Theoretical Computer Science 915 (2022) 74–89

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Resolute control: Forbidding candidates from winning an election is hard

Sushmita Gupta [a,1], Sanjukta Roy [b,*,2], Saket Saurabh [a,c,3], Meirav Zehavi [d,4]

[a] *The Institute of Mathematical Sciences, HBNI, Chennai, India*
[b] *TU Wien, Austria*
[c] *University of Bergen, Bergen, Norway*
[d] *Ben-Gurion University, Beersheba, Israel*

ARTICLE INFO

Article history:
Received 20 October 2021
Accepted 23 February 2022
Available online 2 March 2022
Communicated by F.V. Fomin

Keywords:
Voting
Resolute control
W-hardness
Parameterized complexity

ABSTRACT

We study a set of voting problems where given an election $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$ (where $\mathcal{C}$ is the set of candidates and $\Pi_{\mathcal{V}}$ is a set of votes), and a non-empty subset of candidates J, the question under consideration is: Can we modify the election in a way so that none of the candidates in J wins the election? The modification operations allowed are that of either adding or deleting some candidates. Yang and Wang (2017) [44] introduced these problems as *the* Resolute Control *problem*, a generalization of the destructive control problem where J is a singleton. They studied parameterized complexity of Resolute Control for voting rules Borda (both addition and deletion), Maximin (addition), and Copeland (both addition and deletion). They primarily consider $|J|$ as parameter. In this paper we study Resolute Control parameterized by the other natural parameters viz., the number of candidates added or deleted. We show that the Resolute Control for Borda (both addition and deletion), Maximin (addition) and Copeland (deletion) are W[2]-hard. We complement this by showing that when the number of voters is odd, Copeland (deletion) is FPT parameterized by the sum of the number of deleted candidates and the size of the *feedback arc set* of the majority graph of the election.

1. Introduction

Voting is a prevalent mechanism for collective decision-making in modern society. Standard formats of an *election*, denoted by $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$, consist of a set of candidates $\mathcal{C}$, a set of voters $\mathcal{V}$, and a multiset $\Pi_{\mathcal{V}}$ of *voting profiles* (that is, linear orderings of $\mathcal{C}$), with one voting profile $\succ_v$ for each voter $v \in \mathcal{V}$. Given the significance of the outcomes of elections, they are highly attractive for manipulators. Thus, it is imperative to find solutions that deal with manipulation effectively.

* Corresponding author.
E-mail addresses: sushmitagupta@imsc.res.in (S. Gupta), sanjukta.roy@tuwien.ac.at (S. Roy), saket@imsc.res.in (S. Saurabh), meiravze@bgu.ac.il (M. Zehavi).

[1] Supported by SERB-Starting Research Grant (SRG/2019/001870).
[2] Supported by the WWTF research grant (VRG18-012).
[3] Supported by European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant no. 819416), and Swarnajayanti Fellowship grant DST/SJF/MSA-01/2017-18.
[4] Supported by ISF grant (no. 1176/18) and BSF grant (no. 2018302).

https://doi.org/10.1016/j.tcs.2022.02.031
0304-3975/

Table 1
Our five negative results.

	parameter	Borda	Maximin	Copeland$^{\alpha}$
Control by deleting candidates	k_{DC}	W[2]-hard	P [44]	W[2]-hard
Control by adding candidates	k_{AC}	W[2]-hard	W[2]-hard	W[2]-hard

To this end, we first need to understand the power and limitations of central forms of manipulation by analyzing the computational problems that underlie them [1,2,12,17,30,37,42,44]. Among the most common forms of manipulation are the addition and/or deletion of candidates, and the addition and/or deletion of voters, hence they deserve special attention. These forms of manipulation give rise to two well-studied families of computational problems, termed Constructive Control and Destructive Control.

For over three decades, since the seminal work of Bartholdi, Tovey and Trick [1,2], computational problems of Constructive Control have been extensively studied and are nowadays relatively well understood. Here, the objective of the manipulator is to ensure that the winner is some distinguished candidate j, or, more generally, a candidate from a *set* of distinguished candidates J. While initially, the computational problems (of adding/deleting candidates/voters) were defined for a single distinguished candidate j [1,2], the objective was soon generalized to settings that capture the case of a set of distinguished candidates J [35,36,38,39]. Having a shorter history, yet already introduced more than a decade ago by Conitzer, Sandholm and Lang [15] and Hemaspaandra, Hemaspaandra and Rothe [30], are the computational problems of Destructive Control. Naturally complementing Constructive Control, here the objective of the manipulator is to ensure that the winner is *not* some distinguished candidate j, or, more generally, *not* a candidate from a *set* of distinguished candidates J. Clearly, candidates can be hated just as much as they can be favored, hence Destructive Control is as commonplace as Constructive Control. The study of control problems has been further strengthen by the Resolute Destructive Control problem where the goal is to ensure no candidate from J can win under any tie breaking rule by adding candidates (AC) or deleting candidates (DC). We denote it RCX where $X \in \{\mathrm{AC}, \mathrm{DC}\}$. Indeed, it is easy to think of numerous real-life situations where RCX arises, be it the prevention of any person from a certain party J to be promoted to a position of power, or the avoidance of distant locations when selecting a conference venue. Furthermore, Destructive Control is an alternative to Constructive Control (rather than making j win, make its threats lose) that tends to be computationally easier [3,23,43] and hence tempting to undertake.

In sharp contrast to its constructive counterpart, up until less than two years ago [44], RCX was only studied when J is a singleton. Arguably, when we deal with RCX—unless the entire set of candidates is very small—it is easier to come up with situations where multiple candidates, rather than a single candidate, should be eliminated (see the scenarios above). Yang and Wang [44] were the first to amend this discrepancy. Specifically, they addressed RCX under three of the most well known voting correspondences, called Borda, Maximin and Copeland,[5] where J can be a set. They pointed out that in the non-unique winner model, Destructive Control (DCX-NON) is a special case of Resolute Control. It is known [27,34] that DCX-NON is polynomial time solvable for Borda, Maximin and Copeland. In contrast to this, Yang and Wang asserted that for both addition/deletion of candidates and addition/deletion of voters, all of these voting correspondences, apart from the case of Maximin with deletion of candidates, result in NP-hard problems. The core of their work, though, was the analysis of the parameterized complexity of these problems when parameterized by $|J|$. The computational question for RCX with respect to other parameters remains open.

In this paper, we substantially broaden the scope of our knowledge of RCX where J can be a set. In particular, we initiate the study of two new parameterizations: *(i)* the maximum number of candidates to delete k_{DC}; *(i)* the minimum number of candidates to add k_{AC}. Both of these parameterizations are arguably the most sensible ones when the manipulation is to delete or add candidates, respectively. Indeed, manipulators are likely to have limited budget and power, and may further need to avoid exposure. Then, the manipulation can be conducted only if the number of candidates to be added or deleted is small.

We give several key results concerning both our parameterizations. Our main results state that Borda (for both k_{DC} and k_{AC}), Maximin (for k_{AC} only) and Copeland (for k_{DC} and k_{AC}), are all W[2]-hard for every possible tie-breaking scheme (see Table 1). Then, as a step-stone towards the design of parameterized algorithm despite these results, we complement our study by showing that Copeland (for candidate deletion) is FPT parameterized by the *sum* of k_{DC} and a structural parameter. Next, we will formally define all the terminologies used in the paper. Following that we discuss our contribution and its significance in more details.

1.1. Preliminaries

For any string $x \in \Sigma^\star$, we define $\overline{x}$ to be the reverse of x. In particular, we will use this operation on ordered subsets (*blocks*) of candidates in voting profiles. For example, if a voting profile contains the block $b = c_1c_2c_3$, then $\overline{b} = c_3c_2c_1$.

Conducting an election. Two evaluation procedures, called a *voting correspondence* and a *tie-breaking scheme*, govern the outcome of an election $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$. A voting correspondence is a function that maps an election $\mathcal{E}$ to a subset of candidates

[5] Definitions of standard notions as voting correspondences are deferred to Section 1.1.

in $\mathcal{C}$, and a *tie-breaking scheme* maps a subset of candidates in $\mathcal{C}$ to a single candidate in that subset. The *composition* of a voting correspondence and a tie-breaking scheme defines a *voting rule*. Note that, by its definition, a voting rule maps an election $\mathcal{E}$ to a single candidate in $\mathcal{C}$, called the *winner*. For two candidates $c_1, c_2 \in \mathcal{C}$, $N(c_1, c_2)$ is the number of voters which *prefer* c_1 to c_2 (that is, c_1 appears before c_2 in the linear order). If $N(c_1, c_2) > N(c_2, c_1)$, we say c_1 *beats* c_2, or c_2 is *beaten by* c_1; otherwise, if $N(c_1, c_2) = N(c_2, c_1)$, we say c_1 *ties* c_2. We consider three central voting correspondences:

- **Borda.** Every voter gives 0 points to his/her last-ranked candidate, 1 point to the second-last ranked candidate, and so on. The sum of the points awarded to a candidate c yields his/her Borda score, denoted by $\text{Borda}(c)$.
- **Maximin.** For a candidate c, the Maximin score of c is $\text{Maximin}(c) = \min_{c' \in \mathcal{C} \setminus \{c\}} N(c, c')$.
- **Copeland$^\alpha$.** For a candidate c, let $B(c)$ and $T(c)$ be the sets of candidates who are beaten by c and who tie with c, respectively. The Copeland score of c is $\text{Copeland}(c) = |B(c)| + \alpha|T(c)|$. Here, α is a rational number such that $0 \le \alpha \le 1$.

The winner(s) in each of these correspondences are the candidates with the highest score. Arguably, Borda is the most classic positional voting correspondence extant, while Maximin and Copeland are well-studied *Condorcet-consistent* voting correspondences.[6]

For an election $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$ and subset $C \subset \mathcal{C}$, $(C, \Pi_{\mathcal{V}}^C)$ is the election restricted to C, i.e., an election with candidate set C and vote set $\{\succ^C \mid \succ \in \Pi_{\mathcal{V}}\}$, where $\succ^C$ is the vote $\succ$ restricted to C. For each $X \in \{\text{AC}, \text{DC}\}$, where DC and AC denote the operations of deleting and adding candidates, respectively, a resolute control problem is formally defined as follows.

RCX	**Parameter:** k_X

Input: An election $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$, a subset of candidates $\mathcal{A} \subseteq \mathcal{C}$, a non-empty subset of candidates $J \subseteq \mathcal{C} \setminus \mathcal{A}$, non-negative integers $k_{\text{AC}} \le |\mathcal{A}|$, $k_{\text{DC}} \le |\mathcal{C} \setminus (\mathcal{A} \cup J)|$, a voting correspondence ψ, and a tie-breaking scheme μ.
Question: Are there $A \subseteq \mathcal{A}$, $C \subseteq \mathcal{C} \setminus (\mathcal{A} \cup J)$, such that $|A| \le k_{\text{AC}}$, $|C| \le k_{\text{DC}}$ and no candidate from J can win the election $((\mathcal{C} \setminus \mathcal{A}) \setminus C) \cup A, \Pi_{\mathcal{V}}^{((\mathcal{C} \setminus \mathcal{A}) \setminus C) \cup A})$?

Essentially, we ask whether there is a way to add at most k_{AC} candidates from $\mathcal{A}$ and delete at most k_{DC} candidates from $\mathcal{C} \setminus (\mathcal{A} \cup J)$, so that no candidate in J can win. Throughout this paper, $\mathcal{J} = (\mathcal{C}, \Pi_{\mathcal{V}}, \mathcal{A} \subseteq \mathcal{C}, J \subseteq \mathcal{C} \setminus \mathcal{A}, k_{\text{AC}}, k_{\text{DC}}, \psi)$ denotes the instance of the resolute control problem under consideration. As our results hold irrespective of the tie-breaking scheme used (even if it is probabilistic), we neither define nor specify individual schemes.

Majority graph. The *majority graph* of an election $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$ is the graph $G = (V, E)$ with $V = \mathcal{C}$, and where for every pair of candidates $c, c' \in \mathcal{C}$, there is a directed edge from c to c' in G (denoted by $c >_m c'$) if and only if a strict majority of voters prefer c to c'. The majority graph is asymmetric and irreflexive, but it is not necessarily transitive. Moreover, if the number of voters is odd, G is complete—for all $c, c' \ne c$, either $c >_m c'$ or $c' >_m c$ holds. In this case, G is also called a *tournament on* $\mathcal{C}$ [31]. For a subset of vertices $V' \subseteq V$, $G[V']$ denotes the subgraph of G induced by V'. The *feedback arc set number* of G is the minimum number of arcs required to remove from G to make it acyclic. For background on graph theory we refer to [20] and for background on parameterized complexity we refer to [16].

1.2. Our contribution

Our starting point is the following simple observation[7] where $|I|$ is the size of an instance I.

Observation 1.1. For any tie-breaking scheme and $X \in \{\text{AC}, \text{DC}\}$, Borda, Copeland$^\alpha$ and Maximin RCX admit algorithms with running time $\binom{|\mathcal{C}|}{k_X}|I|^{\mathcal{O}(1)} = |\mathcal{C}|^{k_X}|I|^{\mathcal{O}(1)}$ and hence XP.

In light of this observation, the first question that comes to mind is whether our problems admit algorithms with running time $f(k_X)|I|^{\mathcal{O}(1)}$ rather than $|I|^{f(k_X)}$ as above, for some function f. In the language of Parameterized Complexity, this is equivalent to the following:

Question 1. *Are the problems* FPT *with respect to* k_X*?*

Our main contribution negatively resolves Question 1 for *five* open cases, thereby completely characterizing the parameterized complexity of Borda, Copeland$^\alpha$ and Maximin RCX for both $X \in \{\text{AC}, \text{DC}\}$ (see Table 1). Specifically, we exhibit the hardness of these problems (except Maximin RCDC) with respect to the complexity class W[2] [16,22]. This means that under the standard complexity-theoretic assumption of FPT $\ne$ W[2], *none* of these is FPT.

[6] A voting correspondence is *Condorcet-consistent* if it picks the *Condorcet-winner*—the candidate preferred to any other candidate by a majority of voters—whenever it exists.

[7] To see this, iterate over every possible selection of a subset of candidates to delete (or add) and test whether no candidate in J can win the resulting election.

Theorem 1. *For any tie-breaking scheme,* Borda RCDC *and* RCAC*,* Copeland$^\alpha$ RCDC *and* RCAC*, and* Maximin RCAC *are* W[2]-hard*.*

Each of these hardness results is exhibited by a reduction from Red-Blue Dominating Set but the ideas to construct the gadget are very different. This theorem does not end our venture into the discovery of the limits of parameterized algorithms for these problems. Indeed, a follow-up question arises:

Question 2. *Do the problems admit algorithms with running time* $|\mathcal{C}|^{o(k_X)}|I|^{\mathcal{O}(1)}$*?*

That is, not expecting to attain running times of $f(k_X)|I|^{\mathcal{O}(1)}$ (*for any function* f) does not preclude the possibility of attaining running times that are *substantially* faster than those in Observation 1.1 (e.g., $|\mathcal{C}|^{\sqrt{k_X}}|I|^{\mathcal{O}(1)}$). We carefully design our reductions so that the parameter in the output will have *linear* dependence on the parameter in the input. Therefore, our reductions prove Theorem 1 and the following theorem simultaneously.

Theorem 2. *For any tie-breaking scheme,* Borda RCDC *and* RCAC*,* Copeland$^\alpha$ RCDC *and* Maximin RCAC *do not admit any algorithm with running time* $f(k_X)|I|^{o(k)}$*, for any function* f*, unless the Exponential Time Hypothesis (ETH) fails.*

In plain words, under a complexity-theoretic assumption stronger than FPT $\neq$ W[2], called the ETH [16,22], for Borda RCDC and RCAC, Copeland$^\alpha$ RCDC and RCAC, and Maximin RCAC, *the naive algorithms in Observation 1.1 are essentially optimal.* In the reduction the resulting parameter is linear in the input parameter. Hence, we have shown that in addition to Borda RCDC (AC), Maximin RCAC, and Copeland$^\alpha$ RCDC (AC) also do not admit any algorithm of the form $f(k)n^{o(k)}$ unless ETH fails. Moreover, our design of the reductions also proves that there is no polynomial size kernel[8] for these problems when parameterized by $|J|$. It is known that Red-Blue Dominating Set, parameterized by $|B|$, does not admit a polynomial kernel unless NP $\subseteq coNP/poly$ [16]. Since in our reductions $|J| = |B|$, it implies that Borda RCDC and RCAC, Maximin RCAC, and Copeland$^\alpha$ RCDC and RCAC, parameterized by $|J|$ does not admit polynomial kernel unless NP $\subseteq coNP/poly$, although these problems are FPT parameterized by $|J|$ [44]. Furthermore, all our reductions work for any tie-breaking rule.

Corollary 1.1. Borda RCDC *(*AC*),* Copeland$^\alpha$ RCDC *(*AC*), and* Maximin RCAC *does not admit polynomial size kernel with respect to the parameter* $|J|$ *unless* NP $\subseteq$ *co*NP$/poly$.

Furthermore, we note that in our reductions a fixed candidate $c^\star$ wins the "modified" election. Hence, reductions for RCAC imply hardness for Constructive Control (CC) variants as well.[9] This leads us to consider parameters that are the sum of k_X and a *structural parameter*—such parameterizations may give rise to efficient parameterized algorithms.

Question 3. *For which structural parameters t are the problems* FPT *with respect to* $k_X + t$ *(or t alone)?*

We end our paper with a positive (or rather negative) note. Specifically, as a new structural parameter, we propose the *feedback arc set number* (**fasn**), denoted by k_{FAS}, of the majority graph of the input election. We also note that the majority graph produced in our reduction for Copeland$^\alpha$ RCDC is acyclic. That is, k_{FAS} is 0. Hence, under complexity theoretic assumptions, Copeland$^\alpha$ RCDC cannot admit a FPT algorithm parameterized by k_{FAS}. We remark that k_{FAS} is a well-studied parameter in Parameterized Complexity [16,22] that has already received attention in Computational Social Choice [28,29,40]. The motivation behind our introduction of this parameter to our settings stems from the observation that the candidates in an election can usually be (roughly) ordered from *strongest* to *weakest* so that, more often than not, a candidate c stronger than a candidate c' will satisfy $c >_m c'$. This observation directly implies that the feedback arc set number of the majority graph of an election may often be small and hence serve as a sensible parameter. For this parameter, we prove the next algorithmic result. In a voting system where the number of voters is large, it is unlikely that exactly equal number of votes would be received by a pair of candidates in a pairwise election between them. So, we assume the majority graph to be a tournament.

Theorem 3. Copeland$^\alpha$ RCDC *on majority graphs that are tournaments has a parameterized algorithm running in* $c^{k_{\text{DC}}+k_{\text{FAS}}}|I|^{\mathcal{O}(1)}$ *time for some constant* $c < (2e)^2$.

Directions for future research are discussed in Section 5.

Related Works. For an overview of related results we refer to two comprehensive surveys and few more related works [24,26,27,41] that discuss several constructive and destructive control problems are NP-hard but admit FPT algorithms with various natural parameters such as number of candidates.

---

[8] For definition of kernel and lower bounds for kernel we refer to [16].

[9] However, we point out here that the reverse is not true. That is, a hardness for CCX may not show hardness for RCX for $X \in \{\text{AC}, \text{DC}\}$. For example in CCDC, the distinguished candidate can win due to deletion of a candidate from J which is forbidden in RCDC. Relations between known results of CCX and RCX are discussed in related works section.

Reduction to Borda RCDC: Let $\mathcal{I} = (G = (V_B \cup V_R, E), k)$ be an instance of Red-Blue Dominating Set. To simplify calculations ahead, we assume that every vertex $v \in V_B$ has the same degree, denoted by Δ. The problem remain W[2]-hard under this assumption. We create an instance of Borda RCDC $\mathcal{J} = (\mathcal{C}, \Pi_\mathcal{V}, \mathcal{A} = \emptyset, J, k_{\text{AC}} = 0, k_{\text{DC}} = k, \text{Borda})$ as follows.

Candidates: To every vertex $b \in V_B$, we introduce a candidate $b \in B$, and to every vertex $r \in V_R$, we introduce a candidate $r \in R$. Thus, $|B| = |V_B|$ and $|R| = |V_R|$. Let $B = \{b_1, \dots, b_{|B|}\}$. Then, we introduce $|B|$ "dummy" candidates $D = \{d_1, \dots, d_{|B|}\}$, where d_i is called the *twin* of b_i. Lastly, we add a special candidate $c^\star$. Thus, the set of candidates $\mathcal{C} = \{c^\star\} \cup B \cup R \cup D$.

Voting Profile: We fix an ordering of B, R and D. In every vote, the candidates in B, R and D (or their reversals) will appear according to this order. For any vertex $v_x \in V_B \cup V_R$, we use $N(v_x)$ to represents the neighborhood of v_x in G. We abuse notation to use $N(x)$ (for a candidate $x \in B \cup R$) to represent the set of candidates that correspond to the neighbors of vertex v_x in G. Let $\alpha = |B| + 1$, $\beta = (|B| + 1)(2\Delta + |R|) - 1$ and $\gamma = |B|^3|R|^3$. We will have 2α many "X-type" voters for each vertex in V_R, β many "Y-type" voters to increase the score of $c^\star$, and γ many "Z-type" other voters to ensure dummy candidates do not win. Specifically, $\Pi_\mathcal{V}$ is the multiset of votes in Table 2. The set $D_{B \setminus N(r)}$ denotes the subset of dummy candidates containing the twins of the candidates in $B \setminus N(r)$. Lastly, define $J = B \cup D$. This completes the reduction.

Fig. 1. Reduction from Red-Blue Dominating Set to Borda RCDC.

Control problems are mainly concerned with voting correspondences instead of voting rules; in the former there can be tied winners and in the later the winner must be unique. A candidate can win an election uniquely or along with other(s) (known as the unique-winner model and the nonunique-winner model, resp.). As pointed out by Yang and Wang [44] even though the problems seem related, Constructive Control in the unique winner model (CCX-UNI), cannot be reduced to RCX trivially by setting $J = C \setminus \{p\}$. When $X = \text{AC}$, someone in the set of unregistered candidates $\mathcal{A}$ instead of the distinguished candidate p can also prevent the candidates in J from winning and p does not get the highest score. When $X = DC$ we are not allowed to delete candidates in J in RCX. Hence, results about Constrictive Control do not directly imply results about RCX. For the non-unique winner model, Destructive Control is a special case of Resolute Control [44].

Liu and Zhu [33] showed that Constructive Control for Maximin is W[2]-hard. This implies that Destructive Control with a fixed tie breaking rule (say "Fixed Order") is W[2]-hard whereas Theorem 1 implies that Destructive Control is W[2]-hard with any tie breaking rule. Liu et al. [32] showed W-hardness results for Constructive and Destructive control problems for Plurality voting rule. A highlight of related results can be found in the survey [4]. Problems on Constructive Control have been reduced to graph theoretic problems e.g., Betzler and Uhlmann [5] studied Constructive Control by adding/deleting candidates for Copeland$^\alpha$ and show that the problem is W[2]-complete with respect to number of candidates added (resp. deleted) in the unique winner model when the majority graph is a tournament. Our results for Copeland$^\alpha$ RCX hold even when the majority graph is acyclic. Betzler et al. [6] studied control in Lull voting when the parameters are treewidth and feedback vertex set number.

In the last several years, it has become increasingly common to use parameterized algorithms to resolve problems in Social Choice Theory [21]; having proven to be particularly fruitful in Voting Theory (see, e.g., [8–11,13,14,18,19]). For more information on the current state-of-the-art in this regard, we refer to excellent surveys such as [7,21,25].

2. Borda: deletion of candidates

In this section, we prove that Borda RCDC and RCAC are W[2]-hard. Towards this, we give a parameterized reduction from Red-Blue Dominating Set. The Red-Blue Dominating Set problem is defined as follows.

Red-Blue Dominating Set　　**Parameter:** k
Input: A bipartite graph $G = (V_B \cup V_R, E)$, and an integer k.
Question: Does there exist $S \subseteq V_R$ such that $|S| \leq k$ and for every $v \in V_B$, S contains at least one neighbor of v?

It is known that Red-Blue Dominating Set is W[2]-hard with respect to k [16]. The reduction from Red-Blue Dominating Set to Borda RCDC is described in Fig. 1. A very similar reduction from Red-Blue Dominating Set to Borda RCAC can be shown.

Two crucial design choices in our reduction are to introduce dummy candidates, and to have both $X_{r,i}$ and $X'_{r,i}$. The first choice ensures that the size of the two blocks flanking r in every voting profile of an X-voter is *exactly* $|B|$. The second choice ensures that irrespective of the position of a candidate $b \in B$ in the ordering of B, every vertex $v_r \in N(b)$ contributes to its score the *same* number of points, $3(|B| + 1)$, and every vertex $v_r \notin N(b)$ contributes to its score the *same* number of points, $(B + 1)$. Due to these design choices, a trivial calculation results in Lemma 2.1. In particular, for $b \in B$, we have

$$\text{Borda}(b) = (3(B+1) \cdot |N(b)| + (B+1) \cdot |B \setminus N(b)|) \cdot \alpha + (3|B| + 2|R| - 1) \cdot \beta + (3|B| + 2|R|) \cdot \gamma,$$

which evaluates to the number in Table 3.

Lemma 2.1. *Let $\mathcal{I}$ be an instance of* Red-Blue Dominating Set*. Then, Table 3 specifies the Borda score of each candidate in $\mathcal{J}$. In particular, the scores of the candidates in B are highest.*

Proof. The order of the candidates in the blocks $R \setminus \{r\}$, $N(r)$, $B \setminus N(r)$, $D_{B \setminus N(r)}$ and $D_{N(r)}$ in X' is the reverse of that in X. Similar property holds for blocks B, R and D in Y' (Z') and $Y(Z)$. This is a very commonly used trick when constructing

Table 2
Voting Profile used in the reduction of Borda RCDC. For each vertex $v_r \in V_R$ and $i \in [\alpha]$, we have two votes $X_{r,i}$ and $X'_{r,i}$. Additionally, we have β Y-voters, and γ Z-voters.

$X_{r,i}$	$R \setminus \{r\}$	$N(r)$	$D_{B\setminus N(r)}$	r	$B \setminus N(r)$	$D_{N(r)}$	$c^\star$
$X'_{r,i}$	$\overline{R \setminus \{r\}}$	$\overline{D_{B\setminus N(r)}}$	$\overline{N(r)}$	r	$\overline{D_{N(r)}}$	$\overline{B \setminus N(r)}$	$c^\star$

Y_i	$c^\star$	B	R	D
Y'_i	$c^\star$	$\overline{B}$	$\overline{D}$	$\overline{R}$

Z_i	$c^\star$	B	R	D
Z'_i	$\overline{B}$	$c^\star$	$\overline{D}$	$\overline{R}$

Table 3
Borda scores of the candidates in the instance $\mathcal{J}$. The third column refers to the set S in Lemma 2.2 with Bo = Borda.

$\mathcal{C}$	Borda scores in $\mathcal{J}$	Scores After Deleting S
b	$(3\alpha\|N(b)\| + \alpha\|B \setminus N(b)\|)\alpha + (3\|B\| + 2\|R\| - 1)\beta + (3\|B\| + 2\|R\|)\gamma$	$\leq \text{Bo}(b) - (2\alpha + 2k(\beta+\gamma))$
$c^\star$	$2(2\|B\| + \|R\|)\beta + (3\|B\| + 2\|R\|)\gamma$	$= \text{Bo}(c^\star) - 2k(\beta+\gamma)$
r	$(2\alpha + (\|R\| - 1 + 4\|B\| + 2)(\|R\| - 1))\alpha + (\|R\| - 1 + \|B\|)(\beta+\gamma)$	$\leq \text{Bo}(r) - k((\|R\| - 1)\alpha + (\beta+\gamma))$
d (twin of b)	$((B+1)\|N(b)\| + 3(\|B\|+1)\|B \setminus N(b)\|)\alpha + (\|R\| + \|B\| - 1)(\beta+\gamma)$	$\leq \text{Bo}(d) - (2\alpha + k(\beta+\gamma))$

votes. For example, consider a candidate $r \in R$ in the votes Y_i and Y'_i, where $i \in [\beta]$. Suppose candidate r is at the first position in the block R in Y_i. Hence, candidate r is at the last position in the block $\overline{R}$ in Y'_i. Reversal of the block R ensures that candidate r receives $|R| - 1 + |D|$ points form the votes Y_i and Y'_i irrespective of the position of r in the block. In the following calculation of scores we use this phenomenon without stating it again.

Consider a candidate $b \in B$. For each vertex $v_r \in N(b)$ and for each $i \in [\alpha]$, form the votes $X_{r,i}$ and $X'_{r,i}$, candidate b receives $3(|B|+1)$ points because b is in the $N(r)$ block in the votes $X_{r,i}$ and $X'_{r,i}$, $|D_{B\setminus N(r)}| = |B \setminus N(r)|$, and $|D_{N(r)}| = |N(r)|$. If candidate $b \notin N(r)$, then b is in the $B \setminus N(r)$ block in the votes $X_{r,i}$ and $X'_{r,i}$. Since $|D_{N(r)}| = |N(r)|$, b receives $(|B|+1)$ points from $X_{r,i}$ and $X'_{r,i}$, for each $i \in [\alpha]$. Hence, candidate b receives $(3|B|+1) \cdot |N(b)| + (B|+1) \cdot |B \setminus N(b)|) \cdot \alpha$ points from X-type votes. Clearly, candidate b receives $2|R| + 2|D| + |B| - 1$ points from the votes Y_i and Y'_i, for each $i \in [\beta]$ and $2|R| + 2|D| + (|B| - 1) + 1$ points from the votes Z_i, and Z'_i, for each $i \in [\gamma]$. Since $|D| = |B|$, the score b receives is $(3(B+1) \cdot |N(b)| + (B+1) \cdot |B \setminus N(b)|) \cdot \alpha + (3|B| + 2|R| - 1) \cdot \beta + (3|B| + 2|R|) \cdot \gamma$.

Next, consider the candidate $c^\star$. Clearly, $c^\star$ receives 0 points from X-type votes. For each $i \in [\beta]$, from each vote Y_i (Y'_i), $c^\star$ receives $|B| + |R| + |D|$ points. Moreover, $c^\star$ receives $|B| + 2|R| + 2|D|$ points from votes Z_i and Z'_i, for each $i \in [\gamma]$. Since $|D| = |B|$, the score of $c^\star$ is $2(2|B| + |R|) \cdot \beta + (3|B| + 2|R|) \cdot \gamma$.

Next, consider a candidate $r \in R$. For each $i \in [\alpha]$, from $X_{r,i}$ and $X'_{r,i}$ (votes corresponding to the vertex $v_r \in V_R$), r receives $(|B|+1)$ points. For each vertex $v_{\bar{r}} \in V_R \setminus \{v_r\}$, candidate r is in the $R \setminus \{\bar{r}\}$ block in the votes $X_{\bar{r},i}$ and $X'_{\bar{r},i}$, for each $i \in [\alpha]$. Hence, r receives $2|B| + 2|D| + 2 + |R| - 1$ points from $X_{\bar{r},i}$ and $X'_{\bar{r},i}$, for $i \in [\alpha], \bar{r} \in R \setminus \{r\}$. Since $|D| = |B|$, in total candidate r receives $(2(|B|+1) + (4|B|+2+|R|-1) \cdot (|R|-1)) \cdot \alpha$ points from X-type votes. From each of Y and Z -type votes r receives $|D| + |R| - 1$ points. Hence, the score of r is $(2(|B|+1) + (4|B|+2+|R|-1) \cdot (|R|-1)) \cdot \alpha + (|B| + |R| - 1) \cdot (\beta+\gamma)$.

Finally, consider a candidate $d \in D$. Let candidate b be the twin of the dummy candidate d. For each vertex $v_r \in N(b)$, since candidate d is in the $D_{N(r)}$ block in the votes $X_{r,i}$ and $X'_{r,i}$, d receives $(|B|+1)$ points form $X_{r,i}$ and $X'_{r,i}$, for each $i \in [\alpha]$. Suppose candidate $b \in B \setminus N(r)$, i.e., it's twin candidate d is in the $D_{B\setminus N(r)}$ block in the votes $X_{r,i}$ and $X'_{r,i}$. Then, d receives $3(|B|+1)$ points from $X_{r,i}$ and $X'_{r,i}$, for each $i \in [\alpha]$. Hence, in total, d receives $((B+1) \cdot |N(b)| + 3(|B|+1) \cdot |B \setminus N(b)|) \cdot \alpha$ points from X-type votes. From each of Y and Z-type votes candidate d receives $|R| + |D| - 1$ points. Therefore, since $|D| = |B|$, the score of d is $((B+1) \cdot |N(b)| + 3(|B|+1) \cdot |B \setminus N(b)|) \cdot \alpha + (|R| + |B| - 1) \cdot (\beta+\gamma)$. □

For the reverse direction of the proof, we will critically rely on Lemma 2.1 as well as the following observation.

Observation 2.1. Let $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$, $\mathcal{C}' \subseteq \mathcal{C}$, and $\mathcal{E}' = (\mathcal{C} \setminus \mathcal{C}', \Pi_\mathcal{V}^{\mathcal{C}\setminus\mathcal{C}'})$. For any $c \in \mathcal{C} \setminus \mathcal{C}'$, $\text{Borda}_\mathcal{E}(c) > \text{Borda}_{\mathcal{E}'}(c)$ if and only if there is a voter $v \in \mathcal{V}$ that prefers c to a candidate in $\mathcal{C}'$.

Lemma 2.2. *If $\mathcal{I}$ is a* Yes*-instance of* Red-Blue Dominating Set, *then $\mathcal{J}$ is a* Yes *-instance of* Borda RCDC.

Proof sketch. Suppose that $\mathcal{I} = (G = (V_B \cup V_R, E), k)$ is a Yes -instance of Red-Blue Dominating Set. Thus, there exists a subset $S \subseteq V_R$ of cardinality at most k that dominates all the vertices in V_B. Without loss of generality, $|S| = k$. Let $\widehat{S}$ denote the set of candidates corresponding to S, i.e., $\widehat{S} = \{r \in R | v_r \in S\}$. Let the election $\mathcal{E}' = (\mathcal{C} \setminus \widehat{S}, \Pi_\mathcal{V}^{\mathcal{C}\setminus\widehat{S}})$. To conclude the proof, it remains to show that no candidate in $B \cup D$ can win $\mathcal{E}'$. For the Borda score of any candidate $x \in R \cup D \cup \{c^\star\}$, we refer the reader to Table 3. Thus, we immediately have that $\text{Borda}(c^\star) > \text{Borda}(d)$. Towards this, consider a candidate $b \in B$. We recompute the Borda score of b in the election $\mathcal{E}'$ as follows. Since S dominates V_B, from definition of $\widehat{S}$, we have $N(b) \cap \widehat{S} \neq \emptyset$. In particular, there exists $\widehat{r} \in \widehat{S} \cap N(b)$. The deletion of $\widehat{r}$ from the votes of $\cup_{i=1}^{\alpha}\{X_{\widehat{r},i}, X'_{\widehat{r},i}\}$ decreases $\text{Borda}(b)$ by 2α. Further, the deletion of $\widehat{S}$ from the votes of $\cup_{i=1}^{\beta}\{Y_i, Y'_i\}$ decreases $\text{Borda}(b)$ by $2k\beta$, and from the votes of $\cup_{i=1}^{\gamma}\{Z_i, Z'_i\}$ by

The Reduction: Let $\mathcal{I} = (G = (V_B, V_R), k)$ be an instance of Red-Blue Dominating Set. We will construct an instance $\mathcal{J}$ of Maximin RCAC as follows. The instance $\mathcal{J}$ consists of $|V_R| + |V_B| + 1$ candidates and $6|V_R| + 2$ votes.

Candidates: The set of candidates $\mathcal{C} = \{c^\star\} \cup B \cup R$ where R and B denote the set of candidates representing the vertices in V_R and V_B, respectively. Let $v_x \in V_R \cup V_B$. The candidate representing the vertex v_x is denoted by x, and for a candidate $c \in R$, we will abuse notation to define $N(c) = \{b \in B \mid v_b \in N(v_r)\}$, and we will refer to the candidates in $N(c)$ as the *neighbors* of candidate c.

Voting profile: We have two types of votes. First, for every vertex $v_r \in V_R$, we have six votes $X_{r,i}$ and $X'_{r,i}$, for $i \in [3]$. Second, we have two more votes Y_1 and Y_2. Thus, the voting profile $\Pi_{\mathcal{V}} = \cup_{r \in R, i \in [3]}\{X_{r,i}, X'_{r,i}\} \cup \{Y_1, Y_2\}$, is the multiset of votes in Table 4.
The instance $\mathcal{J} = (\mathcal{C}, \Pi_{\mathcal{V}}, \mathcal{A} = R, J = B, k_{\mathrm{AC}} = k, k_{\mathrm{DC}} = 0, \text{Maximin})$.

Fig. 2. Reduction from Red-Blue Dominating Set to Maximin RCAC.

Table 4
Voting profile used in the reduction of Maximin RCAC. For every $r \in R$, we have $X_{r,i}$ and $X'_{r,i}$, where $i \in [3]$. Also, we have two other votes Y_1, Y_2.

$X_{r,i}$	$c^\star$	$B \setminus N(r)$	r	$N(r)$	$R \setminus \{r\}$
$X'_{r,i}$	$\overline{R \setminus \{r\}}$	r	$\overline{N(r)}$	$\overline{B \setminus N(r)}$	$c^\star$

Y_1	B	R	$c^\star$
Y_2	$\overline{B}$	$\overline{R}$	$c^\star$

$2k\gamma$. Thus, $\text{Borda}_{\mathcal{E}'}(b) \le \text{Borda}_{\mathcal{E}}(b) - (2\alpha + 2k(\beta+\gamma))$. In particular, $\text{Borda}_{\mathcal{E}'}(c^\star) - \text{Borda}_{\mathcal{E}'}(b) \ge |B|\beta + 2\alpha + \beta - (\beta+1)\alpha \ge 1$. Hence, no candidate in $B \cup D$ can win $\mathcal{E}'$. $\square$

Lemma 2.3. *If $\mathcal{J}$ is a* Yes*-instance of* RCDC Borda, *then $\mathcal{I}$ is a* Yes*-instance of* Red-Blue Dominating Set.

Proof. Let $\mathcal{J}$ be a Yes-instance of Borda RCDC. Thus, there exists a subset of candidates $S \subseteq \mathcal{C} \setminus (B \cup D)$ such that $|S| \le k_{\mathrm{DC}}$ and the deletion of S ensures that some candidate in $\mathcal{C} \setminus (B \cup D \cup S)$ gets the highest Borda score. As we observed in Lemma 2.1 the scores of the candidates in B are highest. Hence S is not an empty set. Without loss of generality, let $|S| = k_{\mathrm{DC}}$, i.e., $|S| = k$. Let $\mathcal{E}'$ denote the election $(\mathcal{C} \setminus S, \Pi_{\mathcal{V}}^{\mathcal{C}\setminus S})$.

We begin by showing that $c^\star$ does not belong to S. Suppose that $c^\star \in S$. Let $R_S = R \cap S$. Therefore, since $S \subseteq \mathcal{C} \setminus (B \cup D)$, we have $|R_S| = k - 1$. Deletion of $c^\star$ reduces every candidate's score by 1 from the votes $X_{r,i}, X'_{r,i}$, for each vertex $v_r \in V_R$ and each $i \in [\alpha]$; and further brings down the score of every candidate $b \in B$ by 1 from the vote Z'_i, where $i \in [\gamma]$. This implies that if $c^\star \in S$, then $\text{Borda}_{\mathcal{E}'}(b) = \text{Borda}_{\mathcal{E}}(b) - (2\alpha \cdot |R| + \gamma) - (\alpha \cdot |N(b) \cap R_S| + 2(k-1)\cdot(\beta+\gamma))$. The first negative term is due to deletion of $c^\star$. The second negative term is due to deletion of R_S. The score $\text{Borda}_{\mathcal{E}'}(b)$ is minimum when $N(b) \cap R_S$ is maximum, i.e., $R_S \subseteq N(b)$. Suppose that candidate $\widehat{b} \in B$ such that $R_S \subseteq N(\widehat{b})$. We show that even $\widehat{b}$ receives higher score than the score of the candidates in $\mathcal{C} \setminus S$. This would contradict that S is a solution to the instance $\mathcal{J}$ of Borda RCDC. Now consider any candidate $\widehat{r} \in R \setminus R_S$. The score of $\widehat{r}$ in the election $\mathcal{E}'$ is computed as follows: the candidate $\widehat{r}$ loses 2α points from the votes $\cup_{i=1}^{\alpha}\{X_{\widehat{r},i}, X'_{\widehat{r},i}\}$, at least $k\alpha$ points from the votes $\cup_{i=1}^{\alpha}\{X_{r,i}, X'_{r,i}\}$ where $r \in R \setminus (R_S \cup \{\widehat{r}\})$. Additionally, it loses $(k-1)$ points from Y and Z votes. Thus, $\text{Borda}_{\mathcal{E}'}(\widehat{r}) \le \text{Borda}_{\mathcal{E}}(\widehat{r}) - (2\alpha + k\cdot\alpha\cdot(|R|-1) + (k-1)\cdot(\beta+\gamma))$. Since $\gamma = |B|^3|R|^3$, it is easy to check that $\text{Borda}_{\mathcal{E}'}(\widehat{b}) > \text{Borda}_{\mathcal{E}'}(\widehat{r})$, a contradiction. Therefore, $c^\star$ is not in S.

Next, we prove that the set S corresponds to a solution for the instance $\mathcal{I}$ of Red-Blue Dominating Set. For the sake of contradiction, suppose that there exists a candidate $\widehat{b} \in B$ such that $S \cap N(\widehat{b}) = \emptyset$. Let $\{r_1, \ldots, r_{|N(\widehat{b})|}\} \subseteq V_R$ denote the neighbors of the vertex $v_{\widehat{b}} \in V_B$. Then, $S \cap \{r_1, \ldots r_{|N(\widehat{b})|}\} = \emptyset$. Then, it follows that in the votes of $\cup_{i=1}^{\alpha}\{X_{r_j,i}, X'_{r_j,i}\}$ for every $r_j \in S$, all the k_{DC} deleted candidates appeared in the first block $R \setminus r_j$ before deletion. Hence, using Observation 2.1, the contribution from the X type votes to $\widehat{b}$'s score is the same as before deletion. So $\text{Borda}_{\mathcal{E}'}(\widehat{b}) = \text{Borda}(\widehat{b}) - 2k(\beta+\gamma)$. Using the values in Table 2, it is easy to check that $\text{Borda}_{\mathcal{E}'}(b) > \text{Borda}_{\mathcal{E}'}(x)$, for any $x \in R \cup D \cup \{c^\star\}$, a contradiction. Therefore, S corresponds to a solution for the instance $\mathcal{I}$ of Red-Blue Dominating Set. $\square$

Since Red-Blue Dominating Set is W[2]-hard, we have proved Borda RCDC is W[2]-hard.

3. Cordorcet consistent rules

In this section we will show Maximin RCAC, Copeland$^\alpha$ RCDC and RCAC are W[2]-hard parameterized by $k_{\mathrm{AC}}, k_{\mathrm{DC}}$, and k_{AC}, respectively. That is, we prove part of Theorem 1.

3.1. Maximin*: addition of candidates*

We show that Maximin RCAC is W[2]-hard when parameterized by k_{DC}, using a parameterized reduction from the Red-Blue Dominating Set problem given in Fig. 2.

Intuitively, for each vertex $v_r \in V_R$, each $i \in [3]$, the vote $X_{r,i}$ ensures that the set of candidates $N(r)$ (corresponding to v_r's neighbors $N(v_r)$) is separated from the candidates $B \setminus N(r)$ (corresponding to v_r's non-neighbors $V_B \setminus N(v_r)$). This ensures that if the candidate $r \in R$ is added to the election, then the scores of every candidate in $N(r)$ decrease but not for

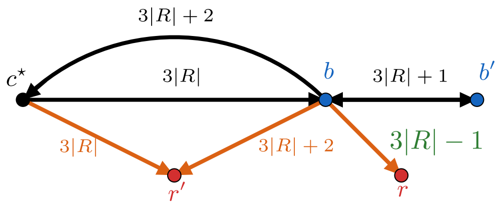

Fig. 3. The figure depicts the computations in Lemma 3.1 (only black arcs) & Lemma 3.2 (black and orange arcs after addition of candidates). The vertices represent the candidates: $\{b,b'\}\subseteq B$, $\{r,r'\}\subseteq R$ such that $b\in N(r)$ and $b\notin N(r')$. For two distinct candidates $c,c'\in\mathcal{C}$, the value $N(c,c')$ is shown on the arc cc'. Note that the Maximin score of a candidate is the minimum number written on its outgoing arcs, e.g., $\text{Maximin}(c^\star)=3|R|$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

those candidates in $B\setminus N(r)$. The purpose of the votes Y_1 and Y_2 is to reduce the score of the candidate $c^\star$ such that every candidate $b\in B$ has the highest Maximin score before addition of candidate(s) from R. Next, we compute the score of each candidate in the election $\mathcal{E}=(C\setminus R,\Pi_{\mathcal{V}}^{\mathcal{C}\setminus R})$.

Lemma 3.1. *Consider the election $\mathcal{E}=(\mathcal{C}\setminus R,\Pi_{\mathcal{V}}^{\mathcal{C}\setminus R})$. Then, for each $b\in B$, $N(b,c^\star)=3|R|+2$, $N(c^\star,b)=3|R|$, and for each $\{b,b'\}\subseteq B$, $N(b,b')=3|R|+1$ in the election $\mathcal{E}$. Moreover,* Maximin$(c^\star)=3|R|$ *and for each $b\in B$,* Maximin$(b)=3|R|+1$.

Proof. For every candidate $b\in B$, candidate $c^\star$ is preferred over b in the vote $X_{r,i}$, for every vertex $v_r\in V_R, i\in[3]$. In all other votes b is preferred to $c^\star$. Therefore, $N(c^\star,b)=3|R|$, and $\text{Maximin}(c^\star)=\min\{N(c^\star,b)|b\in B\}=3|R|$. Similarly, for every $b\in B$, $N(b,c^\star)=3|R|+2$. Let $\{b,b'\}\subseteq B$. For every vertex $v_r\in V_R, i\in[3]$, if b is preferred to b' in the vote $X_{r,i}$, then b' is preferred to b in the vote $X'_{r,i}$. Same holds for the votes Y_1 and Y_2. Therefore, by symmetry, $N(b,b')=N(b',b)$ which is equal to $3|R|+1$. See Fig. 3. Hence, $\text{Maximin}(b)=\min\{N(b,c^\star),\{N(b,b')|b'\in B\setminus\{b\}\}=\min\{3|R|+2,3|R|+1\}=3|R|+1$. □

Lemma 3.1 implies that a non-empty subset of R must be added to ensure $c^\star$ wins. The following observation will be useful for computing scores in a modified election.

Observation 3.1. Let $\mathcal{E}=(\mathcal{C},\Pi_{\mathcal{V}})$, $\mathcal{C}'\subseteq\mathcal{C}$, and $\mathcal{E}'=(\mathcal{C}\setminus\mathcal{C}',\Pi_{\mathcal{V}}^{\mathcal{C}\setminus\mathcal{C}'})$. Then, $N_{\mathcal{E}}(c,c')=N_{\mathcal{E}'}(c,c')$ for every pair of candidates $c,c'\in\mathcal{C}\setminus\mathcal{C}'$.

Let $\mathcal{J}=(\mathcal{C},\Pi_{\mathcal{V}},\mathcal{A}=R,J=B,k_{\text{AC}}=k,k_{\text{DC}}=0,\text{Maximin})$ is a YES-instance of Maximin RCAC. The next lemma completes the proof that Maximin RCAC is W[2]-hard.

Lemma 3.2. *$\mathcal{I}=(G,k)$ is a* YES*-instance of* Red-Blue Dominating Set *if and only if $\mathcal{J}$ is a* YES*-instance of* Maximin RCAC.

Proof. The idea of the proof is as follows. Using Observation 3.1, from Fig. 3 it can be seen that after addition of candidates the score of $b\in B$ is $3|R|-1$ and that of $c^\star$ is $3|R|$. Hence, no candidate in B wins. In the reverse direction, observe that if $r\in N(b)$ is not present in Fig. 3, then Maximin(b) (minimum weight on the out-arcs) is $3|R|+1$ which is the maximum among all candidates. Thus, if b is not dominated b wins the election. Next we argue both the directions in detail.

Let $S\subseteq V_R$ be a solution of Red-Blue Dominating Set for instance $\mathcal{I}$.

Let $S'=\{r\in R\mid v_r\in S\}$. Then, $S'\subseteq R$ and $|S'|=|S|\le k$. We will prove that S' is a solution of Maximin RCAC to $\mathcal{J}$. That is, no candidate from the set B wins the election $\mathcal{E}'=(\mathcal{C}',\Pi_{\mathcal{V}}^{\mathcal{C}'})$, where $\mathcal{C}'=\mathcal{C}\cup S'$.

Consider the value of $N_{\mathcal{E}'}(b,r)$ for any candidate $b\in B$ and candidate $r\in R$. Henceforth, we will only consider election $\mathcal{E}'$, and will drop the subscript $\mathcal{E}'$ from $N(\cdot,\cdot)$. Consider a candidate $b\in B$. Since S dominates all vertices in V_B, candidate b has a neighbor in S'. Let candidate r denote a neighbor of b in S', i.e., $b\in N(r)$. Therefore, candidate r is preferred over candidate b in the vote $X_{r,i}$ (corresponding to the vertex $v_r\in V_R$), for each $i\in[3]$. For every candidate $\bar{r}\in S'\setminus\{r\}$, candidate b is preferred over candidate r in the vote $X_{\bar{r},i}$, for each $i\in[3]$. Moreover, candidate b is preferred over candidate r in the vote Y_1 and Y_2. Hence, $N(b,r)=3(|R|-1)+2=3|R|-1$. Similarly, for every candidate $r\in S'$ such that candidate $b\in B\setminus N(r)$, $N(b,r)=3|R|+2$. Therefore, $\min_{r\in S'}N(b,r)=3|R|-1$. Hence, using Observation 3.1 and Lemma 3.1, $\text{Maximin}(b)=\min\{N(b,c^\star),\min_{b'\in B\setminus\{b\}}N(b,b'),\min_{r\in S'}N(b,r)\}=\min\{3|R|+2,3|R|+1,3|R|-1\}=3|R|-1$ (see Fig. 3).

Now consider Maximin$(c^\star)$ in election $\mathcal{E}'$. For every candidate $r\in S'$, $c^\star$ is preferred over r in the vote $X_{r',i}$, for every $v_{r'}\in V_R$, $i\in[3]$. Therefore, for every $r\in S'$, $N(c^\star,r)=3|R|$. Using Observation 3.1, for every $b\in B$, we have $N_{\mathcal{E}'}(c^\star,b)=N_{\mathcal{E}}(c^\star,b)$. Hence, using Lemma 3.1,

$$\begin{aligned}\text{Maximin}(c^\star)&=\min\{\min_{b\in B}N(c^\star,b),\min_{r\in S'}N(c^\star,r)\}\\&=\min\{3|R|,3|R|\}=3|R|\end{aligned}$$

The Reduction: Let $\mathcal{I} = (G = (V_R, V_B), k)$ denote an instance of Red-Blue Dominating Set. We assume that every vertex $v \in V_B$ has same degree, denoted by Δ. We construct an instance $\mathcal{J} = (\mathcal{C}, \Pi_\mathcal{V}, \mathcal{A} = \emptyset, J, k_{AC} = 0, k_{DC} = k, \text{Copeland}^\alpha)$, where $0 \le \alpha < 1/2$, of Copeland$^\alpha$ RCDC as follows.

Candidates: We have four types of candidates: One candidate for each vertex in V_B, denoted as the set B, one candidate for each vertex in V_R, denoted as the set R. Let $v_x \in V_R \cup V_B$. The candidate representing the vertex v_x is denoted by x. Furthermore, we have a "special" candidate $c^\star$, and a set of $\left\lceil \frac{\alpha}{1-\alpha}|B| \right\rceil + \Delta$ dummy candidates, denoted by D. We assume that $\frac{\alpha}{1-\alpha}|B|$ is not an integer. Thus, the set of candidates $\mathcal{C} = \{c^\star\} \cup B \cup R \cup D$. Note that the candidates in R and B are in one to one correspondence with the vertices of V_R and V_B respectively. We abuse notation to use $N(x)$ (for a candidate $x \in B \cup R$) to represent the set of candidates that correspond to the neighbors of vertex v_x in G, and call the candidates in $N(x)$ as neighbors of x. We fix an ordering of the candidates in B, R and D such that in every vote, candidates in B, R and D appear according to this order. We will use $\overline{B}$, $\overline{R}$ and $\overline{D}$ to denote the reverse of the order in B, R and D respectively.

Voting Profile: The votes can be categorized into three types: (i) for every vertex $v_r \in V_R$, we have two X-votes denoted by X_r and X'_r, (ii) two Y-votes denoted by Y_1 and Y_2, and (iii) $2(\Delta - 1)$ Z-votes, denoted by $Z = \{Z_{i,1}, Z_{i,2} \mid i \in [\Delta - 1]\}$. The voting profile $\Pi_\mathcal{V} = \cup_{r \in R}\{X_r, X'_r\} \cup \{Y_1, Y_2\} \cup_{i \in [\Delta-1]} \{Z_{i,1}, Z_{i,2}\}$, is the multiset of votes in Table 5. Finally, the control set $J = B$. This completes the description of the instance $\mathcal{J}$.

Fig. 4. Reduction from Red-Blue Dominating Set to Copeland$^\alpha$ RCDC.

Table 5
Voting Profile for Copeland$^\alpha$ RCDC. For each $r \in R$, we have votes X_r, X'_r and for each $i \in [\Delta - 1]$, we have $Z_{i,1}, Z_{i,2}$.

X_r	$N(r)$	r	$c^\star$	$B \setminus N(r)$	D	$R \setminus \{r\}$
X'_r	$\overline{R \setminus \{r\}}$	$\overline{N(r)}$	r	$\overline{D}$	$\overline{B \setminus N(r)}$	$c^\star$

Y_1	$c^\star$	D	B	R
Y_2	$c^\star$	$\overline{D}$	$\overline{B}$	$\overline{R}$

$Z_{i,1}$	$c^\star$	D	B	R
$Z_{i,2}$	$\overline{R}$	$\overline{D}$	$\overline{B}$	$c^\star$

Since, for every $b \in B$, $\text{Maximin}(b) < \text{Maximin}(c^\star)$, no candidate from B wins the election $\mathcal{E}'$. Hence, S' is a solution of Maximin RCAC to $\mathcal{J}$.

For the other direction, let $S \subseteq R$ be a set of candidates added to the election $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V}^\mathcal{C})$. Consequently, no candidate from B wins the election $\mathcal{E}' = (\mathcal{C}', \Pi_\mathcal{V}^{\mathcal{C}'})$, where $\mathcal{C}' = \{c^\star\} \cup B \cup S$. We show that S corresponds to a solution of Red-Blue Dominating Set to $\mathcal{I}$.

Suppose S does not correspond to a solution. Then, there is a candidate $b^\star \in B$ such that $N(b^\star) \cap S = \emptyset$. Therefore, for every candidate $r \in S$, candidate $b^\star \notin N(r)$. Hence, $b^\star$ is preferred to candidate r in the vote $X_{r',i}$, for every vertex $v_{r'} \in V_R, i \in [3]$. Also, $b^\star$ is preferred to candidate r in Y_1 and Y_2. So, $N_{\mathcal{E}'}(b^\star, r) = 3|R| + 2$, for every $r \in S$. Using Observation 3.1, for every $b \in B \setminus \{b^\star\}$, $N_{\mathcal{E}'}(b^\star, b) = N_\mathcal{E}(b^\star, b)$ and $N_{\mathcal{E}'}(b^\star, c^\star) = N_\mathcal{E}(b^\star, c^\star)$. Hence, using Lemma 3.1,

$$\begin{aligned}\text{Maximin}(b^\star) &= \min\{N(b^\star, c^\star), \min_{b \in B\setminus\{b^\star\}} N(b^\star, b), \min_{r \in S} N(b^\star, r)\} \\ &= \min\{3|R| + 2, 3|R| + 1, 3|R| + 2\}.\end{aligned}$$

Therefore, $\text{Maximin}(b^\star) = 3|R| + 1$.

For a candidate $r \in S$, we compute the value of $\text{Maximin}(r)$. For every vertex $v_{r'} \in V_R, i \in [3]$, candidate r is preferred to $c^\star$ in the vote $X'_{r',i}$. Additionally, r is preferred to $c^\star$ in the votes Y_1 and Y_2. Therefore, for every $r \in S$, $N(r, c^\star) = 3|R| + 2$. Next, consider a candidate $b \in B$ and a candidate $r \in S$, and the votes that prefer r over b. For every vertex $v_{r'} \in V_R \setminus \{v_r\}, i \in [3]$, in the vote $X'_{r',i}$, candidate r is preferred over candidate b. Additionally, if $b \in N(r)$, then r is preferred to b in the vote $X_{r,i}$ (corresponding to the vertex $v_r \in V_R$), for each $i \in [3]$. Therefore, if $b \in B \cap N(r)$, $N(r, b) = 3|R| + 3$; if $b \in B \setminus N(r)$, $N(r, b) = 3|R|$. Note that for each $r \in S$, candidate $b^\star \notin N(r)$. Hence, $\text{Maximin}(r) = \min\{3|R| + 3, 3|R|\} = 3|R|$. Similarly, we get $\text{Maximin}(c^\star) = 3|R|$. Consequently, $\text{Maximin}(b^\star)$ is strictly more than $\text{Maximin}(c)$ for each candidate $c \in S \cup \{c^\star\}$. Therefore, a candidate in B wins $\mathcal{E}'$, a contradiction. □

Thus, we have proved that Maximin RCAC is W[2]-hard.

3.2. Copeland$^\alpha$: *deletion of candidates*

In this section we show that Copeland$^\alpha$ RCDC is W[2]-hard parameterized by k_{DC}. We give a polynomial time reduction from the Red-Blue Dominating Set problem to Copeland$^\alpha$ RCDC, described in Fig. 4.

Consider the election $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$ constructed in the reduction, i.e., the election where all candidates, registered and unregistered, are present. Intuitively, the X-votes separate the neighbors and non-neighbors of each vertex $v_r \in V_R$. In particular, candidate r is placed between the sets $N(r)$ and $B \setminus N(r)$ in both X_r and X'_r votes. Moreover, the candidates in $R \setminus \{r\}$ appear once at the front (in X'_r) and once at the end (in X_r). Hence, cumulatively, every candidate $b \in B$ beats each one of its neighbors in R, but ties with each of it's non-neighbors in R in the election $\mathcal{E}$. The Y_1 and Y_2 votes help $c^\star$ to beat the candidates in D. The Z-votes ensure that for every candidate $b \in B$ and candidate $d \in D$, b and d are tied in a pairwise election. The purpose of the "dummy" candidates is to increase the score of $c^\star$ so that the difference between the scores of $c^\star$ and every $b \in B$ is less than two; and the number of dummy candidates $|D|$ is set based on this criterion. Next we formally prove the correctness of the reduction. For an election $\mathcal{E}$ and any candidate $c \in \mathcal{C}$, we will use $\text{Copeland}^\alpha_\mathcal{E}(c)$ to denote the Copeland$^\alpha$ score of c in $\mathcal{E}$. The votes computed in Fig. 5 gives the next lemma.

Table 6
The table shows votes received by candidates in the rows in all possible pairwise elections given by the preference profile in Table 5. That is, the cell marked by $(b, c^\star)$ denotes that b receives $2\Delta + |R| - 1$ votes in the pairwise election against $c^\star$. Here, candidates $\{b, b'\} \subseteq B$, $\{r, r'\} \subseteq R$, and $\{d, d'\} \subseteq D$ such that $b \neq b'$, $r \neq r'$, and $d \neq d'$.

	b'	$c^\star$	r'	d'
b	$\|R\| + \Delta$	$2\Delta + \|R\| - 1$	if $b \in N(r')$, $\|R\| + \Delta + 2$ else $\|R\| + \Delta$	$\|R\| + \Delta$
$c^\star$	$\|R\| + 1$	-	$\|R\| + \Delta$	$\|R\| + \Delta + 2$
r	if $b' \in N(r)$, $\|R\| + \Delta - 2$ else $\|R\| + \Delta$	$\|R\| + \Delta$	$\|R\| + \Delta$	$\|R\| + \Delta$
d	$\|R\| + \Delta$	$\|R\| + \Delta - 2$	$\|R\| + \Delta$	$\|R\| + \Delta$

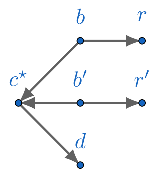

Fig. 5. Majority graph drawn from the values in Table 6 where $\{b, b'\} \in B$, $r \in N(b) \setminus N(b')$, $r' \in N(b') \setminus N(b)$, and $d \in D$.

Lemma 3.3. *Consider the election $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$. Then,*

- *for every candidate $b \in B$, $\text{Copeland}^\alpha_\mathcal{E}(b) = 1 + \Delta + \alpha(|B| - 1 + |D| + |R| - \Delta)$;*
- $\text{Copeland}^\alpha_\mathcal{E}(c^\star) = |D| + \alpha|R|$;
- *for every $r \in R$, $\text{Copeland}^\alpha_\mathcal{E}(r) = \alpha(|B| - |N(r)| + |R| + |D|)$;*
- *for every $d \in D$, $\text{Copeland}^\alpha_\mathcal{E}(d) = \alpha(|B| + |R| + |D| - 1)$.*

Proof. Let $\mathcal{E} = (\mathcal{C}, \Pi_\mathcal{V})$. We calculate the Copeland^α score for every $c \in \mathcal{C}$. Towards this, we compute the number of candidates that are beaten by c and are tied with c in a pairwise election. Recall that for any two distinct candidates c and $c'(\neq c)$, $N_\mathcal{E}(c, c')$ denotes the number of votes that prefer c to c' in $\Pi_\mathcal{V}$. Note that there are $2(|R| + \Delta)$ votes in $\mathcal{E}$. So, for any pair of candidates $c, c' \in \mathcal{C}$, if $N_\mathcal{E}(c, c') > |R| + \Delta$, then c beats c', and if $N_\mathcal{E}(c, c') = |R| + \Delta$, then c and c' are tied. We present the outcome of the pairwise elections, that is, the values of $N_\mathcal{E}(\cdot, \cdot)$, for every pair of candidates in Table 6.

Next, we show how to compute an entry of Table 6. For $r \in R$ and $b \in N(r)$, we will compute $N_\mathcal{E}(b, r)$ (1st row, 3rd column of Table 6) as follows. Candidate b is preferred to r in the votes X_r and X'_r(corresponding to the vertex $v_r \in V_R$). So b gets 2 votes against r from them. For each vertex $v'_r \in V_R \setminus \{v_r\}$, b is preferred to r in $X_{r'}$. Hence, b gets a total of $|R| - 1$ votes from $\cup_{v'_r \in V_R \setminus \{v_r\}} X_{r'}$. Additionally, each vote in $Y \cup Z$-votes, prefers b to r. Hence, in total b gets $2 + |R| - 1 + 2 + \Delta - 1 = |R| + \Delta + 2$ votes. Similarly, we can compute the other entries of the Table 6; we will skip the details of the calculation.

The values in Table 6 immediately give the majority graph shown in Fig. 5. For each $b \in B$, since $N_\mathcal{E}(b, c^\star) > N_\mathcal{E}(c^\star, b)$, b beats $c^\star$. Moreover, b beats every candidate $r \in N(b)$. Recall that the degree of a vertex in V_B is Δ, i.e., $|N(b)| = \Delta$. Therefore, b beats $1 + \Delta$ candidates. For any candidate $c \in \mathcal{C} \setminus (N(b) \cup \{c^\star\})$, we have $N_\mathcal{E}(b, c) = N_\mathcal{E}(c, b)$. Hence, b ties with every candidate in $\mathcal{C} \setminus (N(b) \cup \{c^\star\})$. Therefore, we get that $\text{Copeland}^\alpha{}_\mathcal{E}(b) = 1 + \Delta + \alpha(|B| - 1 + |D| + |R| - \Delta)$. Similar arguments can derive the scores of other candidates. $\square$

Note that deleting candidates do not change the ordering of the remaining candidates in any of the votes. Thus, the outcome of the pairwise election between any two candidates (which are not deleted) is unaltered after deletion. This allows us to use Observation 3.1 in order to compute the scores in a modified election. The following property about a solution of Copeland^α RCDC for the instance $\mathcal{J}$ is crucial in the proof of correctness of the reduction.

Lemma 3.4. *Let $\mathcal{C}' \subseteq \mathcal{C}$, and $|\mathcal{C}'| \leq k_{\text{DC}}$. Then, $\mathcal{C}'$ is a solution of Copeland^α RCDC for $\mathcal{J}$ if and only if for each $b \in B$, $N(b) \cap \mathcal{C}' \neq \emptyset$.*

Proof. Let $\mathcal{E}' = (\mathcal{C} \setminus \mathcal{C}', \Pi_\mathcal{V}^{\mathcal{C} \setminus \mathcal{C}'})$, that is the election after deletion of $\mathcal{C}'$. Let $|\mathcal{C}'| = k' \leq k_{\text{DC}}$. For the reverse direction, let $b \in B$. Since $N(b) \cap \mathcal{C}' \neq \emptyset$, let $r \in \mathcal{C}' \cap N(b)$. We use Observation 3.1 to find the scores in $\mathcal{E}'$. Recall that b beats $c^\star$ and every candidate r in $N(b)$ (see Fig. 5). That is, b beats r. Hence, the score of b reduces by 1 due to deletion of r. Also, deletion of every non-neighbor of b decreases score of b by α. It may be the case that each candidate in $\mathcal{C}_S \setminus \{r\}$ is a non-neighbor of b. So, we have $\text{Copeland}^\alpha_{\mathcal{E}'}(b) \leq \text{Copeland}^\alpha_\mathcal{E}(b) - 1 - \alpha(k' - 1)$. Similarly, $\text{Copeland}^\alpha_{\mathcal{E}'}(c^\star) = \text{Copeland}^\alpha_\mathcal{E}(c^\star) - \alpha k'$.

Table 7
Voting Profile for Copeland$^\alpha$ RCAC. For each $r \in V_R$, we have votes X_r, X'_r.

X_r	$c^\star$	$B \setminus N(r)$	r	$N(r)$	$R \setminus \{r\}$	D_B	d_1	d_2	Y_1	D_B	d_2	B	$c^\star$	R	d_1
X'_r	$\overline{R \setminus \{r\}}$	d_2	r	$\overline{N(r)}$	$\overline{B \setminus N(r)}$	d_1	$c^\star$	$\overline{D_B}$	Y_2	d_1	$c^\star$	$\overline{D_B}$	d_2	$\overline{B}$	$\overline{R}$

Since $|D| = \Delta + \lceil \frac{\alpha}{1-\alpha}|B|\rceil$, it is easy to verify that $\text{Copeland}^\alpha_{\mathcal{E}'}(c^\star) > \text{Copeland}^\alpha_{\mathcal{E}'}(b)$. Hence, no candidate from B wins the election $\mathcal{E}'$. Since $|\mathcal{C}'| \le k_{\text{DC}} = k$, $\mathcal{C}'$ is a solution of Copeland$^\alpha$ RCDC for $\mathcal{J}$.

For the forward direction, suppose towards contradiction, there exists a $b^\star \in B$ such that $N(b^\star) \cap \mathcal{C}' = \emptyset$. We show $b^\star$ has the highest score contradicting the fact that $\mathcal{C}'$ is a solution for Copeland$^\alpha$ RCDC. Recall that $b^\star$ beats $c^\star$ and $N(b^\star)$. We consider two cases based on whether $c^\star \in \mathcal{C}'$.

Case A: $c^\star \in \mathcal{C}'$. Using Observation 3.1 (Fig. 5) score $b^\star$ in $\mathcal{E}'$ decreases by 1. That is, since $N(b^\star) \cap \mathcal{C}' = \emptyset$, we have that $\text{Copeland}^\alpha_{\mathcal{E}'}(b^\star) = \Delta + \alpha(|B| - 1 + |D| + |R| - \Delta - k' - 1)$. Using Observation 3.1, for each $r \in R \setminus \mathcal{C}'$, $\text{Copeland}^\alpha_{\mathcal{E}'}(r) = \alpha(|B| - |N(r)| + |R| + |D| - k')$, and for each $d \in D \setminus \mathcal{C}'$, $\text{Copeland}^\alpha_{\mathcal{E}'}(d) = \alpha(|B| + |R| + |D| - 1 - k')$. Therefore, $\text{Copeland}^\alpha_{\mathcal{E}'}(b^\star) - \text{Copeland}^\alpha_{\mathcal{E}'}(r)$ is $(1-\alpha)\Delta + \alpha|N(r)|$. Since $\alpha < 1$, $b^\star$ has higher score than r. Similarly, we can show $b^\star$ gets higher score than any $d \in D \setminus \mathcal{C}'$. Hence, $b^\star$ has the highest score, contradicting $\mathcal{C}'$ is a solution of Copeland$^\alpha$ RCDC for $\mathcal{J}$.

Case B: $c^\star \notin \mathcal{C}'$. Since $b^\star$ beats $c^\star$ and $N(b^\star)$ and ties with every other candidate, no candidate that is beaten by $b^\star$ is deleted. That is, using Observation 3.1, $\text{Copeland}^\alpha_{\mathcal{E}'}(b^\star) = 1 + \Delta + \alpha(|B| - 1 + |D| + |R| - \Delta - k')$. Notice that the score of each $x \in (R \cup D) \setminus \mathcal{C}'$ is same as in the previous case. Hence, $b^\star$ gets higher score than any x in $(R \cup D) \setminus \mathcal{C}'$. We show that the score of $b^\star$ is higher than that of $c^\star$. Let $|\mathcal{C}' \cap D|$ be denoted by k_d (possibly, 0). Then, using Observation 3.1, $\text{Copeland}^\alpha_{\mathcal{E}}(c^\star) = |D| - k_d + \alpha(|R| - k' + k_d)$. Then,

$$\begin{aligned}\text{Copeland}^\alpha_{\mathcal{E}'}(b^\star) - \text{Copeland}^\alpha_{\mathcal{E}'}(c^\star) &= 1 - \alpha + (1-\alpha)\Delta + (1-\alpha)k_d + \alpha|B| - (1-\alpha)|D| \\ &\ge (1-\alpha)(k_d + 1) > 0.\end{aligned}$$

The first inequality follows from the fact that $|D| = \Delta + \lceil \frac{\alpha}{1-\alpha}|B|\rceil$. The second inequality follows because $\alpha < 1$. Hence, $b^\star$ has higher score than $c^\star$. Hence, $b^\star$ has highest score, contradicting $\mathcal{C}'$ is a solution of Copeland$^\alpha$ RCDC for $\mathcal{J}$. □

The above lemma implies if $\mathcal{C} = \emptyset$, that is, if no candidate is deleted then there exists a $b \in B$ who wins (see Fig. 5). Therefore, we proved that $\mathcal{J}$, created in the reduction, is not a trivial YES instance. The following equivalence can be proved using Lemma 3.4 which completes the proof that Copeland$^\alpha$ RCDC is W[2]-hard.

Lemma 3.5. *If $\mathcal{I}$ is a* YES*-instance of* Red-Blue Dominating Set, *then $\mathcal{J}$ is a* YES*-instance of* Copeland$^\alpha$ RCDC.

Proof. Since $\mathcal{I}$ is a YES-instance of Red-Blue Dominating Set, there exists a subset $S \subseteq V_R$ of size at most k that dominates all the vertices in V_B. Let $|S| = k$. We delete the set of candidates $\mathcal{C}_S$ corresponding to S. That is, for every vertex $v_r \in S$, we delete the candidate r. Since S is a dominating set in G, for every candidate $b \in B$, the set $\mathcal{C}_S$ has a non-empty intersection with its neighborhood. Hence, using Lemma 3.4, $\mathcal{C}_S$ is a solution for $\mathcal{J}$, proving that $\mathcal{J}$ is a YES-instance. □

For the converse, we prove the following.

Lemma 3.6. *If $\mathcal{J}$ is a* YES*-instance of* Copeland$^\alpha$ RCDC, *then $\mathcal{I}$ is a* YES*-instance of* Red-Blue Dominating Set.

Proof. Let $\mathcal{J}$ be a YES-instance of Copeland$^\alpha$ RCDC. Thus, there exists a subset of candidates $\mathcal{C}_S \subseteq \mathcal{C} \setminus B$ such that $|\mathcal{C}_S| \le k_{DC}$ and a candidate in $\mathcal{C} \setminus (B \cup \mathcal{C}_S)$ has highest score in $\mathcal{E}'$, where $\mathcal{E}' = (\mathcal{C} \setminus B \setminus \mathcal{C}_S, \Pi_{\mathcal{V}}^{\mathcal{C} \setminus \mathcal{C}_S})$. To complete the proof, next, we show that the set $S = \{v_r \in V_R \mid r \in \mathcal{C}_S\}$ is a dominating set for V_B in G. Since $\mathcal{C}_S$ is a solution of Copeland$^\alpha$ RCDC for $\mathcal{J}$, from Lemma 3.4, $N(b) \cap \mathcal{C}_S \neq \emptyset$ for every $b \in B$. Recall that $N(b)$ is the set of candidates that correspond to the neighbors of vertex v_b in V_B. Hence, S dominates V_B. Since $|S| \le k_{DC}$, we have that $|S| \le k$ i.e., it is a solution for Red-Blue Dominating Set for (G, k). □

3.3. Copeland$^\alpha$: *addition of candidates*

In this section we show that Copeland$^\alpha$ RCAC is W[2]-hard parameterized by K_{AC}. Towards this, we give a polynomial time reduction from the Red-Blue Dominating Set problem as described in Fig. 6.

Consider the election $\mathcal{E} = (\mathcal{C}, \Pi_{\mathcal{V}})$ constructed in Fig. 6, i.e., the election where all the candidates are present. Intuitively speaking, the X-votes separate the neighbors and non-neighbors of each vertex $v_r \in V_R$. In particular, candidate r is placed before it's neighbors $N(r)$ in both X_r and X'_r votes. Moreover, the candidates in $R \setminus \{r\}$ appear once at the front (in X'_r) and once at the end (in X_r). Hence, cumulatively, every candidate $b \in B$ is beaten by each one of it's neighbors in R, but ties with each of it's non-neighbors in R in the election $\mathcal{E}$. The correctness proof is similar to the proof in the previous section.

The Reduction: Let $\mathcal{I} = (G = (V_R, V_B), k)$ denote an instance of Red-Blue Dominating Set. We construct an instance $\mathcal{J} = (\mathcal{C}, \Pi_{\mathcal{V}}, \mathcal{A} = \emptyset, J, k_{\text{AC}} = k, k_{\text{DC}} = 0, \text{Copeland}^{\alpha})$, where $0 \le \alpha \le 1$, of Copeland$^{\alpha}$ RCDC as follows.

Candidates: We have five types of candidates. To every vertex $v_b \in V_B$, we associate a candidate $b \in B$, and to every vertex $v_r \in V_R$, a candidate $r \in R$. Thus, $|B| = |V_B|$ and $|R| = |V_R|$. We introduce a spacial candidate $c^{\star}$, a set of $|B|$ dummy candidates, denoted by D_B, and two more dummy candidates d_1 and d_2. Thus, the set of candidates $\mathcal{C} = \{c^{\star}\} \cup B \cup R \cup \{d_1, d_2\} \cup D_B$.
Note that the candidates in R and B are in one-to-one correspondence with the vertices of V_R and V_B, respectively. We abuse notation to use $N(x)$ (for a candidate $x \in B \cup R$) to represent the set of candidates that correspond to the neighbors of the vertex v_x in G. Moreover, we will refer to the candidates in $N(x)$ as neighbors of the candidate x. We fix an ordering of the candidates in B and R such that in every vote, candidates in B and R appear according to this order. We will use $\overline{B}$ and $\overline{R}$ to denote the reverse of the order in B and the order in R, respectively.

Voting Profile: We have two X-votes for each vertex $v_r \in V_R$, denoted by X_r and X'_r; two votes Y_1 and Y_2. Thus, the voting profile $\Pi_{\mathcal{V}} = \cup_{r \in R}\{X_r, X'_r\} \cup \{Y_1, Y_2\}$, is the multiset of votes in Table 7. Finally, $\mathcal{A} = R$ and the control set $J = B$. This completes the description of the instance $\mathcal{J}$.

Fig. 6. Reduction from Red-Blue Dominating Set to Copeland$^{\alpha}$ RCAC.

Lemma 3.7. *Consider the election* $\mathcal{E} = (\mathcal{C} \setminus R, \Pi_{\mathcal{V}}^{\mathcal{C} \setminus R})$. *Then,*

- *for every candidate* $b \in B$, $\text{Copeland}^{\alpha}_{\mathcal{E}}(b) = \alpha(|B| - 1) + \alpha + 1 + |B|$;
- $\text{Copeland}^{\alpha}_{\mathcal{E}}(c^{\star}) = \alpha|B| + 2\alpha + |B|$;
- *for every* $d \in D_B$, $\text{Copeland}^{\alpha}{}_{\mathcal{E}}(d) = \alpha|B| + \alpha + 1$;
- $\text{Copeland}^{\alpha}_{\mathcal{E}}(d_1) = \alpha|B| + \alpha$;
- $\text{Copeland}^{\alpha}_{\mathcal{E}}(d_2) = \alpha + |B|$.

Proof. First, we calculate the Copeland$^{\alpha}$ score for every candidate $b \in B$ in election $\mathcal{E}$. Towards this, we compute the number of candidates that are beaten by b and are tied with b in a pairwise election. Recall that for a pair of distinct candidates c and $c'(\neq c)$, $N_{\mathcal{E}}(c, c')$ denotes the number of votes that prefer c to c' in $\Pi_{\mathcal{V}}$. Note that in the election $\mathcal{E}$, there are total $2|R| + 2$ votes in $\mathcal{E}$. So, for any pair of candidates $c, c' \in \mathcal{C}$, if $N_{\mathcal{E}}(c, c') > |R| + 1$, then c beats c'; and if $N_{\mathcal{E}}(c, c') = |R| + 1$, then c and c' are tied.

Let $\{b, b'\} \subseteq B$. For a vertex $v_r \in V_R$, consider the corresponding votes X_r and X'_r. Observer that the ordering of the candidates in B in X_r is reverse of the ordering of those candidates in X'_r. Hence if b is preferred to b' in X_r (Y_1), then the opposite is true in X'_r. Furthermore, the same is true for b and b' in Y_1 and Y_2 as well. Thus, $N_{\mathcal{E}}(b, b') = N_{\mathcal{E}}(b', b) = |R| + 1$ implying that b and b' are tied. Similar argument implies $c^{\star}$ is tied with every candidate $b \in B$. Note that in all of the X-votes b is preferred to any candidate in $D_B \cup \{d_1\}$. Hence, b beats each candidate in $D_B \cup \{d_1\}$. We note that d_2 is preferred to each candidate $b \in B$, the X'_r votes (for any voter $v_r \in V_R$) as well as both the Y-votes. Hence, d_2 beats b. Hence, we get $\text{Copeland}^{\alpha}_{\mathcal{E}}(b) = \alpha(|B| - 1) + \alpha + 1 + |B|$. Similar arguments can derive the scores of other candidates. □

Observe that the outcome of the pairwise election between any two candidates in B is unaltered after addition of any subset of candidates in R. This allows us to use Observation 3.1 in order to compute the scores in a modified election. The following property about a solution of Copeland$^{\alpha}$ RCAC for the instance $\mathcal{J}$ is crucial in the proof of correctness of the reduction.

Lemma 3.8. *Let* $R' \subseteq R$, *and* $|R'| \le k_{\text{AC}}$. *The set* R' *is a solution of* Copeland$^{\alpha}$ RCDC *for the instance* $\mathcal{J}$ *if and only if for each* $b \in B$, $N(b) \cap R' \neq \emptyset$.

Proof. Let $\mathcal{E}' = ((\mathcal{C} \setminus R) \cup R', \Pi_{\mathcal{V}}^{(\mathcal{C} \setminus R) \cup R'})$, denoting the election after addition of candidates in R'. Let $|R'| = k' \le k_{\text{AC}}$. For the reverse direction, let $\widehat{b} \in B$. Since $N(\widehat{b}) \cap R' \neq \emptyset$, let $\widehat{r} \in N(b) \cap R'$. Since $\widehat{b} \in N(\widehat{r})$, the candidate $\widehat{r}$ is preferred to $\widehat{b}$ in $X_{\widehat{r}}$ as well as in X'_r, where $v_r \in V_R$. Moreover, $\widehat{b}$ is preferred to $\widehat{r}$ in both Y_1 and Y_2. Consequently, $N_{\mathcal{E}'}(\widehat{r}, \widehat{b}) = |R| + 1$, i.e., $\widehat{r}$ and $\widehat{b}$ are tied. For each candidate $r \in R \setminus N(\widehat{b})$, $\widehat{b}$ is preferred to r in X_r, where v_r is a vertex in V_R, but in both Y_1 and Y_2, candidate $\widehat{b}$ is preferred to r. Therefore, $\widehat{b}$ beats every candidate r that is not it's neighbor. Thus, Observation 3.1 yields that $\text{Copeland}^{\alpha}_{\mathcal{E}'}(\widehat{b}) = \alpha(|B| - 1) + \alpha + 1 + |B| + \alpha|N(\widehat{b}) \cap R'| + |R' \setminus N(\widehat{b})|$. Note that $|R' \setminus N(\widehat{b})| = |R'| - |R' \cap N(\widehat{b})|$. Hence, $\text{Copeland}^{\alpha}_{\mathcal{E}'}(\widehat{b}) = \alpha(|B| - 1) + \alpha + 1 + |B| - (1 - \alpha)|N(\widehat{b}) \cap R'| + |R'|$. Since $|N(\widehat{b}) \cap R'| \ge 1$, $\text{Copeland}^{\alpha}_{\mathcal{E}'}(\widehat{b}) \le \alpha(|B| - 1) + \alpha + 1 + |B| + \alpha + |R'| - 1 = \alpha|B| + \alpha + |B| + |R'|$. It is easy to check that $\text{Copeland}^{\alpha}_{\mathcal{E}'}(c^{\star}) = \alpha|B| + 2\alpha + |B| + |R'|$. Hence, no candidate from B gets the highest score in the election $\mathcal{E}'$. Since $|R'| \le k_{\text{DC}} = k$, R' is a solution of Copeland$^{\alpha}$ RCDC for $\mathcal{J}$.

For the forward direction, suppose towards contradiction, there exists a $b^{\star} \in B$ such that $N(b^{\star}) \cap R' = \emptyset$. Therefore, $\text{Copeland}^{\alpha}_{\mathcal{E}'}(b^{\star}) = \alpha(|B| - 1) + \alpha + 1 + |B| + |R'|$. Since $\alpha < 1/2$, $\text{Copeland}^{\alpha}_{\mathcal{E}'}(b^{\star}) > \text{Copeland}^{\alpha}_{\mathcal{E}'}(c^{\star})$.

Observe that for each candidate $r \in R'$, similar arguments as in Lemma 3.7 yield that r ties with every other $r' \in R' \setminus \{r\}$ and beat all the dummy candidates in $D_B \cup \{d_1\}$. Without loss of generality we may assume that $|R| \ge 3$, since otherwise the instance can be solved in polynomial time. The candidate d_2 is preferred to r in X'_r, Y_1 and Y_2. Since $|R| \ge 3$, d_2 is beaten by r. Therefore, $\text{Copeland}^{\alpha}_{\mathcal{E}'}(r) = \alpha(|R'| - 1) + \alpha|N(r)| + |B| + 2$ which is at most $\alpha(|R'| - 1) + \alpha|B| + |B| + 2$ because $N(r) \subseteq B$. Since $\alpha < 1$, $\text{Copeland}^{\alpha}_{\mathcal{E}'}(b^{\star}) > \text{Copeland}^{\alpha}_{\mathcal{E}'}(r)$, for every $r \in R'$. Note that in the election $\mathcal{E}'$, the dummy candidates are beaten by every $r \in R'$. Hence, they have the same score in the elections $\mathcal{E}'$ and $\mathcal{E}$. Hence, $b^\star$ has highest score, contradiction to R' being a solution of Copeland$^\alpha$ RCAC for the instance $\mathcal{J}$. $\square$

As in the previous section, the above lemma shows the following equivalence and this completes the proof.

Lemma 3.9. *$\mathcal{I}$ is a* Yes*-instance of* Red-Blue Dominating Set *if and only if $\mathcal{J}$ is a* Yes*-instance of* Copeland$^\alpha$ RCDC.

4. Algorithm for copeland RCDC

In this section we present an algorithm that solves Copeland$^\alpha$ RCDC for the special case where there are no ties between pairs of candidates, and thereby prove Theorem 3. Note that this case subsumes the one where the number of voters is odd. We recast our problem in graph theoretic terms and is formally presented below. The problem is reduced to deciding the existence of a vertex with the maximum out-degree in an appropriately defined subtournament, [5]. But prior to that we will introduce some definitions.

For a tournament $\mathscr{T} = (V, A)$ let $\widehat{A} \subseteq A$ denote a smallest feedback arc set in $\mathscr{T}$. We can find $\widehat{A}$ in time $3^{\mathcal{O}(k)}$ using a simple 3-way branching algorithm that exploits the property that a tournament has a (directed) cycle if and only if it has a (directed) triangle. We define the *affected set* (of vertices) in $\mathscr{T}$, denoted by $V_{\widehat{A}}$, to be the subset of vertices incident on the edges in $\widehat{A}$. Clearly, $|V_{\widehat{A}}| \le 2k_{\text{FAS}}$. From now on, we will assume that we have the affected set $V_{\widehat{A}}$ at our disposal. We solve the following problem.

CT RCDC	**Parameter:** $k_{\text{DC}} + k_{\text{FAS}}$
Input: A tournament $\mathscr{T} = (V, A)$, a subset $J \subseteq V$, and positive integers k_{DC} and k_{FAS}.	
Question: Does there exist a subset $S \subseteq V \setminus J$, $|S| \le k_{\text{DC}}$ such that there is a vertex $v \in V \setminus (S \cup J)$ with the maximum out-degree in the (induced) tournament $\mathscr{T}[V \setminus S]$?	

An algorithm for CT RCDC will invoke as a subroutine an algorithm for a related problem, Disjoint Copeland Tournament (DCT), described below.

Lemma 4.1. *Let $(\mathscr{T} = (V, A), J, k_{\text{DC}}, k_{\text{FAS}})$ be an instance of* CT RCDC*. Then, the following is true: $(\mathscr{T}, J, k_{\text{DC}}, k_{\text{FAS}})$ is a* Yes*-instance if and only if there exist subset $F \subseteq V_{\widehat{A}} \setminus J$, subset $R \subseteq V \setminus (V_{\widehat{A}} \cup J)$ where $|F| + |R| = k_{\text{DC}}$, and vertex $\widehat{v} \in V \setminus (J \cup F \cup R)$ such that $\widehat{v}$ has maximum out-degree in the tournament $\mathscr{T}[V \setminus (F \cup R)]$.*

DCT	**Parameter:** $k_{\text{DC}} + k_{\text{FAS}}$
Input: A tournament $\mathscr{T} = (V, A)$, a subset $J \subseteq V$, a subset $F \subseteq V_{\widehat{A}} \setminus J$ of the affected set, a special vertex $\widehat{v} \in V \setminus J$, and positive integers k_{DC} and k_{FAS}.	
Question: Does there exist a subset $R \subseteq V \setminus (V_{\widehat{A}} \cup J \cup \{\widehat{v}\})$ such that $|R| \le k_{\text{DC}} - |F|$ and $\widehat{v}$ has the maximum out-degree in the (induced) tournament $\mathscr{T}[V \setminus (F \cup R)]$	

For the ease of exposition of the algorithm for DCT and its subsequent analysis, we will begin by introducing some terminology.

Coloring and ordering the vertices: Consider an instance $(\mathscr{T}, J, F, \widehat{v}, k_{\text{DC}}, k_{\text{FAS}})$ of DCT. We will refer to the vertices in $V \setminus (V_{\widehat{A}} \cup J)$ as the **red vertices** and those in $J \cup (V_{\widehat{A}} \setminus F)$ as the **blue vertices**. Thus, the blue and red vertices define a partition of the vertex set $V \setminus F$. Intuitively speaking, the solution set S for the instance $(\mathscr{T}, J, F, \widehat{v}, k_{\text{DC}}, k_{\text{FAS}})$ of DCT contains a subset of the red vertices but none of the blue vertices. In other words, the red vertices can be deleted from the tournament $\mathscr{T}_{-F} = \mathscr{T}[V \setminus F]$ but not the blue vertices.

A *topological ordering* of a directed graph is a linear ordering of its vertices such that for every directed edge uv from vertex u to vertex v, u appears to the left of v in the ordering. Let $\widehat{A}_{-F} \subseteq \widehat{A}$ denote the subset of arcs in $\widehat{A}$ that exist in the tournament $\mathscr{T}_{-F}$. We have a *topological order* of the vertices in $\mathscr{T}_{-F}$ obtained from the directed acyclic graph created by reversing the arcs of $\widehat{A}_{-F}$. Thus, we have a linear ordering, σ of the vertices in $\mathscr{T}_{-F}$, such that for any arc $ij \in A \setminus \widehat{A}_{-F}$, i is to the left of j, denoted by $i <_\sigma j$. For an arc $ji \in \widehat{A}_{-F}$ if $i <_\sigma j$, then ji is called a *back arc*.

4.1. Algorithm for DCT

Let $\mathcal{I} = (\mathscr{T}, J, F, \widehat{v}, k_{\text{DC}}, k_{\text{FAS}})$ be an instance of DCT. The algorithm for DCT is quite simple: We define $k_1 = k_{\text{DC}} - |F|$ and $k = |V_{\widehat{A}}| - |F|$. Consider the first $k_1 + k + 1$ red vertices in σ, denoted by $\mathcal{R}$. For *every* subset of the red vertices in $\mathcal{R}$ of size k_1, denoted by R, algorithm for DCT checks whether $\widehat{v}$ has the maximum out-degree in $\mathscr{T}[V \setminus (F \cup R)]$. If yes, then the algorithm returns Yes. Else, if none of subsets of $\mathcal{R}$ yields a yes-answer, then the algorithm outputs No. The time complexity is calculated in a straightforward manner: Since $\binom{k_1+k+1}{k_1} \le 2^{k_1+k+1} \le 2^{k_{\text{DC}}+k_{\text{FAS}}+1}$, the aforementioned algorithm runs in time $2^{(k_{\text{DC}}+k_{\text{FAS}})}$poly(n). The proof of correctness is presented in Lemmata 4.2 and 4.3.

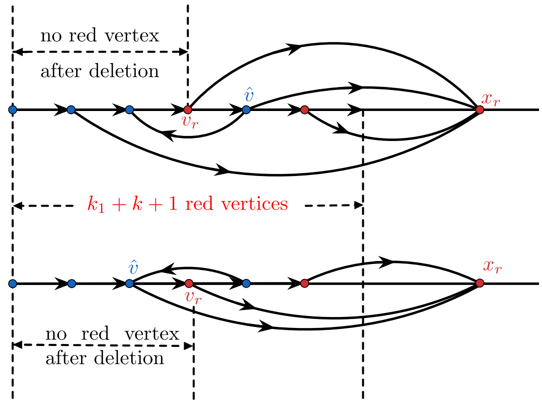

Fig. 7. Depicts proof of Lemma 4.2 when $\hat{v}$ is blue. Two topological ordering and relative position of v_r, $\hat{v}$ and x_r is shown, based on whether v_r appears before $\hat{v}$ or after $\hat{v}$.

Lemma 4.2. *Suppose that $\mathcal{I} = (\mathscr{T}, J, F, \widehat{v}, k_{\mathrm{DC}}, k_{\mathrm{FAS}})$ is a* Yes*-instance of* DCT*. Then, in the graph $\mathscr{T}[V \setminus F]$, there are at most $k_1 + k + 1$ red vertices that are to the left of $\widehat{v}$, where $k_1 = k_{\mathrm{DC}} - |F|$ and $k = |V_{\widehat{A}}| - |F|$. Moreover, there exists a minimal subset of red vertices, denoted by S, that lies among the first $k_1 + k + 1$ red vertices such that $|R| \le k_1$ and $\widehat{v}$ has the maximum out-degree in $\mathscr{T}[V \setminus (F \cup S)]$.*

Proof. Let $S \subseteq V \setminus (J \cup V_{\widehat{A}})$ denote a subset of red vertices such that $|S| \le k_{\mathrm{DC}} - |F|$, and $\widehat{v}$ has the maximum degree in $\mathscr{T}[V \setminus (F \cup J)]$. Suppose that in the graph $\mathscr{T}[V \setminus F]$ there are at least $k_1 + k + 2$ red vertices to the left of $\widehat{v}$. Thus, it follows that in the graph $\mathscr{T}[V \setminus (F \cup S)]$, there exists at least $k + 2$ red vertices to the left of $\widehat{v}$. Among the red vertices in $V \setminus (F \cup S)$, we use v_r to denote the left most red vertex in the ordering σ. Thereby implying that there are at least $k+1$ red vertices between v_r and $\widehat{v}$ in $\mathscr{T}[V \setminus (F \cup S)]$. From now onwards, any reference to "left" or "right" should be understood to be in reference to the ordering σ.

For any vertex x, we will use $\mathrm{Red}_L(x)$ and $\mathrm{Red}_R(x)$ ($\mathrm{Blue}_L(x)$ and $\mathrm{Blue}_R(x)$) to denote the set of red (blue) vertices to the left and right of x, respectively. Thus, in the graph $\mathscr{T}[V \setminus (F \cup S)]$, $\delta^+(\widehat{v}) \le \mathrm{Red}_R(\widehat{v}) + \mathrm{Blue}_R(\widehat{v}) + k$ because $\widehat{v}$ may have at most k out-neighbors to its left. However, we note that in the graph $\mathscr{T}[V \setminus (F \cup S)]$, $\delta^+(v_r) \ge \mathrm{Red}_R(\widehat{v}) + k + 1 + \mathrm{Blue}_R(\widehat{v})$, a contradiction to the property that $\widehat{v}$ has the maximum degree in $\mathscr{T}[V \setminus (F \cup S)]$.

It is not hard to see that if $\widehat{v}$ is a red vertex then it must be the $k_1 + 1$st vertex in σ, and the set S contains only those red vertices that appear to the left of $\widehat{v}$. Thus, the second property only remains to be proved for the case that $\widehat{v}$ is a blue vertex.

For the sake of contradiction, let S denote a solution for the instance $\mathcal{I}$ such that there is a red vertex $x_r \in S$ which is not among the first $k_1 + k + 1$ red vertices in σ (refer to Fig. 7). Note that without loss of generality we may assume that S is minimal.

We begin by observing that a vertex in $V \setminus (F \cup S)$ that is to the right of x_r has the exact same out-degree in both $\mathscr{T}[V \setminus (F \cup S)]$ and $\mathscr{T}[V \setminus (F \cup S \setminus \{x_r\})]$. Hence, the out-degree of such a vertex cannot exceed the out-degree of $\widehat{v}$ in $\mathscr{T}[V \setminus (F \cup S \setminus \{x_r\})]$. Thus, we only need to argue about the vertices that are to the left of x_r. Let v denote an arbitrary vertex in $V \setminus (F \cup S)$ to the left of x_r that is distinct from $\widehat{v}$. We refer to Fig. 7 for illustration. The fact that the out-degree of $\widehat{v}$ is at least the out-degree of v in the graph $\mathscr{T}[V \setminus (F \cup S)]$ implies that the same relation holds in the graph $\mathscr{T}[V \setminus (F \cup S \setminus \{x_r\})]$ because x_r is an out-neighbor of both. This completes our argument that $\widehat{v}$ has the maximum out-degree in the $\mathscr{T}[V \setminus (F \cup S \setminus \{x_r\})]$ thereby contradicting the minimality of S. $\square$

Conversely, the following holds for the instance $\mathcal{I} = (\mathscr{T}, J, F, \widehat{v}, k_{\mathrm{DC}}, k_{\mathrm{FAS}})$ of DCT.

Lemma 4.3. *Suppose that S is a subset of red vertices and that $\widehat{v}$ has the maximum out-degree in $\mathscr{T}[V \setminus (F \cup S)]$. Then, $\mathcal{I}$ is a* Yes*-instance of* DCT.

Thus, we may conclude that there exists an algorithm for CT RCDC that invokes the algorithm for DCT $2^{|V_{\widehat{A}}|} \cdot |V|$ times, thereby requiring time $2^{\mathcal{O}(k_{\mathrm{DC}} + k_{\mathrm{FAS}})}\mathrm{poly}(n)$. This completes the proof of Theorem 3.

5. Conclusion

in this paper, we substantially broadened the scope of research of Resolute Destructive Control where J is a set. In particular, we studied two parameters, namely, the number of candidates to delete, and to add. We also introduced a structural parameter that depends on the majority graph of the election. We derived lower bounds and hardness results for three central voting correspondences under every possible tie-breaking scheme. Additionally, we designed one parameterized algorithm. Our work gives rise to multiple research avenues.

First, we can ask what is the parameterized complexity of Resolute Control, where J is a set, with respect to other voting correspondences such as Bucklin and Approval. Moreover, unlocking the potential in studying voting problems in terms of graphs and thereby parameterizing their structural properties appears tantalizing. Are Borda RCDC (AC), Copeland$^{\alpha}$ RCAC, and Maximin RCAC FPT with respect to $k_X + k_{\mathrm{FAS}}$ (or k_{FAS} alone) on tournaments? What can be said in the case where the majority graph is not a tournament? Moreover, what structural parameters apart from k_{FAS} are of interest for these problems?

Lastly, we note that we believe that W[2]-hardness results are not a dead-end, in fact we want to know, do these problems admit *parameterized approximation algorithms* with good approximation ratios?; what are the best exact exponential-time algorithms possible?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] J.J. Bartholdi III, C.A. Tovey, M.A. Trick, The computational difficulty of manipulating an election, Soc. Choice Welf. 6 (3) (1989) 227–241.
[2] J.J. Bartholdi III, C.A. Tovey, M.A. Trick, How hard is it to control an election?, Math. Comput. Model. 16 (8–9) (1992) 27–40.
[3] D. Baumeister, G. Erdélyi, E. Hemaspaandra, L.A. Hemaspaandra, J. Rothe, Handbook on Approval Voting. Computational Aspects of Approval Voting, Springer, Berlin, 2010.
[4] N. Betzler, R. Bredereck, J. Chen, R. Niedermeier, Studies in computational aspects of voting, in: The Multivariate Algorithmic Revolution and Beyond, Springer, 2012, pp. 318–363.
[5] N. Betzler, J. Uhlmann, Parameterized complexity of candidate control in elections and related digraph problems, Theor. Comput. Sci. 410 (52) (2009) 5425–5442.
[6] N. Betzler, H.L. Bodlaender, R. Bredereck, R. Niedermeier, J. Uhlmann, On making a distinguished vertex of minimum degree by vertex deletion, Algorithmica 68 (3) (2014) 715–738.
[7] R. Bredereck, J. Chen, P. Faliszewski, J. Guo, R. Niedermeier, G.J. Woeginger, Parameterized algorithmics for computational social choice, Tsinghua Sci. Technol. 19 (4) (2014) 358.
[8] R. Bredereck, J. Chen, P. Faliszewski, A. Nichterlein, R. Niedermeier, Prices matter for the parameterized complexity of shift bribery, in: 28th AAAI, 2014, pp. 1398–1404.
[9] R. Bredereck, J. Chen, S. Hartung, S. Kratsch, R. Niedermeier, O. Suchý, G.J. Woeginger, A multivariate complexity analysis of lobbying in multiple referenda, J. Artif. Intell. Res. 50 (2014) 409–446.
[10] R. Bredereck, P. Faliszewski, A. Kaczmarczyk, R. Niedermeier, P. Skowron, N. Talmon, Robustness among multiwinner voting rules, in: 10th SAGT, 2017, pp. 80–92.
[11] R. Bredereck, P. Faliszewski, R. Niedermeier, N. Talmon, Complexity of shift bribery in committee elections, in: 30th AAAI, 2016, pp. 2452–2458.
[12] R. Bredereck, P. Faliszewski, R. Niedermeier, N. Talmon, Large-scale election campaigns: combinatorial shift bribery, J. Artif. Intell. Res. 55 (2016) 603–652.
[13] L. Bulteau, J. Chen, P. Faliszewski, R. Niedermeier, N. Talmon, Combinatorial voter control in elections, Theor. Comput. Sci. 589 (2015) 99–120.
[14] J. Chen, P. Faliszewski, R. Niedermeier, N. Talmon, Elections with few voters: candidate control can be easy, J. Artif. Intell. Res. 60 (2017) 937–1002.
[15] V. Conitzer, T. Sandholm, J. Lang, When are elections with few candidates hard to manipulate?, J. ACM 54 (3) (2007).
[16] M. Cygan, F. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, S. Saurabh, Parameterized Algorithms, Springer, 2015.
[17] P. Dey, N. Misra, Y. Narahari, Frugal bribery in voting, in: 30th AAAI, 2016, pp. 2466–2472, http://www.aaai.org/ocs/index.php/AAAI/AAAI16/paper/view/12133.
[18] P. Dey, N. Misra, Y. Narahari, Kernelization complexity of possible winner and coalitional manipulation problems in voting, Theor. Comput. Sci. 616 (2016) 111–125.
[19] P. Dey, N. Misra, Y. Narahari, Parameterized dichotomy of choosing committees based on approval votes in the presence of outliers, in: 16th AAMAS, 2017, pp. 42–50.
[20] R. Diestel, Graph Theory, 4th edition, Graduate Texts in Mathematics, vol. 173, Springer, 2012.
[21] B. Dorn, I. Schlotter, Having a hard time? Explore parameterized complexity, in: U. Endriss (Ed.), Trends in Computational Social Choice, AI Access, 2017.
[22] R. Downey, M.R. Fellows, Fundamentals of Parameterized Complexity, Texts in Computer Science, Springer, 2013.
[23] G. Erdélyi, M. Nowak, J. Rothe, Sincere-strategy preference-based approval voting fully resists constructive control and broadly resists destructive control, Math. Log. Q. 55 (4) (2009) 425–443.
[24] P. Faliszewski, E. Hemaspaandra, L.A. Hemaspaandra, Using complexity to protect elections, Commun. ACM 53 (11) (2010) 74–82.
[25] P. Faliszewski, R. Niedermeier, Parameterization in computational social choice, in: Encyclopedia of Algorithms, 2016, pp. 1516–1520.
[26] P. Faliszewski, J. Rothe, H. Moulin, Control and Bribery in Voting, Cambridge University Press, 2016, pp. 146–168.
[27] P. Faliszewski, E. Hemaspaandra, L.A. Hemaspaandra, Multimode control attacks on elections, J. Artif. Intell. Res. 40 (2011) 305–351.
[28] S. Gupta, S. Roy, S. Saurabh, M. Zehavi, When rigging a tournament, let greediness blind you, in: 27th IJCAI, 2018, pp. 275–281.
[29] S. Gupta, S. Roy, S. Saurabh, M. Zehavi, Winning a tournament by any means necessary, in: 27th IJCAI 2018, 2018, pp. 282–288.
[30] E. Hemaspaandra, L.A. Hemaspaandra, J. Rothe, Anyone but him: the complexity of precluding an alternative, Artif. Intell. 171 (5–6) (2007) 255–285.
[31] J.F. Laslier, Tournament Solutions and Majority Voting. Studies in Economic Theory, Springer Berlin Heidelberg, 2011, https://books.google.co.in/books?id=2bJ_MQEACAAJ.
[32] H. Liu, H. Feng, D. Zhu, J. Luan, Parameterized computational complexity of control problems in voting systems, Theor. Comput. Sci. 410 (27–29) (2009) 2746–2753.
[33] H. Liu, D. Zhu, Parameterized complexity of control problems in maximin election, Inf. Process. Lett. 110 (10) (2010) 383–388.
[34] A. Loreggia, N. Narodytska, F. Rossi, K.B. Venable, T. Walsh, Controlling elections by replacing candidates: theoretical and experimental results, in: Workshops at the Twenty-Eighth AAAI Conference on Artificial Intelligence, 2014.

[35] R. Meir, A.D. Procaccia, J.S. Rosenschein, A. Zohar, Complexity of strategic behavior in multi-winner elections, J. Artif. Intell. Res. 33 (2008) 149–178.
[36] R. Meir, A.D. Procaccia, J.S. Rosenschein, A broader picture of the complexity of strategic behavior in multi-winner elections, in: 7th AAMAS, 2008, pp. 991–998.
[37] S. Obraztsova, E. Elkind, Optimal manipulation of voting rules, in: 11th AAMAS, 2012, pp. 619–626.
[38] S. Obraztsova, Y. Zick, E. Elkind, On manipulation in multi-winner elections based on scoring rules, in: 12th AAMAS, 2013, pp. 359–366.
[39] A.D. Procaccia, J.S. Rosenschein, A. Zohar, Multi-winner elections: complexity of manipulation, control and winner-determination, in: 16th IJCAI, vol. 7, 2007, pp. 1476–1481.
[40] M.S. Ramanujan, S. Szeider, Rigging nearly acyclic tournaments is fixed-parameter tractable, in: 31st AAAI, 2017, pp. 3929–3935.
[41] J. Rothe, L. Schend, Challenges to complexity shields that are supposed to protect elections against manipulation and control: a survey, Ann. Math. Artif. Intell. 68 (1) (2013) 161–193.
[42] Y. Yang, D. Dimitrov, How hard is it to control a group?, in: 17th AAMAS, vol. 32(5), 2018, pp. 672–692.
[43] Y. Yang, J. Guo, Controlling elections with bounded single-peaked width, in: AAMAS, 2014, pp. 629–636.
[44] Y. Yang, J. Wang, Anyone but them: the complexity challenge for a resolute election controller, in: 16th AAMAS, 2017.