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Tore Fjetland Øgaard

THE WEAK VARIABLE SHARING PROPERTY

Abstract

An algebraic type of structure is shown forth which is such that if it is a characteristic matrix for a logic, then that logic satisfies Meyer's weak variable sharing property. As a corollary, it is shown that **RM** and all its odd-valued extensions $\mathbf{RM}_{2n-1}$ satisfy the weak variable sharing property. It is also shown that a proof to the effect that the "fuzzy" version of the relevant logic **R** satisfies the property is incorrect.

Keywords: characteristic matrix, relevant logics, variable sharing properties.

1. Introduction

The variable sharing property—that $A \to B$ is a logical theorem of a logic only if A and B share a propositional variable—is a hallmark of relevant logics. The property was first shown to hold for the logic **E**—Anderson and Belnap's logic of entailment—as well as Ackermann's logic of "rigorous implication" by Belnap in [2]. One of the logics that this property rather surprisingly turned out *not* to hold for is the logic **RM**—Anderson and Belnap's logic **R** augmented by the *mingle* axiom $A \to (A \to A)$; Meyer and Dunn discovered that $\sim(A \to A) \to (B \to B)$ is a theorem of **RM** (cf. [6]).

Even though Meyer did acknowledge that such theorems do undermine the *raison d'être* of the enterprise of relevant logics, Meyer thought that **RM** was "good enough, when some relevance is desirable" [1, p. 393]. Relevant logics allow for no relevance exceptions: If $A \to B$ is a logical theorem, then A must be relevant to B in the sense that A and B must share a propositional variable. Logics like classical logic, on the other

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hand, allow for exceptions: As a consequence of the interpolation theorem we have that if $A \supset B$ is a logical theorem of classical logic, then either A and B will share a propositional variable, or either $\sim A$ or B are logical theorems. The notion of relevance ensured to hold for logics like **RM** is somewhere in between these two, and is brought out by the *weak variable sharing property* (WVSP), that if $A \to B$ is a logical theorem, then either A and B share a propositional variable, or *both* $\sim A$ and B are logical theorems. This property, then, allows for relevance exceptions, but only for antecedents and consequents which are, respectively, logically rejected and logically forced, as it were.

Meyer showed that **RM** does indeed satisfy (WVSP). Unlike Belnap's original variable sharing property, however, (WVSP) does not automatically extend to any sublogic of a logic for which it holds. Neither does Meyer's original proof of the property easily generalize to other logics. Between classical logic and **RM** there are the n-valued logics $\mathbf{RM}_n$, where $n > 2$. In fact, classical logic can be identified as $\mathbf{RM}_2$. Dunn showed in [5] that any such logic $\mathbf{RM}_n$ for even n's, fail to satisfy (WVSP), and stated, albeit without giving a proof, that every odd-valued $\mathbf{RM}_n$ satisfy (WVSP). Robles and Méndez gave a (WVSP)-proof in [9] which covers the four-valued logic $\mathbf{BN}_4$ as well as an "entailment" version of that logic.[1] This paper generalizes that proof so as to make it also apply to **RM** and all the odd-valued $\mathbf{RM}_n$'s (as well as other logics satisfying certain conditions).

There are two interesting sublogics of **RM** which both fail to satisfy the variable sharing property, but for which the status of the weak version is unsettled, namely **RUE** and **RD**—**R** augmented by, respectively, the axiom $A \wedge \sim A \to B \vee \sim B$ and $(A \to B) \vee (B \to A)$.[2] A proof to the effect that **RD**—"fuzzy **R**"—satisfies (WVSP) was put forth by Yang in [14]. That proof, however, is faulty. This paper ends inconclusively by pointing out the error and thus reopens the question as to whether **RD** satisfies (WVSP). In light of the general (WVSP)-proof, however, one way of making progress on whether **RUE** and **RD** do satisfy (WVSP) is pointed out as interesting.

[1] I am very grateful to Yaroslav Shramko who pointed out that my original proof was quite similar to that given in [9].

[2] The first axiom is sometimes called the axiom of *unrelated extremes*, hence the name ***RUE***, whereas $(A \to B) \vee (B \to A)$ is often called *Dummett's axiom*, hence the name ***RD***.

R	Ax1–Ax12; R1–R2	**RUE**	**R** +Ax13
RD	**R** +A14	**RM**	**R** +Ax15

Table 1. RM and three related logics

2. Logics defined

The consequence relation dealt with in this paper is exclusively the standard Hilbertian one. The following list of axioms and rules are used to define some of the logics in the vicinity of **RM**. Their defining details are found in Tab. 1.

Ax1	$A \to A$
Ax2	$A \to A \vee B$ and $B \to A \vee B$
Ax3	$A \wedge B \to A$ and $A \wedge B \to B$
Ax4	$\neg\neg A \to A$
Ax5	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
Ax6	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
Ax7	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
Ax8	$(A \to \neg B) \to (B \to \neg A)$
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$
Ax11	$A \to ((A \to B) \to B)$
Ax12	$(A \to (A \to B)) \to (A \to B)$
Ax13	$A \wedge {\sim}A \to B \vee {\sim}B$
Ax14	$(A \to B) \vee (B \to A)$
Ax15	$A \to (A \to A)$
R1	$A, B \vdash A \wedge B$
R2	$A, A \to B \vdash B$

Schechter showed in [10] that **R** $\prec$ **RUE** $\prec$ **RD** $\prec$ **RM**, where $\prec$ is the strict sublogic relation. For instance, **R** does not have **RUE**'s defining axiom (Ax13) as a logical theorem, whereas **RD** does, but does not suffice for the "mingle axiom" (Ax15). Lastly, **RM** suffices for deriving Dummett's axiom (Ax14), but **RUE** does not. Only **R** amongst these logics, then, satisfies Belnap's variable sharing property since (Ax13) is an obvious example of a theorem which violates it.

DEFINITION 2.1. The DUGUNDJI SENTENCES (cf. [4, 5, p. 10]) are the following formulas where any p_i is distinct from p_k for $i \neq k$.

$$\begin{array}{ll}
(P_2) & (p_1 \leftrightarrow p_2) \\
(P_3) & (p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_3) \\
(P_4) & (p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_1 \leftrightarrow p_4) \vee \\
 & \quad (p_2 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_4) \vee \\
 & \qquad (p_3 \leftrightarrow p_4) \\
\vdots & \qquad \vdots \\
(P_n) & \bigvee_{1 \leq i < k \leq n} (p_i \leftrightarrow p_k)
\end{array}$$

DEFINITION 2.2. The logic $\mathbf{RM}_n$ for $n \geq 1$ is obtained from $\mathbf{RM}$ by adding every substitutional instance of (P_{n+1}).

Logics in the vicinity of **RM** are sometimes outfitted with truth-constants like the *Church constants* $\bot$ and $\top$, or the *Ackermann constants* $\mathbf{t}$ and $\mathbf{f}$. This paper follows the common practice of defining variable sharing properties for the truth-constant-free fragment of the language.[3]

DEFINITION 2.3. A logic $\mathbf{L}$ has the WEAK VARIABLE SHARING PROPERTY (WVSP) just in case for every truth-constant-free formula A and B, $\vdash_{\mathbf{L}} A \to B$ only if either A and B share a propositional variable, or both $\vdash_{\mathbf{L}} {\sim}A$ and $\vdash_{\mathbf{L}} B$.

To non-trivially satisfy the (WVSP), a logic must have a conditional as a logical constant, and if it is to satisfy (WVSP) while *not* satisfying the full variable sharing property, it must also have a negation. Since the main aim of the paper is to determine some general conditions which are sufficient for a logic to satisfy (WVSP), I have tried to keep the assumptions of the main theorem and lemma to a minimal so that they will also apply to logics with other sets of logical constants.

3. Matrices fit for weak variable sharing

Algebraic structures are in this paper used to provide interpretations for logics, and to do so such structures must provide interpretations for all the logical constants of the logic at hand. A m-ary logical constant $\flat$ will be

[3] See [13] for a different approach, and [7, § 6] for a discussion.

interpreted using a m-ary function $\natural$ on the algebra in question. The arity of such constants and functions will be left to context.

DEFINITION 3.1. A MATRIX for a logic $\mathbf{L}$ with logical constants

$$\langle \sim, \rightarrow, \flat_1, \ldots, \flat_n \rangle,$$

is a structure

$$\mathfrak{A} = \langle \mathcal{K}, \mathcal{D}, \neg, \rightsquigarrow, \natural_1, \ldots, \natural_n \rangle$$

for which

- $\varnothing \neq \mathcal{D} \subseteq \mathcal{K}$
- $\neg$ is a unary function on $\mathcal{K}$
- $\rightsquigarrow$ a binary function on $\mathcal{K}$
- If $\flat_i$ is a m-ary logical constant, then $\natural_i$ is a m-ary function on $\mathcal{K}$.

The elements in $\mathcal{D}$ are the *designated* or "true" elements of $\mathfrak{A}$'s value-space $\mathcal{K}$. $\neg, \rightsquigarrow, \natural_1, \ldots, \natural_n$ are the *defined propositional functions* on $\mathfrak{A}$.

DEFINITION 3.2. An ASSIGNMENT FUNCTION for a matrix $\mathfrak{A}$ is a function I such that for any propositional variable p, $I(p) \in \mathcal{K}$. I is extended to an INTERPRETATION on $\mathfrak{A}$ by letting

$$\begin{aligned} I(\sim A) &=_{df} \neg I(A) \\ I(A \rightarrow B) &=_{df} I(A) \rightsquigarrow I(B) \\ I(\flat_i(A_1, \ldots, A_m)) &=_{df} \natural_i(I(A_1), \ldots, I(A_m)) \end{aligned}$$

- A formula A IS TRUE IN $\mathfrak{A}$ UNDER I just in case $I(A) \in \mathcal{D}$.
- A formula A IS VALID IN $\mathfrak{A}$ just in case it is true in $\mathfrak{A}$ under every assignment function I.

DEFINITION 3.3. A matrix $\mathfrak{A}$ is called a CHARACTERISTIC MATRIX for a logic $\mathbf{L}$ just in case $\vdash_{\mathbf{L}} A$ if and only if A is valid in $\mathfrak{A}$.

DEFINITION 3.4. A WVSP-MATRIX $\mathfrak{W}$ for a logic $\mathbf{L}$ is a matrix for $\mathbf{L}$ for which there exists sets $\mathcal{S}_1$ and $\mathcal{S}_2$ such that

- $\varnothing \neq \mathcal{S}_i \subseteq \mathcal{K}$, for $i \in \{1, 2\}$

- $\mathcal{S}_1$ and $\mathcal{S}_2$ are both closed under all the defined propositional functions of $\mathfrak{W}$
- $a \in \mathcal{R}$ & $b \in \mathcal{S}_1 \Longrightarrow a \rightsquigarrow b \in \mathcal{U}$
- $a \in \mathcal{S}_2$ & $b \in \mathcal{U} \Longrightarrow a \rightsquigarrow b \in \mathcal{U}$,

where $\mathcal{R} =_{df} \{x \in \mathcal{K} \mid \neg x \notin \mathcal{D}\}$ and $\mathcal{U} =_{df} \mathcal{K} \setminus \mathcal{D}$.

THEOREM 3.5. *If a logic has a WVSP-matrix as a characteristic matrix, then it satisfies (WVSP).*

PROOF: Assume that $\mathbf{L}$ has $\mathfrak{W}$ as a characteristic WVSP-matrix. Furthermore, let $\vdash_{\mathbf{L}} A \to B$, where A and B are truth-constant free formulas which share no propositional variables. For contradiction, then, assume that either $\nvdash_{\mathbf{L}} {\sim}A$ or $\nvdash_{\mathbf{L}} B$. The theorem is proven by showing that both disjuncts lead to a contradiction.

Assume first that $\nvdash_{\mathbf{L}} {\sim}A$. Since $\mathfrak{W}$ is a characteristic matrix for $\mathbf{L}$, there is an assignment function I such that $I({\sim}A) \notin \mathcal{D}$. It follows that $I(A) \in \mathcal{R}$. Let I' be just like I, except that $I'(p) \in \mathcal{S}_1$ for every propositional variable p occurring in B. Since $\mathcal{S}_1$ is closed under every propositional function, it follows by an easy induction that $I'(B) \in \mathcal{S}_1$. I' is well-defined since A and B do not share any propositional variables. Furthermore, $I'(A) = I(A)$. Since, then, $I'(A) \in \mathcal{R}$ and $I'(B) \in \mathcal{S}_1$, it follows by the definition of a WVSP-matrix that $I'(A) \rightsquigarrow I'(B) \in \mathcal{U}$, and so $A \to B$ is not true in $\mathfrak{W}$ under I'. However, $A \to B$ is a logical theorem of $\mathbf{L}$ and so valid in $\mathfrak{W}$. Contradiction.

Secondly, assume that $\nvdash_{\mathbf{L}} B$. Since $\mathfrak{W}$ is a characteristic matrix there is an assignment function I such that $I(B) \notin \mathcal{D}$. By definition, then, $I(B) \in \mathcal{U}$. Let I' be just like I, except that $I'(p) \in \mathcal{S}_2$ for every propositional variable p occurring in A. As above it follows from the fact that $\mathcal{S}_2$ is closed under every propositional function, that $I'(A) \in \mathcal{S}_2$. I' is well-defined since A and B do not share any propositional variables. Furthermore, $I'(B) = I(B)$. Since, then, $I'(A) \in \mathcal{S}_2$ and $I'(B) \in \mathcal{U}$, it follows by the definition of a WVSP-matrix that $I'(A) \rightsquigarrow I'(B) \in \mathcal{U}$, and so $A \to B$ is not true in $\mathfrak{W}$ under I'. However, $A \to B$ is a logical theorem of $\mathbf{L}$ and so valid in $\mathfrak{W}$. Contradiction. $\square$

Definition 3.6. A propositional fixed-point of a matrix

$$\mathfrak{A} = \langle \mathcal{K}, \mathcal{D}, \neg, \rightsquigarrow, \natural_1, \ldots, \natural_n \rangle$$

is any point $\mathfrak{f} \in \mathcal{K}$ such that

$$\begin{array}{ll} (1) & \neg\mathfrak{f} = \mathfrak{f} \\ (2) & \mathfrak{f} \rightsquigarrow \mathfrak{f} = \mathfrak{f} \\ (3) & \natural_i(\mathfrak{f}, \ldots, \mathfrak{f}) = \mathfrak{f} \quad (i \leq n) \end{array}$$

Lemma 3.7. *A matrix $\mathfrak{W}$ is a WVSP-matrix if it satisfies the following three conditions, where a, b are any elements in $\mathcal{K}$:*

- *(Fixed-point) There exists a propositional fixed-point $\mathfrak{f}$ such that $\mathfrak{f} \in \mathcal{D}$*
- *($MT_{\mathfrak{f}}$) $a \rightsquigarrow \mathfrak{f} \in \mathcal{D} \Longrightarrow \neg a \in \mathcal{D}$*
- *($MP_{\mathfrak{f}}$) $\mathfrak{f} \rightsquigarrow b \in \mathcal{D} \Longrightarrow b \in \mathcal{D}$*

Proof: Let $\mathcal{S}_1 = \mathcal{S}_2 = \{\mathfrak{f}\}$. We then only need to show that if $a \in \mathcal{R}$, then $a \rightsquigarrow \mathfrak{f} \in \mathcal{U}$, and that $\mathfrak{f} \rightsquigarrow b \in \mathcal{U}$ for every $b \in \mathcal{U}$.

Assume first, then, that $a \in \mathcal{R} =_{df} \{x \in \mathcal{K} \mid \neg x \notin \mathcal{D}\}$. If $a \rightsquigarrow \mathfrak{f} \notin \mathcal{U}$, then by definition $a \rightsquigarrow \mathfrak{f} \in \mathcal{D}$. It follows then from $(MT_{\mathfrak{f}})$ that $\neg a \in \mathcal{D}$ which contradicts the assumption that $a \in \mathcal{R}$.

Assume now that $b \in \mathcal{U}$. If $\mathfrak{f} \rightsquigarrow b \notin \mathcal{U}$, $\mathfrak{f} \rightsquigarrow b \in \mathcal{D}$. It then follows from $(MP_{\mathfrak{f}})$ that $b \in \mathcal{D}$. This, however, contradicts the assumption that $b \in \mathcal{U} =_{df} \mathcal{K} \setminus \mathcal{D}$. □

The above lemma, then, captures three properties which together are sufficient for making a matrix into a WVSP-matrix, namely the existence of a designated propositional fixed-point, and that the algebraic equivalent of both modus ponens and modus tollens are validated at least with regards to the propositional fixed-point. These properties, as we shall see, are satisfied by one of the characteristic matrices for **RM** as well as the characteristic matrices for its odd-valued extensions.

Definition 3.8. Let $n \geq 1$. The $2n$-element Sugihara matrix $\mathfrak{S}_{2n}$ consists of the elements $\mathcal{K} = \{-n, \ldots, -1, 1, \ldots, n\}$. The $2n-1$-element Sugihara matrix $\mathfrak{S}_{2n-1}$, on the other hand, has value-space

$$\mathcal{K} = \{-(n-1), \ldots, -1, 0, 1, \ldots, n-1\}.$$

The $\mathbb{Z}$-element Sugihara matrix $\mathfrak{S}_{\mathbb{Z}}$ has $\mathcal{K} = \mathbb{Z}$. The set of designated elements is in each case defined as $\mathcal{D} =_{df} \{n \in \mathcal{K} \mid 0 \leq n\}$. The propositional functions $\neg, \rightsquigarrow, \sqcap, \sqcup$ are for every Sugihara matrix defined as follows:

$$\begin{array}{lll} \neg a & =_{df} & -a \\ a \sqcap b & =_{df} & min\{a, b\} \\ a \sqcup b & =_{df} & max\{a, b\} \\ a \rightsquigarrow b & =_{df} & \begin{cases} \neg a \sqcup b & \text{if} \quad a \leq b \\ \neg a \sqcap b & \text{else} \end{cases} \end{array}$$

Dunn showed in [5] that each $\mathbf{RM}_n$, for $n \geq 1$, has the n-valued Sugihara matrix as a characteristic matrix (cf. [5, thm. 9 & cor. 2]).[4] Furthermore, Meyer showed that $\mathfrak{S}_{\mathbb{Z}}$ is a characteristic matrix for $\mathbf{RM}$ (cf. [1, p. 415, thm. 4]).[5]

As noted in [5, p. 10], each Dugundji sentence P_n, for $n \geq 2$, is invalid in $\mathfrak{S}_i$ for $i \geq n$. Furthermore, it is easy to verify that $\mathfrak{S}_2$ is in fact the two-element Boolean algebra, and so $\mathbf{RM}_2$ simply amounts to classical logic. $\mathbf{RM}_1$, on the other hand, amounts to the trivial logic since every substitutional instance of $p_1 \leftrightarrow p_2$ is a logical axiom of $\mathbf{RM}_1$, and the logic validates modus ponens. It follows, then, that there are infinitely many $\mathbf{RM}$-logics which can be ordered according to strength as follows:

$$\mathbf{RM} \prec \ldots \mathbf{RM}_n \prec \mathbf{RM}_{n-1} \prec \ldots \prec \mathbf{RM}_1.$$

Dunn showed that for (WVSP) fails to hold for every $\mathbf{RM}_{2n}$ $(n \geq 1)$ on account of

$$(p \wedge {\sim}p) \rightarrow (q_1 \vee (q_1 \rightarrow q_2) \vee (q_2 \rightarrow q_3) \vee \ldots \vee (q_{n-1} \rightarrow q_n))$$

being valid in $\mathfrak{S}_{2n}$. It is easy to verify that the consequent is not valid in $\mathfrak{S}_{2n}$, however: By assigning $-n$ to q_n, the consequent will be evaluated to -1. Since the antecedent and consequent do not share any propositional variables, it follows, therefore, that $\mathbf{RM}_{2n}$—all the even-valued extensions

[4] $\mathbf{RM}_3$ is often axiomatized as $\mathbf{RM}$ augmented by the axiom $A \vee (A \rightarrow B)$. That these axiomatizations, then, are equivalent, follows from Dunn's result, and Brady's result in [3] that $\mathfrak{S}_3$ is characteristic also for $\mathbf{RM}_3$ axiomatized with the other axiom.

[5] Dunn, modifying an example by Meyer, showed that $\mathfrak{S}_{\mathbb{Z}}$ is not *strongly* characteristic for $\mathbf{RM}$. Thus $\mathbf{RM}$ is not strongly complete with regards to interpretations over $\mathfrak{S}_{\mathbb{Z}}$. He showed, however, that the Sugihara matrix over $\mathbb{Q}$ is strongly characteristic for $\mathbf{RM}$ (cf. [5, p. 12]).

of **RM**—cannot satisfy (WVSP).

Dunn also stated, albeit without proof, that the odd-valued extensions $\mathbf{RM}_{2n+1}$ for $n \geq 1$ satisfy (WVSP) (cf. [5, cor. 5]).[6] That this is indeed correct, is an easy consequence of the above lemma and theorem:

COROLLARY 3.9. **RM** and every $\mathbf{RM}_{2n-1}$, satisfy (WVSP).

PROOF: 0 is a propositional fixed-point for $\mathfrak{S}_{\mathbb{Z}}$ as well as of each $\mathfrak{S}_{2n-1}$, where $n \geq 1$. Furthermore, every such Sugihara matrix validates both modus ponens and modus tollens generally, and so also with regards to the propositional fixed-point. By Lem. 3.7, then, these matrices are WVSP-matrices. Since they are also characteristic matrices for **RM** and $\mathbf{RM}_{2n-1}$, it follows from Thm. 3.5 that these logics satisfy (WVSP). □

3.1. Meyer's WVSP-proof in comparison

As we shall soon see, there are **RM**-related logics for which it is currently unknown whether (WVSP) holds. With that in mind it is important to get clear on which features are utilized in the two types of WVSP-proof available—the one displayed in this paper, and that used in Meyer's original proof for **RM**.[7] This subsection briefly outlines Meyer's proof and compares it with the one displayed in this paper.

As already mentioned, the method used in above theorem is a generalization of that found in Robles and Méndez' [9, prop. 8.5].[8] The above corollary shows, then, that the method is quite powerful as it generalizes

[6]Dunn, however, stated that (WVSP) *fails* to hold for $\mathbf{RM}_1$ (cf. [5, cor. 5]). This is evidently incorrect since $\vdash_{\mathbf{RM}_1} A$ for *every* formula A.

[7]Meyer's proof can be found as *RM84* in [1, p. 417].

[8]I should also mention that Robles' gave in [8] a proof that $\mathbf{RM}_3$ satisfies (WVSP) which also uses the same type of approach as in [9]. That proof, however, contains a regrettable flaw. The following (nitpickingly) explains the error:

Robles' proof is a proof by contradiction wherein it is assumes (1) that $\vdash_{\mathbf{RM}_3} A \to B$, and (2) that A and B are such as to share no propositional variable, yet (3) either $\nvdash_{\mathbf{RM}_3} {\sim}A$ or $\nvdash_{\mathbf{RM}_3} B$. The heart of her error is that she takes the latter assumption to yield that there are interpretations I and I' over the $\mathbf{RM}_3$ matrix such that either $I({\sim}A) \notin \mathcal{D}$ or $I'(B) \notin \mathcal{D}$. The proof is then split into two cases with the latter one left to the reader. In the first, however—where $I({\sim}A) \notin \mathcal{D}$ is the leading assumption—she uses *both* I and I' to construct an interpretation I'' which is such that $I''(A \to B) \notin \mathcal{D}$ where the fact appealed to is that $I(A) = 1$ *and* $I'(B) = -1$. The existence of I, however, is conditioned upon $\nvdash_{\mathbf{RM}_3} {\sim}A$ being the case, and the existence of I' is similarly conditioned upon $\nvdash_{\mathbf{RM}_3} B$ being the case, and so unless both these hold, one cannot assume that both I and I' exist.

to cover many logics. This contrasts to Meyer's original proof which so far at least, has not been made to work for other logics.

The method used here relies on the availability of propositionally closed substructure—subsets of the value-space of the algebra which are closed under all the operations used for interpreting the propositional connectives of our language. In the case of the **RM**-logics, this is realized by the presence of a fixed-point: 0 is a fixed-point for every propositional function in both $\mathfrak{S}_{\mathbb{Z}}$ as well as in the odd-numbered Sugihara matrices. Meyer's original proof that **RM** satisfies (WVSP) in contrast, does not rely on such a fixed-point. Rather, it relies on a certain sort of translation being possible. As I will show, however, it can be seen as a variant of the main theorem presented in this paper.

As in the main theorem, Meyer proof relies on the logic having a characteristic matrix. $\mathfrak{S}_{\mathbb{Z}^*}$, Meyer showed, is yet another characteristic matrix for **RM**, where $\mathbb{Z}^*$ is $\mathbb{Z} \setminus \{0\}$. An outline of Meyer's proof, then, goes as follows: Assume that $A \to B$ is a logical theorem and that A and B fail to share any propositional variables. For contradiction it is then assumed that there is some assignment function which makes A true, i.e., that there is some I such that $I(A) \geq 1$. From I a new interpretation I' is defined which assigns to any propositional variable not occurring in A the value 1, and to any p occurring in A the value $I(p) + I(p)$. A little calculation will then show that $I'(A) > 1$ and $I'(B) = \pm 1$, and therefore that $I'(A \to B) = \neg I'(A) \sqcap I'(B) = \neg I'(A) < -1$ contradicting the assumption that $A \to B$ is a logical theorem and hence valid in $\mathfrak{S}_{\mathbb{Z}^*}$. "By parity of reasoning," as Meyer put it, one similarly obtains a contradiction from the assumption that there is some I which fails to make B true.

Notice that $\mathcal{S}_1 =_{df} \{-1, 1\}$ and $\mathcal{S}_2 =_{df} \mathbb{Z}^* \setminus \mathcal{S}_1$ are both closed under the propositional function corresponding to all the logical constants of **RM**. As in the above theorem, let

$$\mathcal{U} =_{df} \mathcal{K} \setminus \mathcal{D} = \mathbb{Z}^* \setminus \{x \in \mathbb{Z}^* \mid x \geq 1\} = \{x \in \mathbb{Z}^* \mid x \leq -1\},$$

and let $\mathcal{U}' =_{df} \mathcal{U} \setminus \mathcal{S}_1$ and $\mathcal{R}' =_{df} \mathcal{R} \setminus \mathcal{S}_1 = \{x \in \mathbb{Z}^* \mid x \geq 2\}$. It is then easy to verify that if $a \in \mathcal{R}'$ and $b \in \mathcal{S}_1$, then $a \rightsquigarrow b \in \mathcal{U}$, and that if $a \in \mathcal{S}_2$ and $b \in \mathcal{U}'$, then $a \rightsquigarrow b \in \mathcal{U}$.

Meyer's proof, then, relies on the fact that if $I(A) \in \mathcal{R}$, then by *translating* the interpretation I by setting $I'(p) = I(p) + I(p)$, $I'(A) \in \mathcal{R}'$. Similarly, if $I(B) \in \mathcal{U}$, one needs to prove that the translated interpretation I' is such that $I'(B) \in \mathcal{U}'$. Of course, translating thus does work in case of $\mathfrak{S}_{\mathbb{Z}^*}$, but it is not evident that such a translation will work in other cases. A case in point is the finite Sugihara matrices for which $I(p) + I(p)$ will simply not be an element of the matrix in many cases.

Meyer's proof, then, is very much alike the one shown forth in this paper. Whereas the latter, however, works effortlessly when the matrix in question has a propositional fixed-point, a Meyer-type translation may make the presence of such a point redundant. In the search for a suitable characteristic matrix for a logic, however, it might at least be easier to try to find one with a propositional fixed-point, rather than one admitting of Meyer's type of translation.[9]

Although the proof offered here does contribute towards a more general way of proving that a logic satisfies (WVSP), the fact that **RM** and its odd-valued extensions satisfy (WVSP) is not news. What is a more recent claim, however, is that the weaker logic **RD** also satisfied (WVSP). The next section goes through an incorrect WVSP-proof and affirms the unsettled nature of the question as to whether either **RUE** or **RD** do in fact satisfy the weak variable sharing property.

4. An incorrect WVSP-proof

Yang has offered a proof to the effect that **RD** satisfies (WVSP). This section explains why that proof is incorrect.

Yang's proof can be found as theorem 2.ii in [14]. As it stands it *is* correct had it only been claimed to hold for $\mathbf{RM}_3$ rather than for **RD**.[10]

[9] A further cause for thinking that making Meyer's translation-approach work for other logics will be difficult is the fact that the propositionally closed substructure $\{-1, 1\}$ of $\mathfrak{S}_{\mathbb{Z}^*}$ contains the values any assignment function must assign to the Ackermann constant $\mathbf{t}$ and its negation $\mathbf{f}$. The Ackermann constant is axiomatized using the axioms $\mathbf{t}$ and $\mathbf{t} \to (A \to A)$. A characteristic matrix for a logic will suffices for showing that $\mathbf{t}$ can be added conservatively, and so one might hope that $\{I(\mathbf{f}), I(\mathbf{t})\}$ would be the needed propositionally closed substructure of a characteristic matrix for, say, **RD** as well. However, it cannot be a propositionally closed substructure of the characteristic matrix for *any* logic weaker than **RM** yet contained in **R** as it would require that $\mathbf{f} \to \mathbf{t}$ be a logical theorem of the logic, and adding $\mathbf{f} \to \mathbf{t}$ as a logical axiom to **R** yields the logic **RM** ($\mathbf{f} \to \mathbf{t}$ yields in **R** $\sim(A \to B) \to (B \to A)$ (cf. [12, p. 33]), which yields the mingle axiom $A \to (A \to A)$ if added to **R** (cf. [10, pp. 122f])). Thus the propositionally closed substructure needed to make Meyer's proof work cannot be identified as $\{I(\mathbf{f}), I(\mathbf{t})\}$ which makes the search for a suitable translation even harder.

[10] I should note that Yang's definition of **RD**—his name for it is ***FR***, "fuzzy **R**"—

Yang notes that the axioms of **RD** are all true on every interpretation over the $\mathbf{RM}_3$ algebra, which is true, but insufficient for deriving the wanted conclusion. Yang assumes that A and B are formulas which do not share any propositional variables and that either $\nvdash_{\mathbf{RD}} \sim A$ or $\nvdash_{\mathbf{RD}} B$. The goal, then, is to show that there is an interpretation in which $A \to B$ fails to be true, and therefore that $A \to B$ fails to be a theorem of the logic. The proof is split into three cases with all of them making the same mistake: from the assumption that $\nvdash_{\mathbf{RD}} C$ to infer that there is a $\mathbf{RM}_3$-interpretation I such that $I(C) = -1$. The proof, then, fails to provide an interpretation in which $A \to B$ fails to hold, and therefore also that $A \to B$ fails to be a theorem of **RD**.

Let's briefly look at an example where Yang's proof goes wrong: Let A be the formula $r \wedge \sim r$ and B the formula $\sim(p \to p) \to (q \to q)$, where r, then, is distinct from both p and q. Now it is easy to verify that $\nvdash_{\mathbf{RD}} \sim(p \to p) \to (q \to q)$ for distinct propositional variables p and q.[11] However, there are no $\mathbf{RM}_3$-interpretation I such that $I(\sim(p \to p) \to (q \to q)) = -1$, nor any I' such that $I'((r \wedge \sim r) \to (\sim(p \to p) \to (q \to q))) = -1$ since both these formulas are theorems of **RM** and so are both valid in the $\mathbf{RM}_3$-matrix.

This, then, reopens the question whether logics like **RD**, as well as the other logics [14] calls "relevant fuzzy logics," do in fact satisfy (WVSP). Additionally, whether **RUE** satisfies (WVSP) is also an open question.

The heart of the error in Yang's proof is easily seen to be that the $\mathbf{RM}_3$-matrix is not a *characteristic* matrix of **RD**. Both Meyer's original proof, as well as that shown forth in this paper rely on the logic in question having a characteristic matrix of a certain sort. As far as I know, neither **RD** nor **RUE** have been shown to have a characteristic matrix. As noted

is different in that Yang defines it as including the Ackermann constants $\mathbf{t}$ and $\mathbf{f}$ and defines $\sim A$ as $A \to \mathbf{f}$. If one only allows $\mathbf{f}$ to occur thus, it is easy to show, however, that the logics are theorem-wise identical. Yang also states the linearity axiom as $((A \to B) \wedge \mathbf{t}) \vee ((B \to A) \wedge \mathbf{t})$, but notes (cf. [14, prop. 2.iii.3]) that $(A \to B) \vee (B \to A)$ is a theorem of all the logics that he considers. Yang also defines the logics to have the fusion connective as a primitive one. In **RD**, however, it is definable using negation and the conditional, and so adding it yields a conservative extension. Lastly, I should also note that his proof is stated to hold not only for **RD**, but for eight different logics in total—see [14, def. 5]—amongst them **RM** and its distributionless variant. His proof does not hold for any of these logics for the same reason as it doesn't work for **RD**.

[11] A model is easily found using `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney [11].

above, then, finding one with a propositional fixed-point would suffice to show that the logic in question satisfies (WVSP). Neither of the available WVSP-proofs, I should stress, indicate that such a characteristic matrix is *required* for the property to hold true, and so it might be possible to find a WVSP-proof which utilizes different properties. Alas, this paper must end inconclusively on this matter, but leaves both the status of a characteristic matrix and that of (WVSP) for both **RUE** and **RD** as interesting open questions for further research.

5. Summary

This paper has shown forth a certain algebraic structure which was used to prove Meyer's weakened version of the variable sharing property—that if $A \to B$ is a logical truth then *either* do A and B share a propositional variable, or both $\sim A$ and B are logical theorems. It was shown that if a logic has such a structure as its characteristic matrix, then it satisfies Meyer's property. As a consequence of results by Meyer and Dunn for the logics **RM** as well as its odd-valued extensions $\mathbf{RM}_{2n-1}$ (for $n \geq 1$), it was then shown that these logics have such algebraic structures as their characteristic matrices and therefore satisfy Meyer's property. The paper also showed that a proof of Meyer's property for the "fuzzy" extension of the relevant logic **R** is incorrect.

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The Weak Variable Sharing Property

Tore Fjetland Øgaard
University of Bergen
Department of Philosophy
Postboks 7805, 5020 Bergen
Bergen, Norway
e-mail: Tore.Ogaard@uib.no