Curvature and Harmonic Forms on Complete Manifolds

Andreas Myklebust

UNIVERSITY OF BERGEN
Department of Mathematics

Master's thesis in Mathematical Analysis
January 2022

Acknowledgements

I want to thank my supervisor Erlend Grong for being so helpful in the process of writing this thesis. He has set aside a lot of time for me, and given me a lot of great advice and feedback in the last year and a half. I am also thankful to him giving introducing me to the scientific community, by giving me the opportunity to attend two conferences. This thesis concludes five years of study, and I want to thank my friends and family for their support during these years. Among them, Anton and Steffen deserve a special mention for being great study buddies.

Contents

— 1 —

Introduction

On smooth manifolds, de Rham cohomology is a a topological invariant of the manifold. In general a topological invariant is a property of a topological space which it shares with all spaces which are the same in the topological sense. Explicitly, homeomorphic spaces share topological invariants. Even though de Rham cohomology is defined via the smooth structure on the manifold, it is actually isomorphic with the singular cohomology [12, Thm 18.14], a result from 1931 by de Rham. The definition of the k'th de Rham cohomology is the vector space of closed k-forms modulo the exact k-forms.

On Riemannian manifolds, we can define the Laplace operator on k-forms. The Lacplace operator is an elliptic operator which depends on the Riemannian metric on the manifold. It is therefore unsuspected that on compact manifolds, the k'th de Rham cohomology is isomorphic to the harmonic k-forms, i.e. the forms that evaluate to zero. It is remarkable that the harmonic forms, which depends both on the smooth structure of the manifold, and the Riemannian metric is actually a topological invariant. Furthermore, we have a decomposition of the k-forms on the manifold. Every k-form can be decomposed into an exact form, a co-exact form and a harmonic form. This is the Hodge Decomposition, the proof of which involves a great deal of analysis, especially theory about Sobolev spaces. For a compact oriented Riemannian manifolds $\mathbb{M}$, the classical Hodge theorem relates the k'th de Rham cohomology, via an isomorphism to the harmonic k-forms on $\mathbb{M}$. This is a exceptional result, connecting the topology of $\mathbb{M}$ with analysis of the Laplace operator on $\mathbb{M}$. It was W.V.D. Hodge who in the 1930's defined a generalization to the Beltrami Laplace operator, to differential forms, in order to study the cohomology on manifolds [7].

For non-compact manifolds we do not have an inner-product on the space of smooth forms on the manifold since there can exist forms that are not integrable. However, if we do restrict to the integrable forms, then we can get a similar result. The difference is that we take the closure of the exact forms, to get the reduced L^2-cohomology. There is also an un-reduced cohomology, however the unreduced cohomology is often not as nice, while the reduced cohomology is a Hilbert space, the unreduced is not necessarily a Hausdorff space[19, Prop 4.5]. If the subspace of exact k-forms is closed, the reduced and unreduced cohomology coincide. The reduced and non-reduced cohomologies are invariant under bi-Lipschitz maps, which in contrast to homeomorphisms, factor in the Riemannian structure of the manifolds. This says that for complete there is much more diversity in the space of forms that are harmonic, which, in contrast to compact manifolds, depends on the geometry of $\mathbb{M}$.

In both compact and the non-compact case, then curvature of the manifold has impact on the space of harmonic forms on $\mathbb{M}$. The Laplace operator can be decomposed into an elliptic differential operator and a curvature operator called the Weitenböck curvature. This is called the Weitzenböck decomposition. Via the Weitzenböck decomposition, there are several result that connect the curvature of a Riemannian manifold, with cohomology see e.g. [16, 1, 13], where the Weitzenböck curvature is positive. The k'th Weitzenböck operator $\mathscr{R}_k$ is a generalization of the Ricci curvature, since $\mathscr{R}_1 = \mathrm{Ric}$. When the Ricci curvature is strictly greater than a positive constant, the manifold is compact, by the Bonnet-Myers theorem. Because it is difficult for the Weitzenböck operator to be positive without the manifold being compact, most results involves compact manifolds. There is a result by S-T. Yau [24] which relate non-negative Ricci curvature on a complete manifold with the space of harmonic forms. We generalize this theorem to include the case where $\mathscr{R}_k \geq 0$.

The structure of this thesis is as follows:

In Chapter 2 we review some basic concepts from Riemannian geometry, on which the rest of the material is built upon. We define the Laplace operator, and decompose it into the Rough Laplace operator plus a linear map involving the curvature; this is the Weitenböck formula.

Chapter 3 is devoted to the Hodge theorem, which relates the de Rham Cohomology, and the Harmonic forms on $\mathbb{M}$. We give a brief introduction to Fourier analysis on the torus, and some useful theorems about elliptic operators. The results are necessary for proving the Hodge theorem on compact manifolds. We also show that if the Weitzenböck curvature operator is positive, then the cohomology is trivial.

In Chapter 4 we consider manifolds that are not compact, but which are complete. We investigate if and how the results in Chapter 3 generalize to complete manifolds. This involves investigating the L^2-cohomologies. The end of the chapter includes my generalization of the mentioned theorem by S-T. Yau [24], which we have not found stated explicitly in the literature.

The Appendix serves as a refresher of some facts involving Hilbert spaces. It could be read first if the reader is not too impatient. Although the appendix contains most of the theory we need from functional analysis, we assume the reader to have familiarity with basic functional analysis. However, the most important prerequisite is a good knowledge of smooth manifolds.

— 2 —

Riemannian Geometry and the Laplace Operator

2.1 Linear Algebra

In differential geometry, linear algebra plays an important role. Each fibre of a vector bundle is a vector space. It is therefore necessary to elaborate on the theory before we introduce Riemannian manifolds. Throughout this section, let V be an n-dimensional inner product space over the real numbers. The inner product on V will be denoted by g, but we will often use angled brackets $\langle\cdot,\cdot\rangle_g$ with g as a subscript in equations, or even just $\langle\cdot,\cdot\rangle$, if it does not cause any confusion.

2.1.1 Extension of inner products

The inner product gives us a natural way to identify V with its dual V^*, namely via the **musical isomorphisms** $\flat\colon V\to V^*$ and $\sharp\colon V^*\to V$, defined by $u^\flat(v)=\langle u,v\rangle_g$ and $\langle\alpha^\sharp,u\rangle_g=\alpha(u)$. We use the musical isomorphisms to extend the inner product to tensors. For elements in $V^{\otimes a}\otimes(V^*)^{\otimes b}$ the inner product is given by

$$\langle v_1\otimes\cdots\otimes v_a\otimes\alpha^1\otimes\cdots\otimes\alpha^b, w_1\otimes\cdots\otimes w_a\otimes\beta^1\otimes\cdots\otimes\beta^b\rangle=\prod_{i=1}^a\langle v_i,w_i\rangle_g\prod_{j=1}^b\langle(\alpha^j)^\sharp,(\beta^j)^\sharp\rangle_g\,.$$

We extend the inner product to the algebra of all tensors, by defining $\langle S,T\rangle=0$ when S and T are tensors of different type, and extend linearly. We include the tensors of order zero, i.e. the scalars, and such that for $x,y\in\mathbb{R}$ we have $\langle x,y\rangle=xy$.

In general, orthonormal bases for our vector spaces are to be favoured because it simplifies the inner product $\langle x,y\rangle$ to the sum of the components $x_1y_1+\cdots+x_ny_n$. If $e_1,\dots,e_n$ is an orthonormal basis for V, and $\theta^1,\dots,\theta^n$ is the dual basis, then we can construct an orthonormal basis for the (a,b)-tensor space. The elements of the form

$$e_{i_1}\otimes\cdots\otimes e_{i_a}\otimes\theta^{j_1}\otimes\cdots\otimes\theta^{j_b}$$

constitutes a basis for $V^{\otimes a}\otimes(V^*)^{\otimes b}$, and the elements are mutually orthogonal:

$$\begin{aligned}\langle e_I\otimes\theta^J,e_K\otimes\theta^L\rangle&=\langle e_{i_1}\otimes\cdots\otimes e_{i_a}\otimes\theta^{j_1}\otimes\cdots\otimes\theta^{j_b},e_{k_1}\otimes\cdots\otimes e_{k_a}\otimes\theta^{\ell_1}\otimes\cdots\otimes\theta^{\ell_b}\rangle\\&=\prod_{r=1}^a\langle e_{i_r},e_{k_r}\rangle_g\prod_{s=1}^b\langle(\theta^{j_s})^\sharp,(\theta^{\ell_s})^\sharp\rangle_g\\&=\prod_{r=1}^a\langle e_{i_r},e_{k_r}\rangle_g\prod_{s=1}^b\langle e_{j_s},e_{\ell_s}\rangle_g=\delta^{IJ}_{KL}\end{aligned}$$

where $\delta_\bullet^\bullet$ is the Kronecker delta, with respect to the multi-indices I, J, K, L of the form $I = \{i_1, \dots, i_a\}$.

We can define the inner product on the exterior power $\Lambda^a(V)$ in a slightly different way. Since permuting the vectors only changes the element by the sign of the permutation, we want the inner product to give the same result after permutation, times the sign of the permutation. This motivates the definition

$$\langle v_1 \wedge \cdots \wedge v_a, w_1 \wedge \cdots \wedge w_a \rangle_{g^{\wedge a}} = \sum_{\sigma \in S_a} (\operatorname{sgn} \sigma) \prod_{i=1}^{a} \langle v_{\sigma(i)}, w_i \rangle = \det[\langle v_i, w_j \rangle]_{i,j}^a .$$

where S_a is the the set of permutations of the set $\{1, \dots, a\}$. We extend this inner product to $\Lambda(V) = \bigoplus_{a=0}^n \Lambda^a(V)$ as we did for the tensor algebra.

Let I denote a multi-index $I = \{i_1 < \dots < i_a\}$, and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_a}$. The set $\{e_I\}$ indexed over multi-indices I constitute a basis of $\Lambda^k(V)$. This basis is orthonormal; if e_I and e_J are elements in the basis, then

$$\langle e_I, e_J \rangle = \det[\langle e_{i_r}, e_{j_s} \rangle]_{r,s}^a = \delta_{IJ}$$

because determinant vanishes when a row is the zero vector, which is the case for non-equal multi-indices. The inner product of two elements $u = \sum_I u^I e_I$ and $v = \sum_J v^J e_J$ with respect to an orthonormal basis is a simple sum

$$\langle u, v \rangle = \sum_{I,J} u^I v^J \langle e_I, e_J \rangle = \sum_I u^I v^I$$

Remark. The inner products for regarding tensor and exterior product are essentially the same, differing only by an integer multiple.

2.1.2 Wedge Product Inequality

If we take the product $u \wedge v$ of an a-vector u and a b-vector v, where $a + b \leq n$, then the magnitude of of the product is bounded by a multiple of the magnitude of the factors. If we write u and v with respect to an orthonormal basis, and let the basis of $\Lambda^{a+b}(V)$ be indexed by M, we have

$$u \wedge v = \sum_{I,J} u^I v^J e_I \wedge e_J = \sum_M \left(\sum_{e^I \wedge e^J = \sigma e^M} (\operatorname{sgn} \sigma) u^I v^J \right) e_M$$

where σ is a permutation applied to e_M which makes the equality below Σ hold. Computing the magnitude by using the formula and using Schwartz inequality yields

$$|u \wedge v|^2 = \sum_M \left(\sum_{e^I \wedge e^J = \sigma e^M} (\operatorname{sgn} \sigma) u^I v^J \right)^2 \leq \sum_M \left(\sum_{e^I \wedge e^J = \sigma e^M} (u^I)^2 \right) \left(\sum_{e^I \wedge e^J = \sigma e^M} (v^J)^2 \right)$$

where the last sum is less than $C|u|^2|v|^2$, where C is a positive integer which depends only on a, b and the dimension n of V. So there is a constant K such that the inequality

$$|u \wedge v| \leq K|u||v| \tag{2.1}$$

holds for all $u, v \in \Lambda(V)$.

In Section 2.2 we define Riemannian manifolds, and we will see that we can do all these constructions on the tangent spaces $T_p\mathbb{M}$, where the inner product g is the Riemannian metric on the manifold $\mathbb{M}$.

2.1.3 Hodge Star Operator

The one-dimensional vector space $\Lambda^n(V)$ has basis a element $\mathrm{vol_g} = e_1 \wedge \cdots \wedge e_n$, composed from an orthonormal basis on V, called the **volume element**. Let $\tilde{e} = [\tilde{e}_1, \ldots, \tilde{e}_n]$ be another orthonormal basis and $e = \tilde{e}A$ for some matrix A. The matrix A has inverse $A^{-1} = A^T$:

$$\delta_{ij} = \langle e_i, e_j\rangle = A_i^k A_j^\ell \langle \tilde{e}_k, \tilde{e}_\ell\rangle = \sum_k A_i^k A_j^k = \sum (A^T)_k^i A_j^k$$

and thus $A^T A = I = AA^T$. By basic properties of determinants $\det A = \pm 1$. Going back to the volume element, we see that

$$\begin{aligned} e_1 \wedge \cdots \wedge e_n =& A_1^{i_1} \ldots A_n^{i_n} \tilde{e}_{i_1} \wedge \cdots \wedge \tilde{e}_{i_n} \\ =& \sum_{\sigma \in S_n} A_1^{\sigma(1)} \ldots A_n^{\sigma(n)} (\operatorname{sgn} \sigma) \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n \\ =& (\det A) \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_n \end{aligned}$$

which means that that any two volume elements differ only by a sign. A choice of volume element in $\Lambda^n(V)$equivalent to choosing an orientation on the vector space V. Let $\mathrm{vol_g}$ define our choice of orientation.

Theorem 1. *There exists a unique linear operator $\star : \Lambda^a(V) \to \Lambda^{n-a}(V)$ with the property that $\langle\alpha, \beta\rangle \mathrm{vol_g} = \alpha \wedge \star\beta$.*

Proof. Suppose that $\star$ is an operator which satisfies the last property. Then $\star$ is linear since

$$\alpha \wedge \star(\beta + \gamma) = \langle \alpha, \beta + \gamma\rangle \mathrm{vol_g} = \langle\alpha, \beta\rangle \mathrm{vol_g} + \langle\alpha, \gamma\rangle \mathrm{vol_g} = \alpha \wedge \star\beta + \alpha \wedge \star\gamma = \alpha \wedge (\star\beta + \star\gamma)\,.$$

Suppose $\diamond$ has the same property as $\star$. Then

$$\langle \star\beta - \diamond\beta, \alpha\rangle \mathrm{vol_g} = (\star\beta - \diamond\beta) \wedge \star\alpha = (-1)^{a(n-a)} \langle \star\alpha, \beta - \beta\rangle = 0$$

which shows that $\star$ is unique. In the orthonormal basis $\{e_I\}$ constructed from the orthonormal vector $e_1, \dots, e_n$ then we define $\star$ on the basis elements by

$$\star(e_{i_1} \wedge \cdots \wedge e_{i_a}) = \operatorname{sgn}(\pi) e_{i_{(a+1)}} \wedge \cdots \wedge e_{i_n}$$

giving the rest of the basis elements e_i, where π is the permutation $\pi = (i_1, \dots, i_n)$ of $\{1, \dots, n\}$. If $\alpha, \beta \in \Lambda^a(V)$ then

$$\alpha \wedge \star\beta = \sum_{I,J} \alpha_I \beta_J e_I \wedge \star e_J = \sum_I \alpha_I \beta_I \operatorname{vol}_{\mathrm{g}} = \langle \alpha, \beta \rangle \operatorname{vol}_{\mathrm{g}}$$

because $e_I \wedge \star e_J = \delta_{IJ}$. □

The linear operator $\star\colon \Lambda^a(V) \to \Lambda^{n-a}(V)$ is called the **Hodge star operator**. From its definition, it is readily seen that $\star 1 = \operatorname{vol}_{\mathrm{g}}$ and $\star \operatorname{vol}_{\mathrm{g}} = 1$. Taking $\star$ two times gives $\star^2 = (-1)^{a(n-a)}$, as we have to make $a(n-a)$ transpositions to obtain $\operatorname{vol}_{\mathrm{g}}$. The Hodge star operator is an isometry: If $\alpha, \beta \in \Lambda^a(V)$, then

$$\langle \star\alpha, \star\beta \rangle = \star\big((\star\alpha) \wedge \star^2\beta\big) = (-1)^{2a(n-a)} \star (\beta \wedge \star\alpha) = \langle \alpha, \beta \rangle .$$

Definition. Let $F\colon V \to V$ be an endomorphism, and $e = [e_1, \dots, e_n]$ be an orthonormal basis for V. Then we define the **trace** of F by

$$\operatorname{tr} F = \sum_{i=1}^{n} \langle F e_i, e_i \rangle .$$

The trace does not depend on choice of orthonormal basis. Let $\tilde{e} = [\tilde{e}_1, \dots, \tilde{e}_n]$ be another orthonormal basis and define $\tilde{\operatorname{tr}}$. Let the two bases be related by $\tilde{e} = eA$, where A is a matrix. Then

$$\operatorname{tr} F = \sum_{i=1}^{n} \langle F\tilde{e}_i, \tilde{e}_i \rangle = \sum_{i,j,k=1}^{n} \langle F A_i^j e_j, A_i^k e_k \rangle = \sum_{i,j,k=1}^{n} A_i^j (A^T)_k^i \langle F e_j, e_k \rangle = \sum_{i=1}^{n} \langle F e_i, e_i \rangle$$

2.2 Riemannian Manifolds

A central part of this thesis will revolve around the de Rham cohomology of a manifold. Before we can define this cohomology theory, we must first make some definitions. Throughout this thesis, $\mathbb{M}$ will denote smooth manifold of dimension n. Unless specified otherwise, we assume that $\mathbb{M}$ has an orientation, i.e. there exists a nowhere-vanishing smooth n-from ν on $\mathbb{M}$ [20, Thm 21.5], and we have chosen $-\nu$ or ν to specify the orientation on $\mathbb{M}$. We say that an ordered basis $e_1, \dots, e_n$ of the tangent space $T_p\mathbb{M}$ is consistent with the orientation if $\nu(e_1, \dots, e_n) > 0$.

The vector space of real smooth maps on $\mathbb{M}$ are denoted by $\mathscr{E}(\mathbb{M})$, and the subspace consisting only of maps with compact support is denoted by $\mathscr{D}(\mathbb{M})$. We let $C^\infty\Lambda^k(\mathbb{M})$

denote the $\mathscr{E}(\mathbb{M})$-module of smooth k-forms on $\mathbb{M}$, and $C^\infty\Lambda(\mathbb{M}) = \bigoplus_{k=1}^n C^\infty\Lambda^k(\mathbb{M})$. The subspace of $C^\infty\Lambda^k(\mathbb{M})$ consisting only of forms with compact support is denoted by $C_0^\infty\Lambda^k(\mathbb{M})$. If there is no possibility for confusion, we will omit "($\mathbb{M}$)" and write just $C^\infty\Lambda^k$, and do the same for other entities depending on $\mathbb{M}$. The set of vector fields on $\mathbb{M}$ is denoted by $\mathfrak{X}(\mathbb{M})$.

2.2.1 Exterior derivative

Even without specifying an orientation, there is natural way to differentiate differential forms. The **exterior derivative** is the unique [20, Thm 19.4] anti-derivation of degree 1 on the graded algebra $C^\infty\Lambda$, with the property that $d^2 = 0$ and $df(X) = Xf$ for 0-forms. It is defined locally by

$$d(a_I dx^I) = \frac{da_I}{dx^i} dx^i \wedge dx^I .$$

Here and throughout the text, we will use the Einstein summation convention; that if an index or multi-index appears both as an upper and a lower index, we sum over that index. This rule also applies to multi-indices, such as in the example above. If we do use the summation sign and multiple indices are involved, only those that appear in a summand are to be summed over.

There is also a coordinate-free description of the exterior derivative. Let ω a smooth k-form and $X_0, \ldots, X_k$ smooth vector fields. Then

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = &\sum_{i=0}^k (-1)^i X_i \omega(X_0 \ldots, \widehat{X}_i, \ldots, X_k) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k) \end{aligned}$$

where the caret "$\widehat{\ }$" means that we omit the element. For a proof that the two formulas for the exterior derivative coincide, again see [20, Thm 20.14].

Giving the manifold $\mathbb{M}$ Riemannian structure, allows us to define more in a sense natural differential operators on $\mathbb{M}$.

2.2.2 Riemannian Metric

Definition. A **Riemannian manifold** is a smooth manifold $\mathbb{M}$ together with a symmetric non-degenerate smooth section $g\colon \mathbb{M} \to T^*\mathbb{M} \otimes T^*\mathbb{M}$ called a **Riemannian metric**.

More conveniently, $g_p\colon T_p\mathbb{M} \times T_p\mathbb{M} \to \mathbb{R}$ is an inner product on the tangent space $T_p\mathbb{M}$ at the point $p \in \mathbb{M}$, which varies smoothly with p. Restricting ourselves to Riemannian manifolds is not very strict at all, since all manifolds can be endowed with a Riemannian metric [21, Thm 1.12]. When evaluating on tangent vectors $u, v \in T_p\mathbb{M}$, we often write $\langle u, v\rangle$ instead of $g(u, v)$. Given a basis $e_1, \dots, e_n$ of $T_p\mathbb{M}$, the metric is defined completely by specifying it on basis elements. Let $g_{ij} := g(e_i, e_j)$. Then the matrix $[g_{ij}]$ determines the metric. Let $[g^{ij}]$ be the inverse matrix of $[g_{ij}]$. If $\theta^1, \dots, \theta^n$ is the dual basis of $e_1 \dots, e_n$, then $\theta^j = A^k_j e_k$, and

$$\delta^j_i = \langle e_i, (\theta^j)^\sharp\rangle = \langle e_i, A^k_j e_k\rangle = g_{ik}A^k_j$$

which shows that $(\theta^j)^\sharp = g^{jk}e_k$. We also have $\langle(\theta^i)^\sharp, (\theta^j)^\sharp\rangle = \theta^i(g^{jk}e_k) = g^{ij}$.

2.2.3 Distance on Riemannian Manifolds

Definition. For two points p and q in a connected Riemannian manifold, we can define the distance between then

$$d_g(p, q) := \inf_\gamma \int_{[a,b]} |\gamma'(t)|\, dt$$

where the infimum is taken over all piecewise smooth curves $\gamma\colon [a, b] \to \mathbb{M}$ from p to q. We call d_g the **distance function** with respect to g. We say that a manifold is **metrically complete**, if every Cauchy sequence with respect to d_g converges.

The topology defined by the metric d_g is the same as the topology already defined on the manifold [11, Thm 13.29]. Examples of metrically complete manifolds are abundant, for example regular Euclidean space and compact manifolds. We give an example of a manifold which is not complete.

Example 2. Let $\mathbb{M} = \mathbb{R}^2 - 0$, with the regular euclidean metric. Then for any point $p \in M$, the sequence $p_n = p/n$ is a Cauchy sequence, with no limit point in $\mathbb{M}$.

2.2.4 Integrating Functions on Riemannian Manifolds

Let $E_1, \dots, E_n$ be an orthonormal frame in a neighbourhood U in $\mathbb{M}$ which is consistent with the orientation on $\mathbb{M}$, and $\theta_1, \dots, \theta_n$ the dual co-frame. The volume form on $\mathbb{M}$ is the n-form defined by

$$\mathrm{vol_g} = \theta_1 \wedge \dots \wedge \theta_n$$

locally. The definition is independent of choice of orthonormal co-frame with the same orientation. A similar calculation as in the beginning of 2.1.3 yields

$$\mathrm{vol_g} = \sqrt{\det g}\, dx^1 \wedge \dots \wedge dx^n\,.$$

The volume form gives us a way of integrating real or complex functions on $\mathbb{M}$ with compact support. Let $f\colon \mathbb{M} \to \mathbb{C}$ have compact support. Then we define the integral of f on $\mathbb{M}$ by

$$\int_{\mathbb{M}} f\, dx := \int_{\mathbb{M}} \operatorname{Re} f \operatorname{vol}_{\mathrm{g}} + i \int_{\mathbb{M}} \operatorname{Im} f \operatorname{vol}_{\mathrm{g}}$$

where $\operatorname{Re} f \operatorname{vol}_{\mathrm{g}}$ and $\operatorname{Im} f \operatorname{vol}_{\mathrm{g}}$ are n-forms with compact support, which we already know how to integrate by using a partition of unity sub-ordinate to an atlas, and pulling each summand back to $\mathbb{R}^n$ on the coordinate patches, see [20, Sec 23].

For every $\omega, \tau \in C_0^\infty \Lambda^k$, the real map $\langle \omega, \tau \rangle$ has compact support, and is therefore integrable. We define the L^2 inner product on $C_0^\infty \Lambda^k$ by

$$\langle\!\langle \omega, \tau \rangle\!\rangle = \int_{\mathbb{M}} \langle \omega, \tau \rangle \, dx \, ,$$

and corresponding norm

$$\|\omega\| = \left\{ \int_{\mathbb{M}} |\omega|^2 \, dx \right\}^{1/2}$$

where $|\omega| = \langle \omega, \omega \rangle^{1/2}$.

2.2.5 Co-differential Operator

Definition. The **co-differential** is a differential operator $\delta_{k+1} \colon C^\infty \Lambda^{k+1} \to C^\infty \Lambda^k$ defined by

$$\delta_{k+1} = (-1)^{nk+1} \star d_{n-k-1} \star$$

Theorem 3. *The co-differential δ_{k+1} is the formal adjoint of d_k in the the sense that*

$$\langle\!\langle d_k \omega, \tau \rangle\!\rangle = \langle\!\langle \omega, \delta_{k+1} \tau \rangle\!\rangle \, .$$

for all $\omega \in C_0^\infty \Lambda^k$ and $\tau \in C_0^\infty \Lambda^{k+1}$.

Proof. Let ω and τ be as in the statement of the theorem, then

$$\begin{aligned} d(\omega \wedge \star\tau) =& (d\omega) \wedge \star\tau + (-1)^k \omega \wedge (d \star \tau) \\ =& (d\omega) \wedge \star\tau + (-1)^k \omega \wedge (\star^{-1} \star d \star \tau) \\ =& (d\omega) \wedge \star\tau + (-1)^s \omega \wedge (\star\delta\tau) \\ =& \langle d\omega, \tau \rangle - \langle \omega, \delta\tau \rangle \end{aligned}$$

where $s = k(k+1) + 1$, which is odd. Let S be an n-dimensional sub-manifold of $\mathbb{M}$ with boundary such that $\operatorname{supp} \omega, \operatorname{supp} \tau \subset S$. By Stokes theorem

$$0 = \int_{\partial S} \alpha \wedge \star\tau = \int_{\mathbb{M}} d(\alpha \wedge \star\tau) = \langle\!\langle d\alpha, \tau \rangle\!\rangle - \langle\!\langle \alpha, \delta\tau \rangle\!\rangle$$

□

We can use the co-differential to define the **divergence** of a vector field. For a vector field X, let $\operatorname{div} X \coloneqq -\delta X^\flat$. There is another definition of the divergence of a vector field X, namely

$$(\operatorname{div} X)\operatorname{vol_g} = d\iota_X \operatorname{vol_g}$$

where $\operatorname{vol_g}$ is the volume form. These definitions coincide. Let $\theta^1 \ldots, \theta^n$ be an orthonormal frame in the neighbourhood U, then

$$\begin{aligned} d\iota_X \operatorname{vol_g} =& d(X^i \theta^1 \wedge \cdots \wedge \widehat{\theta^i} \wedge \cdots \wedge \theta^n)(-1)^{i-1} \\ =& dX^i \star \theta^i \\ =& d \star X^\flat \end{aligned}$$

such that $\operatorname{div} X = \star d \star X^\flat = -\delta X^\flat$. The divergence is used to pass derivatives to the vector fields under the integral sign in the following sense.

Proposition 4. *If either $f \in \mathscr{E}$ or $X \in \mathfrak{X}$ have compact support, then we have the integration by parts formula*

$$\int_{\mathbb{M}} Xf \, dx = -\int_{\mathbb{M}} f \operatorname{div} X \, dx$$

Proof. Let $\operatorname{vol_g}$ denote the volume form. Let $E_1, \ldots, E_n$ be an orthonormal frame about p, then

$$Xf \operatorname{vol_g} = (\iota_X df)\operatorname{vol_g} = \iota_X(df \wedge \operatorname{vol_g}) + df \wedge \iota_X \operatorname{vol_g} = d\iota_X(f \operatorname{vol_g}) - f d\iota_X \operatorname{vol_g}$$

By Stokes's theorem

$$\int_{\mathbb{M}} Xf \operatorname{vol_g} = \int_{\mathbb{M}} d\iota_X(f \operatorname{vol_g}) - \int_{\mathbb{M}} f d\iota_X \operatorname{vol_g} = -\int_{\mathbb{M}} f(\operatorname{div} X)\operatorname{vol_g}$$

□

If $f = gh$ we have $\langle\!\langle Xg, h\rangle\!\rangle + \langle\!\langle g, Xh\rangle\!\rangle = -\langle\!\langle gh, \operatorname{div} X\rangle\!\rangle$ such that the formal adjoint of ∇_X is $-(\nabla_X + \operatorname{div} X)$, where ∇_X is the operator defined by $f \mapsto Xf$.

2.2.6 The Laplace Operator

We can now define the **Laplace operator** $\Delta\colon C^\infty\Lambda^k \to C^\infty\Lambda^k$ by $\Delta = -(\delta d + d\delta)$. The Laplace operator is symmetric when restricted to forms with compact support in the sense that if $\omega, \tau \in C_0^\infty\Lambda^k$, then

$$\langle\!\langle \Delta\omega, \tau\rangle\!\rangle = \langle\!\langle -(d\delta + \delta d)\omega, \tau\rangle\!\rangle = -\left(\langle\!\langle d\omega, d\tau\rangle\!\rangle + \langle\!\langle \delta\omega, \delta\tau\rangle\!\rangle\right) = \langle\!\langle \omega, \Delta\tau\rangle\!\rangle\,.$$

Differential forms ω for which $\Delta\omega = 0$ are called **harmonic**, or just $\mathscr{H}$ for short. The subspace of $C^\infty\Lambda^k$ consisting of harmonic forms is denoted by $\mathscr{H}\Lambda^k$.

2.3 Connections and Curvature

Definition. Let $\pi\colon B \to \mathbb{M}$ be a vector bundle over a Riemannian manifold $\mathbb{M}$. For any such bundle, let $\Gamma(B)$ denote the $\mathscr{E}$-module of smooth sections of B. A **connection** on B is a map

$$\begin{aligned} \nabla\colon \mathfrak{X}(\mathbb{M}) \times \Gamma(B) &\longrightarrow \Gamma(B) \\ (X, s) &\longmapsto \nabla_X s \end{aligned}$$

which is $\mathscr{E}$-linear in the first argument, $\mathbb{R}$-linear in the second argument, and satisfies the Leibniz rule in its second argument, in the sense that $\nabla_X(fs) = (Xf)s + f\nabla_X s$ for $f \in \mathscr{E}$.

2.3.1 Connections as Bundle Maps

The connection is a type of derivative, where at each point $p \in \mathbb{M}$ we take the derivative in the X_p-direction. The derivative at one point, does not depend on the vector field X at other points. We can therefore instead take the equivalent point of view that for each section $s \in \gamma(B)$, $\nabla s\colon$ assigns at each point $p \in \mathbb{M}$ a linear map $\nabla_{X_p} s : T_p\mathbb{M} \to B_p$. This point of view is justified by the following theorem

Theorem 5. [21, Thm 7.26] *There is a one-to-one correspondence*

$$\begin{aligned} \{\, C^\infty \text{ bundle maps } \varphi\colon E \to F\} &\longleftrightarrow \{\mathscr{E}\text{-linear maps } \alpha\colon \Gamma(E) \to \Gamma(F)\} \\ \varphi &\longmapsto \varphi_\# \end{aligned}$$

where $\varphi_\#(s)_p = \varphi(s_p)$.

For each $s \in \Gamma(B)$, we have an $\mathscr{E}$-linear map $\nabla s\colon \mathfrak{X}(\mathbb{M}) \to \Gamma(B)$, and by Theorem 5, a smooth bundle map $\nabla s\colon T\mathbb{M} \to B$. For any real vector spaces V, W we have an isomorphism $\mathrm{Hom}(V, W) \cong W \otimes V^*$. Using this relation on each fibre, Theorem 5 generalizes to tensor fields on the left and $\mathscr{E}$-multilinear maps on the right in [21, Thm 21.11]. Substitute tensors involving T_pM for tensors involving E_p and F_p in . We will therefore equate the two notions for the most part.

2.3.2 Connections Locally

We will see investigate how $\nabla_X Y \in B$ is represented on the trivializing subset U with respect to the local frame $E_1, \ldots, E_n$ of the bundle $\pi\colon B \to \mathbb{M}$. In U the sections

X, Y can be written as $X = X^i E_i$, $Y = Y^j e_j$ which yields:

$$\begin{aligned}\nabla_X Y =& \nabla_X Y^j E_j \\ =& (XY^j)E_j + Y^j \nabla_X E_j \\ =& (XY^j)E_j + Y^j \omega_j^k(X) E_k \\ =& (XY^j)E_j + Y^j X^i (\omega_i)_j^k E_k \,, \end{aligned} \tag{2.2}$$

where the two last lines, are just $\nabla_X e_j$ written out in the chosen basis. The coefficients $\omega_j^k(X)$ define smooth 1-forms on U. The $\mathscr{E}$-linearity in the second argument of ∇ ensures that $\omega_j^k(fX) = f\omega_j^k(X)$ for all $f \in \mathscr{E}$ and $X \in \mathfrak{X}$. We call ω_j^k the **connection forms**, with respect to the trivialization. The smooth maps $(\omega_i)_j^k \coloneqq \omega_j^k(E_i)$ are the **connection coefficients**. From (2.2), we see that any collection of smooth maps $(\omega_i)_j^k$ define an affine connection on U, by covering the manifold with such open sets we can patch the connections to one that is defined globally, using a partition of unity. This leaves us many choices of connections.

2.3.3 Levi-Civita Connection

Of most interest are the **affine connections**, i.e. connections on the tangent bundle. We define two $\mathscr{E}$-linear operators on $T\mathbb{M}$.

Definition. Let the tangent bundle $\pi \colon T\mathbb{M} \to \mathbb{M}$ be endowed with a connection ∇. We define the **torsion** $T \colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ and the **curvature** $R \colon \mathfrak{X} \times \mathfrak{X} \to \mathrm{End}(\mathfrak{X})$ by

$$\begin{aligned} T(X,Y) =& \nabla_X Y - \nabla_Y X - [X,Y] \\ R(X,Y) =& \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \end{aligned}$$

The curvature will be important later, since it is the building block for what we will call curvature operators. The torsion is useful in narrowing down which connections are appropriate. A connection is called **torsion-free** if $T(X,Y) = 0$ for all $X, Y \in \mathfrak{X}$.

Definition. An affine connection is **compatible** with the metric if the equality

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

holds for all $X, Y, Z \in \mathfrak{X}$.

If ∇ is compatible with the metric, the connection forms $(\omega_i)_k^j$ with respect to a local orthonormal frame are anti-symmetric in j and k. If ∇ is torsion-free, and the frame is the partial derivatives relative to a chart, then $(\omega_i)_j^k$ are called the **Christoffel symbols** and are then denoted by $\Gamma_{ij}^k \coloneqq (\omega_i)_j^k$. The Christoffel symbols are symmetric in the lower indices. We will mostly work with either a coordinate frame, or an orthonormal frame, and using one over the other $(\omega_i)_j^k$ or Γ_{ij}^k communicates in which indices we have symmetry.

Theorem 6. [21, Thm 6.6] *There exists an affine connection ∇ which is both compatible and torsion-free. Moreover this connection is unique.*

We call the unique connection ∇ from the theorem above the **Levi-Civita connection**.

2.3.4 Connection on Tensor Bundles

If ∇ is an affine connection, then ∇ can be extended to all tensor fields T , i.e. T is a smooth section of $(T\mathbb{M})^{\otimes a} \otimes (T^*\mathbb{M})^{\otimes b}$. First we define $\nabla_X \omega$ for 1-forms ω by $\nabla_X \omega(Y) = X\omega(Y) - \omega(\nabla_X Y)$. It is defined for a tensor field T by

$$\begin{aligned}(\nabla_X T)(\omega^1, \dots, \omega^a, Y_1, \dots, Y_b) =& \nabla_X T(\omega^1, \dots, \omega^a, Y_1, \dots, Y_b) \\ & - \sum_i^a T(\omega^1, \dots, \nabla_X \omega^i, \dots, \omega^a, Y_1, \dots, Y^b) \\ & - \sum_j^b T(\omega^1, \dots, \omega^a, Y_1, \dots, \nabla_X Y_j, \dots, Y_b)\end{aligned}$$

It also satisfies $\nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T$, for any tensor fields S and T [12], and therefore also $\nabla_X(\alpha \wedge \beta) = \nabla_X \alpha \wedge \beta + \alpha \wedge \nabla_X \beta$ for differential forms.

The map ∇T defined by $\nabla T(\dots, X) = \nabla_X T(\dots)$ is a another tensor field of covariant order one more than T. From now on, we will always let ∇ denote the Levi-Civita connection. Recall that the Levi-Civita connection is compatible with the metric g, that is

$$\nabla_X g)X, Y) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0\,.$$

In other words $\nabla g = 0$. The connection is also compatible with the musical isomorphisms in the following sense

Lemma 7. *Let $X, Y, Z \in \mathfrak{X}$ and $\omega \in C^\infty \Lambda^1$. Then*

$$(\nabla_X Y)^\flat = \nabla_X Y^\flat \quad \textit{and} \quad (\nabla_X \alpha)^\sharp = \nabla_X \alpha^\sharp.$$

Proof. We have

$$\begin{aligned}0 = \nabla_X g(Z, Y) =& Xg(Z, Y) - g(\nabla_X Z, Y) - g(Y, \nabla_X Z) \\ =& XY^\flat(Z) - Y^\flat(\nabla_X Z) - (\nabla_X Y)^\flat(Z) \\ =& (\nabla_X Y^\flat)(Z) - (\nabla_X Y)^\flat(Z)\end{aligned}$$

so $(\nabla_X Y)^\flat = \nabla_X Y^\flat$. If $Y = \omega^\sharp$, we have the second equality via the isomorphism $\sharp$ on both sides. $\square$

2.3.5 Curvature Operators

The curvature can be used to define several$\mathscr{E}$-linear operators on the tangent bundle.

Definition. The **Riemannian curvature** Rm is defined by

$$\mathrm{Rm}(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$$

Let U be an open set in which $E_1, \ldots, E_n$ is an orthonormal frame. Let $X, Y \in \mathfrak{X}$, we define the **Ricci curvature** by

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} \mathrm{Rm}(E_i, X, Y, E_i)\,.$$

For two linearly independent tangent vectors $u, v \in T_p\mathbb{M}$, their **sectional curvature** is

$$\sec(u, v) = \frac{\mathrm{Rm}(u, v, v, u)}{|u \wedge v|^2} = \frac{\langle R(u, v)v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}\,.$$

This is interpreted as the sectional curvature of the linear subspace spanned by u and v in the tangent space, since $\sec(u, v)$ is independent of which basis we choose for the subspace. If we choose u and v orthonormal, the denominator is just 1, and the formula simplifies to $\sec(u, v) = \langle R(u, v)v, u\rangle$. If $e_1, \ldots, e_n$ is a basis of $T_p\mathbb{M}$, we can find the sectional curvature for all the $\binom{n}{2}$ subspaces spanned by pairs of basis vectors. Taking a kind of average of the sectional curvatures produces the **scalar curvature**. It is given by the formula

$$\sum_{i,j} \sec(e_i, e_j)\,.$$

The scalar curvature is also the trace of the endomorphism defined by the Ricci curvature, which should convince the reader that the sectional curvature does not depend on the basis.

The Ricci curvature is completely defined by the sectional curvature. For any two tangent vectors $u, v \in T_p\mathbb{M}$, we have

Theorem 8. *If $e_1 \ldots, e_n$ is an orthonormal basis of $T_p\mathbb{M}$, then*

$$\mathrm{Ric}(e_1, e_1) = \sum_{i=2}^{n} \sec(e_i e_1)$$

Proof.

$$\mathrm{Ric}(e_1, e_1) = \sum_{i=1}^{n} \mathrm{Rm}(e_i, e_1, e_1, e_i) = \sum_{i=2}^{n} \sec(e_i, e_1)$$

because the the term $\mathrm{Rm}(e_i, e_i, e_i, e_i)$ is zero by symmetries of the Riemannian curvature tensor, and $|e_i \wedge e_1| = 1$ when $i \neq 1$. □

This is enough to describe the Ricci curvature because

$$\mathrm{Ric}(u, v) = \frac{1}{4}(\mathrm{Ric}(u+v, u+v) - \mathrm{Ric}(u-v, u-v))$$

2.4 Parallel translation

The Levi-Civita connection gives us a way of differentiating vector fields. If a vector field is defined on the image of a smooth curve γ, the connection also gives a way of differentiating it in the direction of γ. Differentiating vector fields gives an idea how the vector field twists and stretches along the curve. A vector fields that does not is said to be parallel along the curve. Most results in this section are from [21, Section 13–15].

2.4.1 Covariant Differentiation

Theorem 9. *Let $\mathbb{M}$ be a manifold with an affine connection ∇, and $\gamma\colon [a, b] \to \mathbb{M}$ a smooth curve in $\mathbb{M}$. Then there is a unique map*

$$\frac{D}{dt}\colon \Gamma(T\mathbb{M}|_\gamma) \to \Gamma(T\mathbb{M}|_\gamma)$$

such that for $V \in \Gamma(T\mathbb{M}|_\gamma)$

(i) $\frac{D}{dt}(cV) = c\frac{DV}{dt}$ for any real number c.

(ii) For any smooth function f on $[a, b]$,

$$\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$$

(iii) If $V(t) = \tilde{V}(\gamma(t))$ for some C^∞ vector field $\tilde{V} \in \mathfrak{X}$, then

$$\frac{DV}{dt} = \nabla_{\gamma'(t)}\tilde{V}$$

The operation $\frac{D}{dt}$ is called **covariant differentiation**. A smooth vector field $V(t)$ along a $\gamma\colon I \to \mathbb{M}$ is **parallel** if $\frac{DV}{dt} = 0$ for all $t \in I$. If the vector field $V(t) = \gamma'(t)$, and $V(t)$ is parallel, then γ is a **geodesic**. If $V(t)$ is parallel on the curve $\gamma\colon [a, b] \to \mathbb{M}$ from p to q , and $v = V(a) \in T_p\mathbb{M}$, $w = V(b) \in T_q\mathbb{M}$, then w is a **parallel translate** of v along γ. A geodesic $\gamma\colon [a, b] \to \mathbb{M}$ is **minimal** if its length is equal to the distance between the endpoints, with respect to d_g. It is **maximal** if its domain cannot be extended to a larger interval.

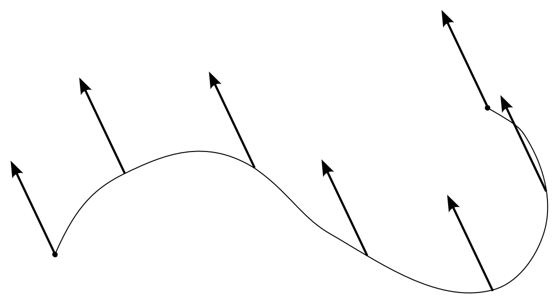

Figure 2.1: Sketch of parallel translation along a curve

Suppose $V(t)$ is a vector field along the curve γ. Let U be a coordinate neighbourhood about $\gamma(t_0)$. In coordinates $V(t) = v^i(t)\partial_i$. By the defining properties of the covariant derivative

$$\begin{aligned}\frac{D}{dt}v^i(t)\partial_i =& \frac{dv^i}{dt}\partial_i + v^i\frac{D}{dt}\partial_i \\ =& \frac{dv^i}{dt}\partial_i + v^i\nabla_{\gamma'(t)}\partial_i \\ =& \left(\frac{dv^k}{dt} + v^i(\gamma^j)'(t)\Gamma^k_{ij}\right)\partial_k\end{aligned}$$

which shows that $V(t)$ is parallel if and only if the equations $\frac{dv^k}{dt} + v^i(\gamma^j)'(t)\Gamma^k_{ij} = 0$ are satisfied.

2.4.2 Existence of Geodesics

By using the theory of ordinary differential equations, one can prove the following theorems.

Theorem 10. *Let $\mathbb{M}$ be a manifold with connection ∇. Given a point $p \in \mathbb{M}$, and a tangent vector $v \in T_p\mathbb{M}$, there is a geodesic $\gamma(t)$ with initial conditions: $\gamma(0) = p$ and $\gamma'(0) = v$. The geodesic is unique in the sense that any other geodesic which satisfies the initial conditions, agrees with $\gamma(t)$ on the intersection of their domains.*

For a tangent vector v let γ_v denote the unique maximal geodesic defined by the theorem above. We want to define a map $\exp_p$ whose domain $D(\exp_p)$ is a subset of the tangent space at p, and codomain will be a neighbourhood of p. Let $D(\exp_p) = \{v \in T_p\mathbb{M} : 1 \text{ is in the domain of } \gamma_v\}$.

Definition. Let $\exp_p \colon D(\exp_p) \to \mathbb{M}$ be defined by $\exp_p(v) = \gamma_v(1)$. Let $D(\exp) = \bigcup_{p\in\mathbb{M}} D(\exp_p)$, and $\exp \colon D(\exp) \to \mathbb{M}$ be defined as the natural extension of $\exp_p$ to $\bigcup_{p\in\mathbb{M}} D(\exp_p)$.

In the Riemannian case where we use the Levi-Civita connection, we can, in a small enough neighbourhood of p, and for small enough tangent vectors, find a geodesic defined on $(-c, c)$, $c > 0$.

Theorem 11. *For any point p of a Riemannian manifold $\mathbb{M}$ and any $c > 0$, there is a neighbourhood U about p, and $\varepsilon > 0$ such that for any $q \in U$ and $v \in T_q\mathbb{M}$ with $|v| < \varepsilon$, there is a unique geodesic $\gamma\colon (-c, c) \to \mathbb{M}$ with $\gamma(0) = q$ and $\gamma'(0) = v$.*

The last theorem ensures that at each point $(p, 0) \in T\mathbb{M}$ there exist a neighbourhood $U \times B_\varepsilon(0)$ where exp is defined.

Theorem 12. *Let $\mathbb{M}$ be a Riemannian manifold with connection ∇ and let $\gamma : [a, b] \to \mathbb{M}$ be a smooth curve in $\mathbb{M}$. There exists for every vector $v_0 \in T_{\gamma(a)}\mathbb{M}$ a vector filed along γ which parallel translates v_0 to a vector v_1 in $T_{\gamma(b)}$. Let $\varphi_{a,b}\colon T_{\gamma(a)}\mathbb{M} \to T_{\gamma(b)}\mathbb{M}$ be the map that takes tangent vectors to its parallel translate along γ. Then $\varphi_{a,b}$ is an $\mathbb{R}$-linear isomorphism.*

2.4.3 Complete Manifolds

Definition. A Riemannian manifold $\mathbb{M}$ is said to be **geodesically complete** if the domain of every geodesic in $\mathbb{M}$ can be extended to the entire real line.

There are equivalent notions of completeness.

Theorem 13 (Hopf–Rinow)**.** [15, Thm 5.7.1] *On a connected Riemannian manifold $\mathbb{M}$, the following statements are equivalent:*

1. *$\mathbb{M}$ is geodesically complete*
2. *$\mathbb{M}$ is metrically complete*
3. *Every closed and bounded subset of $\mathbb{M}$ with respect to the metric is compact*

Because geodesic and metric completeness are equivalent, there is no ambiguity in saying that a connected Riemannian manifold is complete. The third property is very useful, for example we can deduce that the connected compact manifolds are exactly the complete manifolds with finite diameter.

Remark. If a manifold is not connected, but is geodesically complete, we can substitute $\mathbb{M}$ for "each connected component of $\mathbb{M}$" in the two last statements.

2.4.4 Global Results from Pointwise Operators

On complete connected manifolds, there are many results that relate properties of curvature operators, which are defined on tensor products of the tangent spaces, with global properties of the manifold. We will concentrate on the cases where the curvature is "more or less" positive. We mention a result where the sectional curvature is negative.

Theorem 14 (Cartan–Hadamard)**.** [12, Thm 12.8] *If* $\mathbb{M}$ *is a connected complete Riemannian manifold with non-positive sectional curvature, then for every point* p *in* $\mathbb{M}$*, the map* $\exp_p\colon T_p\mathbb{M} \to \mathbb{M}$ *is a smooth covering map. Thus the universal covering space of* $\mathbb{M}$ *is diffeomorphic to* $\mathbb{R}^n$*. If* $\mathbb{M}$ *is simply connected, then* $\mathbb{M}$ *itself is diffeomorphic to* $\mathbb{R}^n$*.*

On the other hand, when the sectional curvatures are more positive:

Theorem 15 (Bonnet–Myers)**.** [12, Thm 12.24] *Let* $\mathbb{M}$ *be a complete, connected Riemannian manifold, and suppose there is a positive constant* r *such that the Ricci curvature of* $\mathbb{M}$ *satisfies*

$$\mathrm{Ric}(v, v) \geq \frac{n-1}{r^2}$$

for all unit vectors v*. Then* $\mathbb{M}$ *is compact, with diameter less than or equal to* πr*, and its fundamental group is finite.*

This theorem divides the investigation of manifolds with non-negative Ricci curvature into two classes, compact manifolds, and manifolds where the greatest lower bound of the ricci curvature is zero. The first class will be dealt with in Chapter 3, and the second in Chapter 4.

2.5 Weitzenböck Formula

In this sections we will define the Laplace operator, which we will decompose into a differential operator and an $\mathscr{E}$-linear map which depends on the curvature of $\mathbb{M}$.

2.5.1 Local Frame Parallel at a Point

First we show that about each point there is an orthonormal frame, which makes a lot of computations easier.

Lemma 16. [23, Lem 1.1] *Let* p *be a point in* $\mathbb{M}$*. There is an orthonormal frame* $E_1, \dots, E_n$ *in a neighbourhood of* p *such that* $\nabla_X E_j(p) = 0$ *for all* $X \in \mathfrak{X}$*.*

Proof. Let (U, ϕ) be a coordinate chart centred at p with $\phi(U)$ star-shaped. For any $q \in U$ we define a smooth curve $\gamma_q(t) = \phi^{-1}(t\phi(q))$ from p to q. For a basis $e_1, \ldots, e_r$ of B_p, let $E_i(q)$ be the parallel translate of e_i to B_q along γ_q. We claim that $E_1, \ldots, E_r$ is a smooth frame in U. The covariant derivative of $E_i(t)$ along γ_q is zero from the definition of parallel translation, and we get

$$0 = \frac{D}{dt} E_i|_{t=1} = \nabla_{\gamma_q'(1)} E_i = x^j \nabla_{\partial_j} E_i|_q = x^j \omega_i^k(\partial j) E_k|_q$$

where we abuse notation and let E_i also be the vector field along γ. The real maps $x^j \omega_i^k(\partial_j) = 0$ on U for each k. We can differentiate each in the ∂_ℓ direction.

and evaluating at p on both sides gives

$$\omega_i^k(\partial_\ell)(p) + x^j(p) \frac{\partial}{\partial x^\ell} \omega_i^k(\partial_j)(p) = \omega_i^k(\partial_\ell)(p) = 0\,.$$

This still holds for any k; and since ∂_ℓ and E_i was arbitrary the equation holds for any i, k and ℓ. We now have that for any $X = X^j \partial j$

$$\nabla_X E_i(p) = X^j \nabla_{\partial_j} E_i(p) = X^j \omega_i^k(\partial_j)|_p E_k = 0$$

By the Gram-Schmidt process, we can choose $e_1, \ldots, e_r$ to be orthonormal. Because ∇ is compatible with g, we have that

$$\frac{d}{dt}\langle E_i, E_j\rangle = \left\langle \frac{D}{dt} E_i, E_j \right\rangle + \left\langle E_i, \frac{D}{dt} E_j \right\rangle = \langle 0, E_j\rangle + \langle E_i, 0\rangle = 0\,.$$

where E_i, E_j are vector fields along some γ_q. Therefore $\langle E_i, E_j\rangle$ is constantly zero. $\square$

This is the same as saying that the connection forms vanish at p, $\omega_i^j(p) = 0$. Since the Levi-Civita connection is torsion-free, $0 = T(E_i, E_j) = \nabla_{E_i} E_j - \nabla_{E_j} E_i - [E_i, E_j]$, so $[E_i, E_j] = 0$ at the point p. Let $\theta^i = (E_i)^\flat$. Then we also have $\nabla_X \theta^i = 0$ at p by Lemma 7 . This extends to the basis elements of $\Lambda^k(T_p\mathbb{M})$ of the form $\theta^I = \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}$ since

$$\nabla_X \theta^I = \sum_r \theta^{i_1} \wedge \cdots \wedge \underbrace{\nabla_X \theta^{i_r}}_{=0} \wedge \cdots \wedge \theta^{i_k} = 0$$

As a demonstration of the usefulness of this kind of frame, we show that the Levi-Civita connection is compatible with the metric on forms, in the sense that $\nabla_X\langle \alpha, \beta\rangle = \langle \nabla_X \alpha, \beta\rangle + \langle \alpha, \nabla_X \beta\rangle$. Let $E_1, \ldots, E_n$ be a frame parallel at p. Then $\nabla_X \alpha_I \theta^I = (\nabla_X \alpha_I)\theta^I + \alpha_i \nabla_X \theta^I = (\nabla_X \alpha_I)\theta^I$ and

$$\nabla_X\langle \alpha, \beta\rangle = \nabla_X \sum_I \alpha_I \beta_I = \sum_I (\nabla_X \alpha_I)\beta_I + \alpha_I(\nabla_X \beta_I) = \langle \nabla_X \alpha, \beta\rangle + \langle \alpha, \nabla_X \beta\rangle$$

2.5.2 Local Formula for the Exterior Derivative and Co-differential

Theorem 17. *Let $E_1, \dots E_n$ be an orthonormal frame in a an open subset U of $\mathbb{M}$. Then we can write the exterior derivative and the co-differential as*

$$d\alpha = \sum_{j=1}^{n} \theta^j \wedge (\nabla_{E_j} \alpha)$$

and

$$\delta\alpha = -\sum_{j=1}^{n} \iota_{E_j} \nabla_{E_j} \alpha$$

in U.

Proof. In the proof, let d denote the operator defined as above. We have to check the three defining properties of the exterior derivative which uniquely defines it.

1. That $df(X) = Xf$ for $f \in \mathscr{E}$: Let $f \in \mathscr{E}$, and $X \in \mathscr{X}$, then

$$df(X) = \theta^j \wedge \nabla_{E_j} f(X) = (E_j f)\theta^j(X) = X^j E_j f = Xf$$

2. That d is an anti-derivation of degree 1: Let $\omega \in C^\infty \Lambda^k$ and $\tau \in C^\infty \Lambda^\ell$

$$d(\omega \wedge \tau) = \theta^j \wedge \nabla_{E_j}(\omega \wedge \tau) = \theta^j \wedge (\nabla_{E_j}\omega) \wedge \tau + \theta^j \wedge \omega \wedge (\nabla_{E_j}\tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau$$

3. That $d^2 = 0$: Let $\tilde{E}_1, \dots, \tilde{E}_n$ be a frame as in Lemma 16, and $\tilde{\theta}^1, \dots, \tilde{\theta}^n$ the dual frame. If A is the matrix such that $\tilde{E} = EA$, then $A^T A = 1$. At any point in U we can substitute orthonormal frame because

$$\tilde{\theta}^j \wedge \nabla_{\tilde{E}_j} = A^k_j \theta^k \wedge \nabla_{A^\ell_j E_\ell} = A^k_j A^\ell_j \theta^k \wedge \nabla_{E_\ell} = (AA^T)^k_\ell \theta^k \wedge \nabla_{E_\ell} = \theta^k \wedge \nabla_{E_k}$$

   We can therefore assume that $E_1, \dots, E_n$ is of the type as defined in Lemma 16. Let $f \in C^\infty \Lambda^0$.

$$d^2 f = \sum_i \theta^i \wedge \nabla_{E_i}(\sum_j (E_j f)\theta^j) = \sum_{i,j} (E_i E_j f)\theta^j \wedge \theta^i$$

   where all terms with $i = j$ vanish, so the ones remaining are

$$\sum_{i<j} (E_i E_j f)\theta^j \wedge \theta^i + \sum_{j<i} (E_i E_j f)\theta^j \wedge \theta^i$$

   which after a change of index in the right sum is

$$= \sum_{i<j} ([E_i, E_j] f)\theta^j \wedge \theta^i = 0$$

since ∇ is torsion-free, and $[E_i, E_j]$ vanishes at p. Suppose $d_{k+1}d_k = 0$ where $k < \ell$. If $\omega \in C^\infty\Lambda^\ell$, then

$$
\begin{aligned}
d^2\omega =& d\Big(d(\omega_I dx^{I_1} \wedge \cdots \wedge dx^{I_{\ell-1}}) \wedge dx^{I_\ell} + (-1)^{\ell-1}(\omega_I dx^{I_1} \wedge \cdots \wedge dx^{I_{\ell-1}}) \wedge d^2x^{I_\ell}\Big) \\
=& (d^2(\omega_I dx^{I_1} \wedge \cdots \wedge dx^{I_{\ell-1}})) \wedge dx^{I_\ell} + (-1)^{\ell}(d(\omega_I dx^{I_1} \wedge \cdots \wedge dx^{I_{\ell-1}})) \wedge d^2x^{I_\ell}) \\
=& 0
\end{aligned}
$$

and by induction $d^2 = 0$

In the second equation of the theorem we may also assume that the frame $E_1 \ldots, E_n$ is such that $\nabla E_i = [E_i, E_j] = 0$ at p, and the computations are carried out at this point. Recall that $\delta_{k+1} = (-1)^{nk+1} \star d\star$. By using our equivalent definition of d, we can write

$$\delta_{k+1}\omega = (-1)^{nk+1} \star \sum_j \theta^j \wedge \nabla_{E_j} \star \omega$$

We see that the new formula for δ coincides with the old.

$$
\begin{aligned}
-\sum_j \iota_{E_j}\nabla_{E_j}\omega =& -\sum_j \iota_{E_j}(E_j\omega_I)\theta^I = \\
=& -\sum_j\sum_r (E_j\omega_I)\delta_j^{i_r}(-1)^{r-1}\theta^{i_1} \wedge \cdots \wedge \widehat{\theta}^{i_r} \wedge \cdots \wedge \theta^{i_{k+1}} \\
=& -\sum_j\sum_r (E_j\omega_I)\delta_j^{i_r}(-1)^{(n-(k+1))k} \star (\theta^{i_r} \wedge \star\theta^I) \\
=& \star\sum_j (-1)^{nk+1}\theta^j \wedge (E_j\omega_I) \star \theta^I \\
=& (-1)^{nk+1} \star \sum_j \theta^j \wedge \nabla_{E_j} \star \omega
\end{aligned}
$$

□

2.5.3 The Rough Laplacian

Let ∇ be an affine connection. We have seen that for any tensor field T, we have covariant derivative ∇T which is another tensor field. Applying ∇ to ∇T gives another tensor field $\nabla^2 T$ called the **Hessian** of T. The Hessian of T defines an $\mathscr{E}$-linear map $(X, Y) \mapsto \nabla^2_{X,Y}T$, where

$$\nabla^2_{X,Y}T = (\nabla_X\nabla_Y - \nabla_{\nabla_X Y})T\,.$$

Linearity in the first variable is trivial. Let $f \in \mathscr{E}$,

$$
\begin{aligned}
\nabla^2_{X,fY}\alpha =& \nabla_X\nabla_{fY}\alpha - \nabla_{\nabla_X fY}\alpha \\
=& (Xf)\nabla_Y\alpha + f\nabla_X\nabla_Y\alpha - \nabla_{(Xf)Y + f\nabla_X Y}\alpha \\
=& f(\nabla_X\nabla_Y - \nabla_{\nabla_X Y})\alpha
\end{aligned}
$$

If ∇ is torsion-free, then $(\nabla^2_{X,Y} - \nabla^2_{Y,X})\alpha = R(X,Y)\alpha$.

The **Rough Laplacian** L is the trace of the Hessian. If α is a k-form, and $E_1, \ldots, E_n$ is an orthonormal frame in an open subset U of $\mathbb{M}$, then $L\alpha$ is given by

$$L\alpha = \sum_i \nabla^2_{E_i,E_i}\alpha$$

2.5.4 Decomposition of the Laplace Operator

We need a small lemma in order to prove the decomposition of the Laplace operator.

Lemma 18. *Let $X, Y \in \mathfrak{X}$ and $\omega \in C^\infty\Lambda^k$. Covariant differentiation of $\iota_Y\omega$ satisfies the Leibniz rule in sense that*

$$\nabla_X(\iota_Y\omega) = \iota_{\nabla_X Y}\omega + \iota_Y\nabla_X\omega$$

Proof.

$$\begin{aligned}\nabla_X(\iota_Y\omega)(Z_1, \ldots, Z_k) =& X\iota_Y\omega(Z_1, \ldots, Z_k) - \sum_{j=1}^k \iota_Y\omega(Z_1, \ldots, \nabla_X Z_j, \ldots, Z_k)\\ =& X\omega(Y, Z_1, \ldots, Z_k) - \sum_{j=1}^k \omega(Y, Z_1, \ldots, \nabla_X Z_j, \ldots, Z_k)\\ =& \iota_Y\nabla_X\omega(Z_1, \ldots, Z_k) + \iota_{\nabla_X Y}\omega(Z_1, \ldots, Z_k)\end{aligned}$$

□

Theorem 19 (Weitzenböck Formula)**.** *The Laplace operator Δ can be written as*

$$\Delta = L + \sum_{i,j} \theta^i \wedge \iota_{E_j} R(E_i, E_j) = L - \mathscr{R}$$

where $\mathscr{R}$ is an $\mathscr{E}$-module endomorphism on $C^\infty\Lambda^k(\mathbb{M})$

Proof. Let $E_1, \ldots, E_n$ be an orthonormal frame about p which is parallel at p. Let $\theta^1, \ldots, \theta^n$ be the dual co-frame. At the point p, we have

$$\begin{aligned}\Delta\omega =& \sum_{i,j} \iota_{E_j}\nabla_{E_j}(\theta^i \wedge \nabla_{E_i}\omega) + \theta^i \wedge (\nabla_{E_i}\iota_{E_j}\nabla_{E_j}\omega)\\ =& \sum_{i,j} \iota_{E_j}(\theta^i \wedge \nabla_{E_j}\nabla_{E_i}\omega) + \theta^i \wedge (\nabla_{E_i}\iota_{E_j}\nabla_{E_j}\omega)\\ =& \sum_i \nabla_{E_i}\nabla_{E_i}\omega + \theta^i \wedge (\nabla_{E_i}\iota_{E_j}\nabla_{E_j}\omega - \iota_{E_j}\nabla_{E_j}\nabla_{E_i}\omega)\\ =& \sum_i \nabla_{E_i}\nabla_{E_i}\omega + \sum_{i,j} \theta^i \wedge \iota_{E_j}\left(R(E_i, E_j) + \nabla_{[E_i,E_j]}\right)\omega\end{aligned}$$

And since both $\nabla_{\nabla_{E_i}E_i}$ and $\nabla_{[E_i,E_j]}$ vanish at p we have our desired equality at that point. That $\mathscr{R}$ is an $\mathscr{E}$-module endomorphism is clear since $R(E_i, E_j)$ is $\mathscr{E}$-linear. □

— 3 —

Hodge Theory

3.1 De Rham cohomology

A topological invariant, is a property of topological spaces which are common to spaces that are homeomorphic. For smooth manifolds, one topological invariant is the de Rham cohomology. The k'th de Rham cohomology of $\mathbb{M}$, denoted by $H^k_{dR}(\mathbb{M})$ is a real vector space, and is a topological invariant of $\mathbb{M}$ [11, Cor 17.12]

3.1.1 De Rham Complexes

The de Rham cohomologies are constructed by taking quotients in a complex of vector spaces. Since the exterior derivative d has the property that $d_{k+1}d_k = 0$, we can form the complex

$$C^\infty\Lambda^0 \xrightarrow{d_0} C^\infty\Lambda^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C^\infty\Lambda^k \xrightarrow{d_k} C^\infty\Lambda^{k+1} \xrightarrow{d_{k+1}} \cdots$$

called the **de Rham complex**. At each $C^\infty\Lambda^k$ we define the k'th **de Rham cohomology** by

$$H^k_{dR}(\mathbb{M}) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}}$$

We can also the define the compact de Rham cohomology $H^k_{0,dR}(\mathbb{M})$ by taking quotients in the complex

$$C_0^\infty\Lambda^0 \xrightarrow{d_0} C_0^\infty\Lambda^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C_0^\infty\Lambda^k \xrightarrow{d_k} C_0^\infty\Lambda^{k+1} \xrightarrow{d_{k+1}} \cdots$$

where $C_0^\infty\Lambda^k$ is the space of k-forms with compact support. The quotient

$$H^k_{0,dR}(\mathbb{M}) := \operatorname{Ker} \frac{d(C^\infty\Lambda^k}{\operatorname{Im} d(C_0^\infty\Lambda^{k+1})}$$

makes sense since for any $(k+1)$-form ω, $\operatorname{supp} d\omega$ is a closed subset of $\operatorname{supp}\omega$. Difference between $C^\infty\Lambda^k$ and $C_0^\infty\Lambda^k$ exists, of course, only when $\mathbb{M}$ is not compact; and the same is true for $H^k_{0,dR}(\mathbb{M})$ and $H^k_{dR}(\mathbb{M})$.

3.1.2 Hodge Theorem

In the rest of the chapter we will study only compact Riemannian manifolds $\mathbb{M}$, with an orientation, and no boundary. The main result in this chapter is a decomposition of the differential forms on for manifolds of this type.

Theorem 20 (Hodge Decomposition)**.** *Let $\mathbb{M}$ be a compact oriented Riemannian manifold. For each integer $0 \leq k \leq n$, the k'th de Rham cohomology $H^k_{dR}(\mathbb{M})$ is finite dimensional, and we have the following orthogonal decomposition of $C^\infty\Lambda^k$:*

$$\begin{aligned} C^\infty\Lambda^k =&\Delta(C^\infty\Lambda^k) \oplus \mathscr{H}\Lambda^k \\ =&\delta d(C^\infty\Lambda^k) \oplus d\delta(C^\infty\Lambda^k) \oplus \mathscr{H}\Lambda^k \\ =&\delta(C^\infty\Lambda^{k+1}) \oplus d(C^\infty\Lambda^{k-1}) \oplus \mathscr{H}\Lambda^k \end{aligned}$$

We postpone the proof, to first explain the decomposition's relation with the de Rham cohomology. For $\omega \in C^\infty\Lambda^k$, let $\mathscr{H}_k\colon C^\infty\Lambda^k \to C^\infty\Lambda^k$ be the projection taking ω onto its harmonic part of $\mathscr{H}_k\omega$. We can restrict $\mathscr{H}_k$ to the closed forms, defining a surjective map $\mathscr{H}_k\colon \operatorname{Ker} d_k \to \mathscr{H}\Lambda^k$. The kernel of $\mathscr{H}_k$ is clearly the set of forms ω that are of the form $\omega = d\alpha + \delta\beta$. Since we know $d\omega = 0$ we have

$$0 = \langle\!\langle d\omega, \beta\rangle\!\rangle = \langle\!\langle d^2\alpha + d\delta\beta, \beta\rangle\!\rangle = \|\delta\beta\|^2$$

and therefore $\omega = d\alpha$, i.e. $\omega \in \operatorname{Im} d_{k-1}$. We therefore have isomorphism $\mathscr{H}\Lambda^k \cong H^k_{dR}(\mathbb{M})$. The isomorphism between de Rham cohomology and the harmonic forms is known as the **Hodge Theorem**.

3.2 Differential operators

To prove the Hodge decomposition theorem, need two analytical result from the theory of Fourier Analysis. Most of the material in this sections can be found in [22, Chap 6], where the reader can also find proofs for various statements in this section.

3.2.1 Fourier Series

We need some notation for higher derivatives.

Definition. Let $\alpha \in \mathbb{Z}^n$, then $[\alpha] = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We define

$$D^\alpha := \left(\frac{1}{i}\right)^{[\alpha]} \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \text{and} \quad \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

when $\xi \in \mathbb{R}^n$. The $1/i$ therm is there to simplify formulas by cancelling the imaginary unit when differentiating $e^{-ix\cdot\xi}$.

Definition. Let $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ denote the smooth maps $f\colon \mathbb{R}^n \to \mathbb{C}^m$, which are (2π)-periodic, i.e. $f(x^1, \ldots, x^i + 2\pi, \ldots, x^n) = f(x^1, \ldots x^i, \ldots, x^n)$. The elements of $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ can be identified with smooth maps on the cube $Q = [0, 2\pi]^n$ with opposite sides identified.

Each function f in $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ can be represented as an infinite sum

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} f_\xi e^{ix\cdot\xi} \tag{3.1}$$

where $f_\xi \in \mathbb{C}^m$, and the sum converges uniformly to f. The j'th coordinate of f_ξ is the Fourier transform of the j'th component

$$(f_\xi)^j = \frac{1}{(2\pi)^n} \int_Q f^j(x) e^{-ix\cdot\xi}\, dx \tag{3.2}$$

evaluated at ξ. We call $(f_\xi)_{\xi\in\mathbb{Z}^n}$ the **Fourier coefficients** of f, and the whole series the **Fourier series** of f. We can now identify each f in $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ with a sequence $(f_\xi)_{\xi\in\mathbb{Z}^n}$ of complex vectors $f_\xi \in \mathbb{C}^m$.

The space of all sequences $(a_\xi)_{\xi\in\mathbb{Z}^n} \subset \mathbb{C}^m$ is denoted by $\mathscr{S}$. There is an injection $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m) \hookrightarrow \mathscr{S}$, by $f \mapsto (f_\xi)_{\xi\in\mathbb{Z}^n}$.

The Fourier coefficient of the derivative $D^k f$ have components

$$(D^k f_\xi)^j = \frac{1}{(2\pi)^n}\int_Q (D^k f^j(x)) e^{-ix\cdot\xi}\, dx = -\frac{1}{(2\pi)^n}\int_Q f^j(x) D^k e^{-ix\cdot\xi}\, dx = \xi_k (f_\xi)^j$$

where we integrate by parts in the second step. Higher derivatives therefore are of the form

$$D^\alpha f(x) = \sum_{\xi\in\mathbb{Z}^n} \xi^\alpha f_\xi e^{ix\cdot\xi}\,.$$

Differentiation of elements in $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ is therefore done, simply by substituting D^α for ξ^α.

3.2.2 Sobolev Spaces

The Fourier Series representation of the derivatives $D^\alpha f$ motivates a definition of the derivative on $\mathscr{S}$, simply by defining $D^\alpha(a_\xi)_{\xi\in\mathbb{Z}^n} = (\xi^\alpha a_\xi)_{\xi\in\mathbb{Z}^n}$. The s'th **Sobolev space** H^s is the subspace of $\mathscr{S}$ containing elements $(a_\xi)_{\xi\in\mathbb{Z}^n}$ such that the squared norm of the derivatives $|D^\alpha|^2 = \sum_{\xi\in\mathbb{Z}^n} |\xi^\alpha f_\xi|^2$ converge for each α with $[\alpha] \le s$. The elements of the s'th Sobolev space is interpreted as the elements that are s times differentiable.

There are positive constants c, C such that

$$c(1+|\xi|^2)^s \le \sum_{[\alpha]\le s} |\xi^\alpha|^2 \le C(1+|\xi|^2)^s \tag{3.3}$$

holds for each $\xi \in \mathbb{Z}^n$, and therefore an equivalent description of H^s is all elements $(a_\xi)_{\xi\in\mathbb{Z}^n}$ in $\mathscr{S}$ such that

$$\sum_\xi (1+|\xi|^2)^s |a_\xi|^2 < +\infty\,.$$

Each Sobolev space H^s have inner product

$$\langle\!\langle (a_\xi), (b_\xi) \rangle\!\rangle_{H^s} = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s a_\xi \cdot b_\xi$$

making it a Hilbert space, that all contain $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ as a dense subspace. For elements in $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$, the inner product in H^0 is the same as the usual L^2 inner product, i.e.

$$\frac{1}{(2\pi)^n} \int f \cdot g \, dx = \sum_\xi f_\xi \cdot g_\xi$$

for all $f, g \in C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$. We also define $H^{-\infty} := \bigcup_{s \in \mathbb{Z}} H^s$.

The two next lemmas are important in the proof of the Hodge Decomposition. Since the s'th Sobolev Spaces are formally functions that are s-times differentiable, the following theorem makes sense.

Lemma 21 (Sobolev)**.** *If $u \in H^s$ for each $s \in \mathbb{Z}$, then $u \in C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$.*

The Sobolev theorem can be stated in greater generality [22, Lem 6.22], but for our purpose this is sufficient.

Lemma 22 (Rellich)**.** *The injections $H^{s+\ell} \to H^s$, where $\ell > 0$, are compact.*

3.2.3 Differential Operators in Euclidean Space

We define differential operators on Euclidean space first. Later we will see how this generalizes to manifolds.

Definition. A **differential operator** P of order ℓ on an open set $U \subset \mathbb{R}^n$ consists of a matrix $[P_{ij}]$, where

$$P_{ij} = \sum_{[\alpha] \leq \ell} a^{ij}_\alpha D^\alpha \tag{3.4}$$

where each a^{ij}_α is a C^∞ function on U. A differential operator on $\mathbb{R}^n$ is a differential operator on $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ if each a^{ij}_α is (2π) periodic.

We can decompose P as $P = P_\ell + P_{<\ell}$ such that $(P_\ell)_{ij} = \sum_{[\alpha]=\ell} a^{ij}_\alpha D^\alpha$, and $(P_{<\ell})_{ij} = \sum_{[\alpha]<\ell} a^{ij}_\alpha D^\alpha$. The operator P_ℓ is called the **principal part** of P. Substituting D^α for ξ^α in P_ℓ yields a matrix of polynomials $P_\ell(\xi)$ with ξ as indeterminate. At each point $x \in U$ we define a map $\sigma_x \colon \mathbb{R}^n \to \mathbb{R}^{m \times m}$ defined by the polynomial $P_\ell(\xi)$ is called the **symbol** of P. We say that the operator P is **elliptic** at x if $\xi \neq 0$ implies $\sigma_x(\xi) \in \mathrm{GL}(m, \mathbb{C})$, i.e. $\sigma_x(\xi)$ is non-degenerate, for each point p in U.

3.2.4 Differential Operators on Manifolds

Definition. Let $\pi\colon B \to \mathbb{M}$ be a smooth vector bundle of rank r. A **differential operator** P of order ℓ on the bundle $\pi\colon B \to \mathbb{M}$ is an $\mathbb{R}$-linear map $P\colon \Gamma(B) \to \Gamma(B)$, and for each point there is a trivializing coordinate patch U such that on $\pi^{-1}(U) \cong U \times \mathbb{R}^r$, P can be written as a differential operator on $\phi(U)$. That is to say that if s is a C^∞ section of B, which by the local trivialization is equivalent to being a smooth map $s\colon \phi(U) \to \mathbb{R}^r$, then Ps is defined by a differential operator on $\phi(U)$.

We allow the fibers of B to be complex vector spaces. If they are not, we must make sure that the differential operator does not involve any complex numbers.

An operator P is **elliptic** if it is defined locally by an elliptic operator on $\phi(U)$. This definition of ellipticity is independent of trivialization. Let A_p be the matrix such that $s_p = A_p\tilde{s}_p$ where s_p and $\tilde{s}_p$ are coordinate vector with respect to two different trivializations. Let $(Ps)_p = Q_p s_p = \tilde{Q}_p\tilde{s}_p$ where Q and $\tilde{Q}$ are matrices of the form described in (3.4). Then $Q_p(A_p\tilde{s}_p) = \tilde{Q}_p\tilde{s}_p$. By the Leibniz rule, the principal part of $\tilde{Q}_p$ is Q_pA_p where $(Q_pA_p)_{ij} = a^{ik}_\alpha A^k_j D^\alpha$. Therefore $\sigma_{\tilde{Q}}(\xi) = \sigma_Q(\xi)A$, where $A \in GL(r, \mathbb{R})$. Similarly, a coordinate change substitutes $\frac{\partial}{\partial x^i}$ with $\sum \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j}$, and therefore substitutes ξ for $J\xi$ where J is the Jacobian matrix, which is non-degenerate since coordinated changes are diffeomorphisms. So if Q and $\tilde{Q}$ define P with respect to x and $\tilde{x}$, then $\sigma_{\tilde{Q}}(\xi) = \sigma_Q(J\xi)$, such that $\tilde{Q}$ defineds an elliptic operator if and only if Q defines an elliptic operator.

For us, the most important example of a an elliptic operator is the Laplace operator:

Proposition 23. *The Laplace operator Δ is elliptic on $\mathbb{M}$.*

Proof. Let U a coordinate chart where we also have an orthonormal frame $E_1, \dots, E_n$. Then by the Weitzenböck formula, the Laplace operator is given by

$$\Delta = L - \mathscr{R} = \sum_i \nabla^2_{E_i,E_i} + \sum_{i,j} \theta^i \wedge \iota_{E_j} R(E_i, E_j)\,.$$

The frames $\{E_i\}$ and $\{\partial_j\}$ are related by a matrix A, such that $E_i = A^j_i\partial_j$. Notice that $(AA^T)^i_j = \sum_k A^i_k A^j_k$, and that

$$\begin{aligned}
\delta^i_j =& g^{ik} g_{kj} \\
=& g^{ik}\langle \partial_k, \partial_j\rangle_g \\
=& g^{ik}\langle (A^{-1})^r_k E_r, (A^{-1})^s_j E_s\rangle_g \\
=& g^{ik}((A^{-1})^T)^k_r (A^{-1})^r_j \\
=& g^{ik}((AA^T)^{-1})^k_j \\
=& (g^{-1}(AA^T)^{-1})^i_j\,,
\end{aligned}$$

which shows that $g^{ij} = \sum_k A^i_k A^j_k$. Clearly the principal part of Δ is contained in L. Let α be a smooth k-form.

$$\begin{aligned}
L\alpha =& \sum_i \nabla^2_{E_i,E_i}\alpha \\
=& \sum_i A^k_i A^\ell_i \nabla^2_{\partial_k,\partial_\ell}\alpha \\
=& g^{k\ell}(\nabla_{\partial_k}, \nabla_{\partial_\ell}\alpha_J\theta^J - \nabla_{\nabla_{\partial_k}\partial_\ell}\alpha_J\theta^J) \\
=& g^{k\ell}(\partial_k\partial_\ell\alpha_J\theta^J + \partial_k\alpha_J\nabla_{\partial_\ell}\theta^J + \partial_\ell\nabla_{\partial_k}\theta^J + \alpha_J\nabla_{\partial_i}\nabla_{\partial_\ell}\theta^J \\
&- (\nabla_{\partial_k}\partial_\ell\alpha_J)\theta^J - \alpha\nabla_{\nabla_{\partial_k}\partial_\ell}\theta^J)
\end{aligned}$$

which means that Δ has principal part which is non-zero only on the diagonal. The determinant of the symbol is therefore of the form $-(\xi^T g^{-1}\xi)^r$, which equals zero if and only if $\xi = 0$, because g^{-1} is positive definite. □

3.2.5 Elliptic Operators on Sobolev Functions

For real maps $a(x), b(x)$ with common domain D, the statement "$a(x) \lesssim b(x)$" for $x \in D$" means that there is some constant $C > 0$ such that $a(x) \leq Cb(x)$ for all $x \in D$. For example, a linear map M on a normed space X is bounded if $\|Mx\| \lesssim \|x\|$ for $x \in X$.

On the Sobolev spaces, elliptic operators have some nice properties. These are needed to prove properties for elliptic operators on manifolds, especially the Laplace operator. A differential operator P of order ℓ on $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ defines a bounded operator $P\colon H^{s+\ell} \to H^s$ by defining P on the dense subset $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ and extend by continuity.

Theorem 24. [22, Prop 6.29] *Let P be an elliptic operator on $C^\infty_{2\pi}(\mathbb{R}, \mathbb{C}^m)$ of order ℓ, and let s be an integer. Then*

$$\|u\|_{H^{s+\ell}} \lesssim \|Pu\|_{H^s} + \|u\|_{H^s}$$

for $u \in H^{s+\ell}$.

The other theorem we need is

Theorem 25. [22, Thm 6.30] *Let P be an elliptic operator on $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ of order ℓ. If $u \in H^{-\infty}$, $v \in H^s$ and $Pu = v$; then $u \in H^{s+\ell}$.*

The spirit of the last theorem is that if $Pu = v$ is s-times differentiable, then u is $s + \ell$ times differentiable.

3.2.6 Properties of the Laplace operator

Let $L^2\Lambda^k$ be the completion [19, p.97] of $C^\infty\Lambda^k$ with respect to the L^2-norm on $C_0^\infty\Lambda^k$. The Laplace operator will be considered as an unbounded operator whose domain $\operatorname{Dom}\Delta$ is $C_0^\infty\Lambda^k$. Let Δ^* be the adjoint of Δ. We say that $\lambda \in L^2\Lambda^k$ is a **weak solution** to the equation $\Delta\alpha = \omega$, if

$$\langle\!\langle \Delta^*\lambda, \varphi \rangle\!\rangle = \langle\!\langle \lambda, \Delta\varphi \rangle\!\rangle = \langle\!\langle \omega, \varphi \rangle\!\rangle$$

for each φ in $C^\infty\Lambda^k$. By Riesz representation theorem, λ can also be considered as an element of $(L^2\Lambda^k)'$.

Theorem 26. [22, Thm 6.5] *Let $\omega \in C^\infty\Lambda^k$, and λ a weak solution of the equation $\Delta\alpha = \omega$. Then there exists $\alpha \in C^\infty\Lambda^k$ such that $\lambda = \langle\!\langle \alpha, \cdot \rangle\!\rangle$, and $\Delta\alpha = \omega$.*

Proof. We will show that at each point p of the manifold, there is a C^∞ k-form α_p defined in a small neighbourhood N_p of p, such that for each $\varphi \in C^\infty\Lambda^k$ with support in N_p, $\lambda(\varphi) = \langle\!\langle \alpha_p, \varphi \rangle\!\rangle$. We see that if $p \neq q$ and $N_{pq} := N_p \cap N_q$ is non-empty, then for any φ with support in N_{pq}, we have

$$\langle\!\langle \alpha_p - \alpha_q, \varphi \rangle\!\rangle = \langle\!\langle \alpha_p, \varphi \rangle\!\rangle - \langle\!\langle \alpha_q, \varphi \rangle\!\rangle = \lambda(\varphi) - \lambda(\varphi) = 0$$

and therefore $\alpha_p = \alpha_q$ in N_{pq}. Let $\{\rho_p\}$ be a partition of unity sub-ordinate to $\{N_p\}$. The C^∞ k-form $\alpha := \sum_p \rho_p \alpha_p$ agrees with λ since for any $\varphi \in C_0^\infty\Lambda^k$

$$\langle\!\langle \alpha, \varphi \rangle\!\rangle = \sum \langle\!\langle \alpha_p, \rho_p\varphi \rangle\!\rangle = \sum \lambda(\rho_p\varphi) = \lambda(\varphi)$$

The rest of the proof is showing that we can find α_p for each $p \in \mathbb{M}$. Let p be a fixed point in $\mathbb{M}$, and ϕ be a coordinate chart about p. We can find an open neighbourhood $U \subset \mathbb{R}^n$ of $q := \phi(p)$ which is contained in a (2π)-cube in $\mathbb{R}^n$, small enough that $\phi^{-1}(U)$ is an open trivializing neighbourhood of p. Let V be a smaller neighbourhood of q whose closure is contained in U. There exists a C^∞ map $\eta : \mathbb{R}^n \to [0,1]$ which is identically 1 on V and zero outside U [10, Cor 2.14]. Because $\phi^{-1}(U)$ is trivializing we can find an orthonormal frame $E_1, \dots, E_k$ for $T\mathbb{M}$ and corresponding frame $\theta^{I_1}, \dots, \theta^{I_m}$ for $\Lambda^k T^*\mathbb{M}$ where $m = \binom{n}{k}$. We have a correspondence

$$\begin{aligned} \{f \in C^\infty(\mathbb{R}^n, \mathbb{C}^m) : \operatorname{supp} f \subset U\} &\leftrightarrow \left\{\alpha \in C^\infty\Lambda^k : \operatorname{supp}\alpha \subset \phi^{-1}(U)\right\} \\ (f^1, \dots, f^m) &\mapsto (f^1 \circ \phi)\theta^{I_1} + \dots + (f^m \circ \phi)\theta^{I_m} \end{aligned} \tag{3.5}$$

We can therefore, for example treat a function $f\colon \mathbb{R}^n \to \mathbb{C}^m$ with support in U as a form with support in $\phi^{-1}(U)$, and vice versa. For example, if $f \in C_{2\pi}^\infty(\mathbb{R}^n, \mathbb{C}^m)$ define

$$\|\eta f\| := \|(\eta \circ \phi\ )((f^1 \circ \phi)\theta^{I_1} + \dots + (f^m \circ \phi)\theta^{I_m})\|$$

In addition to the H^0-inner product

$$\langle\!\langle \varphi, \psi \rangle\!\rangle_{H^0} \coloneqq \frac{1}{(2\pi)^n} \int_U \sum_i \varphi^i \psi^i \, dr$$

we can also take into account the volume scaling induced by the metric on $\phi^{-1}(U)$ which yields

$$\langle\!\langle \varphi, \psi \rangle\!\rangle = \int_{\phi^{-1}(U)} \langle \varphi, \psi \rangle \, dx = \int_U \sum \varphi^i \psi^i \sqrt{g} \, dr \leq \{\sup_{r \in U} (2\pi)^n \sqrt{g}(r)\} \langle\!\langle \varphi, \psi \rangle\!\rangle_{H^0}$$

and therefore

$$\langle\!\langle \varphi, \psi \rangle\!\rangle_{H^0} \lesssim \langle\!\langle \varphi, \psi \rangle\!\rangle \lesssim \langle\!\langle \varphi, \psi \rangle\!\rangle_{H^0}$$

for φ, ψ in $C^\infty(\mathbb{R}^n, \mathbb{C}^m)$ with support in U. Let $A = (2\pi)^n \sqrt{g}$, then we have the relations

$$\langle\!\langle \varphi, A\psi \rangle\!\rangle_{H^0} = \langle\!\langle \varphi, \psi \rangle\!\rangle \quad \text{and} \quad \langle\!\langle \varphi, A^{-1}\psi \rangle\!\rangle = \langle\!\langle \varphi, \psi \rangle\!\rangle_{H^0}$$

Because $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ is dense in H^0, we can for each $u \in H^0$ find a sequence (u_j) of C^∞ periodic maps converging to u. Let $\tilde{\lambda}$ be the extension of λ to H^0 by $\tilde{\lambda}(u) = \lim \lambda(\eta u_j)$. Since $\tilde{\lambda}$ is a continuous linear functional on the Hilbert space H^0, there is by Riesz' theorem some element $y \in H^0$ such that $\langle\!\langle y, \cdot \rangle\!\rangle_{H^0} = \tilde{\lambda}$. The Laplace operator Δ defines a differential operator $\tilde{\Delta}$ for functions $f \in C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$ with compact support in U by

$$\tilde{\Delta} f = ((\tilde{\Delta} f)_1, \ldots, (\tilde{\Delta} f)_m)$$

where

$$(\tilde{\Delta} f)_j = \langle \Delta(f_1 \theta^{I_1} + \cdots f_m \theta^{I_m}), \theta^{I_j} \rangle$$

The operator $\tilde{\Delta}$ is elliptic on U. We would like to substitute $\tilde{\Delta}$ with an elliptic operator which is periodic on $\mathbb{R}^n$ in order to apply Theorem 25. An obvious example of such an operator is the operator $M = \left(\sum_{i=1}^n \frac{\partial}{\partial r^i} \frac{\partial}{\partial r^i}\right) I$ where I is the identity matrix. The differential operator $P = \eta \tilde{\Delta} + (1 - \eta) M$ agrees with $\tilde{\Delta}$ on V, but also agree with M around the boundary of Q. We see that P is elliptic since

$$(-1)^m ((\eta \xi^T g \xi)^m + ((1 - \eta)|\xi|^2)^m) \neq 0$$

for every $\xi \in \mathbb{R}$, at every point in Q. By extending P periodically, we have a periodic elliptic operator on $C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$. There exists an adjoint operator P^* such that

$$\langle\!\langle P\varphi, \psi \rangle\!\rangle_{H^0} = \langle\!\langle \varphi, P^* \psi \rangle\!\rangle_{H^0}$$

for every $\varphi, \psi \in C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$. If either φ or ψ have support in V, then

$$\langle\!\langle \varphi, P^* \psi \rangle\!\rangle_{H^0} = \langle\!\langle P\varphi, \psi \rangle\!\rangle_{H^0} = \langle\!\langle \Delta\varphi, A^{-1}\psi \rangle\!\rangle = \langle\!\langle \varphi, \Delta A^{-1} \psi \rangle\!\rangle = \langle\!\langle \varphi, A \Delta A^{-1} \psi \rangle\!\rangle_{H^0}$$

Let $(y_j) \subset C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$, $y_j \to y$ in H^0 and $\varphi \in C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$. Then

$$\begin{aligned}\langle\!\langle Py, \varphi\rangle\!\rangle_{H^0} =& \langle\!\langle P(y-y_j), \varphi\rangle\!\rangle_{H^0} + \langle\!\langle Py_j, \varphi\rangle\!\rangle_{H^0}\\ =& \langle\!\langle P(y-y_j), \varphi\rangle\!\rangle_{H^0} + \langle\!\langle y_j, P^*\varphi\rangle\!\rangle_{H^0}\\ =& \langle\!\langle P(y-y_j), \varphi\rangle\!\rangle_{H^0} - \langle\!\langle y-y_j, P^*\varphi\rangle\!\rangle_{H^0} + \langle\!\langle y, P^*\varphi\rangle\!\rangle_{H^0}\end{aligned} \tag{3.6}$$

and since $P\colon H^0 \to H^{-2}$ is a bounded operator

$$|\langle\!\langle P(y-y_j), \varphi\rangle\!\rangle_{H^0} - \langle\!\langle y-y_j, P^*\varphi\rangle\!\rangle_{H^0}| \lesssim \|y-y_j\|_{H^0}(\|\varphi\|_{H^{-2}} + \|P^*\varphi\|_{H^0})$$

which goes to zero as $j \to \infty$, and we have the equality

$$\langle\!\langle Py, \varphi\rangle\!\rangle_{H^0} = \langle\!\langle y, P^*\varphi\rangle\!\rangle_{H^0}$$

and if φ has support in V, then $P^* = A\Delta A^{-1}$ in (3.6), and

$$\langle\!\langle Py, \varphi\rangle\!\rangle = \langle\!\langle y, A\Delta A^{-1}\varphi\rangle\!\rangle$$

Via the identification (3.5), $\eta\omega : U \to \mathbb{C}^m$. We claim that $\eta Py = \eta\omega$, so ηPy is C^∞. Let $\varphi \in C^\infty_{2\pi}(\mathbb{R}^n, \mathbb{C}^m)$, then

$$\begin{aligned}\langle\!\langle \eta Py - \eta\omega, \varphi\rangle\!\rangle_{H^0} =& \langle\!\langle Py, \eta\varphi\rangle\!\rangle_{H^0} - \langle\!\langle \eta\omega, \varphi\rangle\!\rangle_{H^0}\\ =& \langle\!\langle y, \Delta A^{-1}\eta\varphi\rangle\!\rangle_{L^2} - \langle\!\langle \omega, A^{-1}\eta\varphi\rangle\!\rangle_{L^2}\\ =& \lambda(\Delta A^{-1}\eta\varphi) - \lambda(\Delta A^{-1}\eta\varphi) = 0\end{aligned}$$

Now $P(\eta y) = \eta Py + [P, \eta]y$ where $[P, \eta]$ is a first order operator. Therefore $[P, \eta]y \in H^{-1}$ and so is $P(\eta y)$. Let W be neighbourhood of q, with $\overline{W} \subset V$. There exists a sequence $(S_j)_{j\in\mathbb{N}}$ of open neighbourhoods of q such that

$$\overline{W} \subset S_{j+1} \subset \overline{S}_{j+1} \subset S_j \subset \overline{S}_j \subset V$$

for each $j \in \mathbb{N}$, and corresponding C^∞ functions $\eta_j\colon Q \to [0,1]$ that are identically 1 on S_j and vanish outside S_{j+1}. They have the property that $\eta_{j+1}\eta_j = \eta_{j+1}$ for each j. By the elliptic regularity theorem, $\eta_1 y \in H^1$. By induction, $\eta_j y = \eta_j\eta_{j-1}\dots\eta_1\eta$ is an element of H^j. Let η_∞ be a smooth map that is identically equal to 1 on Ω and 0 outside W. Then $\eta_\infty y = \eta_\infty\eta_j y$ for each j and is therefore contained in each H^j and must therefore be C^∞ by Lemma 21. Let α_p be the C^∞ k-form defined on $N_p = \phi^{-1}(\Omega)$ by $\alpha_p = A^{-1}\eta_\infty y$. Then for any φ with support in N_p,

$$\langle\!\langle \alpha_p, \varphi\rangle\!\rangle_{L^2} = \langle\!\langle A^{-1}\eta_\infty y, \varphi\rangle\!\rangle_{L^2} = \langle\!\langle y, \eta_\infty\varphi\rangle\!\rangle_{H^0} = \lambda(\varphi)$$

□

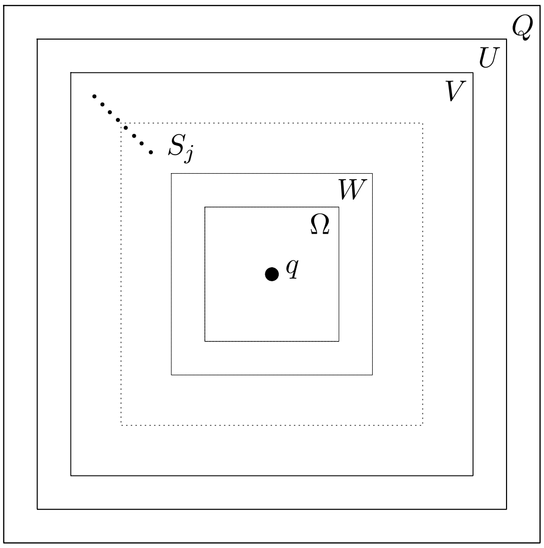

Figure 3.1: Sketch of the neighbourhoods around q.

Theorem 27. *Let (α_i) be a sequence of smooth k-forms on the compact manifold $\mathbb{M}$ such that $\|\alpha_i\| + \|\Delta\alpha_i\| < C$ for some $C > 0$. Then a sub-sequence of (α_n) is a Cauchy sequence in $C^\infty\Lambda^k$.*

Proof. We use the same set up as in the proof above. Around $\phi(p)$, there is a neighbourhood V such that Δ agrees with a periodic second order operator P. Let $B_p = \phi(V)$. Because $\mathbb{M}$ is compact, there is a finite sub-cover $\{B_\kappa\}$ of $\{B_p\}$ covering $\mathbb{M}$. Let $\{\rho_\kappa\}$ be a partition of unity subordinate to the covering $\{B_\kappa\}$. By Theorem 24

$$\begin{aligned}
\|\rho_\kappa\alpha_i\|_{H^1} &\lesssim \|P\rho_\kappa\alpha_i\|_{H^{-1}} + \|\rho_\kappa\alpha_i\|_{H^{-1}} \\
&\lesssim \|\rho_\kappa P\alpha_i\|_{H^{-1}} + \|[P,\rho_\kappa](\eta\alpha_i)\|_{H^{-1}} + \|\rho_\kappa\eta\alpha_i\|_{H^0} \\
&\lesssim \|\rho_\kappa\Delta\alpha_i\|_{H^0} + \|\eta\alpha_i\|_{H^0} + \|\rho_\kappa\|_\infty\|\eta\alpha_i\|_{H^0} \\
&\lesssim \|\Delta\alpha_i\| + \|\alpha_i\| \\
&\lesssim C
\end{aligned}$$

for each α_n. By Rellich lemma, the injections $H^{s+1} \to H^s$ are compact, i.e. there is some sub-sequence $(\rho_\kappa\alpha_n)$ which converges in H^0. We can find a sub-sequence which converges for each κ, and thus α_n is a Cauchy sequence.

$$\|\alpha_i - \alpha_j\|_{L^2} \lesssim \sum_\kappa \|\rho_\kappa(\alpha_i - \alpha_j)\|_{H^0} \lesssim \varepsilon \tag{3.7}$$

for large enough i, j. □

Notice that the the compactness of $\mathbb{M}$ is necessary for the last theorem since this allows us to add up a finite number of terms in (3.7).

3.3 Hodge Decomposition

3.3.1 Proof of Hodge Decomposition

Proof. There are some advantages with assuming $C^\infty\Lambda^k$ are complex forms. By abuse of notation $C^\infty\Lambda^k$ is the space of complex k-forms. Of course the real forms are contained in $C^\infty\Lambda^k$. There is a natural way to extend the inner product to the complex forms, such that the new inner product is Hermitian. Since Δ is densely defined, we have the decomposition $L^2\Lambda^k = \overline{\operatorname{Im}\Delta} \oplus \operatorname{Ker}\Delta^*$. However, if $\Delta^*\alpha = 0$, then α is a weak solution to the equation $\Delta\alpha = 0$. This in turn means that α is smooth by Theorem 26 . We know from before that on smooth forms $\Delta^* = \Delta$. Therefore we have $L^2\Lambda^k = \overline{\operatorname{Im}\Delta} \oplus \mathscr{H}\Lambda^k$. We will show that $\mathscr{H}\Lambda^k$ is finite dimensional. Assume the contrary. Then there exists an orthonormal sequence of harmonic forms (ε_i). Clearly

$$\|\varepsilon_j\| + \|\Delta\varepsilon_j\| = \|\varepsilon_j\| \leq 1\,,$$

and by Theorem 27, (ε_j) is a Cauchy sequence, which is absurd since $\|\varepsilon_i - \varepsilon_j\|^2 = 2$. Therefore $\mathscr{H}_k$ must be finite dimensional. Suppose ω is a smooth form. By the decomposition $\omega = \lim \Delta\alpha_i + \beta$ where β is harmonic. The element $\lim \Delta\alpha_i$ is a smooth form, that is to say that there is a smooth form γ such that for all $\varphi \in C^\infty\Lambda^k$, we have $\lim\langle\!\langle \Delta\alpha_i, \varphi\rangle\!\rangle = \langle\!\langle \gamma, \varphi\rangle\!\rangle$. Then again by regularity, there is a smooth form α such that $\Delta\alpha = \gamma$. So for smooth forms we have decomposition $C^\infty\Lambda^k = \operatorname{Im}\Delta \oplus \mathscr{H}_k$. The rest follow from the fact that $\operatorname{Im} d \perp \operatorname{Im}\delta$ and that $\Delta\omega = 0$ if and only if $d\omega = \delta\omega = 0$. $\square$

We can see immediately that if $f \in \mathscr{E}$ is $\mathscr{H}$, then

$$\int_{\mathbb{M}} |df|^2\, dx = \int_{\mathbb{M}} \langle -\Delta f, f\rangle\, dx = 0$$

which by continuity of df means that f is constant on each connected component. By the Hodge Decomposition $H^0_{dR}(\mathbb{M}) \cong \mathscr{H}\Lambda^0(\mathbb{M}) \cong \mathbb{R}^p$ where p is the number of connected component of $\mathbb{M}$. We have a similar theorem where we only assume that f is **sub-harmonic**, i.e. $\Delta f \geq 0$.

Lemma 28. *Suppose $f \in \mathscr{E}(\mathbb{M})$ and $f, \Delta f \geq 0$. Then f is constant on each connected component.*

Proof. Let f be as in the statement, then

$$0 \leq \|df\|^2 = \langle\!\langle \delta df, f\rangle\!\rangle = -\int_{\mathbb{M}} (\Delta f) f\, dx \leq 0$$

so f is constant . $\square$

3.3.2 Positive Weitzenböck Curvature

We shall see that information about the Weitzenböck operator $\mathscr{R}_k \colon \Lambda^k T^*\mathbb{M} \to \Lambda^k T^*\mathbb{M}$ determines much about the de Rham cohomologies. We say that $\mathscr{R}_k > 0$ at the point $p \in \mathbb{M}$ if $\langle \mathscr{R}_k \alpha_p, \alpha_p \rangle > 0$ for every non-zero $\alpha_p \in \Lambda^k T_p\mathbb{M}$. Similarly we can have $\mathscr{R}_k \geq 0$.

Theorem 29. *If $\mathbb{M}$ is a connected compact manifold with $\mathscr{R}_k \geq 0$ almost everywhere, and $\mathscr{R}_k > 0$ at some point p, then $H^k_{dR}(\mathbb{M}) = 0$.*

Proof. By the Weitzenböck formula $\Delta_k = L - \mathscr{R}_k$. Now, let $\omega \in C^\infty \Lambda^k$, and $E_1, \ldots, E_n$ an orthonormal frame parallel at p, then computation at p yields

$$\Delta|\omega|^2 = L|\omega|^2 = \sum_i \nabla_{E_i} \nabla_{E_i} \langle \omega, \omega \rangle = 2 \sum_i |\nabla_{E_i} \omega|^2 + 2\langle \Delta_k \omega, \omega \rangle + 2\langle \mathscr{R}_k \omega, \omega \rangle$$

If ω is harmonic, then

$$\frac{1}{2}\Delta|\omega|^2 = \sum_i |\nabla_{E_i}\omega|^2 + \langle \mathscr{R}\omega, \omega \rangle \geq 0$$

By Lemma 28, $|\omega|^2$ is constant, all its derivatives vanish and hence $\Delta|\omega|^2 = 0$. This gives

$$0 = \sum_i |\nabla_{E_i}\omega|^2 + \langle \mathscr{R}\omega, \omega \rangle$$

We necessarily have $|\nabla\omega| = 0$, everywhere. Because $\mathscr{R} > 0$ at one point p, $\omega_p = 0$. And since $|\nabla\omega| = 0$, ω is identically zero. ☐

3.3.3 Poincaré Duality

Theorem 30. *The bilinear map $(\cdot, \cdot) \colon H^k_{dR}(\mathbb{M}) \times H^{n-k}_{dR}(\mathbb{M}) \to \mathbb{R}$ defined by*

$$([\alpha], [\omega]) = \int_{\mathbb{M}} \alpha \wedge \omega$$

is a non-degenerate pairing, and therefore determines an isomorphism between $H^k_{dR}(\mathbb{M})$ and $\left(H^{n-k}_{dR}(\mathbb{M})\right)^*$.

Proof. First, notice that the if α and β are co-homologous closed k-forms, and $\alpha - \beta = d\gamma$. Then for any closed $(n-k)$-form ω,

$$\int_{\mathbb{M}} \alpha \wedge \omega = \int_{\mathbb{M}} (\beta + d\gamma) \wedge \omega = \int_{\mathbb{M}} \beta \wedge \omega + \int_{\mathbb{M}} d(\gamma \wedge \omega) \pm \int_{\mathbb{M}} \gamma \wedge d\omega = \int_{\mathbb{M}} \beta \wedge \omega$$

The bi-linear map $(\cdot, \cdot)$ is therefore well-defined. To show that the bi-linear map is non-degenerate, it suffices to show that any non-zero element of $H^k_{dR}(\mathbb{M})$ defines a functional which is not identically zero. Let $\alpha \in H^k_{dR}(\mathbb{M})$ be the harmonic representative of a non-zero cohomology class. Because α is harmonic, $0 = \delta\alpha = \pm \star d \star \alpha$. Since the Hodge star is an isometry, $d{\star}\alpha = 0$ and therefore represents the class $[\star\alpha]$ in $H^{n-k}_{dR}(\mathbb{M})$. Evaluating gives

$$([\alpha], [\star\alpha]) = \int_{\mathbb{M}} \alpha \wedge \star\alpha = \|\alpha\|^2 > 0$$

which shows that $([\alpha], \cdot)$ is not identically zero. $\square$

Since the cohomologies of the compact manifold $\mathbb{M}$ are finite dimensional, we also have isomorphisms $H^k_{dR}(\mathbb{M}) \cong H^{n-k}(\mathbb{M})$.

De Rham cohomology is isomorphic to the singular cohomology with coefficients in $\mathbb{R}$, a fact which is called the De Rham theorem [11, 18.14].

— 4 —

Cohomology on Non-Compact Complete Manifolds

4.1 Stokes' Theorem on Complete Manifolds

When $\mathbb{M}$ is a complete connected manifold, we have a Stoke's theorem for L^2-forms. The proof is dependent of the existence of a sequence of maps whose support exhausts the manifold, and have controlled derivatives.

4.1.1 Cut-off Function on $\mathbb{M}$

Let o be a fixed point in $\mathbb{M}$, the function $\rho\colon \mathbb{M} \to [0, \infty)$ defined by $\rho(p) = d(o, p)$ gives the distance from our chosen origin o. The closed sets $B_r = \rho^{-1}([0, r])$ form a sequence which exhaust $\mathbb{M}$. For each pair B_r and B_s $r < s$ there exists a smooth function $\chi_{r<s}\colon \mathbb{M} \to [0, 1]$ with the property that $\chi_{r<s} = 1$ on B_r, and $\chi_{r<s} = 0$ on $\mathbb{M} \setminus B_s$. They also have the property that $|d\chi_{r<s}|_{L^\infty} \leq c/(s - r)$ for some constant c. By using the "Gaffney cut-off trick" [8] one can show that the above theorem has

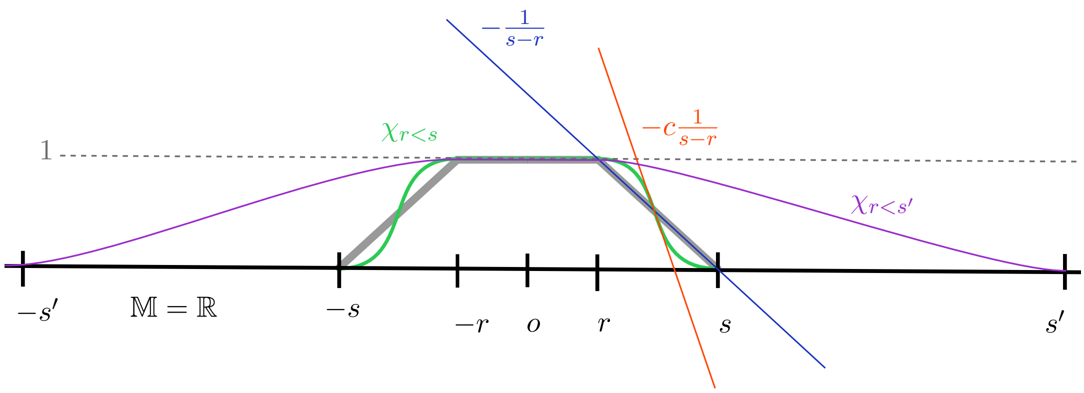

Figure 4.1: Sketch of cut-off function on the real line. The idea is the same on any complete manifold.

an analogue where we do not require any of the forms to have compact support. Let the L^1-norm be defined on forms by $\|\omega\|_{L^1} = \int_{\mathbb{M}} |\omega|\, dx$. The main theorem in [8] is a Stokes' theorem for complete manifolds

Theorem 31 (Stokes)**.** *Let $\mathbb{M}$ be a complete Riemannian manifold. If $\omega \in C^\infty\Lambda^{n-1}$ and $\|\omega\|_{L^1}, \|d\omega\|_{L^1} < \infty$, then*

$$\int_{\mathbb{M}} d\omega = 0$$

Proof. Let $0 < r < s$. Now

$$\left|\int_{\mathbb{M}} \chi_{r<s} d\omega\right| = \left|\int_{\mathbb{M}} d(\chi_{r<s}\omega) - (d\chi_{r<s}) \wedge \omega\right| \leq K\|d\chi_{r<s}\|_{L^\infty}\|\omega\|_{L^2} \leq K\|\omega\|_{L^2}\frac{1}{s-r}$$

and since $\chi_{r<s}d\omega \to d\omega$ as $r \to \infty$

$$\|d\omega\| = \|d\omega - \chi_{r<s}d\omega\| + \|\chi_{r<s}d\omega\| < \varepsilon$$

for large enough r and s. □

Theorem 31 gives us the necessary means to prove a generalization of Theorem 3 to complete manifolds.

Corollary 32. *Let $\mathbb{M}$ be a complete manifold. If $\alpha \in C^\infty\Lambda^k \cap L^2\Lambda^k$, $d\alpha \in L^2\Lambda^{k+1}$ and $\beta \in C^\infty\Lambda^{k+1} \cap L^2\Lambda^{k+1}$, $\delta\beta \in L^2\Lambda^k$, then*

$$\langle\!\langle d\alpha, \beta\rangle\!\rangle = \langle\!\langle \alpha, \delta\beta\rangle\!\rangle$$

Proof. Now,

$$\langle d\alpha, \beta\rangle \operatorname{vol_g} = d\alpha \wedge \star\beta = d(\alpha \wedge \star\beta) + \langle \alpha, \delta\beta\rangle \operatorname{vol_g} \tag{4.1}$$

We claim that $\alpha\wedge\star\beta$ satisfies the requirements to use Theorem 31. There is a constant K from (2.1), for which the inequality

$$|\alpha \wedge \star\beta| \leq K|\alpha||\beta| = K\langle|\alpha|, |\beta|\rangle$$

holds at every point $p \in \mathbb{M}$. Integrating and using the Schwartz-inequality gives

$$\|\alpha \wedge \star\beta\|_{L^1} \leq K\|\alpha\|_{L^2}\|\beta\|_{L^2} < \infty$$

similarly

$$|d(\alpha \wedge \star\beta)| = |d\alpha \wedge \star\beta - \alpha \wedge \star\delta\beta| \leq K\left(|d\alpha||\beta| + |\alpha||\delta\beta|\right)$$

such that

$$\|d(\alpha \wedge \star\beta)\|_{L^1} \leq K\left(\|d\alpha\|_{L^2}\|\beta\|_{L^2} + \|\alpha\|_{L^2}\|\delta\beta\|_{L^2}\right) < \infty\,.$$

Integrating (4.1) gives

$$\langle\!\langle d\alpha, \beta\rangle\!\rangle = \int_{\mathbb{M}} d(\alpha \wedge \star\beta) + \langle\!\langle \alpha, \delta\beta\rangle\!\rangle$$

where the middle term vanishes. □

4.1.2 Eigenvalues of the Laplace operator

In the next section, we will follow [18] to show that on the completion $\overline{C_0^\infty \Lambda^k} = L^2\Lambda^k$, the Laplace operator has unique closed extension. The first is that Δ has only negative eigenvalues.

Proposition 33. *Suppose $\Delta\omega = \lambda\omega$ for some non-zero $\omega \in C^\infty\Lambda^k \cap L^2\Lambda^k$. Then $\lambda < 0$.*

Proof.

$$\lambda\|\omega\|^2 = \langle\!\langle \Delta\omega, \omega \rangle\!\rangle = -(\|d\omega\|^2 + \|\delta\omega\|^2) \tag{4.2}$$

such that $\lambda < 0$. □

The last theorem gives us sufficient conditions of the Laplace operator to show that it is essentially self-adjoint, i.e. it has a unique closed extension which is self-adjoint.

4.2 L^2-cohomology

In this section we will investigate if the Hodge theorem extends in some way to also hold for non-compact complete Riemannian manifolds. It is possible to define de Rham cohomology groups on complete Riemannian manifolds, however it does not capture information at infinity [6]. Therefore we extend the class of forms we are investigating, to include forms that are **square-integrable**, in the sense that

$$\int_{\mathbb{M}} |\omega|^2 \, dx < +\infty \, .$$

4.2.1 L^2-Hilbert Complex

We want to employ techniques from the theory of Hilbert spaces, as we did in the previous chapter. We repeat the process and define $L^2\Lambda^k = \overline{C_0^\infty\Lambda^k}$, where the completion is taken with respect to the norm on $C_0^\infty\Lambda^k$. We could also complete the space of all square-integrable forms, which would produce an isometric Hilbert space. It is convenient to have a complex Hilbert space, and therefore, we abuse notation and let $C_0^\infty\Lambda^k$ denote the smooth sections of the complex vector bundle $\Lambda^k T\mathbb{M}$ where each fibre is $\Lambda^k T_p\mathbb{M} \otimes \mathbb{C}$, with the canonical Hermitian inner product. For any smooth $\omega \in L^2$, there is an injection of the smooth k-forms ω for which the integral $\int_{\mathbb{M}} |\omega|^2 \, dx < \infty$, into L^2 since each such form defines a linear functional, which is identified by Riesz's theorem.

The exterior derivative d is not defined on the whole of $L^2\Lambda^k$. It is defined on the dense subspace $C_0^\infty\Lambda^k$, and is therefore an unbounded operator on $L^2\Lambda^k$. Genereally, linear operators define on a subspace D of a Hilbert space is called an unbounded operator, see the Appendix. The operator d has a priori many closed extensions. The domain of the various extensions gives an oredering between them, where $d_1 \leq d_2$ if $\operatorname{Dom} d_1 \subset d_2$. The largest extension of d is the adjoint $d_{max} \coloneqq d^*$. The smallest is the closure $d_{min} = \overline{d}$, defined for elements $\omega = \lim \omega_j$, such that $d\omega_j$ converges.

Let $W\Lambda^k$ be the domain of any closed extension $\bar{d}_k$ of d_k, which will also satisfy $\bar{d}_k \bar{d}_{k-1} = 0$. The resulting complex

$$W\Lambda^0 \xrightarrow{\bar{d}_0} W\Lambda^1 \xrightarrow{\bar{d}_1} \cdots \xrightarrow{\bar{d}_{k-1}} W\Lambda^k \xrightarrow{\bar{d}_k} W\Lambda^{k+1} \xrightarrow{\bar{d}_{k+1}} \cdots \tag{4.3}$$

is a **Hilbert complex**. A choice of closed extension $\bar{d}$ is an **ideal boundary condition**. On complete manifolds, there is only one choice for the closed extension [2], i.e. the domains of d_{min} and d_{max} coincide and $d_{min} = d_{max}$, which we will come back to. We will denote by $W\Lambda^k$ the domain of $\overline{d} \coloneqq d_{min} = d_{max}$. The space $W\Lambda^k$ is the closure of $C_0^\infty \Lambda^k$ with respect to the norm

$$\|\omega\|_W^2 \coloneqq \|\omega\|_{L^2}^2 + \|d\omega\|_{L^2}^2.$$

We can form the unreduced L^2-cohomologies by

$$H^k_{L^2}(\mathbb{M}) = \frac{\operatorname{Ker} \bar{d}_k}{\operatorname{Im} \bar{d}_{k-1}}$$

and the reduced L^2-cohomologies

$$\tilde{H}^k_{L^2}(\mathbb{M}) = \frac{\operatorname{Ker} \bar{d}_k}{\operatorname{cl}(\operatorname{Im} \bar{d}_{k-1})}$$

where we quotient out by the closure of $\operatorname{Im} \bar{d}_{k-1}$ with respect to the L^2-norm. The unreduced and reduced cohomologies different from each other. The reduced cohomology is a Hilbert space, while the unreduced is not necessarily Hausdorff [19, Prop 4.5].

4.2.2 Lipschitz Invariance of L^2-Cohomology

The unreduced cohomologies are preserved under bi-Lipschitz maps between manifolds. A continuous map $F\colon (\mathbb{M}, g_1) \to (\mathbb{L}, g_2)$ is Lipschitz if $(F_*)_p\colon T_p\mathbb{M} \to T_p\mathbb{L}$ is defined almost everywhere on $\mathbb{M}$, and there is a constant $C > 0$ such that at all points where $(F_*)_p$ is defined the equation

$$|(F_*)_p v|_{g_2} \leq C|v|_{g_1}$$

holds for each $v \in T\mathbb{M}$. A homeomorphism $F\colon \mathbb{M} \to \mathbb{L}$ is called bi-Lipschitz if both F and F^{-1} are Lipschitz map. The pull-back map F^* preserve integrable forms. Let

$F\colon \mathbb{M} \to \mathbb{L}$ be a Lipschitz map, and $\omega \in L^2\Lambda^k(\mathbb{L})$. Let $E_1, \dots, E_n$ be an orthonormal frame around $p \in \mathbb{M}$ and $\tilde{E}_1, \dots, \tilde{E}_n$ and orthonormal frame around $F(p)$. Let $(i_1, \dots, i_k)$ range over ascending multi-indices, then

$$|F^*\omega|^2 = \sum_{(i_1,\dots,i_k)} F^*\omega(E_{i_1}, \dots, E_{i_k})^2 = \sum_{(i_1,\dots,i_k)} \omega(F_*E_{i_1}, \dots, F_*E_{i_k})^2$$

where

$$\omega(F_*E_{i_1}, \dots, F_*E_{i_k}) = \sum_{s_1,\dots,s_k} (F_*E_{i_1})^{s_1} \cdots (F_*E_{i_1})^{s_k} \omega_{s_1,\dots,s_k}$$

By Schwartz inequality, and the Lipschitz property of F

$$\begin{aligned} |F^*\omega|^2 \leq& K' \sum_{(i_1,\dots,i_k)} \left(|F_*E_{i_1}|^2 \cdots |F_*E_{i_k}|^2 \sum_{s_1,\dots,s_k} \omega^2_{s_1,\dots,s_k} \right) \\ \leq& K|\omega|^2 \end{aligned}$$

where K is independent of p. Integrating yields $\|F^*\omega\| \leq K\|\omega\|$. We of course also have $dF^*\omega = F^*d\omega$ which ensures that closed forms are mapped to closed forms, and exact forms are mapped to exact forms. This gives an isomorphism $H^k_{L^2}(\mathbb{L}) \cong H^k_{L^2}(\mathbb{M})$. This extends to the reduced cohomology as well, suppose $\omega = \lim d\tau_j$, then $\lim F^*d\tau_j = \lim dF^*\tau_j$. Actually, we can also generalize to Lipschitz maps that are Lipschitz-homotopic [14]. This is analogues to how de Rham cohomology is invariant under homotopy equivalence.

4.2.3 Relation between L^2 and de Rham cohomology

G.Carron has proved a connection between the L^2-cohomology, and the de Rham cohomology for complete manifolds with one flat end. One flat end means that for any compact subset $S \subset \mathbb{M}$, the complement $\mathbb{M} \setminus S$ has only one unbounded connected component, or end, denoted by E. Let $\mathbb{M}$ be a complete manifold with zero cuvature outside a compact subset S. We have two cases depending on the volume growth [3]:

1. If the volume growth $\mathrm{Vol}(r) = \int_{B(p,r)} \mathrm{vol_g}$ is at most quadratic, i.e . $\lim_{r\to\infty} \mathrm{Vol}(r)/r^2 < \infty$ then

$$\mathscr{H}\Lambda^k(\mathbb{M}) \cong \mathrm{Im}\left(H^k_{0,dR}(\mathbb{M}) \to H^k_{dR}(\mathbb{M}) \right) .$$

2. If $\lim_{r\to\infty} \mathrm{Vol}(r)/r^2 = \infty$, then the boundary of E has a finite covering space diffeomorphic to the product $S^{\nu-1} \times T$ where T is a flat $(n-\nu)$-torus. Let $\pi\colon T \to \partial E$ be the induced immersion, then

$$\mathscr{H}k(\mathbb{M}) \cong H^k(\mathbb{M} \setminus E, \mathrm{Ker}\,\pi^*)$$

where

$$H^k(\mathbb{M} \smallsetminus E, \operatorname{Ker} \pi^*) := \frac{C^\infty \Lambda^k(\mathbb{M} \smallsetminus E) \cap \operatorname{Ker} d_k \cap \operatorname{Ker} \pi^*}{C^\infty \Lambda^k(\mathbb{M} \smallsetminus E) \cap \operatorname{Im} d_{k-1} \cap \operatorname{Ker} \pi^*}$$

4.2.4 Self-adjointness of the Laplace Operator

We will use the fact that the Laplace operator has only negative eigenvalues to show that it is **essentially self-adjoint**, i.e. it has a unique closed extension which is self-adjoint.

Theorem 34. [17, Prop 3.9] *Let T be a densely defined symmetric operator on the Hilbert space H, which is bounded below by m. If $\operatorname{Ker}(T^* - \lambda) = 0$ for some $\lambda < m$, then T is essentially self -adjoint.*

We will use this criterion to show that $-\Delta$, which is bounded below by 0, is essentially self-adjoint.

Theorem 35. *The Laplace operator Δ is essentially self-adjoint.*

Proof. Suppose $\omega \in \operatorname{Ker}(-\Delta^* - \lambda)$ then

$$0 = \langle\!\langle (-\Delta^* - \lambda)\omega, \alpha \rangle\!\rangle = \langle\!\langle \omega, -\Delta\alpha \rangle\!\rangle - \langle\!\langle \lambda\omega, \alpha \rangle\!\rangle$$

for all $\alpha \in C^\infty\Lambda^k \cap L^2\Lambda^k$ This means that ω is a weak solution to the equation $\Delta f = -\lambda\omega$.

By using the method in the proof of the regularity theorem, we see by induction that ω is C^∞. We may therefore use the symmetry of Δ on its domain to infer that $-\Delta\omega - \lambda\omega = 0$ for some negative λ which by Proposition 33 implies that $\omega = 0$. $\square$

The

Theorem 36. [2, Lemma 3.8.] *If the elliptic differential operator $d\delta + \delta d$ associated to the elliptic complex*

$$C_0^\infty E_0 \xrightarrow{d_0} C_0^\infty E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C_0^\infty E_k \xrightarrow{d_k} C_0^\infty E_{k+1} \xrightarrow{d_{k+1}} \cdots \tag{4.4}$$

is essentially self-adjoint, then there is a unique Hilbert complex

$$W\Lambda^0 \xrightarrow{\bar{d}_0} W\Lambda^1 \xrightarrow{\bar{d}_1} \cdots \xrightarrow{\bar{d}_{k-1}} W\Lambda^k \xrightarrow{\bar{d}_k} W\Lambda^{k+1} \xrightarrow{\bar{d}_{k+1}} \cdots$$

associated to (4.4) i.e. the elliptic complex has a unique boundary condition. In particular $d_{min} = d_{max}$.

4.2.5 Decomposition on L^2-forms

The following theorem is called the **weak** decomposition.

Theorem 37. *The square-integrable forms have decomposition*

$$L^2\Lambda^k = \mathscr{H}\Lambda^k \oplus \overline{\operatorname{Im}\overline{d}} \oplus \overline{\operatorname{Im}\overline{d}^*}$$

Proof. This is [2, Lem 2.1] because $\Delta\omega = 0$ if and only if $d\omega = d^*\omega = 0$ by Corollary 32. $\square$

We also have **strong** decomposition by Gromov [9]

Theorem 38. *If there exist some constant $C > 0$ such that for every $\omega \in C^\infty\Lambda^k \cap L^2\Lambda^k$*

$$\|\omega\|^2 \leq C\langle\!\langle -\Delta\omega, \omega\rangle\!\rangle$$

then $\operatorname{Im}\overline{d}$ and $\operatorname{Im}\overline{d}^*$ is closed. And we have decomposition

$$L^2\Lambda^k = \mathscr{H}\Lambda^k \oplus \operatorname{Im}\overline{d} \oplus \operatorname{Im}\overline{d}^*$$

This theorem can be extended to include some instances of L^p integrable forms, see [13]

The weak decomposition gives a Hodge theorem. The kernel of the map $\mathscr{H}_k\colon \operatorname{Ker} d_k \to \mathscr{H}\Lambda^k$ are all the elements ω of the form

$$\omega = \lim_j (d\alpha_j + \delta\beta_j)$$

and since

$$0 = \lim_j (d^2\alpha_j + d\delta\beta_j) = \lim_j d\delta\beta_j$$

we have

$$0 = \lim_j \langle\!\langle d\delta\beta_j, \beta_j\rangle\!\rangle = \lim_j \|\delta\beta_j\|^2$$

so $\delta\beta = 0$. Therefore $\omega = \lim_j d\alpha_j$ for some $(\alpha_j) \in \operatorname{Dom} d_{k-1}$. So we have an isomorphism $\tilde{H}^k_{L^2}(\mathbb{M}) \cong \mathscr{H}\Lambda^k(\mathbb{M})$.

4.3 Non-Negative Weitzenböck Curvature

The next theorem is my generalization of [24, Thm 6] by S-T. Yau.

Theorem 39. *Let α be a smooth harmonic k-form which is square-integrable on the complete manifold $\mathbb{M}$. If $\mathscr{R}_p \geq 0$ for all p in $\mathbb{M}$, then α is parallel. If $\mathscr{R}_q > 0$ at some point q, then α vanishes.*

S-T. Yau proved this theorem for 1-forms and $(n-1)$-forms. For 1-forms, $\langle \mathscr{R}\alpha, \alpha\rangle = \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp)$, and similarly for $(n-1)$-forms via the Hodge star. We need a few result before the proof.

Theorem 40. *If $f\colon \mathbb{M} \to \mathbb{R}$ is a smooth square integrable function, and $f, \Delta f \geq 0$, then f is constant.*

Proof. Because the Stokes theorem holds for square-integrable forms, we can use the same technique as in the proof of Lemma 28 □

Theorem 41. *If α is a smooth square integrable harmonic form, then α is both closed and co-closed.*

Proof. Let ω be harmonic, and square integrable. Then

$$0 = \langle\!\langle \Delta\omega, \omega \rangle\!\rangle = -(\|d\omega\|^2 + \|\delta\omega\|^2)\,,$$

and ω is therefore both closed and co-closed. □

We can now prove our main theorem.

Proof of Theorem 39. In the proof of Theorem 29 showed that for any C^∞ form α we have equality

$$\frac{1}{2}\Delta|\alpha|^2 = \langle \Delta\alpha, \alpha\rangle_g + |\nabla\alpha|^2 + \langle \mathscr{R}\alpha, \alpha\rangle_g\,. \tag{4.5}$$

We will show that when α is harmonic and square integrable, then $\Delta|\alpha| \geq 0$, and $|\alpha|$ is therefore constant, by Theorem 40. For such forms α, equation (4.5) reduces to $0 = |\nabla\alpha| + \langle \mathscr{R}\alpha, \alpha\rangle_g$, and the hypothesis that $\mathscr{R} \geq 0$ implies that $|\nabla\alpha| = 0$. By adding to the hypothesis that, $\mathscr{R}_p > 0$ at the point p, we have that $\alpha_p = 0$ and since α is parallel, α is identically 0. Now we will show that $\Delta|\alpha| \geq 0$, when α is harmonic. First of all, we prove, that (4.5) holds. Let $E_1, \ldots, E_n$ be an orthonormal frame parallel at p, and θ^i is the dual to E_i. Then

$$\Delta|\alpha|^2 = \Delta\left(\sum_J \alpha_J^2\right) = 2\sum_{J,i} \nabla_{E_i}(\alpha_J \nabla_{E_i}\alpha_J) = 2\sum_{J,i} (\nabla_{E_i}\alpha_J)^2 + \alpha_J \nabla_{E_i}\nabla_{E_i}\alpha_J$$

which is just the local expression for

$$2|\nabla\alpha|^2 + 2\langle L\alpha, \alpha\rangle = 2|\nabla\alpha|^2 + 2\langle \Delta\alpha, \alpha\rangle + 2\langle \mathscr{R}\alpha, \alpha\rangle$$

where the equality comes from the Weitzenböck formula. By the chain rule we have

$$\Delta|\alpha|^2 = 2|\nabla|\alpha||^2 + 2|\alpha|\Delta|\alpha| \tag{4.6}$$

and

$$|\nabla|\alpha||^2 = \sum_i (\nabla_{E_i}|\alpha|)^2 = \frac{1}{|\alpha|^2} \sum_i \left(\sum_J (\nabla_{E_i}\alpha_J)\alpha_J \right)^2 . \tag{4.7}$$

Recall that

$$d\alpha = \sum_i \theta^i \wedge \nabla_{E_i}\alpha \quad \text{and} \quad \delta\alpha = -\sum_i \iota_{E_i}\nabla_{E_i}\alpha$$

Since α is harmonic, $d\alpha = 0$ and $\delta\alpha = 0$ which we write out in bases at p

$$\sum_J \sum_{i \notin J} (\nabla_{E_i}\alpha_J)\theta^i \wedge \theta^J = 0 \quad \text{and} \quad \sum_J \sum_{i \in J} (\nabla_{E_i}\alpha_J)\theta^{j_1} \wedge \cdots \wedge \widehat{\theta^i} \wedge \cdots \wedge \theta^{j_k} = 0$$

and we see that $\nabla_{E_i}\alpha_J$ vanishes for all $i \notin J$, because $d\alpha = 0$, and for all $i \in J$ because $\delta\alpha = 0$. Therefore each term in the sum in (4.7) is zero. All we have to worry about is the case in which $|\alpha| = 0$. We can lift the function $|\alpha|^2$ a little which yields

$$\Delta(|\alpha|^2 + \varepsilon)^{1/2} = -(|\alpha|^2 + \varepsilon)^{-3/2}|\alpha|^2|\nabla|\alpha||^2 + (|\alpha|^2 + \varepsilon)^{-1/2}(|\nabla|\alpha||^2 + |\alpha|\Delta|\alpha|) .$$

where the middle term vanishes, and by (4.6)

$$\Delta(|\alpha|^2 + \varepsilon)^{1/2} = \frac{1}{\sqrt{|\alpha|^2 + \varepsilon}}(|\nabla\alpha|^2 + \langle \mathscr{R}\alpha, \alpha\rangle) \geq 0 .$$

By Theorem 40 the integrable function $(|\alpha|^2 + \varepsilon)^{1/2}$ is constant, and so is $|\alpha|$. □

4.3.1 Final Words

By Theorem 38, the strong decomposition follows if we can find a constant $C > 0$ such that

$$\|\omega\|^2 \leq C\langle\!\langle -\Delta\omega, \omega\rangle\!\rangle$$

for every $\omega \in C^\infty\Lambda^k \cap L^2\Lambda^k$. By (4.5)

$$\frac{1}{2}\Delta|\alpha|^2 = \langle \Delta\alpha, \alpha\rangle_g + |\nabla\alpha|^2 + \langle \mathscr{R}\alpha, \alpha\rangle_g$$

If $\mathscr{R}_p > 0$, then

$$\int |\alpha|^2\, dx \leq \int \frac{1}{\mathscr{R}_p}\langle \mathscr{R}\alpha, \alpha\rangle\, dx \leq \|\frac{1}{\mathscr{R}_p}\|_{L^\infty} \int \langle \mathscr{R}\alpha, \alpha\rangle\, dx$$

$$= \frac{1}{2}\int \Delta|\alpha|^2\, dx + \langle\!\langle -\Delta\alpha, \alpha\rangle\!\rangle - \|\nabla\alpha\|^2$$

if $\Delta|\alpha|^2$ is integrable

$$\int |\star d \star d|\alpha|^2|\, dx = \int |d \star d|\alpha|^2|\, dx < \infty$$

and by Theorem 31, $\int \Delta|\alpha|^2\,dx = 0$. We therefore have inequality

$$\|\alpha\|^2 \leq \|\frac{1}{\mathscr{R}_p}\|_{L^\infty}\langle\!\langle -\Delta\alpha, \alpha\rangle\!\rangle$$

and hence we have strong decomposition as long as the Weitzenböck curvature is bounded below by a constant $c > 0$. It is however difficult to have $\mathscr{R}_k > c_1 > 0$ without also $\mathrm{Ric} > c_2 > 0$, which would imply that $\mathbb{M}$ is compact by Theorem 15.

Since the Weitzenböck curvature reduces to the Ricci curvature on for 1-forms, the condition that $\mathscr{R}_1 > c > 0$ does not bring any new information to the table because $\mathrm{Ric} > c > 0$ implies again that $\mathbb{M}$ is compact. It is also showed by Chen [4] that if $\mathbb{M}$ is a non-compact complete Riemannian manifold with $\mathrm{Ric} \geq 0$, then the bottom of the spectrum of the Laplace operator on 0-forms is zero. It would therefore be difficult to prove that the reduced cohomology is isomorphic to the unreduced. It is not clear if the the same is clear for k-forms where $k > 0$.

Lastly we give an example of a non-compact manifold with $\mathscr{R}_1 = \mathrm{Ric} > 0$ where we can apply Theorem 39. Let $\mathbb{M}$ be the surface defined by the equation $x = -\frac{1}{2}(y^2 + z^2)$. The Ricci curvature is related to the Gaussian curvature by $\mathrm{Ric} = Kg$[12, Cor 8.28], and the Gaussian curvature K of $\mathbb{M}$ is given by

$$K(y, z) = 1/(1 + y^2 + z^2)^2\,.$$

Hence $\mathrm{Ric} > 0$, but there is on constant such that $\mathrm{Ric} \geq c > 0$, which by Myers theorem would mean that the manifold is compact. By Theorem 39, $\mathscr{H}\Lambda^1(\mathbb{M}) = 0$.

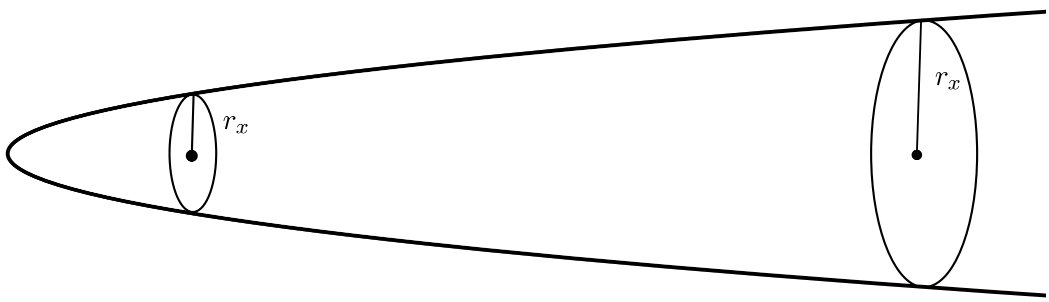

Figure 4.2: Sketch of the surface defined by $x = \frac{1}{2}(y^2 + z^2)$. Notice that $r_x \to \infty$, for $x \to \infty$.

— Appendix —

Unbounded Operators on Hilbert Spaces

This appendix is supposed to give some basic facts on Hilbert spaces, and operators between Hilbert spaces. A general reference for the material covered can be found in [5] and [10].

Definition. A Hilbert space V is a vector space over the real or complex numbers, which is endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm defined by

$$\|v\| = \sqrt{\langle v, v \rangle}$$

such that every Cauchy sequnece with respect to the norm converges.

Proposition A.1 (Schwartz inequality)**.** *For any two vectors v, w in V we have inequality*

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

Proof. If either v or w are zero, the proof is trivial. We therefore assume they are non-zero vectors, and first assume $\|w\| = 1$. Let $\lambda \in \mathbb{C}$, then

$$0 \leq \|v - \lambda w\|^2 = \|v\|^2 - \overline{\lambda} \langle v, w \rangle - \lambda \overline{\langle v, w \rangle} + |\lambda|^2 \|w\|^2$$

Setting $\lambda = \langle v, w \rangle / 2$ gives us

$$0 \leq \|v\|^2 - |\langle v, w \rangle|^2 \iff |\langle v, w \rangle| \leq \|v\|$$

If w is a general non-zero vector, then

$$|\langle v, w \rangle| = \|w\| |\langle v, w / \|w\| \rangle| \leq \|v\| \|w\|$$

□

Example A.2. The complex vector space $\mathbb{C}^n$ with inner product

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$$

is a complex Hilbert space. The Schwartz inequality takes the form

$$\left| \sum_{i=1}^{n} x_i \overline{y_i} \right| \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2}$$

Definition. A set S of vectors in V is said to be orthonormal, if every vector has unit length and for any two vectors v and w in S, $\langle v, w\rangle = 0$. The set S is an orthonormal basis if S is maximal.

Theorem A.3. *Every Hilbert space has an orthonormal basis.*

Proof. Let $\mathscr{C}$ be a chain of orthonormal sets S_j, where $S_j \subset S_{j+1}$. The union $U = \bigcup_{S_j \in \mathscr{C}} S_j$ is clearly an upper bound of $\mathscr{C}$, if it is orthonormal. For any two elements $v, w \in U$, there is some orthonormal S_j which contains both, and they are therefore orthonormal. The partially ordered set of the orthonormal sets, satisfies the conditions to apply Zorn's lemma, which states that the collection of orthonormal sets contains a maximal set. ☐

Theorem A.4. *For every continuous linear functional $\phi \in V'$, there is a unique element $u \in V$ such that $\phi(v) = \langle v, u\rangle$ for all v in V. This correspondence defines an isometry between V and V'.*

If S is a subset of V, the orthogonal complement of S is

$$S^{\perp} = \{v \in V : \langle u, v\rangle = 0 \text{ for all } u \in S\}\,.$$

For any closed subspace S of V, we have orthogonal decomposition $V = S^{\perp} \oplus S$. Any vector v can be written uniquely as $v = u + w$ where $v \in S^{\perp}$ and $w \in S$, and $\langle u, w\rangle = 0$. The map $Pv := u$ is called the **orthogonal projection** onto $S^{\perp}$. The element Pv is the vector in $S^{\perp}$ closest to v. Similarly we have projection onto S.

Definition. A linear operator $A\colon D(\subset V) \to W$ between Hilbert spaces defined on a subset D of V is called an **unbounded operator**. The unbounded operator A is said to be **closed** if and only if for every sequence (v_i) in D converging to an element v in V, and $Av_i \to w$ in W; then $v \in D$ and $Tv = w$.

We will write $A\colon V \to W$ even though A is not defined on the whole space V. By $\operatorname{Dom} A$ we will always mean the domain D where A is defined.

Definition. Let $A\colon V \to W$ be a densely defined unbounded operator. The **adjoint** of A is a closed unbounded operator $A^*\colon W \to V$, with the property that

$$\langle Av, w\rangle_W = \langle v, A^* w\rangle_V$$

for all $v \in \operatorname{Dom} A$, when $w \in \operatorname{Dom} A^*$. The domain $\operatorname{Dom} A^*$ are the elements $w \in W$ such that the map $\phi_w : v \mapsto \langle Av, w\rangle_W$ is bounded. Since A is defined on a dense subset, we can extend ϕ_w to the whole of V. By Theorem A.4, there is a unique element $u \in V$ such that $\phi_w = \langle \cdot, u\rangle$. We define A^*w to be u, $A^*w := u\,$. When $A\colon V \to V$ and $A^* = A$, we say that A is **self-adjoint**.

Theorem A.5. *If* $A\colon V \to W$ *is a densely defined unbounded operator, then*

$$W = \operatorname{Ker} A^* \oplus \overline{\operatorname{Im} A}\,.$$

Proof. We have decomposition $W = \overline{\operatorname{Im} A}^{\perp} \oplus \overline{\operatorname{Im} A}$. All we need to show is that $\overline{\operatorname{Im} A}^{\perp} = \operatorname{Ker} A^*$. Let $u \in \operatorname{Ker} A^*$, $w = \lim Av_j$. Then

$$\langle u, w\rangle = \lim\langle u, Av_j\rangle = \lim\langle A^*u, v_j\rangle = 0$$

which shows that $\operatorname{Ker} A^* \subset \overline{\operatorname{Im} A}^{\perp}$. Conversely, if $w \in \overline{\operatorname{Im} A}^{\perp}$ and $v = \lim u_j$ where $u_j \in \operatorname{Dom} A$, then

$$\langle A^*w, v\rangle = \lim\langle A^*w, u_j\rangle = \lim\langle w, Au_j\rangle = 0$$

for all $v \in V$. Thus A^*v must be 0 by the non-degeneracy of the inner product. $\square$

Bibliography

[1] S. Bochner. Curvature and Betti numbers. *Ann. of Math. (2)*, 49:379–390, 1948.

[2] J. Brüning and M. Lesch. Hilbert complexes. *J. Funct. Anal.*, 108(1):88–132, 1992.

[3] G. Carron. L^2-cohomology of manifolds with flat ends. *Geom. Funct. Anal.*, 13(2):366–395, 2003.

[4] J. C. Chen. A note on Riesz potentials and the first eigenvalue. *Proc. Amer. Math. Soc.*, 117(3):683–685, 1993.

[5] W. Cheney. *Analysis for applied mathematics*, volume 208 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[6] X. Dai. An introduction to L^2 cohomology. In *Topology of stratified spaces*, volume 58 of *Math. Sci. Res. Inst. Publ.*, pages 1–12. Cambridge Univ. Press, Cambridge, 2011.

[7] J. Dieudonné. *A history of algebraic and differential topology. 1900–1960*. Birkhäuser Boston, Inc., Boston, MA, 1989.

[8] M. P. Gaffney. A special Stokes's theorem for complete Riemannian manifolds. *Ann. of Math. (2)*, 60:140–145, 1954.

[9] M. Gromov. Kähler hyperbolicity and L_2-Hodge theory. *J. Differential Geom.*, 33(1):263–292, 1991.

[10] G. Grubb. *Distributions and operators*, volume 252 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[11] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[12] J. M. Lee. *Introduction to Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer, Cham, 2018. Second edition of [MR1468735].

[13] X.-D. Li. On the strong L^p-Hodge decomposition over complete Riemannian manifolds. *J. Funct. Anal.*, 257(11):3617–3646, 2009.

[14] J. Lott. L^2-cohomology of geometrically infinite hyperbolic 3-manifolds. *Geom. Funct. Anal.*, 7(1):81–119, 1997.

[15] P. Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.

[16] W. A. Poor. A holonomy proof of the positive curvature operator theorem. *Proc. Amer. Math. Soc.*, 79(3):454–456, 1980.

[17] K. Schmüdgen. *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2012.

[18] R. S. Strichartz. Analysis of the Laplacian on the complete Riemannian manifold. *J. Functional Analysis*, 52(1):48–79, 1983.

[19] F. Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.

[20] L. W. Tu. *An introduction to manifolds*. Universitext. Springer, New York, second edition, 2011.

[21] L. W. Tu. *Differential geometry*, volume 275 of *Graduate Texts in Mathematics*. Springer, Cham, 2017. Connections, curvature, and characteristic classes.

[22] F. W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

[23] H.-H. Wu. *The Bochner technique in differential geometry*, volume 6 of *CTM. Classical Topics in Mathematics*. Higher Education Press, Beijing, 2017. New edition of the 1988 original [MR1079031], Expanded version of [MR0714349].

[24] S. T. Yau. Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana Univ. Math. J.*, 25(7):659–670, 1976.