LONG TIME WELL-POSEDNESS OF WHITHAM-BOUSSINESQ SYSTEMS

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Abstract. Consideration is given to three different full dispersion Boussinesq systems arising as asymptotic models in the bi-directional propagation of weakly nonlinear surface waves in shallow water. We prove that, under a non-cavitation condition on the initial data, these three systems are well-posed on a time scale of order $\mathcal{O}(\frac{1}{\varepsilon})$, where ε is a small parameter measuring the weak non-linearity of the waves. For one of the systems, this result is new even for short time. The two other systems involve surface tension, and for one of them, the non-cavitation condition has to be sharpened when the surface tension is small. The proof relies on suitable symmetrizers and the classical theory of hyperbolic systems. However, we have to track the small parameters carefully in the commutator estimates to get the long time well-posedness.

Finally, combining our results with the recent ones of Emerald provide a full justification of these systems as water wave models in a larger range of regimes than the classical (a, b, c, d)-Boussinesq systems.

1. Introduction

1.1. **Full dispersion models.** The Korteweg-de Vries (KdV) equation is an asymptotic model for the unidirectional propagation of small amplitude, long waves on the surface of an ideal fluid of constant depth. It was introduced in [8, 32] to model the propagation of solitary waves in shallow water with a wide range of applications both mathematically and physically. However, its dispersion is too strong in high frequencies when compared to the full water wave system. In particular, the KdV equation does not feature wave breaking or peaking waves. To overcome these shortcomings, Whitham introduced in [52] an equation with an improved dispersion relation. He replaced the KdV dispersion with the exact dispersion of the linearized water wave system obtaining the equation

$$\partial_t \zeta + \sqrt{\mathcal{K}_\mu}(D)\partial_x \zeta + \varepsilon \zeta \partial_x \zeta = 0, \tag{1.1}$$

for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, where the function $\zeta(x, t) \in \mathbb{R}$ denotes the surface elevation and the operator $\sqrt{\mathcal{K}_\mu}(D)$ is the square root of the Fourier multiplier $\mathcal{K}_\mu(D)$ defined in frequency by

$$K_\mu(\xi) = \frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}\left(1 + \beta\mu|\xi|^2\right). \tag{1.2}$$

Moreover, μ and ε are small parameters related to the level of dispersion and nonlinearity, and β is a nonnegative parameter related to the surface tension[1,2].

Whitham conjectured in [52] that equation (1.1) would allow, in addition to the KdV traveling-wave regime, the occurrence of waves of greatest height with a sharp crest as well as the formation of shocks. However, it was not until recently that these phenomena

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[1]Actually, Whitham introduced the equation formally without the parameters μ and ε.

[2]He also did not include surface tension, *i.e.* $\beta = 0$.

were rigorously proved. We mention among others the existence of periodic waves [18], the existence and stability of traveling waves [17, 4, 48, 27], the formation of shocks [24, 45], Benjamin-Feir instabilities [47, 25], the existence of periodic waves of greatest height [20] and solitary waves of greatest height [50]. Note that in the case of surface tension ($\beta > 0$), the dynamics appear to be rather different (see e.g. [31] and the references therein).

These results illustrate some mathematical properties uniquely related to an improved dispersion relation, though there are some phenomena that the Whitham equation does not feature due to its unidirectionality. For instance, the Euler equations admit non-modulational instabilities of small-amplitude periodic traveling waves [36], but the unidirectional nature of the Whitham equation is believed to prohibit such instabilities [10].

Regarding the two-way propagation of waves at the surface of a fluid and in the long wave regime, Bona, Chen, and Saut derived a three-parameter family of Boussinesq systems [5]

$$\begin{cases}(1-b\mu\partial_x^2)\partial_t\zeta + (1+a\mu\partial_x^2)\partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ (1-d\mu\partial_x^2)\partial_t v + (1+c\mu\partial_x^2)\partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.3}$$

where a, b, c and d are real parameters satisfying $a+b+c+d=\frac{1}{3}$, $\zeta(x,t)\in\mathbb{R}$ is the deviation of the free surface with respect to its rest state, and $v(x,t)\in\mathbb{R}$ approximates the fluid velocity at some height in the fluid domain. Like the KdV equation, the Boussinesq systems are celebrated models for surface waves in coastal oceanography. Analogously to the unidirectional case, one could replace the dispersion with the linearized dispersion of the water wave equations in (1.3). These improved dispersion versions are expected to lead to a more "accurate" description of the full water wave system. Those systems are commonly referred to as the Whitham-Boussinesq systems or full dispersion Boussinesq systems.

Actually, there are different possibilities of full dispersion Boussinesq systems. This paper will focus on three important ones, linking them to some specific cases of the Boussinesq systems without BBM terms ($b=d=0$). To be precise, we introduce the operator $\mathcal{T}_\mu(D)$ corresponding to $\mathcal{K}_\mu(D)$ for $\beta=0$, and whose Fourier symbol is defined by

$$T_\mu(\xi) = \frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}. \tag{1.4}$$

First, we consider the system

$$\begin{cases}\partial_t\zeta + \mathcal{K}_\mu(D)\partial_x v + \varepsilon\partial_x(\zeta v) = 0\\ \partial_t v + \partial_x\zeta + \varepsilon v\partial_x v = 0,\end{cases} \tag{1.5}$$

introduced in [33, 1, 38] without surface tension and in [31] with surface tension. Here, as above, ζ denotes the elevation of the surface around its equilibrium position, while v approximates the fluid velocity at the free surface. We also consider its two-dimensional counterpart

$$\begin{cases}\partial_t\zeta + \mathcal{K}_\mu(D)\nabla\cdot\mathbf{v} + \varepsilon\nabla\cdot(\zeta\mathbf{v}) = 0\\ \partial_t\mathbf{v} + \nabla\zeta + \frac{\varepsilon}{2}\nabla|\mathbf{v}|^2 = \mathbf{0},\end{cases} \tag{1.6}$$

where $x\in\mathbb{R}^2$ and $\mathbf{v}(x,t)\in\mathbb{R}^2$ approximates the fluid velocity at the surface in two space dimensions. In the case zero surface tension, it is proved that (1.5) models solitary waves [39] and admit high-frequency (non-modulational) instabilities of small-amplitude periodic traveling waves [19]. We also observe that (1.5) is related to (1.3) by expanding (1.2) in low frequencies. Indeed, since $K_\mu(\xi)\simeq 1+\mu(\beta-\frac{1}{3})\xi^2$ by a Taylor expansion we see that (1.5) reduce to (1.3) with $(a,b,c,d)=(\frac{1}{3}-\beta,0,0,0)$.

A second system is obtained by applying the operator (1.4) to $\partial_x \zeta$, which gives

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon \partial_x(\zeta v) = 0 \\ \partial_t v + \mathcal{T}_\mu(D)\partial_x \zeta + \varepsilon v \partial_x v = 0, \end{cases} \tag{1.7}$$

and in two dimensions reads

$$\begin{cases} \partial_t \zeta + \nabla \cdot \mathbf{v} + \varepsilon \nabla \cdot (\zeta \mathbf{v}) = 0 \\ \partial_t \mathbf{v} + \mathcal{T}_\mu(D)\nabla \zeta + \frac{\varepsilon}{2} \nabla |\mathbf{v}|^2 = \mathbf{0}. \end{cases} \tag{1.8}$$

This system was first introduced in [26], where it is proved that (1.7) features Benjamin-Feir (modulational) instabilities. Note that while ζ plays the same role as for system (1.5), it is $\mathcal{T}_\mu^{-1}(D)v$ which approximates the velocity potential at the free surface in this case (and $\mathcal{T}_\mu^{-1}(D)\mathbf{v}$ in two dimensions). We also observe that (1.7) reduces in the formal limit $\sqrt{\mu}|\xi| \to 0$ to the Boussinesq system (1.3) with $(a,b,c,d) = (0,0,\frac{1}{3},0)$ in low frequencies.

Finally, we will also consider a full dispersion version of (1.3) when $\mathcal{T}_\mu(D)$ is applied to the nonlinear terms, while $\mathcal{K}_\mu(D)$ is applied on the $\partial_x \zeta$. This system reads

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon \mathcal{T}_\mu(D) \partial_x(\zeta v) = 0 \\ \partial_t v + \mathcal{K}_\mu(D)\partial_x \zeta + \varepsilon \mathcal{T}_\mu(D)(v \partial_x v) = 0, \end{cases} \tag{1.9}$$

and in two dimensions is given by

$$\begin{cases} \partial_t \zeta + \nabla \cdot \mathbf{v} + \varepsilon \mathcal{T}_\mu(D) \nabla \cdot (\zeta \mathbf{v}) = 0 \\ \partial_t v + \mathcal{K}_\mu(D)\nabla \zeta + \frac{\varepsilon}{2} \mathcal{T}_\mu(D) \nabla |\mathbf{v}|^2 = \mathbf{0}. \end{cases} \tag{1.10}$$

Here ζ and v play the same roles as for system (1.7) (similarly, $\mathbf{v}$ has the same role as in (1.8)). It was introduced in [13] and has the advantage of being Hamiltonian. Moreover, the existence of solitary waves is proved in [14].

1.2. **Full justification.** A fundamental question in the derivation of an asymptotic model is whether its solution converges to the solution of the original physical system. In particular, we say that an asymptotic model is a valid approximation of the Euler equations with a free surface if we can answer the following points in the affirmative [33]:

1. The solutions of the water wave equations exist on the relevant scale $\mathcal{O}(\frac{1}{\varepsilon})$.
2. The solutions of the asymptotic model exist (at least) on the scale $\mathcal{O}(\frac{1}{\varepsilon})$.
3. Lastly, we must establish the *consistency* between the asymptotic model and the water wave equations, and then show that the error is of order $\mathcal{O}(\mu\varepsilon t)$ when comparing the two solutions.

The first point was proved by Alvarez-Samaniego and Lannes [2] for surface gravity waves and Ming, Zhang and Zhang [37] for gravity-capillary waves in the weakly transverse regime, while points 2. and 3. are specific to the asymptotic model under consideration. For instance, in the case of the Whitham equation, Klein *et al.* [31] compared its solution rigorously with those of the KdV equation. In particular, they proved that the difference between two solutions evolving from the same initial datum is bounded by $\mathcal{O}(\varepsilon^2 t)$ for all $0 \leq t \lesssim \varepsilon^{-1}$ with ε, μ in the KdV-regime:

$$\mathcal{R}_{KdV} = \{(\varepsilon, \mu) \, : \, 0 \leq \mu \leq 1, \quad \mu = \varepsilon\},$$

which justified the Whitham equation as a water wave model in this regime by relying on the justification of the KdV equation [9, 33].

On the other hand, due to the improved dispersion relation of (1.1), Emerald [22] was able to decouple the parameters (ε, μ) and prove an error estimate between the Whitham equation and the water wave system with a precision $\mathcal{O}(\mu\varepsilon t)$ for $0 \le t \lesssim \varepsilon^{-1}$ in the shallow water regime:

$$\mathcal{R}_{SW} = \{(\varepsilon,\mu) \, : \, 0 \le \mu \le 1, \quad 0 \le \varepsilon \le 1\}. \tag{1.11}$$

Moreover, Emerald decoupled the small parameters for the KdV equation and proved its precision to be $\mathcal{O}(\mu^2 + \mu\varepsilon)t$ for $0 \le t \lesssim \varepsilon^{-1}$. Consequently, the Witham equation is valid for a larger set of small parameters when compared to the KdV equation. Specifically, when $\varepsilon \ll \mu$, these estimates imply that (1.1) equation is a better approximation of the water wave equations.

In the case of the Boussinesq systems (1.3), consistency was first proved in [6] for $(\varepsilon,\mu) \in \mathcal{R}_{KdV}$ by relying on intermediate symmetric systems for which the long time well-posedness follows by classical arguments. However the long time well-posedness for the (a,b,c,d) Boussinesq is far from trivial. This result was proved[3] later by Saut, Xu and Wang [42, 46]. The proof relies on suitable symmetrizers and hyperbolic theory.

The natural next step is to consider the Whitham-Boussinesq systems for $(\varepsilon,\mu) \in \mathcal{R}_{SW}$. *In particular, the goal of this paper is to establish the well-posedness of* (1.5)-(1.10)*, with uniform bounds, on time intervals of size* $\mathcal{O}(\frac{1}{\varepsilon})$. Since point 1. of the justification is already established, the long-time existence and consistency remain. Using the method of Emerald, one can prove the consistency of any Whitham-Boussinesq system with the water wave system (see also [21] for other full dispersion shallow water models). Therefore, having the long time well-posedness theory for (1.5)-(1.10) will provide the final step for the full justification of these systems.

1.3. **Former well-posedness results.** Regarding system (1.5) and (1.6), we know from previous studies that surface tension plays a fundamental role in the well-posedness theory. In fact, when $\beta = 0$ the initial value problem associated to system (1.5) is probably ill-posed unless $\zeta > 0$ (see the formal argument in Section 4 in [31]). We refer to [40] for a well-posedness under the non-physical condition $\zeta \ge c_0 > 0$. When surface tension is taken into account, system (1.5) was proved to be locally well-posed by Kalisch and Pilod [28] for $(\zeta, v) \in H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})$, $s > \frac{5}{2}$ (and $s > 3$ in two dimensions), by using a modified energy method. We also refer to the work by Wang [51] for an alternative proof using a nonlocal symmetrizer. However, it is worth noting that all these well-posedness resutls were proved on a short time without considering the small parameters ε and μ. Finally, in the formal limit $\sqrt{\mu}|\xi| \to 0$, one recovers the Boussinesq system corresponding to $(a,b,c,d) = (\frac{1}{3} - \beta, 0, 0, 0)$. This system has been proved in [46] to be well-posed on large time for $\beta > \frac{1}{3}$, while it is known to be ill-posed for $\beta < \frac{1}{3}$ [3]. This is a formal indication that the threshold $\beta = \frac{1}{3}$ will play an important role for the long well-posedness of (1.5) and (1.6). We will come back to this issue in the next section (see Figure 1).

As far as we know, there are no well-posedness results for system (1.7) and (1.8) even on short time. In the formal limit $\sqrt{\mu}|\xi| \to 0$, system (1.7) reduces to the Boussinesq system corresponding to $(a,b,c,d) = (0,0,\frac{1}{3},0)$, which is believed to be ill-posed [31].

Next, attention is turned to (1.9) and (1.10). There are several results when $\beta = 0$. In this case, Dinvay [12] proved short time local well-posedness for $(\zeta, v) \in H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+1}(\mathbb{R})$, $s \ge 0$ in the one-dimensional case. The proof is based on standard hyperbolic theory that involves a modified energy similar to [28]. This result was then extended in [15] by exploiting the smoothing effect of the linear flow using dispersive techniques improving the regularity

[3]In the most dispersive case $(a,b,c,d) = (\frac{1}{6}, \frac{1}{6}, 0, 0)$, the relevant time scale $\mathcal{O}(\varepsilon^{-1})$ is still missing; the best results being on a time scale $\mathcal{O}(\varepsilon^{-\frac{2}{3}})$ [43, 44], (see also [35] on a time scale $\mathcal{O}(\varepsilon^{-\frac{1}{2}})$ by using dispersive techniques).

to $H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})$, $s > -\frac{1}{10}$. Furthermore, when considering small data, the system is globally well-posed due to the control of the Hamiltonian. The estimates derived in the aforementioned papers are not uniform in μ. However, a recent study by Tesfahun [49] proved that the corresponding 2-dimensional system (1.10) without surface tension is well-posed on a time interval of order $\mathcal{O}(\frac{1}{\sqrt{\varepsilon}})$ in the KdV−regime. Indeed, dispersive techniques are tailored-made for short waves and therefore seem not to be well suited to capture the long wave regime (see for instance [35] for similar results for the Boussinesq system in the KdV-KdV case). Finally, in the case of surface tension $\beta > 0$, Dinvay proved in [11] the short time local well-posedness of (1.9) and (1.10) by using modified energy techniques. This result also implies the small data global well-posedness in this case.

Lastly,[4] we would like to comment on a recent work by Emerald [23]. Here he considered a class of non-local quasi-linear systems in one and two dimensions that include the following family of Whitham-Boussinesq systems,

$$\begin{cases} \partial_t \zeta + \mathcal{T}_\mu(D)\nabla\cdot\mathbf{v} + \varepsilon(\mathcal{T}_\mu)^\alpha(D)\nabla\cdot(\zeta(\mathcal{T}_\mu)^\alpha(D)\mathbf{v}) = 0 \\ \partial_t\mathbf{v} + \nabla\zeta + \varepsilon((\mathcal{T}_\mu)^\alpha(D)\mathbf{v}\cdot\nabla)((\mathcal{T}_\mu)^\alpha(D)\mathbf{v}) = \mathbf{0}, \end{cases} \tag{1.12}$$

with $\alpha \geq \frac{1}{2}$. In the paper, the author proves the long time well-posedness of (1.12), and deomonstrate that the error between the water wave system is of order $\mathcal{O}(\mu\varepsilon t)$. Also, note that in the case $\alpha = 0$, then (1.12) corresponds to system (1.6) in the case $\beta = 0$. This case is still an open problem. However, combining the results of [23] with the ones in this paper, accounts for many of the possible Whitham-Boussinesq systems, and thus complete each other well.

1.4. **Main results.** In the current paper, we take into account the small parameters (ε, μ) and prove the well-posedness of (1.5), (1.7), (1.9), and their two-dimensional versions, on a time scale $\mathcal{O}(\frac{1}{\varepsilon})$.

In the case of systems (1.7)-(1.8) and (1.9)-(1.10), we will work under the standard non-cavitation condition.

Definition 1.1 (Non-cavitation condition)**.** *Let $d = 1$ or 2 with $s > \frac{d}{2}$ and $\varepsilon \in (0,1)$. We say the initial surface elevation $\zeta_0 \in H^s(\mathbb{R}^d)$ satisfies the "non-cavitation condition" if there exist $h_0 \in (0,1)$ such that*

$$1 + \varepsilon\zeta_0(x) \geq h_0, \quad \textit{for all } \ x \in \mathbb{R}^d. \tag{1.13}$$

In the case of system (1.5) and (1.6), we will distinguish between the cases $\beta \geq \frac{1}{3}$ and $0 < \beta < \frac{1}{3}$. More precisely, for $\beta \geq \frac{1}{3}$, we will also assume the non-cavitation condition in Definition 1.1, while for $0 < \beta < \frac{1}{3}$, we have to impose the following $\beta-$dependent surface condition.

Definition 1.2 ($\beta-$dependent surface condition)**.** *Let $d = 1$ or 2 with $s > \frac{d}{2}$, $\varepsilon \in (0,1)$ and $\beta \in (0, \frac{1}{3})$. We say the initial surface elevation $\zeta_0 \in H^s(\mathbb{R}^d)$ satisfy the "$\beta-$dependent surface condition" if*

$$1 + \varepsilon\zeta_0(x) \geq h_\beta, \quad \textit{for all } \ x \in \mathbb{R}^d, \tag{1.14}$$

where $h_\beta = 1 - \frac{\beta}{2}$.

Remark 1.3. *For $0 < \beta < \frac{1}{3}$, $K_\mu(\xi)$ is not a monotone function for positive frequencies, as we can be seen in the figure below. This is why we choose to impose condition* (1.14) *in this case.*

[4]See also [16] for a survey on resent developments in the field.

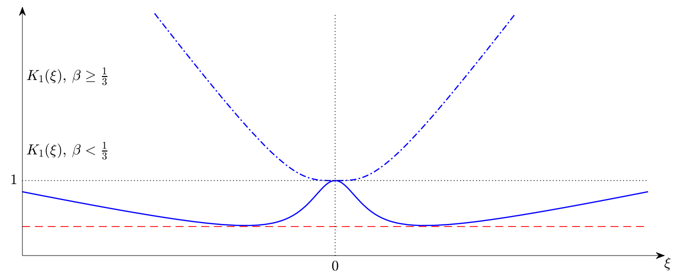

FIGURE 1. The multipiler $K_1(\xi)$ in the case when $\beta \geq \frac{1}{3}$ (dash-dot) and $\beta < \frac{1}{3}$ (line). The horizontal line (dashed) specifies the minimum.

Remark 1.4. *One can see the $\beta-$dependent surface condition as a constraint on the initial data that is related to the minimum of the function $K_\mu(\xi)$. For instance, if we consider the multiplier in Figure 1, then an admissible initial datum must satisfy the constraint in the figure below.*

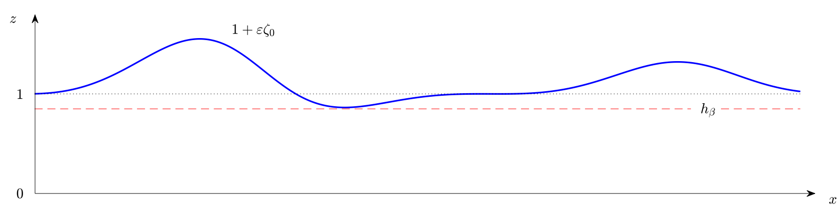

FIGURE 2. The blue line denotes the initial surface elevation $1+\varepsilon\zeta_0$, and is restricted by h_β when $0<\beta<\frac{1}{3}$.

Before we state the main results, we define a natural solution space for systems (1.5)-(1.6) and (1.7)-(1.8).

Definition 1.5. *We define the norm on the function space $V^s_\mu(\mathbb{R}^d)$ to be*

$$\|(\zeta,\mathbf{v})\|^2_{V^s_\mu} := \|\zeta\|^2_{H^s} + \|\mathbf{v}\|^2_{H^s} + \sqrt{\mu}\|D^{\frac{1}{2}}\mathbf{v}\|^2_{H^s}.$$

Theorem 1.6. *Let $d=1$ or 2 with $s>\frac{d}{2}+\frac{3}{2}$, $\beta>0$ and $\varepsilon,\mu\in(0,1)$. Assume that $(\zeta_0,\mathbf{v}_0)\in V^s_\mu(\mathbb{R}^d)$ satisfies either the non-cavitation condition (1.13) in the case $\beta\geq 1/3$ or the $\beta-$dependent surface condition (1.14) in the case $0<\beta<\frac{1}{3}$, where curl $\mathbf{v}_0=\mathbf{0}$ if $d=2$. Moreover, we assume that*

$$0<\varepsilon\leq\frac{1}{k^2_\beta\|(\zeta_0,\mathbf{v}_0)\|_{V^s_\mu}} \quad \text{for} \quad k^2_\beta=\begin{cases}\frac{c}{\beta} & \text{for } \ 0<\beta<\frac{1}{3}\\ c\beta & \text{for } \ \beta\geq\frac{1}{3}\end{cases} \tag{1.15}$$

for some $c>0$. Then there exists a positive T given by

$$T=\frac{1}{k^1_\beta\|(\zeta_0,\mathbf{v}_0)\|_{V^s_\mu}} \quad \text{with} \quad k^1_\beta=\begin{cases}\frac{c}{\beta} & \text{for } 0<\beta<\frac{1}{3}\\ c\beta^2 & \text{for } \beta\geq\frac{1}{3}\end{cases} \tag{1.16}$$

such that (1.5) and (1.6) admits a unique solution

$$(\zeta,\mathbf{v})\in C([0,T/\varepsilon]:V^s_\mu(\mathbb{R}^d))\cap C^1([0,T/\varepsilon]:V^{s-\frac{3}{2}}_\mu(\mathbb{R}^d)),$$

that satisfies

$$\sup_{t\in[0,T/\varepsilon]} \|(\zeta,\mathbf{v})\|_{V^s_\mu} \lesssim \|(\zeta_0,\mathbf{v}_0)\|_{V^s_\mu}. \tag{1.17}$$

Furthermore, there exists a neighborhood $\mathcal{U}$ *of* $(\zeta_0,\mathbf{v}_0)$ *such that the flow map*

$$F^s_{T,\varepsilon,\mu} : V^s_\mu(\mathbb{R}^d) \;\to\; C([0,\tfrac{T}{2\varepsilon}];V^s_\mu(\mathbb{R}^d)), \quad (\zeta_0,\mathbf{v}_0) \;\mapsto\; (\zeta,\mathbf{v}),$$

is continuous.

Remark 1.7. *The proof of the continuous dependence on long time of order* $\mathcal{O}(\frac{1}{\varepsilon})$ *seems to be new for Boussinesq type systems. It relies on the Bona-Smith argument* [7] *and could be easily adapted for the* (a,b,c,d)*-Boussinesq systems.*

Remark 1.8. *A heuristic argument can be made to argue that the physical solutions appear when the initial data is of order one in terms of* ε [41]. *To illustrate this point, take the Burgers equation*

$$u_t - \varepsilon u u_x = 0,$$

a simple model that can describe an inviscid fluid in shallow water theory. Then by the energy method, it is easy to deduce that the time of existence is of order $T \sim \frac{1}{\varepsilon\|u_0\|_{H^s}}$ *for* $s > \frac{3}{2}$. *As a consequence, we have that* $T \sim \frac{1}{\varepsilon}$ *if the initial data is of size* $\mathcal{O}_\varepsilon(1)$.

Remark 1.9. *If* $\beta \sim 1$ *then* $\varepsilon \lesssim 1$ *by* (1.15), *and so* (1.16) *implies that* $T/\varepsilon \sim 1/\varepsilon$. *On the other hand, in the case of having* $\beta \ll 1$, (1.15) *would impose* $\varepsilon \lesssim \beta$, *and by* (1.16) *we have the existence on the timescale* $T/\varepsilon \sim \beta/\varepsilon$.

Remark 1.10. *Regarding the* $\beta-$*dependent surface condition, we demonstrate that the solution will persist for a long time and satisfy* $\varepsilon\zeta(x,t) \geq -c\beta$ *for some constant* $c > 0$. *One should also note that this is coherent since* $0 < \varepsilon \lesssim \beta$ *as explained in the previous remark. For a related discussion on this physical condition see Subsection 1.3.*

Next, we state a well-posedness result for (1.7) and (1.8). These systems does not feature any surface tension and is well-posed for a long time under the standard non-cavitation condition.

Theorem 1.11. *Let* $d = 1$ *or* 2 *with* $s > \frac{d}{2}+1$ *and* $\mu \in (0,1)$. *Assume that* $(\zeta_0, v_0) \in V^s_\mu(\mathbb{R})$ *satisfies the non-cavitation condition* (1.13), *where curl* $\mathbf{v}_0 = \mathbf{0}$ *if* $d = 2$. *Also assume that for some* $c > 0$ *that* $0 < \varepsilon \leq c\big(\|(\zeta_0,\mathbf{v}_0)\|_{V^s_\mu}\big)^{-1}$. *Then there exists* $T = c\big(\|(\zeta_0,\mathbf{v}_0)\|_{V^s_\mu}\big)^{-1}$ *such that* (1.7) *and* (1.8) *admits a unique solution*

$$(\zeta,\mathbf{v}) \in C([0,T/\varepsilon] : V^s_\mu(\mathbb{R}^d)) \cap C^1([0,T/\varepsilon] : V^{s-1}_\mu(\mathbb{R}^d)),$$

that satisfies

$$\sup_{t\in[0,T/\varepsilon]} \|(\zeta,\mathbf{v})\|_{V^s_\mu} \lesssim \|(\zeta_0,\mathbf{v}_0)\|_{V^s_\mu}.$$

In addition, the flow map is continuous with respect to the initial data.

Remark 1.12. *As far as we know, Theorem 1.11 is the first well-posedness result for systems* (1.7)-(1.8).

Similarly, we can combine the techniques used to prove Theorem 1.6 and Theorem 1.11 to establish the long time well-posedness of (1.9)-(1.10) in the space:

Definition 1.13. *Define the norm on the function space* $X^s_{\beta,\mu}(\mathbb{R}^d)$ *to be*

$$\|(\zeta,\mathbf{v})\|^2_{X^s_{\beta,\mu}} := \|\zeta\|^2_{H^s} + \beta\mu\|D^1\zeta\|^2_{H^s} + \|\mathbf{v}\|^2_{H^s} + \sqrt{\mu}\|D^{\frac{1}{2}}\mathbf{v}\|^2_{H^s}.$$

Theorem 1.14. *Let $d = 1$ or 2 with $s > \frac{d}{2} + 1$, $\beta \geq 0$ and $\mu \in (0,1)$. Assume that $(\zeta_0, \mathbf{v}_0) \in X^s_{\beta,\mu}(\mathbb{R})$ satisfies the non-cavitation condition (1.13), where curl $\mathbf{v}_0 = \mathbf{0}$ if $d = 2$. Also assume that for some $c > 0$ that $0 < \varepsilon \leq c\big(\|(\zeta_0, \mathbf{v}_0)\|_{X^s_{\beta,\mu}}\big)^{-1}$. Then there exists $T = c\big(\|(\zeta_0, \mathbf{v}_0)\|_{X^s_{\beta,\mu}}\big)^{-1}$ such that (1.9) and (1.10) admits a unique solution*

$$(\zeta, \mathbf{v}) \in C([0, T/\varepsilon] : X^s_{\beta,\mu}(\mathbb{R}^d) \cap C^1([0, T/\varepsilon] : X^{s-1}_{\beta,\mu}(\mathbb{R}^d)),$$

that satisfies

$$\sup_{t\in[0,T/\varepsilon]} \|(\zeta, \mathbf{v})\|_{X^s_{\beta,\mu}} \lesssim \|(\zeta_0, \mathbf{v}_0)\|_{X^s_{\beta,\mu}}.$$

In addition, the flow map is continuous with respect to the initial data.

Remark 1.15. *Including $\beta > 0$ in the norm in the definition of $X^s_{\beta,\mu}(\mathbb{R}^d)$ will allow us to obtain a long time well-posedness result under the non-cavitation condition. Additionally, when $0 < \beta < \frac{1}{3}$ then ε is independent from the surface tension parameter, and in the case $\beta = 0$ we have that $X^s_{0,\mu}(\mathbb{R}^d)$ is equal to $V^s_\mu(\mathbb{R}^d)$.*

Remark 1.16. *For the sake of clarity, we will mainly focus on the one-dimensional case. Theorems 1.6, 1.11 and 1.14 can be easily extended to the 2-dimensional case by following the same methods since the symbols $\mathcal{K}_\mu(D)$ and $\mathcal{T}_\mu(D)$ are radial. We give a brief outline of what would be the main changes in Section 6.*

1.5. **Strategy and outline.** The proof of Theorem 1.6 relies mainly on energy estimates similar to the ones provided in [28] on a fixed time. Though, we use the idea of Wang [51], who included the nonlocal operator $\mathcal{K}_\mu(D)$ in the definition of the energy[5]:

Definition 1.17. *Let $(\eta, u) = \varepsilon(\zeta, v)$ and J^s be the bessel potential of order $-s$. Then we define the energy associated to (1.5) in the one-dimensional case to be:*

$$E_s(\eta, u) := \int_{\mathbb{R}} \Big((J^s\eta)^2 + \eta(J^s u)^2 + (\sqrt{\mathcal{K}_\mu}(D) J^s u)^2 \Big)\, dx.$$

This energy formulation will free us to cancel out specific nonlinear terms that appear naturally in the computations yielding the estimate

$$\frac{d}{dt} E_s(\eta, u) \lesssim_\beta \big(E_s(\eta, u)\big)^{\frac{3}{2}}. \tag{1.18}$$

Combined with the coercivity of the energy, then by a standard bootstrap argument, one deduces a solution with the lifespan of $T_0 = \mathcal{O}(\frac{1}{\varepsilon})$. We refer the reader to Proposition 3.1 and Lemma 5.3 for these results. The proof of the energy estimate is similar to the one presented in [51], but we keep track of the small parameters. We should also note that estimate (1.18) is applied to a regularized version of (1.5), where we recover the original system using a Bona-Smith argument.

To run the Bona-Smith argument for $s > 2$, one classically needs to estimate the difference between two solutions at the $V^0_\mu(\mathbb{R})-$level. These estimates will be the most technical point of the paper and are specific to the dependence of the small parameters. In short, the technical difficulty is related to the apparent need for 'generalized' Kato-Ponce type commutator estimates on $\mathcal{K}_\mu(D)$ (see Lemma 2.9 and the generalization for $\mathcal{K}_\mu(D)$ in Lemma 2.11). Whereas for the case $\mu = 1$, one can use Calderón type estimates to simplify $\mathcal{K}_\mu(D)$ directly (see [28] and the reformulated system (2.1)). The main idea will be to split $\mathcal{K}_\mu(D)$

[5] Wang actually used this multiplier in the case $\mu = 1$.

in high and low frequencies, and then derive new commutator estimates that allow us to obtain the necessary order of μ in the estimates related to the energy.

For the proof of Theorem 1.11, we follow the same strategy, but in this case, the dispersion operator (1.4) is regularizing. The trick will be to introduce a scaled Bessel potential in the energy, allowing us to mimic the properties of (1.2). The energy is given by:

Definition 1.18. *Let* $(\eta, u) = \varepsilon(\zeta, v)$ *and* $J_\mu^{\frac{1}{2}}$ *be the scaled Bessel potential defined by the symbol* $\xi \mapsto (1+\mu\xi^2)^{\frac{1}{4}}$ *in frequency. Then the energy associated to* (1.7) *in the one-dimensional case reads:*

$$\mathcal{E}_s(\eta,u) := \int_{\mathbb{R}} \left((\sqrt{\mathcal{T}_\mu}(D) J_\mu^{\frac{1}{2}} J^s \eta)^2 + (1+\eta)(J_\mu^{\frac{1}{2}} J^s u)^2 \right) dx.$$

The energy formulated in Definition 1.18 is new and will require commutator estimates specific to the equation. This will, in turn, allow us to decouple the parameters μ and ε in the estimates and, by extension, provide an estimate in the form of (1.18).

In the same spirit, we define a modified energy for system (1.9):

Definition 1.19. *Let* $(\eta, u) = \varepsilon(\zeta, v)$ *and* $\beta > 0$*. Then the energy associated to* (1.9) *in the one-dimensional case reads:*

$$\mathscr{E}_s(\eta,u) := \int_{\mathbb{R}} \left((J^s\eta)^2 + \beta\mu(D^1 J^s \eta)^2 + \eta (J^s u)^2 + (\sqrt{\mathcal{T}}_\mu^{-1}(D) J^s u)^2 \right) dx.$$

Note also that the energy includes the surface tension parameter β and will allow us to deduce an estimate on the form (1.18), where the coercivity estimate will be uniform in β. In turn, this will provide the long time well-posedness for $\beta \ll 1$ and $T/\varepsilon \sim 1/\varepsilon$ as pointed out in Remark 1.15.

The paper is organized as follows. In Section 2, we introduce some important technical results whose proofs will be postponed to the appendix. In the same section, we also present new commutator estimates needed to treat the nonlinear terms when estimating the energy in Sections 3 and 4. Then we conclude in Section 5 by combining the results obtained in the former sections to prove Theorem 1.6 in full detail in the one-dimensional case. Lastly, we comment briefly on the changes to adapt the proof in the two-dimensional setting, while the proof of Theorem 1.11 and Theorem 1.14 will follow by the same arguments.

1.6. **Notation.**

- We let c denote a positive constant independent of μ, ε that may change from line to line. Also, as a shorthand, we use the notation $a \lesssim b$ to mean $a \leq c\, b$. Similarly, if the constant depends on β, we write $a \lesssim_\beta b$. In particular, we define the constants depending on β,

$$c_\beta^1 = \begin{cases} c\beta & \text{for } 0<\beta<\frac{1}{3} \\ c & \text{for } \beta \geq \frac{1}{3} \end{cases} \quad \text{and} \quad c_\beta^2 = \begin{cases} c & \text{for } 0<\beta<\frac{1}{3} \\ c\beta & \text{for } \beta \geq \frac{1}{3} \end{cases} \tag{1.19}$$

- Let $(V, \|\cdot\|_V)$ be a vector space. Then for $\alpha \geq 0$, $\lambda > 0$ and $f_\lambda \in V$ be a function depending on λ, we define the "big$-\mathcal{O}$" notation to be

$$\|f_\lambda\|_V = \mathcal{O}(\lambda^\alpha) \iff \lim_{\lambda \to 0} \lambda^{-\alpha}\|f_\lambda\|_V < \infty.$$

  Similarly, we define the "small$-o$" notation to be

$$\|f_\lambda\|_V = o(\lambda^\alpha) \iff \lim_{\lambda \to 0} \lambda^{-\alpha}\|f_\lambda\|_V = 0.$$

- Let $L^2(\mathbb{R})$ be the usual space of square integrable functions with norm $\|f\|_{L^2} = \sqrt{\int_{\mathbb{R}} |f(x)|^2\, dx}$. Also, for any $f, g \in L^2(\mathbb{R})$ we denote the scalar product by $\big(f, g\big)_{L^2} = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$.
- For any tempered distribution f, the operator $\mathcal{F}$ denoting the Fourier transform, applied to f, will be written as $\hat{f}(\xi)$ or $\mathcal{F}f(\xi)$.
- Let $m : \mathbb{R} \to \mathbb{R}$ be a smooth function. Then we will use the notation $m(D)$ for a multiplier defined in frequency by $\widehat{m(D)f}(\xi) = m(\xi)\hat{f}(\xi)$.
- For any $s \in \mathbb{R}$ we call the multiplier $\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi)$ the Riesz potential of order $-s$. One should note that $D^1 = \mathcal{H}\partial_x$, where $\widehat{\mathcal{H}f}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$ is the Hilbert transform.
- For any $s \in \mathbb{R}$ we call the multiplier $J^s = (1 + D^2)^{\frac{s}{2}} = \langle D\rangle^s$ the Bessel potential of order $-s$. Moreover, the Sobolev space $H^s(\mathbb{R})$ is equivalent to the weighted L^2-space; $\|f\|_{H^s} = \|J^s f\|_{L^2}$. We also find it convenient to define $J_\mu^{\frac{1}{2}}$ which is a multiplier assosiated to the symbol:
$$\mathcal{F}(J_\mu^{\frac{1}{2}} f)(\xi) = (1 + \mu\xi^2)^{\frac{1}{4}} \hat{f}(\xi). \tag{1.20}$$
- We say f is a Schwartz function $\mathscr{S}(\mathbb{R})$, if $f \in C^\infty(\mathbb{R})$ and satisfies for all $\alpha, \beta \in \mathbb{N}$,
$$\sup_x |x^\alpha \partial_x^\beta f| < \infty.$$
- If A and B are two operators, then we denote the commutator between them to be $[A, B] = AB - BA$.

2. Preliminary results

2.1. **Pointwise estimates.** The first result concerns the properties of the dispersive part of the equation. Namely, we deduce pointwise estimates for the multipliers (1.2) and (1.4) that are needed to obtain the coercivity of the energy (see, for instance, equation (3.7) below). Moreover, these estimates will prove essential when dealing with the nonlinear parts of the equation that appear in the energy estimates.

Lemma 2.1. *Let $\mu \in (0, 1)$. Then we have the following pointwise estimates on the kernel $K_\mu(\xi)$:*

- *For $\beta \geq 0$, we have the upper bound*
$$K_\mu(\xi) \lesssim 1 + \beta(1 + \beta\sqrt{\mu}|\xi|). \tag{2.1}$$
- *If $\beta \geq \frac{1}{3}$, then for all $h_0 \in (0, 1)$ we have the lower bound*
$$K_\mu(\xi) \geq (1 - \frac{h_0}{2}) + c\sqrt{\mu}|\xi|, \tag{2.2}$$
whereas, if $0 < \beta < \frac{1}{3}$, we have the lower bound
$$K_\mu(\xi) \geq \beta + c\beta\sqrt{\mu}|\xi|. \tag{2.3}$$
- *The derivative of the symbol $K_\mu(\xi)$ satisfies*
$$\Big|\frac{d}{d\xi}\sqrt{K_\mu(\xi)}\Big| \lesssim \langle\xi\rangle^{-1} + \sqrt{\beta}\mu^{\frac{1}{4}}\langle\xi\rangle^{-\frac{1}{2}}. \tag{2.4}$$
- *We have the following comparison of $\sqrt{\mathcal{K}_\mu(\xi)}$ by*
$$\big|\sqrt{K_\mu(\xi)} - \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\big| \lesssim \sqrt{\beta} + \beta. \tag{2.5}$$

- *There holds*

$$\sqrt{K_\mu(\xi)}\langle\xi\rangle^{s-1}|\xi| \lesssim (\sqrt{\beta}+\beta)\langle\xi\rangle^s + \sqrt{\beta}\mu^{\frac{1}{4}}\langle\xi\rangle^s|\xi|^{\frac{1}{2}}. \tag{2.6}$$

Remark 2.2. *For inequality* (2.3)*, it is crucial to specify the dependence in β as it will provide the coercivity of the energy when $0<\beta<\frac{1}{3}$. The same is true for* (2.2)*, whose importance will be revealed in the proof of Proposition 3.1 below. Though, we note that* (2.3) *does not agree with* (2.2) *when $\beta=\frac{1}{3}$. This is because the lower bound in* (2.3) *is not optimal, but it does not play a role in the overall result.*

Remark 2.3. *We also trace the dependence in β for the first pointwise estimate* (2.1)*, and it will sometimes be replaced with c_β^2 given by* (1.19)*. This constant will again appear when we prove the energy estimates which will provide the size of the time of existence (see Lemma 5.3 in the proof Theorem 1.6).*

The proof of Lemma 2.1 is technical and postponed to the Appendix in Section A.2. A corollary of Proposition 2.1 may now be stated.

Corollary 2.4. *Take $f\in\mathscr{S}(\mathbb{R})$, $\mu\in(0,1)$ and $s\in\mathbb{R}$. Then in the case $\beta\geq\frac{1}{3}$ and for all $h_0\in(0,1)$ we have*

$$(1-\frac{h_0}{2})\|f\|_{H^s}^2 + c\sqrt{\mu}\|D^{\frac{1}{2}}f\|_{H^s}^2 \leq ||\sqrt{\mathcal{K}_\mu}(D)f\|_{H^s}^2 \leq c_\beta^2\|f\|_{H^s}^2 + c\beta\sqrt{\mu}\|D^{\frac{1}{2}}f\|_{H^s}^2. \tag{2.7}$$

Similarly, in the case $0<\beta<\frac{1}{3}$ there holds

$$\beta\|f\|_{H^s}^2 + c\beta\sqrt{\mu}\|D^{\frac{1}{2}}f\|_{H^s}^2 \leq ||\sqrt{\mathcal{K}_\mu}(D)f\|_{H^s}^2 \leq c_\beta^2\|f\|_{H^s}^2 + c\sqrt{\mu}\|D^{\frac{1}{2}}f\|_{H^s}^2. \tag{2.8}$$

Proof. The upper bound in (2.7) follows by Plancherel's identity and the pointwise estimate (2.1), while the lower bound is a consequence of (2.2).

In the same way, for $0<\beta<\frac{1}{3}$, then (2.8) is deduced from (2.3). □

Similarly, we state some useful pointwise estimates on $T_\mu(\xi)$ and the scaled Bessel potential $J_\mu^{\frac{1}{2}}$, where the proof is presented in Appendix A.2.

Lemma 2.5. *Let $\mu\in(0,1)$. Then we have the following pointwise estimates on the kernel $T_\mu(\xi)$:*

- *For all $h_0\in(0,1)$ there holds*

$$(1-\frac{h_0}{2}) + c\sqrt{\mu}|\xi| \leq (T_\mu(\xi))^{-1} \lesssim 1+\sqrt{\mu}|\xi|. \tag{2.9}$$

- *There holds*

$$1 \lesssim T_\mu(\xi)\langle\sqrt{\mu}\xi\rangle \lesssim 1. \tag{2.10}$$

- *For $s\in\mathbb{R}$ there holds*

$$\left|\frac{d}{d\xi}\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\right| \lesssim \langle\xi\rangle^{s-1}\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}. \tag{2.11}$$

- *For $s\in\mathbb{R}$ there holds*

$$\left|\frac{d}{d\xi}\sqrt{T_\mu(\xi)}\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\right| \lesssim \langle\xi\rangle^{s-1}. \tag{2.12}$$

- *There holds*

$$\left|\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\right| \lesssim 1. \tag{2.13}$$

A direct consequence of the above estimates can now be given.

Corollary 2.6. *Let $f \in \mathscr{S}(\mathbb{R})$, $\mu \in (0,1)$, $s \in \mathbb{R}$ and $c > 0$. Then for all $h_0 \in (0,1)$ there holds*

$$\|\sqrt{\mathcal{T}_\mu}(D)f\|_{L^2} \leq \|f\|_{L^2}. \tag{2.14}$$

$$(1 - \frac{h_0}{2})\|f\|^2_{H^s} + c\sqrt{\mu}\|D^{\frac{1}{2}}f\|^2_{H^s} \leq \|\sqrt{\mathcal{T}_\mu}^{-1}(D)f\|^2_{H^s} \lesssim \|f\|^2_{H^s} + c\sqrt{\mu}\|D^{\frac{1}{2}}f\|^2_{H^s}. \tag{2.15}$$

$$\|f\|_{H^s} \lesssim \|\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}f\|_{H^s} \lesssim \|f\|_{H^s}. \tag{2.16}$$

$$\|f\|^2_{H^s} + \sqrt{\mu}\|D^{\frac{1}{2}}f\|^2_{H^s} \lesssim \|J_\mu^{\frac{1}{2}}f\|^2_{H^s} \lesssim \|f\|^2_{H^s} + \sqrt{\mu}\|D^{\frac{1}{2}}f\|^2_{H^s}. \tag{2.17}$$

2.2. **Commutator estimates.** To handle derivatives in the nonlinear parts of the equations, we need commutator estimates on $\mathcal{K}_\mu(D)$ and $\mathcal{T}_\mu(D)$.

Lemma 2.7. *Let $f, g \in \mathscr{S}(\mathbb{R})$, $\mu \in (0,1)$, $s \geq 1$, and $t_0 > \frac{1}{2}$. Then we have the following commutator estimate*

$$\begin{aligned}\|[\sqrt{\mathcal{K}_\mu}(D)J^s, f]\partial_x g\|_{L^2} &\lesssim (c_\beta^2\|f\|_{H^s} + \sqrt{\beta}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}f\|_{H^s})\|\partial_x g\|_{H^{t_0}} \\ &\quad + (c_\beta^2\|g\|_{H^s} + \sqrt{\beta}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}g\|_{H^s})\|\partial_x f\|_{H^{t_0}}. \end{aligned}\tag{2.18}$$

In the high regularity setting, the proof will follow the same lines as in [51], but we track the dependence in μ and β using the pointwise estimates above.

Proof. First, write the commutator as a bilinear form:

$$\big\|[\sqrt{\mathcal{K}_\mu}(D)J^s, f]\partial_x g\big\|_{L^2} = \bigg\| \int_{\mathbb{R}} \Big(\sqrt{K_\mu(\xi)}\langle\xi\rangle^s - \sqrt{K_\mu(\rho)}\langle\rho\rangle^s\Big)\hat{f}(\xi-\rho)\widehat{\partial_x g}(\rho)\, d\rho\bigg\|_{L^2_\xi}.$$

Then if $a = \min\{\xi, \rho\}$ and $b = \max\{\xi, \rho\}$, we can use the mean value theorem, leaving us to estimate the following terms

$$\Big|\sqrt{K_\mu(\xi)}\langle\xi\rangle^s - \sqrt{K_\mu(\rho)}\langle\rho\rangle^s\Big| \lesssim \sup_{\omega\in(a,b)} |m(\omega)|\,|\xi - \rho|,$$

where

$$m(\omega) = m_1(\omega) + m_2(\omega) = \langle\omega\rangle^s \frac{d}{d\omega}\sqrt{K_\mu(\omega)} + \langle\omega\rangle^{s-1}\sqrt{K_\mu(\omega)}.$$

But using (2.5) to estimate $m_1(\omega)$ and (2.4) to treat $m_2(\omega)$, we deduce

$$m(\omega) \lesssim c_\beta^2\langle\omega\rangle^{s-1} + \sqrt{\beta}\mu^{\frac{1}{4}}\langle\omega\rangle^{s-1}|\omega|^{\frac{1}{2}}, \tag{2.19}$$

where the upper bound is increasing for $s \geq 1$. Therefore an upper bound is attained at $|\rho|$ or $|\xi| \leq |\xi - \rho| + |\rho|$. In particular, if $\omega = |\xi - \rho|$ then we may conclude by Minkowski integral inequality, the Cauchy-Schwarz inequality and (2.19) that

$$\begin{aligned}\big\|[\sqrt{\mathcal{K}_\mu}(D)J^s, f]\partial_x g\big\|_{L^2} &\lesssim c_\beta^2\bigg\|\int_{\mathbb{R}}\langle\xi-\rho\rangle^{s-1}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho\bigg\|_{L^2_\xi} \\ &\quad + \sqrt{\beta}\mu^{\frac{1}{4}}\bigg\|\int_{\mathbb{R}}\langle\xi-\rho\rangle^{s-1}|\xi-\rho|^{\frac{1}{2}}|\xi-\rho|\,|\hat{f}(\xi-\rho)|\,|\widehat{\partial_x g}(\rho)|\,d\rho\bigg\|_{L^2_\xi} \\ &\lesssim (c_\beta^2\|f\|_{H^s} + \sqrt{\beta}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}f\|_{H^s})\|\partial_x g\|_{H^{t_0}},\end{aligned}$$

for $t_0 > \frac{1}{2}$. On the other hand, if $\omega = |\rho|$, then we make a change of coordinates and argue similarly to deduce,

$$\begin{aligned}\big\|[\sqrt{\mathcal{K}_\mu}(D)J^s, f]\partial_x g\big\|_{L^2} &\lesssim c_\beta^2 \Big\| \int_{\mathbb{R}} \langle \xi-\nu\rangle^{s-1} |\widehat{\partial_x g}(\xi-\nu)|\, |\nu|\, |\hat{f}(\nu)|\, d\nu \Big\|_{L^2_\xi} \\ &\quad + \sqrt{\beta}\mu^{\frac14} \Big\| \int_{\mathbb{R}} \langle \nu\rangle^{s-1} |\xi-\nu|^{\frac12} |\widehat{\partial_x g}(\xi-\nu)|\, |\nu|\, |\hat{f}(\nu)|\, d\nu \Big\|_{L^2_\xi} \\ &\lesssim (c_\beta^2 \|g\|_{H^s} + \sqrt{\beta}\mu^{\frac14}\|D^{\frac12} g\|_{H^s})\|\partial_x f\|_{H^{t_0}}.\end{aligned}$$

Adding the two scenarios, we may conclude that (2.18) holds.

□

We will also need a commutator estimates on $\mathcal{T}_\mu(D)$ and $J_\mu^{\frac12}$.

Lemma 2.8. *Let $f, g \in \mathscr{S}(\mathbb{R})$, $s \geq 1$, $t_0 > \frac12$, $\mu \in (0,1)$ and $J_\mu^{\frac12}$ as defined in* (1.20).

- *Then we have a Kato-Ponce type estimate*

$$\begin{aligned}\|[J^s J_\mu^{\frac12}, f]\partial_x g\|_{L^2} &\lesssim (\|f\|_{H^s} + \mu^{\frac14}\|D^{\frac12} f\|_{H^s})\|\partial_x g\|_{H^{t_0}} \\ &\quad + (\|g\|_{H^s} + \mu^{\frac14}\|D^{\frac12} g\|_{H^s})\|\partial_x f\|_{H^{t_0}}. \end{aligned} \tag{2.20}$$

- *There holds*

$$\|[\sqrt{\mathcal{T}_\mu}(D)J^s J_\mu^{\frac12}, f]\partial_x g\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{H^{t_0+1}} + \|f\|_{H^{t_0+1}}\|g\|_{H^s}. \tag{2.21}$$

Proof. The proof is similar to the one of Lemma 2.7 and relies on the pointwise estimates established in Lemma 2.5. Indeed, for (2.20) we define $a_1(D)(f,g) := [J^s J_\mu^{\frac12}, f]\partial_x g$ and use the mean value theorem combined with (2.11) to deduce

$$\begin{aligned}|\hat{a}_1(\xi)(f,g)| &\leq \int_{\mathbb{R}} \Big| \langle\xi\rangle^s \langle\sqrt{\mu}\xi\rangle^{\frac12} - \langle\rho\rangle^s\langle\sqrt{\mu}\rho\rangle^{\frac12}\Big| |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\rho)|\, d\rho \\ &\lesssim \int_{\mathbb{R}} \langle \xi-\rho\rangle^{s-1}\langle\sqrt{\mu}(\xi-\rho)\rangle^{\frac12}|\xi-\rho|\, |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\eta)|\, d\rho \\ &\quad + \int_{\mathbb{R}} \langle\rho\rangle^{s-1}\langle\sqrt{\mu}\rho\rangle^{\frac12}|\xi-\rho|\, |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\rho)|\, d\rho.\end{aligned}$$

Then if we apply the $L^2(\mathbb{R})-$norm with respect to ξ, we can argue as in Lemma 2.7 that

$$\|\hat{a}_1(\xi)(f,g)\|_{L^2_\xi} \lesssim \|J_\mu^{\frac12} f\|_{H^s} \int_{\mathbb{R}} |\widehat{\partial_x g}(\rho)|\, d\rho + \|J_\mu^{\frac12} g\|_{H^s}\int_{\mathbb{R}} |\rho|\, |\hat{f}(\rho)|\, d\rho.$$

Then use the definiton of $a_1(D)(f,g)$ and (2.17) to conclude.

The proof of (2.21) is the same, with $a_2(D)(f,g) := [\sqrt{\mathcal{T}_\mu}(D)J^s J_\mu^{\frac12}, f]\partial_x g$. We use (2.12) to find that

$$\begin{aligned}|\hat{a}_2(\xi)(f,g)| &\leq \int_{\mathbb{R}} \Big| \sqrt{T_\mu(\xi)}\langle\xi\rangle^s\langle\sqrt{\mu}\xi\rangle^{\frac12} - \sqrt{T_\mu(\rho)}\langle\rho\rangle^s\langle\sqrt{\mu}\rho\rangle^{\frac12}\Big| |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\rho)|\, d\rho \\ &\lesssim \int_{\mathbb{R}} \langle\xi-\rho\rangle^{s-1}|\xi-\rho|\, |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\rho)|\, d\rho \\ &\quad + \int_{\mathbb{R}} \langle\rho\rangle^{s-1}|\xi-\rho|\, |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\rho)|\, d\rho,\end{aligned}$$

and the result follows.

□

Next, we state the classical Kato-Ponce commutator estimate. We will use it repeatedly to commute the Bessel potential with functions to obtain the desired energy estimates in the coming sections.

Lemma 2.9 (Kato - Ponce commutator estimates [29]). *Let $s \geq 0$, $p, p_2, p_3 \in (1, \infty)$ and $p_1, p_4 \in (1, \infty]$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$. Then*

$$\|J^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}} \tag{2.22}$$

and

$$\|[J^s, f]g\|_{L^p} \lesssim \|\partial_x f\|_{L^{p_1}} \|J^{s-1} g\|_{L^{p_2}} + \|J^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}. \tag{2.23}$$

Similar commutator estimates also hold for more general multipliers. In fact, by splitting the frequency domain into two parts using smooth cut-off functions defined in frequency, we can obtain sharper commutator estimates specific to equation (1.5).

Definition 2.10. *We define the smooth cut-off functions $\chi^{(i)} \in \mathscr{S}(\mathbb{R})$ as Fourier multipliers*

$$\mathcal{F}(\chi^{(i)}(D)f)(\xi) = \chi^{(i)}(|\xi|)\hat{f}(\xi),$$

for any $f \in \mathscr{S}(\mathbb{R})$ with the following properties:

$$0 \leq \chi^{(i)}(\xi) \leq 1, \qquad (\chi^{(1)}(\xi))^2 + (\chi^{(2)}(\xi))^2 = 1 \quad \text{on} \quad \mathbb{R},$$

and

$$\text{supp}\, \chi^{(1)} \subset [-1, 1], \quad \text{supp}\, \chi^{(2)} \subset \mathbb{R} \backslash \big[-\frac{1}{2}, \frac{1}{2}\big].$$

Moreover, we denote the scaled version in μ by $\chi_\mu^{(i)}(\xi) = \chi^{(i)}(\sqrt{\mu}\xi)$.

We have the results:

Lemma 2.11. *Let $s > \frac{3}{2}$, $\mu \in (0, 1)$ and $f, g \in \mathscr{S}(\mathbb{R})$.*

- *Let $(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu})(D)$ be the multiplier of the symbol $(\chi_\mu^{(1)} \sqrt{K_\mu})(\xi)$. Then*
$$\|(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu})(D) f\|_{L^2} \lesssim_\beta \|f\|_{L^2}, \tag{2.24}$$
and
$$\|[(\chi_\mu^{(1)} \sqrt{\mathcal{K}_\mu})(D), f] \partial_x g\|_{L^2} \lesssim_\beta \|f\|_{H^s} \|g\|_{L^2}. \tag{2.25}$$
- *We define the symbol*
$$\sigma_{\mu, \frac{1}{2}}(D) := \Big(\frac{1}{\sqrt{\mu}|D|} + \beta\sqrt{\mu}|D| \Big)^{\frac{1}{2}}. \tag{2.26}$$
Then
$$\|(\chi_\mu^{(2)} \sigma_{\mu, \frac{1}{2}})(D) f\|_{L^2} \lesssim_\beta \|f\|_{L^2} + \mu^{\frac{1}{4}} \|D^{\frac{1}{2}} f\|_{L^2} \tag{2.27}$$
and
$$\|[(\chi_\mu^{(2)} \sigma_{\mu, \frac{1}{2}})(D), f] \partial_x g\|_{L^2} \lesssim_\beta \mu^{\frac{1}{4}} \|f\|_{H^s} \|g\|_{H^{\frac{1}{2}}}. \tag{2.28}$$
- *Lastly, we define the symbol $\sigma_{\mu,0}(D)$ to be*
$$\sigma_{\mu,0}(D) := \Big(\frac{1}{\sqrt{\mu}|D|} + \beta\sqrt{\mu}|D| - \mathcal{K}_\mu(D) \Big)^{\frac{1}{2}}. \tag{2.29}$$
Then
$$\|(\chi_\mu^{(2)} \sigma_{\mu,0})(D) f\|_{L^2} \lesssim_\beta \|f\|_{L^2} \tag{2.30}$$
and
$$\|[(\chi_\mu^{(2)} \sigma_{\mu,0})(D), f] \partial_x g\|_{L^2} \lesssim_\beta \|f\|_{H^s} \|g\|_{L^2}. \tag{2.31}$$

The proof is postponed to Appendix A.3, where we also will prove the following commutator estimates at the $L^2(\mathbb{R})-$level:

Lemma 2.12. *Let $s > \frac{3}{2}$, $\mu \in (0,1)$ and $f, g \in \mathscr{S}(\mathbb{R})$.*

- *For the composition of $\sqrt{\mathcal{T}_\mu}(D)$ and $J_\mu^{\frac{1}{2}}$ there holds,*
$$\|[\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}, f]\partial_x g\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{L^2}. \tag{2.32}$$
- *While for the usual Bessel potential there holds,*
$$\|[\sqrt{\mathcal{T}_\mu}(D)J^s, f]\partial_x g\|_{L^2} \lesssim \|f\|_{H^s}\|J^s g\|_{L^2}. \tag{2.33}$$
- *Similarly, when the operator J^s is the identity, we have*
$$\|[\sqrt{\mathcal{T}_\mu}(D), f]\partial_x g\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{L^2}. \tag{2.34}$$
- *The derivative of the following commutator satisfies*
$$\|\partial_x[\sqrt{\mathcal{T}_\mu}(D), f]g\|_{L^2} \lesssim \|f\|_{H^s}\|g\|_{L^2}. \tag{2.35}$$
- *Lastly, we can commute $J_\mu^{\frac{1}{2}}$ by*
$$\|[J_\mu^{\frac{1}{2}}, f]\partial_x g\|_{L^2} \lesssim \|f\|_{H^s}\|J_\mu^{\frac{1}{2}} g\|_{L^2}. \tag{2.36}$$

2.3. **Classical estimates.** Before turning to the proof of the energy estimates, we state some necessary results that will also be used throughout the paper. First, recall the embeddings (see, for example [34]).

Lemma 2.13 (Sobolev embeddings)**.** *Let $f \in \mathscr{S}(\mathbb{R})$ and $s \in (0, \frac{1}{2})$. Then $H^s(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$ with $p = \frac{2}{1-2s}$, and there holds*
$$\|f\|_{L^p} \lesssim \|D^s f\|_{L^2}. \tag{2.37}$$
Moreover, In the case $s > \frac{1}{2}$, then $H^s(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$.

We also will use the Leibniz rule for the Riesz potential on multiple occasions.

Lemma 2.14 (Fractional Leibniz rule [30])**.** *Let $\sigma = \sigma_1 + \sigma_2 \in (0,1)$ with $\sigma_i \in [0, \sigma]$ and $p, p_1, p_2 \in (1, \infty)$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, for $f, g \in \mathscr{S}(\mathbb{R})$*
$$\|D^\sigma(fg) - fD^\sigma g - gD^\sigma f\|_{L^p} \lesssim \|D^{\sigma_1} f\|_{L^{p_1}}\|D^{\sigma_2} g\|_{L^{p_2}}. \tag{2.38}$$
Moreover, the case $\sigma_2 = 0$, $p_2 = \infty$ is also allowed.

Finally, we recall the following results for the Bona-Smith argument (provided in the classical paper [7]) on the multiplier $\varphi_\delta(D)$ defined by:

Definition 2.15. *Let $\varphi \in \mathscr{S}(\mathbb{R})$ such that $\int \varphi = 1$ and for $\delta > 0$ define the regularization operators $\varphi_\delta(D)$ in frequency by*
$$\forall f \in L^2(\mathbb{R}), \quad \forall \xi \in \mathbb{R}, \quad \widehat{\varphi_\delta f}(\xi) := \varphi(\delta\xi)\hat{f}(\xi),$$
where φ is a real valued and $\varphi(0) = 1$.

We give the version of the regularization estimates as presented in [34] (Proposition 9.1).

Proposition 2.16. *Let $s > 0$, $\delta > 0$ and $f \in \mathscr{S}(\mathbb{R})$. Then*
$$\|\varphi_\delta(D)f\|_{H^{s+\alpha}} \lesssim \delta^{-\alpha}\|f\|_{H^s}, \quad \forall \alpha > 0, \tag{2.39}$$
and
$$\|\varphi_\delta(D)f - f\|_{H^{s-\beta}} \lesssim \delta^{\beta}\|f\|_{H^s}, \quad \forall \beta \in [0, s]. \tag{2.40}$$

Moreover, there holds

$$\|\varphi_\delta(D)f - f\|_{H^{s-\beta}} \underset{\delta\to 0}{=} o(\delta^\beta), \qquad \forall \beta \in [0,s]. \tag{2.41}$$

3. A priori estimates

In this section, we give *a priori* estimates for solutions of the three systems (1.5), (1.7), and (1.9).

3.1. Estimates for system (1.5).

As noted in the introduction, we revisit the energy estimate in [51] to keep track of the parameters β, ε and μ. For simplicity, we adopt the notation $\mathbf{U} = (\eta, u)^T = \varepsilon(\zeta, v)^T$, where we write (1.5) on the compact form:

$$\partial_t \mathbf{U} + M(\mathbf{U}, D)\mathbf{U} = \mathbf{0}, \tag{3.1}$$

with

$$M(\mathbf{U}, D) = \begin{pmatrix} u\partial_x & (\mathcal{K}_\mu(D) + \eta)\partial_x \\ \partial_x & u\partial_x \end{pmatrix}. \tag{3.2}$$

Also, we simplify the notation for the energy given in Definition 1.17 by introducing the symmetrizer

$$Q(\mathbf{U}, D) = Q^{(1)}(\mathbf{U}, D) + Q^{(2)}(\mathbf{U}, D) = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{K}_\mu(D) \end{pmatrix}. \tag{3.3}$$

Then the energy given in Definition 1.17 can be rewritten as

$$E_s(\mathbf{U}, D) = \big(J^s\mathbf{U}, Q(\mathbf{U}, D)J^s\mathbf{U}\big)_{L^2}.$$

Proposition 3.1. *Let $s > 2$, $\varepsilon, \mu \in (0,1)$ and $(\eta, u) = \varepsilon(\zeta, v) \in C([0,T_0]; V^s_\mu(\mathbb{R}))$ be a solution to (3.1) on a time interval $[0,T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} \|(\eta,u)\|_{H^s\times H^s} \leq h_1, \tag{3.4}$$

when $\beta \geq \frac{1}{3}$, and that

$$-\frac{\beta}{2} \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]} \|(\eta,u)\|_{H^s\times H^s} \leq h_1, \tag{3.5}$$

when $0 < \beta < \frac{1}{3}$.

Then, for the energy given in Definition 1.17 and c^i_β defined by (1.19),

$$\frac{d}{dt}E_s(\mathbf{U}) \leq c^2_\beta \left(E_s(\mathbf{U})\right)^{\frac{3}{2}}, \tag{3.6}$$

for all $0 < t < T_0$, and

$$c^1_\beta \|(\eta,u)\|^2_{V^s_\mu} \leq E_s(\mathbf{U}) \leq c^2_\beta \|(\eta,u)\|^2_{V^s_\mu}, \tag{3.7}$$

for all $0 < t < T_0$.

Remark 3.2. *Note that we aim to prove (3.6) with power $\frac{3}{2}$ on the right-hand side. This result will prove essential in getting the time of existence $T \sim \frac{1}{\varepsilon}$ in the proof of Theorem 1.6. One should also note that if we have (3.7), then it is enough to show*

$$\frac{d}{dt}E_s(\mathbf{U}) \lesssim_\beta \|(\eta,u)\|^3_{V^s_\mu},$$

to obtain (3.6). With this in mind, in the proof of the proposition, we will repeatedly use assumption (3.4)–(3.5) to discard higher powers in the norm of the solution than 3. Meaning *the terms of form $\|(\eta,u)\|_{V^s_\mu}^{3+n}$ for $n\in\mathbb{N}$ will be bounded by $\|(\eta,u)\|^3_{V^s_\mu}$ since this seems to be the best we can hope for when using the current method.*

Proof of Proposition 3.1. We first prove estimate (3.7) in the case $\beta\geq 1/3$. By definition, we have that

$$E_s(\mathbf{U})=\|J^s\eta\|^2_{L^2}+\big(J^su,(\mathcal{K}_\mu(D)+\eta)J^su\big)_{L^2}.$$

Thus, as a result of the non-cavitation condition (3.4) and the estimate (2.7), there holds

$$\big(J^su,(\mathcal{K}_\mu(D)+\eta)J^su\big)_{L^2}\geq\frac{h_0}{2}\|u\|^2_{H^s}+c\sqrt{\mu}\|D^{\frac12}u\|^2_{H^s}.$$

The reverse inequality holds for any $\beta>0$ and is a consequence of (2.7), Hölder's inequality, the Sobolev embedding with $s>\frac32$, and conditions $(3.4)-(3.5)$. Indeed, we observe that

$$E_s(\mathbf{U})\leq\|\eta\|^2_{H^s}+\|\sqrt{\mathcal{K}_\mu}(D)u\|^2_{H^s}+\|\eta\|_{L^\infty}\|u\|^2_{H^s}\leq c_\beta\|(\eta,u)\|^2_{V^s_\mu}.$$

In the case $0<\beta<\frac13$, we impose the $\beta-$dependent surface condition (3.5), leaving less to be absorbed for the coercivity and in conjunction with (2.8). This implies

$$\big(J^su,(\mathcal{K}_\mu(D)+\eta)J^su\big)_{L^2}\geq\frac{\beta}{2}\|u\|^2_{H^s}+c\sqrt{\mu}\|D^{\frac12}u\|^2_{H^s}.$$

As a consequence, we have that (3.7) is established for all $\beta>0$.

Next, we prove (3.6). By using (3.1) and the fact that $Q(\mathbf{U},D)$ is self-adjoint, we compute

$$\begin{aligned}\frac12\frac{d}{dt}E_s(\mathbf{U})&=\big(J^s\partial_t\mathbf{U},Q(\mathbf{U},D)J^s\mathbf{U}\big)_{L^2}+\frac12\big(J^s\mathbf{U},(\partial_tQ(\mathbf{U},D))J^s\mathbf{U}\big)_{L^2}\\&=-\big(J^sM(\mathbf{U},D)\mathbf{U},Q(\mathbf{U},D)J^s\mathbf{U}\big)_{L^2}+\frac12\big(J^s\mathbf{U},(\partial_tQ(\mathbf{U},D))J^s\mathbf{U}\big)_{L^2}\\&=:-I+II.\end{aligned}$$

<u>Control of I</u>. We may write

$$\begin{aligned}I&=\big([J^s,M(\mathbf{U},D)]\mathbf{U},Q^{(1)}(\mathbf{U},D)J^s\mathbf{U}\big)_{L^2}+\big(Q^{(1)}(\mathbf{U},D)M(\mathbf{U},D)J^s\mathbf{U},J^s\mathbf{U}\big)_{L^2}\\&\quad+\big(J^sM(\mathbf{U},D)\mathbf{U},Q^{(2)}(\mathbf{U},D)J^s\mathbf{U}\big)_{L^2}\\&=:I_1+I_2+I_3.\end{aligned}$$

Control of I_1. It follows from the Cauchy-Schwarz inequality that

$$|I_1|\leq\|[J^s,M(\mathbf{U},D)]\mathbf{U}\|_{L^2}\|Q^{(1)}(\mathbf{U},D)J^s\mathbf{U}\|_{L^2}.$$

The second term is easily treated,

$$\|Q^{(1)}(\mathbf{U},D)J^s\mathbf{U}\|_{L^2}\lesssim\|J^s\eta\|_{L^2}+\|\eta\|_{L^\infty}\|J^su\|_{L^2}\lesssim\|(\eta,u)\|_{V^s_\mu},$$

by Hölder's inequality, the Sobolev embedding with $s>\frac12$, and assumption (3.4). Furthermore, using the Kato-Ponce commutator estimate (2.23) yields

$$\begin{aligned}\|[J^s,M(\mathbf{U},D)]\mathbf{U}\|_{L^2}&\leq\|[J^s,u]\partial_x\eta\|_{L^2}+\|[J^s,\eta]\partial_xu\|_{L^2}+\|[J^s,u]\partial_xu\|_{L^2}\\&\leq\|\eta\|_{H^s}\|u\|_{H^s}+\|u\|^2_{H^s}\\&\leq\|(\eta,u)\|^2_{V^s_\mu}.\end{aligned}$$

The desired bound on I_1 follows:

$$|I_1|\lesssim\|(\eta,u)\|^3_{V^s_\mu}.$$

Control of I_2+I_3. First note that $(a_{ij}) = Q^{(1)}(\mathbf{U},D)M(\mathbf{U},D)$ is given by,

$$(a_{ij}) = \begin{pmatrix} u\partial_x & (\mathcal{K}_\mu(D)+\eta)\partial_x \\ \eta\partial_x & \eta u\partial_x \end{pmatrix}.$$

We must estimate each piece below,

$$\begin{aligned}
&\big(Q^{(1)}(\mathbf{U},D)M(\mathbf{U},D)J^s\mathbf{U}, J^s\mathbf{U}\big)_{L^2} \\
&= \big(a_{11}J^s\eta, J^s\eta\big)_{L^2} + \big(a_{12}J^s u, J^s\eta\big)_{L^2} + \big(a_{21}J^s\eta, J^s u\big)_{L^2} + \big(a_{22}J^s u, J^s u\big)_{L^2} \\
&=: A_{11} + A_{12} + A_{21} + A_{22}.
\end{aligned}$$

As we will shortly see, $A_{12}+A_{21}$ needs to be compensated by B_{21}, that is defined by the remaining part:

$$\begin{aligned}
\big(J^s M(\mathbf{U},D)\mathbf{U}, Q^{(2)}(\mathbf{U},D)J^s\mathbf{U}\big)_{L^2} &= \big(\partial_x J^s\eta, \mathcal{K}_\mu(D)J^s u\big)_{L^2} + \big(J^s(u\partial_x u), \mathcal{K}_\mu(D)J^s u\big)_{L^2} \\
&=: B_{21} + B_{22},
\end{aligned}$$

while B_{22} is the price we pay for symmetry.

Control of A_{11}. Integration by part and the Sobolev embedding yields

$$|A_{11}| \leq \frac{1}{2}\big|\big(\partial_x u J^s\eta, J^s\eta\big)_{L^2}\big| \leq \frac{1}{2}\|\partial_x u\|_{L^\infty}\|\eta\|_{H^s}^2 \lesssim \|(\eta,u)\|_{V_\mu^s}^3.$$

Control of $A_{12}+A_{21}+B_{21}$. By definition, consideration is given to the expression

$$A_{12}+A_{21}+B_{21} = \big((\mathcal{K}_\mu(D)+\eta)\partial_x J^s u, J^s\eta\big)_{L^2} + \big((\mathcal{K}_\mu(D)+\eta)\partial_x J^s\eta, J^s u\big)_{L^2}.$$

Observe, after integration by parts that

$$A_{12} = -\big(J^s u, (\mathcal{K}_\mu(D)+\eta)\partial_x J^s\eta\big)_{L^2} - \big(J^s u, \partial_x\eta J^s\eta\big)_{L^2}.$$

The first term cancels with $(A_{21}+B_{21})$, while the Sobolev embedding easily controls the remaining part,

$$|\big(J^s u, \partial_x\eta J^s\eta\big)_{L^2}| \leq \|\partial_x\eta\|_{L^\infty}\|\eta\|_{H^s}\|u\|_{H^s} \lesssim \|(\eta,u)\|_{V_\mu^s}^3.$$

Control of A_{22}. We simply use integration by parts as above together with (3.4)−(3.5) to deduce

$$|A_{22}| \leq |\big(\eta u\partial_x J^s u, J^s u\big)_{L^2}| \leq c_\beta^2\|(\eta,u)\|_{V_\mu^s}^3.$$

Control of B_{22}. We observe, after integrating by parts that

$$\begin{aligned}
B_{22} &= \big(J^s(u\partial_x u), \mathcal{K}_\mu(D)J^s u\big)_{L^2} \\
&= \big([\sqrt{\mathcal{K}_\mu}(D)J^s, u]\partial_x u, \sqrt{\mathcal{K}_\mu}(D)J^s u\big)_{L^2} - \frac{1}{2}\big((\partial_x u)\sqrt{\mathcal{K}_\mu}(D)J^s u, \sqrt{\mathcal{K}_\mu}(D)J^s u\big)_{L^2}.
\end{aligned}$$

Thus, we deduce by using Hölder's inequality, estimates (2.18) and (2.7) that

$$|B_{22}| \leq c_\beta^2\|(\eta,u)\|_{V_\mu^s}^3.$$

<u>Control of II.</u> First we claim that $\|\mathcal{K}_\mu(D)\partial_x u\|_{L^\infty} \lesssim_\beta \|(\eta,u)\|_{V_\mu^s}$ for $s>2$. Indeed, it follows from (2.1) and the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ that

$$\begin{aligned}
\|\mathcal{K}_\mu(D)\partial_x u\|_{L^\infty} &\leq \|\partial_x u\|_{H^{s-\frac{3}{2}}} + \beta\sqrt{\mu}\|D^1\partial_x u\|_{H^{s-\frac{3}{2}}} \\
&\leq c_\beta^2(\|u\|_{H^s} + \sqrt{\mu}\|D^{\frac{1}{2}}u\|_{H^s}). \qquad (3.8)
\end{aligned}$$

Then we observe by using equation (3.1) yields,

$$II = \big(J^s u, (\partial_t\eta)J^s u\big)_{L^2} = -\big(J^s u, (\mathcal{K}_\mu(D)\partial_x u)J^s u\big)_{L^2} - \big(J^s u, (\partial_x(\eta u))J^s u\big)_{L^2}.$$

Consequently, the desired estimate follows from Hölder's inequality, the Sobolev embedding, and the above claim that,

$$|II| \lesssim \|\mathcal{K}_\mu(D)\partial_x u\|_{L^\infty}\|u\|_{H^s}^2 + \|\partial_x(\eta u)\|_{L^\infty}\|u\|_{H^s}^2 \lesssim_\beta \|(\eta,u)\|_{V_\mu^s}^3. \tag{3.9}$$

Adding together all the estimates, combined with (3.7) yields,

$$\frac{d}{dt}E_s(\mathbf{U}) \leq c_\beta^2\left(E_s(\mathbf{U})\right)^{\frac{3}{2}},$$

and completes the proof of Proposition 3.1.

□

3.2. **Estimates for system** (1.7)**.** As in the former subsection we define $\mathbf{U} = (\eta,u)^T = \varepsilon(\zeta,v)^T$ and we write the system on a compact form:

$$\partial_t\mathbf{U} + \mathcal{M}(\mathbf{U},D)\mathbf{U} = \mathbf{0}, \tag{3.10}$$

with

$$\mathcal{M}(\mathbf{U},D) = \begin{pmatrix} u\partial_x & (1+\eta)\partial_x \\ \mathcal{T}_\mu(D)\partial_x & u\partial_x \end{pmatrix}. \tag{3.11}$$

We define the symmetrizer associated to (3.10) to be

$$\mathcal{Q}(\mathbf{U},D) = \begin{pmatrix} \mathcal{T}_\mu(D) & 0 \\ 0 & 1+\eta \end{pmatrix}. \tag{3.12}$$

Then the energy given in Definition 1.18 can be written as

$$\mathcal{E}_s(\mathbf{U}) = \left(J^sJ_\mu^{\frac{1}{2}}\mathbf{U}, \mathcal{Q}(\mathbf{U},D)J^sJ_\mu^{\frac{1}{2}}\mathbf{U}\right)_{L^2}, \tag{3.13}$$

and the *a priori estimate* for (1.7) is stated in the following proposition.

Proposition 3.3. *Let $s > \frac{3}{2}$, $\varepsilon,\mu \in (0,1)$ and $(\eta,u) = \varepsilon(\zeta,v) \in C([0,T_0];V_\mu^s(\mathbb{R}))$ be a solution to* (1.7) *on a time interval $[0,T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \textit{and} \quad \sup_{t\in[0,T_0]}\|(\eta,u)\|_{H^s\times H^s} \leq h_1. \tag{3.14}$$

Then, for the energy given in Definition 1.18, there holds

$$\frac{d}{dt}\mathcal{E}_s(\mathbf{U}) \lesssim \left(\mathcal{E}_s(\mathbf{U})\right)^{\frac{3}{2}}, \tag{3.15}$$

for all $0 < t < T_0$, and

$$\|(\eta,u)\|_{V_\mu^s}^2 \lesssim \mathcal{E}_s(\mathbf{U}) \lesssim \|(\eta,u)\|_{V_\mu^s}^2, \tag{3.16}$$

for all $0 < t < T_0$.

Proof of Proposition 3.3. We begin by proving (3.16). By Definition (1.18) of the energy, the non-cavitation condition (3.14), (2.17), and (2.16) we obtain the lower bound

$$\begin{aligned}
\mathcal{E}_s(\mathbf{U}) &= \|\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\eta\|_{H^s}^2 + \left(J_\mu^{\frac{1}{2}}J^su, (1+\eta)J_\mu^{\frac{1}{2}}J^su\right)_{L^2} \\
&\geq c\|\eta\|_{H^s}^2 + h_0\|J_\mu^{\frac{1}{2}}u\|_{H^s}^2 \\
&\geq c\|\eta\|_{H^s}^2 + c\cdot h_0(\|u\|_{H^s}^2 + \sqrt{\mu}\|D^{\frac{1}{2}}u\|_{H^s}^2),
\end{aligned}$$

for some $c > 0$. The reverse inequality follows by the estimates (2.16), (2.17), Hölder's inequality, the Sobolev embedding, and (3.14):

$$\mathcal{E}_s(\mathbf{U}) \leq \|\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\eta\|_{H^s}^2 + \|J_\mu^{\frac{1}{2}}u\|_{H^s}^2 + \|\eta\|_{L^\infty}\|J_\mu^{\frac{1}{2}}u\|_{H^s}^2 \lesssim \|(\eta,u)\|_{V_\mu^s}^2.$$

Next, we prove (3.15). There follows by using (3.10) and the self-adjointness of $\mathcal{Q}(U,D)$ that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathcal{E}_s(\mathbf{U}) &= -\big(J^sJ_\mu^{\frac{1}{2}}\mathcal{M}(\mathbf{U},D)\mathbf{U}, \mathcal{Q}(\mathbf{U},D)J^sJ_\mu^{\frac{1}{2}}\mathbf{U}\big)_{L^2} \\ &\quad + \frac{1}{2}\big(J^sJ_\mu^{\frac{1}{2}}\mathbf{U}, (\partial_t\mathcal{Q}(\mathbf{U},D))J^sJ_\mu^{\frac{1}{2}}\mathbf{U}\big)_{L^2} \\ &=: -\mathcal{I} + \mathcal{II}.\end{aligned}$$

Control of $\mathcal{I}$. By definition of (3.13) we decompose $\mathcal{I}$ in four pieces,

$$\begin{aligned}\mathcal{I} &= \big(J^sJ_\mu^{\frac{1}{2}}(u\partial_x\eta), J^sJ_\mu^{\frac{1}{2}}\mathcal{T}_\mu(D)\eta\big)_{L^2} + \big(J^sJ_\mu^{\frac{1}{2}}\big((1+\eta)\partial_x u\big), J^sJ_\mu^{\frac{1}{2}}\mathcal{T}_\mu(D)\eta\big)_{L^2} \\ &\quad + \big(J^sJ_\mu^{\frac{1}{2}}\mathcal{T}_\mu(D)\partial_x\eta, (1+\eta)J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2} + \big(J^sJ_\mu^{\frac{1}{2}}u\partial_x u, (1+\eta)J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2} \\ &=: \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}.\end{aligned}$$

Control of $\mathcal{A}_{11}$. We aim to exploit symmetries, and we first write $\mathcal{A}_{11}$ as

$$\begin{aligned}\mathcal{A}_{11} &= \big([J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D), u]\partial_x\eta, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2} \\ &\quad + \big(uJ^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\partial_x\eta, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2} \\ &=: \mathcal{A}_{11}^1 + \mathcal{A}_{11}^2.\end{aligned}$$

The first term is treated by the commutator estimate (2.21) with $s > \frac{3}{2}$, the Cauchy-Schwarz inequality and (2.16). Thus, there holds

$$|\mathcal{A}_{11}^1| \leq \|[J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D), u]\partial_x\eta\|_{L^2}\|J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\|_{L^2} \lesssim \|u\|_{H^s}\|\eta\|_{H^s}^2.$$

Similar to previous estimates, we use integration by parts and exploit the symmetries of $\mathcal{A}_{11}^2$, then conclude by (2.16), and the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ that

$$|\mathcal{A}_{11}| \leq \frac{1}{2}\big|\big((\partial_x u)J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2}\big| \lesssim \|(\eta,u)\|_{V_\mu^s}^3.$$

Control of $\mathcal{A}_{12} + \mathcal{A}_{21}$. We first decompose $\mathcal{A}_{12}$ in two parts

$$\begin{aligned}\mathcal{A}_{12} &= \big([J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D), \eta]\partial_x u, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2} \\ &\quad + \big((1+\eta)J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\partial_x u, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2} \\ &= \mathcal{A}_{12}^1 + \mathcal{A}_{12}^2.\end{aligned}$$

We estimate $\mathcal{A}_{12}^1$ the same way we did for $\mathcal{A}_{11}^1$ and obtain

$$|\mathcal{A}_{12}^1| \lesssim \|u\|_{H^s}\|\eta\|_{H^s}^2.$$

For the second term, after integration by parts, we find

$$\begin{aligned}\mathcal{A}_{12}^2 &= -\big((\partial_x\eta)J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)u, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2} \\ &\quad - \big((1+\eta)J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)u, J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\partial_x\eta\big)_{L^2} \\ &= \mathcal{A}_{12}^{2,1} + \mathcal{A}_{12}^{2,2}.\end{aligned}$$

By using the Sobolev embedding and (2.16), we find that

$$|\mathcal{A}_{12}^{2,1}| \leq \|\partial_x\eta\|_{L^\infty}\|J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}u\|_{H^s}\|J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}\eta\|_{H^s} \lesssim \|u\|_{H^s}\|\eta\|_{H^s}^2.$$

On the other hand, we cannot estimate $\mathcal{A}_{12}^{2,2}$ on its own. We must therefore cancel it with $\mathcal{A}_{21}$. Observe

$$\begin{aligned}\mathcal{A}_{21} &= \big(J^s J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\partial_x\eta, [\sqrt{\mathcal{T}_\mu}(D),\eta]J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2}\\ &\quad+\big(J^s J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\partial_x\eta,(1+\eta)J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)u\big)_{L^2}\\ &=\mathcal{A}_{21}^1+\mathcal{A}_{21}^2.\end{aligned}$$

First, by using integration by parts, the Cauchy-Schwarz inequality, (2.16), (2.35) and (2.17) we find that

$$|\mathcal{A}_{21}^1| = \|J^sJ_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\eta\|_{L^2}\|\partial_x[\sqrt{\mathcal{T}_\mu}(D),\eta]J^sJ_\mu^{\frac{1}{2}}u\|_{L^2}\lesssim\|(\eta,u)\|^3_{V^s_\mu}. \tag{3.17}$$

On the other hand, we observe that $\mathcal{A}_{21}^2=-\mathcal{A}_{21}^{2,2}$. We may therefore conclude that the sum satisfies:

$$|\mathcal{A}_{12}+\mathcal{A}_{21}|\lesssim\|(\eta,u)\|^3_{V^s_\mu}.$$

Control of $\mathcal{A}_{22}$. Similar to $\mathcal{A}_{11}$ we write the expression with the good commutator:

$$\begin{aligned}\mathcal{A}_{22}&=\big([J^sJ_\mu^{\frac{1}{2}},u]\partial_xu,(1+\eta)J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2}+\big(uJ^sJ_\mu^{\frac{1}{2}}\partial_xu,(1+\eta)J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2}\\&=\mathcal{A}_{22}^1+\mathcal{A}_{22}^2.\end{aligned}$$

Then use the Cauchy-Schwarz inequality, (3.14), (2.20) with $s>\frac{3}{2}$, and the Sobolev embedding to get

$$|\mathcal{A}_{22}^1|\lesssim\|[J^sJ_\mu^{\frac{1}{2}},u]\partial_xu\|_{L^2}(1+\|\eta\|_{L^\infty})\|J^sJ_\mu^{\frac{1}{2}}u\|_{L^2}\lesssim\|(\eta,u)\|^3_{V^s_\mu}.$$

While for $\mathcal{A}_{22}^2$ we integrate by parts, apply the Sobolev embedding, and again bound each term by the $V^s_\mu-$norm of (η,u) to obtain that

$$|\mathcal{A}_{22}^2|\lesssim(\|\partial_xu\|_{L^\infty}+\|\partial_x\eta\|_{L^\infty})\|J_\mu^{\frac{1}{2}}u\|^2_{H^s}\lesssim\|(\eta,u)\|^3_{V^s_\mu}.$$

Gathering all these estimates, we conclude that

$$|\mathcal{I}|\lesssim\|(\eta,u)\|^3_{V^s_\mu}. \tag{3.18}$$

<u>Control of $\mathcal{II}$.</u> By defintion of (3.12) and (3.10) we get that,

$$\begin{aligned}\mathcal{II}&=\big(J^sJ_\mu^{\frac{1}{2}}u,(\partial_t\eta)J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2}\\&=-\big(J^sJ_\mu^{\frac{1}{2}}u,(\partial_xu)J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2}-\big(J^sJ_\mu^{\frac{1}{2}}u,(\partial_x(\eta u))J^sJ_\mu^{\frac{1}{2}}u\big)_{L^2}\end{aligned}$$

Then, by using Hölder's inequality, the Sobolev embedding, (3.14) and (2.17), we deduce that

$$|\mathcal{II}|\lesssim(\|\partial_xu\|_{L^\infty}+\|\partial_x(\eta u)\|_{L^\infty})\|J_\mu^{\frac{1}{2}}u\|^2_{H^s}\lesssim\|(\eta,u)\|^3_{V^s_\mu}. \tag{3.19}$$

Consequently, we may add (3.18) and (3.19), then apply (3.16) to conclude the proof of estimate (3.15).

□

3.3. **Estimates for system** (1.9)**.** As in the former subsections we let $\mathbf{U} = (\eta, u)^T = \varepsilon(\zeta, v)^T$ and write the system on the form

$$\partial_t \mathbf{U} + \mathscr{M}(\mathbf{U}, D)\mathbf{U} = \mathbf{0}, \tag{3.20}$$

with

$$\mathscr{M}(\mathbf{U}, D) = \begin{pmatrix} \mathcal{T}_\mu(D)(u\partial_x \cdot) & \partial_x + \mathcal{T}_\mu(D)(\eta\partial_x \cdot) \\ \mathcal{K}_\mu(D)\partial_x & \mathcal{T}_\mu(D)(u\partial_x \cdot) \end{pmatrix}. \tag{3.21}$$

The symmetrizer is defined by

$$\mathscr{Q}(\mathbf{U}, D) = \begin{pmatrix} \mathcal{T}_\mu^{-1}(D)\mathcal{K}_\mu(D) & 0 \\ 0 & \mathcal{T}_\mu^{-1}(D) + \eta \end{pmatrix}. \tag{3.22}$$

Then the energy given in Definition 1.19 can be written as

$$\mathscr{E}_s(\mathbf{U}) = \big(J^s\mathbf{U}, \mathscr{Q}(\mathbf{U}, D)J^s\mathbf{U}\big)_{L^2}, \tag{3.23}$$

and the *a priori estimate* for (1.9) is stated in the following proposition.

Proposition 3.4. *Let $s > \frac{3}{2}$, $\varepsilon, \mu \in (0,1)$, $\beta \geq 0$, and let $(\eta, u) = \varepsilon(\zeta, v) \in C([0, T_0]; X^s_{\beta,\mu}(\mathbb{R}))$ be a solution to* (1.9) *on a time interval $[0, T_0]$ for some $T_0 > 0$. Moreover, assume that there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \textit{and} \quad \sup_{t \in [0,T_0]} \|(\eta, u)\|_{H^s \times H^s} \leq h_1. \tag{3.24}$$

Then, for the energy given in Definition 1.19, there holds,

$$\frac{d}{dt}\mathscr{E}_s(\mathbf{U}) \lesssim c_\beta^2 \big(\mathscr{E}_s(\mathbf{U})\big)^{\frac{3}{2}}, \tag{3.25}$$

and the energy is coercive:

$$\|(\eta, u)\|^2_{X^s_{\beta,\mu}} \lesssim \mathscr{E}_s(\mathbf{U}) \lesssim \|(\eta, u)\|^2_{X^s_{\beta,\mu}}. \tag{3.26}$$

Proof of Proposition 3.4. We will first provide the coercivity estimate (3.26). By Definition 1.19 for the energy, the non-cavitation condition (3.24) and (2.15) we obtain the lower bound

$$\begin{aligned} \mathscr{E}_s(\mathbf{U}) &= \|\langle \sqrt{\beta\mu} D^1 \rangle \eta\|^2_{H^s} + \big(J^s u, (\mathcal{T}_\mu^{-1}(D) + \eta)J^s u\big)_{L^2} \\ &\geq \|\eta\|^2_{H^s} + \beta\mu\|D^1\eta\|^2_{H^s} + \frac{h_0}{2}\|u\|^2_{H^s} + c\sqrt{\mu}\|D^{\frac{1}{2}}u\|^2_{H^s}, \end{aligned}$$

for some $c > 0$ and $\beta \geq 0$. The reverse inequality follows by the upper bound in (2.15), the Sobolev embedding and (3.24):

$$\mathscr{E}_s(\mathbf{U}) \leq \|\langle \sqrt{\beta\mu} D^1 \rangle \eta\|^2_{H^s} + \|\sqrt{\mathcal{T}_\mu}^{-1}(D)J^s u\|^2 + \|\eta\|_{L^\infty}\|J^s u\|^2 \lesssim \|(\eta, u)\|^2_{X^s_{\beta,\mu}}.$$

We may now prove (3.25). To do so, we use (3.20) and the self-adjointness of $\mathscr{Q}(\mathbf{U}, D)$ to write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathscr{E}_s(\mathbf{U}) &= -\big(J^s\mathscr{M}(\mathbf{U}, D)\mathbf{U}, \mathscr{Q}(\mathbf{U}, D)J^s\mathbf{U}\big)_{L^2} + \frac{1}{2}\big(J^s\mathbf{U}, (\partial_t\mathscr{Q}(\mathbf{U}, D))J^s\mathbf{U}\big)_{L^2} \\ &=: \mathscr{I} + \mathscr{I}\mathscr{I}. \end{aligned}$$

<u>Control of $\mathscr{I}$.</u> By definition of (3.23) we must estimate the following terms:

$$\begin{aligned} \mathscr{I} &= \big(J^s(u\partial_x\eta), \mathcal{K}_\mu(D)J^s\eta\big)_{L^2} + \big(J^s\partial_x u + J^s\mathcal{T}_\mu(D)(\eta\partial_x u), \mathcal{T}_\mu^{-1}(D)\mathcal{K}_\mu(D)J^s\eta\big)_{L^2} \\ &\quad + \big(J^s\mathcal{K}_\mu(D)\partial_x\eta, (\mathcal{T}_\mu^{-1}(D) + \eta)J^s u\big)_{L^2} + \big(J^s\mathcal{T}_\mu(D)(u\partial_x u), (\mathcal{T}_\mu^{-1}(D) + \eta)J^s u\big)_{L^2} \\ &=: \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}. \end{aligned}$$

Control of $\mathscr{A}_{11}$. We rewrite $\mathscr{A}_{11}$ as

$$\begin{aligned}\mathscr{A}_{11} &= \big([J^s\sqrt{\mathcal{K}_\mu}(D),u]\partial_x\eta, J^s\sqrt{\mathcal{K}_\mu}(D)\eta\big)_{L^2} + \big(uJ^s\sqrt{\mathcal{K}_\mu}(D)\partial_x\eta, J^s\sqrt{\mathcal{K}_\mu}(D)\eta\big)_{L^2}\\ &=: \mathscr{A}_{11}^1 + \mathscr{A}_{11}^2.\end{aligned}$$

Then in the case $\beta > 0$, we first observe by interpolation and Young's inequality that

$$\sqrt{\beta}\mu^{\frac{1}{4}}\|\eta\|_{H^{s+\frac{1}{2}}} \le \|\eta\|_{H^s}^{\frac{1}{2}}\big(\beta\sqrt{\mu}\|\eta\|_{H^{s+1}}\big)^{\frac{1}{2}} \lesssim_\beta \|\eta\|_{H^s} + \beta\sqrt{\mu}\|D^1\eta\|_{H^s}, \tag{3.27}$$

and thus $\mathscr{A}_{11}^1$ is treated by the Cauchy-Schwarz inequality, the commutator estimate (2.18) with $s > \frac{3}{2}$, (2.7), and (3.27):

$$|\mathscr{A}_{11}^1| \lesssim c_\beta^2(\|u\|_{H^s} + \mu^{\frac{1}{2}}\|u\|_{H^{s+\frac{1}{2}}})(\|\eta\|_{H^s} + \sqrt{\beta}\mu^{\frac{1}{4}}\|\eta\|_{H^{s+\frac{1}{2}}})^2 \lesssim c_\beta^2\|(\eta,u)\|^3_{X^s_{\beta,\mu}}.$$

On the other hand, for $\mathscr{A}_{11}^2$ we conclude by integration by parts, (2.7), (3.27), and the Sobolev embedding with $s > \frac{3}{2}$ that

$$|\mathscr{A}_{11}| \lesssim |\mathscr{A}_{11}^1| + \|\partial_x u\|_{L^\infty}\|\sqrt{\mathcal{K}_\mu}(D)\eta\|_{H^s}\|\sqrt{\mathcal{K}_\mu}(D)\eta\|_{H^s} \lesssim c_\beta^2\|(\eta,u)\|^3_{X^s_{\beta,\mu}},$$

for $\beta > 0$. Moreover, in the case $\beta = 0$, then $\mathcal{K}_\mu(D)$ is equal to $\mathcal{T}_\mu(D)$ and we simply use Hölder's inequality, (2.33), (2.14), the Sobolev embedding, and integration by parts to deduce the estimate

$$\begin{aligned}|\mathscr{A}_{11}| &\le |\big([J^s\sqrt{\mathcal{T}_\mu}(D),u]\partial_x\eta, J^s\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2}| + |\big(uJ^s\sqrt{\mathcal{T}_\mu}(D)\partial_x\eta, J^s\sqrt{\mathcal{T}_\mu}(D)\eta\big)_{L^2}|\\ &\lesssim \|u\|_{H^s}\|\eta\|^2_{H^s}.\end{aligned}$$

Control of $\mathscr{A}_{12} + \mathscr{A}_{21}$. By using integration by parts we write,

$$\begin{aligned}\mathscr{A}_{12} &= \big(J^s(\eta\partial_x u), \mathcal{K}_\mu(D)J^s\eta\big)_{L^2} - \big(J^s u, \mathcal{T}_\mu^{-1}(D)\mathcal{K}_\mu(D)J^s\partial_x\eta\big)_{L^2}\\ &= \mathscr{A}_{12}^1 + \mathscr{A}_{12}^2.\end{aligned}$$

For $\mathscr{A}_{12}^1$, observe

$$\begin{aligned}\mathscr{A}_{12}^1 &= \big([J^s,\eta]\partial_x u, \mathcal{K}_\mu(D)J^s\eta\big)_{L^2} - \big((\partial_x\eta)J^s u, \mathcal{K}_\mu(D)J^s\eta\big)_{L^2} - \big(\eta J^s u, \mathcal{K}_\mu(D)J^s\partial_x\eta\big)_{L^2}\\ &= \mathscr{A}_{12}^{1,1} + \mathscr{A}_{12}^{1,2} + \mathscr{A}_{12}^{1,3}.\end{aligned}$$

Then in the case $\beta > 0$ we use the Kato-Ponce commutator estimate (2.9), the Sobolev embedding, and the pointwise estimate (2.1) combined with Plancherel imply that

$$|\mathscr{A}_{12}^{1,1} + \mathscr{A}_{12}^{1,2}| \lesssim c_\beta^2\|u\|_{H^s}\|\eta\|_{H^s}(\|\eta\|_{H^s} + \beta\sqrt{\mu}\|D^1\eta\|_{H^s}) \lesssim c_\beta^2\|(\eta,u)\|^3_{X^s_{\beta,\mu}}.$$

While for the case $\beta = 0$, we simply use the boundedness of $\mathcal{T}_\mu(D)$ on $L^2(\mathbb{R})$ to deduce,

$$|\mathscr{A}_{12}^{1,1} + \mathscr{A}_{12}^{1,2}| \lesssim \|u\|_{H^s}\|\eta\|^2_{H^s}.$$

However, in either case the contribution of remaining terms, $\mathscr{A}_{12}^{1,3} + \mathscr{A}_{12}^2$, will be canceled by $\mathscr{A}_{21}$. Indeed, we observe that

$$\mathscr{A}_{21} = \big(J^s\mathcal{K}_\mu(D)\partial_x\eta, \eta J^s u\big)_{L^2} + \big(J^s\mathcal{K}_\mu(D)\partial_x\eta, \mathcal{T}_\mu^{-1}(D)J^s u\big)_{L^2} = -\mathscr{A}_{12}^{1,3} - \mathscr{A}_{12}^2.$$

Hence, combining these identities and estimates gives the bound

$$|\mathscr{A}_{12} + \mathscr{A}_{21}| \lesssim c_\beta^2\|(\eta,u)\|^3_{X^s_{\beta,\mu}}.$$

Control of $\mathscr{A}_{22}$. Consider the two terms:

$$\mathscr{A}_{22} = \big(J^s(u\partial_x u), J^s u\big)_{L^2} + \big(J^s\mathcal{T}_\mu(D)(u\partial_x u), \eta J^s u\big)_{L^2} = \mathscr{A}_{22}^1 + \mathscr{A}_{22}^2.$$

The control of $\mathscr{A}_{22}^1$ is a direct consequence of the Kato-Ponce commutator estimate (2.9) and integration by parts. Since $s > \frac{3}{2}$, we have that

$$|\mathscr{A}_{22}^1| \leq \big|\big([J^s, u]\partial_x u, J^s u\big)_{L^2}\big| + \frac{1}{2}\big|\big((\partial_x u) J^s u, J^s u\big)_{L^2}\big| \lesssim \|u\|_{H^s}^3.$$

To deal with $\mathscr{A}_{22}^2$, we make the decomposition

$$\begin{aligned}\mathscr{A}_{22}^2 &= \big([J^s\sqrt{\mathcal{T}_\mu}(D), u]\partial_x u, \sqrt{\mathcal{T}_\mu}(D)\eta J^s u\big)_{L^2} + \big(uJ^s\sqrt{\mathcal{T}_\mu}(D)\partial_x u, [\sqrt{\mathcal{T}_\mu}(D), \eta]J^s u\big)_{L^2} \\ &\quad + \big(uJ^s\sqrt{\mathcal{T}_\mu}(D)\partial_x u, \eta\sqrt{\mathcal{T}_\mu}(D)J^s u\big)_{L^2} \\ &= \mathscr{A}_{22}^{2,1} + \mathscr{A}_{22}^{2,2} + \mathscr{A}_{22}^{2,3}.\end{aligned}$$

Then for $\mathscr{A}_{22}^{2,1}$ we employ the Cauchy-Schwarz inequality, (2.33), (3.24), (2.14), and the Sobolev embedding to deduce

$$|\mathscr{A}_{22}^{2,1}| \leq \|[J^s\sqrt{\mathcal{T}_\mu}(D), u]\partial_x u\|_{L^2}\|\sqrt{\mathcal{T}_\mu}(D)(\eta J^s u)\|_{L^2} \lesssim \|(\eta, u)\|_{X_{\beta,\mu}^s}^3.$$

Before we treat $\mathscr{A}_{22}^{2,2}$, we note that $\|[\sqrt{\mathcal{T}_\mu}(D), \eta]J^s u\|_{L^2} \lesssim \|\eta\|_{H^s}\|u\|_{H^s}$. Indeed, using (2.14) and the Sobolev embedding we find that

$$\|[\sqrt{\mathcal{T}_\mu}(D), \eta]J^s u\|_{L^2} \lesssim \|\eta\|_{L^\infty}\|J^s u\|_{L^2} \lesssim \|\eta\|_{H^s}\|u\|_{H^s}. \tag{3.28}$$

Consequently, using integration by parts, the Cauchy-Schwarz inequality, (3.28) and (2.35) we get

$$\begin{aligned}|\mathscr{A}_{22}^{2,2}| &= \big|\big((\partial_x u)J^s\sqrt{\mathcal{T}_\mu}(D)u, [\sqrt{\mathcal{T}_\mu}(D), \eta]J^s u\big)_{L^2}\big| \\ &\quad + \big|\big(uJ^s\sqrt{\mathcal{T}_\mu}(D)u, \partial_x[\sqrt{\mathcal{T}_\mu}(D), \eta]J^s u\big)_{L^2}\big| \\ &\lesssim \|\eta\|_{H^s}\|u\|_{H^s}^3,\end{aligned}$$

then use (3.24) on one term. Similarly, for $\mathscr{A}_{22}^{2,3}$ we use integration by parts, the Sobolev embedding, and (2.14) to get the bound

$$|\mathscr{A}_{22}^{2,3}| \lesssim \|\partial_x(\eta u)\|_{L^\infty}\|u\|_{H^s}^2.$$

Therefore, we conclude by (3.24) and gathering all these estimates that

$$|\mathscr{A}_{22}| \lesssim \|(\eta, u)\|_{X_{\beta,\mu}^s}^3,$$

and by extension, we have the bound

$$|\mathscr{I}| \lesssim_\beta \|(\eta, u)\|_{X_{\beta,\mu}^s}^3.$$

<u>Control of $\mathscr{I}\mathscr{I}$.</u> By defintion of (3.22) and (3.20) we get that,

$$\begin{aligned}\mathscr{I}\mathscr{I} &= \big(J^s u, (\partial_t \eta)J^s u\big)_{L^2} \\ &= -\big(J^s u, (\partial_x u)J^s u\big)_{L^2} - \big(J^s u, (\mathcal{T}_\mu(D)\partial_x(\eta u))J^s u\big)_{L^2},\end{aligned}$$

Then the final estimate follows by the Cauchy-Schwarz inequality, (3.24) and the fact that $\mathcal{T}_\mu(D)$ is bounded on $L^2(\mathbb{R})$, then apply Hölder's inequality, and the Sobolev embedding to deduce

$$|\mathscr{I}\mathscr{I}| \leq \|\partial_x u\|_{L^\infty}\|u\|_{H^s}^2 + \|\mathcal{T}_\mu(D)\partial_x(\eta u)\|_{L^\infty}\|u\|_{H^s}^2 \lesssim \|(\eta, u)\|_{X_{\beta,\mu}^s}^3.$$

□

4. Estimates for the difference of two solutions

4.1. **Estimates for system** (1.5)**.** We will now estimate the difference between two solutions of (1.5) given by $\mathbf{U}_1 = (\eta_1, u_1)^T = \varepsilon(\zeta_1, v_1)^T$ and $\mathbf{U}_2 = (\eta_2, u_2)^T = \varepsilon(\zeta_2, v_2)^T$. For convenience, we define $(\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then $\mathbf{W} = (\psi, w)^T$ solves

$$\partial_t \mathbf{W} + M(\mathbf{U}_1, D)\mathbf{W} = \mathbf{F}, \tag{4.1}$$

with M defined as in (3.2) and $\mathbf{F} = -\big(M(\mathbf{U}_1, D) - M(\mathbf{U}_2, D)\big)\mathbf{U}_2$. Specifically, the source term is given by

$$\mathbf{F} = -\begin{pmatrix} w\partial_x \eta_2 + \psi \partial_x u_2 \\ w \partial_x u_2 \end{pmatrix}. \tag{4.2}$$

The energy associated to (4.1) is given in terms of the symmetrizer $Q(\mathbf{U}_1, D)$ defined in (3.3) and reads

$$\tilde{E}_s(\mathbf{W}) := \big(J^s\mathbf{W}, Q(\mathbf{U}_1, D)J^s\mathbf{W}\big)_{L^2}. \tag{4.3}$$

The main result of this section reads:

Proposition 4.1. *Take $s > 2$ and $\varepsilon, \mu \in (0,1)$. Let $(\eta_1, u_1), (\eta_2, u_2) \in C([0, T_0] : V^s_\mu(\mathbb{R}))$ be two solutions of* (1.5) *on a time interval $[0, T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \text{and} \quad \sup_{t \in [0,T_0]} \|(\eta_1, u_1)\|_{H^s \times H^s} \leq h_1, \tag{4.4}$$

when $\beta \geq \frac{1}{3}$, and that

$$-\frac{\beta}{2} \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \text{and} \quad \sup_{t \in [0,T_0]} \|(\eta_1, u_1)\|_{H^s \times H^s} \leq h_1, \tag{4.5}$$

when $0 < \beta < \frac{1}{3}$.

Define the difference to be $\mathbf{W} = (\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then, for the energy defined by (4.3)*, there holds*

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\beta \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu}, \tag{4.6}$$

and

$$\|(\psi, w)\|^2_{V^0_\mu} \lesssim_\beta \tilde{E}_0(\mathbf{W}) \lesssim_\beta \|(\psi, w)\|^2_{V^0_\mu}. \tag{4.7}$$

Furthermore, we have the following estimate at the $V^s_\mu-$ level:

$$\frac{d}{dt}\tilde{E}_s(\mathbf{W}) \lesssim_\beta \big|\big(J^s\mathbf{F}, Q(\mathbf{U}_1, D)J^s\mathbf{W}\big)_{L^2}\big| + \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^s_\mu}, \tag{4.8}$$

and

$$\|(\psi, w)\|^2_{V^s_\mu} \lesssim_\beta \tilde{E}_s(\mathbf{W}) \lesssim_\beta \|(\psi, w)\|^2_{V^s_\mu}. \tag{4.9}$$

Remark 4.2. *The source term corresponding to* $\mathbf{F}$ *given by* (4.8) *will be treated in the proof of Theorem 1.6 by using regularization estimates and a classical Bona-Smith argument* [7].

Proof of Proposition 4.1. First, the proofs of (4.7) and (4.9) are similar to the one of (3.7).

Next, we only prove (4.6), where (4.8) is more straightforward and follows the same line, utilizing similar estimates to those applied for the proof of Proposition 3.1.

To prove (4.6), we use (4.1) and the self-adjointness of $Q(\mathbf{U}_1, D)$ to write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\tilde{E}_0(\mathbf{W}) &= \frac{1}{2}\big(\mathbf{W}, (\partial_t Q(\mathbf{U}_1, D))\mathbf{W}\big)_{L^2} + \big(\mathbf{F}, Q(\mathbf{U}_1, D)\mathbf{W}\big)_{L^2} \\ &\quad - \big(M(\mathbf{U}_1, D)\mathbf{W}, Q(\mathbf{U}_1, D)\mathbf{W}\big)_{L^2} \\ &=: I - II - III.\end{aligned}$$

Control of I. We estimate the first term for $s > 2$ by arguing similarly to estimate (3.9). Indeed, we have that

$$I = (w, (\partial_t \eta_1)w) \lesssim \|\partial_t \eta_1\|_{L^\infty}\|w\|_{L^2}^2 \lesssim_\beta \|(u_1, \eta_1)\|_{V_\mu^s}\|(\psi, w)\|_{V_\mu^0}^2.$$

Control of II. For II, we write

$$\begin{aligned}II &= \big(w\partial_x\eta_2, \psi\big)_{L^2} + \big(\psi\partial_x\eta_2, \psi\big)_{L^2} + \big(w\partial_x u_2, \eta_1 w\big)_{L^2} + \big(w\partial_x u_2, \mathcal{K}_\mu(D)w\big)_{L^2} \\ &=: II_1 + II_2 + II_3 + II_4.\end{aligned}$$

The first three terms are treated by the Cauchy-Schwarz inequality and the Sobolev embedding. Take, for instance, II_1:

$$|II_1| \lesssim \|w\partial_x\eta_2\|_{L^2}\|\psi\|_{L^2} \lesssim \|\eta_2\|_{H^s}\|(\psi, w)\|_{V_\mu^0}^2,$$

for $s > \frac{3}{2}$. Then estimating $II_2 + II_3$ similarly gives

$$|II_1 + II_2 + II_3| \lesssim \max_{i=1,2}\|(\eta_i, u_i)\|_{V_\mu^s}\|(\psi, w)\|_{V_\mu^0}^2.$$

Regarding the term containing the multiplier $\mathcal{K}_\mu(D)$, we write

$$II_4 \leq \|\sqrt{\mathcal{K}_\mu}(D)(w\partial_x u_2)\|_{L^2}\|\sqrt{\mathcal{K}_\mu}(D)w\|_{L^2} =: II_4^1 \cdot II_4^2,$$

and make the observation

$$\begin{aligned}II_4^1 &\leq \|(\sqrt{\mathcal{K}_\mu}(D) - \sqrt{\beta}\mu^{\frac{1}{4}}D^{\frac{1}{2}})(w\partial_x u_2)\|_{L^2} + \sqrt{\beta}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}(w\partial_x u_2)\|_{L^2} \\ &=: II_4^{1,1} + \sqrt{\beta}II_4^{1,2}.\end{aligned}$$

For the first term, we note that $(\sqrt{\mathcal{K}_\mu}(D) - \sqrt{\beta}\mu^{\frac{1}{4}}D^{\frac{1}{2}})$ is bounded on $L^2(\mathbb{R})$ by (2.5), and we can conclude by the Sobolev embedding that

$$II_4^{1,1} \lesssim_\beta \|w\partial_x\eta_2\|_{L^2} \lesssim_\beta \|\eta_2\|_{H^s}\|(\psi, w)\|_{V_\mu^0}.$$

For the remaining term, $II_4^{1,2}$, we first make an observation. Let $\nu = \frac{1}{2}^-$ and $(p_1, p_2) = (\frac{1}{\nu}, \frac{2}{1-2\nu})$ then by (2.37) there holds

$$\|D^{\frac{1}{2}}\partial_x u_2\|_{L^{p_1}}\mu^{\frac{1}{4}}\|w\|_{L^{p_2}} \lesssim \|u_2\|_{H^{2-\nu}}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}w\|_{L^2}. \tag{4.10}$$

Moreover, by the fractional Leibniz rule (2.38), the triangle inequality and Hölder's inequality yields the bound

$$\begin{aligned}II_4^{1,2} &\lesssim \mu^{\frac{1}{4}}\|D^{\frac{1}{2}}(w\partial_x u_2) - wD^{\frac{1}{2}}\partial_x u_2 - (\partial_x u_2)D^{\frac{1}{2}}w\|_{L^2} + \mu^{\frac{1}{4}}\|wD^{\frac{1}{2}}\partial_x u_2\|_{L^2} \\ &\quad + \mu^{\frac{1}{4}}\|(\partial_x u_2)D^{\frac{1}{2}}w\|_{L^2} \\ &\lesssim \|D^{\frac{1}{2}}\partial_x u_2\|_{L^{p_1}}\mu^{\frac{1}{4}}\|w\|_{L^{p_2}} + \|w\|_{L^2}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}\partial_x u_2\|_{L^\infty} + \|\partial_x u_2\|_{L^\infty}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}w\|_{L^2}.\end{aligned}$$

Now, since $\frac{1}{p_1} + \frac{1}{p_2} = \nu + \frac{1-2\nu}{2} = \frac{1}{2}$, we may apply (4.10) to deal with the first term, and combined with the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we deduce that

$$II_4^{1,2} \lesssim \|u_2\|_{H^s}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}w\|_{L^2} + \|w\|_{L^2}\mu^{\frac{1}{4}}\|D^{\frac{1}{2}}u_2\|_{H^s},$$

with $s > \frac{3}{2}$. Consequently, the bound on II_4 is given by

$$II_4 \lesssim_\beta \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu},$$

which allows us to conclude that

$$II \lesssim_\beta \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu}.$$

Control of III. By definition, we must estimate:

$$\begin{aligned} III &= \big(u_1\partial_x\psi, \psi\big)_{L^2} + \big((\mathcal{K}_\mu(D) + \eta_1)\partial_x w, \psi\big)_{L^2} \\ &\quad + \big(\partial_x\psi, (\mathcal{K}_\mu(D) + \eta_1)w\big)_{L^2} + \big(u_1\partial_x w, (\mathcal{K}_\mu(D) + \eta_1)w\big)_{L^2} \\ &= A_1 + A_2 + A_3 + A_4. \end{aligned}$$

The first term is handled by integration by parts and the Sobolev embedding

$$|A_1| \lesssim \|\partial_x u_1\|_{L^\infty} \|\psi\|^2_{L^2} \lesssim \|u_1\|_{H^s} \|\psi\|^2_{L^2}.$$

Next, we observe a cancelation in the off-diagonal terms due to the symmetry. Indeed, we see after integrating by parts that

$$A_2 = -\big((\partial_x\eta_1)w, \psi\big)_{L^2} - A_3.$$

Consequently, we observe after using Hölder's inequality and the Sobolev embedding that

$$|A_2 + A_3| \lesssim \|\partial_x\eta_1\|_{L^\infty} \|w\|_{L^2} \|\psi\|_{L^2}.$$

The only term remaining is A_4, which contains the multiplier that will need some more care. In particular, we write

$$\begin{aligned} A_4 &= \big(u_1\partial_x w, \eta_1 w\big)_{L^2} + \big(u_1\partial_x w, \mathcal{K}_\mu(D)w\big)_{L^2} \\ &=: A_4^1 + A_4^2. \end{aligned}$$

The first term is again treated by integration by parts, and we obtain the bound

$$|A_4^1| \lesssim \|u_1\|_{H^s} \|\eta_1\|_{H^s} \|w\|^2_{L^2}.$$

Lastly, to estimate A_4^2, we split the kernel $\mathcal{K}_\mu(D)$ into several pieces that are localized in low and high frequencies:

$$\mathcal{K}_\mu(D) = ((\chi^{(1)}_\mu)^2\mathcal{K}_\mu)(D) + ((\chi^{(2)}_\mu)^2(\sigma_{\mu,\frac{1}{2}})^2)(D) - ((\chi^{(2)}_\mu)^2(\sigma_{\mu,0})^2)(D), \tag{4.11}$$

where $\sigma_{\mu,\frac{1}{2}}(D)$ is defined in (2.26), $\sigma_{\mu,0}(D)$ is defined in (2.29) and $\chi^{(i)}_\mu(D)$ with its porperties given by Definition 2.10. Then, we get that

$$\begin{aligned} \big(u_1\partial_x w, \mathcal{K}_\mu(D)w\big)_{L^2} &= \big((\chi^{(1)}_\mu\sqrt{\mathcal{K}_\mu})(D)(u_1\partial_x w), (\chi^{(1)}_\mu\sqrt{\mathcal{K}_\mu})(D)w\big)_{L^2} \\ &\quad + \big((\chi^{(2)}_\mu\sigma_{\mu,\frac{1}{2}})(D)(u_1\partial_x w), (\chi^{(2)}_\mu\sigma_{\mu,\frac{1}{2}})(D)w\big)_{L^2} \\ &\quad - \big((\chi^{(2)}_\mu\sigma_{\mu,0})(D)(u_1\partial_x w), (\chi^{(2)}_\mu\sigma_{\mu,0})(D)w\big)_{L^2} \\ &=: A_4^{2,1} + A_4^{2,2} - A_4^{2,3}. \end{aligned}$$

We treat each term individually using the commutator estimates in Lemma 2.11, where the remaining part is symmetric and is treated by using integration by parts and the Sobolev embedding in the usual way.

Control of $A_4^{2,1}$. Proceeding as explained above, we have that

$$\begin{aligned} A_4^{2,1} &= \big([(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D), u_1]\partial_x w, (\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)w\big)_{L^2} \\ &\quad + \big(u_1(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)\partial_x w, (\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)w\big)_{L^2} \\ &= A_4^{2,1,1} + A_4^{2,1,2}. \end{aligned}$$

For $A_4^{2,1,1}$ we use the Cauchy-Schwarz inequality, (2.24), and (2.25) to obtain the bound

$$\begin{aligned} |A_4^{2,1,1}| &\lesssim \|[(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D), u_1]\partial_x w\|_{L^2}\|(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)w\|_{L^2} \\ &\lesssim \|u_1\|_{H^s}\|w\|_{L^2}^2. \end{aligned}$$

For the remaining term, we deduce from (2.24) that

$$\begin{aligned} |A_4^{2,1,2}| &= \frac{1}{2}|\big((\partial_x u_1)(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)w, (\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)w\big)_{L^2}| \\ &\lesssim \|\partial_x u_1\|_{L^\infty}\|w\|_{L^2}^2. \end{aligned}$$

Control of $A_4^{2,2}$. Similarly we get from the estimates (2.27), (2.28) and the Sobolev embedding that

$$\begin{aligned} |A_4^{2,2}| &\lesssim |\big([(\chi_\mu^{(2)}\sigma_{\mu,\frac{1}{2}})(D), u_1]\partial_x w, (\chi_\mu^{(2)}\sigma_{\mu,\frac{1}{2}})(D)w\big)_{L^2}| \\ &\quad + \frac{1}{2}|\big((\partial_x u_1)(\chi_\mu^{(2)}\sigma_{\mu,\frac{1}{2}})(D)w, (\chi_\mu^{(2)}\sigma_{\mu,\frac{1}{2}})(D)w\big)_{L^2}| \\ &\lesssim \|u_1\|_{H^s}(\|w\|_{L^2} + \mu^{\frac{1}{4}}\|D^{\frac{1}{2}}w\|_{L^2})^2. \end{aligned}$$

Control of $A_4^{2,3}$. By the same approach as above, combined with estimates (2.30) and (2.31) leaves us with the bound

$$\begin{aligned} |A_4^{2,3}| &\lesssim |\big([(\chi_\mu^{(2)}\sigma_{\mu,0})(D), u_1]\partial_x w, (\chi_\mu^{(2)}\sigma_{\mu,0})(D)w\big)_{L^2}| \\ &\quad + \frac{1}{2}|\big((\partial_x u_1)(\chi_\mu^{(2)}\sigma_{\mu,0})(D)w, (\chi_\mu^{(2)}\sigma_{\mu,0})(D)w\big)_{L^2}| \\ &\lesssim \|u_1\|_{H^s}\|w\|_{L^2}^2 + \|\partial_x u_1\|_{L^\infty}\|w\|_{L^2}^2. \end{aligned}$$

Gathering all these estimates, we obtain the result

$$|A_4| = |A_4^1 + A_4^2 + A_4^3| \lesssim_\beta (\|u_1\|_{H^s} + \|\eta_1\|_{H^s})\|(\psi, w)\|_{V_\mu^0}^2.$$

Adding $I + II + III$ concludes the proof. □

4.2. **Estimates for system** (1.7)**.** As in the prvious subsection, we let $\mathbf{U}_1 = (\eta_1, u_1)^T = \varepsilon(\zeta_1, v_1)^T$ and $\mathbf{U}_2 = (\eta_2, u_2)^T = \varepsilon(\zeta_2, v_2)^T$ be two solutions of (1.7) and define the difference $(\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then $\mathbf{W} = (\psi, w)^T$ solves

$$\partial_t \mathbf{W} + \mathcal{M}(\mathbf{U}_1, D)\mathbf{W} = \mathbf{F}, \tag{4.12}$$

with $\mathcal{M}$ defined as in (3.11) and $\mathbf{F}$ will remain the same as previously defined by (4.2). Then the energy associated to (4.12) is given in terms of the symmetrizer (3.12):

$$\tilde{\mathcal{E}}_s(\mathbf{W}) := \big(J_\mu^{\frac{1}{2}}J^s\mathbf{W}, \mathcal{Q}(\mathbf{U}_1, D)J_\mu^{\frac{1}{2}}J^s\mathbf{W}\big)_{L^2}. \tag{4.13}$$

Proposition 4.3. *Take* $s > \frac{3}{2}$ *and* $\varepsilon, \mu \in (0,1)$. *Let* $(\eta_1, u_1), (\eta_2, u_2) \in C([0, T_0] : V_\mu^s(\mathbb{R}))$ *be two solutions of* (1.7) *on a time interval* $[0, T_0]$ *for some* $T_0 > 0$. *Moreover, assume there exists* $h_0 \in (0,1)$ *and* $h_1 > 0$ *such that*

$$h_0 - 1 \leq \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0, T_0] \quad \text{and} \quad \sup_{t\in[0,T_0]} \|(\eta_1, u_1)\|_{H^s \times H^s} \leq h_1. \tag{4.14}$$

Define the difference to be $\mathbf{W} = (\psi, w) = (\eta_1 - \eta_2, u_1 - u_2)$. *Then, for the energy defined by* (4.13), *there holds*

$$\frac{d}{dt}\tilde{\mathcal{E}}_0(\mathbf{W}) \lesssim \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu}, \tag{4.15}$$

and

$$\|(\psi, w)\|^2_{V^0_\mu} \lesssim \tilde{\mathcal{E}}_0(\mathbf{W}) \lesssim \|(\psi, w)\|^2_{V^0_\mu}. \tag{4.16}$$

Furthermore, we have the following estimate at the $V^s_\mu-$ *level:*

$$\frac{d}{dt}\tilde{\mathcal{E}}_s(\mathbf{W}) \lesssim |\big(J^s\mathbf{F}, \mathcal{Q}(\mathbf{U}_1, D)J^s\mathbf{W}\big)_{L^2}| + \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^s_\mu}, \tag{4.17}$$

and

$$\|(\psi, w)\|^2_{V^s_\mu} \lesssim \tilde{\mathcal{E}}_s(\mathbf{W}) \lesssim \|(\psi, w)\|^2_{V^s_\mu}. \tag{4.18}$$

Proof. The proofs of (4.16) and (4.18) are similar to the proof of (3.16).

Also, we only prove (4.15) since the control of (4.17) follows by the proof of Proposition 3.3.

To prove (4.15), we use (4.12) and the self-adjointness of $\mathcal{Q}(\mathbf{U}_1, D)$ to write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\tilde{\mathcal{E}}_0(\mathbf{W}) &= \frac{1}{2}\big(J^{\frac{1}{2}}_\mu \mathbf{W}, (\partial_t \mathcal{Q}(\mathbf{U}_1, D))J^{\frac{1}{2}}_\mu \mathbf{W}\big)_{L^2} + \big(J^{\frac{1}{2}}_\mu \mathbf{F}, \mathcal{Q}(\mathbf{U}_1, D)J^{\frac{1}{2}}_\mu \mathbf{W}\big)_{L^2} \\ &\quad - \big(J^{\frac{1}{2}}_\mu \mathcal{M}(\mathbf{U}_1, D)\mathbf{W}, \mathcal{Q}(\mathbf{U}_1, D)J^{\frac{1}{2}}_\mu \mathbf{W}\big)_{L^2} \\ &=: \mathcal{I} - \mathcal{II} - \mathcal{III}.\end{aligned}$$

<u>Control of $\mathcal{I}$</u>. Using (4.12), (2.17), the Sobolev embedding and (4.14) yields

$$|\mathcal{I}| = \frac{1}{2}|\big(J^{\frac{1}{2}}_\mu w, (\partial_t \eta_1)J^{\frac{1}{2}}_\mu w\big)_{L^2}| \lesssim \|(\eta_1, u_1)\|_{V^s_\mu} \|J^{\frac{1}{2}}_\mu w\|^2_{L^2},$$

since $s > \frac{3}{2}$.

<u>Control of $\mathcal{II}$.</u> The contribution of the source term is given by

$$\begin{aligned}\mathcal{II} &= \big(J^{\frac{1}{2}}_\mu (w\partial_x \eta_2), \mathcal{T}_\mu(D)J^{\frac{1}{2}}_\mu \psi\big)_{L^2} + \big(J^{\frac{1}{2}}_\mu (\psi\partial_x u_2), \mathcal{T}_\mu(D)J^{\frac{1}{2}}_\mu \psi\big)_{L^2} \\ &\quad + \big(J^{\frac{1}{2}}_\mu (w\partial_x u_2), J^{\frac{1}{2}}_\mu w\big)_{L^2} + \big(J^{\frac{1}{2}}_\mu (w\partial_x u_2), \eta_1 J^{\frac{1}{2}}_\mu w\big)_{L^2} \\ &=: \mathcal{II}_1 + \mathcal{II}_2 + \mathcal{II}_3 + \mathcal{II}_4.\end{aligned}$$

Control of $\mathcal{II}_1 + \mathcal{II}_2$. The estimate of $\mathcal{II}_1$ is a direct consequence of the Cauchy-Schwarz inequality, (2.16) and the Sobolev embedding. Indeed, since $s > \frac{3}{2}$, we get

$$|\mathcal{II}_1| \leq \|\sqrt{\mathcal{T}_\mu}(D)J^{\frac{1}{2}}_\mu (w\partial_x \eta_2)\|_{L^2} \|\sqrt{\mathcal{T}_\mu}(D)J^{\frac{1}{2}}_\mu \psi\|_{L^2} \lesssim \|\eta_2\|_{H^s} \|w\|_{L^2} \|\psi\|_{L^2}.$$

Next, the control of $\mathcal{II}_2$ follows by the same estimates and gives

$$|\mathcal{II}_2| \lesssim \|\eta_2\|_{H^s} \|\psi\|^2_{H^1}.$$

Control of $\mathcal{II}_3 + \mathcal{II}_4$. We first deduce from (2.13) that

$$\|J^{\frac{1}{2}}_\mu (w\partial_x u_2)\|_{L^2} \leq \|w\partial_x u_2\|_{L^2} + \mu^{\frac{1}{4}} \|D^{\frac{1}{2}}(w\partial_x u_2)\|_{L^2}.$$

The first term is estimated by the Sobolev embedding, while the second term is equal to the term $II_4^{1,2}$ in the proof of Proposition 4.1. Since the terms w and u_2 in $II_4^{1,2}$ belong to the same function space, we can apply the same estimates. Thus, there holds for $s > \frac{3}{2}$ that

$$\|J_\mu^{\frac{1}{2}}(w\partial_x u_2)\|_{L^2} \lesssim \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu}. \tag{4.19}$$

Therefore, by using the Cauchy-Schwarz inequality, (4.19), (2.17), (4.14), and the Sobolev embedding implies

$$|\mathcal{II}_3| + |\mathcal{II}_4| \lesssim (1 + \|\eta_1\|_{L^\infty})\|J_\mu^{\frac{1}{2}}(w\partial_x u_2)\|_{L^2}\|J_\mu^{\frac{1}{2}} w\|_{L^2} \lesssim \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu}.$$

<u>Control of $\mathcal{III}$.</u> Lastly, the symmetrized term reads:

$$\begin{aligned}
\mathcal{III} &= \big(J_\mu^{\frac{1}{2}}(u_1\partial_x\psi), \mathcal{T}_\mu(D)J_\mu^{\frac{1}{2}}\psi\big)_{L^2} + \big(J_\mu^{\frac{1}{2}}\big((1+\eta_1)\partial_x w\big), \mathcal{T}_\mu(D)J_\mu^{\frac{1}{2}}\psi\big)_{L^2} \\
&\quad + \big(\mathcal{T}_\mu(D)J_\mu^{\frac{1}{2}}\partial_x\psi, (1+\eta_1)J_\mu^{\frac{1}{2}} w\big)_{L^2} + \big(J_\mu^{\frac{1}{2}}(u_1\partial_x w), (1+\eta_1)J_\mu^{\frac{1}{2}} w\big)_{L^2} \\
&= \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}.
\end{aligned}$$

Each term is treated by using integration by parts and suitable commutator estimates.

Control of $\mathcal{A}_{11}$. For $\mathcal{A}_{11}$, we use integration by parts to find that

$$\mathcal{A}_{11} = \big([\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}, u_1]\partial_x\psi, \sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi\big)_{L^2} - \frac{1}{2}\big((\partial_x u_1)\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi, \sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi\big)_{L^2}.$$

Thus, it follows from the commutator estimate (2.32) with $s > \frac{3}{2}$ and estimate (2.16) that

$$|\mathcal{A}_{11}| \lesssim \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|\psi\|^2_{L^2}.$$

Control of $\mathcal{A}_{12} + \mathcal{A}_{21}$. Treating the off-diagonal terms we first observe,

$$\begin{aligned}
\mathcal{A}_{12} &= \big([\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}, \eta_1]\partial_x w, \sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi\big)_{L^2} \\
&\quad + \big((1+\eta_1)\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\partial_x w, \sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi\big)_{L^2} \\
&= \mathcal{A}_{12}^1 + \mathcal{A}_{12}^2.
\end{aligned}$$

The commutator estimate (2.32) and estimate (2.16) deals with the first term. Indeed, we get the bound

$$|\mathcal{A}_{12}^1| \leq \|[\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}, \eta_1]\partial_x w\|_{L^2}\|\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi\|_{L^2} \lesssim \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \|(\psi, w)\|^2_{V^0_\mu}.$$

Next, we integrate $\mathcal{A}_{12}^2$ by parts to obtain two new terms

$$\begin{aligned}
\mathcal{A}_{12}^2 &= -\big((\partial_x\eta_1)J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)w, J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\psi\big)_{L^2} \\
&\quad - \big((1+\eta_1)J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)w, J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\partial_x\psi\big)_{L^2} \\
&= \mathcal{A}_{12}^{2,1} + \mathcal{A}_{12}^{2,2}.
\end{aligned}$$

Arguing as above, we find that

$$|\mathcal{A}_{12}^{2,1}| \leq \|\partial_x\eta_1\|_{L^\infty}\|J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)w\|_{L^2}\|J_\mu^{\frac{1}{2}}\sqrt{\mathcal{T}_\mu}(D)\psi\|_{L^2} \lesssim \|\eta_1\|_{H^s}\|(\psi, w)\|^2_{V^0_\mu},$$

for $s > \frac{3}{2}$. On the other hand, the term $\mathcal{A}_{12}^{2,2}$, is absorbed by $\mathcal{A}_{21}$. Indeed,

$$\begin{aligned}\mathcal{A}_{21} &= -\big(\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi, \partial_x[\sqrt{\mathcal{T}_\mu}(D), \eta_1]J_\mu^{\frac{1}{2}}w\big)_{L^2} \\ &\quad + \big(\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\partial_x\psi, (1+\eta_1)\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}w\big)_{L^2} \\ &= \mathcal{A}_{21}^1 + \mathcal{A}_{21}^2,\end{aligned}$$

with $\mathcal{A}_{21}^2 = -\mathcal{A}_{12}^{2,2}$. We estimate $\mathcal{A}_{21}^1$ by using the Cauchy-Schwarz inequality, (2.16), (2.35), and (2.17) to get

$$|\mathcal{A}_{21}^1| \le \|\sqrt{\mathcal{T}_\mu}(D)J_\mu^{\frac{1}{2}}\psi\|_{L^2}\|\partial_x[\sqrt{\mathcal{T}_\mu}(D), \eta_1]J_\mu^{\frac{1}{2}}w\|_{L^2} \lesssim \|\eta_1\|_{H^s}\|(\psi,w)\|_{V_\mu^0}^2.$$

Thus, we deduce by gathering all these estimates that

$$|\mathcal{A}_{12} + \mathcal{A}_{21}| \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{V_\mu^s}\|(\psi,w)\|_{V_\mu^0}^2.$$

Control of $\mathcal{A}_{22}$. Lastly, the term $\mathcal{A}_{22}$ is estimated by (2.36) for $s > \frac{3}{2}$, (4.14), and integration by parts

$$\begin{aligned}|\mathcal{A}_{22}^2| &\le |\big([J_\mu^{\frac{1}{2}}, u_1]\partial_x w, (1+\eta_1)J_\mu^{\frac{1}{2}}w\big)_{L^2}| + |\big(u_1 J_\mu^{\frac{1}{2}}\partial_x w, (1+\eta_1)J_\mu^{\frac{1}{2}}w\big)_{L^2}| \\ &\lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{V_\mu^s}\|(\psi,w)\|_{V_\mu^0}^2.\end{aligned}$$

Therefore, we deduce that

$$\frac{d}{dt}\tilde{\mathcal{E}}_0(\mathbf{W}) \lesssim |\mathcal{I}| + |\mathcal{II}| + |\mathcal{III}| \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{V_\mu^s}\|(\psi,w)\|_{V_\mu^0}^2,$$

which concludes the proof of Proposition 4.3.

□

4.3. **Estimates for system** (1.9)**.** Again, we let $\mathbf{U}_1 = (\eta_1,u_1)^T = \varepsilon(\zeta_1,v_1)^T$ and $\mathbf{U}_2 = (\eta_2,u_2)^T = \varepsilon(\zeta_2,v_2)^T$ be two solutions of (1.9) and define the difference $(\psi,w) = (\eta_1 - \eta_2, u_1 - u_2)$. Then $\mathbf{W} = (\psi,w)^T$ solves

$$\partial_t\mathbf{W} + \mathscr{M}(\mathbf{U}_1, D)\mathbf{W} = \mathbf{F}, \tag{4.20}$$

with $\mathscr{M}$ defined as in (3.21) and $\mathbf{F}$ is defined by

$$\mathbf{F} = -\begin{pmatrix}\mathcal{T}_\mu(D)(w\partial_x\eta_2) + \mathcal{T}_\mu(D)(\psi\partial_x u_2) \\ \mathcal{T}_\mu(D)(w\partial_x u_2)\end{pmatrix}. \tag{4.21}$$

The energy associated with (4.20) is given in terms of the symmetrizer (3.22) by

$$\tilde{\mathscr{E}}_s(\mathbf{W}) := \big(J^s\mathbf{W}, \mathscr{Q}(\mathbf{U}_1,D)J^s\mathbf{W}\big)_{L^2}. \tag{4.22}$$

Proposition 4.4. *Take $s > \frac{3}{2}$, $\varepsilon,\mu \in (0,1)$ and $\beta \ge 0$. Let $(\eta_1,u_1),(\eta_2,u_2) \in C([0,T_0] : X_{\beta,\mu}^s(\mathbb{R}))$ be two solutions of (1.9) on a time interval $[0,T_0]$ for some $T_0 > 0$. Moreover, assume there exist $h_0 \in (0,1)$ and $h_1 > 0$ such that*

$$h_0 - 1 \le \eta_1(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,T_0] \quad \text{and} \quad \sup_{t\in[0,T_0]}\|(\eta_1,u_1)\|_{H^s\times H^s} \le h_1. \tag{4.23}$$

Define the difference to be $\mathbf{W} = (\psi,w) = (\eta_1-\eta_2, u_1-u_2)$. Then, for the energy defined by (4.22), there holds

$$\frac{d}{dt}\tilde{\mathscr{E}}_0(\mathbf{W}) \lesssim_\beta \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|(\psi,w)\|_{X_{\beta,\mu}^0}^2, \tag{4.24}$$

and

$$\|(\psi,w)\|^2_{X^0_{\beta,\mu}} \lesssim \tilde{\mathscr{E}}_0(\mathbf{W}) \lesssim \|(\psi,w)\|^2_{X^0_{\beta,\mu}}. \tag{4.25}$$

Furthermore, we have the following estimate at the $X^s_{\beta,\mu}-$ *level:*

$$\frac{d}{dt}\tilde{\mathscr{E}}_s(\mathbf{W}) \lesssim_\beta |\big(J^s\mathbf{F}, \mathscr{Q}(\mathbf{U}_1,D)J^s\mathbf{W}\big)_{L^2}| + \max_{i=1,2}\|(\eta_i,u_i)\|_{X^s_{\beta,\mu}}\|(\psi,w)\|^2_{X^s_{\beta,\mu}}, \tag{4.26}$$

and

$$\|(\psi,w)\|^2_{X^s_{\beta,\mu}} \lesssim \tilde{\mathscr{E}}_s(\mathbf{W}) \lesssim \|(\psi,w)\|^2_{X^s_{\beta,\mu}}. \tag{4.27}$$

Proof. By previous arguments, we note that the proofs of (4.25) and (4.27) are similar to the proof of (3.26).

Moreover, we will only prove (4.24) since the control of (4.26) follows by the proof of Proposition 3.4.

We will now prove (4.24). Then we first use (4.20) and the self-adjointness of $\mathscr{Q}(\mathbf{U}_1,D)$ to write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\tilde{\mathscr{E}}_0(\mathbf{W}) &= \frac{1}{2}\big(\mathbf{W},(\partial_t\mathscr{Q}(\mathbf{U}_1,D))\mathbf{W}\big)_{L^2} + \big(\mathbf{F},\mathscr{Q}(\mathbf{U}_1,D)\mathbf{W}\big)_{L^2}\\ &\quad - \big(\mathscr{M}(\mathbf{U}_1,D)\mathbf{W},\mathscr{Q}(\mathbf{U}_1,D)\mathbf{W}\big)_{L^2}\\ &=: \mathscr{I} - \mathscr{I}\mathscr{I} - \mathscr{I}\mathscr{I}\mathscr{I}.\end{aligned}$$

<u>Control of $\mathscr{I}$.</u> By (4.20), Hölder's inequality, the Sobolev embedding and (4.23) we deduce

$$|\mathscr{I}| = \frac{1}{2}|\big(w,(\partial_t\eta_1)w\big)_{L^2}| \lesssim \|u_1\|_{H^s}(1+\|\eta_1\|_{H^s})\|w\|^2_{L^2} \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|(\psi,w)\|^2_{X^0_{\beta,\mu}}.$$

<u>Control of $\mathscr{I}\mathscr{I}$.</u> The contribution from the source term is given by,

$$\begin{aligned}\mathscr{I}\mathscr{I} &= \big(w\partial_x\eta_2,\mathcal{K}_\mu(D)\psi\big)_{L^2} + \big(\psi\partial_x\eta_2,\mathcal{K}_\mu(D)\psi\big)_{L^2}\\ &\quad + \big(w\partial_x u_2,w\big)_{L^2} + \big(\mathcal{T}_\mu(D)(w\partial_x u_2),\eta_1 w\big)_{L^2}\\ &=: \mathscr{I}\mathscr{I}_1 + \mathscr{I}\mathscr{I}_2 + \mathscr{I}\mathscr{I}_3 + \mathscr{I}\mathscr{I}_4.\end{aligned}$$

Control of $\mathscr{I}\mathscr{I}_1 + \mathscr{I}\mathscr{I}_2$. For $\beta > 0$, we first apply the Cauchy-Schwarz inequality, (2.1), and the Sobolev embedding to deduce that for $s > \frac{3}{2}$

$$\begin{aligned}|\mathscr{I}\mathscr{I}_1| + |\mathscr{I}\mathscr{I}_2| &\lesssim (\|w\|_{L^2} + \|\psi\|_{L^2})\|\eta_2\|_{H^s}\|\mathcal{K}_\mu(D)\psi\|_{L^2}\\ &\lesssim \|\eta_2\|_{H^s}\|(\psi,w)\|^2_{X^0_{\beta,\mu}}.\end{aligned}$$

The case $\beta = 0$, is similar where we instead use the boundedness of $\mathcal{T}_\mu(D)$ on $L^2(\mathbb{R})$ to obtain

$$|\mathscr{I}\mathscr{I}_1| + |\mathscr{I}\mathscr{I}_2| \lesssim \|\eta_2\|_{H^s}\|(\psi,w)\|^2_{L^2}.$$

Control of $\mathscr{I}\mathscr{I}_3 + \mathscr{I}\mathscr{I}_4$. Both terms are treated with the Cauchy-Schwatz inequality, (2.14) and the Sobolev embedding. Consequently, for $s > \frac{3}{2}$ and $\beta \geq 0$ there holds

$$|\mathscr{I}\mathscr{I}_3| + |\mathscr{I}\mathscr{I}_4| \lesssim (1+\|\eta_1\|_{H^s})\|u_2\|_{H^s}\|w\|^2_{L^2} \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|(\psi,w)\|^2_{X^0_{\beta,\mu}}.$$

Gathering all these estimates yields

$$|\mathscr{I}\mathscr{I}| \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|(\psi,w)\|^2_{X^0_{\beta,\mu}}.$$

Control of $\mathscr{I}\mathscr{I}\mathscr{I}$. The symmetrized term $\mathscr{I}\mathscr{I}\mathscr{I}$ reads:

$$\begin{aligned}\mathscr{I}\mathscr{I}\mathscr{I} =& \big(u_1\partial_x\psi, \mathcal{K}_\mu(D)\psi\big)_{L^2} + \big((1+\mathcal{T}_\mu(D)\eta_1)\partial_x w, \mathcal{T}_\mu^{-1}(D)\mathcal{K}_\mu(D)\psi\big)_{L^2}\\ &+ \big(\mathcal{K}_\mu(D)\partial_x\psi, (\mathcal{T}_\mu^{-1}(D)+\eta_1)w\big)_{L^2} + \big(\mathcal{T}_\mu(D)(u_1\partial_x w), (\mathcal{T}_\mu^{-1}(D)+\eta_1)w\big)_{L^2}\\ &= \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}.\end{aligned}$$

Control of $\mathscr{A}_{11}$. In the case $\beta > 0$, we decompose $\mathscr{A}_{11}$ as

$$\begin{aligned}\mathscr{A}_{11} =& \big(u_1\partial_x\psi, ((\chi_\mu^{(1)})^2\mathcal{K}_\mu)(D)\psi\big)_{L^2} + \big(u_1\partial_x\psi, ((\chi_\mu^{(2)})^2(\sigma_{\mu,\frac{1}{2}})^2)(D)\psi\big)_{L^2}\\ &- \big(u_1\partial_x\psi, ((\chi_\mu^{(2)})^2(\sigma_{\mu,0})^2)(D)\psi\big)_{L^2},\end{aligned}$$

where we have divided the multiplier $\mathcal{K}_\mu(D)$ into three pieces in the same way as we did in (4.11). We may therefore apply the same estimates as for A_4^3 in the proof of Proposition 4.1, where we change the role of ψ and w to obtain

$$|\mathscr{A}_{11}| = |\big(u_1\partial_x\psi, \mathcal{K}_\mu(D)\psi\big)_{L^2}| \lesssim_\beta \|u_1\|_{H^s}(\|\psi\|_{L^2}^2 + \sqrt{\beta}\mu^{\frac{1}{4}}\|\psi\|_{H^{\frac{1}{2}}}^2).$$

Then use inequality (3.27) to conclude that

$$|\mathscr{A}_{11}| \lesssim \|u_1\|_{H^s}\|(\psi,w)\|_{X_{\beta,\mu}^0}^2.$$

In the case $\beta = 0$, we simply use Hölder's inequality, (2.14), (2.34), to obtain

$$\begin{aligned}|\mathscr{A}_{11}| &\lesssim |\big((\partial_x u_1)\sqrt{\mathcal{T}_\mu}(D)\psi, \sqrt{\mathcal{T}_\mu}(D)\psi\big)_{L^2}| + |\big([\sqrt{\mathcal{T}_\mu}(D), u_1]\partial_x\psi, \sqrt{\mathcal{T}_\mu}(D)\psi\big)_{L^2}|\\ &\lesssim \|u_1\|_{H^s}\|\psi\|_{L^2}^2.\end{aligned}$$

Control of $\mathscr{A}_{12} + \mathscr{A}_{21}$. Treating the off-diagonal terms we first observe by integrating by parts that

$$\mathscr{A}_{12} = -\big((\partial_x\eta_1)w, \mathcal{K}_\mu(D)\psi\big)_{L^2} - \mathscr{A}_{21}.$$

Therefore, we may apply Hölder's inequality, the Sobolev embedding, and (2.1) for $\beta \geq 0$, to deduce

$$|\mathscr{A}_{12} + \mathscr{A}_{21}| \lesssim \|\eta_1\|_{H^s}\|(\psi,w)\|_{X_{\beta,\mu}^0}.$$

Control of $\mathscr{A}_{22}$. We decompose $\mathscr{A}_{22}$ into two terms

$$\begin{aligned}\mathscr{A}_{22} &= \big(u_1\partial_x w, w\big)_{L^2} + \big(\mathcal{T}_\mu(D)(u_1\partial_x w), \eta_1 w\big)_{L^2}\\ &= \mathscr{A}_{22}^1 + \mathscr{A}_{22}^2.\end{aligned}$$

We see that $\mathscr{A}_{22}^1$ is easily treated by the Cauchy-Schwarz inequality, integration by parts, the Sobolev embedding, and (4.23). Indeed, there holds

$$|\mathscr{A}_{22}^1| \lesssim \|u_1\|_{H^s}\|w\|_{L^2}^2.$$

Next, we decompose $\mathscr{A}_{22}^2$ into three parts

$$\begin{aligned}\mathscr{A}_{22}^2 =& \big([\sqrt{\mathcal{T}_\mu}(D), u_1]\partial_x w, \sqrt{\mathcal{T}_\mu}(D)(\eta_1 w)\big)_{L^2} + \big(u_1\sqrt{\mathcal{T}_\mu}(D)\partial_x w, [\sqrt{\mathcal{T}_\mu}(D), \eta_1]w\big)_{L^2}\\ &+ \big(u_1\sqrt{\mathcal{T}_\mu}(D)\partial_x w, \eta_1\sqrt{\mathcal{T}_\mu}(D)w\big)_{L^2}.\\ =& \mathscr{A}_{22}^{2,1} + \mathscr{A}_{22}^{2,2} + \mathscr{A}_{22}^{2,3}.\end{aligned}$$

For $\mathscr{A}_{22}^{2,1}$, we simply apply Hölder's inequality, (2.34), (2.14), the Sobolev embedding to find that

$$|\mathscr{A}_{22}^{2,1}| \leq \|[\sqrt{\mathcal{T}_\mu}(D), u_1]\partial_x w\|_{L^2}\|\sqrt{\mathcal{T}_\mu}(D)(\eta_1 w)\|_{L^2} \lesssim \|u_1\|_{H^s}\|\eta_1\|_{H^s}\|w\|_{L^2}^2.$$

For $\mathscr{A}_{22}^{2,2}$, we first remark that

$$\|[\sqrt{\mathcal{T}_\mu}(D),\eta_1]w\|_{L^2} \lesssim \|\eta_1\|_{L^\infty}\|w\|_{L^2}, \tag{4.28}$$

simply by using Hölder's inequality and (2.14). Then after integrating by parts, we use Hölder's inequality, the Sobolev embedding, (2.35), (4.23), and (4.28) to deduce that

$$\begin{aligned}|\mathscr{A}_{22}^{2,2}| &\leq \|\partial_x u_1\|_{L^\infty}\|\sqrt{\mathcal{T}_\mu}(D)w\|_{L^2}\|[\sqrt{\mathcal{T}_\mu}(D),\eta_1]w\|_{L^2}\\ &\quad + \|u_1\|_{L^\infty}\|\sqrt{\mathcal{T}_\mu}(D)w\|_{L^2}\|\partial_x[\sqrt{\mathcal{T}_\mu}(D),\eta_1]w\|_{L^2}\\ &\lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|w\|_{L^2}^2.\end{aligned}$$

Lastly, we use integration by parts, then apply Hölder's inequality, (2.14), the Sobolev embedding, and (4.23) to obtain that

$$|\mathscr{A}_{22}^{2,3}| \leq \frac{1}{2}\|\partial_x(u_1\eta_1)\|_{L^\infty}\|\sqrt{\mathcal{T}_\mu}(D)w\|_{L^2}^2 \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|w\|_{L^2}^2.$$

We may now gather these estimates to conclude that

$$|\mathscr{A}_{22}| \lesssim \max_{i=1,2}\|(\eta_i,u_i)\|_{H^s}\|(\psi,w)\|_{X^0_{\beta,\mu}}^2,$$

and as a result the proof of Proposition 4.4 is now complete. □

5. Proof of Theorem 1.6 in the one-dimensional case

Proof. The proof is divided into eight steps, utilizing the results above.

Step 1: *Existence of solutions for a regularized system.* Let $s > \frac{1}{2}$, $0 < \nu < 1$ and $\alpha = \frac{3}{2}^+$. Then, for any initial data $\mathbf{U}_0 := (\eta_0, u_0) \in V^s_\mu(\mathbb{R})$, we claim that there exist $c_\beta > 0$, and a time

$$0 < T_\nu := T_\nu\big(\|(\eta_0,u_0)\|_{V^s_\mu}\big) = \left(\frac{c_\beta \nu^{\frac{2}{3\alpha}}}{1+\|(\eta_0,u_0)\|_{V^s_\mu}}\right)^{\frac{1}{1-\frac{2}{3\alpha}}} \tag{5.1}$$

such that $\mathbf{U}^\nu := (\eta^\nu, u^\nu)^T \in C([0,T_\nu];V^s_\mu(\mathbb{R}))$ is a unique solution of the regularized Cauchy problem:

$$\begin{cases}\partial_t\eta^\nu + u^\nu\partial_x\eta^\nu + (\mathcal{K}_\mu(D)+\eta^\nu)\partial_x u^\nu = -\nu\langle D\rangle^\alpha\eta^\nu\\ \partial_t u^\nu + \partial_x\eta^\nu + u^\nu\partial_x u^\nu = -\nu\langle D\rangle^\alpha u^\nu.\end{cases} \tag{5.2}$$

The proof of the existence of a unique solution is a consequence of the contraction mapping principle. First, we find the diagonalisation of the linear part, $S^\nu(t)$, of (5.2) to be

$$S^\nu(t) = \frac{1}{2}\begin{pmatrix}-\sqrt{\mathcal{K}_\mu}(D) & \sqrt{\mathcal{K}_\mu}(D)\\ 1 & 1\end{pmatrix}\begin{pmatrix}\exp(-t\mathcal{L}^\nu_-(D)) & 0\\ 0 & \exp(-t\mathcal{L}^\nu_+(D))\end{pmatrix}\begin{pmatrix}-\mathcal{K}_\mu^{-\frac{1}{2}}(D) & 1\\ \mathcal{K}_\mu^{-\frac{1}{2}}(D) & 1\end{pmatrix}$$

where $\mathcal{L}^\nu_\pm(D) = \pm iD\sqrt{\mathcal{K}_\mu}(D) + \nu\langle D\rangle^\alpha$. Then we shall show that

$$\Phi_{\mathbf{U}_0}(\mathbf{U}^\nu)(t) := S^\nu(t)\mathbf{U}_0 - \int_0^t S^\nu(t-s)\partial_x\begin{pmatrix}\eta^\nu u^\nu\\ \frac{(u^\nu)^2}{2}\end{pmatrix}(s)\,ds, \tag{5.3}$$

defines a contraction on the closed subspace $B(a)$ of $C([0,T];V^s_\mu(\mathbb{R}))$, whose norm is bounded by a, and is centered at the point $S^\nu(t)\mathbf{U}_0$. However, we note by Plancherel that for $|\xi| > 1$ there holds,

$$\|S^\nu(t)\partial_x\mathbf{U}\|_{L^2} \lesssim_\beta \||\xi|^{\frac{3}{2}}e^{-\nu|\xi|^\alpha t}\hat{\mathbf{U}}\|_{L^2} \lesssim_\beta \frac{1}{(\nu t)^{\frac{2}{3\alpha}}}\|\mathbf{U}\|_{L^2}. \tag{5.4}$$

The same is trivially true for $|\xi| \leq 1$. Now, combining (5.4) with the fact that $\frac{2}{3\alpha} < 1$ and the algebra property of $H^{\frac{1}{2}^+}(\mathbb{R})$ we deduce that

$$\|\Phi_{\mathbf{U}_0}(\mathbf{U}^\nu) - S^\nu(t)\mathbf{U}_0\|_{H^s} \lesssim_\beta T^{1-\frac{2}{3\alpha}}\nu^{-\frac{2}{3\alpha}}\|\mathbf{U}^\nu\|_{H^s}^2$$

and

$$\|\Phi_{\mathbf{U}_0}(\mathbf{U}_1^\nu) - \Phi_{\mathbf{U}_0}(\mathbf{U}_2^\nu)\|_{H^s} \lesssim_\beta T^{1-\frac{2}{3\alpha}}\nu^{-\frac{2}{3\alpha}}\|\mathbf{U}_1^\nu - \mathbf{U}_2^\nu\|_{H^s}(\|\mathbf{U}_1^\nu\|_{H^s} + \|\mathbf{U}_2^\nu\|_{H^s})$$

Therefore, by choosing $a = \|\mathbf{U}_0\|_{L^\infty_T V^s_\mu}$ and T as in (5.1) we can use the above estimates to conclude by the Fixed Point Theorem that there exist a unique solution of (5.2) in $C([0,T_\nu]; V^s_\mu(\mathbb{R}))$.

Remark 5.1. *A consequence of Step* 1*, is the continuity of the flow map associated with* (5.2)*. But this is only for the 'short' time T_ν given by* (5.1)*, and is therefore not useful for the limit equation.*

Step 2: *The blow-up alternative.* We define the maximal time of existence to be

$$T_\nu^\star = \sup\Big\{T_\nu > 0 \ : \ \exists!\, \mathbf{U}^\nu = (\eta^\nu, u^\nu)^T \text{ solution of (5.2) in } C([0,T_\nu]; V^s_\mu(\mathbb{R}))\Big\}.$$

Then we claim that the solution of (5.2) satisfies the blow-up alternative:

$$\text{If} \quad T_\nu^\star < \infty, \quad \text{then} \quad \lim_{t\nearrow T_\nu^\star}\|(\eta^\nu, u^\nu)(t)\|_{V^s_\mu} = \infty. \tag{5.5}$$

First, we argue by contradiction that $T_\nu^\star < \infty$ and there exist $A \in \mathbb{R}^+$ such that

$$\sup_{t\in[0,T_\nu^\star)}\|(\eta^\nu, u^\nu)(t)\|_{V^s_\mu} = A. \tag{5.6}$$

We use (5.1) to define $\tau_{\nu,A} = T_\nu^\star - \frac{T_\nu(A)}{2}$. Then we have that

$$a := \|(\eta^\nu, u^\nu)(\tau_{\nu,A})\|_{V^s_\mu} \leq A.$$

Therefore, if we let $\mathbf{V}_0^\nu = (\eta^\nu, u^\nu)^T(\tau_{\nu,A})$ serve as initial data, then (5.2) has a unique solution given by

$$\mathbf{V}^\nu(t) = S^\nu(t)\mathbf{V}_0^\nu - \int_0^t S^\nu(t-s)\partial_x\begin{pmatrix} v_1^\nu v_2^\nu \\ \frac{(v_2^\nu)^2}{2}\end{pmatrix}(s)\,ds \tag{5.7}$$

with $\mathbf{V}^\nu = (v_1^\nu, v_2^\nu) \in C([0,T_\nu(a)]; V^s_\mu(\mathbb{R}))$. Here, $T_\nu(a)$ is given by (5.1) due to Step 1. Moreover, we observe that $T_\nu(a) \geq T_\nu(A)$ by definition (5.1), and implies $\tau_{\nu,A} + T_\nu(a) \geq T_\nu^\star + \frac{T_\nu(A)}{2}$. Thus, we define the extension of $\mathbf{U}^\nu = (\eta^\nu, u^\nu)^T$ by the function

$$\mathbf{Z}^\nu(t) = \begin{cases} \mathbf{U}^\nu(t), & \text{if} \quad 0 \leq t < \tau_{\nu,A} \\ \mathbf{V}^\nu(t-\tau_{\nu,A}), & \text{if} \quad \tau_{\nu,A} \leq t \leq \tau_{\nu,A} + T_\nu(a), \end{cases}$$

and one can verify that it is a solution of (5.2) for all $t \in [0, T_\nu^\star + \frac{T_\nu(A)}{2}] \subset [0, \tau_{\nu,A} + T_\nu(a)]$. This contradicts the definition of $T_\nu^\star$. Thus, we conclude that if $T_\nu^\star < \infty$, then necessarily $A = \infty$ in (5.6), and implies

$$\limsup_{t\nearrow T_\nu^\star}\|(\eta^\nu, u^\nu(t))\|_{V^s_\mu} = \infty. \tag{5.8}$$

To conclude the proof of the claim, we use (5.8) to verify that for all $R > 0$ there exists an open interval $(t_R, T_\nu^\star)$ such that $\|(\eta^\nu, u^\nu(t))\|_{V_\mu^s} > R$, for all $t \in (t_R, T_\nu^\star)$. Indeed, we argue by contradiction that there exists $R \in \mathbb{R}^+$ such that for all $0 < t_R < T_\nu^\star$, we have

$$\|(\eta^\nu, u^\nu)(t)\|_{V_\mu^s} \leq R, \tag{5.9}$$

for some $t \in (t_R, T_\nu^\star)$. By (5.8) there is a time such that $\tau_{R,0} > T_\nu^\star - \frac{T_\nu(R)}{2}$ and satisfying

$$\|(\eta^\nu, u^\nu)(\tau_{R,0})\|_{V_\mu^s} > R.$$

On the other hand, by assumption (5.9) we can take $t_R = \tau_{R,0}$ and use the fact that there is a time $\tau_{R,1} \in (t_R, T_\nu^\star)$ such that

$$\|(\eta^\nu, u^\nu)(\tau_{R,1})\|_{V_\mu^s} \leq R.$$

Thus, by the same argument as above we can take $(\eta^\nu, u^\nu)(\tau_{R,1})$ as initial data of (5.2) to find an extended solution defined on $[0, T_\nu^\star + \frac{T_\nu(R)}{2}] \subset [0, \tau_{R,1} + T_\nu(R)]$. This contradicts the definition of $T_\nu^\star$. As a result, we conclude that (5.5) holds true.

Step 3: *The existence time is independent of $\nu > 0$.* We claim that there exists

$$T = \frac{1}{k_\beta^1 \|(\zeta_0, v_0)\|_{V_\mu^s}},$$

as in (1.16), such that the regularized solution $\varepsilon(\zeta^\nu, v^\nu) = (\eta^\nu, u^\nu)$ exists on the interval $[0, \frac{T}{\varepsilon}]$.

The proof relies on a bootstrap argument similar to the proof of Lemma 5.1 in [28]. In fact, the long time existence is a direct consequence of the following remark and lemma.

Remark 5.2. *We will invoke the estimates in Proposition 3.1 for system* (5.2)*. However, due to the parabolic regularisation, we must also control the additional terms given by*

$$\frac{d}{dt}E_s(\mathbf{U}^\nu) \lesssim_\beta \big(E_s(\mathbf{U}^\nu)\big)^{\frac{3}{2}} - \nu\big(J^{s+\alpha}\mathbf{U}^\nu, Q(\mathbf{U}^\nu, D)J^s\mathbf{U}^\nu\big)_{L^2}.$$

But decomposing the last term, we note that

$$\begin{aligned}\big(J^{s+\alpha}\mathbf{U}^\nu, Q(\mathbf{U}^\nu, D)J^s\mathbf{U}^\nu\big)_{L^2} &= \big(J^{s+\frac{\alpha}{2}}\eta^\nu, J^{s+\frac{\alpha}{2}}\eta^\nu\big)_{L^2} + \big(J^{s+\frac{\alpha}{2}}u^\nu, (\mathcal{K}_\mu(D) + \eta^\nu)J^{s+\frac{\alpha}{2}}u^\nu\big)_{L^2} \\ &\quad + \big(J^{s+\frac{\alpha}{2}}u^\nu, [J^{\frac{\alpha}{2}}, \eta^\nu]J^s u^\nu\big)_{L^2} \\ &= I + II + III.\end{aligned}$$

The first two terms has a positive sign, while the third term, III, can be absorbed by the second term by using Cauchy-Schwarz, (A.9) *and Young's inequality:*

$$|III| \leq 2c_1\|u^\nu\|_{H^{s+\frac{\alpha}{2}}}\|\eta^\nu\|_{H^s}\|u^\nu\|_{H^s} \leq c_2\|u^\nu\|^2_{H^{s+\frac{\alpha}{2}}} + \frac{c_1}{c_2}\|\eta^\nu\|^2_{H^s}\|u^\nu\|^2_{H^s},$$

by choosing $0 < c_2 < \min\{\frac{h_0}{2}, \frac{\beta}{2}\}$. Indeed, by (2.7), (2.8), (3.4), (3.5) *and* (3.7) *we get the bound*

$$\begin{aligned}-\nu(I + II + III) \leq -\nu\|\eta^\nu\|^2_{H^{s+\frac{\alpha}{2}}} + \nu(c_2 - \min\{\frac{h_0}{2}, \frac{\beta}{2}\})\|u^\nu\|^2_{H^{s+\frac{\alpha}{2}}} \\ + \frac{c_1}{c_2}c_\beta\big(E_s(\mathbf{U}^\nu)\big)^{\frac{3}{2}}.\end{aligned}$$

Therefore, we have that Proposition 3.1 holds for the regularised system.

Lemma 5.3. *Let $s>2$ and ε be as in (1.15). Let $(\eta^\nu,u^\nu)=\varepsilon(\zeta^\nu,v^\nu)\in C([0,T^\star_\nu);V^s_\mu(\mathbb{R}))$ be a solution of (5.2) with initial data $\varepsilon(\zeta_0,v_0)=(\eta_0,u_0)\in V^s_\mu(\mathbb{R})$, defined on its maximal time of existence and satisfying the blow-up alternative (5.5). Moreover, let $\eta_0=\varepsilon\zeta_0$ satisfy either the non-cavitation condition (1.13) or the $\beta-$dependent surface condition (1.14), depending on whether $\beta\geq\frac{1}{3}$ or $0<\beta<\frac{1}{3}$, respectively. Then there exists a time*

$$T_0=\frac{1}{k^1_\beta\|(\eta_0,u_0)\|_{V^s_\mu}}, \tag{5.10}$$

such that $T^\star_\nu>T_0$ and

$$\sup_{t\in[0,T_0]}\|(\eta^\nu,u^\nu)(t)\|_{V^s_\mu}\leq 4k^2_\beta\|(\eta_0,u_0)\|_{V^s_\mu}. \tag{5.11}$$

The constants are on the form

$$k^2_\beta=\frac{c^2_\beta}{c^1_\beta}\quad \text{and}\quad k^1_\beta=\begin{cases}\frac{C_1}{\beta} & \text{for } 0<\beta<\frac{1}{3}\\ C_2\beta^2 & \text{for } \beta\geq\frac{1}{3}\end{cases}$$

where C_1 and C_2 are two positive constants to be fixed in the proof.

Proof. We define the set

$$\tilde{T}_\nu=\sup\Big\{T_\nu\in(0,T^\star_\nu)\ :\ \sup_{t\in[0,T_\nu]}\|(\eta^\nu,u^\nu)(t)\|_{V^s_\mu}\leq 4k^2_\beta\|(\eta_0,u_0)\|_{V^s_\mu}\Big\}. \tag{5.12}$$

Then we first note that $\tilde{T}_\nu<T^\star_\nu$, or else it would contradict the blow-up alternative (5.5). For the proof we argue by contradiction that $\tilde{T}_\nu\leq T_0$.

The main idea is to improve the estimate given in (5.12). First, we verify that the solution (η^ν,u^ν) satisfy (3.4). Indeed, recalling assumption (1.15):

$$0<\varepsilon\leq\frac{1}{k^2_\beta\|(\zeta_0,v_0)\|_{V^s_\mu}},$$

implies

$$\|(\eta^\nu,u^\nu)\|_{H^s}\leq k^2_\beta\|(\eta^\nu_0,u^\nu_0)\|_{V^s_\mu}=4\varepsilon k^2_\beta\|(\zeta_0,v_0)\|_{V^s_\mu}\leq 4, \tag{5.13}$$

for all $t\in[0,\tilde{T}_\nu]$. Next, the solution (η^ν,u^ν) satisfy the non-cavitation condition. We will prove this as a consequence of the bound

$$\sup_{\tau\in[0,\tilde{T}_\nu]}|\partial_t\eta^\nu(\tau)|\leq k^2_\beta\beta^2.$$

Indeed, by similar argument as for (3.8), we use (5.2), (5.13), and $H^{\frac{1}{2}^+}(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$ to find

$$\|\partial_t\eta^\nu\|_{L^\infty}\leq c^2_\beta\|(\eta^\nu,u^\nu)\|_{V^s_\mu}+\|\eta^\nu\|_{H^s}\|u^\nu\|_{H^s}\leq 4k^2_\beta c^2_\beta\|(\eta_0,u_0)\|_{V^s_\mu}.$$

Then, by the Fundamental Theorem of Calculus we obtain

$$1+\eta^\nu(x,t)=1+\eta_0+\int_0^t\partial_t\eta^\nu(x,s)\,ds\geq h_0-k^2_\beta\beta^2\tilde{T}_\nu, \tag{5.14}$$

for all $t\in[0,\tilde{T}_\nu]$. On the one hand, if $\beta\geq\frac{1}{3}$, then

$$k^2_\beta=\frac{c^2_\beta}{c^1_\beta}=c\beta.$$

Thus, for $C_2 > 0$ large enough, we get that $k^1_\beta \geq C_2\beta^2 \geq \frac{c\beta^2}{h_0}$. Moreover, by the assumption $\tilde{T}_\nu \leq T_0$, we conclude from (5.14) that

$$1 + \eta^\nu(x,t) \geq h_0 - k^2_\beta \beta^2 T_0 \geq \frac{h_0}{2},$$

for all $t \in [0, \tilde{T}_\nu]$. On the other hand, in the case when $\beta \in (0, \frac{1}{3})$ we need to verify (3.5). But this can be done the same way by choosing $k^1_\beta \geq \frac{C_1}{\beta} \geq \frac{c}{\beta h_\beta}$ for $C_1 > 0$ large enough.

Having remark 5.2 in mind, the hypotheses of Proposition 3.1 are now verified, leaving us (3.6) and (3.7) at our disposal. With this at hand, we observe that

$$E_s(\mathbf{U}^\nu)(t) \leq E_s(\mathbf{U}^\nu)(0) + c^2_\beta \int_0^t \big(E_s(\mathbf{U}^\nu)(s')\big)^{\frac{3}{2}}\, ds' =: \psi(t).$$

By the above inequality, we then have $\psi'(t) \leq c^2_\beta \big(E_s(\mathbf{U}^\nu)(t)\big)^{\frac{3}{2}} \leq c^2_\beta \left(\psi(t)\right)^{\frac{3}{2}}$. We solve the differential inequality and use (3.7) to relate the energy with the $V^s_\mu-$norm of the solution and deduce that

$$c^1_\beta \|(\eta^\nu, u^\nu)(t)\|_{V^s_\mu} \leq \frac{c^2_\beta \|(\eta_0, u_0)\|_{V^s_\mu}}{1 - \frac{(c^2_\beta)^2}{2} t \|(\eta_0, u_0)\|_{V^s_\mu}}, \tag{5.15}$$

for all $t \in [0, \tilde{T}_\nu]$. Finally, if $C_1, C_2 > 0$ is large enough then since $\tilde{T}_\nu \leq T_0$ we have that

$$\|(\eta^\nu, u^\nu)(t)\|_{V^s_\mu} \leq 2k^2_\beta \|(\eta^\nu_0, u^\nu_0)\|_{V^s_\mu}.$$

Though, by continuity of the solution in time $t \in [0, T^\star_\nu)$, there exists $\tau > 0$ such that $\|(\eta^\nu, u^\nu)(\tau)\|_{V^s_\mu} \leq 3k^2_\beta \|(\eta_0, u_0)\|_{V^s_\mu}$ for $\tilde{T}_\nu < \tau < T^\star_\nu$. This contradicts the definition of $\tilde{T}_\nu$. Thus, we may conclude $T_0 < \tilde{T}_\nu$ for all $\nu > 0$ and that T_0 is independent from ν by its definition in (5.10).

□

Remark 5.4. *For $0 < \beta < \frac{1}{3}$ we observe that $k^1_\beta \sim k^2_\beta \sim \frac{1}{\beta}$ and is due to the appearance of c^1_β in the coercivity estimate* (3.7)*. This will impact the size of the time interval when β is small (see Remark 1.9). On the other hand, for system* (1.9) *the coercivity estimate* (3.26) *is independent of β and therefore gives a longer time of existence, as noted in Remark 1.15.*

Step 4: *Uniqueness.* Given a solution of (1.5), then we claim that it must be unique.

We consider two solutions

$$\varepsilon(\zeta_1, v_1) = (\eta_1, u_1) \text{ and } \varepsilon(\zeta_1, v_1) = (\eta_1, u_1) \text{ in } C([0, T_0]; V^s_\mu(\mathbb{R})),$$

with the same initial data. Then define $\mathbf{W} = (\eta_1 - \eta_2, u_1 - u_2)^T$, which is associated to the initial datum $\mathbf{W}(0) = \mathbf{0}$. Since $(\eta_1, u_1) \in H^s(\mathbb{R})$, there exist a number $h_1 > 0$ such that $\|(\eta_1, u_1)\|_{H^s \times H^s} \leq h_1$. Moreover, η_1 satisfies the non-cavitation condition by the Fundamental Theorem of Calculus and the argument made in the proof of Lemma 5.3. Thus, we may use Proposition 4.1 to deduce

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\beta \max_{i=1,2} \|(\eta_i, u_i)\|_{V^s_\mu} \tilde{E}_0(\mathbf{W}).$$

Then Grönwall's lemma and (4.7) implies that $\|(\eta_1 - \eta_2, u_1 - u_2)(t)\|_{V^s_\mu} = 0$ for all $t \in [0, T_0]$. We therefore conclude the proof of the uniqueness.

Step 5: *Existence of solutions.* We claim that for all $0 \leq s' < s$ there exists a solution $(\zeta, v) = \varepsilon^{-1}(\eta, u) \in C([0, T_0]; V^{s'}_\mu(\mathbb{R})) \cap L^\infty([0, T_0]; V^s_\mu(\mathbb{R}))$ of (1.5) with $T_0 = \mathcal{O}(\frac{1}{\varepsilon})$ defined by (5.10).

Using the change of variable $(\zeta, v) = \varepsilon^{-1}(\eta, u)$, we see that the claim in Step 5 is equivalent to proving that (η^ν, u^ν) solving (5.2) will satisfy system (3.1) in the limit $\nu \searrow 0$ on $[0, T_0]$. In fact, the main idea is to prove the convergence of $\{(\eta^\nu, u^\nu)\}$ as $\nu \searrow 0$ by considering the difference between two solutions

$$\mathbf{W} = (\psi, w) := (\eta^{\nu'} - \eta^\nu, u^{\nu'} - u^\nu).$$

with $0 < \nu' < \nu < \mu$ and where $(\eta^{\nu'}, u^{\nu'}), (\eta^\nu, u^\nu)$ are two sets of solutions to system (5.2), obtained in Step 1. Then for $\alpha = \frac{3}{2}^+$ we have that (ψ, w) satisfies a regularized version of (4.1):

$$\partial_t \mathbf{W} + M(\mathbf{U}^{\nu'}, D)\mathbf{W} = \mathbf{F}^\nu - \nu' J^\alpha \mathbf{W} + (\nu - \nu') J^\alpha \mathbf{U}^\nu,$$

with

$$\mathbf{F}^\nu = -\begin{pmatrix} w\partial_x \eta^\nu + \psi \partial_x u^\nu \\ w \partial_x u^\nu \end{pmatrix}, \tag{5.16}$$

and the same initial data.

The system also satisfies the estimates of Proposition 4.1 by simply noting that

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\beta \tilde{E}_0(\mathbf{W}) - \nu' \big(J^\alpha \mathbf{W}, Q(\mathbf{U}^{\nu'}, D)\mathbf{W}\big)_{L^2} + (\nu - \nu')\big(J^\alpha \mathbf{U}^\nu, Q(\mathbf{U}^{\nu'}, D)\mathbf{W}\big)_{L^2}$$

with

$$\begin{aligned}\big(J^\alpha \mathbf{W}, Q(\mathbf{U}^{\nu'}, D)\mathbf{W}\big)_{L^2} = {} & \big(J^{\frac{\alpha}{2}}\psi, J^{\frac{\alpha}{2}}\psi\big)_{L^2} + \big(J^{\frac{\alpha}{2}} w, (\mathcal{K}_\mu(D) + \eta^{\nu'}) J^{\frac{\alpha}{2}} w\big)_{L^2} \\ & + \big(J^{\frac{\alpha}{2}} w, [J^{\frac{\alpha}{2}}, \eta^{\nu'}] w\big)_{L^2}.\end{aligned}$$

The two first term has a positive sign, while the last term can be absorbed arguing exactly as in remark 5.2. On the other hand, we have directly that for $s > \frac{3}{2}$

$$\begin{aligned}|\big(J^\alpha \mathbf{U}^\nu, Q(\mathbf{U}^{\nu'}, D)\mathbf{W}\big)_{L^2}| \lesssim {} & \|\eta^\nu\|_{H^s}\|\psi\|_{L^2} + \|\eta^{\nu'}\|_{H^s}\|u^\nu\|_{H^s}\|w\|_{L^2} \\ & + \|\sqrt{\mathcal{K}_\mu}(D) u^\nu\|_{H^s}\|\sqrt{\mathcal{K}_\mu}(D) w\|_{L^2}.\end{aligned}$$

Thus, gathering these estimates with (1.17), (2.7) and (4.7) we find that

$$\frac{d}{dt}\tilde{E}_0(\mathbf{W}) \lesssim_\beta \|(\eta_0, u_0)\|_{V^s_\mu}(\tilde{E}_0(\mathbf{W}) + (\nu - \nu')(\tilde{E}_0(\mathbf{W}))^{\frac{1}{2}}). \tag{5.17}$$

Step 5.1: *Convergence in $C([0, T_0]; V^0_\mu(\mathbb{R}))$.* Define the difference (ψ, w) as above, then use (5.17) and (4.7), combined with Grönwall's inequality and (5.11) to find the estimate

$$\sup_{t \in [0, T_0]} \|(\psi, w)(t)\|_{V^0_\mu} \lesssim_\beta \|(\eta_0, u_0)\|_{V^s_\mu}(\nu - \nu').$$

Consequently, $\{(\eta^\nu, u^\nu)\}_{0 < \nu \leq 1}$ defines a Cauchy sequence in $C([0, T_0]; V^0_\mu(\mathbb{R}))$ and we conclude that there exists a limit $(\eta, u) \in C([0, T_0]; V^0_\mu(\mathbb{R}))$ by completeness.

Step 5.2: *Solution in* $C([0,T_0];V_\mu^{s'}(\mathbb{R}))\cap L^\infty([0,T_0];V_\mu^s(\mathbb{R}))$ *for* $s'\in[0,s)$. As a direct consequence of (5.11) and the previous step, we deduce by interpolation

$$\begin{aligned}\|(\psi,w)\|_{L^\infty_{T_0}V_\mu^{s'}} &\lesssim_\beta \|(\psi,w)\|^{\frac{s'}{s}}_{L^\infty_{T_0}V_\mu^s}\|(\psi,w)\|^{1-\frac{s'}{s}}_{L^\infty_{T_0}V_\mu^0}\\ &\lesssim_\beta (\nu-\nu')^{1-\frac{s'}{s}}\|(\eta_0,u_0)\|_{V_\mu^s}\underset{\nu\to 0}{\longrightarrow} 0. \end{aligned}\tag{5.18}$$

Step 6: *The solution is bounded by the initial data.* We claim that the solution obtained in Step 5 satisfies (1.17).

Indeed, using the notation from the previous step, we deduce by (5.11) that

$$\{(\eta^\nu,u^\nu)\}_{0<\nu\leq 1}\subset C([0,T_0];V_\mu^s(\mathbb{R}))$$

is a bounded sequence in a reflexive Banach space. As a result, we have by Eberlein-Šmulian's Theorem that $(\eta^\nu,u^\nu)\underset{\nu\to 0}{\rightharpoonup}(\eta,u)$ weakly in $V_\mu^s(\mathbb{R})$ for all $t\in[0,T_0]$ and implies

$$\sup_{t\in[0,T_0]}\|(\eta,u)\|_{V_\mu^s}\lesssim_\beta \|(\eta_0,u_0)\|_{V_\mu^s}.\tag{5.19}$$

Remark 5.5. *For smooth data* $(\eta_0,u_0)\in H^\infty(\mathbb{R})$ *of* (3.1) *we could reapply the arguments above to deduce the existence of a smooth solution* $(\eta,u)\in C([0,T_0];H^\infty(\mathbb{R}))$*, who satisfy the bound* (5.19) *for any* $s>2$ *and with* T_0 *as defined in* (5.10).

Step 7: *Presistence of the solution.* We claim that there exists a unique solution $(\zeta,v)=\varepsilon^{-1}(\eta,u)\in C([0,T_0];V_\mu^s(\mathbb{R}))$ of (1.5).

We consider (η^δ,u^δ), solving (3.1) with regularised initial data: $(\eta_0^\delta,u_0^\delta)=(\varphi_\delta(D)\eta_0,\varphi_\delta(D)u_0)$ and with $\varphi_\delta(D)$ as in definition 2.15. Then for any $\delta>0$ we have by remark 5.5 that the solution is smooth and satisfy

$$\|(\eta^\delta,u^\delta)\|_{L^\infty_{T_0}V_\mu^s}\lesssim\|(\eta_0^\delta,u_0^\delta)\|_{V_\mu^s},\tag{5.20}$$

for $t\in[0,T_0]$. To conclude the proof, we let $0<\delta'<\delta<1$ and again consider the difference

$$\mathbf{W}=(\psi,w):=(\eta^{\delta'}-\eta^\delta,u^{\delta'}-u^\delta),$$

which also satisfy

$$\partial_t\mathbf{W}+M(\mathbf{U}^{\delta'},D)\mathbf{W}=\mathbf{F}^\delta$$

with

$$\mathbf{F}^\delta=-\begin{pmatrix}w\partial_x\eta^\delta+\psi\partial_x u^\delta\\ w\partial_x u^\delta\end{pmatrix},\tag{5.21}$$

and with initial data

$$(\psi(0),w(0))=((\varphi_{\delta'}(D)-\varphi_\delta(D))\eta_0,(\varphi_{\delta'}(D)-\varphi_\delta(D))u_0).\tag{5.22}$$

The system satisfies the estimates of Proposition 4.1 and we use (4.6) and (4.7), combined with Grönwall's inequality and (5.20) to first find the estimate

$$\sup_{t\in[0,T_0]}\|(\psi,w)(t)\|_{V_\mu^0}\leq e^{c_\beta^2\|(\eta_0,u_0)\|_{V_\mu^s}T_0}\|(\psi(0),w(0))\|_{V_\mu^0},$$

with $\|(\eta_0,u_0)\|_{V_\mu^s}T_0\lesssim_\beta 1$ by definition (5.10). As a result, we use (5.22), the triangle inequality and (2.40) to deduce that

$$\sup_{t\in[0,T_0]}\|(\psi,w)(t)\|_{V_\mu^0}\lesssim_\beta \delta^s\|(\eta_0,u_0)\|_{V_\mu^s}\underset{\delta\to 0}{\longrightarrow}0.\tag{5.23}$$

Moreover, as a direct consequence of (5.11) and (5.23), we deduce by interpolation

$$\begin{aligned}\|(\psi,w)\|_{L^\infty_{T_0}V^{s'}_\mu} &\lesssim_\beta \|(\psi,w)\|^{\frac{s'}{s}}_{L^\infty_{T_0}V^s_\mu}\|(\psi,w^\delta)\|^{1-\frac{s'}{s}}_{L^\infty_{T_0}V^0_\mu} \\ &\lesssim_\beta \delta^{s-s'}\|(\eta_0,u_0)\|_{V^s_\mu} \underset{\delta\to 0}{\longrightarrow} 0. \end{aligned} \tag{5.24}$$

To conclude, we keep these estimates in mind where we aim to apply estimate (4.8), following the Bona-Smith argument [7]. But first we must control $\mathbf{F}^\delta$ in $V^s_\mu(\mathbb{R})$. The elements of $\mathbf{F}^\delta$ are given in (4.2), and we must therefore control the terms given by:

$$\begin{aligned}&\big(J^s\mathbf{F}^\delta, Q(\mathbf{U}^{\delta'},D)J^s\mathbf{W}\big)_{L^2} \\ &\quad= \big(J^s(w\partial_x\eta^\delta), J^s\psi\big)_{L^2} + \big(J^s(\psi\partial_x\eta^\delta), J^s\psi\big)_{L^2} \\ &\qquad+ \big(J^s(w\partial_x u^\delta), \eta^{\delta'}J^s w\big)_{L^2} + \big(J^s(w\partial_x u^\delta), \mathcal{K}_\mu(D)J^s w\big)_{L^2} \\ &\quad=: A_1 + A_2 + A_3 + A_4.\end{aligned}$$

The terms A_1, A_2 and A_3 are treated similarly. For instance, take A_1. Then we observe that

$$A_1 \le \|J^s(w\partial_x\eta^\delta)\|_{L^2}\|J^s\psi\|_{L^2}.$$

Furthermore, using (2.22) and the Sobolev embedding, we obtain that

$$\|J^s(w\partial_x\eta^\delta)\|_{L^2} \lesssim \|w\|_{L^\infty}\|J^s\partial_x\eta^\delta\|_{L^2} + \|J^s w\|_{L^2}\|\eta^\delta\|_{H^s}. \tag{5.25}$$

Using the triangle inequality, (2.39), and (5.20), we observe that

$$\|J^s\partial_x\eta^\delta\|_{L^2} \le \delta^{-1}\|\eta_0\|_{H^s}, \tag{5.26}$$

which needs to be compensated to close the estimate. With this in mind, we use the Sobolev embedding $H^{\frac{1}{2}^+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (5.24) to deduce

$$\|w\|_{L^\infty_{T_0}L^\infty} \lesssim \|(\psi,w)\|_{L^\infty_{T_0}V^{\frac{1}{2}^+}_\mu} \lesssim \delta^{s-\frac{1}{2}^+}\|(\eta_0,u_0)\|_{V^s_\mu}. \tag{5.27}$$

Thus, combining (5.25) with (5.26) and (5.27) we get that

$$|A_1| \lesssim \sup_{t\in[0,T_0]}\big(\|w\|_{H^s}\|\eta^\delta\|_{H^s} + \delta^{s-\frac{3}{2}^+}\|(\eta_0,u_0)\|_{V^s_\mu}\big)\|\psi\|_{H^s},$$

as $\delta \searrow 0$. Arguing similarly, and using estimate (5.20), we deduce that

$$|A_1| + |A_2| + |A_3| \lesssim \sup_{t\in[0,T_0]}\|(\eta_0,u_0)\|_{V^s_\mu}\big(\|(\psi,w)\|^2_{V^s_\mu} + \delta^{s-\frac{3}{2}^+}\|(\psi,w)\|_{V^s_\mu}\big).$$

For A_4, we write

$$\begin{aligned}A_4 &= \big([J^s\sqrt{\mathcal{K}_\mu}(D), w]\partial_x u^\delta, J^s\sqrt{\mathcal{K}_\mu}(D)w\big)_{L^2} \\ &\quad+ \big(wJ^s\sqrt{\mathcal{K}_\mu}(D)\partial_x u^\delta, J^s\sqrt{\mathcal{K}_\mu}(D)w\big)_{L^2}.\end{aligned}$$

The commutator is treated by (2.18). While in the second term, we use (2.7) and argue as above, giving the estimate

$$|A_4| \lesssim_\beta \sup_{t\in[0,T_0]}\|(\eta_0,u_0)\|_{V^s_\mu}\big(\|(\psi,w)\|^2_{V^s_\mu} + \delta^{s-\frac{3}{2}^+}\|(\psi,w)\|_{V^s_\mu}\big).$$

We may therefore conclude by (4.8):

$$\frac{d}{dt}\tilde{E}_s(\mathbf{W}) \lesssim_\beta \|(\eta_0,u_0)\|_{V^s_\mu}\big(\tilde{E}_s(\mathbf{W}) + \delta^{s-\frac{3}{2}^+}\tilde{E}_s(\mathbf{W})^{\frac{1}{2}}\big).$$

Then Grönwall's inequality and (4.9) implies

$$\|(\psi, w)\|_{L^\infty_{T_0} V^s_\mu} \lesssim_\beta \delta^{s-\frac{3}{2}^+} \|(\eta_0, u_0)\|_{V^s_\mu} \underset{\delta\to 0}{\longrightarrow} 0.$$

Thus, (η^δ, u^δ) is a Cauchy sequence in $C([0,T_0]; V^s_\mu(\mathbb{R}))$ and we conclude by the uniqueness of the limit that the solution $(\eta, u) \in C([0,T_0]; V^s_\mu(\mathbb{R}))$.

Step 8: *Continuous dependence of the flow map data solution.* Consider two sets of initial data $(\zeta_1, v_1)(0), (\zeta_2, v_2)(0) \in V^s_\mu(\mathbb{R})$. Then we claim that for all $\lambda > 0$, there exists $\gamma > 0$ such that having

$$\|(\zeta_1 - \zeta_2, v_1 - v_2)(0)\|_{V^s_\mu} < \gamma,$$

implies

$$\|(\zeta_1 - \zeta_2, v_1 - v_2)\|_{L^\infty_{\frac{T_0}{2}} V^s_\mu} < \lambda.$$

Equivalently, we will prove that for $\varepsilon(\zeta_1, \zeta_2, v_1, v_2) = (\eta_1, \eta_2, u_1, u_2)$ such that

$$\|(\eta_1 - \eta_2, u_1 - u_2)(0)\|_{V^s_\mu} < \varepsilon\gamma,$$

implies

$$\|(\eta_1 - \eta_2, u_1 - u_2)\|_{L^\infty_{\frac{T_0}{2}} V^s_\mu} < \varepsilon\lambda.$$

Using the notation in Step 7, we let $0 < \delta < 1$ to be fixed, and $(\eta_1^\delta, u_1^\delta), (\eta_2^\delta, u_2^\delta) \in C([0, \frac{T_0}{2}]; V^s_\mu(\mathbb{R}))$ be two solutions of (5.2) on large time with corresponding initial data $(\varphi_\delta(D)\eta_1, \varphi_\delta(D)u_1)(0)$ and $(\varphi_\delta(D)\eta_2, \varphi_\delta(D)u_2)(0)$. Then observe

$$\begin{aligned} \|(\eta_1 - \eta_2, u_1 - u_2)\|_{V^s_\mu} &\leq \|(\eta_1 - \eta_1^\delta, u_1 - u_1^\delta)\|_{V^s_\mu} + \|(\eta_2 - \eta_2^\delta, u_2 - u_2^\delta)\|_{V^s_\mu} \\ &\quad + \|(\eta_1^\delta - \eta_2^\delta, u_1^\delta - u_2^\delta)\|_{V^s_\mu} \\ &=: B_1 + B_2 + B_3. \end{aligned} \tag{5.28}$$

For the first two terms we use that $\varepsilon^{-1}(\eta^\delta, u^\delta) = (\zeta^\delta, v^\delta) \to (\zeta, v) = \varepsilon^{-1}(\eta, u)$ as $\delta \searrow 0$ by Step 6. Therefore it follows that B_1 and B_2 must at least satisfy the estimate,

$$\sup_{t\in[0,\frac{T_0}{2}]} (B_1 + B_2)(t) \lesssim_\beta \varepsilon o_\delta(1). \tag{5.29}$$

While for B_3, we need the continuity of the flow map of the regularized system (5.2) on a long time (see Remark 5.1).

We let $\tilde{\mathbf{W}} = (\tilde{\psi}, \tilde{w}) = (\eta_1^\delta - \eta_2^\delta, u_1^\delta - u_2^\delta)$. Then staying consistent with previous notation, we have that the difference between two regularized solutions will satisfy the equation:

$$\partial_t \tilde{\mathbf{W}} + M(\mathbf{U}_1^\delta, D)\tilde{\mathbf{W}} = \tilde{\mathbf{F}}^\delta, \tag{5.30}$$

with

$$\tilde{\mathbf{F}}^\delta = -\begin{pmatrix} \tilde{w}\partial_x\, \eta_2^\delta + \tilde{\psi}\partial_x u_2^\delta \\ \tilde{w}\partial_x u_2^\delta \end{pmatrix},$$

and initial data

$$(\tilde{\psi}, \tilde{w})(0) = (\varphi_\delta(D)\eta_1 - \varphi_\delta(D)\eta_2, \varphi_\delta(D)u_1 - \varphi_\delta(D)u_2)(0).$$

We will use this information to estimate B_3 by suitable energy estimates at $V^0_\mu(\mathbb{R})$ and $V^s_\mu(\mathbb{R})-$level.

Similar to Step 6, we first obtain the estimate in $V^0_\mu(\mathbb{R})$ by using (4.7) and (4.9). Indeed, there holds

$$\frac{d}{dt}\tilde{E}_0(\tilde{\mathbf{W}}) \lesssim_\beta \max_{i=1,2} \|(\eta_i^\delta, u_i^\delta)\|_{V^s_\mu} \tilde{E}_0(\tilde{\mathbf{W}}). \tag{5.31}$$

For simplicity we let $\|(\eta_1,u_1)(0)\|_{V^s_\mu}=\varepsilon K$. Moreover, we observe that if $\varepsilon\gamma<\frac{1}{2}\|(\eta_1,u_1)(0)\|_{V^s_\mu}$, then we have by (5.20) that

$$\begin{aligned}\|(\eta_1^\delta,u_1^\delta)\|_{L^\infty_{\frac{T_0}{2}}V^s_\mu}+\|(\eta_2^\delta,u_2^\delta)\|_{L^\infty_{\frac{T_0}{2}}V^s_\mu}&\lesssim_\beta \|(\eta_1,u_1)(0)\|_{V^s_\mu}+\|(\eta_2,u_2)(0)\|_{V^s_\mu}\\&\lesssim_\beta \varepsilon K.\end{aligned}\tag{5.32}$$

As a result, we have an estimate of the difference in $V^0_\mu(\mathbb{R})$. Indeed, by Grönwall's inequality, (5.31), (5.32), the triangle inequality, and (2.40) implies

$$\|(\tilde{\psi},\tilde{w})\|_{V^0_\mu}\lesssim_\beta \|(\tilde{\psi},\tilde{w})(0)\|_{V^0_\mu}\lesssim_\beta \varepsilon K(\delta^s+K^{-1}\gamma).\tag{5.33}$$

We will now use this decay estimate to deal with (4.8), which is at the $V^s_\mu(\mathbb{R})-$level. Similar to Step 6, we decompose the source term (4.2) in four pieces

$$\begin{aligned}\tilde{A}&:=\big(J^s\tilde{\mathbf{F}}^\delta,Q(\mathbf{U}_1^\delta,D)J^s\tilde{\mathbf{W}}^\delta\big)_{L^2}\\&=\big(J^s(\tilde{w}\partial_x\eta_2^\delta),J^s\tilde{\psi}\big)_{L^2}+\big(J^s(\tilde{\psi}\partial_x\eta_2^\delta),J^s\tilde{\psi}\big)_{L^2}\\&\quad+\big(J^s(\tilde{w}\partial_x u_2^\delta),\eta_2^\delta J^s\tilde{w}\big)_{L^2}+\big(J^s(\tilde{w}\partial_x u_2^\delta),\mathcal{K}_\mu(D)J^s\tilde{w}\big)_{L^2}\\&=:\tilde{A}_1+\tilde{A}_2+\tilde{A}_3+\tilde{A}_4.\end{aligned}$$

To estimate $\tilde{A}_1$, we first obtain a bound similar to (5.24). Indeed, using the Sobolev emebedding, interpolation, (5.32), and (5.33) yields

$$\begin{aligned}\sup_{t\in[0,T_0]}\|\tilde{w}\|_{L^\infty}&\lesssim\sup_{t\in[0,T_0]}\|(\tilde{\psi},\tilde{w})\|_{V_\mu^{\frac{1}{2}^+}}\\&\lesssim\|(\tilde{\psi},\tilde{w})\|^{\frac{1}{2s}^+}_{L^\infty_{T_0}V^s_\mu}\|(\tilde{\psi},\tilde{w})\|^{1-\frac{1}{2s}^+}_{L^\infty_{T_0}V^0_\mu}\\&\lesssim\varepsilon K(\delta^{s-\frac{1}{2}^+}+(K^{-1}\gamma)^{1-\frac{1}{2s}^+})\end{aligned}$$

where $1-\frac{1}{2s}^+>0$ for $s>\frac{1}{2}^+$. Then arguing as we did for A_1 in Step 7, we obtain that

$$\begin{aligned}|\tilde{A}_1|&\lesssim\sup_{t\in[0,T_0]}\big(\|\tilde{w}\|_{H^s}\|\eta_2^\delta\|_{H^s}+\|\tilde{w}\|_{L^\infty}\delta^{-1}\|\eta_2^\delta\|_{H^s}\big)\|\tilde{\psi}\|_{H^s}\\&\lesssim\varepsilon K\sup_{t\in[0,T_0]}\big(\|\tilde{w}\|_{H^s}+\delta^{s-\frac{3}{2}^+}+\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}\big)\|\tilde{\psi}\|_{H^s}.\end{aligned}$$

Moreover, for the remaining terms, we can use similar estimates, recalling that for $\tilde{A}_4$ we also need to deal with the non-local operator $\mathcal{K}_\mu(D)$ (see step 7 for details). Indeed,

$$\tilde{A}\lesssim_\beta\varepsilon K\sup_{t\in[0,T_0]}\big(\|(\tilde{\psi},\tilde{w})\|_{V^s_\mu}+\delta^{s-\frac{3}{2}^+}+\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}\big)\|(\tilde{\psi},\tilde{w})\|_{V^s_\mu}.\tag{5.34}$$

Consequently, combining estimates (4.8) and (4.9) with (5.34) yields

$$\frac{d}{dt}\tilde{E}_s(\tilde{\mathbf{W}})\lesssim_\beta\varepsilon K\big(\tilde{E}_s(\tilde{\mathbf{W}})+(\delta^{s-\frac{3}{2}^+}+\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+})\tilde{E}_s(\tilde{\mathbf{W}})^{\frac{1}{2}}\big).$$

Thus, we have an estimate on B_3 by the energy estimate (4.9), Grönwall's inequality and (2.41). Indeed, there holds

$$\begin{aligned}B_3=\|(\tilde{\psi},\tilde{w})\|_{V^s_\mu}&\lesssim_\beta\|(\tilde{\psi},\tilde{w})(0)\|_{V^s_\mu}+\varepsilon K(\delta^{s-\frac{3}{2}^+}+\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+})\\&\lesssim_\beta\varepsilon o_\delta(1)+\varepsilon\gamma+\varepsilon K(\delta^{s-\frac{3}{2}^+}+\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}).\end{aligned}\tag{5.35}$$

Returning to (5.28), we may conclude the proof of the continuous dependence. We first fix $0<\delta<1$ to be small enough and satisfying

$$o_\delta(1) + K\delta^{s-\frac{3}{2}^+} < \frac{\lambda}{2c_\beta},$$

for some constant c_β depending on β. Then let γ verify the restriction:

$$\varepsilon\gamma < \frac{1}{2}\|(\eta_1, u_1)(0)\|_{V^s_\mu},$$

such that $\gamma + K\delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+} < \frac{\lambda}{2c_\beta}$. Consequently, we have by (5.28), (5.29) and (5.35) that

$$\begin{aligned}\sup_{t\in[0,\frac{T_0}{2}]} \|(\eta_1-\eta_2, u_1-u_2)(t)\|_{V^s_\mu} &\leq \varepsilon c_\beta(o_\delta(1) + \gamma + K(\delta^{s-\frac{3}{2}^+} + \delta^{-1}(K^{-1}\gamma)^{1-\frac{1}{2s}^+}))\\ &< \varepsilon\lambda.\end{aligned}$$

As a result, we have demonstrated that the solution of (1.5) depends continuously on the initial data and thus completes the proof of Theorem 1.6.

□

6. The two-dimensional case

Let $x=(x_1,x_2)\in\mathbb{R}^2$ and $\mathbf{v}=(v_1,v_2)^T$. Then we observe that under the curl-free condition on the initial datum in Theorem 1.6 that (1.6) enjoys a similar structure to (1.5) (see also Lemma 4.2 in [35]). Indeed, since $\operatorname{curl}\mathbf{v}_0=\mathbf{0}$ we can take the curl of the second equation in (1.6) and find that $\operatorname{curl}\mathbf{v}=\mathbf{0}$, courtesy of the Fundamental Theorem of Calculus. We therefore have the relation

$$\partial_{x_1}v_2 = \partial_{x_2}v_1. \tag{6.1}$$

Now, let $\mathbf{u}=\varepsilon\mathbf{v}$ and define $\mathbf{U}=(\eta,\mathbf{u})^T=\varepsilon(\zeta,\mathbf{v})^T$. Then use (6.1) to rewrite system (1.6) as

$$\partial_t\mathbf{U} + M(\mathbf{U},D)\mathbf{U} = \mathbf{0}, \tag{6.2}$$

with

$$M(\mathbf{U},D) = \begin{pmatrix} \mathbf{u}\cdot\nabla & (\mathcal{K}_\mu(D)+\eta)\partial_{x_1} & (\mathcal{K}_\mu(D)+\eta)\partial_{x_2}\\ \partial_{x_1} & (\mathbf{u}\cdot\nabla)\,\boldsymbol{\cdot} & 0\\ \partial_{x_2} & 0 & (\mathbf{u}\cdot\nabla)\,\boldsymbol{\cdot}\end{pmatrix}.$$

Then the natural symmetrizer is given by

$$Q(\mathbf{U},D) = \begin{pmatrix} 1 & 0 & 0\\ 0 & (\mathcal{K}_\mu(D)+\eta) & 0\\ 0 & 0 & (\mathcal{K}_\mu(D)+\eta)\end{pmatrix},$$

and by extension, an energy associated to (6.2) reads

$$E_s(\mathbf{U},D) = \big(J^s\mathbf{U}, Q(\mathbf{U},D)J^s\mathbf{U}\big).$$

The energy estimates are similar to the one-dimensional case. Indeed, for $(\eta,\mathbf{u})\in V^s_\mu(\mathbb{R}^2)$ and $s>\frac{5}{2}$, we observe that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_s(\mathbf{U}) = &-\big(J^s(\mathbf{u}\cdot\nabla\eta), J^s\eta\big)_{L^2} - \big(J^s((\mathcal{K}_\mu(D)+\eta)\nabla\cdot\mathbf{u}), J^s\eta\big)_{L^2}\\ &-\big(J^s\nabla\eta, (\mathcal{K}_\mu(D)+\eta)J^s\mathbf{u}\big)_{L^2} - \big(J^s((\mathbf{u}\cdot\nabla)\mathbf{u}), (\mathcal{K}_\mu(D)+\eta)J^s\mathbf{u}\big)_{L^2}\\ &+\frac{1}{2}\big(J^s\mathbf{u}, (\partial_t\eta)J^s\mathbf{u}\big)_{L^2}.\end{aligned}$$

An estimate analogous to the ones of Proposition 3.1 is a consequence of two-dimensional versions of estimates in Section 2. However, these are easily extended to 2-d by noting that $\mathcal{K}_\mu(D)$ and $\mathcal{T}_\mu(D)$ is radial.

The estimate of the difference between two solutions is similar to the proof of Proposition 4.1.

Appendix A

A.1. **Pointwise estimates for $\sqrt{K_\mu(\xi)}$ and $\sqrt{T_\mu(\xi)}$.** Before turning to the proof of the pointwise estimates in Lemma 2.1 and Lemma 2.5, we make an important observation. Let $\sqrt{\mathcal{T}_\mu}(D)$ be the Fourier multiplier associated with the symbol

$$\sqrt{T_\mu(\xi)} = \sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}.$$

Then the operator is regularizing for $\mu > 0$ on $L^2(\mathbb{R})$, and acts similar to the scaled Bessel potential $J_\mu^{-\frac{1}{2}}$ defined by the symbol $\xi \mapsto (1+\mu\xi^2)^{-\frac{1}{4}}$. While $\sqrt{K_\mu(\xi)}$ has a similar behaviour in low frequency for $\beta < \frac{1}{3}$, but acts like $J_\mu^{\frac{1}{2}}$ in high frequencies.

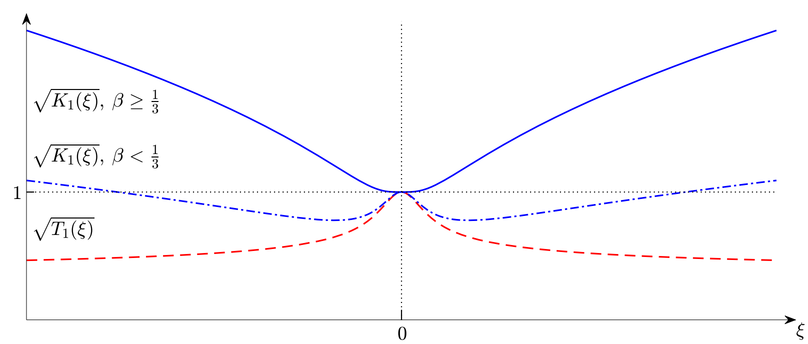

Figure 3. The multiplier $\sqrt{K_1(\xi)}$ in the cases $\beta \geq \frac{1}{3}$ (line) and $\beta < \frac{1}{3}$ (dash-dots). The red curve is a plot of $\sqrt{T_1(\xi)}$ (dash).

Lemma A.1. *Let $\mu > 0$ and take any $n \in \mathbb{N}$.*

- *Then $T_\mu(\xi)$ satisfies*

$$\Big|\frac{d^n}{d\xi^n}\sqrt{T_\mu(\xi)}\Big| \lesssim \mu^{\frac{n}{2}} \langle\sqrt{\mu}\xi\rangle^{-\frac{1}{2}-n}. \tag{A.1}$$

- *Similarly, $K_\mu(\xi)$ satisfies*

$$\Big|\frac{d^n}{d\xi^n}\sqrt{K_\mu(\xi)}\Big| \lesssim_\beta \mu^{\frac{n}{2}} \langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}-n}. \tag{A.2}$$

Proof. The proof is a generalization of Lemma 8 in [15]. Following their arguments, we observe that since $T_1(\sqrt{\mu}r) = T_\mu(r)$ for $r > 0$, it is sufficient to show that

$$\Big|\frac{d^n}{dr^n}\sqrt{T_1(r)}\Big| \lesssim \langle r\rangle^{-\frac{1}{2}-n}.$$

We divide the proof into two steps.

First, let $0 < r < \frac{1}{2}$ and prove that any derivative of $\sqrt{T_1(r)}$ is bounded. We have that $\sqrt{T_1(r)}$ is bounded in this region. By direct computation, we observe

$$\frac{d}{dr}\sqrt{T_1(r)} = \frac{\mathrm{sech}^2(r)}{(2r)^2\sqrt{T_1(r)}}\left(2r + \frac{e^{-2r} - e^{2r}}{2}\right) =: \frac{\mathrm{sech}^2(r)}{\sqrt{T_1(r)}}G(r), \tag{A.3}$$

where $G(r)$ can be written as a series by expanding the exponentials. Indeed, we have that

$$G(r) = \frac{1}{(2r)^2}\left(2r + \frac{e^{-2r} - e^{2r}}{2}\right) = -\sum_{k=0}^{\infty}\frac{(2r)^{2k+1}}{(2k+3)!}.$$

The series is uniformly convergent for $r \geq 0$. Moreover, $G(r)$ and its derivatives are bounded for $0 < r < \frac{1}{2}$. By extension, since for all $n \geq 0$ there holds $\frac{d^n}{dr^n} sech^2(r) \lesssim e^{-2r}$ we have that

$$\Big|\frac{d^n}{dr^n}\sqrt{T_1(r)}\Big| \lesssim 1.$$

Now, we let $r \geq \frac{1}{2}$ and prove the necessary decay estimate. We use the identity

$$\tanh(r) = 1 - \frac{2}{e^{2r}+1}, \tag{A.4}$$

and deduce by the chain rule that

$$\Big|\frac{d^n}{dr^n}\sqrt{T_1(r)}\Big| \lesssim \sum_{k=0}^{n}\Big(\frac{1}{r}\big(1 - \frac{2}{e^{2r}+1}\big)\Big)^{\frac{1}{2}-k} r^{-k-n} \lesssim \langle r\rangle^{-\frac{1}{2}-n}.$$

Lastly, we have that (A.2) follows by the Leibniz rule. Indeed, we observe

$$\begin{aligned}\Big|\frac{d^n}{dr^n}\sqrt{K_1(r)}\Big| &= \Big|\frac{d^n}{dr^n}\sqrt{T_1(r)(1+\beta r^2)}\Big| \\ &\lesssim \sum_{k=0}^{n}\Big|\frac{d^{n-k}}{dr^{n-k}}\sqrt{T_1(r)}\Big|\Big|\frac{d^k}{dr^k}\sqrt{1+\beta r^2}\Big| \\ &\lesssim_\beta \langle r\rangle^{\frac{1}{2}-n},\end{aligned}$$

which concludes the proof of Lemma A.1. □

A.2. **Proof of Lemmas 2.1 and 2.5.**

Proof of Lemma 2.1. First, we again make the observation that $K_1(\sqrt{\mu}\xi) = K_\mu(\xi)$. Therefore, we simply let $r > 0$ and consider $K_1(r)$. To establish the upper bound given in (2.1), we note that for $r < 1$ we have

$$K_1(r) \leq 1 + \beta.$$

This is because $\frac{\tanh(r)}{r} \leq 1$. On the other hand, when $r \geq 1$ then $\tanh(r) < 1$ and it follows that

$$K_1(r) \leq 1 + \beta r.$$

Consequently, for all $r > 0$ there holds $K_1(r) \leq c_\beta^2(1 + r)$ with c_β^2 as defined in (1.19).

Next, we prove the lower bound given by (2.2) with $\beta \geq \frac{1}{3}$. We will again split r into two intervals, where we aim to prove

$$K_1(r) = \frac{\tanh(r)}{r}(1 + \beta r^2) \geq (1 - \frac{h_0}{2}) + cr, \tag{A.5}$$

for some positive constant $c > 0$ and any $h_0 \in (0, 1)$. We prove (A.5) by considering two cases for r. When $0 \leq r \leq \frac{h_0}{4}$ we use that

$$\tanh(r) = \int_0^r (1 - \tanh^2(x))\, dx \geq r - \frac{r^3}{3}, \tag{A.6}$$

since $\tanh^2(x) \leq x^2$ by the mean value theorem. Therefore, we have that

$$K_1(r) \geq \Big(1 - \frac{r^2}{3}\Big)\Big(1 + \frac{r^2}{3}\Big) \geq \Big(1 - \frac{h_0}{2}\Big) + \Big(\frac{h_0}{4} - \frac{r^4}{9}\Big) + r,$$

which implies (A.5) since $0 \leq r \leq \frac{h_0}{4}$.

For the remaining part, we use the identity (A.4) and show that (A.5) holds for $r \geq \frac{h_0}{4}$ if:

$$\begin{aligned}
&\tanh(r)\Big(1 + \frac{r^2}{3}\Big) - r(1 + cr) > 0 \\
\iff &\Big(1 + \frac{r^2}{3} - r(1 + cr)\Big) - \frac{2}{e^{2r} + 1}\Big(1 + \frac{r^2}{3}\Big) > 0 \\
\iff &\Big(1 + \frac{r^2}{3} - r(1 + cr)\Big)\big(e^{2r} + 1\big) - 2\Big(1 + \frac{r^2}{3}\Big) := G(r) > 0.
\end{aligned}$$

But this holds since

$$\begin{aligned}
G'''(r) &= \frac{4}{3}e^{2r}\big(2r^2 - 3c(2r^2 + 6r + 3)\big) \\
&\geq \frac{4}{3}e^{2r}\Big(r^2(1 - 6c) + r(\frac{h_0}{8} - 18c) + (\frac{h_0^2}{32} - 9c)\Big),
\end{aligned}$$

and is positive for $0 < c < 10^{-3}h_0^2$ with $r \geq \frac{h_0}{4}$. Indeed, as a consequence we have the following chain of implications

$$\begin{aligned}
&0 < G''\Big(\frac{h_0}{2}\Big) \leq G''(r) = \frac{2}{3}\big(e^{2r}(1 - 2r + 2r^2 - 3c(2r^2 + 4r + 1)) - 3c - 1\big) \\
\Longrightarrow\ &0 < G'\Big(\frac{h_0}{2}\Big) \leq G'(r) = \frac{1}{3}\big(e^{2r}(3 - 4r + 2r^2 - 6cr(r + 1)) - 2r - 3 - 6cr\big) \\
\Longrightarrow\ &0 < G\Big(\frac{h_0}{2}\Big) \leq G(r).
\end{aligned}$$

We have therefore verified (A.5) for all $r \geq 0$ and we conclude that (2.2) holds true.

Similarly, for $0 < \beta < \frac{1}{3}$, we have that (2.3) is a consequence of the inequality

$$\frac{\tanh(r)}{r}(1 + \beta r^2) \geq \beta + cr.$$

One should note that we do not require sharp estimates. In fact, we simply need to obtain the estimate

$$\Big(1 + \beta r^2 - r(\beta + cr)\Big)\big(e^{2r} + 1\big) - 2\Big(1 + \beta r^2\Big) =: H(r) \geq 0,$$

for $r \geq 0$. On the other hand, we observe that

$$H'''(r) = 4e^{2r}\Big(2 + 2\beta r(2 + r) - 3c + 2cr(3 + r)\Big) > 0$$

for all $r \geq 0$ if

$$(2 - 3c) + 2r(2\beta - 3c) + 2r^2(\beta - c) > 0$$

and is ensured for $0 < c \leq 10^{-3}\beta$. Consequently,

$$0 < H''(0) \leq H''(r) = 2e^{2r}\Big(2 + \beta(2r^2 + 2r - 1) - c(2r^2 + 4r + 1)\Big) - 2(\beta + c)$$
$$0 < H'(0) \leq H'(r) = e^{2r}\Big(2 + \beta(2r^2 - 1) - 2c(r^2 + 2r)\Big) - \beta(2r + 1) - 2cr$$
$$0 < H(0) \leq H(r),$$

and we argue as above to conclude.

The proof of estimate (2.4) is a direct consequence of Lemma A.1 and (A.2) with $n = 1$ if we trace the dependence in β:

$$\Big|\frac{d}{dr}\sqrt{K_1(r)}\Big| \lesssim \langle r\rangle^{-\frac{1}{2}-1}(1 + \beta r^2)^{\frac{1}{2}} + \langle r\rangle^{-\frac{1}{2}}\frac{\beta r}{(1 + \beta r^2)^{\frac{1}{2}}} \lesssim \langle r\rangle^{-1} + \sqrt{\beta}\langle r\rangle^{-\frac{1}{2}}.$$

Estimate (2.5) concerns the following bound on the difference:

$$\Big|\sqrt{K_\mu(\xi)} - \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big| = \Big|\Big(\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}(1 + \beta\mu\xi^2)\Big)^{\frac{1}{2}} - \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big|.$$

For $\beta\sqrt{\mu}|\xi| \leq 1$ there holds trivially by using the triangle inequality that

$$\Big|\sqrt{K_\mu(\xi)} - \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big| \lesssim 1.$$

While for $\beta\sqrt{\mu}|\xi| > 1$ we observe by direct calculations that

$$\begin{aligned}
\Big|\sqrt{K_\mu(\xi)} - \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big| &= \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big|\Big(\frac{\tanh(\sqrt{\mu}|\xi|)}{\beta\mu\xi^2} + \tanh(\sqrt{\mu}|\xi|)\Big)^{\frac{1}{2}} - 1\Big| \\
&\lesssim \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big|\frac{\tanh(\sqrt{\mu}|\xi|)}{\beta\mu\xi^2} + (\tanh(\sqrt{\mu}|\xi|) - 1)\Big| \\
&\lesssim \frac{1}{(\beta\sqrt{\mu}|\xi|)^{\frac{1}{2}}}\frac{1}{\sqrt{\mu}|\xi|} + \sqrt{\beta}(\sqrt{\mu}|\xi|)^{\frac{1}{2}}e^{-2\sqrt{\mu}|\xi|} \\
&\lesssim \beta + \sqrt{\beta},
\end{aligned}$$

where we used the triangle inequality and that $\beta\sqrt{\mu}|\xi| > 1$.

Lastly, we prove (2.6) by using (2.5):

$$\begin{aligned}
\sqrt{K_\mu(\xi)}\langle\xi\rangle^{s-1}|\xi| &= \Big(\sqrt{K_\mu(\xi)} - \sqrt{\beta}\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big)\langle\xi\rangle^{s-1}|\xi| + \sqrt{\beta}\mu^{\frac{1}{4}}\langle\xi\rangle^{s-1}|\xi|^{\frac{3}{2}} \\
&\lesssim (\beta + \sqrt{\beta})\langle\xi\rangle^{s} + \sqrt{\beta}\mu^{\frac{1}{4}}\langle\xi\rangle^{s}|\xi|^{\frac{1}{2}}.
\end{aligned}$$

□

Proof of Lemma 2.5. To prove (2.9), since $T_1(\sqrt{\mu}\xi) = T_\mu(\xi)$, we only need to establish the following inequality:

$$1 - \frac{h_0}{2} + cr \leq \frac{r}{\tanh(r)} \lesssim 1 + r, \tag{A.7}$$

for all $r > 0$ and some $c > 0$. We also note that the upper bound is trivial, so we only prove the lower bound. Let $h_0 \in (0, 1)$. By the mean value theorem we find that $tanh(r) \leq r$ and observe

$$\frac{r}{\tanh(r)} = (1 - \frac{h_0}{2})\frac{r}{\tanh(r)} + \frac{h_0}{2}\frac{r}{\tanh(r)} \geq 1 - \frac{h_0}{2} + \frac{h_0}{2}r.$$

Next, we consider (2.10). For $\sqrt{\mu}|\xi| \leq 1$ we have that $T_\mu(\xi) \sim 1$ and $\langle \sqrt{\mu}\xi \rangle \sim 1$. On the other hand, when $\sqrt{\mu}|\xi| \geq 1$ then $T_\mu(\xi) \sim \frac{1}{\sqrt{\mu}|\xi|}$ and $\langle \sqrt{\mu}\xi \rangle \sim \sqrt{\mu}|\xi|$. Multiplying the two functions, we obtain the desired result.

We estimate the derivative (2.11) directly and using that $\mu \in (0,1)$:

$$\Big|\frac{d}{d\xi}\langle \xi \rangle^s \langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}}\Big| \lesssim \langle \xi \rangle^{s-1}\langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}} + \langle \xi \rangle^{s}\langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}}\sqrt{\mu}\langle \sqrt{\mu}\xi \rangle^{-1} \lesssim \langle \xi \rangle^{s-1}\langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}},$$

since $\sqrt{\mu}\langle \sqrt{\mu}\xi \rangle^{-1} \leq \langle \xi \rangle^{-1}$.

Similarly, we have that (2.12) follows by the same argument after using (A.1) and (2.11):

$$\begin{aligned}\Big|\frac{d}{d\xi}\sqrt{T_\mu(\xi)}\langle \xi \rangle^s \langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}}\Big| &\lesssim \sqrt{\mu}\langle \xi \rangle^{-\frac{3}{2}}\langle \xi \rangle^{s}\langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}} + \langle \sqrt{\mu}\xi \rangle^{-\frac{1}{2}}\langle \xi \rangle^{s-1}\langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}} \\ &\lesssim \langle \xi \rangle^{s-1}.\end{aligned}$$

For estimate (2.13), we observe that

$$\langle \sqrt{\mu}\xi \rangle^{\frac{1}{2}} - \mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}} = \mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}\Big((\frac{1}{\mu\xi^2}+1)^{\frac{1}{4}} - 1\Big) \lesssim 1.$$

□

A.3. **Proof of Lemmas 2.11 and 2.12.** For the proof of Lemma 2.11 and Lemma 2.12, we need a "generalized" version of the Kato-Ponce commutator estimate which holds for symbols defined by:

Definition A.2 (Symbol class [33] Def. *B*.7)**.** *We say that a symbol $\sigma(D)$ is a member of the symbol class $\mathcal{S}^s$ with $s \in \mathbb{R}$, if $\xi \mapsto \sigma(\xi) \in \mathbb{C}$ is smooth and satisfies*

$$\forall \alpha \in \mathbb{N}, \quad \sup_{\xi \in \mathbb{R}} \langle \xi \rangle^{\alpha - s}\Big|\frac{d^\alpha}{d\xi^\alpha}\sigma(\xi)\Big| < \infty.$$

One also associates the following seminorm on $\mathcal{S}^s$:

$$\mathcal{N}^s(\sigma) = \sup_{\alpha \in \mathbb{N},\ \alpha \leq 4} \ \sup_{\xi \in \mathbb{R}} \langle \xi \rangle^{\alpha - s}\Big|\frac{d^\alpha}{d\xi^\alpha}\sigma(\xi)\Big|. \tag{A.8}$$

The following results is found in Appendix B of [33].

Lemma A.3. *Let $t_0 > 1/2$, $s \geq 0$ and $\sigma \in \mathcal{S}^s$. If $f \in H^s \cap H^{t_0+1}(\mathbb{R})$, then for all $g \in H^{s-1}(\mathbb{R})$,*

$$\|[\sigma(D), f]g\|_{L^2} \lesssim \mathcal{N}^s(\sigma)\|f\|_{H^{\max\{t_0+1,s\}}}\|g\|_{H^{s-1}}. \tag{A.9}$$

With this at hand, we may give the proof.

Proof of Lemma 2.11. To prove (2.24) and (2.25), it suffices to verify for all $n \in \mathbb{N}$ that

$$\sup_{\xi \in \mathbb{R}} \langle \xi \rangle^n \left|\frac{d^n}{d\xi^n}(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(\xi)\right| \lesssim_\beta 1, \tag{A.10}$$

for any $0 < \mu < 1$. Indeed, in agreement with Definition A.2, then $(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu}) \in \mathcal{S}^0$ and (2.25) holds true due to Lemma A.3. Moreover, using Plancherel and (A.10) with $n = 0$ we have

$$\|(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu})(D)f\|_{L^2} \lesssim_\beta \|f\|_{L^2},$$

proving (2.24). Now, let us prove (A.10). We observe that

$$\mu^{\frac{k}{2}}\langle \xi \rangle^k \Big|(\frac{d^k}{d\xi^k}\chi^{(1)})(\sqrt{\mu}\xi)\Big| \lesssim 1, \quad k \geq 0, \tag{A.11}$$

since $\sqrt{\mu}|\xi| \lesssim 1$ on the support of $\chi_\mu^{(1)}(\xi)$. Moreover, we observe by Lemma A.1 and $\mu \in (0,1)$ that

$$\chi_\mu^{(1)}(\xi)\left|\frac{d^k}{d\xi^k}\sqrt{K_\mu(\xi)}\right| \lesssim_\beta \chi_\mu^{(1)}(\xi)\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\mu^{\frac{k}{2}}\langle\sqrt{\mu}\xi\rangle^{-k} \lesssim_\beta \langle\xi\rangle^{-k}.$$

Combining these estimates with the Leibniz rule yields

$$\begin{aligned}\langle\xi\rangle^n\Big|\frac{d^n}{d\xi^n}\big(\chi_\mu^{(1)}(\xi)\sqrt{K_\mu(\xi)}\big)\Big| &\lesssim \langle\xi\rangle^n\sum_{k=0}^n\Big|\frac{d^{n-k}}{d\xi^{n-k}}\big(\chi^{(1)}(\sqrt{\mu}\xi)\big)\ \frac{d^k}{d\xi^k}\big(\sqrt{K_\mu(\xi)}\big)\Big| \\ &\lesssim_\beta \langle\xi\rangle^n\sum_{k=0}^n\mu^{\frac{n-k}{2}}\big|\big(\frac{d^{n-k}}{d\xi^{n-k}}\chi^{(1)}\big)(\sqrt{\mu}\xi)\big|\langle\xi\rangle^{-k} \\ &\lesssim_\beta \sum_{k=0}^n\mu^{\frac{n-k}{2}}\langle\xi\rangle^{n-k}\big|\big(\frac{d^{n-k}}{d\xi^{n-k}}\chi^{(1)}\big)(\sqrt{\mu}\xi)\big| \lesssim_\beta 1.\end{aligned}$$

Hence, $(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu}) \in \mathcal{S}^0$ and $\mathcal{N}^0(\chi_\mu^{(1)}\sqrt{\mathcal{K}_\mu}) \lesssim_\beta 1$ independently from μ, proves (A.10).

Next, we consider estimates (2.27) and (2.28). Recalling (2.26) we define

$$\tilde{\sigma}_{\mu,\frac{1}{2}}(\xi) = \mu^{-\frac{1}{4}}\sigma_{\mu,\frac{1}{2}}(\xi) = \frac{1}{\mu^{\frac{1}{4}}}\ \frac{1}{\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}}\ (1+\mu\beta\xi^2)^{\frac{1}{2}}. \tag{A.12}$$

Then, it suffices to prove that

$$\sup_{\xi\in\mathbb{R}}\langle\xi\rangle^{n-\frac{1}{2}}\Big|\frac{d^n}{d\xi^n}(\chi_\mu^{(2)}\tilde{\sigma}_{\mu,\frac{1}{2}})(\xi)\Big| \lesssim_\beta 1, \tag{A.13}$$

for all $n \in \mathbb{N}$ and any $\mu \in (0,1)$. Indeed, if we assume (A.13) and take $n = 0$ we deduce from Plancherel's identity that

$$\|(\chi_\mu^{(2)}\sigma_{\mu,\frac{1}{2}})(D)f\|_{L^2} \lesssim_\beta \mu^{\frac{1}{4}}\|f\|_{H^{\frac{1}{2}}} \lesssim_\beta \|f\|_{L^2} + \mu^{\frac{1}{4}}\|D^{\frac{1}{2}}f\|_{L^2}.$$

which proves (2.27). Moreover, (A.13) also implies that $(\chi_\mu^{(2)}\tilde{\sigma}_{\mu,\frac{1}{2}}) \in \mathcal{S}^{\frac{1}{2}}$ with $\mathcal{N}^{\frac{1}{2}}(\chi_\mu^{(2)}\tilde{\sigma}_{\mu,\frac{1}{2}}) \lesssim_\beta 1$ so that

$$\|[(\chi_\mu^{(2)}\sigma_{\mu,\frac{1}{2}})(D), f]\partial_x g\|_{L^2} \lesssim_\beta \mu^{\frac{1}{4}}\|f\|_{H^s}\|g\|_{H^{\frac{1}{2}}},$$

by Lemma A.3. Now we prove (A.13). First, we consider the functions,

$$a_\mu(\xi) = \frac{1}{\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}} \quad \text{and} \quad b_\mu(\xi) = (1+\mu\beta\xi^2)^{\frac{1}{2}}.$$

Then, since $|\xi| > \sqrt{\mu}|\xi| > 1$ on the support of $\chi_\mu^{(2)}$, we observe that

$$\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}a_\mu(\xi)\Big| \lesssim \mu^{\frac{1}{4}}\chi_\mu^{(2)}(\xi)\frac{1}{\sqrt{\mu}|\xi|}\frac{1}{|\xi|^{k-\frac{1}{2}}} \lesssim \mu^{\frac{1}{4}}\langle\xi\rangle^{\frac{1}{2}-k}\langle\sqrt{\mu}\xi\rangle^{-1}. \tag{A.14}$$

While $b_\mu(\xi) \lesssim_\beta \langle\sqrt{\mu}\xi\rangle$ and its deriatives satisfy the bound,

$$\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}b_\mu(\xi)\Big| \lesssim_\beta \chi_\mu^{(2)}(\xi)\mu^{\frac{k}{2}}\langle\sqrt{\mu}\xi\rangle^{1-k} \lesssim_\beta \langle\sqrt{\mu}\xi\rangle\langle\xi\rangle^{-k}. \tag{A.15}$$

Thus, if all derivatives falls on $\tilde{\sigma}_{\mu,\frac{1}{2}}$, the Leibniz rule, (A.14) and (A.15) imply

$$\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}\tilde{\sigma}_{\mu,\frac{1}{2}}(\xi)\Big| \lesssim \mu^{-\frac{1}{4}}\chi_\mu^{(2)}(\xi)\sum_{j=0}^k\Big|\frac{d^{k-j}}{d\xi^{k-j}}a_\mu(\xi)\Big|\,\Big|\frac{d^j}{d\xi^j}b_\mu(\xi)\Big| \lesssim_\beta \langle\xi\rangle^{\frac{1}{2}-k}.$$

On the other hand, when derivatives fall the cut-off function, we observe

$$\mu^{\frac{k}{2}}\langle\xi\rangle^k\Big|(\frac{d^k}{d\xi^k}\chi^{(2)})(\sqrt{\mu}\xi)\Big| \lesssim 1, \quad k\geq 1, \tag{A.16}$$

since the support of $\frac{d^k}{d\xi^k}\chi_\mu^{(2)}$ is contained in the support of $\chi_\mu^{(1)}$. As a result, there holds

$$\begin{aligned}\langle\xi\rangle^{n-\frac{1}{2}}\Big|\frac{d^n}{d\xi^n}\big(\chi_\mu^{(2)}(\xi)\tilde{\sigma}_{\mu,\frac{1}{2}}(\xi)\big)\Big| &\lesssim \langle\xi\rangle^{n-\frac{1}{2}}\sum_{k=0}^n\Big|\frac{d^{n-k}}{d\xi^{n-k}}\big(\chi^{(2)}(\sqrt{\mu}\xi)\big)\Big|\,\Big|\frac{d^k}{d\xi^k}\big(\tilde{\sigma}_{\mu,\frac{1}{2}}(\xi)\big)\Big| \\ &\lesssim_\beta \langle\xi\rangle^{n-\frac{1}{2}}\sum_{k=0}^n\mu^{\frac{n-k}{2}}\big|(\frac{d^{n-k}}{d\xi^{n-k}}\chi^{(2)})(\sqrt{\mu}\xi)\big|\,\langle\xi\rangle^{\frac{1}{2}-k} \\ &\lesssim_\beta \sum_{k=0}^n\mu^{\frac{n-k}{2}}\langle\xi\rangle^{n-k}\big|(\frac{d^{n-k}}{d\xi^{n-k}}\chi^{(2)})(\sqrt{\mu}\xi)\big| \\ &\lesssim_\beta 1.\end{aligned}$$

The estimate is uniform in $\mu\in(0,1)$, and (A.13) is proved, which provides the desired result.

Lastly, we prove (2.30) and (2.31) arguing in the same vein. First, we write:

$$\begin{aligned}\chi_\mu^{(2)}(\xi)\sigma_{\mu,0}(\xi) &= \chi_\mu^{(2)}(\xi)\cdot\frac{1}{\mu^{\frac{1}{4}}|\xi|^{\frac{1}{2}}}\cdot\Big(1+\mu\beta\xi^2\Big)^{\frac{1}{2}}\cdot\Big(\frac{2}{e^{2\sqrt{\mu}|\xi|}+1}\Big)^{\frac{1}{2}} \\ &=: \chi_\mu^{(2)}(\xi)\,a_\mu(\xi)\,b_\mu(\xi)\,c_\mu(\xi),\end{aligned}$$

making use of the identity (A.4). Then we observe for all $N\in\mathbb{N}$ that

$$\langle\xi\rangle^k\langle\sqrt{\mu}\xi\rangle^N\Big|\frac{d^k}{d\xi^k}c_\mu(\xi)\Big| \lesssim \mu^{\frac{k}{2}}\langle\xi\rangle^k\langle\sqrt{\mu}\xi\rangle^N e^{-\sqrt{\mu}|\xi|}\lesssim 1. \tag{A.17}$$

As a result, we deduce by (A.15) and (A.17) with $N=1$ that

$$\begin{aligned}\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}(b_\mu(\xi)\,c_\mu(\xi))\Big| &\lesssim \chi_\mu^{(2)}(\xi)\sum_{j=0}^k\big|\frac{d^{k-j}}{d\xi^{k-j}}\big(b_\mu(\xi)\big)\big|\,\big|\frac{d^j}{d\xi^j}\big(c_\mu(\xi)\big)\big| \\ &\lesssim_\beta \sum_{j=0}^k\langle\xi\rangle^{-(k-j)}\langle\sqrt{\mu}\xi\rangle\,\langle\xi\rangle^{-j}\langle\sqrt{\mu}\xi\rangle^{-1} \\ &\lesssim_\beta \langle\xi\rangle^{-k}.\end{aligned}$$

Moreover, we use (A.14) to deduce

$$\chi_\mu^{(2)}(\xi)\Big|\frac{d^k}{d\xi^k}\sigma_{\mu,0}(\xi)\Big| \lesssim \chi_\mu^{(2)}(\xi)\sum_{j=0}^k\Big|\frac{d^{k-j}}{d\xi^{k-j}}\big(a_\mu(\xi)\big)\Big|\,\Big|\frac{d^j}{d\xi^j}\big(b_{\mu,\beta}(\xi)\,c_\mu(\xi)\big)\Big| \lesssim_\beta \langle\xi\rangle^{-k},$$

from which we find

$$\langle\xi\rangle^n\Big|\frac{d^n}{d\xi^n}\big(\chi_\mu^{(2)}(\xi)\sigma_{\mu,0}(\xi)\big)\Big| \lesssim \langle\xi\rangle^n\sum_{k=0}^n\Big|\frac{d^{n-k}}{d\xi^{n-k}}\big(\chi_\mu^{(2)}(\xi)\big)\Big|\,\Big|\frac{d^k}{d\xi^k}\big(\sigma_{\mu,0}(\xi)\big)\Big| \lesssim_\beta 1,$$

by (A.16). Arguing as above, we may conclude that the estimates (2.30) and (2.31) hold. □

Proof of Lemma 2.12. In order to prove (2.32), we simply verify that $\sqrt{\mathcal{T}_\mu}J_\mu^{\frac{1}{2}} \in \mathcal{S}^0$ and $\mathcal{N}^0(\sqrt{\mathcal{T}_\mu}J_\mu^{\frac{1}{2}}) \lesssim 1$ uniformly in $\mu \in (0,1)$. But this is a direct consequence of Lemma A.1 and the Leibniz rule:

$$\begin{aligned}
\langle\xi\rangle^n\Big|\frac{d^n}{d\xi^n}\sqrt{T_\mu(\xi)}J_\mu^{\frac{1}{2}}\Big| &\lesssim \langle\xi\rangle^n\sum_{k=0}^n\Big|\frac{d^{n-k}}{d\xi^{n-k}}\sqrt{T_\mu(\xi)}\Big|\,\Big|\frac{d^k}{d\xi^k}\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}\Big| \\
&\lesssim \langle\xi\rangle^n\sum_{k=0}^n\mu^{\frac{n-k}{2}}\langle\sqrt{\mu}\xi\rangle^{-\frac{1}{2}-(n-k)}\mu^{\frac{k}{2}}\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}-k} \\
&\lesssim \langle\xi\rangle^n\mu^{\frac{n}{2}}\langle\sqrt{\mu}\xi\rangle^{-n},
\end{aligned}$$

and is bounded by a constant independent from $\mu \in (0,1)$. Hence, we may conclude by Lemma A.3 that (2.32) holds true.

A similar approach is used for the proof of (2.33). Indeed, we observe that

$$\langle\xi\rangle^{n-s}\Big|\frac{d^n}{d\xi^n}\sqrt{T_\mu(\xi)}J^s\Big| \lesssim \langle\xi\rangle^{n-s}\sum_{k=0}^n\mu^{\frac{n-k}{2}}\langle\sqrt{\mu}\xi\rangle^{-\frac{1}{2}-(n-k)}\langle\xi\rangle^{s-k} \lesssim 1.$$

Hence, $\sqrt{\mathcal{T}_\mu}J^s \in \mathcal{S}^s$ and $\mathcal{N}^s(\sqrt{\mathcal{T}_\mu}J^s) \lesssim 1$ uniformly in $\mu \in (0,1)$, allowing us to conclude by Lemma A.3.

The proof of (2.34) is the same, by a direct application of (A.1) we deduce that $\sqrt{\mathcal{T}_\mu} \in \mathcal{S}^0$ uniformly in $\mu \in (0,1)$.

Next, we consider (2.35). Define the bilinear form: $a_1(D)(f,g) = \partial_x[\sqrt{\mathcal{T}_\mu}(D), f]g$. Then we may use Plancherel to write

$$|\hat{a}_1(\xi)(f,g)| \leq \int_{\mathbb{R}}|\xi|\Big|\sqrt{T_\mu(\xi)} - \sqrt{T_\mu(\rho)}\Big||\hat{f}(\xi-\rho)|\,|\hat{g}(\rho)|\,d\rho.$$

Clearly, if we can prove that

$$b_1(\xi,\rho) := |\xi|\Big|\sqrt{T_\mu(\xi)} - \sqrt{T_\mu(\rho)}\Big| \lesssim 1 + |\xi-\rho|, \tag{A.18}$$

then we can conclude as we did for the proof of Lemma 2.7. Indeed, assuming the claim (A.18), then there holds

$$\|\partial_x[\sqrt{\mathcal{T}_\mu}(D), f]g\|_{L^2} = \|\hat{a}_1(\xi)(f,g)\|_{L^2} \lesssim (\|f\|_{H^{t_0}} + \|\partial_x f\|_{H^{t_0}})\|g\|_{L^2}.$$

Now, in order to estimate $b_1(\xi,\rho)$ we consider three cases. First, if $|\rho| \leq 1$, then we have by the triangle inequality,

$$b_1(\xi,\rho) \leq (1 + |\xi-\rho|)\Big(\sqrt{T_\mu(\xi)} + \sqrt{T_\mu(\rho)}\Big) \lesssim 1 + |\xi-\rho|,$$

since $\xi \mapsto \sqrt{T_\mu(\xi)}$ is bounded by one. Secondly, consider the region where $|\rho| > 1$ and $|\xi| \geq |\rho|$. Then since $\xi \mapsto \tanh(\sqrt{\mu}|\xi|)$ is increasing and $\xi \mapsto T_\mu(\xi)$ is decreasing, we have that

$$\frac{|\rho|}{|\xi|} \leq \Big(\frac{|\rho|}{|\xi|}\Big)^{\frac{1}{2}} \leq \Big(\frac{T_\mu(\xi)}{T_\mu(\rho)}\Big)^{\frac{1}{2}} \leq 1.$$

Thus, there holds

$$b_1(\xi,\rho) = |\xi|\Big(1 - \Big(\frac{T_\mu(\xi)}{T_\mu(\rho)}\Big)^{\frac{1}{2}}\Big)\sqrt{T_\mu(\rho)} \leq |\xi| - |\rho| \leq |\xi-\rho|.$$

For $|\rho| > 1$ and $|\xi| < |\rho|$ we use a similar argument to find,

$$b_1(\xi,\rho) = |\xi|\Big(1 - \Big(\frac{T_\mu(\rho)}{T_\mu(\xi)}\Big)^{\frac{1}{2}}\Big)\sqrt{T_\mu(\xi)} \le \frac{|\xi|}{|\rho|}(|\rho| - |\xi|) \le |\xi - \rho|.$$

Finally, we estimate (2.36) using a similar approach. We define the bilinear form $a_2(D)(f,g) = [J_\mu^{\frac{1}{2}}, f]\partial_x g$ and look in frequency:

$$|\hat{a}_2(\xi)(f,g)| \le \int_{\mathbb{R}} \Big|\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}\Big| |\hat{f}(\xi-\rho)|\, |\widehat{\partial_x g}(\rho)|\, d\rho.$$

Then by the same argument as above, we only need to prove that

$$b_2(\xi,\rho) = \Big|\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}} - \langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}\Big| \frac{|\rho|}{\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}} \lesssim 1 + |\xi - \rho|. \tag{A.19}$$

We consider three cases. If $|\rho| \le 1$ then since $\mu \in (0,1)$, there holds by the triangle inequality:

$$b_2(\xi,\rho) \lesssim 1 + \langle\xi - \rho\rangle^{\frac{1}{2}}.$$

In the case $|\xi| \ge |\rho| > 1$, observe that

$$\frac{1+\mu\rho^2}{1+\mu\xi^2} \le \frac{(1+\mu\rho^2)^{\frac{1}{4}}}{(1+\mu\xi^2)^{\frac{1}{4}}} \le 1, \tag{A.20}$$

and we have

$$\frac{\xi^2 - \rho^2}{|\xi-\rho|\,|\xi|} \le \frac{|\xi| + |\rho|}{|\xi|} \lesssim 1. \tag{A.21}$$

As a consequence, recalling $\mu \in (0,1)$, we have that

$$\begin{aligned} b_2(\xi,\rho) &\le \Big(1 - \frac{1+\mu\rho^2}{1+\mu\xi^2}\Big)\frac{\langle\sqrt{\mu}\xi\rangle^{\frac{1}{2}}}{\langle\sqrt{\mu}\rho\rangle^{\frac{1}{2}}}|\rho| \\ &\le \frac{\mu(\xi^2-\rho^2)}{\mu^{\frac{1}{4}}(1+\mu\xi^2)^{\frac{3}{4}+\frac{1}{4}}}\frac{(1+\mu\xi^2)^{\frac{1}{4}}}{\langle\rho\rangle^{\frac{1}{2}}}|\rho| \\ &\le \frac{\xi^2-\rho^2}{|\xi|}\frac{|\rho|}{|\xi|^{\frac{1}{2}}\langle\rho\rangle^{\frac{1}{2}}} \\ &\lesssim |\xi-\rho|. \end{aligned}$$

Lastly, in the case $|\rho| > 1$ and $|\xi| < |\rho|$, we can simply change the role of ξ and ρ in (A.20) and (A.21). As result, we get

$$b_2(\xi,\rho) \le \Big(1 - \frac{1+\mu\xi^2}{1+\mu\rho^2}\Big)|\rho| \le \frac{\rho^2-\xi^2}{|\rho|} \lesssim |\xi-\rho|.$$

We may therefore conclude that (A.19) holds and the estimate (2.36) follows. □

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