism from a digraph G to H as a q-partition $(X_1, \ldots, X_q)$ of $V(G)$ such that we can only have an edge from X_i to X_j if the edge (i, j) is present in H. See Table 2. The Oriented k-Coloring problem, introduced by Sopena [41], asks whether there is a homomorphism to some orientation of a complete graph on at most k vertices, and can therefore be reduced to a series of directed LCVP problems. Removing the requirement that the color classes have to be independent sets, Smolíková [40] introduced the notion of a *simple k-coloring*, requiring however that the number of colors is at least two, to avoid trivial solutions. Several works in the literature concern problems of 2-partitioning the vertex sets of digraphs into parts with degree constraints either inside or between the parts of the partition [1, 3, 4, 5, 6]. All of these problems can be observed to be LCVP problems as well, see Table 2. Note that in the DLCVP-framework, we can consider q-partitions for any fixed $q \geq 2$, for all problems apart from 2-Out-Coloring. This fails for q-Out-Coloring, since this problem asks for a q-coloring with no monochromatic out-neighborhood.

▶ **Theorem 1.** *Directed LCVS and LCVP problems represented by finite or co-finite sets can be solved in time XP parameterized by bi-mim-width, when a branch decomposition is given.*

Furthermore, we show that the distance variants of DLCVS problems, for instance Distance-r Dominating Set can be solved in polynomial time on digraphs of bounded bi-mim-width. Another natural variant is the k-Kernel problem (see [7, Section 8.6.2]), which asks for a kernel in the $(k-1)$-th power of a given digraph. To show this, we prove that the r-th power of a digraph of bi-mim-width w has bi-mim-width at most rw (Lemma 12). For undirected graphs, there is a bound that does not depend on r [26], but we were not able to obtain such a bound for the directed case.

▶ **Theorem 2.** *Distance variants of directed LCVS problems represented by finite or co-finite sets can be solved in time XP parameterized by bi-mim-width, when a branch decomposition is given.*

Classes of intersection digraphs and their bi-mim-width. We provide various classes of digraphs of bounded bi-mim-width. We first summarize our results in the following theorem and give the background below. We illustrate the bounds in Figure 1.

▶ **Theorem 3.**

1. *Given a reflexive interval digraph, one can output a linear branch decomposition of bi-mim-width at most* 2 *in polynomial time. On the other hand, interval digraphs have unbounded bi-mim-width.*
2. *Given a representation of an adjusted permutation digraph G, one can construct in polynomial time a linear branch decomposition of G of bi-mim-width at most* 4*. Permutation digraphs have unbounded bi-mim-width.*
3. *Given a representation of an adjusted rooted directed path digraph G, one can construct in polynomial time a branch decomposition of G of bi-mim-width at most* 2*. Rooted directed path digraphs have unbounded bi-mim-width and adjusted rooted directed path digraphs have unbounded linear bi-mim-width.*
4. *Let H be an undirected graph. Given a representation of a reflexive H-digraph G, one can construct in polynomial time a linear branch decomposition of G of bi-mim-width at most* $12|E(H)|$*. P_2-digraphs, which are interval digraphs, have unbounded bi-mim-width.*
5. *Let H be an undirected graph. Given a nice H-convex digraph G with its bipartition (A, B), one can construct in polynomial time a linear branch decomposition of G of bi-mim-width at most* $12|E(H)|$*. P_2-convex digraphs have unbounded bi-mim-width.*
6. *Tournaments and directed acyclic graphs have unbounded bi-mim-width.*

(1. Interval digraphs) Recall that Müller [33] devised a recognition algorithm for interval digraphs, which also outputs a representation. By testing the reflexivity of a digraph, we can recognize reflexive interval digraphs, and output its representation. We convert it into a linear branch decomposition of bi-mim-width at most 2. On the other hand, interval digraphs generally have unbounded bi-mim-width. By Theorem 1, we can solve all DLCV problems on reflexive interval digraphs in polynomial time. This extends the polynomial-time algorithms for Independent Dominating Set and Kernel on interval nest digraphs given by Prisner [35], and includes polynomial-time algorithms for Absorbing Set, Dominating Set, and Kernel by Francis, Hell, and Jacob [22].

(2. Permutation digraphs) A *permutation digraph* is an intersection digraph of pairs of line segments whose endpoints lie on two parallel lines. Müller [33] considered permutation digraphs under the name "matching diagram digraph", and observed that every interval digraph is a permutation digraph. Therefore, permutation digraphs have unbounded bi-mim-width. We say that a permutation digraph is *adjusted* if there exists one of the parallel lines, say Λ, such that for all $v \in V(G)$, S_v and T_v have the same endpoint in Λ. We show that every adjusted permutation digraph has linear mim-width at most 4.

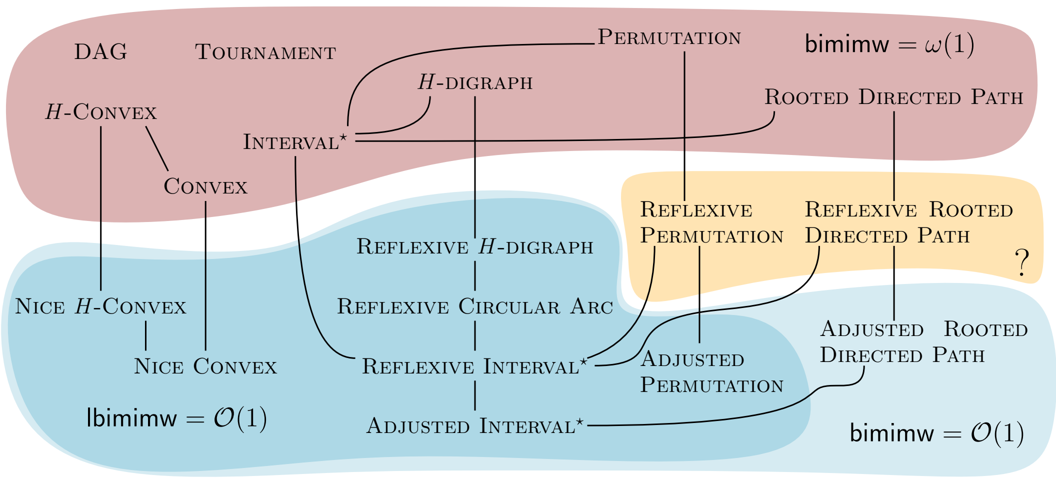

Figure 1 Digraph classes with bounds on their (linear) bi-mim-width. For graph classes marked with * there are polynomial-time algorithms to compute representations of their members. If digraph class A is depicted above B and there is an edge between A and B then $B \subseteq A$.

(3. Rooted directed path digraphs) It is known that chordal graphs have unbounded mim-width [28, 32]. As restrictions of chordal graphs, it has been shown that rooted directed path graphs, and more generally, leaf power graphs have mim-width at most 1 [26], while they have unbounded linear mim-width. A *rooted directed path digraph* is an intersection digraph of pairs of directed paths in a rooted directed tree (every node is reachable from the root), and it is *adjusted* if for every vertex v, the endpoint of S_v that is farther from the root is the same as the endpoint of T_v that is farther from the root. We show that every adjusted rooted directed path digraph has bi-mim-width at most 2. Since this class includes the biorientations of trees, it has unbounded linear bi-mim-width.

(4. H-digraphs) For an undirected graph H, an H*-graph* is an undirected intersection graph of connected subgraphs in an H-subdivision, introduced by Bíró, Hujter, and Tuza [11]. For example, interval graphs and circular-arc graphs are P_2-graphs and C_3-graphs, respectively. Fomin, Golovach, and Raymond [21] showed that H-graphs have linear mim-width at most $2|E(H)|+1$. Motivated by H-graphs, we introduce an H*-digraph* that is the intersection digraph of pairs of connected subgraphs in an H-subdivision (where H and its subdivision are undirected). We prove that reflexive H-digraphs have linear bi-mim-width at most $12|E(H)|$. This extends the linear bound of Fomin et al. [21] for H-graphs.

(5. H-convex digraphs) For an undirected graph H, a bipartite digraph G with bipartition (A,B) is an H*-convex digraph*, if there exists a subdivision F of H with $V(F)=A$ such that for every vertex b of B, each of the set of out-neighbors and the set of in-neighbors of v induces a connected subgraph in F. We say that an H-convex digraph is *nice* if for every vertex b of B, there is a bi-directed edge between b and some vertex of A. Note that H-convex graphs, introduced by Bonomo-Braberman et al. [12], can be seen as nice H-convex digraphs, by replacing every edge with bi-directed edges. We prove that nice H-convex digraphs have linear bi-mim-width at most $12|E(H)|$. This implies that H-convex graphs have linear mim-width at most $6|E(H)|$. For the special case when T is a tree with maximum degree Δ and t branching nodes, Bonomo-Braberman et al. [12] showed an improved bound of $2+t(\Delta-2)$ on the mim-width of T-convex graphs.

(6. Directed acyclic graphs and tournaments) We show that if H is the underlying undirected graph of a digraph G, then the bi-mim-width of G is at least the mim-width of H. Using this, we can show that acyclic orientations of grids have unbounded bi-mim-width. We also prove that tournaments have unbounded bi-mim-width. This refines an argument that they have unbounded bi-rank-width [7, Lemma 9.9.11].

We can summarize our algorithmic results as follows.

▶ **Corollary 4.** *Given a reflexive interval digraph, or a representation of either an adjusted permutation digraph, or an adjusted rooted directed path digraph, or a reflexive H-digraph, or a nice H-convex digraph, we can solve all DLCV problems represented by finite or co-finite sets, and their distance variants, in polynomial time.*

Related work. Intersection digraphs have first been considered by Beineke and Zamfirescu in 1982 [9]. Sen et al. [38] introduced the class of interval digraphs and Sen et al. [39] the class of circular-arc digraphs. Permutation digraphs were first studied under the name "matching diagram digraphs" by Müller [33]. Prisner [35] showed that the problems Clique, Chromatic Number, Independent Set, Partition into Cliques, Kernel, and Independent Dominating Set are polynomial-time solvable on interval *nest* digraphs, a subclass of interval digraphs G having a representation $\{(S_v, T_v) : v \in V(G)\}$ where for each vertex $v \in V(G)$, either $S_v \subseteq T_v$ or $T_v \subseteq S_v$. Very recently, and independently of this work, Francis, Hell, and Jacob [22] showed that Absorbing Set, Dominating Set, and Kernel are polynomial-time solvable on *reflexive* interval digraphs, a superclass of interval nest digraphs. They also showed that these problems remain hard on interval digraphs, even when all intervals are single points. Feder et al. [20] considered the List H-Homomorphism problem, but posing a structural restriction on H rather than the input graph. They showed that if H is an *adjusted* interval digraph, i.e. an interval digraph with a representation where both intervals associated with each vertex have the same left endpoint, then List H-Homomorphism is polynomial-time solvable.

The algorithmic result for undirected graphs analogous to ours is that all (undirected) locally checkable vertex problems are polynomial-time solvable if the input graph is given together with a decomposition of constant mim-width. This has been shown by Bui-Xuan, Telle, and Vatshelle [14]. In their work, the runtime of the algorithms is stated in terms of the number of equivalence classes of the *d-neighborhood equivalence* relation, and the connection between this notion and mim-width was made explicit by Belmonte and Vatshelle [10].

Organization of the paper. The paper is organized as follows. In Section 2, we introduce basic notations. In Section 3, we formally introduce bi-mim-width and compare with other known width parameters. In Section 4, we prove Theorem 3, and in Section 5, we prove Theorems 1 and 2. Proofs of statements marked with "⋆" are deferred to the full version.

2 Preliminaries

For a positive integer n, we use the shorthand $[n] := \{1, \ldots, n\}$.

Undirected Graphs. We use standard notions of graph theory and refer to [19] for an overview. All undirected graphs considered in this work are finite and simple. For a graph G, we denote by $V(G)$ the vertex set of G and $E(G)$ the edge set of G. For an edge $\{u, v\} \in E(G)$, we may use the shorthand "uv".

For two vertices $u, v \in V(G)$, the *distance* between u and v, denoted by $\mathrm{dist}_G(u,v)$ or simply $\mathrm{dist}(u,v)$, is the length of the shortest path between u and v. For $u \in V(G)$ and $A \subseteq V(G)$, we let $\mathrm{dist}_G(u,A) = \min_{v\in A} \mathrm{dist}_G(u,v)$.

Let G be a graph and $e = uv \in E(G)$. The *(edge) subdivision* of e is the operation of removing the edge e and adding a new vertex x and the edges ux and xv to G. A graph H is a *subdivision* of G if H can be obtained from G by a series of edge subdivisions. If H is a subdivision of G, then each vertex in $V(G)$ is called a *branching vertex* in H. A path P in H is called a *branching path* if its endpoints are branching vertices and no other vertices in P are branching vertices.

Digraphs. All digraphs considered in this work are finite and have no multiple edges, but may have loops. For a digraph G, we denote by $V(G)$ its vertex set and by $E(G) \subseteq V(G) \times V(G)$ its edge set. We say that an edge $(u,v) \in E(G)$ is directed from u to v. For a vertex v of G, we denote by $N^+_G(v)$ the set of out-neighbors of v, and by $N^-_G(v)$ the set of in-neighbors of v. If G is clear from the context, then we allow to remove G from the subscript.

A *rooted directed tree* is a digraph obtained from an undirected tree by selecting a root and directing all edges away from the root.

For a digraph G and two disjoint vertex sets $A, B \subseteq V(G)$, we denote by $G[A \to B]$ the bipartite digraph on bipartition (A,B) with edge set $E(G[A,B]) = E(G) \cap (A \times B)$, and denote by $G[A,B]$ the bipartite digraph on bipartition (A,B) with edge set $E(G[A \to B]) \cup E(G[B \to A])$. A set M of edges in a digraph G is a *matching* if no two edges share an endpoint, and it is an *induced matching* if there are no edges in G meeting two distinct edges in M. We denote by $\nu(G)$ the maximum size of an induced matching of G. For a vertex set A of G, we denote by $\overline{A} := V(G) \setminus A$. A vertex bipartition $(A, \overline{A})$ of G for some vertex set A of G is called a *cut*.

For two vertices $u, v \in V(G)$, the *distance* between u and v, denoted by $\mathrm{dist}_G(u,v)$ or simply $\mathrm{dist}(u,v)$, is the length of the shortest directed path from u to v. For a positive integer d, we denote by G^d the graph obtained from G by, for every pair (x,y) of vertices in G, adding an edge from x to y if there is a path of length at most d from x to y in G. We call it the *d-th power* of G.

3 Bi-mim-width

Throughout this section, definitions of concepts that are only touched on briefly can be found in Appendix A.

▶ **Definition 5** (Branch Decomposition). *Let Ω be a set. A* branch decomposition over Ω *is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L}$ from Ω to the leaves of T. If T is a caterpillar, then $(T, \mathcal{L})$ is called a* linear branch decomposition *of G. For $e \in E(T)$, let T_A, T_B be the components of $T - e$. Let (A_e, B_e) be the cut of Ω where A_e is the set of elements that $\mathcal{L}$ maps to the leaves in T_A and B_e is the set of elements that $\mathcal{L}$ maps to the leaves in T_B.*

We introduce the bi-mim-width of a digraph. For a digraph G and $A \subseteq V(G)$, let $\mathrm{mim}^+_G(A) := \nu(G[A \to \overline{A}])$, $\mathrm{mim}^-_G(A) := \nu(G[\overline{A} \to A])$, and $\mathrm{bimim}_G(A) := \mathrm{mim}^+_G(A) + \mathrm{mim}^-_G(A)$. A *branch decomposition of a digraph G* is a branch decomposition over $V(G)$.

▶ **Definition 6** (Bi-mim-width). *Let G be a digraph and $(T, \mathcal{L})$ be a branch decomposition of G. The* bi-mim-width *of $(T, \mathcal{L})$ is $\mathrm{bimimw}(T, \mathcal{L}) := \max_{e\in E(T)} (\mathrm{bimim}_G(A_e))$. The* bi-mim-width *of G, denoted by $\mathrm{bimimw}(G)$, is the minimum bi-mim-width of any branch decomposition of G. The* linear bi-mim-width *of G, denoted by $\mathrm{lbimimw}(G)$, is the minimum bi-mim-width of any linear branch decomposition of G.*

For an undirected graph G, we denote by $\mathrm{mimw}(G)$ its mim-width and by $\mathrm{lmimw}(G)$ its linear mim-width. The following two lemmas are clear by definition.

▶ **Lemma 7.** *Let G be a digraph and let H be an induced subdigraph of G. Then $\mathrm{bimimw}(H) \leq \mathrm{bimimw}(G)$ and $\mathrm{lbimimw}(H) \leq \mathrm{lbimimw}(G)$.*

▶ **Lemma 8.** *Let G be an undirected graph and let H be the biorientation of G. Then $\mathrm{mimw}(G) = \frac{\mathrm{bimimw}(H)}{2}$.*

We show that if a digraph G has small bi-mim-width, then its underlying undirected graph has small mim-width. But the other direction does not hold; the class of tournaments has unbounded bi-mim-width. We also argue that directed tree-width [27] and bi-mim-width are incomparable.

▶ **Lemma 9** (⋆). *Let G be a digraph and let H be the underlying undirected graph of G. Then $\mathrm{mimw}(H) \leq \mathrm{bimimw}(G)$ and $\mathrm{lmimw}(H) \leq \mathrm{lbimimw}(G)$. On the other hand, the class of tournaments has unbounded bi-mim-width, while their underlying undirected graphs have linear mim-width 1.*

▶ **Lemma 10** (⋆). *Directed tree-width and bi-mim-width are incomparable.*

We compare the bi-mim-width with the bi-rank-width of a digraph, introduced by Kanté [29]. Kanté and Rao [30] later generalized this notion to edge-colored graphs. For a digraph G, we denote by $\mathrm{birw}(G)$ its bi-rank-width and by $\mathrm{lbirw}(G)$ its linear bi-rank-width. We can verify that for every digraph G, $\mathrm{bimimw}(G) \leq \mathrm{birw}(G)$. Interestingly, we can further show that for every positive integer r, the bi-mim-width of the r-th power of G is at most the bi-rank-width of G. This does not depend on the value of r.

▶ **Lemma 11** (⋆). *Let r and w be positive integers. If $(T, \mathcal{L})$ is a branch-decomposition of a digraph G of bi-rank-width w, then it is a branch-decomposition of G^r of bi-mim-width at most w.*

Next, we show that the r-th power of a digraph of bi-mim-width w has bi-mim-width at most rw. This will be used to prove Theorem 2.

▶ **Lemma 12.** *Let r and w be positive integers. If $(T, \mathcal{L})$ is branch-decomposition of a digraph G of bi-mim-width w, then it is a branch-decomposition of G^r of bi-mim-width at most rw.*

Proof. It is sufficient to prove that for every ordered vertex partition (A, B) of G, we have $\nu(G^r[A \to B]) \leq r\nu(G[A \to B])$. Assume $\nu(G[A \to B]) = t$ and suppose for contradiction that $\nu(G^r[A \to B]) \geq rt + 1$.

Let $\{(a_i, b_i) : i \in [rt{+}1]\}$ be an induced matching of $G^r[A \to B]$ with $\{a_i : i \in [rt{+}1]\} \subseteq A$. For each $i \in [rt + 1]$, let P_i be a directed path of length at most r from a_i to b_i in G. We choose an edge (c_i, d_i) in each P_i where $c_i \in A$ and $d_i \in B$. For each $i \in [rt + 1]$, let ℓ_i be the length of the subpath of P_i from a_i to c_i. Observe that $0 \leq \ell_i \leq r - 1$.

By the pigeonhole principle, there exists a subset I of $[rt + 1]$ of size at least $t + 1$ such that for all $i_1, i_2 \in I$, $\ell_{i_1} = \ell_{i_2}$. Since $\nu(G[A \to B]) = t$, there exist distinct integers $i_1, i_2 \in I$ such that there is an edge from c_{i_1} to d_{i_2}. Then there is a path of length at most d from a_{i_1} to b_{i_2}, contradicting the assumption that there is no edge from a_{i_1} to b_{i_2} in G^r. ◀

4 Classes of digraphs of bounded bi-mim-width

In this section we present several digraph classes of bounded bi-mim-width. Recall that a digraph G is an *intersection digraph* if there is a family of ordered pairs of sets $\{(S_v, T_v) : v \in V(G)\}$, called a *representation* of G, such that $(u, v) \in E(G)$ if and only if $S_u \cap T_v \neq \emptyset$. G

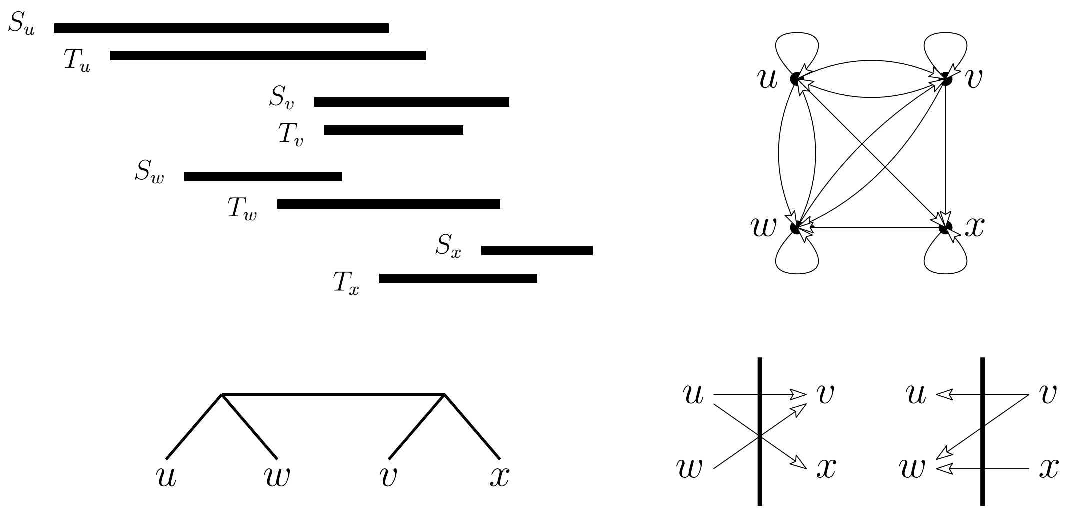

Figure 2 An example of a reflexive interval digraph. On the top left is its representation, on the top right one of its drawings, on the bottom left a linear branch decomposition and the bottom right shows that the cut associated with the "middle" edge of the branch decomposition has bi-mim-width value 2.

is called *reflexive* if for each $v \in V(G)$, $S_v \cap T_v \neq \emptyset$. Let H be a fixed undirected graph. A digraph G is an *H-digraph* if there is a subdivision F of H such that G is an intersection digraph of pairs of vertex sets inducing connected components in F.

▶ **Proposition 13.** *Let H be an undirected graph. Given a representation of a reflexive H-digraph G, one can construct in polynomial time a linear branch decomposition of G of bi-mim-width at most* $12|E(H)|$.

Proof. Let $m := |E(H)|$. We may assume that H is connected. If H has no edge, then it is trivial. Thus, we may assume that $m \geq 1$. Let G be a reflexive H-digraph, let F be a subdivision of H, and let $\mathcal{M} := \{(S_v, T_v) : v \in V(G)\}$ be a given reflexive H-digraph representation of G with underlying graph F. For each $v \in V(G)$, choose a vertex α_v in $S_v \cap T_v$. We may assume that vertices in $(\alpha_v : v \in V(G))$ are pairwise distinct and they are not branching vertices, by subdividing F more and changing $\mathcal{M}$ accordingly, if necessary.

We may assume that F has a branching vertex r, and we obtain a BFS ordering of F starting from r. We denote by $v <_B w$ if v appears before w in the BFS ordering. We give a linear ordering L of G such that for all $v, w \in V(G)$, if $\alpha_v <_B \alpha_w$, then v appears before w in L. This can be done in linear time. We claim that L has width at most $12m$. We choose a vertex v of G arbitrarily, and let A be the set of vertices in G that are v or a vertex appearing before v in L, and let $B := V(G) \setminus A$. It suffices to show $\text{bimim}_G(A) \leq 12m$. Let A^* be the set of vertices of F that are α_v or a vertex appearing before α_v, and let $B^* := V(F) \setminus A^*$. Let $\mathcal{P}$ be the set of paths in F such that

- for every $P \in \mathcal{P}$, P is a subpath of some branching path of F and it is a maximal path contained in one of A^* and B^*,
- $\bigcup_{P \in \mathcal{P}} V(P) = V(F)$.

Because of the property of a BFS ordering, it is easy to see that each branching path of F is partitioned into at most 3 vertex-disjoint paths in $\mathcal{P}$. Thus, we have $|\mathcal{P}| \leq 3m$. Note that two paths in $\mathcal{P}$ from two distinct branching paths may share an endpoint.

We first show that $\text{mim}^+_G(A) \leq 6m$. Suppose for contradiction that $G[A \to B]$ contains an induced matching M of size $6m + 1$. By the pigeonhole principle, there is a subset $M_1 = \{(x_i, y_i) : i \in [3]\}$ of M of size 3 and a path P in $\mathcal{P}$ such that for every $(x, y) \in M_1$, S_x and T_y meet on P. Let p_1, p_2 be the endpoints of P.

Observe that $V(P) \subseteq A^*$ or $V(P) \subseteq B^*$. So, for each $i \in [3]$, it is not possible that α_{x_i} and α_{y_i} are both contained in $V(P)$. It implies that each connected component of $(S_{x_i} \cup T_{y_i}) \cap P$ contains an endpoint of P, as $S_{x_i} \cup T_{y_i}$ is connected. Therefore, there are at least two integers $j_1, j_2 \in [3]$ and a connected component C_1 of $(S_{x_{j_1}} \cup T_{y_{j_1}}) \cap P$ and a connected component C_2 of $(S_{x_{j_2}} \cup T_{y_{j_2}}) \cap P$ so that (1) C_1 and C_2 contain the same endpoint of P, and (2) for each $i \in [2]$, C_i contains a vertex of $S_{x_{j_i}}$ and a vertex of $T_{y_{j_i}}$. However, it implies that (x_{j_1}, y_{j_2}) or (x_{j_2}, y_{j_1}) is an edge, a contradiction.

We deduce that $\text{mim}^+_G(A) \leq 6m$. By a symmetric argument, we get $\text{mim}^-_G(A) \leq 6m$. Therefore, we have $\text{bimim}_G(A) \leq 12m$, as required. ◀

Interval digraphs are intersection digraphs of pairs of intervals over the real line, or, equivalently, P_2-digraphs. We first obtain a bound on the bi-mim-width of reflexive interval digraphs that improves the bound due to Proposition 13.

▶ **Proposition 14** (⋆). *Given a reflexive interval digraph, one can output a linear branch decomposition of bi-mim-width at most* 2 *in polynomial time.*

▶ **Proposition 15** (⋆). *Interval digraphs have unbounded bi-mim-width.*

A *permutation digraph* is an intersection digraph of pairs of line segments whose endpoints lie on two parallel lines. A permutation digraph G with representation $\{(S_v, T_v) : v \in V(G)\}$ is *adjusted* if for one of the two parallel lines, say Λ, it holds that all for all $v \in V(G)$, S_v and T_v have the same endpoint on Λ. We show that adjusted permutation digraphs have linear bi-mim-width at most 4.

▶ **Proposition 16** (⋆). *Given a representation of an adjusted permutation digraph G, one can construct in polynomial time a linear branch decomposition of G of bi-mim-width at most* 4.

Proof Sketch. Let $\Lambda_1 := \{(x, 0) : x \in \mathbb{R}\}$ and $\Lambda_2 := \{(x, 1) : x \in \mathbb{R}\}$ be two lines. Let G be a given adjusted permutation digraph with its representation $\{(S_v, T_v) : v \in V(G)\}$ where S_v and T_v are line segments whose endpoints lie on Λ_1 and Λ_2 and they have a common endpoint in Λ_1, say $(\alpha_v, 0)$. For each $v \in V(G)$, let $(\beta_v, 1)$ be the endpoint of S_v in Λ_2 and $(\gamma_v, 1)$ be the endpoint of T_v in Λ_2. We give a linear ordering L of G such that for all $v, w \in V(G)$, if $\alpha_v < \alpha_w$, then v appears before w in L.

We claim that L has bi-mim-width at most 4. We choose a vertex v of G arbitrarily, and let A be the set of vertices in G that are v or a vertex appearing before v in L, and let $B := V(G) \setminus A$. We verify that $\text{mim}^+_G(A) \leq 2$. By a symmetric argument, we have $\text{mim}^-_G(A) \leq 2$. Suppose for contradiction that $G[A \to B]$ has an induced matching $\{(v_i, w_i) : i \in [3]\}$ with $v_1, v_2, v_3 \in A$. Without loss of generality, we assume that $\alpha_{v_1} \leq \alpha_{v_2} \leq \alpha_{v_3}$. Observe that $\alpha_{w_1}, \alpha_{w_2} > \alpha_{v_3}$ and $\alpha_{w_3} \geq \alpha_{v_3}$. Let $w \in \{w_i : i \in [3]\}$ such that $|\alpha_w - \alpha_{v_3}|$ is minimum.

If $\alpha_w = \alpha_{v_3}$ and $\gamma_{w_3} > \beta_{v_3}$, then it is not difficult to verify that $\beta_{v_3} < \beta_{v_1}, \beta_{v_2}, \gamma_{w_1}, \gamma_{w_2} < \gamma_{w_3}$, as $\{(v_i, w_i) : i \in [3]\}$ is an induced matching. If $\beta_{v_1} \leq \beta_{v_2}$, then T_{w_1} has to meet S_{v_2}, a contradiction. We can deal with other cases similarly. ◀

A *rooted directed path digraph* is an intersection digraph of pairs of directed paths in a rooted directed tree, and it is *adjusted* if for every vertex v, the endpoint of S_v that is farther from the root is the same as the endpoint of T_v that is farther from the root. We prove that adjusted rooted directed path digraphs have bounded bi-mim-width. We obtain a desired branch decomposition by attaching a leaf node corresponding to a vertex v to the common endpoint of S_v and T_v that is farther from the root, after finding an equivalent representation where the underlying directed tree has out-degree at most 2 and there are no two vertices v and w for which S_v and S_w share the endpoint farther from the root.

▶ **Proposition 17** (⋆). *Given a representation of an adjusted rooted directed path digraph G, one can construct in polynomial time a branch decomposition of G of bi-mim-width at most 2. Adjusted rooted directed path digraphs have unbounded linear bi-mim-width.*

For an undirected graph H, a bipartite digraph G with bipartition (A, B) is an *H-convex digraph* if there is a subdivision F of H with $V(F) = A$ such that for every $b \in B$, both the set of out-neighbors of b and the set of in-neighbors of b induce a connected subgraph in F. An H-convex digraph is *nice* if every vertex of B is incident to some bi-directed edge. We prove that nice H-convex digraphs have linear bi-mim-width at most $12|E(H)|$. This proof resembles the proof of Proposition 13.

▶ **Proposition 18** (⋆). *Let H be an undirected graph. Given a nice H-convex digraph G with its bipartition (A, B), one can construct in polynomial time a linear branch decomposition of G of bi-mim-width at most $12|E(H)|$.*

▶ **Proposition 19** (⋆). *P_2-convex digraphs have unbounded bi-mim-width.*

5 Algorithmic applications

In this section we give the algorithmic applications of bi-mim-width. We show that all directed locally checkable vertex subset and all directed locally checkable vertex partitioning problems can be solved in XP time parameterized by the bi-mim-width of a given branch decomposition of the input digraph. We do so by adapting the framework of the d-neighborhood equivalence relation introduced by Bui-Xuan et al. [14] to digraphs.

d-Bi-neighhorhood-equivalence. The subsets of natural numbers that characterize locally checkable vertex subset/partitioning problems can be fully characterized when counting in- and out-neighbors up to some constant d. Therefore, if a vertex v has more than d for instance out-neighbors in two sets X and Y, then these two sets look the same to v in terms of its out-neighborhood. This is the main motivation for the following definition.

▶ **Definition 20.** *Let $d \in \mathbb{N}$. Let G be a digraph and $A \subseteq V(G)$. For two sets $X, Y \subseteq A$, we say that X and Y are d-*bi-neighbor* equivalent, written $X \equiv^{\pm}_{d,A} Y$, if* [1]

$$\forall u \in V(G) \setminus A\colon\ \min\{d, |N^-(u) \cap X|\} = \min\{d, |N^-(u) \cap Y|\} \text{ and}$$
$$\min\{d, |N^+(u) \cap X|\} = \min\{d, |N^+(u) \cap Y|\}.$$

We denote the number of equivalence classes of $\equiv^{\pm}_{d,A}$ by $\text{nec}(\equiv^{\pm}_{d,A})$. If $(T, \mathcal{L})$ is a branch decomposition of G, we let $\text{nec}_d(T, \mathcal{L}) = \max_{t \in V(T)} \max\{\text{nec}(\equiv^{\pm}_{d,V_t}), \text{nec}(\equiv^{\pm}_{d,\overline{V_t}})\}$.

The enumeration of equivalence classes is based on pairs of vectors called *d-bi-neighborhoods* of a subset X of A.

[1] Since the definition is given in terms of vertices from $\overline{A}$, we consider the directions of the edges in reverse, i.e., we consider $N^-(v)$ for $v \in \overline{A}$ when defining $\equiv^+$.

▶ **Definition 21.** *Let G be a digraph, $X \subseteq A \subseteq V(G)$, and $d \in \mathbb{N}$. The d-*out-neighborhood *of X, denoted by $U^+_{d,A}(X)$, and the d-*in-neighborhood *of X, denoted by $U^-_{d,A}(X)$ are the following vectors in $\{0, 1, \ldots, d\}^{\overline{A}}$:*

$$U^+_{d,A}(X) = (\min\{d, |N^-(v) \cap X|\})_{v \in \overline{A}} \quad U^-_{d,A}(X) = (\min\{d, |N^+(v) \cap X|\})_{v \in \overline{A}}$$

*We refer to the pair $(U^+_{d,A}(X), U^-_{d,A}(X))$ as the d-*bi-neighborhood *$U^\pm_{d,A}(X)$; and we denote the set of all d-bi-neighborhoods as $\mathcal{U}^\pm_{d,A}$.*

There is a natural bijection between the d-bi-neighborhoods and the equivalence classes of $\equiv^\pm_{d,A}$.

▶ **Observation 22.** *Let G be a digraph and $X, Y \subseteq A \subseteq V(G)$. Then, $X \equiv^\pm_{d,A} Y$ if and only if $U^\pm_{d,A}(X) = U^\pm_{d,A}(Y)$.*

▶ **Lemma 23** ($\star$)**.** *Let G be a digraph on n vertices, $A \subseteq V(G)$, and $d \in \mathbb{N}$. There is an algorithm that enumerates all members of $\mathcal{U}^\pm_{d,A}$ in time $\mathcal{O}(\text{nec}(\equiv^\pm_{d,A}) \log \text{nec}(\equiv^\pm_{d,A}) \cdot dn^2)$. Furthermore, for each $Y \in \mathcal{U}^\pm_{d,A}$, the algorithm can provide some $X \subseteq A$ with $U^\pm_{d,A}(X) = Y$.*

5.1 Generalized Directed Domination Problems

The algorithm in this section is bottom-up dynamic programming along the given branch decomposition $(T, \mathcal{L})$ of the input digraph G, which we assume to be rooted in an arbitrary degree two node. For a node $t \in V(T)$, we let V_t be the vertices of G that are mapped to a leaf in the subtree of T rooted at t. We recall the formal definition of $(\sigma^+, \sigma^-, \rho^+, \rho^-)$-sets.

▶ **Definition 24.** *Let $\sigma^+, \sigma^-, \rho^+, \rho^- \subseteq \mathbb{N}$, and let $\Sigma = (\sigma^+, \sigma^-)$ and $\mathrm{R} = (\rho^+, \rho^-)$. Let G be a digraph and $S \subseteq V(G)$. We say that S $(\sigma^+, \sigma^-, \rho^+, \rho^-)$-dominates G, or simply that S (Σ, R)-dominates G, if:*

$$\forall v \in V(G)\colon |N^+(v) \cap S| \in \begin{cases} \sigma^+, & \text{if } v \in S \\ \rho^+, & \text{if } v \notin S \end{cases} \text{ and } |N^-(v) \cap S| \in \begin{cases} \sigma^-, & \text{if } v \in S \\ \rho^-, & \text{if } v \notin S \end{cases}$$

▶ **Definition 25.** *Let $d(\mathbb{N}) = 0$. For a finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min\{\max_{x \in \mathbb{N}} x \in \mu, \max_{x \in \mathbb{N}} x \notin \mu\}$. For finite or co-finite $\sigma^+, \sigma^-, \rho^+, \rho^- \subseteq \mathbb{N}$, $\Sigma = (\sigma^+, \sigma^-)$ and $\mathrm{R} = (\rho^+, \rho^-)$: $d(\sigma^+, \sigma^-, \rho^+, \rho^-) = d(\Sigma, \mathrm{R}) = \max\{d(\sigma^+), d(\sigma^-), d(\rho^+), d(\rho^-)\}$.*

As our algorithm progresses, it keeps track of partial solutions that may become a (Σ, R)-set once the computation has finished. This does not necessarily mean that at each node $t \in V(T)$, such a partial solution $X \subseteq V_t$ has to be a (Σ, R)-dominating set of $G[V_t]$. Instead, we additionally consider what is usually referred to as the "expectation from the outside" [14] in form of a subset Y of $\overline{V_t}$ such that $X \cup Y$ is a (Σ, R)-dominating set of $G[V_t]$.

▶ **Definition 26.** *Let $\mu^+, \mu^- \subseteq \mathbb{N}$ and let $\mathrm{M} = (\mu^+, \mu^-)$. Let G be a digraph, $A \subseteq V(G)$ and $X \subseteq V(G)$. We say that X M-dominates A if for all $v \in A$, we have that $|N^+(v) \cap X| \in \mu^+$ and $|N^-(v) \cap X| \in \mu^-$. Let Σ and R be as above. For $X \subseteq A$ and $Y \subseteq \overline{A}$, we say that (X, Y) (Σ, R)-dominates A, if $X \cup Y$ Σ-dominates X and $X \cup Y$ R-dominates $A \setminus X$.*

To describe an equivalence class $\mathcal{Q}$ of $\equiv^\pm_{d,A}$ we use the d-bi-neighbohoods of its members, which we denote by $\mathsf{desc}(\mathcal{Q})$. Note that by Observation 22, this is well-defined.

▶ **Definition 27.** *Let $\sigma^+, \sigma^-, \rho^+, \rho^- \subseteq \mathbb{N}$ be finite or co-finite, let $\Sigma = (\sigma^+, \sigma^-)$, $\mathrm{R} = (\rho^+, \rho^-)$, and $d = d(\Sigma, \mathrm{R})$. Let opt stand for $\min$ if we consider a minimization problem and for $\max$ if we consider a maximization problem. Let G be a digraph with branch decomposition $(T, \mathcal{L})$ and let $t \in V(T)$. For an equivalence class $\mathcal{Q}_t$ of $\equiv^{\pm}_{d,V_t}$, and an equivalence class $\mathcal{Q}_{\bar{t}}$ of $\equiv^{\pm}_{d,\overline{V_t}}$, we let:*

$$Tab_t[\mathsf{desc}(\mathcal{Q}_t), \mathsf{desc}(\mathcal{Q}_{\bar{t}})] = \begin{cases} \mathsf{opt}_{S \subseteq V_t} |S| : & S \in \mathcal{Q}_t \text{ and for any } S_{\bar{t}} \in \mathcal{Q}_{\bar{t}}: \\ & (S, S_{\bar{t}})\ (\Sigma, \mathrm{R})\text{-dominates } V_t \\ \infty & \text{if } \mathsf{opt} = \min \text{ and no such } S \text{ exists} \\ -\infty & \text{if } \mathsf{opt} = \max \text{ and no such } S \text{ exists} \end{cases}$$

We use the shorthand "$Tab_t[\mathcal{Q}_t, \mathcal{Q}_{\bar{t}}]$" for "$Tab_t[\mathsf{desc}(\mathcal{Q}_t), \mathsf{desc}(\mathcal{Q}_{\bar{t}})]$". For all such t, $\mathcal{Q}_t$, and $\mathcal{Q}_{\bar{t}}$, we initialize $Tab_t[\mathcal{Q}_t, \mathcal{Q}_{\bar{t}}]$ to be $-\infty$ if $\mathsf{opt} = \max$ and ∞ if $\mathsf{opt} = \min$.

Leaves of T. For a leaf $\ell \in V(T)$, let $v \in V(G)$ be such that $\mathcal{L}(v) = \ell$. Clearly, $\equiv_{d,\{v\}}$ has only two equivalence classes, namely the one containing $\emptyset$ and the one containing $\{v\}$. For each equivalence class $\mathcal{Q}$ of $\equiv_{d,V(G)\setminus\{v\}}$, let $R \in \mathcal{Q}$ which we can assume is given to us by Lemma 23. If $|N^+(v) \cap R| \in \sigma^+$ and $|N^-(v) \cap R| \in \sigma^-$, then $Tab_\ell[\{\{v\}\}, \mathcal{Q}] = 1$. If $|N^+(v) \cap R| \in \rho^+$ and $|N^-(v) \cap R| \in \rho^-$, then $Tab_\ell[\{\emptyset\}, \mathcal{Q}] = 0$.

Internal nodes of T. Let $t \in V(T)$ be an internal node with children a and b.

1. Consider each triple $\mathcal{Q}_a, \mathcal{Q}_b, \mathcal{Q}_{\bar{t}}$ of equivalence classes of $\equiv^{\pm}_{d,V_a}$, $\equiv^{\pm}_{d,V_b}$, and $\equiv^{\pm}_{d,\overline{V_t}}$, respectively.
2. Let $R_a \in \mathcal{Q}_a$, $R_b \in \mathcal{Q}_b$, and $R_{\bar{t}} \in \mathcal{Q}_{\bar{t}}$. Determine:
   - $\mathcal{Q}_{\bar{a}}$, the equivalence class of $\equiv^{\pm}_{d,\overline{V_a}}$ containing $R_b \cup R_{\bar{t}}$.
   - $\mathcal{Q}_{\bar{b}}$, the equivalence class of $\equiv^{\pm}_{d,\overline{V_b}}$ containing $R_a \cup R_{\bar{t}}$.
   - $\mathcal{Q}_t$, the equivalence class of $\equiv^{\pm}_{d,V_t}$ containing $R_a \cup R_b$.
3. Update $Tab_t[\mathcal{Q}_t, \mathcal{Q}_{\bar{t}}] = \mathsf{opt}\{Tab_t[\mathcal{Q}_t, \mathcal{Q}_{\bar{t}}], Tab_a[\mathcal{Q}_a, \mathcal{Q}_{\bar{a}}] + Tab_b[\mathcal{Q}_b, \mathcal{Q}_{\bar{b}}]\}$.

▶ **Theorem 28** ($\star$). *Let $\sigma^+, \sigma^-, \rho^+, \rho^- \subseteq \mathbb{N}$ be finite or co-finite, $\Sigma = (\sigma^+, \sigma^-)$, $\mathrm{R} = (\rho^+, \rho^-)$, and $d = d(\Sigma, \mathrm{R})$. There is an algorithm that given a digraph G on n vertices together with one of its branch decompositions $(T, \mathcal{L})$, computes and optimum-size (Σ, R)-dominating set in time $\mathcal{O}(\mathrm{nec}_d(T, \mathcal{L})^3 \cdot n^3 \log n)$. For $n \le \mathrm{nec}_d(T, \mathcal{L})$, the algorithm runs in time $\mathcal{O}(\mathrm{nec}_d(T, \mathcal{L})^3 \cdot n^2)$.*

▶ **Observation 29** ($\star$). *For $d \in \mathbb{N}$, a digraph G, and $A \subseteq V(G)$: $\mathrm{nec}(\equiv^{\pm}_{d,A}) \le n^{d \cdot \mathrm{bimim}_G(A)}$.*

▶ **Corollary 30.** *Let $\sigma^+, \sigma^-, \rho^+, \rho^- \subseteq \mathbb{N}$ be finite or co-finite, $\Sigma = (\sigma^+, \sigma^-)$, $\mathrm{R} = (\rho^+, \rho^-)$, and $d = d(\Sigma, \mathrm{R})$. Let G be a digraph on n vertices with branch decomposition $(T, \mathcal{L})$ of bi-mim-width $w \ge 1$. There is an algorithm that given any such G and $(T, \mathcal{L})$ computes an optimum-size (Σ, R)-dominating set in time $\mathcal{O}(n^{3dw+2})$.*

5.2 Directed Vertex Partitioning Problems

We now show that the locally checkable vertex partitioning problems can be solved in XP time parameterized by the bi-mim-width of a given branch decomposition. In analogy with [14], we lift the d-bi-neighborhood equivalence to q-tuples over vertex sets, which allows for devising the desired dynamic programming algorithm. The resulting algorithm follows a very similar strategy to the one for (Σ, R)-problems; the details are deferred to the full version ($\star$).

▶ **Definition 31.** *A* bi-neighborhood-constraint *matrix is a* $(q \times q)$*-matrix* D_q *over pairs of finite or co-finite sets of natural numbers. Let* G *be a digraph, and* $\mathcal{X} = (X_1, \ldots, X_q)$ *be a* q*-partition of* $V(G)$*. We say that* $\mathcal{X}$ *is a* D-partition *if for all* $i, j \in \{1, \ldots, q\}$ *with* $D_q[i,j] = (\mu^+_{i,j}, \mu^-_{i,j})$*, we have that for all* $v \in X_i$*,* $|N^+(v) \cap X_j| \in \mu^+_{i,j}$ *and* $|N^-(v) \cap X_j| \in \mu^-_{i,j}$*. The* d-value *of* D_q *is* $d(D_q) = \max_{i,j}\{d(\mu^+_{i,j}), d(\mu^-_{i,j})\}$*.*

▶ **Theorem 32** (⋆)**.** *Let* D_q *be a bi-neighborhood constraint matrix with* $d = d(D_q)$*. There is an algorithm that given a digraph* G *on* n *vertices together with one of its branch decompositions* $(T, \mathcal{L})$*, determines whether* G *has a* D_q*-partition in time* $\mathcal{O}(\text{nec}_d(T, \mathcal{L})^{3q} \cdot q \cdot n^3 \log n)$*. For* $n \le \text{nec}_d(T, \mathcal{L})$*, the algorithm runs in time* $\mathcal{O}(\text{nec}_d(T, \mathcal{L})^{3q} \cdot q \cdot n^2)$*.*

▶ **Corollary 33.** *Let* D_q *be a bi-neighborhood constraint matrix with* $d = d(D_q)$*. Let* G *be a digraph on* n *vertices with branch decomposition* $(T, \mathcal{L})$ *of bi-mim-width* $w \ge 1$*. There is an algorithm that given any such* G *and* $(T, \mathcal{L})$ *decides whether* G *has a* D_q*-partition in time* $\mathcal{O}(q \cdot n^{3qdw+2})$*.*

6 Conclusion

We introduced the digraph width measure bi-mim-width, and showed that (finitely represented) directed locally checkable vertex problems and their distance-r versions can be solved in polynomial time if the input digraph is given together with a branch decomposition of constant bi-mim-width. A natural next step in the understanding of this new parameter would be to determine the complexity of the Directed Feedback Vertex Set problem on digraphs of bounded bi-mim-width. We showed that several classes of intersection digraphs have constant bi-mim-width which adds a large number of polynomial-time algorithms for locally checkable problems related to domination and independence (given a representation) to the relatively sparse literature on the subject.

Intersection digraph classes such as interval digraphs seem too complex to give polynomial-time algorithms for optimization problems. Our work points to reflexivity as a reasonable additional restriction to give successful algorithmic applications of intersection digraphs, while maintaining a high degree of generality. This was observed independently for interval digraphs by Francis, Hell, and Jacob [22] who studied the Kernel, Absorbing Set, and Dominating Set problems. Apart from giving polynomial-time algorithms for these problems on reflexive interval digraphs, they showed that even for the severely restricted case when the intervals associated with the vertices are single points, the aforementioned problems remain hard.

Reflexivity presents a natural tractability barrier in the case of interval digraphs, or, more generally, H-digraphs for fixed H. The situation is not as clear yet when considering permutation digraphs or rooted directed path digraphs. Both digraph classes contain interval digraphs, therefore the hardness results from [22] apply as well. However, there are no matching polynomial-time algorithms for directed locally checkable vertex problems on reflexive permutation digraphs or reflexive rooted directed path digraphs; in particular, it is not known whether their bi-mim-width is bounded or not. We did show bounds on the bi-mim-width of their *adjusted* subclasses where we additionally require that every pair of objects representing a vertex share a common "endpoint" (where the concrete notion of endpoint depends on the considered type of representation). Arguably, reflexivity is the more natural restriction and one would hope that also in the case of these two digraph classes, it is the right barrier separating the tractable cases from the intractable ones. However, this question remains open for the time being.

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A Additional Definitions

For an undirected graph G and two disjoint vertex sets $A, B \subseteq V(G)$, we denote by $G[A, B]$ the bipartite graph on bipartition (A, B) such that $E(G[A, B])$ is exactly the set of edges of G incident with both A and B.

For a graph G and a set $A \subseteq V(G)$, we let $\text{mim}_G(A) := \nu(G[A, \overline{A}])$. A *branch decomposition of G* is a branch decomposition over $V(G)$ (recall Definition 5).

▶ **Definition 34** (Mim-width). *Let G be a graph and $(T, \mathcal{L})$ be a branch decomposition of G. The* mim-width of $(T, \mathcal{L})$ *is* $\text{mimw}(T, \mathcal{L}) := \max_{e \in E(T)} \text{mim}_G(A_e)$. *The* mim-width of G, *denoted by* $\text{mimw}(G)$, *is the minimum mim-width over all branch decompositions of G. The* linear mim-width of G, *denoted by* $\text{lmimw}(G)$, *is the minimum mim-width over all linear branch decompositions of G.*

Let G be a digraph, and $A, B \subseteq V(G)$ be two disjoint vertex sets. We let $M_G[A \to B]$ the matrix whose columns are indexed by A and whose rows are indexed by B such that for $a \in A$ and $b \in B$, $M_G[A \to B](a, b) = 1$ if $(a, b) \in E(G)$ and $M_G[A \to B](a, b) = 0$ otherwise. For each $A \subseteq V(G)$,we let $\text{cutrk}^+_G(A) := \text{rank}(M_G[A \to \overline{A}])$ and $\text{cutrk}^-_G(A) := \text{rank}(M_G[\overline{A} \to A])$. We let $\text{bicutrk}_G(A) := \text{cutrk}^+_G(A) + \text{cutrk}^-_G(A)$.

▶ **Definition 35** (Bi-rank-width). *Let G be a digraph, and $(T, \mathcal{L})$ be a branch decomposition of G. The* bi-rank-width *of $(T, \mathcal{L})$ is* $\max_{e \in E(T)} \text{bicutrk}_G(A_e)$. *The (linear)* bi-rank-width *of G is the minimum bi-rank-width of any (linear) branch decomposition of G.*

Let T be a rooted directed tree. For a vertex $t \in V(T)$, we denote by T_t the subtree of T containing all vertices v such that there is a directed path from t to v in T.

▶ **Definition 36** (Strong guard). *Let G be a digraph and $X, Y \subseteq V(G)$. We say that Y* is a strong guard for X *if every walk starting and ending in X, and containing a vertex from $V(G) \setminus X$, contains a vertex from Y.*

▶ **Definition 37** (Directed treewidth). *Let G be a digraph. A* directed tree decomposition *is a triple (T, β, γ) of a rooted directed tree T and two maps $\beta \colon V(T) \to 2^{V(G)}$ and $\gamma \colon E(T) \to 2^{V(G)}$,*

1. *The set $\{\beta(t) \colon t \in V(T)\}$ is a partition of $V(G)$.*
2. *For each $e = (u, v) \in E(T)$, $\gamma(e)$ is a strong guard for $\bigcup_{t \in V(T_v)} \beta(t)$.*

For each $t \in V(T)$, we let $\Gamma(t) := \beta(t) \cup \bigcup_{e \sim t} \gamma(e)$, where $e \sim t$ means that e is incident with t. The width *of (T, β, γ) is $\max_{t \in V(T)} |\Gamma(t)| - 1$, and the* directed treewidth *of a digraph G is the minimum width over all its directed tree decompositions.*

We refer to [31] for an introduction to the width measures bi-rank-width and directed treewidth.