Longest Cycle Above Erdős–Gallai Bound

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Abstract

In 1959, Erdős and Gallai proved that every graph G with average vertex degree $\mathsf{ad}(G) \geq 2$ contains a cycle of length at least $\mathsf{ad}(G)$. We provide an algorithm that for $k \geq 0$ in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ decides whether a 2-connected n-vertex graph G contains a cycle of length at least $\mathsf{ad}(G) + k$. This resolves an open problem explicitly mentioned in several papers. The main ingredients of our algorithm are new graph-theoretical results interesting on their own.

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1 Introduction

The circumference of a graph is the length of its longest (simple) cycle. In 1959, Erdős and Gallai [4] gave the following, now classical, lower bound for the circumference of an undirected graph.

▶ **Theorem 1** (Erdős and Gallai [4]). *Every graph with n vertices and more than $\frac{1}{2}(n-1)\ell$ edges ($\ell \geq 2$) contains a cycle of length at least $\ell + 1$.*

We provide an algorithmic extension of the Erdős-Gallai theorem: A fixed-parameter tractable (FPT) algorithm with parameter k, that decides whether the circumference of a graph is at least $\ell + k$. To state our result formally, we need a few definitions. For an undirected graph G with n vertices and m edges, we define $\ell_{EG}(G) = \frac{2m}{n-1}$. Then by the Erdős-Gallai theorem, G always has a cycle of length at least $\ell_{EG}(G)$ if $\ell_{EG}(G) > 2$. The parameter $\ell_{EG}(G)$ is closely related to the *average* degree of G, $\mathsf{ad}(G) = \frac{2m}{n}$. It is easy to see that for every graph G with at least two vertices, $\ell_{EG}(G) - 1 \leq \mathsf{ad}(G) < \ell_{EG}(G)$.

The *maximum average degree* $\mathsf{mad}(G)$ is the maximum value of $\mathsf{ad}(H)$ taken over all induced subgraphs H of G. Note that $\mathsf{ad}(G) \leq \mathsf{mad}(G)$ and $\mathsf{mad}(G) - \mathsf{ad}(G)$ may be arbitrarily large. By Goldberg [13] (see also [12]), $\mathsf{mad}(G)$ can be computed in polynomial time. By Theorem 1, we have that if $\mathsf{ad}(G) \geq 2$, then G has a cycle of length at least $\mathsf{ad}(G)$ and, furthermore, if $\mathsf{mad}(G) \geq 2$, then there is a cycle of length at least $\mathsf{mad}(G)$. Based on this guarantee, we define the following problem.

Longest Cycle Above MAD

Input: A graph G on n vertices and an integer $k \geq 0$.
Task: Decide whether G contains a cycle of length at least $\mathsf{mad}(G) + k$.

Our main result is that this problem is FPT parameterized by k. More precisely, we show the following.

▶ **Theorem 2.** *Longest Cycle Above MAD can be solved in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ on 2-connected graphs.*

While Theorem 2 concerns the decision variant of the problem, its proof may be easily adapted to produce a desired cycle if it exists. We underline this because the standard construction of a long cycle that for every $e \in E(G)$ invokes the decision algorithm on $G - e$, does not work in our case, as edge deletions decrease the average degree of a graph.

Theorem 2 has several corollaries. The following question was explicitly stated in the literature [6, 9]. For a 2-connected graph G and a nonnegative integer k, how difficult is it to decide whether G has a cycle of length at least $\mathsf{ad}(G) + k$? According to [9], it was not known whether the problem parameterized by k is FPT, W[1]-hard, or Para-NP. Even the simplest variant of the question, whether a path of length $\mathsf{ad}(G) + 1$ can be computed in polynomial time, was open. Theorem 2 resolves this question because $\mathsf{mad}(G) \geq \mathsf{ad}(G)$ for every graph G.

▶ **Corollary 3.** *For a 2-connected graph G and a nonnegative integer k, deciding whether G has a cycle of length at least $\mathsf{ad}(G) + k$ can be done in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.*

Similarly, we have the following corollary.

▶ **Corollary 4.** *For a 2-connected graph G and a nonnegative integer k, deciding whether G has a cycle of length at least $\ell_{EG}(G) + k$ can be done in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.*

An undirected graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d, and the *degeneracy* of G is defined to be the minimum value of d for which G is d-degenerate. Since a graph of degeneracy d has a subgraph H with at least $d \cdot |V(H)|/2$ edges, we have that $d \leq \mathsf{ad}(H) \leq \mathsf{mad}(G)$. Therefore, Theorem 2 implies the following corollary, which is the main result of [6].

▶ **Corollary 5** ([6]). *For a 2-connected graph G of degeneracy d, deciding whether G has a cycle of length at least $d + k$ can be done in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.*

Theorem 1 provides the same lower bound on the number of vertices in a longest path. We consider the Longest Path Above MAD problem that, given a graph G and integer k, asks whether G has a path with at least $\mathsf{mad}(G) + k$ vertices. Observe that a graph G has a path with ℓ vertices if and only if the graph G', obtained by adding to G a universal vertex that is adjacent to every vertex of the original graph, has a cycle with $\ell + 1$ vertices. Because $\mathsf{mad}(G') \geq \mathsf{mad}(G)$, Theorem 2 yields the following.

▶ **Corollary 6.** *Longest Path Above MAD can be solved in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ on connected graphs.*

We complement Theorem 2 by observing that the 2-connectivity condition is crucial for tractability due to the fact that the considered properties are not closed under taking biconnected components. In particular, it may happen that every long cycle of a graph is in a biconnected component of small average degree. This yields the following theorem.

▶ **Theorem 7** ($\star$).[1] *It is* NP*-complete to decide whether an n-vertex connected graph G has a cycle of length at least $\ell_{EG}(G) + 1$.*

The single-exponential dependence in k of algorithm in Theorem 2 is asymptotically optimal: it is unlikely that Longest Cycle Above MAD can be solved in $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ time. This immediately follows from the well-known result (see e.g. [2, Chapter 14]) that existence of an algorithm for Hamiltonian Cycle with running time $2^{o(n)}$ would refute the *Exponential Time Hypothesis* (ETH) of Impagliazzo, Paturi, and Zane [14]. Thus Longest Cycle Above MAD cannot be solved in $2^{o(k)} \cdot n^{\mathcal{O}(1)}$ time, unless ETH fails.

Comparison with the previous work. Two of the recent articles on the circumference of a graph above guarantee are most relevant to our work. The first is the paper of Fomin, Golovach, Lokshtanov, Panolan, Saurabh, and Zehavi [6] who gave an algorithm that in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ for a 2-connected graph G of degeneracy d, decides whether G has a cycle of length at least $d+k$. In the heart of their algorithm is the following "rerouting" argument: If a cycle hits a sufficiently "dense" subgraph H of G, then this cycle can be rerouted inside H to cover all vertices of H. The main obstacle on the way of generalizing the result of Fomin et al. [6] "beyond" the average degree was the lack of rerouting arguments in graphs of large average degree.

The rerouting arguments in the proof of Theorem 2 use the structural properties of dense graphs developed in the recent work of Fomin, Golovach, Sagunov, and Simonov [9] (see [8] for the full version) on parameterized complexity of finding a cycle above Dirac's bound. We remind that by the classical theorem of Dirac [3], every 2-connected graph has a cycle of length at least $\min\{2\delta(G), |V(G)|\}$, where $\delta(G)$ is the minimum degree of G. Fomin et al. gave an algorithm that in time $2^{\mathcal{O}(k+|B|)} \cdot n^{\mathcal{O}(1)}$ decides whether a 2-connected graph G contains a cycle of length at least $\min\{2\delta(G-B), |V(G)|-|B|\}+k$, where B is a given subset of vertices which may have "small" degrees. The result of Fomin et al. [8, 9] is "orthogonal" to ours in the following sense: It does not imply Theorem 2 and Theorem 2 does not imply the theorem from [8]. However, the tools developed in [8], in particular the new type of graph decompositions called *Dirac decompositions*, appear to be useful in our case too.

From a more general perspective, our work belongs to a popular subfield of Parameterized Complexity concerning parameterization above/below specified guarantees. In addition to [9, 6], the parameterized complexity of paths and cycles above some guarantees was studied in [1, 15], and [7].

2 Overview of the proof of the main result

Here we outline the critical technical ideas leading to our main result, Theorem 2. We first explain our techniques for the Longest Cycle Above AD problem. Let us remind that in this problem, the task is to decide whether a graph G has a cycle of length at least $\mathsf{ad}(G)+k$. (The difference with mad is that we do not take the maximum over all subgraphs.)

[1] The results with omitted proofs are marked with the "$\star$" sign. Missing proofs can be found in the full version of this paper [10].

The nucleus of our proof is a novel structural analysis of dense subgraphs in graphs with large average degrees. Informally, we prove that if there is a cycle of length at least $\mathsf{ad}(G)+k$ in G, then G contains a dense subgraph H and a long (of length at least $\mathsf{ad}(G)+k$) cycle C that "revolves" around H (see Figure 1). By that, we mean the following. First, the number of times cycle C enters and leaves H is bounded by $\mathcal{O}(k)$. Second, C contains at least $\mathsf{ad}(G) - ck$ vertices of H for some constant c. Moreover, we need a way stronger "routing" property of H. Basically for any possible "points of entry and departure" of cycle C in H, we show that these pairs of vertices could be connected in H by internally vertex-disjoint paths of total length at least $\mathsf{ad}(G) - ck$. Furthermore, such paths could be found in polynomial time. Then everything boils down to the following problem. For a given subgraph H of G, we are looking for at most k internally vertex-disjoint paths outside H of total length $\Omega(k)$, each path starts and ends in H. This task can be done in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ by making use of color-coding. Finally, if we find such paths, then we could complete them to a cycle of length at least $\mathsf{ad}(G)+k$ by augmenting them by the paths inside H.

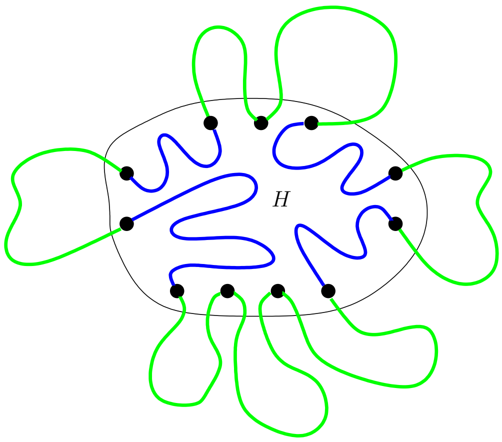

Figure 1 A cycle "revolving" around H. The segments of the cycle outside H are shown in green and the segments inside H are blue.

Identifying dense subgraph H. Notice that we can assume that $\mathsf{ad}(G) \geq \alpha k$ for a sufficiently big positive constant α. Otherwise, we can solve the problem in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time using the known algorithm for Longest Cycle [11]. We start with preprocessing rules "illuminating" some "useless" parts of the graph. If G contains several connected components, it suffices to keep only the densest of them, as its average degree is at least the average degree of G. Similarly, if G is connected but has a cut-vertex, keeping the densest block also suffices. Further, if there is a vertex v of degree less than $\frac{1}{2}\mathsf{ad}(G)$, then v can be safely removed. By applying these reduction rules exhaustively, we find an induced 2-connected subgraph H of G whose minimum degree $\delta(H) \geq \frac{1}{2}\mathsf{ad}(H) \geq \frac{1}{2}\mathsf{ad}(G)$. Similarly to removing sparse blocks, if G contains a vertex separator X of size two such that there is a "sparse" component A of $G - X$, then A can be removed. By applying the last reduction rule we either find a cycle of length at least $\mathsf{ad}(G)+k$ or can conclude that the resulting subgraph H is 3-connected.

If (G, k) is a yes-instance, that is, graph G contains a cycle of length at least $\mathsf{ad}(G)+k$, there are two possibilities. Either in G a cycle of length at least $2\delta(H)+k$ "lives" entirely in H, or it passes through some other vertices of G. If a long cycle is entirely in H, we can employ the recent result of Fomin et al. [8] that finds in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ in a 2-connected graph G a cycle of length at least $2\delta(G)+k \geq \mathsf{ad}(G)+k$. However, if no long cycle lives entirely in H, the result of Fomin et al. is not applicable.

The next step of constructing H crucially benefits from the graph-theoretical result of Fomin et al. [8]. Specifically, we use the theorem about the Dirac decomposition from [8]. The definition of the Dirac decomposition is technical and we give it in Section 4. For 2-connected graphs, the Dirac decomposition imposes a very intricate structure. However, since, thanks to the reduction rules, H is 3-connected, we bypass most of the technical details from [8]. Informally, the Dirac decomposition leads to the following win-win situation. By the Dirac's theorem [3], graph H contains a cycle S of length at least $2\delta(H) \geq \mathsf{ad}(G)$. Moreover, we could find such a cycle in polynomial time. By the result of Fomin et al. [8], if the length of S is less than $2\delta(H) + k$, then either S can be enlarged in polynomial time, or (a) H is small, that is, $|V(H)| < \mathsf{ad}(H) + k$, yielding that H is extremely dense; or (b) H has a vertex cover of size $\frac{1}{2}\mathsf{ad}(H) - \mathcal{O}(k)$. If S got enlarged, we iterate until we achieve cases (a) or (b). If we are in case (a), the construction of H is completed. In case (b), we need to prune the obtained graph a bit more. More specifically, we can delete $\mathcal{O}(k)$ vertices in the vertex cover and select a subset of the independent set to achieve the property that (i) each of remaining vertices in the vertex cover is adjacent to at least $\mathsf{ad}(H) - \mathcal{O}(k)$ vertices in the selected independent subset, and (ii) every vertex of the selected subset of the independent set sees nearly all vertices of the vertex cover. This mean that the obtained induced subgraph is also "dense", albeit in a different sense. Depending on the case, we use different arguments to establish the routing properties of H.

Routing in H. The case (a), when $|V(H)| < \mathsf{ad}(H) + k$, is easier. In this case, the degrees of almost all vertices are close to $|V(H)|$. Let $S = \{x_1y_1, \ldots, x_\ell y_\ell\}$ an arbitrary set of $\mathcal{O}(k)$ pairs of distinct vertices of H forming a linear forest (that is, the union of x_iy_i is a union of disjoint paths). The intuition behind S is that x_i corresponds to the vertex from where the long cycle leaves H and y_i when it enters H again. We show first how to construct a cycle in $H + S$ (that is, the graph obtained from H by turning the pairs of S into edges) containing every pair x_iy_i from S as an edge. This is done by performing constant-length jumps: any two vertices can be connected either by an edge, or through a common neighbor, or through a sequence of two neighbors. Then we extend the obtained cycle to a Hamiltonian cycle in $H + S$ – every vertex of H that is not yet on a cycle can be inserted due to the high degrees of the vertices. The extension of S into a Hamiltonian cycle is shown in Figure 2 (a).

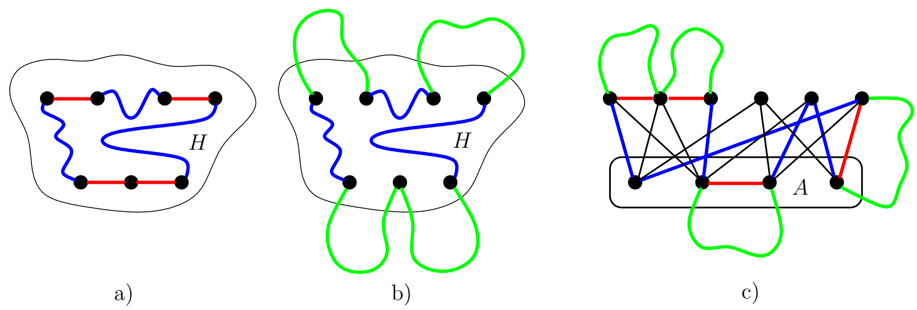

Figure 2 Constructing cycles. The set of pairs S that may be both edges and nonedges of H is shown by red lines and the extension of S into a long cycle is blue. The paths "revolving" around H are green. The vertex cover in c) is denoted by A.

Therefore, if there is a collection of at most k internally vertex disjoint paths going outside from H and returning back, the high density of H allows collecting all of them in a cycle containing all the vertices of H. Together with all the additional vertices these paths visit outside of H we construct a long cycle in G (see Figure 2 (b)). The only condition is that these paths have to form a linear forest. Thus, if we find a collection of such paths with enough internal vertices, we immediately obtain a long cycle "revolving" around H. The crucial part of the proof is to show that if there is *any* cycle of length at least $\mathsf{ad}(H) + k$ in G, then it can be assumed to have this form.

Let us remark that a similar "rerouting" property was used by Fomin et al. [6] in their above-degeneracy study. Actually, for case (a), we need only a minor adjustment of the arguments from [6]. However, in the "bipartite dense" case (b) the structure of the dense subgraph H is more elaborate and this case requires a new approach. Contrarily to case (a), the long cycle that we construct in $H + S$ is not Hamiltonian but visits all the vertices of the vertex cover (see Figure 2 (c)). In this case, the behavior of paths depends on which part of H they hit. Because of that, while establishing the routing properties, we have to take into account the difference between paths connecting vertices from the vertex cover, independent set, and both. Pushing the "rerouting" intuition through, in this case, turns out to be quite challenging.

Final steps. After finalizing the "rerouting" arguments above, it only remains to design an algorithm that checks whether there exists a collection of paths in G that start and end in H and have at least a certain number of internal vertices in total. We do it by a color-coding-style approach. For case (a), such a subroutine has already been developed in the above-degeneracy case [6]. On the other hand, for the "bipartite dense" case (b) we need to impose an additional restriction on the desired paths, as the length of the final cycle also depends on how the paths' end-vertices are distributed between the two parts and we have to incorporate these kinds of constraints in our path-finding subroutine.

Finally, to solve Longest Cycle Above MAD, we use the fact that given a graph G, we can find an induced subgraph F with $\mathsf{ad}(F) = \mathsf{mad}(G)$ in polynomial time by the result of Goldberg [13] (see also [12]). Then we find a dense subgraph H of F with the described properties and use H to find a cycle of length at least $\mathsf{mad}(G) + k$.

3 Preliminaries

In this section, we introduce basic notations, and a series of previously-known results that will be helpful to us.

We consider only finite undirected graphs. For a graph G, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Throughout the paper we use $n = |V(G)|$ and $m = |E(G)|$ whenever the considered graph G is clear from the context. For a graph G and a subset $X \subseteq V(G)$ of vertices, we write $G[X]$ to denote the subgraph of G induced by X. We write $G - X$ to denote the graph $G[V(G) \setminus X]$; for a single-element set $X = \{x\}$, we write $G - x$. Similarly, if Y is a set of pairs of distinct vertices, $G - Y = (V(G), E(G) \setminus Y)$. For a set Y of pairs of distinct vertices of G, $G + Y$ denotes the graph $(V(G), E(G) \cup Y)$, that is, the graph obtained by adding the edges in $Y \setminus E(G)$; slightly abusing notation we may denote the pairs of such a set Y in the same way as edges. For a vertex v, we denote by $N_G(v)$ the *(open) neighborhood* of v, i.e., the set of vertices that are adjacent to v in G. A set of vertices X is a *vertex cover* of G if for every edge xy of G, $x \in X$ or $y \in X$.

A *path* P in G is a subgraph of G with $V(P) = \{v_0, \ldots, v_\ell\}$ and $E(P) = \{v_{i-1}v_i \mid 1 \leq i \leq \ell\}$. We write $v_0v_1 \cdots v_\ell$ to denote P; the vertices v_0 and v_ℓ are *end-vertices* of P, the vertices $v_2, \ldots, v_\ell$ are *internal*, and ℓ is the *length* of P. For a path P with end-vertices s and t, we say that P is an (s, t)-path. Two paths P_1 and P_2 are *internally disjoint* if no internal vertex of one of the paths is a vertex of the other; note that end-vertices may be the same.

For two internally disjoint paths P_1 and P_2 having one common end-vertex, we write P_1P_2 to denote the *concatenation* of P_1 and P_2. A graph F is a *linear forest* if every connected component of F is a path. Let S be a set of pairs of distinct vertices of G; they may be either edges or nonedges. We say that S is *potentially cyclable* if $(V(G), S)$ is a linear forest. A *cycle* is a graph C with $V(C) = \{v_1, \ldots, v_\ell\}$ for $\ell \geq 3$ and $E(C) = \{v_{i-1}v_i \mid 1 \leq i \leq \ell\}$, where $v_0 = v_\ell$. We may write that $C = v_1 \cdots v_\ell$. A cycle C (a path P, respectively) is *Hamiltonian* if $V(C) = V(G)$ ($V(P) = V(G)$, respectively). A graph G is *Hamiltonian* if it has a Hamiltonian cycle.

A set of vertices S is a *separator* of a connected graph G, if $G - S$ is disconnected. For a positive integer k, G is *k-connected* if $|V(G)| > k$ and for every set S of at most $k-1$ vertices, $G - S$ is connected. If $S = \{v\}$ is a separator of size one, then v is called a *cut-vertex*. Note, in particular, that a connected graph with at least three vertices is 2-connected if it has no cut-vertex. A *block* of a connected graph with at least two vertices is an inclusion-wise maximal induced subgraph without cut-vertices, that is, either a 2-connected graph or K_2.

The *degree* of a vertex v in a graph G is $\mathsf{d}_G(v) = |N_G(v)|$. The *minimum degree* of G is $\delta(G) = \min\{\mathsf{d}_G(v) \mid v \in V(G)\}$. For a nonempty set of vertices X, the *average degree* of X is $\mathsf{ad}_G(X) = \frac{1}{|X|}\sum_{v \in X} \mathsf{d}_G(v)$, and the *average degree* of G is $\mathsf{ad}(G) = \mathsf{ad}_G(V(G)) = \frac{2m}{n}$. The *maximum average degree* is $\mathsf{mad}(G) = \max\{\mathsf{ad}(H) \mid H \text{ is induced subgraph of } G\}$.

The following observation about the circumference lower bound $\ell_{EG}(G)$ and the average degree of G is useful for us.

▶ **Observation 8.** *For every graph G with at least two vertices $\ell_{EG}(G) - 1 \leq \mathsf{ad}(G) < \ell_{EG}(G)$.*

Goldberg [13] proved that, given a graph G, an induced subgraph H of maximum *density*, that is, a subgraph with the maximum value $\frac{|E(H)|}{|V(H)|}$, can be found in polynomial time. This result was improved by Gallo, Grigoriadis, and Tarjan [12]. Note that if H is an induced subgraph of maximum density, then $\mathsf{mad}(G) = \mathsf{ad}(H)$.

▶ **Proposition 9** ([12]). *An induced subgraph of maximum density of a given graph G can be found in $\mathcal{O}(nm\log(n^2/m))$ time.*

We use the lower bound on the length of a longest (s,t)-path in a 2-connected graph via the average degree obtained by Fan [5].

▶ **Proposition 10** ([5, Theorem 1]). *Let s and t be two distinct vertices in a 2-connected graph G. Then G has an (s,t)-path of length at least $\mathsf{ad}_G(V(G) \setminus \{s,t\})$.*

Notice that the proof of Proposition 10 in [5] is constructive and a required path can be found in polynomial time.

It is well-known that Longest Cycle, which asks whether a graph has a cycle of length at least k, can be solved in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time. The currently best deterministic algorithm is due to Fomin et al. [11].

▶ **Proposition 11** ([11]). *Longest Cycle can be solved in $4.884^k \cdot n^{\mathcal{O}(1)}$ time.*

The task of Longest (s,t)-Path is, given a graph G with two *terminal* vertices s and t, and a positive integer k, decide whether G has an (s,t)-path with at least k vertices. Fomin et al. [11] proved that this problem is FPT when parameterized by k.

▶ **Proposition 12** ([11]). *Longest (s,t)-Path can be solved in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time.*

4 Finding a dense subgraph

Here we show that given an instance of Longest Cycle Above MAD, we can in polynomial time either solve the problem or find a dense induced subgraph of the input graph. This part crucially depends on structural and algorithmic results obtained by Fomin et al. in [8]. We derive the following structural corollary for 3-connected graphs from [8, Lemma 20].

▶ **Corollary 13** ($\star$). *Let G be a 3-connected graph and k be an integer such that $0 < k \leq \frac{1}{24}\delta(G)$. Then there is an algorithm that, given a cycle C of length less than $2\delta(G) + k$, in polynomial time either*

- *returns a longer cycle in G, or*
- *returns a vertex cover of G of size at most $\delta(G) + 2k$, or*
- *reports that C is Hamiltonian.*

Now, by applying exhaustively the classical reduction rules from the proof of Theorem 1 and a new reduction rule that removes sparse 2-connected components, we reach the situation where we can apply Corollary 13. The result of this process is encapsulated in the next lemma.

▶ **Lemma 14** ($\star$). *There is a polynomial-time algorithm that, given an instance (G, k) of Longest Cycle Above MAD, where $0 < k \leq \frac{1}{80}\mathsf{mad}(G) - 1$, either*

(i) *finds a cycle of length at least $\mathsf{mad}(G) + k$ in G, or*
(ii) *finds an induced subgraph H of G with $\mathsf{ad}(H) \geq \mathsf{mad}(G) - 1$ such that $\delta(H) \geq \frac{1}{2}\mathsf{ad}(H)$ and $|V(H)| < \mathsf{ad}(H) + k + 1$, or*
(iii) *finds an induced subgraph H of G such that there is a partition $\{A, B\}$ of $V(H)$ with the following properties:*
  - *B is an independent set,*
  - *$\frac{1}{2}\mathsf{mad}(G) - 4k \leq |A|$,*
  - *for every $v \in A$, $|N_H(v) \cap B| \geq 2|A|$,*
  - *for every $v \in B$, $\mathsf{d}_H(v) \geq |A| - 2k - 2$.*

5 Covering vertices of dense graphs

In this section, we prove that, given a sufficiently dense graph G and a bounded-size set of pairs of distinct vertices S forming a linear forest, we can find a long cycle in $G + S$ containing all edges from S. First, we consider the case where there is a small number of vertices in the graph compared to the average degree. Then, we deal with the case where one part in a bipartition of a dense bipartite graph has bounded size. The proofs of the next two lemmas follow the strategy of using the high density of the graph to connect an arbitrary small subset of vertices in a cycle via constant-length jumps, and then extend this cycle to a long cycle using similar arguments. Recall that for a set S of pairs of distinct vertices of a graph G, we say that S is potentially cyclable if $(V(G), S)$ is a linear forest.

▶ **Lemma 15** ($\star$). *Let G be a graph and k be an integer such that (i) $0 < k \leq \frac{1}{60}\mathsf{ad}(G)$, (ii) $\delta(G) \geq \frac{1}{2}\mathsf{ad}(G)$, and (iii) $\mathsf{ad}(G) + k > n$. Let also S be a potentially cyclable set of at most k pairs of distinct vertices. Then $G + S$ has a Hamiltonian cycle containing every edge of S.*

Now we consider dense bipartite graphs. Similarly to Lemma 15, we show that for a given set of pairs of vertices forming a linear forest there is a cycle containing all these pairs in the extended graph, and also each vertex of the "high-degree" part of the graph. For an example, see Figure 3.

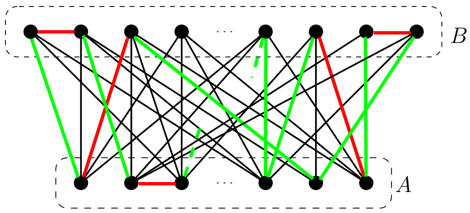

Figure 3 Structure of G and $G' = G + S$. The set of pairs S is shown by red lines and the edges of C that are not in S are green. Note that G' is not required to be bipartite.

▶ **Lemma 16** ($\star$). *Let G be a bipartite graph, $\{A, B\}$ is a bipartition of $V(G)$ with $p = |A|$, and let k be an integer such that (i) $0 < k \leq \frac{1}{10}p$, (ii) for every $v \in A$, $\mathsf{d}_G(v) \geq 2p$, and (iii) for every $v \in B$, $\mathsf{d}_G(v) \geq p - k$. Let S be a potentially cyclable set of at most $\frac{9}{4}k$ pairs of distinct vertices. Then $G' = G + S$ has a cycle C containing every edge of S and every vertex of A. Furthermore, C is a longest cycle in G' containing the edges of S and the length of C is $2p - s + t$, where s is the number of edges of S with both end-vertices in A and t is the number of edges in S with both end-vertices in B.*

6 Rerouting long cycles to dense dubgraphs

In this section, we show that a dense induced subgraph can be used to find a long cycle in a 2-connected graph. Specifically, we show that one can always assume that a long cycle is an extension of a longest cycle in a dense subgraph. To state this more precisely, we need some additional terminology that we introduce next.

Let $T \subseteq V(G)$ for a graph G. A path P is called a *T-segment* if P has length at least two, the end-vertices of P lie in T, and $v \notin T$ for any internal vertex v of P. A set of internally disjoint paths $\mathcal{P} = \{P_1, \ldots, P_r\}$ is a *system of T-segments* if (i) P_i is a T-segment for every $i \in \{1, \ldots, r\}$, and (ii) the union of the paths in $\mathcal{P}$ is a linear forest. Let $A, B \subseteq V(G)$ be disjoint sets of vertices in G. For a pair $\{x, y\}$ of distinct vertices in G, we say that $\{x, y\}$ is an *A-pair* (*B-pair*, respectively) if $x, y \in A$ ($x, y \in B$, respectively), and we say that $\{x, y\}$ is an *(A, B)-pair* if either $x \in A$, $y \in B$ or, symmetrically, $y \in A$, $x \in B$. If $\{A, B\}$ is a partition of $T \subseteq V(G)$, then for a T-segment P with end-vertices x and y, P is an *A-segment* if $\{x, y\}$ is an A-pair, P is a *B-segment* if $\{x, y\}$ is a B-pair, and P is an *(A, B)-segment* if $\{x, y\}$ is an $\{A, B\}$-pair.

First, we consider the case when there is a dense subgraph H with the property that for every potentially cyclable set S of at most k pairs of distinct vertices, $H + S$ has a Hamiltonian cycle containing every edge of S. We show the following lemma whose proof is almost identical to the proof of Lemma 3 in [6].

▶ **Lemma 17** ($\star$). *Let G be a 2-connected graph and let k be a positive integer. Suppose that H is an induced subgraph of G such that $|V(H)| \geq 2k$ and for every potentially cyclable set S of at most k pairs of distinct vertices of H, $H + S$ has a Hamiltonian cycle containing every edge of S. Then G has a cycle of length at least $|V(H)| + k$ if and only if one of the following holds:*

(i) *There are two distinct vertices $s, t \in V(H)$ such there is an (s, t)-path P in G of length at least $k + 1$ whose internal vertices lie in $V(G) \setminus V(H)$.*

(ii) *There is a system of T-segments $\mathcal{P} = \{P_1, \ldots, P_r\}$ for $T = V(H)$ such that $r \leq k$ and the total number of vertices on the paths in $\mathcal{P}$ outside T is at least k and at most $2k - 2$.*

Now we show a related result for dense induced subgraphs of another type. See Figure 4 for an illustration.

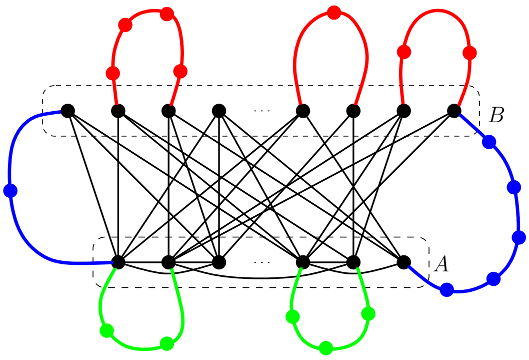

Figure 4 Structure of segments in Case (ii) of Lemma 18. The A-segments are shown by green lines, the B-segments are red, and the (A,B)-segments are blue.

▶ **Lemma 18.** *Let G be a 2-connected graph and let k be a positive integer. Suppose that H is an induced subgraph of G whose set of vertices has a partition $\{A, B\}$ with $|A| \geq \frac{3}{2}k$ and B being an independent set. Suppose also that for every potentially cyclable set S in H of at most k pairs of distinct vertices in H, with s A-pairs and t B-pairs, $H + S$ has a cycle of length at least $2|A| - s + t$. Then G has a cycle of length at least $2|A| + k$ if and only if one of the following holds:*

- **(i)** *There are two distinct vertices $x, y \in V(H)$ such that H has an (x, y)-path P of length at least $k + 2$ whose internal vertices lie in $V(G) \setminus V(H)$.*
- **(ii)** *There is a system of T-segments $\mathcal{P} = \{P_1, \ldots, P_r\}$ for $T = V(H)$ with s A-segments and t B-segments such that*
  - **(a)** *$r \leq k$,*
  - **(b)** *every A-segment has at least two internal vertices,*
  - **(c)** *the total number of internal vertices on the paths in $\mathcal{P}$ is at least $k + s - t$ and at most $3k - 2$.*

Proof. Let $T = V(H)$. First, we show that if either (i) or (ii) is fulfilled, then G has a cycle of length at least $|V(H)| + k$.

Suppose that there are distinct $x, y \in T$ and an (x, y)-path P in G with all internal vertices outside T such that the length of P is at least $k + 2$. Let $S = \{xy\}$. We have that $H + S$ has a cycle C containing xy of length at least $2|A| - 1$. We replace the edge xy in C by the path P. Then the length of the obtained cycle C' is at least $2|A| + k$ as required.

Assume that there is a system of T-segments $\mathcal{P} = \{P_1, \ldots, P_r\}$ for $T = V(H)$ with s A-segments and t B-segments such that (a)–(b) are fulfilled. Let x_i and y_i be the end-vertices of P_i for $i \in \{1, \ldots, r\}$ and define $S = \{x_1y_1, \ldots, x_ry_r\}$. Observe that S is a potentially cyclable set for H and $|S| \leq k$. Then $H + S$ has a cycle C of length at least $2|A|$ that contains every edge of S. We construct the cycle C' from C by replacing x_iy_i by the path P_i for every $i \in \{1, \ldots, r\}$. Because the total number of internal vertices in the paths of $\mathcal{P}$ is at least $k + s - t$, the length of C' is at least $|V(H)| + k$.

For the opposite direction, assume that G has a cycle C of length at least $2|A| + k$. We consider the following three cases.

Case 1. $V(C) \cap T = \emptyset$. Since G is a 2-connected graph, there are pairwise distinct vertices $x, y \in T$ and $x', y' \in V(C)$, and vertex disjoint (x, x') and (y, y')-paths P_1 and P_2 such that the internal vertices of the paths are outside $T \cup V(C)$. The cycle C has length at least $2|A| + k \geq 3k$. Therefore, C contains an (x', y')-path P with at least $k + 1$ vertices. The concatenation of P_1, P and P_2 is an (x, y)-path in G of length at least $k + 2$ whose internal verices are outside T. Hence, (i) is fulfilled.

Case 2. $|V(C) \cap T| = 1$. Let $V(C) \cap T = \{x\}$ for some vertex x. Since G is 2-connected, there is an (y, y')-path P in $G - x$ such that $y' \in V(C)$, $y \in T$, and the internal vertices are outside $T \cup V(C)$. Because the length of C is at least $3k$, C contains an (x', y')-path P' with at least $k + 2$ vertices. The concatenation of P' and P is an (s, t)-path in G of length at least $k + 2$ whose internal verices are outside T. Hence, (i) holds.

Case 3. $|V(C) \cap T| \geq 2$. Observe that because B is an independent set, H has no cycle of length greater that $2|A|$. Therefore, as $k > 0$ and $|V(C)| \geq 2|A| + k$, $V(C) \setminus T \neq \emptyset$. Let $P_1, \ldots, P_\ell$ be the "outside" segments of C with respect to H, that is, $P_1, \ldots, P_\ell$ are paths on C such that $(*)$ for every $i \in \{1, \ldots, \ell\}$, P_i is an (x_i, y_i)-path with at least one internal vertex for some distinct $x_i, y_i \in T$ and the internal verices of P_i are outside T, and $(**)$ $\bigcup_{i=1}^{\ell} V(P_i) \setminus T = V(C) \setminus T$. If P_i has length at least $k + 2$ for some $i \in \{1, \ldots, \ell\}$, then (i) holds. Assume that this is not the case, that is, the length of each P_i is at most $k + 1$. Let $I_A, I_B, I_{AB} \subseteq \{1, \ldots, \ell\}$ be the subsets of indices such that P_i is an B-segment for $i \in I_B$, a B-segment for $i \in I_B$, and an (A, B)-segment for $i \in I_{AB}$; note that some of these sets may be empty.

First, we consider I_B. Suppose that the paths P_i for $i \in I_B$ have at least $k - |I_B|$ internal vertices. Consider an inclusion minimal subset of indices $J \subseteq I_B$ such that the paths P_i for $i \in J$ have at least $k - |J|$ internal vertices and let $S = \{x_i y_i \mid i \in J\}$. Observe that the pairs of S compose either a linear forest or a cycle. Suppose that the pairs in S form a cycle. Then every edge of C is outside H, and we have that C is the concatenation of the paths $P_i \in J$. Note that $|J| \geq 2$ in this case. Let $j \in J$. By the choice of J, the total number of internal vertices on the paths P_i for $i \in J \setminus \{j\}$ is at most $k - |J| - 1$. Because the length of P_j is at most $k + 1$, we have that $|V(C)| \leq (k - |J| - 1) + |J| + k = 2k - 1 < 2|A| + k$; a contradiction. Therefore, S forms a linear forest. We obtain that $\mathcal{P} = \{P_i \mid i \in J\}$ is a system of T segments and $|\mathcal{P}| \leq k$. To see that the total number of internal vertices on the paths in $\mathcal{P}$ is at most $2k$, let $j \in J$. Because the total number of internal vertices on the paths P_i for $i \in J \setminus \{j\}$ is at most $k - |J| - 1$ and the length of P_j is at most $k + 1$, the number of internal vertices on the paths in $\mathcal{P}$ is at most $(k - |J| - 1) + k \leq 3k - 2$. We conclude that (ii) is fulfilled.

Assume from now on that the paths P_i for $i \in I_B$ have at most $k - |I_B| - 1$ internal vertices. Then we analyse I_{AB} in a similar way. Let $t = |I_B|$. Suppose that the paths P_i for $i \in I_{AB} \cup I_B$ have at least $k - t$ internal vertices. Consider an inclusion minimal subset of indices $J \subseteq I_{AB}$ such that the paths P_i for $i \in J \cup I_B$ have at least $k - t$ internal vertices and let $S = \{x_i y_i \mid i \in J \cup I_B\}$. Notice that $|S| \leq k$. Again, we have that the pairs of S compose either a linear forest or a cycle. Then we exclude the possibility that S forms a cycle. If we have a cycle, then C is the concatenation of the paths $P_i \in J \cup I_B$. Pick an arbitrary $j \in J$. We have that the total number of internal vertices on the paths P_i for $i \in (J \setminus \{j\}) \cup I_B$ is at most $k - t - 1$. Because the length of P_j is at most $k + 1$, $|V(C)| \leq (k - t - 1) + (|J| + t) + k = 2k + |J| - 1 < 2|A| + k$ and we get a contradiction. Hence, S forms a linear forest and $\mathcal{P} = \{P_i \mid i \in J \cup I_B\}$ is a system of T segments and

$|\mathcal{P}| \leq k$. To upper bound the total number of internal vertices on the paths in $\mathcal{P}$, let $j \in J$. Because the total number of internal vertices on the paths P_i for $i \in (J \setminus \{j\}) \cup I_B$ is at most $k - t - 1$ and the length of P_j is at most $k + 1$, the number of internal vertices on the paths in $\mathcal{P}$ is at most $2k - t - 1 \leq 3k - 2$. We obtain that (ii) holds.

It remains to consider the case where the paths P_i for $i \in I_{AB} \cup I_B$ have at most $k - t - 1$ internal vertices. For this we analyse I_A. Let $I'_A \subseteq I_A$ be the set of indices $i \in I_A$ such that P_i has at least two internal vertices. Let r be the number of internal vertices on the paths P_i with $i \in I'_A \cup I_B \cup I_{AB}$. Observe that because B is an independent set, $|V(C)| \leq r + t + 2|A| - |I'_A|$. Hence, $r + t - |I'_A| \geq k$. We select an inclusion minimal set of indices $J \subseteq I'_A$ such that the paths P_i for $i \in J \cup I_B \cup I_{AB}$ have at least $k - t + |J|$ internal vertices and let $S = \{x_i y_i \mid i \in J \cup I_B \cup I_{AB}\}$. Let also $s = |J|$. Observe that because P_i has at least two internal vertices for every $i \in I'_A$, $|S| \leq k$. In the same way as above, the pairs of S compose either a linear forest or a cycle, and we show that it should be a linear forest. If the pairs of S form a cycle, then C is the concatenation of the paths $P_i \in J$. Let $j \in J$. By the minimality of J, the total number of internal vertices on the paths P_i for $i \in (J \setminus \{j\}) \cup I_B \cup I_{AB}$ is at most $k + (s - 1) - t - 1$. Because the length of P_j is at most $k + 1$, $|V(C)| \leq (k + (s - 1) - t - 1) + (s + t + |I_{AB}|) + k = 2k + |I_{AB}| + 2s - 2$. Observe that $t + s + |I_{AB}| \leq k$, because if $t + s + |I_{AB}| \geq k + 1$, the total number of the internal vertices on the paths P_i for $i \in (J \setminus \{j\}) \cup I_B \cup I_{AB}$ would be at least $k + s$. Therefore, $|V(C)| \leq 2k + |I_{AB}| + 2s - 2 \leq 4k - 2 < 2|A| + k$; a contradiction. We obtain that S forms a linear forest and $\mathcal{P} = \{P_i \mid i \in J \cup I_B \cup I_{AB}\}$ is a system of T segments, and $|\mathcal{P}| \leq k$. To get the upper bound for the total number of internal vertices on the paths in $\mathcal{P}$, let $j \in J$. Because the total number of internal vertices on the paths P_i for $i \in (J \setminus \{j\}) \cup I_B$ is at most $k + (s - 1) - t - 1$ and the length of P_j is at most $k + 1$, the number of internal vertices on the paths in $\mathcal{P}$ is at most $2k + s - t - 2 \leq 3k - 2$. We conclude that (ii) is fulfilled. This concludes the analysis of Case 3 and the proof of the lemma. ◀

Fomin et al. [6] proved the following algorithmic result about systems of T-segments.

▶ **Proposition 19** ([6, Lemma 4]). *Let G be a graph, $T \subseteq V(G)$, and let p and r be positive integers. Then it can be decided in $2^{\mathcal{O}(p)} \cdot n^{\mathcal{O}(1)}$ time whether there is a system of T-segments $\mathcal{P}$ with r paths having p internal vertices in total.*

However, we need an algorithm for constructing a system of T-segments with additional properties described in Lemma 18. For this, we modify the algorithm from Proposition 19.

▶ **Lemma 20** ($\star$). *Let G be a graph, $T \subseteq V(G)$, and let $\{A, B\}$ be a partition of T. Let also p and r be positive integers, and suppose that s and t are nonnegative integers with $s + t \leq r$. Then it can be decided in $2^{\mathcal{O}(p)} \cdot n^{\mathcal{O}(1)}$ time whether there is a system of T-segments $\mathcal{P}$ with r paths having p internal vertices in total such that (i) $\mathcal{P}$ contains s A-segments, (ii) t B-segments, and (iii) every A-segment has at least two internal vertices.*

7 Proof of the main result

Now we have all ingredients to prove our main result. We restate it here for the reader's convenience.

▶ **Theorem 2.** *Longest Cycle Above MAD can be solved in time $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ on 2-connected graphs.*

Proof. Let (G, k) be an instance of LONGEST CYCLE ABOVE MAD, where G is a 2-connected graph. We use the algorithm from Proposition 9 and compute $\mathsf{mad}(G)$ in polynomial time. If $k = 0$, the problem is trivial, because a cycle of length at least $\mathsf{mad}(G)$ exists by Theorem 1. Hence, we can assume that $k \geq 1$. If $k > \frac{1}{88}\mathsf{mad}(G) - 1$, we use Proposition 11 and solve the problem in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time. From now, we assume that $0 < k \leq \frac{1}{88}\mathsf{mad}(G) - 1$. In particular, $k \leq \frac{1}{80}\mathsf{mad}(G) - 1$. We apply Lemma 14, and in polynomial time either

(i) find a cycle of length at least $\mathsf{mad}(G) + k$ in G, or

(ii) find an induced subgraph H of G with $\mathsf{ad}(H) \geq \mathsf{mad}(G) - 1$ such that $\delta(H) \geq \frac{1}{2}\mathsf{ad}(H)$ and $|V(H)| < \mathsf{ad}(H) + k + 1$, or

(iii) find an induced subgraph H of G such that there is a partition $\{A, B\}$ of $V(H)$ with the following properties:
- B is an independent set,
- $\frac{1}{2}\mathsf{mad}(G) - 4k \leq |A|$,
- for every $v \in A$, $|N_H(v) \cap B| \geq 2|A|$,
- for every $v \in B$, $\mathsf{d}_H(v) \geq |A| - 2k - 2$.

If the algorithm finds a cycle of length at least $\mathsf{mad}(G) + k$, then we return it and stop. In Cases (ii) and (iii), we get a dense induced subgraph H that can be used to find a solution.

Case (ii). The algorithm from Lemma 14 returns an induced subgraph H of G with $\mathsf{ad}(H) \geq \mathsf{mad}(G) - 1$ such that $\delta(H) \geq \frac{1}{2}\mathsf{ad}(H)$ and $|V(H)| < \mathsf{ad}(H) + k + 1$. Let $k' = \lceil \mathsf{mad}(G) \rceil + k - |V(H)|$. We have that G has a cycle of length at least $\mathsf{mad}(G) + k$ if and only if G has a cycle of length at least $|V(H)| + k'$. Note that $k' \leq k + 1 \leq \frac{1}{88}\mathsf{mad}(G) \leq \frac{1}{60}\mathsf{ad}(H)$. By Lemma 15, for every potentially cyclable set S of at most $k + 1$ pairs of distinct vertices of H, $H + S$ has a Hamiltonian cycle containing every edge of S.

Suppose that $k' \leq 0$. Observe that H has a Hamiltonian cycle as we can use Lemma 15 for $S = \{e\}$, where e is an arbitrary edge $e \in E(H)$. Then we conclude that H has a cycle of length at least $\mathsf{mad}(G) + k$ and stop. Assume that $k' > 0$. Note that $|V(H)| \geq \mathsf{ad}(H) \geq 2k'$. Then by Lemma 17, G has a cycle of length at least $|V(H)| + k'$ if and only if one of the following holds:

(a) There are two distinct vertices $s, t \in V(H)$ such that H has an (s, t)-path P of length at least $k' + 1$ whose internal vertices lie in $V(G) \setminus V(H)$.

(b) There is a system of T-segments $\mathcal{P} = \{P_1, \ldots, P_r\}$ for $T = V(H)$ such that $r \leq k'$ and the total number of vertices on the paths in $\mathcal{P}$ outside T is at least k' and at most $2k' - 2$.

First, we check if (a) can be satisfied. For this, we consider all pairs of distinct vertices s and t of H. For every pair, we construct $G' = G[(V(G) \setminus V(H)) \cup \{s, t\}]$ and use Proposition 12 to find an (s, t)-path of length at least $k' + 1$ in G in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time. If we find such a path for some pair, we report the existence of a cycle of length at least $\mathsf{mad}(G) + k$ and stop. Otherwise, we verify (b) using Proposition 19. We use the algorithm from Proposition 19 for $r \in \{1, \ldots, k'\}$ and for $p \in \{k', \ldots, 2k' - 2\}$. If we find a required system of T-segments, then we return that G has a cycle of of length at least $\mathsf{mad}(G) + k$ and stop. If we fail to find such a system for every r and p, we conclude that G has no cycle of length at least $\mathsf{mad}(G) + k$. Note that this can be done in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time. This concludes Case (ii).

Case (iii). The algorithm from Lemma 14 returns an induced subgraph H of G such that there is a partition $\{A, B\}$ of $V(H)$ with the properties:
- B is an independent set,
- $\frac{1}{2}\mathsf{mad}(G) - 4k \leq |A|$,
- for every $v \in A$, $|N_H(v) \cap B| \geq 2|A|$,
- for every $v \in B$, $\mathsf{d}_H(v) \geq |A| - 2k - 2$.

Let $k' = \lceil \mathsf{mad}(G) \rceil + k - 2|A|$. Observe that G has a cycle of length at least $\mathsf{mad}(G) + k$ if and only if G has a cycle of length at least $2|A| + k'$. We have that $2|A| \geq \lceil \mathsf{mad}(G) \rceil - 8k$ and, therefore, $k' \leq 9k$.

Note that $|A| \geq \frac{1}{2}\mathsf{mad}(G) - 4k \geq 40k$, since $k \leq \frac{1}{88}\mathsf{mad}(G) - 1$. Also, we have that for every $v \in B$, $\mathsf{d}_H(v) \geq |A| - 4k$. Therefore, by Lemma 16, for every potentially cyclable set S of at most $9k$ pairs of distinct vertices, $G' = G + S$ has a cycle C containing every edge of S and the length of C is $2|A| - s + t$, where s in the number of edges of S with both end-vertices in A and t is the number of edges in S with both end-vertices in B.

Suppose that $k' \leq 0$. Then we observe that H has a cycle of length $2|A|$ because we can set $S = \{xy\}$, where $xy \in E(H)$ with $x \in A$ and $y \in B$. Then H has a cycle of length at least $2|A| + k'$ and we conclude that G has a cycle of length at least $\mathsf{mad}(G) + k$. Assume that $k' > 0$. Since $|A| \geq 40k \geq \frac{3}{2}k'$, we can apply Lemma 18. We obtain that G has a cycle of length at least $2|A| + k'$ if and only if one of the following holds:

(a) There are two distinct vertices $x, y \in V(H)$ such that H has an (x, y)-path P of length at least $k' + 2$ whose internal vertices are in $V(G) \setminus V(H)$.

(b) There is a system of T-segments $\mathcal{P} = \{P_1, \ldots, P_r\}$ for $T = V(H)$ with s A-segments and t B-segments such that
- $r \leq k' \leq 9k$,
- every A-segment has at least two internal vertices,
- the total number of internal vertices vertices on the paths in $\mathcal{P}$ is at least $k' + s - t$ and at most $3k' - 2 \leq 27k - 2$.

To verify (a), we use the same approach as in Case (ii), that is, we consider all pairs of distinct vertices x and y of H. For every pair, we construct $G' = G[(V(G) \setminus V(H)) \cup \{x, y\}]$ and use Proposition 12 to find an (x, y)-path of length at least $k' + 2$ in G in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time. If we find such a path for some pair, we report the existence of a cycle of length at least $\mathsf{mad}(G) + k$ and stop. Otherwise, we verify (b) using Lemma 20. We use the algorithm from this lemma for $r \in \{1, \ldots, k'\}$, $s \in \{0, \ldots, k'\}$ and $t \in \{0, \ldots, k'\}$ such that $s + t \leq r$, and for $p \in \{k' + s - t, 3k' - 2\}$. If we find a system of T-segments $\mathcal{P} = \{P_1, \ldots, P_r\}$ for $T = V(H)$ with s A-segments and t B-segments with the required properties, then we conclude that G has a cycle of of length at least $2|A| + k'$ and stop. If such a system does not exist for every choice of r, s, t, and p, we have that G has no cycle of length at least $\mathsf{mad}(G) + k$. By Lemma 20, this can be done in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time, because $k' \leq 9k$. This concludes Case (iii).

Because the algorithm from Lemma 14 is polynomial and the other subroutines used in our algorithm for Longest Cycle Above MAD run in $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$, the overall running time is $2^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ and this concludes the proof.

Let us remark that since the algorithms for paths in Propositions 19 and 12 and Lemma 20 are, in fact, constructive, and the same holds for the algorithms for cycles in Lemmas 17 and 18 and Proposition 11, our algorithm is not only able to solve the decision problem, but also can find a cycle of length at least $\mathsf{mad}(G) + k$ if it exists. ◀

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