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Symmetric waves are traveling waves of some shallow water scalar equations

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Following a straightforward proof for symmetric solutions to be traveling waves by Pei (Exponential decay and symmetry of solitary waves to Degasperis-Procesi equation. *Journal of Differential Equations*. 2020;269(10):7730-7749), we prove that classical symmetric solutions of the highly nonlinear shallow water equation recently derived by Quirchmayr (A new highly nonlinear shallow water wave equation. *Journal of Evolution Equations*. 2016;16(3):539-556) are indeed traveling waves, with further information on their steady structures. We also provide a simple proof that symmetric waves are traveling waves to the free surface evolution equation of moderate amplitude waves in shallow water.

KEYWORDS

shallow water equations, symmetric waves, traveling waves

MSC CLASSIFICATION

35Q35, 35C07

1 | INTRODUCTION

We are interested here in the one-dimensional highly nonlinear equation describing at any position $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ the unknown horizontal velocity $u(t, x)$ of motion to unidirectional surface of large amplitude wave a in shallow water and over flat bottom topographies at a specific depth h_0. The typical wave length λ of the wave is always assumed to be larger than the depth. Following the approach first introduced by Johnson,[1] the equation was derived by Quirchmayr[2] under the form

$$\begin{aligned} &u_t + u_x + \frac{3}{2}\varepsilon u u_x - \frac{1}{18}\delta^2(4u_{xxx} + 7u_{xxt}) \\ &- \frac{1}{6}\varepsilon\delta^2(u u_{xxx} + 2u_x u_{xx}) + \frac{1}{96}\varepsilon^2\delta^2(398 u u_x u_{xx} + 45u^2 u_{xxx} + 154u_x^3) = 0\,. \end{aligned} \tag{1}$$

The latter equation arises as an approximation for the one-dimensional (or two-dimensional unidirectional) gravity water waves for a perfect fluid (see Johnson[3]), attained by means of double asymptotic expansions in the shallowness (dispersion) parameter $\delta = h_0/\lambda$, the nonlinearity (amplitude) parameter $\varepsilon = a/h_0$, the pressure, and the free surface. Under the scaling

$$\delta \ll 1\,, \qquad \varepsilon \sim O\left(\sqrt{\delta}\right)\,,$$

the latter equation settle. The *shallow water regime for waves of large amplitude* was referred to this scaling. Unlike the Camassa-Holm regime (see, for instance, (13)), here the nonlinear effects are stronger. In fact, the right-hand side of (1) is of order $O(\varepsilon^3\delta^2, \delta^3)$, and we see the dependence on $\varepsilon^2\delta^2$ in the left-hand side.

Several authors have addressed the local well-posedness of Equation (1) in recent years. In Yang and Xu,[4] the local well-posedness of the corresponding Cauchy problem with initial data in $H^s(\mathbb{R})$ for $s > 3/2$ was established by applying Kato's theory. In previous studies,[5,6] the authors improved the local existence of solution to (1) to the Besov space setting $B^2_{p,q}(\mathbb{R})$ where $p, q \in [0, +\infty]$ and $s > \max(\{3/2, (p+1)/p\}$ and provides some blow-up criterion. Recently, the well-posedness for space-periodic solutions is established in $H^s(\mathbb{R}/\mathbb{Z})$ for $s > 3/2$ in Duruk Mutlubas et al.[7] Furthermore, Khorbatly[8] addresses maximal time existence and wave breaking in terms of ε^{-1} dependence. On the other hand, Geyer and Quirchmayr[9] classify all (weak) traveling wave solutions of (1) in $H^1_{loc}(\mathbb{R})$.

Consider the arbitrary real numbers $(\alpha, \beta) \in \mathbb{R}^2$ and by the scaling

$$t \to \frac{\beta}{\delta}\, t, \qquad x \to \frac{\beta}{\delta}\, x, \qquad u \to \frac{\alpha}{\varepsilon}\, u\,, \tag{2}$$

we can transfer Equation (1) to

$$\begin{aligned} &u_t + u_x + \frac{3}{2}\alpha u u_x - \frac{1}{18}\beta^2(4u_{xxx} + 7u_{xxt}) \\ &- \frac{1}{6}\alpha\beta^2(u u_{xxx} + 2u_x u_{xx}) + \frac{1}{96}\alpha^2\beta^2(398 u u_x u_{xx} + 45u^2 u_{xxx} + 154u_x^3) = 0\,, \end{aligned} \tag{3}$$

or equivalently in the quasi-linear form as

$$u_t + \left(\frac{4}{7} + \frac{3\alpha}{7}u - \frac{135\alpha^2}{112}u^2\right)u_x + K * v = 0\,, \tag{4}$$

with $v = \frac{3}{7}u_x + \frac{15\alpha}{14}uu_x + \frac{\alpha\beta^2}{6}u_x u_{xx} + \frac{2\alpha^2\beta^2}{3}u_x^3 + \frac{4\varepsilon^2\beta^2}{3}uu_xu_{xx} + \frac{135\alpha^2}{112}u^2u_x$. Note that this equation holds in $H^1(\mathbb{R})$. Indeed, denote by K the convolution kernel function $K(x) = \frac{3}{|\beta|\sqrt{14}}\exp\left(-\frac{6}{|\beta|\sqrt{14}}|x|\right)$ associated to the operator $\left(1 - \frac{7\beta^2}{18}\partial_x^2\right)^{-1}$ whose Fourier transform reads $\hat{K}(\omega) = \left(1 + \frac{7\beta^2}{18}\omega^2\right)^{-1}$. We have $\left(1 - \frac{7\beta^2}{18}\partial_x^2\right)^{-1} f = K * f$ and $K * \left(f - \frac{7\beta^2}{18}f_{xx}\right) = f$ for all $f \in L^2(\mathbb{R})$.

In the present note, we are concerned with the classical symmetric solutions of (1). We recall that Ehrnström et al[10] and then Bruell et al[11] introduced a general principle of proving that symmetric waves are traveling waves for large class of evolutionary nonlinear and nonlocal partial differential equations. We conclude that Equation (1) meets the formal requirements of the general principle stated in Theorem 2.2 of Ehrnström et al,[10] the proof of which is based on the construction of a traveling wave whose initial data coincides with that of a symmetric solution.

However, Pei[12] recently provides a more straightforward proof for symmetric solutions to be traveling waves, as well as additional information on how the symmetric structure of waves can be connected with their steady structures. Inspired from Pei's conclusion[12] that symmetric waves are traveling waves for the Degasperis-Procesi equation, we are interested in stating similar result to our Equation (1). We will also mention that this proof applies to the modeling equation of free surface evolution derived in Constantin and Lannes[13] for moderate amplitude waves in shallow water.

2 | SYMMETRIC SOLUTIONS OF (1) TO BE TRAVELING WAVES

Let us start first by recalling the formal definition for symmetric waves.

Definition 1 (Symmetric waves). A solution u is *x-symmetric* if there exists a function $\lambda \in C^1(\mathbb{R}_+)$ such that for every $t \in \mathbb{R}_+$,

$$u(t, x) = u(t, 2\lambda(t) - x)$$

for a.e. $x \in \mathbb{R}_+$. We say that $\lambda(t)$ is the axis of symmetry.

Theorem 1. *Solutions to the highly nonlinear equation (1) with a priori spatial symmetry are steady solutions.*

Proof. In view of Definition 1, assume that $u(t,x)$ is a solution to Equation (1) such that

$$u(t,x) = u(t, 2\lambda(t) - x). \tag{5}$$

where the symmetric axis $\lambda(t) \in C^1(\mathbb{R}_+)$. Denoting by $\dot{\lambda} := \dot{\lambda}(t)$ the derivative of $\lambda(t)$ with respect to t, the spatial and time derivatives of $u(t,x)$ satisfy

$$u_t(t,x) = u_t(t, 2\lambda - x) + 2\dot{\lambda} u_x(t, 2\lambda - x),\ \ u_x(t,x) = -u_x(t, 2\lambda - x),\ \ u_{xx}(t,x) = u_{xx}(t, 2\lambda - x)\ . \tag{6}$$

In addition, we have $v(t,x) = -v(t, 2\lambda - x)$, and since the convolution kernel function K is even, it holds that

$$K * v\ (t,x) = -K * v\ (t, 2\lambda - x)\ . \tag{7}$$

At this stage, inserting (5)-(7) into (4) and in view of the arbitrariness of t and x, we find that for any $(\alpha, \beta) \in \mathbb{R}^2$, u satisfies the following equation

$$u_t + 2\dot{\lambda} u_x - \left(\frac{4}{7} + \frac{3\alpha}{7} u - \frac{135\alpha^2}{112} u^2\right) u_x - K * v = 0\ . \tag{8}$$

The comparison between (8) and (4) then leads to the following constraint conditions

$$u_t + \dot{\lambda} u_x = 0\ , \tag{9}$$

$$-\dot{\lambda} u_x + \left(\frac{4}{7} + \frac{3\alpha}{7} u - \frac{135\alpha^2}{112} u^2\right) u_x + K * v = 0\ . \tag{10}$$

In view of the first-order linear PDE (9) whose coefficients relying only on the time variable, $u(t,x)$ must take the form

$$u(t,x) = u(t, 2\lambda(t) - x) = g(x - \lambda(t)) \tag{11}$$

for some function g, which implies that the shape of the solution will not change in later evolution and the solution propagates with speed $\dot{\lambda}(t)$. Inserting (11) into (10), we get the following differential equation

$$\left[-\dot{\lambda} g' + \left(\frac{4}{7} + \frac{3\alpha}{7} g - \frac{135\alpha^2}{112} g^2\right) g' + K * f\right] |_{x-\lambda(t)} = 0\ , \tag{12}$$

where $f = \frac{3}{7} g' + \frac{15\alpha}{14} g g' + \frac{\alpha\beta^2}{6} g' g'' + \frac{2\alpha^2\beta^2}{3} (g')^3 + \frac{4\varepsilon^2\beta^2}{3} g g' g'' + \frac{135\alpha^2}{112} g^2 g'$. Now, choose arbitrarily two pairs $(t_1, x_1), (t_2, x_2) \in \mathbb{R}_+ \times \mathbb{R}$ (for which the solution exists and makes sense) such that

$$x_1 - \lambda(t_1) = x_2 - \lambda(t_2) =: \xi\ .$$

Evaluating (12) at these two pairs gives

$$\left(\dot{\lambda}(t_1) - \dot{\lambda}(t_2)\right) g'(\xi) = 0\ .$$

Due to the arbitrariness of ξ, $\dot{\lambda}(t)$ has to be a constant so that the wave profile has a constant propagation speed. Therefore, $u(t,x)$, with fixed shape and constant propagation speed, is a traveling wave solution. □

3 | A REMARK ON THE SYMMETRIC WAVES OF THE MODERATE AMPLITUDE SHALLOW WATER EQUATION

Under the following scaling (usually called the Camassa-Holm regime)

$$\delta^2 \ll 1, \qquad \varepsilon \sim O(\delta)\ , \tag{13}$$

an equation for the evolution of the free surface $\zeta(t,x)$ of moderate amplitude in shallow water reads

$$\zeta_t + \zeta_x + \frac{3}{2}\varepsilon\zeta\zeta_x + \frac{1}{12}\delta^2(\zeta_{xxx} - \zeta_{xxt}) - \frac{8}{3}\varepsilon^2\zeta^2\zeta_x + \frac{3}{16}\varepsilon^3\zeta^3\zeta_x + \frac{7}{24}\varepsilon\delta^2(\zeta\zeta_{xxx} + 2\zeta_z\zeta_{xx}) = 0\,. \tag{14}$$

Based on Constantin and Lannes[13] (see eq. (19) in Constantin and Lannes[13]), the latter equation arises as an approximation of the Euler equations for a perfect fluid taking the Green-Naghdi system[14] as a starting point, and it is valid in the same regime as the CH and DP equations. Here, the right-hand side of (14) is of order $O(\varepsilon^4, \delta^4)$, and we see the dependence on $O(\varepsilon^3, \varepsilon\delta^2)$ in the left-hand side.

Following the same scaling (2), the quasi-linear form of (14) is given by

$$\zeta_t + \left(1 - \frac{7\alpha}{2}\zeta\right)\zeta_x + P * \left(2\zeta_x + 5\alpha\zeta\zeta_x - \frac{3\alpha^2}{8}\zeta^2\zeta_x + \frac{3\alpha^3}{16}\zeta^3\zeta_x - \frac{7\alpha\beta^2}{24}\zeta_x\zeta_{xx}\right) = 0\,. \tag{15}$$

Note that this equation holds in $H^1(\mathbb{R})$. Indeed, denote by P the convolution kernel function $P(x) = \frac{\sqrt{3}}{|\beta|}\exp\left(-\frac{2\sqrt{3}}{|\beta|}|x|\right)$ associated to the operator $\left(1 - \frac{\beta^2}{12}\partial_x^2\right)^{-1}$ whose Fourier transform reads $\hat{P}(\omega) = \left(1 + \frac{\beta^2}{12}\omega^2\right)^{-1}$. We have $\left(1 - \frac{\beta^2}{12}\partial_x^2\right)^{-1} f = P * f$ and $P * \left(f - \frac{\beta^2}{12} f_{xx}\right) = f$ for all $f \in L^2(\mathbb{R})$. It is not hard to realize that in Geyer,[15] the author used the latter scaling (2) for specific $(\alpha, \beta) = \left(4, 2\sqrt{3}\right)$ such that the general principle conditions of Ehrnström et al[10] fulfills the weak non-local formulation of (14) introduced in Geyer.[15] Despite the scarcity of knowledge about the steady structure of symmetric waves in Geyer,[15] those waves are in fact traveling waves for any arbitrary numbers $(\alpha, \beta) \in \mathbb{R}^2$, proceeding as shown in Section 2.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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