

POLARIZATIONS OF POWERS OF GRADED MAXIMAL IDEALS

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ABSTRACT. We give a complete combinatorial characterization of all possible polarizations of powers of the graded maximal ideal $(x_1, x_2, \dots, x_m)^n$ of a polynomial ring in m variables. We also give a combinatorial description of the Alexander duals of such polarizations. In the three variable case $m = 3$ and also in the power two case $n = 2$ the descriptions are easily visualized and we show that every polarization defines a (shellable) simplicial ball. We give conjectures relating to topological properties and to algebraic geometry, in particular that any polarization of an Artinian monomial ideal defines a simplicial ball.

1. INTRODUCTION

The notion of polarization of a monomial ideal has been a standard technique in algebraic geometry and combinatorial commutative algebra for many years. It gives a way to “separate” the powers of variables in a monomial ideal while keeping its homological properties. An early success was Hartshorne’s proof of the connectedness of the Hilbert scheme [16] using distractions, which are essentially a form of polarization. Polarization gives a way to make combinatorial objects from any monomial ideal: it gives a squarefree monomial ideal and thus a simplicial complex, from which one may obtain algebraic invariants such as graded Betti numbers in a combinatorial/topological way. Although many authors have considered and used polarizations in various forms, there has been no focused, systematic study of the variety of polarizations, and of the characterizing properties of such ideals. The purpose of this article is to undertake this task. As we shall see, this raises both interesting questions in itself, as well as connections to combinatorial topology and to algebraic geometry manifesting in quite general conjectures.

Before proceeding let us briefly give simple examples and explain the notion of polarization. Consider the ideal

$$(1) \quad I = (x_1, x_2)^2 = (x_1^2, x_1x_2, x_2^2) \subseteq k[x_1, x_2].$$

It polarizes to the ideal

$$J = (x_{11}x_{12}, x_{11}x_{21}, x_{21}x_{22}) \subseteq k[x_{11}, x_{12}, x_{21}, x_{22}].$$

Date: August 18, 2021.

2010 Mathematics Subject Classification. Primary: 13F20, 13F55; Secondary: 55U10, 05E40.

Key words and phrases. polarization, maximal ideal power, Artinian monomial ideal, isotone map, Alexander dual, simplicial ball, polynomial ring.

AA was partially supported by the NSF GRFP under Grant No. DGE-1650441.

The quotient ring $k[x_1, x_2]/I$ then comes from $k[x_{11}, x_{12}, x_{21}, x_{22}]/J$ by cutting down by a regular sequence of variable differences

$$x_{11} - x_{12}, x_{21} - x_{22}.$$

These two graded rings have the same homological properties, such as codimension and codepth, and the same graded Betti numbers.

In general, for a monomial ideal I one gets the polarization J by taking each minimal generator of I

$$(2) \quad x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$$

and making a minimal generator

$$(x_{11}x_{12} \cdots x_{1a_1}) \cdot (x_{21}x_{22} \cdots x_{2a_2}) \cdots (x_{m1} \cdots x_{ma_m})$$

of J . We call this the *standard polarization*.

However, for a monomial ideal I in $k[x_1, \dots, x_m]$, there may be many other ways to get a squarefree monomial ideal J in a larger polynomial ring $k[x_{ij}]$ such that $k[x_i]/I$ comes from $k[x_{ij}]/J$ by dividing out by a regular sequence of variable differences $x_{i,j} - x_{i,j'}$.

For instance, if I is a strongly stable ideal, one has the “b-polarization”, [23], [28] which from the ideal (1) constructs the ideal

$$J^b = (x_{11}x_{12}, x_{11}x_{22}, x_{21}x_{22}) \subseteq k[x_{11}, x_{12}, x_{21}, x_{22}].$$

(In this special case the ideals J and J^b are isomorphic, but this is not so in general.) The b-polarization takes the minimal generator (2) and makes a minimal generator

$$(x_{1,1} \cdots x_{1,a_1}) \cdot (x_{2,a_1+1} \cdots x_{2,a_1+a_2}) \cdots (x_{m,a_1+\dots+a_{m-1}+1} \cdots x_{m,a_1+\dots+a_m})$$

so the second index runs through the integers from 1 to $a_1 + \dots + a_m$.

The notion of shifting operator is a form of polarization. It has been studied by several authors, [21] in the setting of squeezed spheres, and in [28]. Recently letterplace ideals associated to poset ideals of P -partitions, [12] and [13] have provided large classes which are new polarizations of Artinian monomial ideals.

1.1. Polarizations of powers of the graded maximal ideal. We combinatorially describe all possible polarizations of powers of the graded maximal ideal

$$(3) \quad I = (x_1, \dots, x_m)^n \subseteq k[x_1, \dots, x_m].$$

These maximal ideal powers are the ideals in $k[x_1, \dots, x_m]$ with maximal symmetry. They are $GL(n)$ -invariant. A polarization is somehow a way of breaking this symmetry, but still keeping the homological properties. Thus we classify all symmetry breaks of this ideal. (Let us also mention that powers of graded maximal ideals have been studied from another combinatorial perspective in [2], using discrete Morse theory to make cellular resolutions.)

Let $\Delta_m(n)$ be the lattice simplex with vertices given by all $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{N}_0^n$ with $\sum_{i=1}^m a_i = n$. Its vertices are in one-one correspondence with minimal generators of $(x_1, \dots, x_m)^n$. See Figure 1 for $\Delta_3(3)$. It is divided into smaller triangles, and of these triangles, three are pointing down; we refer to these as *down-triangles*. By going

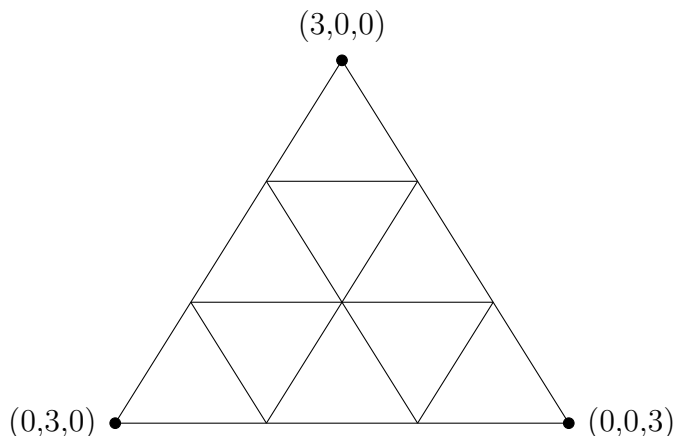


FIGURE 1

upwards along the edges in this diagram, Figure 1, we get a partial order \geq_1 on $\Delta_m(n)$. If the monomial (2) polarizes to a monomial

$$m(\mathbf{a}) = m_1(\mathbf{a}) \cdots m_m(\mathbf{a})$$

where $m_i(\mathbf{a})$ maps to $x_i^{a_i}$, let $X_1(\mathbf{a})$ be the variables in $m_1(\mathbf{a})$. Then X_1 can be considered as a map from $\Delta_m(n)$ to the boolean poset of subsets of the x_{1j} -variables. Similarly, we get maps X_i for each $i = 1, \dots, m$. For any polarization J of I in (3), it turns out that each X_i is an isotone map for a partial order \geq_i on $\Delta_m(n)$. Conversely, given the maps $\{X_i\}$, we can construct monomial generators $m(\mathbf{a})$ of an ideal J . When is this a polarization of I ?

For a given edge in $\Delta_m(n)$ between \mathbf{a} and \mathbf{b} , we call the edge a *linear syzygy edge* if there is a linear syzygy between the monomials $m(\mathbf{a})$ and $m(\mathbf{b})$. Our main result, Theorem 4.8, says that the maps $\{X_i\}$ determine a polarization of I if and only if the *linear syzygy* edges for these maps contain a spanning tree of the edge graph (this is a complete graph) of each higher-dimensional “down-triangle” of $\Delta_m(n)$.

Two special cases are worth attention due to the easy visualization of the various polarizations. First, the polarizations of $(x_1, x_2, x_3)^n$, which are easily visualized in terms of Figure 1, see Corollary 4.10 and Example 4.11. Secondly, the polarizations of $(x_1, x_2, \dots, x_m)^2$, which are in one-to-one correspondence with trees on $(m+1)$ vertices, Theorem 7.1 and Examples 7.2 and 7.3.

1.2. Polarizations of Artinian monomial ideals. A nice result on the topological properties of polarizations by S.Murai [22] is that the standard polarization of any Artinian monomial ideal gives a simplicial ball. The boundary is a simplicial sphere with a simply described Stanley-Reisner ideal, which generalizes Bier spheres, [5]. This construction has again been comprehensively generalized by A. D’Ali, A.Nematbakhsh and the second author in [7], by considering letterplace ideals.

We here give the full simple conjecture, Conjecture 3.2, that via the Stanley-Reisner correspondence, any polarization of an Artinian monomial ideal gives a simplicial ball with a simple and naturally described Stanley-Reisner ideal of its boundary, a simplicial

sphere. We show that all polarizations of $(x_1, x_2, x_3)^n$ and of $(x_1, \dots, x_m)^2$ give simplicial balls, by showing that the Alexander duals of all such polarizations have linear quotients. In a forthcoming paper we show the squeezed spheres of [18] and the generalized Bier spheres both fit into one and the same framework of polarization and deseparation which we give here, see Section 2.

Relating to algebraic geometry we conjecture, Conjecture 3.12, that every first order deformation of any polarization of any Artinian monomial ideal lifts to a global deformation. In particular, such a polarization would be a smooth point on any Hilbert scheme. If true, this would give a plethora of components of the Hilbert scheme even for the same Hilbert polynomial.

1.3. Alexander duals of polarizations. If J is any polarization of an Artinian monomial ideal I of $k[x_1, \dots, x_m]$, then its Alexander dual J^\vee will be generated by “colored” monomials of degree m , monomials of the form

$$x_{1j_1}x_{2j_2} \cdots x_{mj_m}$$

where for each $i = 1, \dots, m$, we have one variable x_{i,j_i} from the class of i -variables (color i). We call these *rainbow monomials*. The class of ideals generated by rainbow monomials and with m -linear resolution is precisely the class which is Alexander dual to the class of polarizations of Artinian monomial ideals in m variables, Proposition 3.6. A concise criterion for ideals generated by rainbow monomials to have linear resolution is given by A.Nematbakhsh in [24], and we recall it in Theorem 3.8.

We describe the Alexander dual of any polarization of the maximal ideal power $(x_1, \dots, x_m)^n$, in terms of the isotone maps X_i . As it turns out, the argument involves only a baby-form χ_i of the isotone maps X_i , where χ_i is an isotone map from $\Delta_m(n)$ to the poset $\{0 < 1\}$. This description of the Alexander dual, however, leaves something to be desired concerning transparency; for instance, it is not obvious that the number of generators is actually always $\binom{n+m-1}{m}$. We discuss problems concerning ideals generated by rainbow monomials in Section 3.

1.4. Organization of the paper. In Section 2 we recall the notions of separations, separated models and polarizations. We show basic results on polarizations of Artinian monomial ideals. In Section 3 we discuss various conjectures and problems that have come up during our investigations. Section 4 defines the setting and notions we work with, and in particular introduces the isotone maps X_i . We also state our main result concerning the complete combinatorial classification of all polarizations of $(x_1, \dots, x_m)^n$. Examples are given for the three-variable case. Section 5 develops the notion of linear syzygy paths, and gives the proof of the main theorem. A short Section 6 classifies for the three-variable the isotone maps $\Delta_3(n) \rightarrow B(n)$. In Section 7 we consider the degree two case and show that polarizations of $(x_1, \dots, x_m)^2$ are in one-to-one correspondence with trees on $(m + 1)$ vertices. Section 8 gives the Alexander duals of polarizations of maximal ideal powers, and in Section 9 we show that polarizations of $(x_1, x_2, x_3)^m$ are shellable, which implies that they define simplicial balls by the Stanley-Reisner correspondence.

Note. The results in Section 7, Section 6, and Theorem 4.8 in the special case of three variables are essentially found in the unpublished preprint [20] by the third author.

Acknowledgement. We thank an anonymous referee for good feedback and suggestions leading to substantial improvements of the paper. Also the proofs of Lemma 2.4 and Lemma 3.1 were simplified by suggestions from the referee.

2. SEPARATIONS OF MONOMIAL IDEALS

We recall the basic notions of separation of a monomial ideals and separated models, as introduced in [13]. We also define a polarization of a monomial ideal as a separation which is a squarefree monomial ideal. We consider Artinian monomial ideals and show that for these the notion of polarization and separated models are the same. We introduce the isotone maps X_i from the lattice simplex $\Delta_m(n)$ to the Boolean poset $B(n)$, which are our main gadgets to classify all the polarizations of maximal ideal powers.

If R is a set, let $k[x_R]$ be the polynomial ring in the variables x_r where $r \in R$. If $S \rightarrow R$ is a map of sets, it induces a k -algebra homomorphism $k[x_S] \rightarrow k[x_R]$ by mapping x_s to x_r if $s \mapsto r$.

2.1. Separations and polarizations.

Definition 2.1. Let $R' \xrightarrow{p} R$ be a surjection of finite sets with the cardinality of R' one more than that of R . Let r_1 and r_2 be the two distinct elements of R' which map to a single element r in R . Let I be a monomial ideal in the polynomial ring $k[x_R]$ and J a monomial ideal in $k[x_{R'}]$. We say J is a *simple separation* of I if the following holds:

- i. The monomial ideal I is the image of J by the map $k[x_{R'}] \rightarrow k[x_R]$.
- ii. Both the variables x_{r_1} and x_{r_2} occur in some minimal generators of J (usually in distinct generators).
- iii. The variable difference $x_{r_1} - x_{r_2}$ is a non-zero divisor in the quotient ring $k[x_{R'}]/J$.

More generally, if $R' \xrightarrow{p} R$ is a surjection of finite sets and $I \subseteq k[x_R]$ and $J \subseteq k[x_{R'}]$ are monomial ideals such that J is obtained by a succession of simple separations of I , J is a *separation* of I . If J has no further separation, we call J a *separated model* (of I).

A simple, but for us significant observation is that the minimal generators of the separation J and the ideal I are in one-one correspondence. More generally the graded Betti numbers of J and I are the same, since we get from $k[x_{R'}]/J$ to $k[x_R]/I$ by dividing out by a regular sequence of linear forms.

In [1] it is shown that simple separations may be considered as deformations of the ideal I .

Any monomial ideal may be separated to its standard polarization. So clearly any separated model is a squarefree monomial ideal. The standard polarization may, however, be further separable, so it may not be a separated model.

Example 2.2. Consider $I = (x^2y^2, x^2z^2, y^2z^2)$ in $k[x, y, z]$. The standard polarization is

$$\tilde{I} = (x_1x_2y_1y_2, x_1x_2z_1z_2, y_1y_2z_1z_2).$$

This may be further separated to

$$J = (x_1x_2y_1y_2, x'_1x'_2z_1z_2, y_1y_2z_1z_2).$$

Definition 2.3. Let $I \subseteq k[x_R]$ be a monomial ideal and $R' \rightarrow R$ be a surjection of finite sets. An ideal $J \subseteq k[x_{R'}]$ is a *polarization* of I if J is squarefree and a separation of I .

This general notion of polarization is likely first defined in [28]. By the example above it is not true that any polarization is a separated model. However, we shall see in Corollary 2.6 that for Artinian monomial ideals, these notions are equivalent.

2.2. Polarizations of Artinian monomial ideals. We consider an Artinian monomial ideal $I \subseteq k[x_1, \dots, x_m]$. This is simply a monomial ideal such that for every index i , some power $x_i^{n_i}$ is a minimal generator of I . Let $\check{X}_i = \{x_{i1}, x_{i2}, \dots, x_{in'_i}\}$ be a set of variables. We get a polynomial ring whose variables are those in the union of all these variables, and a homomorphism

$$\pi : k[\check{X}_1, \check{X}_2, \dots, \check{X}_m] \rightarrow k[x_1, \dots, x_m],$$

by mapping every variable in \check{X}_i to x_i .

In a polarization $J \subseteq k[\check{X}_1, \dots, \check{X}_m]$ of I , we separate each monomial generator $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ to squarefree monomials

$$(4) \quad m(\mathbf{a}) = m_1(\mathbf{a}) \cdot m_2(\mathbf{a}) \cdots m_m(\mathbf{a}),$$

where $m_i(\mathbf{a})$ is a squarefree monomial of degree a_i in variables from \check{X}_i . Considering $x_i^{n_i}$ we see that we must have $n'_i \geq n_i$. We shall shortly show, Remark 2.5, that we may take $n'_i = n_i$.

Starting from $K[\check{X}_1, \dots, \check{X}_m]/J$ we get the quotient ring $k[x_1, \dots, x_m]/I$ by dividing out by a regular sequence consisting of variable differences $x_{ip} - x_{iq}$. For each i we choose $(n_i - 1)$ linearly independent such variable differences. Any such sequence of variable differences in any order will do.

We may get intermediate separations of I as follows. Choose surjections $p_i : \check{X}_i \rightarrow \check{X}'_i$. We get a map of polynomial rings

$$k[\check{X}_1, \dots, \check{X}_m] \rightarrow k[\check{X}'_1, \dots, \check{X}'_m].$$

The image of the polarization J is an ideal I' in $k[\check{X}'_1, \dots, \check{X}'_m]$ and I' is a separation of I . We then get from $k[\check{X}_1, \dots, \check{X}_m]/J$ to $k[\check{X}'_1, \dots, \check{X}'_m]/I'$ by dividing out by a regular sequence of variable differences $x_{ia} - x_{ib}$ where for each i , x_{ia} and x_{ib} are in the same fiber $p_i^{-1}(x')$ of p_i , and we have $|p_i^{-1}(x')| - 1$ linearly independent such variable differences for each fiber.

The following lemma has key consequences in defining our setting, see Subsection 4.1.

Lemma 2.4. *Let $x^{\mathbf{a}}$ and $x^{\mathbf{b}}$ be minimal generators of a monomial ideal I and $m(\mathbf{a})$ and $m(\mathbf{b})$ the corresponding generators in a polarization of I .*

Fix an index i . If $a_i \leq b_i$ and $a_j \geq b_j$ for every $j \neq i$, then the i 'th part $m_i(\mathbf{a})$ divides $m_i(\mathbf{b})$.

Proof. We may assume $i = 1$ and $a_1 < b_1$ and $a_j \geq b_j$ for $j \geq 2$. Let $I' \subseteq k[\check{X}_1, x_2, \dots, x_m]$ be the image of the polarization of I by the k -algebra homomorphism $k[\check{X}_1, \dots, \check{X}_m] \rightarrow k[\check{X}_1, x_2, \dots, x_m]$ which sends each variable x_{ij} in \check{X}_i to x_i for $i \geq 2$ and fixes the variables in \check{X}_1 . The ideal I' is then a separation of I and the generator of I' corresponding to the generator $x^{\mathbf{c}}$ of I is of the form $m_1(\mathbf{c}) \prod_{i=2}^m x_i^{c_i}$. We then need only show that $m_1(\mathbf{a})$ divides $m_1(\mathbf{b})$. Let \mathbf{m}_1 be the greatest common divisor of $m_1(\mathbf{a})$ and $m_1(\mathbf{b})$ and write:

$$m_1(\mathbf{a}) = \mathbf{u} \cdot \mathbf{m}_1, \quad m_1(\mathbf{b}) = \mathbf{v} \cdot \mathbf{m}_1, \quad \mathbf{n} = \mathbf{m}_1 \cdot \prod_{j=2}^m x_j^{a_j}$$

Assume $m_1(\mathbf{a})$ does *not* divide $m_1(\mathbf{b})$ then i) \mathbf{u} and \mathbf{v} are relatively prime monomials of degrees ≥ 1 , ii) \mathbf{n} is not in I' , and iii) $\mathbf{u}\mathbf{n}, \mathbf{v}\mathbf{n} \in I'$. Both \mathbf{u} and \mathbf{v} are then in the colon ideal $I' : \mathbf{n}$ and so in some associated prime \mathfrak{p} of I' . Then \mathfrak{p} contains a variable $x_{1,j}$ from \mathbf{u} and a variable $x_{1,k}$ from \mathbf{v} . These are distinct variables and the difference is in \mathfrak{p} . This contradicts the regularity of $x_{1,j} - x_{1,k}$. \square

Remark 2.5. If $m(\mathbf{a})$ is a minimal generator of J , by the lemma $m_i(\mathbf{a})$ will divide $m_i(0, \dots, n_i, \dots, 0)$ which of course is just $m(0, \dots, n_i, \dots, 0)$. Thus, if the polarization of $x_i^{n_i}$ is $x_{i1}x_{i2} \cdots x_{in_i}$, then every x_i -variable occurring in the minimal generators of J are among these variables, and so we may take $\check{X}_i = \{x_{i1}, \dots, x_{in_i}\}$.

The following is quite particular for Artinian monomial ideals, note Example 2.2.

Corollary 2.6. *Every polarization of an Artinian monomial ideal I is a separated model for I .*

Proof. If the polarization J was not a separated model, then let J' be a further simple separation. Since I in $k[x_1, \dots, x_m]$ is an Artinian monomial ideal, every variable x_i of course occurs in a minimal generator of I , in fact $x_i^{n_i}$ is a minimal generator. Then if J' is in $k[\check{X}'_1, \dots, \check{X}'_m]$ then every variable in this polynomial ring must also occur in a generator of J' , by the definition of a separation. By the above Lemma 2.4 and Remark 2.5, if $x_i^{n_i}$ polarizes to $x_{i1} \cdots x_{in_i}$ then $\check{X}'_i = \{x_{i1}, \dots, x_{in_i}\}$. But J is obtained from J' by dividing out by a variable difference $x_{ia} - x_{ib}$. Then the image of $x_{i1} \cdots x_{in_i}$ in J would not be squarefree, a contradiction. \square

3. CONJECTURES AND PROBLEMS

Before embarking on the main results of the paper we here discuss conjectures and problems on polarizations of Artinian monomial ideals in general that have come up during our investigations. They concern i) the topology of their associated simplicial complexes, ii) their Alexander duals, a class of rainbow monomial ideals (terminology introduced here), iii) their deformations.

3.1. Balls and spheres.

Lemma 3.1. *Let $\Delta(J)$ be the simplicial complex associated to the polarization J of an Artinian monomial ideal I . Then every codimension one face of $\Delta(J)$ is contained in one or two facets. If I is not a complete intersection, then at least once there is a codimension one face contained in exactly one facet.*

Proof. Let Δ_i be the simplex on $\{(i, j) \mid j = 1, \dots, n_i\}$. The squarefree monomial $x_{i1}x_{i2} \cdots x_{in_i}$ in $k[\check{X}_i]$ defines the sphere which is the boundary $\partial\Delta_i$ of this simplex. The natural polarization in $k[\check{X}_1, \dots, \check{X}_m]$ of the complete intersection $(x_1^{n_1}, x_2^{n_2}, \dots, x_m^{n_m})$ then defines the sphere of codimension m which is the join $S = \bigstar_{i=1}^m \partial\Delta_i$. Every codimension one face is here on precisely two facets.

The simplicial complex $\Delta(J)$ is a Cohen-Macaulay subcomplex of S with the same dimension as S . If $\Delta(J)$ is not all of S , let F be a facet of $\Delta(J)$ and G a facet of S not in $\Delta(J)$. Since S is strongly connected, there is a path of facets

$$F = F_0, F_1, \dots, F_r = G$$

such that $F_i \cap F_{i+1}$ has codimension one for each i , [17, Prop.9.1.12]. Let p be maximal such that F_p is in $\Delta(J)$. Then $F_p \cap F_{p+1}$ is only on the facet F_p in $\Delta(J)$. \square

By a result of Björner [4, Thm.11.4] a constructible simplicial complex with the property of Lemma 3.1 above is a simplicial ball or a simplicial sphere (the latter when every codimension one face is on exactly two facets).

Conjecture 3.2. *The simplicial complex $\Delta(J)$ associated to a polarization J of an Artinian monomial ideal I , is a simplicial ball, save for the case when I is a complete intersection, when it is a simplicial sphere.*

By the result of Björner loc.cit. a positive answer to the following question would settle the above conjecture.

Question 3.3. Do polarizations of Artinian monomial ideals have constructible (for instance shellable) simplicial complexes?

That the standard polarization of an Artinian monomial ideal is shellable seems first to have been shown by A.Soleyman Jahan in [25]. In [22] S.Murai uses this to conclude that the standard polarizations give simplicial balls. More generally it is shown that letterplace ideals define simplicial balls, [7], by showing that these simplicial complexes are shellable. Letterplace ideals are introduced in [13] and are polarizations of Artinian monomial ideals. The article [12] discusses such Artinian monomial ideals more in depth.

In our last Section 9 we show in the case of *three* variables that the Alexander dual of any polarization J has linear quotients, see [17, Sec.8.2, Cor.8.2.4] for this notion. In Section 7 we show when the power of the maximal ideal (in any number of variables) is two, then the Alexander dual has linear quotients. Thus in these cases the simplicial complex $\Delta(J)$ is shellable and hence a simplicial ball.

For a letterplace ideal the second author et.al. in [7] get an explicit simple description of the Stanley-Reisner ideal of the boundary of the simplicial ball defined by the

letterplace ideal. In fact a general result, see [6, Section 5], says that the canonical module of a Stanley-Reisner ring $k[\Delta]$ identifies as a multigraded proper ideal of this ring $k[\Delta]$ if and only if Δ is a homology ball. Then the ideal defines the boundary of this homology ball, which is a homology sphere. In [7] an explicit description of this canonical module is given.

Conjecture 3.4. *For polarizations of Artinian monomial ideals, the canonical module identifies (in a simply described and natural way) as a multigraded ideal of the Stanley-Reisner ring of the polarization.*

Consequence 3.5. With Conjecture 3.2, this would give an explicit description of the Stanley-Reisner ring of the boundary of the simplicial ball, defining a simplicial sphere.

3.2. Rainbow monomial ideals with linear resolution. Recall that for a square-free monomial ideal J in a polynomial ring S , the squarefree monomial ideal I is the *Alexander dual* of J if the monomials in I are precisely the monomials in S which have nontrivial common divisor with every monomial in J , or equivalently, every generator of J .

We may consider each set of variables \check{X}_i , $i = 1, \dots, m$ as a *color* class of monomials. A monomial $x_{1i_1}x_{2i_2} \cdots x_{mi_m}$ with one variable of each color is a *rainbow* monomial.

Proposition 3.6. *The class of ideals generated by rainbow monomials and with m -linear resolution is precisely the class which is Alexander dual to the class of polarizations of Artinian monomial ideals in m variables:*

- a. *Let J be a polarization of an Artinian monomial ideal I in $k[x_1, \dots, x_m]$. The Alexander dual ideal of J is generated by rainbow monomials and has m -linear resolution.*
- b. *If an ideal J' is generated by rainbow monomials and has m -linear resolution (and every variable in the ambient ring occurs in some generator of the ideal), then its Alexander dual J is a polarization of an Artinian monomial ideal in m variables.*

Proof. a. Since I is Cohen-Macaulay of codimension m , the same is true for J . Then the Alexander dual of J is generated in degree m and has m -linear resolution [8]. But if \mathbf{m} is a generator for this Alexander dual, it has a common variable with $x_{i_1}x_{i_2} \cdots x_{i_{n_i}}$ (the polarization of $x_i^{n_i}$) for every $i = 1, \dots, m$. Hence \mathbf{m} must have a variable of each of the m colors.

b. By [8], the Alexander dual J of J' is Cohen-Macaulay of codimension m . For each color class $\check{X}_i = \{x_{i_1}, \dots, x_{i_{n_i}}\}$, the ideal J will contain the monomial which is the product of all these variables.

If we for every color class i divide the quotient ring $k[\check{X}_1, \dots, \check{X}_m]/J$ by all the variable differences $x_{i,j} - x_{i,j-1}$ for $j = 2, \dots, n_i$, we get a quotient ring $k[x_1, \dots, x_m]/I$ where I is Artinian since it contains $x_i^{n_i}$ for each i . Hence like J the ideal I is Cohen-Macaulay of codimension m . Then the sequence we divided out by must have been a regular sequence, and so J is a polarization of I . \square

Remark 3.7. Considering the \check{X}_i as color classes, both the Artinian ideal I , the polarization J and its Alexander dual are generated by colored monomials. Such ideals and the

associated simplicial complexes have been considered in various settings, like balanced simplicial complexes by Stanley [26], relating to the colorful topological Helly theorem by Kalai and Meshulam, [19], and resolutions of such ideals by the second author [11].

In Section 8 we describe the Alexander dual of any polarization J of a maximal ideal power. But while the description of the generators of J is rather direct, Sections ?? and 7, the description the Alexander dual is more subtle. For instance, it is not obvious from the description that there are actually always $\binom{n+m-1}{m}$ generators of the Alexander dual.

In [24] A. Nematbakhsh gives a precise description of when an ideal generated by rainbow monomials has linear resolution. His terminology for rainbow monomials of d colors is "edge monomials of d -partite d -uniform clutters". He is able to give a characterization through a remarkable connection to the article [9] where they give a characterization of when a point set in the multiprojective space $(\mathbb{P}^1)^n$ is arithmetically Cohen-Macaulay (meaning that the associated multihomogeneous coordinate ring is a Cohen-Macaulay ring). The characterization given in [24] by translating the one in [9] is the following.

Theorem 3.8. *Let I be generated by rainbow monomials in d colors. Then I has a d -linear resolution iff:*

- a. *Whenever m_1 and m_2 are two rainbow monomials in I (i.e. generators of degree d) with $\text{lcm}(m_1, m_2)$ of degree $\geq d+2$, there is a third distinct rainbow monomial m_3 in I dividing this least common multiple.*
- b. *Whenever m_1 and m_2 are two rainbow monomials not in I with $\text{lcm}(m_1, m_2)$ of degree $\geq d+2$, there is a third distinct rainbow monomial m_3 not in I dividing this least common multiple.*

So this says that a subset A of the product $\check{X}_1 \cdot \check{X}_2 \cdots \check{X}_d$ gives an ideal with d -linear resolution iff *both* A and its complement are in some sense convex.

Fröberg's theorem [15] characterizes when a monomial ideal generated in degree two has linear resolution. It is easily seen that both this theorem and the theorem above give the following criterion when we have rainbow monomials with two colors: If $x_a y_b$ and $x_{a'} y_{b'}$ are in I , then either $x_a y_{b'}$ or $x_{a'} y_b$ is in I . Many attempts have been done to generalize Fröberg's theorem to higher degrees, but none fully successful. For rainbow monomials, however, the above gives such a generalization.

Example 3.9. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$ be three color classes and let I be the ideal generated by the six monomials

$$x_1 y_1 z_2, x_1 y_2 z_1, x_2 y_1 z_1, x_2 y_2 z_1, x_2 y_1 z_2, x_1 y_2 z_2.$$

Then I does not have linear resolution since $x_1 y_1 z_1$ and $x_2 y_2 z_2$ are *not* in I and their least common multiple is not divided by any distinct rainbow monomial *not* in I .

If we remove $x_1 y_1 z_2$ we also do not have linear resolution, since now the least common multiple of $x_2 y_1 z_2$ and $x_1 y_2 z_2$ is not divided by any distinct rainbow monomial in I . If we remove *both* $x_1 y_1 z_2$ and $x_1 y_2 z_2$ then we do get a monomial ideal with linear resolution.

Problem 3.10. Consider an ideal J' generated by rainbow monomials. Is there a direct criterion on this ideal to tell if its Alexander dual is a polarization of a power of a graded maximal ideal?

Since an Artinian monomial ideal has linear resolution iff it is a power of the graded maximal ideal, this is the same as asking for a criterion for the ideal J' to be bi-Cohen-Macaulay: The ideal has both linear resolution and is Cohen-Macaulay.

Such a description would maybe involve reconstructing the isotone maps X_i given in (5), from J' . In [3] a cellular resolution is computed when J' is the Alexander dual of the *standard polarization* J of any Artinian monomial ideal.

Remark 3.11. If we for each color class i have only two variables $\{x_{i0}, x_{i1}\}$, then a rainbow monomial of degree m may be identified with a binary string, say if $m = 6$ then 101011 corresponds to $x_{11}x_{20}x_{31}x_{40}x_{51}x_{61}$. Thus investigating homological properties of ideals generated by rainbow monomials with two variables of each color, corresponds to investigating algebraic and topological properties of sets of binary words.

3.3. Deformations of polarizations. In [14] the second author and A.Nematbakhsh showed that the letterplace ideals $L(2, P)$ (which are polarizations of quadratic Artinian monomial ideals) have unobstructed deformations when the Hasse diagram is a tree. Moreover we computed the full deformation family of these ideals. Together with G.Scattareggia we have also verified that all deformations of various polarizations of quadratic powers $(x_1, x_2, \dots, x_m)^2$ lift to global deformations for $m = 3, 4$ by computing a full global family. We have also verified this for the letterplace ideal $L(3, 5)$ (introduced in [13]), a cubic ideal.

Conjecture 3.12. *Every first order deformation of a polarization of an Artinian monomial ideal (regardless of whether the deformation is homogeneous for the standard grading) lifts to a global deformation, with any grading on the first order deformation respected by the global deformation. In particular, whenever such a polarization is on a (multigraded) Hilbert scheme, it is a smooth point.*

If the homogeneous ideal I in the polynomial ring S corresponds to a point on a Hilbert scheme of subschemes of a projective space, then the tangent space of the Hilbert scheme at this point is the degree zero part $\text{Hom}_S(I, S/I)_0$. By explicit computation with Macaulay 2, one can verify that the dimension of this space varies quite much for different polarizations. For instance, for the polarizations of $(x_1, x_2, x_3)^3$, the lowest dimension of the tangent space occurs for the standard polarization, with dimension 69. The largest dimension occurs for the b-polarization, giving a dimension of 102 (this ideal is a smooth point on the Hilbert scheme component of ideals of maximal minors of 3×5 matrices of linear forms). Furthermore there are also many values of the dimension of this tangent space between 69 and 102. Thus if they are all smooth points, they would be on many different components of the Hilbert scheme.

4. POLARIZATIONS OF POWERS OF THE GRADED MAXIMAL IDEAL

We now consider powers of the maximal ideals

$$M = (x_1, x_2, \dots, x_m)^n \subseteq k[x_1, \dots, x_m].$$

A polarization of this ideal may by Remark 2.5 be taken to live in polynomial ring $k[\check{X}_1, \dots, \check{X}_m]$ where $\check{X}_i = \{x_{i1}, \dots, x_{in}\}$. Our goal is to combinatorially classify all possible polarizations of M in this polynomial ring.

4.1. Setting and notions.

Definition 4.1. $\Delta_m(n)$ is the subset of \mathbb{N}_0^m of all tuples $\mathbf{b} = (b_1, \dots, b_m)$ of non-negative integers with $b_1 + \dots + b_m = n$. For a given \mathbf{b} , its *support* $\text{supp}(\mathbf{b})$ is the set of all i such that $b_i \geq 1$.

The generators of the monomial ideal M are all monomials $x_1^{b_1} \dots x_m^{b_m}$ with $\mathbf{b} \in \Delta_m(n)$. In a polarization J of M we have one minimal generator of J , $m(\mathbf{b})$ for every \mathbf{b} in $\Delta_m(n)$. Now fix an index $1 \leq i \leq m$. Then $\Delta_m(n)$ may be given a partial order \geq_i by letting $\mathbf{b} \geq_i \mathbf{a}$ if $b_i \geq a_i$ and $b_j \leq a_j$ for $j \neq i$. Thus there is one maximal element $(0, \dots, 0, n, 0, \dots, 0)$ where n is in position i , and it has minimal elements all \mathbf{b} with $b_i = 0$. This is a graded partial order with \mathbf{b} of rank b_i .

Now, given any $\mathbf{b} \in \Delta_m(n)$, we get from the polarization J a squarefree monomial $m_i(\mathbf{b})$, see (4). The set of variables dividing this monomial is a subset of \check{X}_i which we denote as $X_i(\mathbf{b})$. Let $B(\check{X}_i)$ be the Boolean poset on \check{X}_i , a Boolean poset on a set of n elements. We get a function

$$(5) \quad \begin{aligned} X_i : \Delta_m(n) &\rightarrow B(\check{X}_i) \\ \mathbf{b} &\mapsto X_i(\mathbf{b}). \end{aligned}$$

The following is an immediate consequence of Lemma 2.4.

Corollary 4.2. *Let J be a polarization of M . Then X_i is an isotone rank-preserving map when $\Delta_m(n)$ has the ordering \geq_i .*

Remark 4.3. Since $\check{X}_i = \{x_{i1}, \dots, x_{in}\}$, the group S_n acts on \check{X}_i . Also the group S_m acts on $k[x_1, \dots, x_m]$ by permutation of variables and hence on the set of maps $\{X_i\}$. In all there is an action of a semi-direct product $S_m \times (S_n)^m$ on $k[\check{X}_1, \dots, \check{X}_m]$ compatible with the action of S_m on $k[x_1, \dots, x_m]$. Since $(x_1, \dots, x_m)^n$ is equivariant for the group action, the isomorphism classes of polarizations of this maximal ideal power are precisely the orbits of $S_m \times (S_n)^m$ on the set of polarizations.

The convex hull of $\Delta_m(n)$ in \mathbb{R}^m is a simplex of dimension $(m-1)$. We shall however only need the graph structure it induces. Given a point \mathbf{c} in $\Delta_m(n+1)$ and i, j in the support of \mathbf{c} , let e_i and e_j be the unit coordinate vectors. Then we get an edge between the points $\mathbf{c} - e_i$ and $\mathbf{c} - e_j$ in $\Delta_m(n)$, denoted $(\mathbf{c}; i, j)$. Every edge in $\Delta_m(n)$ is of this form for unique \mathbf{c}, i and j . A point \mathbf{c} of $\Delta_m(n+1)$ induces a subgraph of $\Delta_m(n)$, the complete *down-graph* $D(\mathbf{c})$ on the points $\mathbf{c} - e_i$ for $i \in \text{supp}(\mathbf{c})$. The *dimension* of $D(\mathbf{c})$ is the dimension of its convex hull, which is one less than the cardinality of $\text{supp}(\mathbf{c})$. When $\text{supp}(\mathbf{c})$ has dimension two we call this a *down-triangle*. In Figure 2 we have three down-triangles.

Let $\Delta_m^+(n+1)$ be the subset of $\Delta_m(n+1)$ consisting of \mathbf{c} with $c_i \geq 1$ for every i . The complete down-graph $D(\mathbf{c})$ has full (maximal) dimension $(m-1)$ iff $\mathbf{c} \in \Delta_m^+(n+1)$. Note that $\Delta_m^+(n+1)$ is in one-one correspondence with $\Delta_m(n+1-m)$ by sending \mathbf{c}

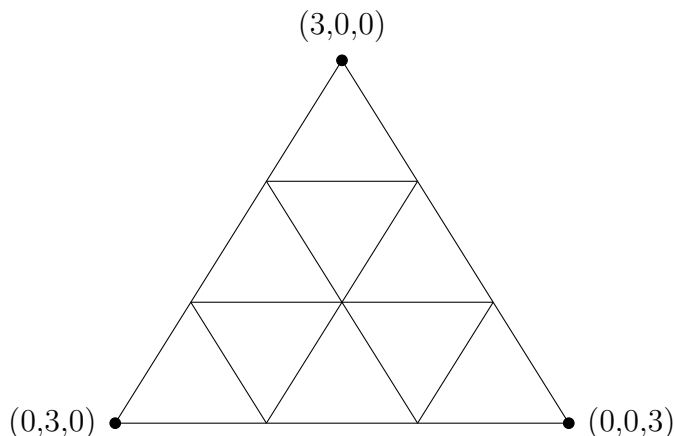


FIGURE 2

to $\mathbf{c} - \mathbf{1}$ where $\mathbf{1} = (1, 1, \dots, 1)$. In Figure 2 there are three full-dimensional complete down-graphs, or in this case down-triangles, corresponding to the three elements of $\Delta_3(\mathbf{1})$, or equivalently of $\Delta_3^+(\mathbf{1})$.

Each \mathbf{a} in $\Delta_m(n-1)$ also determines a subgraph of $\Delta_m(n)$, the complete *up-graph* $U(\mathbf{a})$ consisting of the points $\mathbf{a} + e_i$ for $i = 1, \dots, m$ and with edges $(\mathbf{a} + e_i + e_j; i, j)$ for $i \neq j$. For each \mathbf{a} in $\Delta_m(n-1)$ the convex hull of the up-graph $U(\mathbf{a})$ has full dimension $(m-1)$. When $m = 3$ we call this an *up-triangle*. In Figure 2 there are six up-triangles.

4.2. The main result with examples. We state our main result, the complete combinatorial description, Theorem 4.8, of all polarizations of powers of maximal ideals $(x_1, \dots, x_m)^n$. Write $[m] = \{1, 2, \dots, m\}$.

The essential objects are the rank-preserving isotone maps $X_i : \Delta_m(n) \rightarrow B(n)$ for $i = 1, \dots, m$ and conditions on them. For each $\mathbf{b} \in \Delta_m(n)$ we get a monomial $m_i(\mathbf{b}) = \prod_{j \in X_i(\mathbf{b})} x_{ij}$ in the variables $\check{X}_i = \{x_{i1}, \dots, x_{in}\}$. To the vertex \mathbf{b} we associate the monomial

$$m(\mathbf{b}) = \prod_{i=1}^m m_i(\mathbf{b})$$

and let J be the ideal in $k[\check{X}_1, \dots, \check{X}_m]$ generated by the $m(\mathbf{b})$.

If B is a subset of $[m]$, denote by $\mathbf{1}_B$ the m -tuple $\sum_{i \in B} e_i$. For instance, if $B = [m]$, then $\mathbf{1}_B = (1, 1, \dots, 1)$.

Lemma 4.4. *Let $\mathbf{c} \in \Delta_m(n+1)$ have support $C \subseteq \{1, 2, \dots, m\}$. The monomials associated to the vertices in the down-graph $D(\mathbf{c})$ have a common factor of degree $\mathbf{c} - \mathbf{1}_C$. This common factor is $\prod_{i \in C} m_i(\mathbf{c} - e_i)$.*

Proof. Fix an element $j \in C$. For the order \geq_k we have $\mathbf{c} - e_j \geq_k \mathbf{c} - e_k$ for every $k \in C$. Hence $X_k(\mathbf{c} - e_k)$ is contained in $X_k(\mathbf{c} - e_j)$ for every $k \in C$. Thus $m(\mathbf{c} - e_j)$ has $m_k(\mathbf{c} - e_k)$ as a factor for each $k \in C$. \square

Example 4.5. Let $m = 3$ and $\mathbf{c} = (c_1, c_2, c_3)$ be in $\Delta_3^+(n+1)$. At the left in Figure 3 we illustrate the down triangle $D(\mathbf{c})$. Let

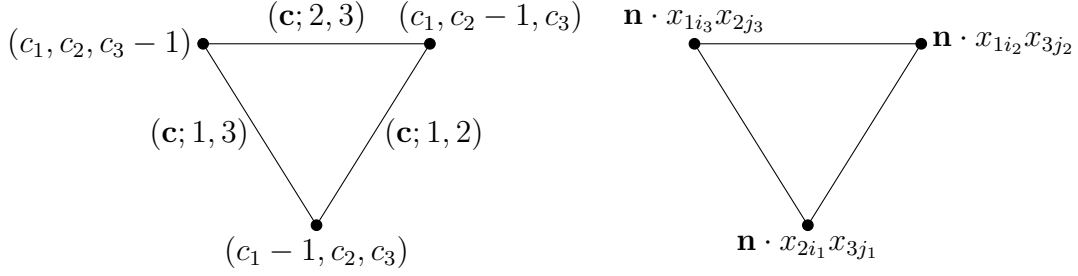


FIGURE 3

$$\mathbf{n} = m_1(\mathbf{c} - e_1) \cdot m_2(\mathbf{c} - e_2) \cdot m_3(\mathbf{c} - e_3).$$

Then the monomials associated to the vertices of this down-triangle are shown to the right in Figure 3.

Definition 4.6. An edge $(\mathbf{c}; i, j)$ in $\Delta_m(n)$ (where $\mathbf{c} \in \Delta_m(n+1)$) is a *linear syzygy edge* (LS-edge) if there is a monomial \mathbf{m} of degree $(n-1)$ such that

$$(6) \quad m(\mathbf{c} - e_i) = x_{j_r} \cdot \mathbf{m}, \quad m(\mathbf{c} - e_j) = x_{i_s} \cdot \mathbf{m}$$

for suitable variables $x_{j_r} \in \check{X}_j$ and $x_{i_s} \in \check{X}_i$. So this edge gives a linear syzygy between the monomials $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$. By Lemma 4.4 both $m_i(\mathbf{c} - e_i)$ and $m_j(\mathbf{c} - e_j)$ are common factors of $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$. So (6) is equivalent to

$$m_p(\mathbf{c} - e_i) = m_p(\mathbf{c} - e_j) \text{ for every } p \neq i, j,$$

or formulated in terms of the isotone maps

$$(7) \quad X_p(\mathbf{c} - e_i) = X_p(\mathbf{c} - e_j) \text{ for every } p \neq i, j.$$

We denote by $\text{LS}(\mathbf{c})$ the set of linear syzygy edges in the complete down-graph $D(\mathbf{c})$.

Example 4.7. In Figure 3 the edge $(\mathbf{c}; 2, 3)$ is a linear syzygy edge when $x_{1i_3} = x_{1i_2}$. Similarly the edge $(\mathbf{c}; 1, 3)$ is a linear syzygy edge when $x_{2j_3} = x_{2i_1}$. In general, when the support of \mathbf{c} has cardinality three and when $(\mathbf{c}; i, j)$ is *not* a linear syzygy edge we call it a *quadratic syzygy (QS) edge*, since we then have a quadratic syzygy between the monomials $m(\mathbf{c} - e_i)$ and $m(\mathbf{c} - e_j)$

Here is the main theorem of this article.

Theorem 4.8. *The isotone maps X_1, \dots, X_m determine a polarization of the ideal $(x_1, \dots, x_m)^n$ if and only if for every $\mathbf{c} \in \Delta_m(n+1)$, the linear syzygy edges $\text{LS}(\mathbf{c})$ contain a spanning tree for the down-graph $D(\mathbf{c})$.*

We prove this towards the end of this section.

Remark 4.9. For every $\mathbf{c} \in \Delta_m(n+1)$ which has support $\{i, j\}$ of cardinality 2, it is automatic by the condition that the X_p are rank-preserving and isotone that the edge $(\mathbf{c}; i, j)$ is a linear syzygy edge in $\text{LS}(\mathbf{c})$. (Condition (7) is empty.) So the conditions in Theorem 4.8 is automatically fulfilled for the \mathbf{c} with support of cardinality 2.

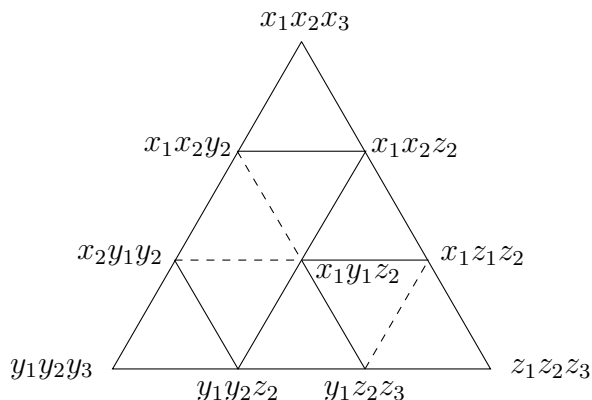


FIGURE 4

Corollary 4.10. *When $m = 3$ the isotone maps X_1, X_2, X_3 give a polarization of $(x_1, x_2, x_3)^n$ iff each down-triangle contains at most one QS-edge.*

Example 4.11. Write x, y, z for x_1, x_2, x_3 . Consider the ideal

$$J = (x_1x_2x_3, x_1x_2y_2, x_1x_2z_2, x_2y_1y_2, x_1y_1z_2, x_1z_1z_2, y_1y_2y_3, y_1y_2z_2, y_1z_2z_3, z_1z_2z_3)$$

which is a polarization of $(x, y, z)^3 \subset k[x, y, z]$. In Figure 4 we denote quadratic syzygy edges by dashed lines. We see that each of the three down-triangles has exactly one quadratic syzygy edge.

Proof of Corollary 4.10. A $\mathbf{c} \in \Delta_3(n+1)$ has support of cardinality 1, 2 or 3. If it is 3, we have a down-triangle, and so at least two of the three edges must be linear syzygy edges. Equivalently at most one of the edges is a QS-edge. If the cardinality is 2, the edge is on the boundary of the simplex induced by $\Delta_3(n)$ and it is a linear syzygy edge by the above Remark 4.9. \square

5. LINEAR SYZYGY PATHS AND THE PROOF OF THE MAIN RESULT

We give the arguments for the main result, Theorem 4.8. The main idea is to show the existence of *linear syzygy paths* under the assumption of Theorem 4.8.

5.1. Linear syzygy paths. Let $R \subseteq [m]$ and $\mathbf{c} \in \Delta_m(n+1)$ with R contained in the support of \mathbf{c} . Let $r, s \in R$. We say $(\mathbf{c}; r, s)$ is an R -linear syzygy edge if

$$X_p(\mathbf{c} - e_r) = X_p(\mathbf{c} - e_s) \text{ for } p \in R \setminus \{r, s\}.$$

Take note that by isotonicity of the X_p , for $p = r, s$:

$$X_r(\mathbf{c} - e_r) \subseteq X_r(\mathbf{c} - e_s), \quad X_s(\mathbf{c} - e_s) \subseteq X_s(\mathbf{c} - e_r).$$

Let $D_R(\mathbf{c})$ be the complete graph with edges $(\mathbf{c}; r, s)$ for $r, s \in R$.

Lemma 5.1. *Let $\mathbf{c} \in \Delta_m(n+1)$. If the set of linear syzygy edges in $\text{LS}(\mathbf{c})$ contains a spanning tree for $D(\mathbf{c})$, then for each $R \subseteq \text{supp}(\mathbf{c})$, the set of R -linear syzygy edges contains a spanning tree for $D_R(\mathbf{c})$.*

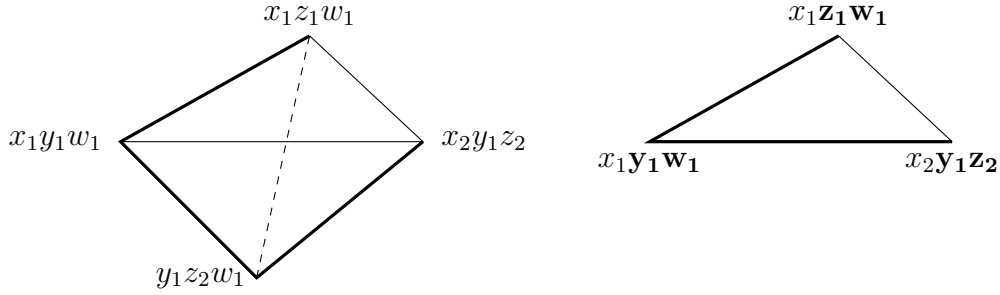


FIGURE 5

Example 5.2. Consider the case of four variables and $\mathbf{c} = (1, 1, 1, 1)$. Write x, y, z, w for x_1, x_2, x_3, x_4 . On the left of Figure 5 is the down-graph $D(\mathbf{c})$ with the three thick edges the linear syzygy edges.

Let $R = \{2, 3, 4\}$. On the right is the down-graph $D_R(\mathbf{c})$ where the two thick edges are the R -linear syzygy edges and the relevant variables marked in bold.

Proof. Let Q be the complement of R in $\text{supp}(\mathbf{c})$. Let r and s be two elements in R . There is a path from $\mathbf{c} - e_r$ to $\mathbf{c} - e_s$ in $D(\mathbf{c})$ consisting of linear syzygy edges. It may be broken up into smaller paths: From $\mathbf{c} - e_r = \mathbf{c} - e_{r_0}$ to $\mathbf{c} - e_{r_1}$, from $\mathbf{c} - e_{r_1}$ to $\mathbf{c} - e_{r_2}$, ..., from $\mathbf{c} - e_{r_{p-1}}$ to $\mathbf{c} - e_{r_p} = \mathbf{c} - e_s$ where on the path from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ the only vertices $\mathbf{c} - e_q$ with $q \in R$ are the end vertices $q = r_{i-1}$ and $q = r_i$ while the in-between vertices $\mathbf{c} - e_q$ all have $q \in Q$. We claim that each edge from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ is an R -linear syzygy edge. This will prove the lemma.

Let the path from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ be

$$\mathbf{c} - e_{r_{i-1}} = \mathbf{c} - e_{q_0}, \mathbf{c} - e_{q_1}, \dots, \mathbf{c} - e_{q_t} = \mathbf{c} - e_{r_i}$$

where q_1, \dots, q_{t-1} are all in Q . We must show that

$$(8) \quad X_p(\mathbf{c} - e_{r_{i-1}}) = X_p(\mathbf{c} - e_{r_i}) \text{ for } p \in R \setminus \{r_{i-1}, r_i\}.$$

But since the edges on the path are linear syzygy edges we have

$$X_p(\mathbf{c} - e_{q_{j-1}}) = X_p(\mathbf{c} - e_{q_j}) \text{ for } p \in \text{supp}(\mathbf{c}) \setminus \{q_{j-1}, q_j\}.$$

Since q_1, \dots, q_{t-1} are not in R we get (8) □

Given two m -tuples $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$ in $\Delta_m(n)$. Let $[m] = A \cup B$ be the disjoint set partition such that $a_i \geq b_i$ for $i \in A$ and $a_i < b_i$ for $i \in B$. We let

$$(9) \quad d(\mathbf{a}, \mathbf{b}) = \sum_{i \in B} (b_i - a_i) = \sum_{i \in A} (a_i - b_i)$$

be a measure for the distance between \mathbf{a} and \mathbf{b} . Note that the distance may be measured using only the index set B which in turn depends on the ordered set (\mathbf{a}, \mathbf{b}) . It should thus really be written $B(\mathbf{a}, \mathbf{b})$. When we measure the distance between two vertices, the first will normally be denoted by a variation on \mathbf{a} and the second a variation on \mathbf{b} .

We have the partial order $\mathbf{a} \leq \mathbf{b}$ if each $a_i \leq b_i$. The least upper bound for \mathbf{a} and \mathbf{b} in this partial order is

$$\mathbf{a} \vee \mathbf{b} = (\max\{a_1, b_1\}, \dots, \max\{a_m, b_m\}).$$

Recall that $m(\mathbf{a})$ and $m(\mathbf{b})$ are the monomials in positions \mathbf{a} and \mathbf{b} respectively. The following is the main and crucial ingredient on the proof of Theorem 4.8.

Proposition 5.3. *Given $\mathbf{a}, \mathbf{b} \in \Delta_m(n)$. Suppose for every $\mathbf{c} \in \Delta_m(n+1)$ the linear syzygy edges $\text{LS}(\mathbf{c})$ contains a spanning tree for the down-graph $D(\mathbf{c})$. Then there is a path*

$$\mathbf{a} = \mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_N = \mathbf{b}$$

in $\Delta_m(n)$ such that:

- (1) Every $\mathbf{b}_i \leq \mathbf{a} \vee \mathbf{b}$,
- (2) Every $m(\mathbf{b}_i)$ divides the least common multiple $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$,
- (3) The edge from \mathbf{b}_{i-1} to \mathbf{b}_i is a linear syzygy edge for each i .

We call such a path a linear syzygy path, for short LS-path, from \mathbf{a} to \mathbf{b} .

We first show this when the distance between \mathbf{a} and \mathbf{b} is one.

Lemma 5.4. *When the distance between \mathbf{a} and \mathbf{b} is one, there is an LS-path from \mathbf{a} to \mathbf{b} .*

Proof. In this case there is a unique $\mathbf{c} \in \Delta_m(n+1)$ such that $\mathbf{a} = \mathbf{c} - e_i$ and $\mathbf{b} = \mathbf{c} - e_j$, and then $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$. Let T in the linear syzygy edges $\text{LS}(\mathbf{c})$ be a spanning tree for $D(\mathbf{c})$. Then there is a unique path from \mathbf{a} to \mathbf{b} in T . We show that for any $m(\mathbf{u})$ on this path, $m(\mathbf{u})$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. It is enough to show for any $k \in \text{supp}(\mathbf{c})$ that any x_k -variable in $m(\mathbf{u})$ is contained in either the x_k -variables of $m(\mathbf{a})$ or the x_k -variables of $m(\mathbf{b})$.

Case 1. If the path from \mathbf{a} to \mathbf{b} does not contain $\mathbf{c} - e_k$, then since all edges on the path are linear syzygy edges, $X_k(\mathbf{b}_{i-1}) = X_k(\mathbf{b}_i)$ for every i .

Case 2. If the path from \mathbf{a} to \mathbf{b} contains $\mathbf{c} - e_k$, say this is \mathbf{b}_t , then:

- $X_k(\mathbf{b}_{i-1}) = X_k(\mathbf{b}_i)$ for $i < t$ and these are all equal to $X_k(\mathbf{a})$,
- $X_k(\mathbf{b}_i) = X_k(\mathbf{b}_{i+1})$ for $i > t$ and these are all equal to $X_k(\mathbf{b})$,
- $X_k(\mathbf{b}_t) = X_k(\mathbf{c} - e_k)$ is contained in both $X_k(\mathbf{a})$ and $X_k(\mathbf{b})$ by the isotonicity of X_k .

□

Proof of Proposition 5.3. We do this in three parts.

Part A. In this part we define the setting. Take distinct \mathbf{a} and \mathbf{b} in $\Delta_m(n)$. Assume the distance $d(\mathbf{a}, \mathbf{b}) \geq 2$ since the case of distance 1 is done above. Let

$$B = B(\mathbf{a}, \mathbf{b}) = \{i \mid b_i > a_i\}, \quad A_{>} = \{i \mid a_i > b_i\}, \quad A_{=} = \{i \mid a_i = b_i\},$$

and $A = A_{>} \cup A_{=}$. We want to consider \mathbf{b}' which are “close” to \mathbf{b} . Let $P(\mathbf{b})$ consist of all $\mathbf{b}' \in \Delta_m(n)$ such that

1. – For i in B , \mathbf{b} and \mathbf{b}' have equal i 'th coordinate.
- For i in A , $b'_i \leq a_i$.

2. There is some LS-path from \mathbf{b}' to \mathbf{b} where the vertices \mathbf{u} on the path satisfy
- $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ (which since we are assuming an LS-path, follows from 1 above),
 - $m(\mathbf{u})$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$.

Now let the subset A_1 of A consist of all coordinate indices i in A such that there is some \mathbf{b}' in $P(\mathbf{b})$ with strict inequality $b'_i < a_i$. Let A_0 be the complement $A \setminus A_1$. It is the intersection of all the $A_=_$ associated to \mathbf{b}' in $P(\mathbf{b})$. In particular note i) that $A_1 \supseteq A_>$ (since $\mathbf{b} \in P(\mathbf{b})$). So A_1 is not empty and $A_0 \subseteq A_=_$.

$$(10) \quad \textit{ii) } b'_i = a_i = b_i \text{ for } i \in A_0, \quad \textit{iii) } d(\mathbf{a}, \mathbf{b}') = d(\mathbf{a}, \mathbf{b}) \text{ for } \mathbf{b}' \in P(\mathbf{b}),$$

the latter because the B -sets $B(\mathbf{a}, \mathbf{b}') = B(\mathbf{a}, \mathbf{b})$ and the distance may be measured by this, confer (9).

Choose $\beta \in B$ and let $R = A_1 \cup \{\beta\}$. Consider the down-graph $D_R(\mathbf{a} + e_\beta)$. With β fixed, there is by Lemma 5.1 an R -linear syzygy edge $(\mathbf{a} + e_\beta; \beta, \alpha)$ for some α in A_1 . This is an edge from \mathbf{a} to $\mathbf{a} + e_\beta - e_\alpha$. Since $\alpha \in A_1$ there is a $\mathbf{b}' \in P(\mathbf{b})$ with $b'_\alpha < a_\alpha$. Then the B -sets (see (9)) $B(\mathbf{a} + e_\beta - e_\alpha, \mathbf{b}') \subseteq B(\mathbf{a}, \mathbf{b}')$, with equality unless $a_\beta + 1 = b_\beta$ in which case the the former set comes from removing β from the latter. In any case the distances

$$d(\mathbf{a} + e_\beta - e_\alpha, \mathbf{b}') = d(\mathbf{a}, \mathbf{b}') - 1 (= d(\mathbf{a}, \mathbf{b}) - 1).$$

By induction on distance there is an LS-path

$$(11) \quad \mathbf{a} + e_\beta - e_\alpha = \mathbf{b}^0, \mathbf{b}^1, \dots, \mathbf{b}^N = \mathbf{b}'.$$

So we have

- each $m(\mathbf{b}^j)$ divides $\text{lcm}(m(\mathbf{a} + e_\beta - e_\alpha), m(\mathbf{b}'))$,
- each $\mathbf{b}^j \leq (\mathbf{a} + e_\beta - e_\alpha) \vee \mathbf{b}' \leq \mathbf{a} \vee \mathbf{b}$.

There might be elements $m(\mathbf{b}^j)$ on this LS-path that do not divide $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. But $m(\mathbf{b}^j)$ will divide if we get sufficiently close to \mathbf{b}' as we show in Fact 5.6 below. If $\mathbf{a} + e_\beta - e_\alpha$ and \mathbf{b}' have equal i 'th coordinates for every $i \in B$, the distance $d(\mathbf{a}, \mathbf{b})$ would be 1, but we are assuming the distance is ≥ 2 . So $\mathbf{a} + e_\beta - e_\alpha$ and \mathbf{b}' do not have equal i 'th coordinate for every $i \in B$. Let \mathbf{b}^p be the last element on the path (11) for which $b_k^p \neq b'_k$ for some $k \in A_0 \cup B$.

Part B. In this part, in Facts 5.5, 5.6, and 5.7, we investigate in detail the path from \mathbf{b}^p to $\mathbf{b}' = \mathbf{b}^N$.

Fact 5.5. There is a unique $k \in A_0 \cup B$ such that $b_k^p \neq b'_k$. For every $i \in A_0 \cup B$ and $j = p, \dots, N$ we have $b_i^j = b'_i = b_i$, save for $j = p$ and $i = k$ when $b_k^p = b_k^{p+1} - 1$ which is $b'_k - 1 = b_k - 1$.

Proof. Clearly, by the definition of \mathbf{b}^p , we have that the b_i^j are all equal to b'_i for $i \in A_0 \cup B$ and $j = p + 1, \dots, N$. If $i \in B$ then $b'_i = b_i$ since $\mathbf{b}' \in P(\mathbf{b})$. If $i \in A_0$ then $b'_i = b_i$ by (10). Furthermore we must have $b_k^p = b_k^{p+1} \pm 1$ which is $b'_k \pm 1$. Note that

$$(12) \quad b_i^p \leq \max\{a_i, b_i\} = b_i, \quad i \in A_0 \cup B.$$

Since the edge from \mathbf{b}^p to \mathbf{b}^{p+1} is an LS-edge, there are exactly two coordinates k, ℓ where \mathbf{b}^p and \mathbf{b}^{p+1} are distinct. Since $b_k^{p+1} = b_k$ we must by (12) have $b_k^p = b_k^{p+1} - 1$. Then we will have $b_\ell^p = b_\ell^{p+1} + 1$. If $\ell \in A_0 \cup B$ then $b_\ell^{p+1} = b'_\ell = b_\ell$ which together

with (12) gives a contradiction. Thus we have a unique k in $A_0 \cup B$. Whence when $i \in A_0 \cup B$ and $i \neq k$ we have $b_i^p = b_i^{p+1} = b'_i = b_i$. \square

Fact 5.6. For all \mathbf{b}^j with $j = p, \dots, N$ we have

- i) $\mathbf{b}^j \leq \mathbf{a} \vee \mathbf{b}$,
- ii) $m(\mathbf{b}^j)$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$.

Proof. Part i) is already noted when we defined the path from $\mathbf{a} + e_\beta - e_\alpha$ to \mathbf{b}' . Now we do part ii). We know that $m_t(\mathbf{b}^j)$ divides $\text{lcm}(m_t(\mathbf{a} + e_\beta - e_\alpha), m_t(\mathbf{b}'))$ for every t .

a. Let $t \in A_1$. There is an $R = A_1 \cup \{\beta\}$ -linear syzygy between $m(\mathbf{a})$ and $m(\mathbf{a} + e_\beta - e_\alpha)$ and so $m_t(\mathbf{a}) = m_t(\mathbf{a} + e_\beta - e_\alpha)$ for $t \in A_1 \setminus \{\alpha\}$. For $t = \alpha$ then $m_\alpha(\mathbf{a} + e_\beta - e_\alpha)$ divides $m_\alpha(\mathbf{a})$ by X_α being isotone. From this and the defining requirements on \mathbf{b}' , it follows that $m_t(\mathbf{b}^j)$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ for $t \in A_1$.

b. Let now $t \in A_0 \cup B$. The edges on the path between \mathbf{b}^p and \mathbf{b}' are LS-edges. It follows then from Fact 5.5 that for each $t \in (A_0 \cup B) \setminus \{k\}$ and $j = p, \dots, N$ that $m_t(\mathbf{b}^j) = m_t(\mathbf{b}')$. When $t = k$, since $b_k^p = b_k^{p+1} - 1$ the part $m_k(\mathbf{b}^p)$ divides $m_k(\mathbf{b}^{p+1})$ and we will further have all $m_k(\mathbf{b}^j)$ equal for $j = p+1, \dots, N$, since these \mathbf{b}^j are related by LS-edges, and have the same k 'th coordinate. The upshot is that also $m_k(\mathbf{b}^j)$ divides $m_k(\mathbf{b}')$. Thus $m_t(\mathbf{b}^j)$ divides $m_t(\mathbf{b}')$ for every $t \in A_0 \cup B$. Since $\mathbf{b}' \in P(\mathbf{b})$, the $m_t(\mathbf{b}')$ divide $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ and we are done. \square

The following is the main technical detail that makes the proof work. It ensures that we can use induction on distance in Part C. To achieve this we needed A_0 as small as possible, and therefore introduced the neighbourhood $P(\mathbf{b})$ of \mathbf{b} . (But this had to be balanced against A_1 , the complement of A_0 in A , not being too big in order to construct the LS-path in (11) by induction.)

Fact 5.7. The coordinate $k \in B$.

Proof. By Fact 5.6 $m(\mathbf{b}^j)$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ and $\mathbf{b}^j \leq \mathbf{a} \vee \mathbf{b}$ for $j = p, \dots, N$. If $k \in A_0$ then by Fact 5.5 $b_i^p = b'_i = b_i$ for each $i \in B$. So \mathbf{b}^p fulfills the requirement to be in $P(\mathbf{b})$. But then by (10) $b_k^p = a_k = b_k$ contradicting Fact 5.5. \square

Part C. In this last part we splice three paths together to make an LS-path from \mathbf{a} to \mathbf{b} . Consider the distance between \mathbf{a} and \mathbf{b}^p . For $i \in A$ we have $b_j^p \leq a_j$ since $\mathbf{b}^p \leq \mathbf{a} \vee \mathbf{b}$. For $i \in B \setminus \{k\}$ we have $b_i^p = b'_i = b_i > a_i$ and when $i = k$ then $a_k < b_k$ and $b_k^p = b_k - 1$. Thus, by looking at the terms with coordinates in B , the distance $d(\mathbf{a}, \mathbf{b}^p) = d(\mathbf{a}, \mathbf{b}) - 1$. By induction there is an LS-path from \mathbf{a} to \mathbf{b}^p . We now have three LS-paths which we splice:

- The LS-path from \mathbf{a} to \mathbf{b}^p . All $m(\mathbf{u})$ on this path have i) $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}^p \leq \mathbf{a} \vee \mathbf{b}$ and ii) $m(\mathbf{u})$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}^p))$ which again divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$ by Fact 5.6.
- The LS-path from \mathbf{b}^p to \mathbf{b}' . We refer to Fact 5.6 concerning the terms here.
- The LS-path from \mathbf{b}' to \mathbf{b} with properties required by the definition of $P(\mathbf{b})$.

Splicing these three LS-paths together we get a path of linear syzygy edges from $m(\mathbf{a})$ to $m(\mathbf{b})$ where all $m(\mathbf{u})$ on this path have i) $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ and ii) $m(\mathbf{u})$ divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. Thus we have an LS-path from \mathbf{a} to \mathbf{b} . \square

5.2. Proofs of the main Theorem 4.8. We show first Part a., that if the isotone maps $\{X_i\}$ give a polarization, then for each $\mathbf{c} \in \Delta_m(n+1)$ the linear syzygy edges $\text{LS}(\mathbf{c})$ of the down-graph $D(\mathbf{c})$ contain a spanning tree for this down-graph.

Proof of Theorem 4.8, Part a. We assume that the isotone maps $\{X_i\}$ give an ideal J which is a polarization. We shall prove that every down-graph $D(\mathbf{c})$ contains a spanning tree of linear syzygy edges. For simplicity we shall assume $\text{supp}(\mathbf{c})$ has full support $[m] = \{1, 2, \dots, m\}$. The arguments work just as well in the other cases. Since by Lemma 4.4

$$\mathbf{m} = \prod_{i=1}^m m_i(\mathbf{c} - e_i)$$

of degree $\mathbf{c} - \mathbf{1}$ is a divisor of $m(\mathbf{c} - e_v)$ for any $\mathbf{c} - e_v$ in $D(\mathbf{c})$, we may write $m(\mathbf{c} - e_v) = \mathbf{m} \cdot n(\mathbf{c} - e_v)$ where $n(\mathbf{c} - e_v)$ has degree $\mathbf{1} - e_v$. For two distinct vertices $\mathbf{c} - e_v$ and $\mathbf{c} - e_w$ in $D(\mathbf{c})$ we define the distance $d(m(\mathbf{c} - e_v), m(\mathbf{c} - e_w))$ to be the number of $k \in [m]$ such that either:

- The (unique) x_k -variables of $n(\mathbf{c} - e_v)$ and of $n(\mathbf{c} - e_w)$ are distinct,
- $k = v$ (then $n(\mathbf{c} - e_v)$ has no x_v -variable),
- $k = w$ (then $n(\mathbf{c} - e_w)$ has no x_w -variable),

Note that if the distance between $m(\mathbf{c} - e_v)$ and $m(\mathbf{c} - e_w)$ is 2, then the set of k 's is $\{v, w\}$ and there is a linear syzygy between these monomials. Suppose now the vertices of $D(\mathbf{c})$ can be divided into two distinct subsets V_1 and V_2 such that there is no linear syzygy edge between a vertex in V_1 and a vertex in V_2 .

Let $\mathbf{c} - e_v$ in V_1 and $\mathbf{c} - e_w$ in V_2 be such that the distance d between $m(\mathbf{c} - e_v)$ and $m(\mathbf{c} - e_w)$ is minimal. We must have $d \geq 3$ and the number of vertices $m \geq 3$. We may as well assume $v = 1$ and $w = 2$ and that we may write

$$n(\mathbf{c} - e_2) = x_{1i_1}x_{3i_3} \cdots x_{mi_m}, \quad n(\mathbf{c} - e_1) = x_{2j_2}x_{3j_3} \cdots x_{mj_m},$$

where $x_{pi_p} \neq x_{pj_p}$ for $p = 3, \dots, d$ and $x_{pi_p} = x_{pj_p}$ for $p > d$ where $d \geq 3$.

Consider the graded ring $k[\check{X}_1, \dots, \check{X}_m]/J$ and divide out by the regular sequence $x_{pi_p} - x_{pj_p}$ for $p = 4, \dots, d$. This is a regular sequence, since we assume we have a polarization. We get a quotient algebra $k[\check{X}'_1, \dots, \check{X}'_m]/J'$ and denote by x_p the class $\overline{x_{pi_p}} = \overline{x_{pj_p}}$ for $p \geq 4$. In J' we have generators

$$\begin{aligned} \overline{m}(\mathbf{c} - e_2) &= \overline{\mathbf{m}} \cdot \overline{n}(\mathbf{c} - e_2), & \overline{n}(\mathbf{c} - e_2) &= x_{1i_1}x_{3i_3}x_4 \cdots x_m \\ \overline{m}(\mathbf{c} - e_1) &= \overline{\mathbf{m}} \cdot \overline{n}(\mathbf{c} - e_1), & \overline{n}(\mathbf{c} - e_1) &= x_{2j_2}x_{3j_3}x_4 \cdots x_m \end{aligned}$$

Now $x_{3i_3} - x_{3j_3}$ is a non-zero divisor of $k[\check{X}'_1, \dots, \check{X}'_m]/J'$. Consider

$$(x_{3i_3} - x_{3j_3})x_{1i_1}x_{2j_2}x_4 \cdots x_m \cdot \overline{\mathbf{m}}.$$

It is zero in this quotient ring, and so

$$\overline{\mathbf{m}'} = x_{1i_1}x_{2j_2}x_4 \cdots x_m \cdot \overline{\mathbf{m}}$$

is zero in this quotient ring and so must be a generator of J' of degree $\mathbf{c} - e_3$. But then the generator of this degree in the polarization J must be

$$\mathbf{m}' = x_{1i_1}x_{2j_2}x_{4k_4} \cdots x_{mk_m} \cdot \mathbf{m}$$

where each k_p is either i_p or j_p . Hence all $k_p = i_p = j_p$ for $p > d$. But then we see that the distance between \mathbf{m}' and $m(\mathbf{c} - e_2)$ is $\leq d - 1$ and similarly the distance between \mathbf{m}' and $m(\mathbf{c} - e_1)$ is $\leq d - 1$. Whether \mathbf{m}' is now in V_1 or in V_2 we see that this contradicts d being the minimal distance. \square

We use the following elementary lemma below in the proof of Theorem 4.8, part b.

Lemma 5.8. *Let I be a monomial ideal in $k[x_0, x_1, \dots, x_m]$ such that each generator of I is squarefree in the x_0 -variable. Then if $(x_0 - x_1) \cdot f$ is in I , then for every monomial m in f we have that x_0m and x_1m are in I .*

Proof. Let $f = x_0^a f_a + x_0^{a-1} f_{a-1} + \dots + f_0$ where each f_p have no x_0 -terms. Then if $(x_0 - x_1)f$ is in I , the only terms with x_0^{a+1} are the terms in $x_0^{a+1} f_a$, and so these are in I since we are in a \mathbb{Z}^m -graded setting. But since I is squarefree in x_0 , we have $x_0 f_a$ in I and so $x_0^a f_a$ in I . In this way we may peel off and get that all terms $x_0^p f_p$ are in I for $p \geq 1$.

Then in $(x_0 - x_1)f_0$, the terms with x_0 are those in $x_0 f_0$. Hence $x_0 f_0$ is in I and so $x_0 f$ is in I . Again since I is multigraded, each monomial term $x_0 m$ is in I . We also get $x_1 f \in I$ and then each $x_1 m \in I$. \square

Proof of Theorem 4.8, Part b. We prove that if each down-graph $D(\mathbf{c})$ contains a spanning tree of linear syzygy edges, then J will be a polarization. Order the variables in each \check{X}_i in a sequence $x_{i1}, x_{i2}, \dots, x_{in}$. Let \check{X}'_i consist of $x_{i1}, \dots, x_{ip_i}, x_i$ so we have a surjection $\check{X}_i \rightarrow \check{X}'_i$ for each i sending x_{ij} to itself for $j \leq p_i$, and to x_i for $j > p_i$. Denote the image of J in $k[\check{X}'_1, \dots, \check{X}'_m]$ by J' and the image of $m(\mathbf{a})$ by $m'(\mathbf{a})$. The quotient ring $k[\check{X}'_1, \dots, \check{X}'_m]/J'$ is obtained from $k[\check{X}_1, \dots, \check{X}_m]/J$ by dividing out by variable differences $x_{ij} - x_{i,j+1}$ for $i = 1, \dots, m$ and $j > p_i$. We assume this is a regular sequence. We show that if we now divide out by $x_{i,p_i} - x_{i,p_i+1}$ this is a non-zero divisor of $k[\check{X}'_1, \dots, \check{X}'_m]/J'$. By continuing we get eventually that $k[x_1, \dots, x_m]/(x_1, \dots, x_m)^n$ is a regular quotient of $k[\check{X}_1, \dots, \check{X}_m]/J$ and so J is a polarization of $(x_1, \dots, x_m)^n$.

We write $x'_i = x_{i,p_i}$. Suppose $(x'_i - x_i) \cdot \mathbf{f} = 0$ in $k[\check{X}'_1, \dots, \check{X}'_m]/J'$ where \mathbf{f} is a polynomial. By Lemma 5.8 (with $x'_i = x_{i,p_i}$ taking the place of x_0), we must show that if \mathbf{m} is a monomial such that $x'_i \cdot \mathbf{m} = 0$ and $x_i \cdot \mathbf{m} = 0$, then $\mathbf{m} = 0$ in the quotient ring. So some generator $m'(\mathbf{a})$ of J' divides $x'_i \mathbf{m}$ and some generator $m'(\mathbf{b})$ divides $x_i \mathbf{m}$.

By Proposition 5.3 there is an LS-path from $m(\mathbf{a})$ to $m(\mathbf{b})$ consisting of linear syzygy edges and such that each \mathbf{u} on this path has $\mathbf{u} \leq \mathbf{a} \vee \mathbf{b}$ and $m(\mathbf{u})$ on this path divides $\text{lcm}(m(\mathbf{a}), m(\mathbf{b}))$. The image $m'(\mathbf{u})$ then divides $x'_i x_i \mathbf{m}$. We will show by induction on the length of the path that some monomial $m'(\mathbf{u})$ on this path divides \mathbf{m} , and so \mathbf{m} is zero in the quotient ring $k[\check{X}'_1, \dots, \check{X}'_m]/J'$.

If the path has length one, there is a linear syzygy edge between $m(\mathbf{a})$ and $m(\mathbf{b})$. Write

$$x'_i \cdot \mathbf{m} = m'(\mathbf{a}) \cdot n^0(\mathbf{a}), \quad x_i \cdot \mathbf{m} = m'(\mathbf{b}) \cdot n^0(\mathbf{b}).$$

Write also $\mathbf{m} = (x'_i)^p (x_i)^q \cdot \mathbf{n}$ where \mathbf{n} does not contain x'_i or x_i . If none of $m'(\mathbf{a})$ or $m'(\mathbf{b})$ divides \mathbf{m} , then

$$m'(\mathbf{a}) = (x'_i)^{p+1} (x_i)^q \cdot n^1(\mathbf{a}), \quad m'(\mathbf{b}) = (x'_i)^p (x_i)^{q+1} \cdot n^1(\mathbf{b}),$$

where $p' \leq p$ and $q' \leq q$ (and $n^1(\mathbf{a})$ and $n^1(\mathbf{b})$ do not contain x'_i or x_i). But since the edge from \mathbf{a} to \mathbf{b} is a linear syzygy edge, we must have $p' = p$, $q' = q$. But a linear syzygy edge involves variables of distinct x_i -type, which is not so here. Thus one of $m'(\mathbf{a})$ or $m'(\mathbf{b})$ must divide \mathbf{m} .

Suppose now the path has length ≥ 2 .

Case $a_i \geq b_i$. Let \mathbf{a} to \mathbf{a}' be the first edge along the path. Then the coordinate $a'_i \leq a_i$.

If i) the coordinates a_i and a'_i are equal, the x_i -type variables of $m(\mathbf{a})$ and $m(\mathbf{a}')$ are the same, since this is a linear syzygy edge. Since $m'(\mathbf{a})$ divides $x'_i \mathbf{m}$ we get that $m'_i(\mathbf{a}')$ divides $x'_i \mathbf{m}$. If ii) $a'_i < a_i$ then when going from $m(\mathbf{a})$ to $m(\mathbf{a}')$ some x_i -type variable drops out from $m_i(\mathbf{a})$ by isotonicity of X_i and so also $m'_i(\mathbf{a}')$ divides $x'_i \mathbf{m}$.

Since the path from \mathbf{a} to \mathbf{b} is an LS-path, $m'(\mathbf{a}')$ divides $x'_i x_i \mathbf{m}$. For $j \neq i$, then $m'_j(\mathbf{a}')$ must divide \mathbf{m} since $m'_j(\mathbf{a}')$ contains no x_i -type variable. The upshot is that $m'(\mathbf{a}')$ divides $x'_i \mathbf{m}$. Considering the LS-path from \mathbf{a}' to \mathbf{b} , by induction on path length, some $m'(\mathbf{u})$ along this path divides \mathbf{m} .

Case $a_i \leq b_i$. Let \mathbf{b} to \mathbf{b}' be the first edge along the path going from \mathbf{b} to \mathbf{a} . Then the coordinate $b'_i \leq b_i$. Exactly the same argument as in the case above works in this case also. \square

6. THREE VARIABLES CASE: CLASSIFICATION OF ISOTONE MAPS

In the three variables case there is a particular nice classification of the isotone rank preserving maps from $\Delta_3(n) \rightarrow B(n)$.

Let $\Delta_3(n)$ have the partial order \geq_1 . The symmetric group S_n acts on the Boolean poset $B(n)$ and so acts on the set of isotone rank-preserving maps

$$X : \Delta_3(n) \rightarrow B(n).$$

Note that for $\mathbf{c} \in \Delta_3^+(n)$, an edge $(\mathbf{c}; 2, 3)$ being a quadratic syzygy edge, see Example 4.7, is a property invariant under the action of the group S_n . The following completely describes the combinatorics of such isotone maps. This is quite nice for the three variables case. For the m -variables case in general it seems harder to see a nice description.

Proposition 6.1. *Let $n \geq 1$. The orbits under the action of S_n of rank-preserving isotone maps $X : \Delta_3(n) \rightarrow B(n)$ are in one to one correspondence with subsets of $\Delta_3^+(n+1)$. An isotone map X corresponds to those $\mathbf{c} \in \Delta_3^+(n+1)$ such that the $(\mathbf{c}; 2, 3)$ -edges of $\Delta_3(n)$ are quadratic syzygy edges for X .*

Since $\Delta_3^+(n+1)$ is in bijection to $\Delta_3(n-2)$ which has cardinality $\binom{n}{2}$, the number of such orbits is $2^{\binom{n}{2}}$.

Proof. In each orbit there is exactly one isotone map where the sets

$$X(0, 0, n) \subseteq X(1, 0, n-1) \subseteq X(2, 0, n-2) \subseteq \cdots \subseteq X(0, 0)$$

are

$$(13) \quad \emptyset \subseteq \{1\} \subseteq \{1, 2\} \subseteq \cdots \subseteq \{1, 2, \dots, n\}$$

so we need only consider isotone maps which fulfill this condition and show that they are in one-one correspondence with subsets Q of $\Delta_3(n-2)$.

Given such an isotone map X we get a subset Q of QS-edges, those (c_1, c_2, c_3) in $\Delta_3^+(n+1)$ such that $X(c_1, c_2 - 1, c_3) \neq X(c_1, c_2, c_3 - 1)$. Conversely given a subset Q of $\Delta_3^+(n+1)$ we show that this determines uniquely an isotone X fulfilling (13) whose associated set of QS-edges is Q . The essential idea is the following simple observation:

Let A be a set of cardinality $p - 1$ where $p \geq 1$, B, C sets containing A of cardinality p , and D a set of cardinality $p + 1$ containing B and C . For fixed A, B, D , then either $C = B$ or $C = A \cup (D \setminus B)$. In the latter case note that $A = B \cap C$ and $D = B \cup C$.

Note now first that $X(0, r, n - r)$ is always the empty set. Let A, B, C, D be the four sets of cardinalities respectively $p - 1, p, p$ and $p + 1$:

$$X(p - 1, 1, n - p), X(p, 1, n - p - 1), X(p, 0, n - p), X(p + 1, 0, n - p - 1).$$

The last two are known by (13). By induction on p the first is known. Then the second is completely determined by whether the edge $((p, 1, n - p); 2, 3)$ is a LS- or QS-edge. Thus given Q we may determine all the $X(p, 1, n - p - 1)$ for $p = 1, \dots, n - 1$. Then we may continue and similarly determine all $X(p, 2, n - p - 2)$ for $p = 1, \dots, n - 2$ and so on. \square

7. DEGREE TWO CASE: POLARIZATIONS OF $(x_1, x_2, \dots, x_m)^2$

We show that polarizations of the second power $(x_1, x_2, \dots, x_m)^2$ of the maximal ideal are in one-to-one correspondence with oriented trees with edges labels $1, 2, \dots, m$. Moreover we show that the isomorphism classes of polarizations are in one-to-one correspondence with trees on $(m + 1)$ vertices.

First we recall a construction given in [10]. Given a directed tree T (the edges are directed) with edges labelled $1, 2, \dots, m$. The label of an edge e is denoted $l(e)$. Let $\check{X}_i = \{x_{i0}, x_{i1}\}$ for $i = 1, \dots, m$. We construct a monomial ideal $K(T)$ in $k[\check{X}_1, \dots, \check{X}_m]$ generated by $(m + 1)$ monomials, one for each vertex of T . Consider a vertex v of the tree T . If an edge e in the tree T is pointing in the direction towards v (this does not mean that e and v have to be incident) let $e_{to}(v) = 1$ and if e is pointing in the opposite direction, let $e_{to}(v) = 0$. To the vertex v of the tree T associate a monomial m_v of degree m

$$m_v = \prod_{e \in T} x_{l(e), e_{to}(v)},$$

and let $I(T)$ be the ideal in $k[\check{X}_1, \dots, \check{X}_m]$ generated by these monomials. We see that the m_v are rainbow monomials. By [10] this is a Cohen-Macaulay monomial ideal of codimension two with m -linear resolution.

We now construct another monomial ideal $J(T)$ in this polynomial ring as follows. For each pair of distinct vertices v, w of T there is a unique path (forgetting the direction of the edges) between v and w . Let e be the edge on this path incident to v and f the edge on this path incident to w . Define the monomial $m_{v,w}$ of degree two

$$m_{v,w} = x_{l(e), e_{to}(w)} x_{l(f), f_{to}(v)},$$

and let $J(T)$ be the ideal generated by all these monomials.

Theorem 7.1. *Let T be a directed tree with edges labelled by $1, 2, \dots, m$.*

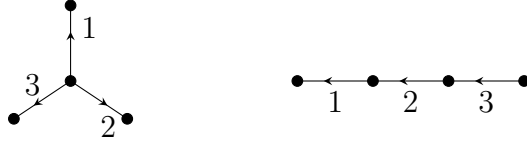


FIGURE 6

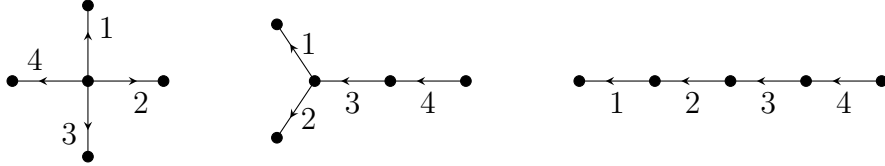


FIGURE 7

- The ideals $J(T)$ and $I(T)$ are Alexander duals.
- The ideal $J(T)$ is a polarization of $(x_1, \dots, x_m)^2$, and every polarization of the latter ideal is isomorphic as a monomial ideal to some $J(T)$.
- Two polarizations $J(T)$ and $J(T')$ are isomorphic as monomial ideals if and only if the underlying (unlabelled, undirected) trees of T and T' are isomorphic.

Example 7.2. The two trees in Figure 6 give the non-isomorphic polarizations of $(x_1, x_2, x_3)^2$. The first tree gives the standard polarization

$$(x_{11}x_{12}, x_{11}x_{21}, x_{21}x_{22}, x_{11}x_{31}, x_{21}x_{31}, x_{31}x_{32}).$$

The second tree gives the b-polarization

$$(x_{11}x_{12}, x_{11}x_{22}, x_{21}x_{22}, x_{11}x_{32}, x_{21}x_{32}, x_{31}x_{32}),$$

which is the letterplace ideal $L(2, 3)$ of [13].

Example 7.3. The trees with five vertices, Figure 7, decorated with direction and labelling, give the three non-isomorphic polarizations of $(x_1, x_2, x_3, x_4)^2$.

The first tree gives the standard polarization. The second tree gives the polarization:

$$(x_{11}x_{12}, x_{21}x_{22}, x_{31}x_{32}, x_{41}x_{42}, x_{11}x_{21}, x_{11}x_{32}, x_{11}x_{42}, x_{21}x_{32}, x_{21}x_{42}, x_{31}x_{42})$$

The third tree gives the b-polarization:

$$(x_{11}x_{12}, x_{21}x_{22}, x_{31}x_{32}, x_{41}x_{42}, x_{11}x_{22}, x_{11}x_{32}, x_{11}x_{42}, x_{21}x_{32}, x_{21}x_{42}, x_{31}x_{42}),$$

which is the letterplace ideal $L(2, 4)$.

Proof of Theorem 7.1. a. Consider the generator $m_{v,w} = x_{l(e),e_{to}(w)}x_{l(f),f_{to}(v)}$ of $J(T)$. Recall that e is incident to v . Let u be another vertex of T . If the first variable of $m_{v,w}$ is not in the monomial m_u then w and u are in distinct directions from v . If the second variable is not in m_u , then v and u are in distinct directions from w , but these two situations together are not possible in a tree.

Hence all the $\binom{m+1}{2}$ monomials $m_{v,w}$ are in the Alexander dual of $I(T)$. But $I(T)$ has m -linear resolution and is Cohen-Macaulay of codimension two by [10]. Such an ideal is easily seen to have multiplicity $\binom{m+1}{2}$. For a squarefree monomial ideal I in a

polynomial ring S , the multiplicity of I (or really of S/I) equals the number of the facets of maximal dimension of $\Delta(I)$, see Section 6.11 and equation (6.4) in [17]. Since $I(T)$ is Cohen-Macaulay all of its facets have the same dimension. So there are $\binom{m+1}{2}$ facets. Since $I(T)$ has codimension two the facets have cardinality $2m-2$. The Alexander dual of $I(T)$ is then generated by $\binom{m+1}{2}$ quadratic monomials by the standard correspondence between facets of a squarefree monomial ideal and the generators of its Alexander dual, see Lemma 1.5.4 and Corollary 1.5.5 of [17]. So these generators must be precisely the generators of $J(T)$ making it the Alexander dual.

b. The ideal $J(T)$ has codimension m , since its Alexander dual $I(T)$ has generators of degree m , loc.cit. Dividing out by the variable differences $x_{i0} - x_{i1}$ we easily see that we get the ideal $(x_1, \dots, x_m)^2$ which also has codimension m . Hence this sequence of variable differences is a regular sequence for $k[\check{X}_1, \dots, \check{X}_m]/J(T)$, and so $J(T)$ is a polarization of $(x_1, \dots, x_m)^2$.

Conversely let J be a polarization of $(x_1, \dots, x_m)^2$. Then J is Cohen-Macaulay of codimension m with 2-linear resolution. The Alexander dual I will then be Cohen-Macaulay of codimension 2 with m -linear resolution, [8]. By [10, Prop.2.4] there is a tree T such that that I is the image of the $I(T)$ by a map of polynomial rings $k[\check{X}_1, \dots, \check{X}_m]$ (the ring where $I(T)$ lives) to $k[\check{X}_1, \dots, \check{X}_m]$ (the ring where I lives), sending the variables in the former ring to monomials in the latter ring. But since the generators of I and $I(T)$ have the same degree, we must map the variables to variables. Hence I is isomorphic to $I(T)$ as a monomial ideal.

c. If the underlying trees are the same then clearly T and T' are related by the action of an element of $S_m \times (\mathbb{Z}_2)^m$, and so also $J(T)$ and $J(T')$. If the underlying trees are different, then $I(T)$ and $I(T')$ are not isomorphic: Their linear syzygies are given precisely by the edges of the trees. Any isomorphism between $I(T)$ and $I(T')$ would induce a bijection of monomial generators, corresponding to a bijection of vertices of the trees, such that the corresponding linear syzygies between generators, would correspond to a bijection between the edges of the trees. \square

Together with the discussion in Subsection 3.1 the following shows that polarizations of $(x_1, \dots, x_m)^2$ define simplicial balls. For the notion of linear quotients consult [17, Section 8.2.1].

Proposition 7.4. *The ideal $I(T)$ has linear quotients, and so equivalently ([17, Prop.8.2.5]) the Alexander dual $J(T)$ defines a shellable simplicial complex.*

Proof. Let r be any vertex in T and orient all the arrows away from r . This gives a partial order on the vertices of the tree. Take a linear extension of this partial order and let I be generated by the monomials in an initial segment for this total order. Let m_u be the subsequent monomial. We show that $I : m_u$ is generated by variables. So let $m \in I : m_u$. Then $m \cdot m_u$ is divisible by some $m_v \in I$. Each of m_u and m_v have one variable for each edge in T . Only the edges on the (undirected) path between u and v give distinct variables in m_u and m_v . Starting from u let e be the first edge on the path to v . Let w be the other end point of e . Since m_v comes before m_u in the total order, we see that m_w must also be in the initial segment which generates I . Of the variables in m_u and m_w , only the e -variable is different. Thus $I : m_u$ contains the

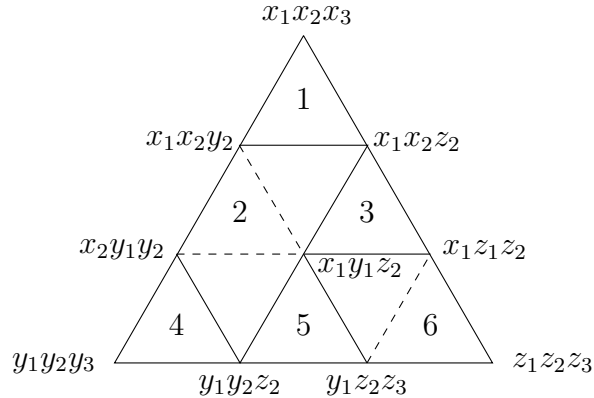


FIGURE 8

e -variable $x_{l(e), e_{to}(w)}$ occurring in m_w . But then this e -variable is also in m_v and so this variable divides m . \square

8. ALEXANDER DUALS

Recall that for a squarefree monomial ideal I in a polynomial ring S , the *Alexander dual* ideal J is the monomial ideal in S whose monomials are precisely those that have nontrivial common divisor with every monomial in I , or equivalently, every generator of I .

We describe the Alexander dual ideal of any polarization of the power $(x_1, \dots, x_m)^n$. The description is a direct construction involving the isotone maps $X_i, i = 1, \dots, m$. The proof is reduced to study certain baby-versions χ_i of the X_i which are isotone maps $\chi_i : \Delta_m(n) \rightarrow \{0 < 1\}$.

8.1. Statement and examples. Let $J \subset k[\check{X}_1, \dots, \check{X}_m]$ be a polarization of the ideal $(x_1, \dots, x_m)^n$ in $k[x_1, \dots, x_m]$. For any $\mathbf{a} \in \Delta_m(n-1)$ we have the up-graph $U(\mathbf{a})$, with vertices $\mathbf{a} + e_j$ for $j = 1, \dots, m$. At the vertex $\mathbf{a} + e_j$ we have the x_j -type variables $X_j(\mathbf{a} + e_j)$. We take the product of all these variable sets:

$$\mathcal{M}(\mathbf{a}) = \prod_{j=1}^m X_j(\mathbf{a} + e_j).$$

It consists of monomials $x_{1i_1}x_{2i_2} \cdots x_{mi_m}$ where x_{ji_j} is in $X_j(\mathbf{a} + e_j)$. Let I be the ideal generated by the monomials in the union of all the $\mathcal{M}(\mathbf{a})$ for $\mathbf{a} \in \Delta_m(n-1)$.

Theorem 8.1. *The ideal I is the Alexander dual of J .*

Example 8.2. Consider again the polarization J from Example 4.11. Its graph of linear syzygies is again given in Figure 8, this time labeling each of the up-triangles in the graph with $1, \dots, 6$. The up-triangle i corresponds to an element $a_i \in \Delta_3(2)$, where $a_1 = (2, 0, 0), a_2 = (1, 1, 0), a_3 = (1, 0, 1), a_4 = (0, 2, 0), a_5 = (0, 1, 1)$, and $a_6 = (0, 0, 2)$. The sets of monomials $\mathcal{M}(\mathbf{a}_i)$ are:

$$\begin{aligned}
\mathcal{M}(\mathbf{a}_1) &= \{\mathbf{x}_1\mathbf{y}_2\mathbf{z}_2, \mathbf{x}_2\mathbf{y}_2\mathbf{z}_2, \mathbf{x}_3\mathbf{y}_2\mathbf{z}_2\} \\
\mathcal{M}(\mathbf{a}_2) &= \{\mathbf{x}_1\mathbf{y}_1\mathbf{z}_2, x_1y_2z_2, \mathbf{x}_2\mathbf{y}_1\mathbf{z}_2, x_2y_2z_2\} \\
\mathcal{M}(\mathbf{a}_3) &= \{\mathbf{x}_1\mathbf{y}_1\mathbf{z}_1, x_1y_1z_2, \mathbf{x}_2\mathbf{y}_1\mathbf{z}_1, x_2y_1z_2\} \\
\mathcal{M}(\mathbf{a}_4) &= \{x_2y_1z_2, x_2y_2z_2, \mathbf{x}_2\mathbf{y}_3\mathbf{z}_2\} \\
\mathcal{M}(\mathbf{a}_5) &= \{x_1y_1z_2, \mathbf{x}_1\mathbf{y}_1\mathbf{z}_3, x_1y_2z_2, \mathbf{x}_1\mathbf{y}_2\mathbf{z}_3\} \\
\mathcal{M}(\mathbf{a}_6) &= \{x_1y_1z_1, x_1y_1z_2, x_1y_1z_3\}
\end{aligned}$$

The boldface monomials are the ten distinct monomials we find from this process, which in fact generate the Alexander dual I of J .

We shall go through several steps in proving the above theorem. It turns out that we will be able to abstract the situation so our arguments will only involve a collection of isotone maps

$$(14) \quad \chi_i : \Delta_m(n) \rightarrow \{0 < 1\}, \quad i = 1, \dots, m$$

where $\Delta_m(n)$ has the partial order \geq_i , and such that $\chi_i(\mathbf{b}) = 0$ whenever $b_i = 0$.

Remark 8.3. When $m = 3$, it is a curious fact that for each i the number of maps χ_i is the Catalan number C_{n+1} : Such maps are in one-one correspondence with "stacking of coins", [27, Exercise 6.19 hhh].

First we establish some notation. For a monomial $\mathbf{m} \in k[\check{X}_1, \dots, \check{X}_m]$, define maps

$$\begin{aligned}
\chi_{i,\mathbf{m}} : \Delta_m(n) &\rightarrow \{0 < 1\} \\
\mathbf{b} &\mapsto \begin{cases} 0, & \text{no variable of } X_i(\mathbf{b}) \text{ is in } \mathbf{m}. \\ 1, & \text{some variable of } X_i(\mathbf{b}) \text{ is in } \mathbf{m}. \end{cases}
\end{aligned}$$

Note i) this is an isotone map and ii) if $(\mathbf{c}; j, k)$ is a linear syzygy edge for the isotone maps $\{X_i\}$, then $\chi_{i,\mathbf{m}}(\mathbf{c} - e_j) = \chi_{i,\mathbf{m}}(\mathbf{c} - e_k)$ for every $i \neq j, k$. Note furthermore:

- That a monomial $\mathbf{m} \in k[\check{X}_1, \dots, \check{X}_m]$ is in the Alexander dual of J means that it has a common variable with every $m(\mathbf{b})$, or for every $\mathbf{b} \in \Delta_m(n)$, a common variable with some $m_i(\mathbf{b})$ for $i = 1, \dots, m$. This holds if and only if for every such \mathbf{b} , there is some i with $\chi_{i,\mathbf{m}}(\mathbf{b}) = 1$.
- The monomial \mathbf{m} is in I if and only if for some $\mathbf{a} \in \Delta_m(n-1)$, it has a common variable with every $X_j(\mathbf{a} + e_j)$ for $j = 1, \dots, m$. Thus for such an \mathbf{a} we have $\chi_{j,\mathbf{m}}(\mathbf{a} + e_j) = 1$ for every j .

We now abstract the negations of the above. Consider for $i = 1, \dots, m$ isotone maps as in (14).

Definition 8.4. A multidegree $\mathbf{b} \in \Delta_m(n)$ is a *full zero point* for the collection $\{\chi_i\}$ if $\chi_i(\mathbf{b}) = 0$ for every $i = 1, \dots, m$. An up-simplex $U(\mathbf{a})$ of $\Delta_m(n)$ has a *zero corner* if $\chi_i(\mathbf{a} + e_i) = 0$ for some i .

An edge $(\mathbf{c}; i, j)$ of $\Delta_m(n)$ is a *linear syzygy edge* for the collection $\{\chi_i\}$ if $\chi_p(\mathbf{c} - e_i) = \chi_p(\mathbf{c} - e_j)$ for every $p \neq i, j$.

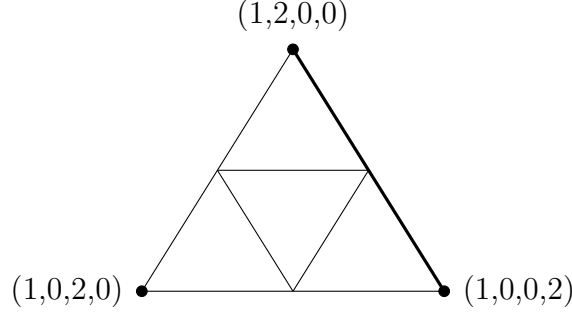


FIGURE 9

We prove the following.

Theorem 8.5. *Given the collection of isotone maps $\{\chi_i\}$ such that for every down-graph of $\Delta_m(n)$ the linear syzygy edges for $\{\chi_i\}$ contains a spanning tree. Then $\{\chi_i\}$ has a full zero-point in $\Delta_m(n)$ if and only if every up-graph of $\Delta_m(n)$ has a zero corner.*

As a consequence we get Theorem 8.1.

Proof of Theorem 8.1. That $\mathbf{m} \notin I$ means that every up-graph in $\Delta_m(n)$ has a zero corner for the $\chi_{i,\mathbf{m}}$'s. That \mathbf{m} is not in the Alexander dual of J means that it has a full zero-point for the $\chi_{i,\mathbf{m}}$'s. Hence by Theorem 8.5 I will be the Alexander dual of J . \square

8.2. Definitions and key lemma. In order to facilitate our arguments we need to have a more flexible framework to work in. For $\mathbf{d} \in \mathbb{N}^m$ with $|\mathbf{d}| \leq n$ and $S \subseteq [m]$, let $\Delta_S(n, \mathbf{d})$ be the induced subgraph of $\Delta_m(n)$ whose vertices are the degrees $\mathbf{b} \geq \mathbf{d}$ such that $\text{supp}(\mathbf{b} - \mathbf{d}) \subseteq S$. This means:

$$i) |\mathbf{b}| = n, \quad ii) b_i = d_i \text{ for } j \in [m] \setminus S, \quad iii) b_i \geq d_i \text{ for } i \in S.$$

(We omit m in $\Delta_S(n, \mathbf{d})$ since m is fixed throughout.) We normally write $\Delta_S(\mathbf{d})$ for $\Delta_S(n, \mathbf{d})$ but on a few occasions we want $(n-1)$ or $(n+1)$ instead of n as argument, and use the full notation.

The convex hull of $\Delta_S(\mathbf{d})$ in \mathbb{R}^S is a simplex of dimension $|S| - 1$ if $|\mathbf{d}| < n$. The size of this simplex is $n - |\mathbf{d}|$. For $\mathbf{d} = \mathbf{0}$, the zero degree, we have $\Delta_{[m]}(\mathbf{0}) = \Delta_m(n)$ as defined earlier. Note that $\Delta_{S'}(\mathbf{d}')$ is a non-empty subset of $\Delta_S(\mathbf{d})$ iff:

$$i) S' \subseteq S, \quad ii) \mathbf{d}' \geq \mathbf{d}, \quad iii) d'_i = d_i \text{ for } i \in [m] \setminus S.$$

Example 8.6. Let $S = \{2, 3, 4\}$ and $\mathbf{d} = (1, 0, 0, 0)$. Then $\Delta_S(\mathbf{d})$ is the induced subgraph of $\Delta_4(3)$ depicted in Figure 9; its convex hull is a simplex of dimension 2, and its size is 2. If $S' = \{2, 4\}$, then $\Delta_{S'}(\mathbf{d})$ is the subgraph of $\Delta_S(\mathbf{d})$ depicted by the thick line in Figure 9. Its convex hull is a simplex of dimension 1, and its size is also 2.

Definition 8.7. For $\mathbf{a} \in \Delta_S(n-1, \mathbf{d})$ we get an induced subgraph $U_S(\mathbf{a}; \mathbf{d})$ of $\Delta_S(\mathbf{d})$ with vertices $\{\mathbf{a} + e_i \mid i \in S\}$. This is a complete graph on $|S|$ vertices whose convex hull is of dimension $|S| - 1$. The graph $U_S(\mathbf{a}; \mathbf{d})$ is an *up-graph*. The $i \in S$ are the *corners* of the up-graph.

For $\mathbf{c} \in \Delta_S(n+1, \mathbf{d})$ we get an induced subgraph $D_S(\mathbf{c}; \mathbf{d})$ of $\Delta_S(\mathbf{d})$ with vertices $\{\mathbf{c} - e_i \mid i \in \text{supp}(\mathbf{c} - \mathbf{d})\}$. This is a complete graph on $|\text{supp}(\mathbf{c} - \mathbf{d})|$ vertices whose convex hull is a simplex of dimension $|\text{supp}(\mathbf{c} - \mathbf{d})| - 1$. The graph $D_S(\mathbf{c}; \mathbf{d})$ is a *down-graph*.

Suppose that for each $j \in S$ we have isotone maps, where we have given $\Delta_S(\mathbf{d} + e_j)$ the \geq_j -ordering:

$$\chi_j : \Delta_S(\mathbf{d} + e_j) \rightarrow \{0 < 1\}.$$

Note that for $\mathbf{b} \in \Delta_S(\mathbf{d})$, the $\chi_j(\mathbf{b})$ are defined precisely for $j \in \text{supp}(\mathbf{b} - \mathbf{d})$.

Definition 8.8. A vertex $\mathbf{b} \in \Delta_S(\mathbf{d})$ is a *full zero point* for $\{\chi_i\}_{i \in S}$ if $\chi_i(\mathbf{b}) = 0$ for every $i \in \text{supp}(\mathbf{b} - \mathbf{d})$. An up-graph $U_S(\mathbf{a}; \mathbf{d})$ has a *zero corner* for $\{\chi_i\}_{i \in S}$ if $\chi_j(\mathbf{a} + e_j) = 0$ for some $j \in S$.

An edge $(\mathbf{c}; r, s)$ of $\Delta_S(\mathbf{d})$ (note that then $\{r, s\} \subseteq S$) is a *linear syzygy edge* for the $\{\chi_j\}_{j \in S}$ if

$$\chi_j(\mathbf{c} - e_r) = \chi_j(\mathbf{c} - e_s), \quad \text{for } j \in \text{supp}(\mathbf{c} - \mathbf{d}) \setminus \{r, s\}.$$

Lemma 8.9. Suppose we have isotone maps $\{\chi_j\}_{j \in S}$ for $\Delta_S(\mathbf{d})$ such that the linear syzygy edges in every down-graph of $\Delta_S(\mathbf{d})$ contains a spanning tree.

Let $\Delta_R(\mathbf{d}')$ be a non-empty subgraph of $\Delta_S(\mathbf{d})$. For $j \in R$ let $\bar{\chi}_j$ be the restriction of χ_j to isotone maps associated to $\Delta_R(\mathbf{d}')$. Then each down-graph of $\Delta_R(\mathbf{d}')$ contains a spanning tree of linear syzygy edges for the $\{\bar{\chi}_j\}_{j \in R}$.

Proof. (This proof is essentially the same as for Lemma 5.1.) Choose a particular down-graph $D_R(\mathbf{c}; \mathbf{d}')$ of $\Delta_R(\mathbf{d}')$. Let $\mathbf{c} - e_r$ and $\mathbf{c} - e_s$ be vertices of $D_R(\mathbf{c}; \mathbf{d}')$ (so $r, s \in \text{supp}(\mathbf{c} - \mathbf{d}')$). We will show that there is a path from $\mathbf{c} - e_r$ to $\mathbf{c} - e_s$ in the graph $D_R(\mathbf{c}; \mathbf{d}')$ consisting of linear syzygy edges for the $\{\bar{\chi}_j\}_{j \in R}$.

Since $\mathbf{c} \geq \mathbf{d}$, $|\mathbf{c}| = n + 1$ and $c_i = d_i$ for i outside of S , we may consider the down graph $D_S(\mathbf{c}; \mathbf{d})$ in $\Delta_S(\mathbf{d})$. We know there is a path in $D_S(\mathbf{c}; \mathbf{d})$ from $\mathbf{c} - e_r$ to $\mathbf{c} - e_s$ consisting of linear syzygy edges for the $\{\chi_j\}_{j \in S}$.

It may be broken up into smaller paths: From $\mathbf{c} - e_r = \mathbf{c} - e_{r_0}$ to $\mathbf{c} - e_{r_1}$, from $\mathbf{c} - e_{r_1}$ to $\mathbf{c} - e_{r_2}$, ..., from $\mathbf{c} - e_{r_{p-1}}$ to $\mathbf{c} - e_{r_p} = \mathbf{c} - e_s$ where on the path from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ the only vertices $\mathbf{c} - e_q$ with $q \in R$ are the end vertices $q = r_{i-1}$ and $q = r_i$ while the in-between vertices all have $q \in Q = \text{supp}(\mathbf{c} - \mathbf{d}) \setminus R$. We claim that each edge from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ is a linear syzygy edge for the $\{\bar{\chi}_j\}_{j \in R}$. This will prove the lemma.

Let the path from $\mathbf{c} - e_{r_{i-1}}$ to $\mathbf{c} - e_{r_i}$ be

$$\mathbf{c} - e_{r_{i-1}} = \mathbf{c} - e_{q_0}, \mathbf{c} - e_{q_1}, \dots, \mathbf{c} - e_{q_{t-1}}, \mathbf{c} - e_{q_t} = \mathbf{c} - e_{r_i}$$

where q_1, \dots, q_{t-1} are all in Q . We must show that

$$(15) \quad \bar{\chi}_p(\mathbf{c} - e_{r_{i-1}}) = \bar{\chi}_p(\mathbf{c} - e_{r_i}) \text{ for } p \in \text{supp}(\mathbf{c} - \mathbf{d}') \setminus \{r_{i-1}, r_i\}.$$

But since the edges on the path are linear syzygy edges for the $\{\chi_j\}_{j \in S}$ we have

$$\chi_p(\mathbf{c} - e_{q_{j-1}}) = \chi_p(\mathbf{c} - e_{q_j}) \text{ for } p \in \text{supp}(\mathbf{c} - \mathbf{d}) \setminus \{q_{j-1}, q_j\}.$$

Hence

$$\chi_p(\mathbf{c} - e_{r_{i-1}}) = \chi_p(\mathbf{c} - e_{r_i}) \text{ for } p \in \text{supp}(\mathbf{c} - \mathbf{d}) \setminus \{r_{i-1}, q_1, \dots, q_{t-1}, r_i\}.$$

Since $\text{supp}(\mathbf{c} - \mathbf{d}') \subseteq R$ and $\{q_1, \dots, q_{t-1}\} \subseteq S \setminus R$, these are disjoint sets. Also $\text{supp}(\mathbf{c} - \mathbf{d}') \subseteq \text{supp}(\mathbf{c} - \mathbf{d})$. Hence

$$\chi_p(\mathbf{c} - e_{r_{i-1}}) = \chi_p(\mathbf{c} - e_{r_i}) \text{ for } p \in \text{supp}(\mathbf{c} - \mathbf{d}') \setminus \{r_{i-1}, r_i\}.$$

This shows (15). \square

8.3. Full zero point implies a zero corner in every up-graph. We prove this direction of Theorem 8.5.

Lemma 8.10. *Let $S \subseteq [m]$ have cardinality ≥ 2 . Suppose every down-graph of $\Delta_S(\mathbf{d})$ contains a spanning tree of linear syzygies for the isotone maps $\{\chi_j\}_{j \in S}$. Let $\mathbf{b} \in \Delta_S(\mathbf{d})$ and $p \in S$. Suppose $\chi_i(\mathbf{b}) = 0$ for every $i \in \text{supp}(\mathbf{b} - \mathbf{d}) \setminus \{p\}$. Then every up-graph $U_S(\mathbf{a}; \mathbf{d})$ in $\Delta_S(\mathbf{d})$ with $a_p \geq b_p$ has a zero corner for the $\{\chi_i\}_{i \in S}$.*

Corollary 8.11. *Suppose the maps $\{\chi_i\}_{i \in S}$ have a full zero-point in $\Delta_S(\mathbf{d})$. Then every up-graph in $\Delta_S(\mathbf{d})$ has a zero corner.*

Proof Lemma 8.10 and Corollary 8.11. We prove these in tandem by induction on the cardinality $|S|$. We prove Corollary 8.11 by assuming Lemma 8.10. Then we prove Lemma 8.10 by assuming Corollary 8.11 has been proven for all $\Delta_{S'}(\mathbf{d})$ when the cardinality $|S'| < |S|$.

Assume we have shown Lemma 8.10. Let \mathbf{b} be a full zero-point for $\Delta_S(\mathbf{d})$ and consider an up-graph $U_S(\mathbf{a}; \mathbf{d})$ in $\Delta_S(\mathbf{d})$. Since $|\mathbf{b}| = n$ and $|\mathbf{a}| = n - 1$, then at least for one p we have $a_p \geq b_p$. Then Lemma 8.10 implies that $U_S(\mathbf{a}; \mathbf{d})$ is an up-graph with a zero corner, proving Corollary 8.11.

We now show Lemma 8.10. We may assume $p = 1$. First we do the case $|S| = 2$, say $S = \{1, 2\}$. Then $\chi_2(\mathbf{b}) = 0$. By isotonicity of χ_2 we have $\chi_2(\mathbf{b} + \lambda e_1 - \lambda e_2) = 0$ for $\lambda \geq 0$. But letting $\lambda = a_1 - b_1$, the point $\mathbf{b} + \lambda e_1 - \lambda e_2 = \mathbf{a} + e_2$ is a zero corner for $U_S(\mathbf{a}; \mathbf{d})$.

Assume now $|S| \geq 3$ and Corollary 8.11 holds for S' with $2 \leq |S'| < |S|$. We argue by induction on the difference $a_1 - b_1$.

Case $a_1 = b_1$. Let $S' = S \setminus \{1\}$ and $\mathbf{d}' = \mathbf{d} + (b_1 - d_1)e_1$. Then $\mathbf{b} \in \Delta_{S'}(\mathbf{d}')$ and \mathbf{b} is a full zero point for the $\{\bar{\chi}_i\}_{i \in S'}$. By Lemma 8.9 the $\{\bar{\chi}_i\}_{i \in S'}$ also fulfill the hypotheses of Lemma 8.10 and Corollary 8.11. Then the corollary gives that every up-graph $U_{S'}(\mathbf{a}; \mathbf{d}')$ in $\Delta_{S'}(\mathbf{d}')$ has a zero corner for $\{\bar{\chi}_i\}_{i \in S'}$. Since $\mathbf{a} \in \Delta_{S'}(n - 1, \mathbf{d}')$ iff $\mathbf{a} \in \Delta_S(n - 1, \mathbf{d})$ with $a_1 = d'_1 = b_1$, up-graphs $U_{S'}(\mathbf{a}; \mathbf{d}')$ in $\Delta_{S'}(\mathbf{d}')$ correspond to up-graphs $U_S(\mathbf{a}; \mathbf{d})$ with $a_1 = b_1$. Then every up-graph $U_S(\mathbf{a}; \mathbf{d})$ with $a_1 = b_1$ has a zero corner for $\{\chi_i\}_{i \in S}$.

Case $a_1 > b_1$. Consider the down-graph $D_S(\mathbf{b} + e_1; \mathbf{d})$. By assumption on the maps χ_i , at least one edge $(\mathbf{b} + e_1; 1, r)$ is a linear syzygy edge, so $\chi_j(\mathbf{b} + e_1 - e_r) = \chi_j(\mathbf{b}) = 0$ for every $j \in \text{supp}(\mathbf{b} - \mathbf{d}) \setminus \{1, r\}$. But $\mathbf{b} + e_1 - e_r \leq_r \mathbf{b}$, so $\chi_r(\mathbf{b} + e_1 - e_r) \leq \chi_r(\mathbf{b}) = 0$. Therefore $\chi_j(\mathbf{b} + e_1 - e_r) = 0$ for every $j \in \text{supp}(\mathbf{b} - \mathbf{d}) \setminus \{1\}$. Since the difference in first coordinates of \mathbf{a} and $\mathbf{b} + e_1 - e_r$ is one less than $a_1 - b_1$, by induction the up-graph $U_S(\mathbf{a}; \mathbf{d})$ has a zero corner for $\{\chi_j\}_{j \in S}$. \square

8.4. Every up-graph has a zero corner implies a full zero point. We now prove the other direction of Theorem 8.5. We need the following specific lemma. It says that you can “pull a point” \mathbf{b} with specific properties in a given direction e_p and into a simplex $\Delta_S(\mathbf{d}')$ of smaller size. Our assumptions in this subsection are those of Theorem 8.5.

Lemma 8.12. *Let $\mathbf{b} \in \Delta_S(\mathbf{d})$ with $|\mathbf{d}| \leq n - 1$, $1 \in S$, and $b_1 \leq d_1 + 1$. Suppose*

$$\chi_i(\mathbf{b}) = 0 \text{ for } i \in \text{supp}(\mathbf{b} - \mathbf{d}) \setminus \{1\}, \quad \chi_1(\mathbf{b}) = 1 \text{ if } 1 \in \text{supp}(\mathbf{b} - \mathbf{d}).$$

Let $p \in S \setminus \{1\}$ and $\mathbf{d}' = \mathbf{d} + e_p$. Then there is some $\mathbf{b}' \in \Delta_S(\mathbf{d}')$ with $b'_1 \leq d'_1 + 1 (= d_1 + 1)$ such that

$$(16) \quad \chi_i(\mathbf{b}') = 0 \text{ for } i \in \text{supp}(\mathbf{b}' - \mathbf{d}') \setminus \{1\}, \quad \chi_1(\mathbf{b}') = 1 \text{ if } 1 \in \text{supp}(\mathbf{b}' - \mathbf{d}').$$

Proof. If $b_p \geq d_p + 1$ we simply let $\mathbf{b}' = \mathbf{b}$. So suppose $b_p = d_p$. Looking at the down-graph $D_S(\mathbf{b} + e_p; \mathbf{d})$ there is a linear syzygy edge $(\mathbf{b} + e_p; p, q)$ for some $q \in S \setminus \{p\}$. It goes from \mathbf{b} to $\mathbf{b}' = \mathbf{b} + e_p - e_q$. Let us show that this \mathbf{b}' has the desired properties.

We have $\mathbf{b}' - \mathbf{d}' = \mathbf{b} - \mathbf{d} - e_q$ and so $\text{supp}(\mathbf{b}' - \mathbf{d}')$ is $\text{supp}(\mathbf{b} - \mathbf{d})$ with q possibly removed. We see that $p \notin \text{supp}(\mathbf{b}' - \mathbf{d}')$, and so $\chi_i(\mathbf{b}') = \chi_i(\mathbf{b})$ for $i \in \text{supp}(\mathbf{b}' - \mathbf{d}') \setminus \{q\}$.

Case $q = 1$. In this case $b'_1 = d_1$ so $1 \notin \text{supp}(\mathbf{b}' - \mathbf{d}')$ and (16) holds.

Case $q \neq 1$. Note $\mathbf{b}' \leq_q \mathbf{b}$. So if q is contained in $\text{supp}(\mathbf{b}' - \mathbf{d}')$ then $\chi_q(\mathbf{b}') \leq \chi_q(\mathbf{b}) = 0$. Furthermore, since $(\mathbf{b} + e_p; p, q)$ is a linear syzygy edge, $\chi_1(\mathbf{b}') = \chi_1(\mathbf{b}) = 1$ if $1 \in \text{supp}(\mathbf{b}' - \mathbf{d}')$. So again (16) holds. \square

Proposition 8.13. *Let $|\mathbf{d}| \leq n - 1$ and $|S| \geq 2$. Suppose every up-graph in $\Delta_S(\mathbf{d})$ has a zero corner for the $\{\chi_i\}_{i \in S}$. Then there is an element of $\Delta_S(\mathbf{d})$ which is a full zero-point.*

Proof. We prove by induction on the size $(n - |\mathbf{d}|)$ and cardinality $|S|$ that the above holds. If $|\mathbf{d}| = n - 1$, then $\Delta_S(\mathbf{d})$ equals the up-graph $U_S(\mathbf{d}; \mathbf{d})$. If we have a zero at corner $p \in S$, so $\chi_p(\mathbf{d} + e_p) = 0$, then $\mathbf{b} = \mathbf{d} + e_p$ is a full zero point of $\Delta_S(\mathbf{d})$ since $\text{supp}(\mathbf{b} - \mathbf{d}) = \{p\}$.

Now pick an element of S , say $1 \in S$. By induction on size, $\Delta_S(\mathbf{d} + e_1)$ has a full zero point \mathbf{b} . If $b_1 > d_1 + 1$ then $\text{supp}(\mathbf{b} - \mathbf{d}) = \text{supp}(\mathbf{b} - (\mathbf{d} + e_1))$ and \mathbf{b} is a full zero point for $\Delta_S(\mathbf{d})$. So suppose $b_1 = d_1 + 1$. If $\chi_1(\mathbf{b}) = 0$, it is a full zero point in $\Delta_S(\mathbf{d})$. Otherwise we have:

$$(17) \quad \chi_i(\mathbf{b}) = 0 \text{ for } i \in \text{supp}(\mathbf{b} - \mathbf{d}) \setminus \{1\}, \quad \chi_1(\mathbf{b}) = 1.$$

First consider when S has cardinality two, say $S = \{1, 2\}$, let $\mathbf{a} = \mathbf{b} - e_1$ and consider the up-graph $U_S(\mathbf{a}; \mathbf{d})$. Since $\chi_1(\mathbf{a} + e_1) = 1$, we must in the other corner of this up-graph have $\chi_2(\mathbf{a} + e_2) = 0$. Since 1 is not in the support of $(\mathbf{a} + e_2) - \mathbf{d}$, then $\mathbf{b} = \mathbf{a} + e_2$ is a full zero point.

So let S have cardinality ≥ 3 and put $S' = S \setminus \{1\}$. By Lemma 8.9, $\Delta_{S'}(\mathbf{d})$ fulfills the assumptions of Theorem 8.5. We show that each up-graph in $\Delta_{S'}(\mathbf{d})$ has a zero corner. By induction we then have a full zero \mathbf{b}^0 in $\Delta_{S'}(\mathbf{d})$. Since 1 is not in the support of $\mathbf{b}^0 - \mathbf{d}$ this \mathbf{b}^0 is also a full zero point in $\Delta_S(\mathbf{d})$ and we are done.

So let $U_{S'}(\mathbf{a}; \mathbf{d})$ be an up-graph in $\Delta_{S'}(\mathbf{d})$. We also have the up-graph $U_S(\mathbf{a}; \mathbf{d})$ in $\Delta_S(\mathbf{d})$. If there is some $p \geq 2$ such that $a_p > b_p$ we apply Lemma 8.12 and "pull" the point \mathbf{b} of (17) to a point \mathbf{b}' in a smaller sized $\Delta_S(\mathbf{d} + e_p)$, which still contains $U_S(\mathbf{a}; \mathbf{d})$. In this way we continue until we have a \mathbf{b} with either i) $a_p \leq b_p$ for every $p \geq 2$, or ii) the size of $\Delta_S(\mathbf{d})$ has become 1. But with this size we have $\mathbf{a} = \mathbf{d}$ and so $a_p \leq b_p$ for every $p \geq 2$ in any case. In both cases we also have $a_1 = d_1 \leq b_1 \leq d_1 + 1$ and recall that $|\mathbf{b}| = n$ and $|\mathbf{a}| = n - 1$.

Case $b_1 = a_1$. Then $\mathbf{b} = \mathbf{a} + e_p$ for some $p \geq 2$ and $p \in \text{supp}(\mathbf{b} - \mathbf{d})$. Then

$$\chi_p(\mathbf{a} + e_p) = \chi_p(\mathbf{b}) = 0$$

and so $U_{S'}(\mathbf{a}; \mathbf{d})$ has a zero corner.

Case $b_1 = a_1 + 1$. Then $\mathbf{b} = \mathbf{a} + e_1$ and $1 \in \text{supp}(\mathbf{b} - \mathbf{d})$ so

$$\chi_1(\mathbf{a} + e_1) = \chi_1(\mathbf{b}) = 1.$$

Since $U_S(\mathbf{a}; \mathbf{d})$ has a zero corner we must have $\chi_p(\mathbf{a} + e_p) = 0$ for some $p \in S'$ and so $U_{S'}(\mathbf{a}; \mathbf{d})$ has a zero corner. \square

9. POLARIZATIONS DEFINE SHELLABLE SIMPLICIAL COMPLEXES

In this section we show that the Alexander dual of any polarization of the power $(x, y, z)^m$ is a monomial ideal with linear quotients. This is equivalent to the polarization defining a shellable simplicial complex via the Stanley-Reisner correspondence, [17, Prop.8.2.5]. By a result of Björner [4, Thm.11.4] this immediately implies that these polarizations define simplicial balls, see Lemma 3.1 and Subsection 3.1.

An element (a, b, c) in $\Delta_3(n - 1)$ corresponds to an up-triangle in $\Delta_3(n)$. If $x_\alpha \in X(a + 1, b, c)$ we say that x_α (or just α) is an x -variable belonging to the up-triangle $U(a, b, c)$, and similarly if $y_\beta \in Y(a, b + 1, c)$ and $z_\gamma \in Z(a, b, c + 1)$. We also say the monomial $x_\alpha y_\beta z_\gamma$ (or just $\alpha\beta\gamma$) belongs to (a, b, c) .

Lemma 9.1. *Suppose $\alpha\beta\gamma$ belongs to the up-triangle $U(a + 1, b, c)$ in $\Delta_3(n - 1)$, see Figure 10.*

a. Then either the up-triangle $U(a, b + 1, c)$ or the up-triangle $U(a, b, c + 1)$ has a monomial $\alpha'\beta\gamma$ belonging to them.

b. If α either belongs to the up-triangle $U(a, b + 1, c)$ or to $U(a, b, c + 1)$, then $\alpha\beta\gamma$ will belong to one of the up-triangles $U(a, b, c + 1)$ or in $U(a, b + 1, c)$.

Proof. Consider the up-triangles in Figure 10. In the middle we have a down-triangle $D(a + 1, b + 1, c + 1) \in \Delta_3(n)$. Note that since Y is isotone, β will be in both the up-triangles $U(a + 1, b, c)$ and $U(a, b + 1, c)$ and since Z isotone γ in both $U(a + 1, b, c)$ and $U(a, b, c + 1)$.

a. If the edge $((a + 1, b + 1, c + 1); 1, 2)$ is a linear syzygy edge, then also γ belongs to $U(a, b + 1, c)$, and if $((a + 1, b + 1, c + 1); 1, 3)$ is a linear syzygy edge then β belongs to $U(a, b, c + 1)$. Since at least one of them is a linear syzygy edge we are done.

b. If $((a, b + 1, c + 1); 2, 3)$ is a linear syzygy edge, α is in both the two lower up-triangles. It then follows by part a that $\alpha\beta\gamma$ belongs to one of these up-triangles.

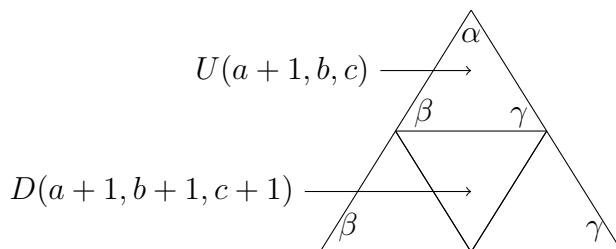


FIGURE 10

If $((a, b + 1, c + 1); 2, 3)$ is not a linear syzygy edge, the two other edges are linear syzygy edges. By the argument in part a, both the lower up-triangles contains β and γ and so at least one of them contains $\alpha\beta\gamma$. \square

Let \check{X} be a set of x -variables (with various indices) and \check{Y} and \check{Z} be sets of y - and z -variables.

Lemma 9.2. *Let I be an ideal generated by a subset of monomials in the product set $\check{X} \cdot \check{Y} \cdot \check{Z}$. Let $x_\alpha y_\beta z_\gamma$ be in $\check{X} \cdot \check{Y} \cdot \check{Z}$ but not in I . Then $I : x_\alpha y_\beta z_\gamma$ is generated by variables iff for every $x_{\alpha'} y_{\beta'} z_{\gamma'} \in I$ one of the variables $x_{\alpha'}$, $y_{\beta'}$ or $z_{\gamma'}$ is in the colon ideal.*

Proof. Note that by the construction of I and definition of $x_\alpha y_\beta z_\gamma$, none of the variables x_α , y_β , or z_γ can be in $I : x_\alpha y_\beta z_\gamma$.

That the first assertion implies the second is easy. Assume the second assertion holds. Then, if say $y_{\beta'} z_{\gamma'}$ is in the colon ideal, then $x_\alpha y_\beta y_{\beta'} z_{\gamma'} z_{\gamma'}$ is in I . So at least some $x_\alpha y_{\tilde{\beta}} z_{\tilde{\gamma}}$ is in I , where $\tilde{\beta} = \beta'$ or $\tilde{\gamma} = \gamma'$. But by assumption then either $y_{\beta'}$ or $z_{\gamma'}$ is in the colon ideal. This implies the colon ideal is generated by variables. \square

We now consider the monomials $x_\alpha y_\beta z_\gamma$ belonging to the up-triangles $U(a, b, c) \in \Delta_3(n - 1)$ and shall provide a total order on these monomials. First consider the pre-order on *triples* where $(a, b, c) \geq (a', b', c')$ if $a \geq a'$ and take any linear extension on this to get a total order \succeq on triples.

Now for each up-triangle $U(a, b, c)$ we shall make a total order on the (degree 3) *monomials* belonging to it. For each $X(a + 1, b, c)$ choose any total order of the x -variables. To order the variables in $Y(a, b + 1, c)$ we have an ascending chain

$$(18) \quad Y(a, 1, n - a - 1) \subseteq Y(a, 2, n - a - 2) \subseteq \cdots \subseteq Y(a, n - a, 0).$$

We order the variables such that each new variable popping up in the chain is less than the foregoing variables. Similarly for the variables in $Z(a, b, c + 1)$ we have a chain

$$Z(a, n - a - 1, 1) \subseteq Z(a, n - a - 2, 2) \subseteq \cdots \subseteq Z(a, 0, n - a),$$

and we order the variables such that each new variables popping up in the chain is less than the foregoing variables. The monomials belonging to $U(a, b, c)$ correspond to

$$X(a + 1, b, c) \times Y(a, b + 1, c) \times Z(a, b, c + 1).$$

We get the partial product order on this and take a linear extension of this partial order.

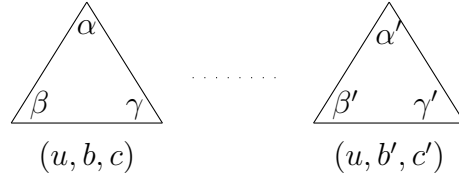


FIGURE 11

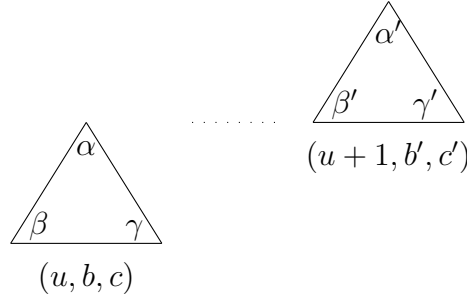


FIGURE 12

We now order the monomials associated to the up-triangles in $\Delta_3(n)$ as follows. If $\alpha'\beta'\gamma'$ occurs first in (a', b', c') and $\alpha\beta\gamma$ occurs first in (a, b, c) , then

$$(19) \quad \alpha'\beta'\gamma' > \alpha\beta\gamma$$

if $(a', b', c') \succ (a, b, c)$, or if $(a', b', c') = (a, b, c)$ and the order of (19) is given by the order on the monomials belonging to the up-triangle $U(a, b, c)$.

Proposition 9.3. *The ideal generated by all the variables belonging to the up-triangles of $\Delta_3(n)$, has linear quotients given by the ordering of the monomials above.*

Proof. Let $\alpha\beta\gamma$ occur for the first time in the up-triangle (u, b, c) and let I be the ideal generated by all the larger monomials. We use Lemma 9.2 and show that $I : x_\alpha y_\beta z_\gamma$ is generated by variables.

1. Let $\alpha'\beta'\gamma'$ be in (u, b', c') where $(u, b', c') \succ (u, b, c)$. Suppose $c' \geq c$ (see Figure 11), and so γ belongs to (u, b', c') , since the map $Z : \Delta_3(n) \rightarrow B(n)$ is isotone.

a. If $\beta \geq \beta'$ then β will be in $U(u, b'', c'')$ where $b'' \leq b'$ so $c'' \geq c'$. Then β will also be in $U(u, b', c')$ and since γ is in $U(u, b', c')$ we will have $\alpha'\beta\gamma$ belonging to $U(u, b', c')$. If $\alpha = \alpha'$ then $\alpha\beta\gamma$ would occur in $U(u, b', c')$ contradicting that $\alpha\beta\gamma$ first occurs in $U(u, b, c)$. So $\alpha \neq \alpha'$ and this gives $x_{\alpha'}$ in the colon ideal.

b. Assume now that $\beta < \beta'$. Note that since $b' \leq b$ we have β' belonging to $U(u, b, c)$. Then $\alpha\beta'\gamma$ is already in I by the ordering on the monomials belonging to $U(u, b, c)$, and hence β' is in the colon ideal.

2. A symmetric argument works when $\alpha'\beta'\gamma'$ is in $U(u, b', c')$ and $b' \geq b$.

3. Assume now that $\alpha'\beta'\gamma'$ belongs to the up-triangle $U(u + 1, b', c')$ where the sum of these coordinates is $n - 1$. Either $b' \geq b$ or $c' \geq c$. Suppose the latter, see Figure 12. Then β' belongs to the up-triangle $U(u, b, c)$ due to Y being isotone.

a. If $\beta' > \beta$ in the order given by (18), then $\alpha\beta'\gamma > \alpha\beta\gamma$ and so the former belongs to I and β' is in the colon ideal $I : x_\alpha y_\beta z_\gamma$.

b. If $\beta' = \beta$ note that $y_\beta = y_{\beta'}$ is in $Y(u+1, b, c)$ since $b > b'$. By Lemma 9.1 (applied in the y -direction, not x -direction) either $\alpha\beta\gamma$ is in $U(u+1, b-1, c)$ or in $U(u, b-1, c+1)$. The latter must be the case since $\alpha\beta\gamma$ first occurs in $U(u, b, c)$.

c. If $\beta' < \beta$, then since β' belongs to $U(u, b, c)$, β must belong to $U(u, b-1, c+1)$ and by Lemma 9.1 $\alpha\beta\gamma$ will belong to $U(u, b-1, c+1)$.

In case b. and c. we may continue like this and push $\alpha\beta\gamma$ stepwise to the right, until we get to $(u, b-r, c+r)$ where $c+r = c'+1$, and $(u, b-r, c+r) = (u, b', c'+1)$, so $\alpha\beta\gamma$ is in both $U(u, b'+1, c')$ and $U(u, b', c'+1)$. Note that by X being isotone α belongs to $(u+1, b', c')$. We show that one of α', β' , or γ' is in the colon ideal.

- i. If $\beta = \beta'$ and $\gamma = \gamma'$ then $\alpha\beta\gamma$ belongs to $U(u+1, b', c')$. But this cannot be so since $\alpha\beta\gamma$ occurs first in $U(u, b, c)$.
- ii. If $\beta \neq \beta'$ and $\gamma = \gamma'$ then $\alpha\beta'\gamma$ belongs to $U(u+1, b', c')$ and so $y_{\beta'}$ is in the colon ideal.
- iii. If $\beta = \beta'$ and $\gamma \neq \gamma'$ then $\alpha\beta\gamma'$ belongs to $U(u+1, b', c')$ and so $z_{\gamma'}$ is in the colon ideal.
- iv. Suppose that $\beta \neq \beta'$ and $\gamma \neq \gamma'$. If the edge $((u+1, b'+1, c'+1); 1, 2)$ is a linear syzygy edge then γ is in $Z(u+1, b', c'+1)$ and so $\alpha\beta'\gamma$ belongs to $U(u+1, b', c')$. If $((u+1, b'+1, c'+1); 1, 3)$ is a linear syzygy edge then $\beta \in Y(u+1, b'+1, c')$ and so $\alpha\beta\gamma'$ is belongs to $U(u+1, b', c')$.

4. Suppose then that $\alpha'\beta'\gamma'$ is in $U(u+r, b', c')$ where $r \geq 2$.

a. If $b' \leq b$ and $c' \leq c$ (then at least one inequality is strict) then $\alpha'\beta'\gamma'$ is in $U(u+r, b', c')$ since the map X is isotone, and α also belongs to either $U(u+r-1, b'+1, c')$ or $U(u+r-1, b', c'+1)$. Hence by Lemma 9.1 $\alpha'\beta'\gamma'$ is in one of these up-triangles. We may continue until either $u+r-1 = u+1$, treated in Case 3., or until $b' > b$ or $c' > c$. Assume $c' > c$.

b. We then assume $\alpha'\beta'\gamma'$ is in $(u+r, b', c')$ where $r \geq 2$ and $c' > c$. Note that by Y being isotone and $b' < b$, β' will belong to $U(u, b, c)$ and to $U(u, b-1, c+1)$.

- b1. If $\beta' > \beta$, then $\alpha\beta'\gamma > \alpha\beta\gamma$ and so $y_{\beta'}$ is in the colon ideal.
- b2. If $\beta = \beta'$ then β belongs to $U(u+r, b', c')$. By Y being isotone, β belongs to $U(u, b-1, c+1)$.
- b3. If $\beta' < \beta$, then β is in the up-triangle $U(u, b-1, c+1)$. In both cases b2 and b3, by Lemma 9.1 $\alpha\beta\gamma$ is either in up-triangle $U(u+1, b-1, c)$, not possible, or in $U(u, b-1, c+1)$.

In this way we may continue going rightwards until we get to $(u, b-t, c+t)$ with $c+t = c'$. Then $(u, b'+r, c')$ contains $\alpha\beta\gamma$ and so α is in $(u+r, b', c')$ and $(u+r-1, b'+1, c')$. Then $\alpha'\beta'\gamma'$ is in $(u+r, b', c')$ and since $\alpha\beta\gamma$ occurs first in $U(a, b, c)$ this is not equal to $\alpha'\beta'\gamma'$. By Lemma 9.1 we may push $\alpha'\beta'\gamma'$ down to level $u+r-1$. In this way we can continue until we get $\alpha'\beta'\gamma'$ on level $u+1$ which is treated in Case 3. \square

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