Long Cycles in Graphs: Extremal Combinatorics Meets Parameterized Algorithms

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Abstract

We discuss recent algorithmic extensions of two classic results of extremal combinatorics about long paths in graphs. First, the theorem of Dirac from 1952 asserts that a 2-connected graph G with the minimum vertex degree $d > 1$, is either Hamiltonian or contains a cycle of length at least $2d$. Second, the theorem of Erdős-Gallai from 1959, states that a graph G with the average vertex degree $D > 1$, contains a cycle of length at least D. The proofs of these theorems are constructive, they provide polynomial-time algorithms constructing cycles of lengths $2d$ and D. We extend these algorithmic results by showing that each of the problems, to decide whether a 2-connected graph contains a cycle of length at least $2d + k$ or of a cycle of length at least $D + k$, is fixed-parameter tractable parameterized by k.

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1 Introduction

The two fundamental theories from graph theory guarantee the existence of long cycles in dense graphs. The first theorem is Dirac's theorem from 1952.

▶ **Theorem 1** (Dirac [2, Theorem 4]). *Every n-vertex 2-connected undirected graph G with minimum vertex degree $\delta(G) \geq 2$, contains a cycle with at least $\min\{2\delta(G), n\}$ vertices.*

The second theorem from 1959 is due to Erdős and Gallai [3].

▶ **Theorem 2** (Erdős and Gallai [3]). *Every undirected graph with n vertices and more than $\frac{1}{2}(n-1)\ell$ edges ($\ell \geq 2$) contains a cycle of length at least $\ell + 1$.*

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The proofs of both theorems are constructive, in the sense that they provide polynomial-time algorithms constructing cycles of lengths $\min\{2\delta(G), n\}$ and $\ell + 1$. This brings us to a natural and "innocent" question: is it possible to extend the algorithms provided by Theorems 1 and 2 by a "tiny" bit? For example, for an integer $k \geq 1$, *is there a polynomial time algorithm comuting a cycle of length at least $2\delta(G) + k$?* Or, *is it possible to identify in polynomial time whether a graph with $\frac{1}{2}(n-1)\ell$ edges contains a cycle of length at least $\ell + k$?*

The methods developed in the extremal Hamiltonian graph theory do not answer such questions. The combinatorial bounds in Theorems 1 and 2 are known to be sharp; that is, there exist graphs that have no cycles of length at least $\min\{2\delta(G) + 1, n\}$ or $\ell + 2$. Since the extremal graph theory studies the existence of a cycle under certain conditions, such type of questions are beyond its applicability. On the other hand, the existing methods of parameterized complexity, see e.g. [1], do not seem to be much of use here either. Such algorithms compute a cycle of length at least k in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$, which in our case is $2^{\mathcal{O}(\delta(G))} \cdot n^{\mathcal{O}(1)}$. Hence when $\delta(G)$ is, for example, at least $n^{1/100}$, these algorithms do not run in polynomial time.

We answer both questions affirmatively and in a much more general way. Our first theorem, this theorem appears in [4], implies that in polynomial time one can decide whether G contains a cycle of length at least $2\delta(G - B) + k$ for $B \subseteq V(G)$ and $k \geq 0$ as long as $k + |B| \in \mathcal{O}(\log n)$. (We denote by $G - B$ the induced subgraph of G obtained by removing vertices of B.) To state our result more precisely, we define the following problem.

Long Dirac Cycle parameterized by $k + |B|$

Input: Graph G with vertex set $B \subseteq V(G)$ and integer $k \geq 0$.
Task: Decide whether G contains a cycle of length at least $\min\{2\delta(G - B), |V(G)| - |B|\} + k$.

In the definition of Long Dirac Cycle we use the minimum of two values for the following reason. The question whether an n-vertex graph G contains a cycle of length at least $2\delta(G - B) + k$ is meaningful only for $\delta(G - B) \leq n/2$. Indeed, for $\delta(G - B) > n/2$, G does not contain a cycle of length at least $2\delta(G - B) + k > n$. However, even when $\delta(G - B) > n/2$, deciding whether G is Hamiltonian, is still very intriguing. By taking the minimum of the two values, we capture both interesting situations.

▶ **Theorem 3.** *On an n-vertex 2-connected graph G, Long Dirac Cycle is solvable in time $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$.*

In other words, Long Dirac Cycle is fixed-parameter tractable parameterized by $k + |B|$ and the dependence on the parameters is single-exponential. This dependence is asymptotically optimal up to the Exponential Time Hypothesis (ETH) of Impagliazzo, Paturi, and Zane [6]. Solving Long Dirac Cycle in time $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ even with $B = \emptyset$ yields recognizing in time $2^{o(n)}$ whether a graph is Hamiltonian. A subexponential algorithm deciding Hamiltonicity would fail ETH. We show that solving Long Dirac Cycle in time $2^{o(|B|)} \cdot n^{\mathcal{O}(1)}$ even for $k = 1$ would contradict ETH as well. It is also NP-complete to decide whether a 2-connected graph G has a cycle of length at least $(2 + \varepsilon)\delta(G)$ for any $\varepsilon > 0$.

The 2-connectivity requirement in the statement of the theorem is important – without it Long Dirac Cycle is already NP-complete for $k = |B| = 0$. Indeed, for an n-vertex graph G construct a graph H by attaching to each vertex of G a clique of size $n/2$. Then H has a cycle of length at least $2\delta(H) \geq n$ if and only if G is Hamiltonian.

Our second theorem, that appears in [5], provides an algorithmic extension of the Erdős-Gallai theorem: A fixed-parameter tractable (FPT) algorithm with parameter k, that decides whether the circumference (the length of the longest cycle) of a graph is at least $\ell + k$. To state our result formally, we need a few definitions. For an undirected graph G with n vertices and m edges, we define $\ell_{EG}(G) = \frac{2m}{n-1}$. Then by the Erdős-Gallai theorem, G always has a cycle of length at least $\ell_{EG}(G)$ if $\ell_{EG}(G) > 2$. The parameter $\ell_{EG}(G)$ is closely related to the *average* degree of G, $\mathsf{ad}(G) = \frac{2m}{n}$. It is easy to see that for every graph G with at least two vertices, $\ell_{EG}(G) - 1 \leq \mathsf{ad}(G) < \ell_{EG}(G)$.

The *maximum average degree* $\mathsf{mad}(G)$ is the maximum value of $\mathsf{ad}(H)$ taken over all induced subgraphs H of G. Note that $\mathsf{ad}(G) \leq \mathsf{mad}(G)$ and $\mathsf{mad}(G) - \mathsf{ad}(G)$ may be arbitrary large. By Theorem 2, we have that if $\mathsf{ad}(G) \geq 2$, then G has a cycle of length at least $\mathsf{ad}(G)$ and, furthermore, if $\mathsf{mad}(G) \geq 2$, then there is a cycle of length at least $\mathsf{mad}(G)$. Based on this guarantee, we define the following problem.

Longest Cycle Above MAD

Input: A graph G on n vertices and an integer $k \geq 0$.
Task: Decide whether G contains a cycle of length at least $\mathsf{mad}(G) + k$.

Our main result is that this problem is FPT parameterized by k. More precisely, we show the following.

▶ **Theorem 4.** *Longest Cycle Above MAD can be solved in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ on 2-connected graphs.*

While Theorems 1 and 2 concern decision problems, their proofs may be adapted to produce desired cycles, if they exist. We underline this because the standard construction of a long cycle that for every $e \in E(G)$ invokes the decision algorithm on $G - e$, does not work in our case, as edge deletions decrease the average degree of a graph.

We also briefly discuss the ideas behind the proofs of both theorems that are based on an interplay between extremal combinatorics and parameterized algorithms. We develop a new graph decomposition that we call *Dirac decomposition* and then show how to use this decomposition algorithmically. Dirac decomposition is defined for a cycle C in a 2-connected graph G. Let C be a cycle of length less than $2\delta(G) + k$. Informally, the components of Dirac decomposition are connected components in $G - V(C)$. Since G is 2-connected, we can reach C by a path starting in such a component in G. One of the essential properties of Dirac decomposition is a limited number of vertices in $V(C)$ that have neighbors outside of C. In fact, we can choose two short paths P_1 and P_2 in C (and short means that their total length is of order k) such that all connections between connected components of $G - V(C)$ and C go through $V(P_1) \cup V(P_2)$. The second important property is that each connected component of $G - (V(P_1) \cup V(P_2))$ is connected with P_i in G in a very restricted way: The maximum matching size between its vertex set and the vertex set of P_i is at most one. Dirac decomposition appears to be very useful for algorithmic purposes. For a cycle C, given a Dirac decomposition for C, in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ we either solve the problem or succeed in enlarging C.

To apply Dirac decomposition, we also design a polynomial time that (except some "extremal" cases) we can either (a) enlarge the cycle C, or (b) compute a vertex cover of G of size at most $\delta(G) + 2k$, or (c) compute a Dirac decomposition. In cases (a) and (c), we can proceed iteratively. For the case (b) we need another algorithm that solves the problem in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

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