Stability is Unnatural

by
Anton Pletten

Master's Thesis in Mathematics

Department of Mathematics
University of Bergen

1st June 2023

Acknowledgements

First and foremost I want to thank my supervisor Bjørn Ian Dundas for suggesting me the topic of the thesis and his invaluable support in writing it. I would also like to extend a thanks to everyone at the topology group I have had the pleasure of working with. I am very grateful for the friends I have made at the university and would like to extend a special thanks to Andreas and Steffen for their companionship through the years.

Last, but not least, I would like to express my gratitude to my family. Especially I would like to thank my brother John Otto, without whom I would never have found the pleasure of mathematics.

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Chapter 1

Introduction

Higher Hochschild homology can be defined as the homology of the Loday functor L, which takes as input a simplicial set X and a commutative algebra A and gives as output a simplicial commutative algebra $L(X, A)$. It can be thought of as a generalization of the classical Hochschild homology of a commutative algebra in the sense that the homology of $L(S^1, A)$ coincides with the Hochschild homology of A. Originally, the term was coined higher order Hochschild homology due to [Pir00] considering the homology of algebras over S^n.

On the other hand, fixing the algebra gives a resulting homology of spaces that has been shown to behave nicely in many settings. If a pair of spaces are homotopy equivalent they will for instance have isomorphic higher Hochschild homology. Furthermore, if the algebra is smooth, then the resulting homology has been shown to only depend on the stable homotopy type of the spaces. However, the stability of higher Hochschild homology may not be assumed in general, with a counter example provided in [Ten16] and [DT18]. Attempts to systematically investigate the stability of more cases has been made by [LR22] with both positive and negative results given.

We approach the subject from the following point of view: We know that the higher Hochschild homology is a stable invariant for a fixed smooth algebra with a counter example to this being true in general provided by [Ten16]. However, this may seem odd, since we can after all make a free simplicial resolution of an algebra, with the resolution in each degree free and consequently giving a stable homology of spaces. This is true even if the algebra we started with does not give a stable homology of spaces. The question then arises how it is possible for a free simplicial resolution of an algebra to possess the stability observed without extending the property to the algebra it resolves. The answer to the question turns out to be that the stability of the resolution is not natural.

The original counter example featured a comparison of the homology of the dual rational numbers over the torus and a wedge sum of spheres. We show that this unstable behaviour can be observed in the attaching map of the torus, which is of course detected by the fundamental group. Even so, we demonstrate that the unstable behaviour still persists in an analogous higher connected case, showing that the instability of the counter example is not merely an artefact of the low connectedness of the components of the torus. Further we build upon the idea that the instability is caused by the algebra by relating it to free algebras with known stable homology of spaces. Through calculations with Greenlees spectral sequence, we investigate the stability by means of the attaching map, showing that the contrasting stable and unstable properties is expressible in our case as a particular lift of the attaching map up to homotopy.

Chapter 2

Simplicial Homotopy Theory

2.1 Simplicial Sets

Definition 2.1.1 (Simplex Category). Let Δ denote the *simplex category* consisting of objects the total ordered sets

$$[n] = (0 \to 1 \to \cdots \to n),$$

where $n \geq 0$ along with morphisms the order preserving functions $f : [n] \to [m]$ satisfying the property

$$i \leq j \implies f(i) \leq f(j).$$

Definition 2.1.2 (Simplicial Set). A *simplicial set* X is a functor

$$X : \Delta^{\mathrm{op}} \to \mathbf{Sets},$$

where **Sets** is the category of sets. We denote the corresponding category of simplicial sets by **sSets**. More generally a functor

$$X : \Delta^{\mathrm{op}} \to \mathcal{C}$$

gives a *simplicial object* in a category $\mathcal{C}$. We denote the corresponding category of simplicial objects by $s\mathcal{C}$.

Simplicial sets are of course the simplicial objects in **Sets**, but they admit a lot more structure than one may assume for a general category of simplicial objects. Specifically they admit the structure of a simplicial model category, and even more they also have the property that weak equivalences are preserved by pullbacks over fibrations and pushouts along cofibrations. In general one cannot assume this to be true for general categories of simplicial objects. However, one can use the structure of the underlying simplicial sets to show that simplicial groups, modules and algebras actually do have the structure of simplicial model categories [GJ09, lemma 5.1.]. For definitions of closed model categories, simplicial model categories and a proper treatment of them, we refer to [GJ09, ch. II].

However, there is an easier way to describe simplicial sets using the fact that the set of cosimplicial face and degeneracy maps generates all the relations of Δ [Mac98, p. VII.5.1]. These satisfy a certain set of cosimplicial identities whose dual in simplicial sets is the following:

Definition 2.1.3 (Simplicial Identities). Let X be a simplicial set. The collection of face maps

$$d_i : X_n \to X_{n-1}, \quad 0 \leq i \leq n$$

and degeneracy maps

$$s_j : X_n \to X_{n+1}, \quad 0 \leq j \leq n$$

satisfy a set of identities

$$\begin{cases} d_i d_j = d_{j-1} d_i, & i < j \\ d_i s_j = s_{j-1} d_i, & i < j \\ d_j s_j = d_{j+1} s_j = 1 & \\ d_i s_j = s_j d_{i-1}, & i > j+1 \\ s_i s_j = s_{j+1} s_i, & i \leq j \end{cases}$$

that we call the *simplicial identities*.

Due to these being dual to the cofaces and codegeneracies generating all relations of Δ, it is sufficient to define a simplicial set by explicitly writing down the sets $X_n = X[n]$ along with the face and degeneracy maps.

Example 2.1.4 (Standard Simplex, Boundary and Horn). There is a simplicial set

$$\Delta^n = \hom_{\Delta}(-, [n])$$

called *the standard n-simplex*. Due to the Yoneda Lemma [Lei14] there exists a natural bijection

$$\hom_{\mathbf{sSets}}(\Delta^n, X) \cong X_n$$

given by taking the standard simplex $1_{[n]} \in \Delta^n_n$ and associating each simplicial map $\phi : \Delta^n \to X$ to the simplex $\phi(1_{[n]})$.

Explicitly we will denote an element $(\alpha : [k] \to [n]) \in \Delta^n_k$ by its image. Suppressing the arrows for the objects of the simplex category, we then characterize $1_{[n]}$ by $(01 \dots n)$ with degeneracy and face maps $s_i(1_n) = (01 \dots ii \dots n)$ and $d_i(1_n) = (01 \dots \hat{i} \dots n)$, where $\hat{i}$ means that i is not in the image of α.

There is a particularly important subcomplex $\partial\Delta^n \subseteq \Delta^n$ called the *boundary.* It is defined as the smallest simplicial set containing all faces $d_i(1_{[n]})$, $0 \leq i \leq n$ of the standard simplex $1_{[n]}$. This complex also has a subcomplex called the k-th *horn* Λ^n_k, which is defined as the smallest subcomplex of Δ^n containing all faces save $d_k(1_{[n]})$.

We will use the standard simplex frequently when constructing simplicial sets later on. We remark that a characteristic property of the standard simplex is that it has exactly one non-degenerate simplex of degree n, that is

$$(01\ldots n) = 1_{[n]} \in \Delta^n_n,$$

Furthermore, any simplicial map $\Delta^n \to X$ is defined by how it acts on this simplex, due to the simplicial maps commuting with face and degeneracy maps. Therefore designating a map by its action on $1_{[n]}$ is well defined, which indeed is part of the proof of the Yoneda lemma and something we will make use of later on.

Example 2.1.5 (Simplicial Circle). We define the *simplicial circle* $S^1 = \Delta^1/\partial\Delta^1$ and more generally the *simplicial sphere* $S^n = \Delta^n/\partial\Delta^n$. They are characterized by only having one nondegenerate simplex in the 0-th and in the n-th degree. Following the latest remark we write explicitly

$$S^1_1 = \{[00], 01\}\,,$$

where the equivalence class is given by relating $00 \sim 11$. As we go forward we will not be too careful with denoting the equivalence classes arising from quotients of simplicial sets, but will instead usually just refer to them by their lowest representative.

Sometimes we need to distinguish a base point $x_0 = (\Delta^0 \to X)$, or equivalently $x_0 \in X_0$, of a simplicial set X. In this case we say that (X, x_0) is *pointed.*

Definition 2.1.6 (Wedge Sum and Smash Product). Given a pair of pointed simplicial sets $(X, x_0), (Y, y_0)$ there is a simplicial set called the *wedge sum* defined by

$$X \vee Y = X \bigsqcup Y/\sim,$$

where the relation is given by $x_0 \sim y_0$. This construction also gives rise to another called the *smash product* defined by

$$X \wedge Y = X \times Y/(X \vee Y).$$

Example 2.1.7 (Suspension). Smashing a pointed simplicial set (X, x_0) with the pointed simplicial circle (S^1, s_0) gives us the *suspension* $\Sigma X = S^1 \wedge X$.

Remark 2.1.8. Note that it is possible to construct the suspension in an equivalent manner without alluding to base points by taking the double cone $CX \bigsqcup_X CX$, where $CX = X \times \Delta^1 / X \times \{0\}$ and where we have made the identification $X \cong X \times \{0\}$ in CX. In topological spaces the reduced suspension and unreduced suspension of CW-complexes are homotopy equivalent.

For CW-complexes we have the following theorem:

Theorem 2.1.9. *If X, Y is a pair of CW-complexes, then*

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y).$$

Proof. See [Hat01, proposition 4I.1]. □

Since the realization of a simplicial set is a CW-complex and the realization preserves finite products and colimits, we have that the theorem is true in the paradigm of simplicial sets as well.

Example 2.1.10 (Nerve). Given a small category $\mathcal{C}$ we have a simplicial set $\mathcal{NC}$ called the *nerve* of $\mathcal{C}$ given by

$$\mathcal{NC}_n = \hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C}),$$

where $\mathbf{n}$ is the ordinal number $[n]$ viewed as a category. An n-simplex of the nerve is then a string of composable arrows

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} c_n$$

in $\mathcal{C}$ of length n.

Example 2.1.11. Identifying a group G with the category with one object $*$ and morphisms $g : * \to *$ for each $g \in G$ with composition given by multiplication, we have that the nerve $\mathcal{N}G$ is a simplicial set whose realization is $K(G, 1)$.

2.2 Homological Algebra

We begin the section with a note that we will generalize some constructions to objects in abelian categories. Since we are mainly interested in abelian groups and sometimes also more generally modules over some commutative ring k, we will for the sake of not digressing too much avoid making a proper definition. The reader is welcome to think of abelian groups and k-modules whenever we make reference to objects in such a category.

Definition 2.2.1 (Chain Complex). We define a *chain complex* C of objects in an abelian category $\mathcal{C}$ to be a sequence

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

of composable arrows such that $\partial_n \circ \partial_{n+1} = 0$ for all n.

Remark 2.2.2. For our purposes we will only consider nonnegative chain complexes, that is, chain complexes with C_n the zero object for $n < 0$. Note that the condition $\partial_n \circ \partial_{n+1} = 0$ only implies that $\operatorname{Im}\partial_{n+1} \subseteq \operatorname{Ker}\partial_n$, whereas when the image and kernel are equal we say that the sequence is *exact*. Furthermore we may measure the deviation from said exactness by the homology groups

$$H_n C = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1},$$

which are of course well defined by the above. We will refer to the category of nonnegative chain complexes in an abelian category $\mathcal{A}$ along with morphisms the chain maps by $\mathbf{Ch}_+(\mathcal{A})$.

Definition 2.2.3 (Chain Map and Homotopy). A *map of complexes* $f : C \to C'$, equivalently a *chain map*, is a collection of maps $f = \{f_n : C_n \to C'_n\}$ so that the diagram

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\partial_{n+1}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \xrightarrow{\partial_{n+1}} & C'_{n+1} & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{\partial_n} & C'_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \end{array}$$

commutes. A *chain homotopy* h between two chain maps $f, g : C \to C'$ is a collection of maps $h = \left\{h_n : C_n \to C'_{n+1}\right\}$ with the property that

$$\partial_{n+1}h_n + h_{n-1}\partial_n = f_n - g_n$$

for all n.

Remark 2.2.4. Given a map $\alpha : C_0 \to C'_0$ we say that the chain map $f : C \to C'$ *extends* α if $f_0 = \alpha$. It is a standard fact that chain maps induce well defined maps on the homology groups and that the induced maps are equal whenever the chain maps are chain homotopic. If all the maps f_n induce isomorphisms on homology, we say that f is a *chain equivalence*.

Definition 2.2.5 (Augmentation and Resolution). A nonnegative chain complex E is *augmented by an object* M if there exists a map $\epsilon : C_0 \to M$ so that $\epsilon \circ \partial_1 = 0$. We call it a *resolution* if the augmented complex

$$\cdots \xrightarrow{\partial_{n+1}} E_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\epsilon} M \longrightarrow 0.$$

is exact.

Remark 2.2.6. Note that we distinguish between the augmented chain complex above and the chain complex E that constitutes the resolution of the augmentation. We say that the resolution E is free, projective, flat and so on whenever all the E_i are. Finding resolutions with some of these additional properties can be of interest since we have $H_n(E) = 0$ for $n \geq 1$ and $\epsilon_* : H_0(E) \to M$ an isomorphism.

As we will see, we may find free resolutions of many different objects.

Example 2.2.7. We show that every k-module M admits a free resolution F. Given a generating set $\{x_i\}_{i\in I} \subseteq M$, i.e. the smallest submodule of M containing the set is M itself, we can define a free module $F_0 = \bigoplus_{i\in I} k\{x_i\}$ with a k-linear surjection f_0 given by $x_i \mapsto x_i$. Repeating the construction with $\ker f_0$ in the place of M gives us then a new module F_1 along with a k-linear map f_1 defined as the composite $F_1 \to \ker f_0 \hookrightarrow F_0$. As such we can inductively construct a free resolution.

We get an even easier example when dealing with modules over a principal ideal domain:

Example 2.2.8. Every abelian group G has a a free resolution of the form

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0.$$

To see this, construct F_0 in a similar way as above and let F_1 be the kernel of the map $F_0 \to G$. Then F_1 is free, being a subgroup of a free abelian group, and has a canonical inclusion $F_1 \hookrightarrow F_0$ giving the result.

To add to our intuition that the homology of resolutions characterizes the structure of the augmented object, we remark that even though a pair of resolutions may at the outset look vastly different, they are often equivalent in the eyes of homology:

Proposition 2.2.9. *If E is a projective resolution of a k-module M and E' is another resolution, then there exists a chain map $f : E \to E'$ extending* id_M *and any two such maps are chain homotopic.*

In particular, this is true for free resolutions since being free precipitates projective.

Proof. A proof is given for the case of abelian groups in lemma 3.1 of [Hat01]. □

There is a standard way of resolving an algebra by the means of the bar complex, closely related to a particularly important homology theory:

Definition 2.2.10 (Hochschild Homology). Let A be an associative unital algebra over a commutative ring k and M be a bimodule over A. Then define a chain complex $C_*(A, M)$ by $C_n(A, M) = M \otimes A^{\otimes n}$ with boundary $\partial = \sum_{i=0}^{n} d_i$, where

$$d_i(m, a_1, \dots, a_n) = \begin{cases} (ma_1, \dots, a_n), & i = 0 \\ (m, a_1, \dots, a_i a_{i+1}, a_n), & 0 < i < n \\ (a_n m, a_1 \dots, a_{n-1}), & i = n \end{cases}$$

The *Hochschild homology* of A is then the homology of the chain complex $HH(A, M) = HC(A, M)$. If $M = A$ we denote the Hochschild homology by $HH(A) = HC(A)$.

Remark 2.2.11. That the alleged chain complex defined above actually constitutes a chain complex, i.e. that $\partial^2 = 0$, is a fact that follows from the d_i satisfying the simplicial identities $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$ [Lod98, lemma 1.0.7.]. We remark also that the construction is functorial. Further connection between Hochschild homology and simplicial sets will be made through the Loday functor later on.

Definition 2.2.12 (Normalized Hochschild Complex). Let D_n be the submodule of $C_n(A, M)$ generated by elements $(m, a_1, \dots, a_n)$ where $a_i = 1$ for at least some i. The modules form an acyclic subcomplex D_* that we call the *subcomplex of degenerate elements*. Note that if one constructs Hochschild homology through the means of a simplicial algebra, these modules are indeed generated by degenerate elements. Writing $\overline{A} = A/k$, we have the *normalized Hochschild complex* $\overline{C}(A, M) = M \otimes \overline{A}^{\otimes n}$, which is equivalent to the $M \otimes A^{\otimes n}/D_n$.

Remark 2.2.13. Due to the acyclicity of D_*, the normalized Hochschild complex is in fact chain equivalent to the Hochschild complex itself. This is a general fact for simplicial abelian categories and we suggest comparison with theorem 2.6.5. A more direct proof can be found in [Lod98, prop. 1.6.5].

Definition 2.2.14 (Tensor Algebra). Let V be a k-module over a ring k and define the *tensor algebra*

$$T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \dots$$

with products the canonical isomorphisms $V^{\otimes n} \times V^{\otimes m} \to V^{\otimes(m+n)}$ given by concatenation

$$(v_1 \otimes \dots \otimes v_n, v_1' \otimes \dots \otimes v_m') \mapsto v_1 \otimes \dots \otimes v_n \otimes v_1' \otimes \dots \otimes v_m'.$$

Definition 2.2.15 (Symmetric and Exterior Algebra). Let I_n be the ideal of $T_n(V)$ generated by all elements of the form

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

where σ is a permutation. The symmetric algebra on V is then defined to be

$$S(V) = \bigotimes_{n=0}^{\infty} T_n(V)/I_n.$$

Similarly, if we let the ideal J_n of $T_n(V)$ be generated by elements of the form

$$v_1 \otimes \cdots \otimes v_n - \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)},$$

we get the exterior algebra as

$$E(V) = \bigotimes_{n=0}^{\infty} T_n(V)/J_n.$$

Remark 2.2.16. Note the following: If V is finite dimensional and free with basis $x_1, \ldots, x_n$, then the symmetric algebra $S(V)$ is isomorphic to the polynomial algebra $k[x_1, \ldots, x_n]$. In particular, if V is free of rank 1 with generator x, then $T(V) = S(V) = k[x]$. The constructions above are also functorial in the sense that they define functors from the category of k-modules to that of graded k-algebras. For a proof we refer to [Lan05, ch. XVI, §8].

Example 2.2.17. To each tensor algebra there corresponds a particularly simple complex $C^{\text{small}}(T(V))$ called the *small complex* given by

$$\cdots \longrightarrow 0 \longrightarrow T(V) \otimes V \longrightarrow T(V),$$

where the non-trivial map is $(\bigotimes_i v_i) \otimes v \mapsto (\bigotimes_i v_i)v - v(\bigotimes_i v_i)$ with multiplication still given by concatenation. What is interesting about the small complex is that it is chain equivalent to the normalized complex $\overline{C}(T(V))$ [Lod98, prop. 3.1.2]. In particular if V is again free of rank 1 with generator x so that $T(V) = k[x]$, we have that this chain equivalence gives an isomorphism

$$k[x] \otimes k[x] \to k[x] \otimes k\{x\}$$

Composing this with the identification $k[x] \otimes k\{x\} \cong k[x]$, using that $k\{x\} \cong k$, the resulting map is in fact $p(x) \otimes q(x) \mapsto p(x)q'(x)$, where the latter factor is the derivative of $q(x)$.

The small complex gives us readily the Hochschild homology of commutative polynomial algebras [Lod98, theorem 3.1.4]

$$HH_n(k[x]) = \begin{cases} k[x], & n = 0 \\ k[x], & n = 1 \\ 0, & n \geq 2 \end{cases}$$

Example 2.2.18. Consider now the Hochschild homology of the polynomial algebra $\mathbb{Q}[t]$ whose homology groups we know from example 2.2.17 above. In order to find the generators of the homology we may look at the boundary $\partial : \mathbb{Q}[t]^{\otimes 3} \to \mathbb{Q}[t]^{\otimes 2}$ in the Hochschild complex. It is clear that for $p, q, r \in \mathbb{Q}[t]$ we have

$$\partial(p \otimes q \otimes r) = rp \otimes q - p \otimes qr + pq \otimes r,$$

but due to commutativity we can factor this as

$$p \cdot (r \otimes q - 1 \otimes qr + q \otimes r)$$

where the multiplication is defined as in definition 2.2.14.

Calculating modulo boundaries for a general cycle in $q \otimes r \in HH_1(\mathbb{Q}[t])$ we thus have that

$$1 \otimes qr \equiv q \otimes r + r \otimes q.$$

This tells us that

$$1 \otimes t^2 \equiv 2t \cdot 1 \otimes t$$

and

$$1 \otimes t^3 \equiv t \otimes t^2 + t^2 \otimes t \equiv t \cdot (1 \otimes t^2 + t \otimes t) \equiv 3t \cdot t \otimes t = 3t^2 \cdot 1 \otimes t.$$

We thus see that $1 \otimes t^n \equiv nt^{n-1} \cdot 1 \otimes t$, which we can prove by induction:

$$1 \otimes t^{n+1} \equiv t \cdot (1 \otimes t^n + t^{n-1} \otimes t) \equiv t \cdot (nt^{n-1} \cdot 1 \otimes t + t^{n-1} \otimes t) \equiv (n+1)t^n \cdot 1 \otimes t.$$

As such we can conclude that $H_1(\mathbb{Q}[t]) = \mathbb{Q}[t] \{1 \otimes t\}$. Note that this is the generator as a $\mathbb{Q}[t]$-algebra.

Example 2.2.19. A short exact sequence of A-bimodules

$$0 \to M' \to M \to M'' \to 0$$

flat over k induces a long exact sequence of homology

$$\cdots \to HH_{n+1}(A, M'') \to HH_n(A, M') \to HH_n(A, M) \to HH_n(A, M'') \to \ldots$$

Hence we can use the short exact sequence

$$0 \to k[t] \xrightarrow{\cdot t} k[t] \xrightarrow{t \mapsto 0} k \to 0$$

and the corresponding long exact sequence of homology to find that

$$HH_n(k[t], k) = \begin{cases} k, & n = 0, 1 \\ 0, & n \geq 2 \end{cases}$$

with generators 1 and $1 \otimes t$ over k, respectively.

Example 2.2.20. Let now k be a field. By computing the resolution

$$\dots \xrightarrow{P(y,z)} Q \xrightarrow{\cdot(y-z)} Q \xrightarrow{P(y,z)} \dots \xrightarrow{\cdot(y-z)} Q \to k[t]/(t^r)$$

where $Q = k[y, z]/y^r = z^r$ and $P(y, z)$ is multiplication with the telescoping sum such that

$$P(y, z)(y - z) = y^r - z^r,$$

one can show that

$$HH_n(k[t]/(t^r)) \cong \begin{cases} k^r, & n = 0 \\ k^{r-1}, & n \geq 1 \end{cases}$$

due to $k[t]/(t^r) \cong k^r$ and k a field.

2.3 Homotopy Theory

It suffices to say that the closed model structure of the simplicial sets is what allows us to define a homotopy theory for simplicial sets. A neat feature of this homotopy theory is that through the adjoint relation with topological spaces, given by the singular and realization functors, one may show that

$$\pi_n(X) \cong \pi_n(|X|)$$

for a simplicial set X [GJ09, prop. 11.1]. This entails that in particular the notions of weak equivalences coincide, meaning $f : X \to Y$ is a weak equivalence if and only if $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces, in fact of CW-complexes.

For these objects one may ask how the homotopy behaves with respect to suspensions. For CW-complexes, the realizations of simplicial sets, we have the likes of the Freudenthal suspension theorem [Hat01, cor. 4.24] showing that $\pi_i(X) \to \pi_i(\Sigma X)$ is a isomorphism for $i < 2n - 1$ if X is an $(n - 1)$-connected CW-complex. This theorem gives rise to the notion of stable homotopy groups. We are interested in the stability of the higher Hochschild homology, to be defined later. For this purpose we define the following notion of stability of functors:

Definition 2.3.1 (Stable Functor). Let $\mathcal{C}$ be a closed model category. A functor $F : \mathbf{sSets} \to \mathcal{C}$ is *stable* if for any pair of simplicial sets X, Y we have

$$\Sigma X \simeq \Sigma Y \implies \pi_*(FX) \cong \pi_*(FY).$$

Remark 2.3.2. The homotopy theory can be defined in terms of a homotopy category $\mathrm{Ho}(\mathcal{C})$, constructed by formally inverting weak equivalences. This construction also makes it so that the converse is true, that the morphisms that induce isomorphisms in the homotopy category are exactly the weak equivalences [GJ09, prop. 1.14]. As such we could have made the equivalent definition that the stability condition should be

$$\Sigma X \simeq \Sigma Y \implies FX \simeq FY.$$

Example 2.3.3 (Stability of Singular Homology). As we will see in example 2.6.2, one can view the singular homology of a topological space as a functor from simplicial sets composed with the singular functor on the topological space, with a small modification giving the reduced homology. In accordance with the definition of stability that we opted for, we have that the reduced homology is indeed a stable invariant since due to excision we have $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(\Sigma X)$.

Having mentioned the adjoint relationship of the singular and realization functors numerous times already, we will give an overdue precise definition.

Definition 2.3.4 (Adjoints). A pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are *adjoint* if there exists a natural isomorphism

$$\hom_{\mathcal{D}}(F(c), d) \cong \hom_{\mathcal{C}}(c, G(d))$$

for all pair of objects $c \in \mathcal{C}$ and $d \in \mathcal{D}$. In this case we say that F is left adjoint to G and similarly G is called right adjoint to F, which we write as $F \dashv G$.

Example 2.3.5 (Singular Functor and Realization). As we have mentioned, the functors $\mathrm{Sing} : \mathbf{Top} \rightleftarrows \mathbf{sSets} : |-|$ form an adjoint relation that induces functors

$$\mathrm{Sing}_* : \mathrm{Ho}(\mathbf{Top}) \rightleftarrows \mathrm{Ho}(\mathbf{sSets}) : |-|_*$$

giving an equivalence of the homotopy categories [GJ09, th. 11.4].

The famous slogan of Saunders Mac Lane saying adjoint functors arise everywhere, keeps being true with certain ones of special interest to us:

Example 2.3.6 (Free Module). In example 2.2.7 we showed that any k-module M admits a free resolution by doing free constructions on generating sets. Implicitly, we used the forgetful functor $\mathcal{U} : \mathbf{kMod} \to \mathbf{Sets}$ taking a k-module M to its underlying set and then did a free construction on some generating subset. In the special case that we use the whole set M as a generating set, we get an example of a free-forgetful adjoint relation with the free functor $\mathcal{F}$: **Sets** $\to$ **kMod** taking a set $S \mapsto kS$, where kS is the module of formal k-linear combinations of elements in S. This free functor is left adjoint to the forgetful functor $\mathcal{U}$.

Example 2.3.7 (Free Simplicial Resolution of an Algebra). We will now show that even commutative algebras may be freely resolved. Indeed, in light of the discussion above, this will prove to extend the construction of the free resolution of a module in example 2.2.7. Let $\mathcal{U}$: **kAlg** $\to$ **Sets** be the forgetful functor taking a commutative k-algebra A to its underlying set. The free functor $\mathcal{F}$: **Sets** $\to$ **kAlg** taking $S \mapsto k[S]$, where $k[S]$ is the polynomial algebra over S, is left adjoint to the forgetful functor. Furthermore, $\mathcal{F}$ is equivalent to the composition

$$\mathbf{Sets} \xrightarrow{\mathcal{F}_1} \mathbf{kMod} \xrightarrow{\mathcal{F}_2} \mathbf{kAlg},$$

where $\mathcal{F}_1$ is the free functor in example 2.3.6 and $\mathcal{F}_2$ is the symmetric algebra of definition 2.2.15.

Now, there is a trick with iteratively doing free-forgetful constructions to make a free simplicial algebra resolving A. To do this, let $B_n = (\mathcal{F}\mathcal{U})^{n+1}(A)$. Since $\mathcal{F} \dashv \mathcal{U}$ we have a natural isomorphism

$$\varphi : \mathrm{Hom}_{\mathbf{Sets}}(X, \mathcal{U}(A)) \to \mathrm{Hom}_{\mathbf{kAlg}}(\mathcal{F}(X), A)$$

for a set X and k-algebra A. In particular, we have that if $X = \mathcal{U}(A)$ then the identity map 1_X gives us a homomorphism

$$\delta = \varphi 1_X : \mathcal{F}\mathcal{U}(A) \to A$$

that we call the counit of the adjunction. Similarly, the identity $1_{\mathcal{F}(X)}$ gives us a map of sets

$$\varphi^{-1} 1_{\mathcal{F}(X)} : \mathcal{U}(A) \to \mathcal{U}\mathcal{F}\mathcal{U}(A)$$

that we refer to as the unit of the adjunction. Although the unit is a map of sets, it defines an algebra homomorphism by $\sigma = \mathcal{F}\varphi^{-1} 1_{\mathcal{F}\mathcal{U}(A)}$ which is then a map of algebras $\sigma : \mathcal{F}\mathcal{U}(A) \to \mathcal{F}\mathcal{U}\mathcal{F}\mathcal{U}(A)$.

Now, let $d_i : B_n \to B_{n-1}$ and $s_i : B_n \to B_{n+1}$ be given by

$$d_i = (\mathcal{F}\mathcal{U})^{n-i} \delta (\mathcal{F}\mathcal{U})^i(A)$$

and

$$s_i = (\mathcal{F}\mathcal{U})^{n-1-i} \sigma (\mathcal{F}\mathcal{U})^i(A).$$

One can verify that these homomorphisms satisfy the simplicial identities of face and degeneracy maps making B a free simplicial algebra. Furthermore there is of course a homomorphism $B_0 \to A$ given by the counit. Regarding A as a discrete simplicial algebra with $A_n = A$ for all n and all face and degeneracy maps isomorphisms, it is then clear that the counit induces a simplicial map $B \to A$. We remark that $\delta^n = d_0 \cdots d_n$ gives a map $B_n \to B_0$ and that this map is canonical in the sense that any other composition of face maps $d_{i_1} \cdots d_{i_n} : B_n \to B_0$ gives the same map due to the simplicial identities, recall that the cosimplicial maps in Δ have unique factorizations.

That the map of simplicial algebras $B \to A$ is in fact a weak equivalence follows from applying the forgetful functor once more and see that we get an extra degeneracy map on the simplicial set $\mathcal{U}(B)$. It is then a standard argument that whenever a simplicial set has such an extra degeneracy map we get a homotopy equivalence $\mathcal{U}(B) \to K(\pi_0(\mathcal{U}(B)), 0)$ [GJ09, lemma III.5.1.]. Now, $\pi_0(\mathcal{U}(A)) = \mathcal{U}(A)$ and from the coequalizer diagram

$$\mathcal{U}\left(\mathcal{F}\mathcal{U}\right)^2(A) \rightrightarrows \mathcal{U}\mathcal{F}\mathcal{U}(A) \to \mathcal{U}(A)$$

we get that $\pi_0(\mathcal{U}\mathcal{F}\mathcal{U}(A)) \cong \mathcal{U}(A)$.

As we will see later in the Dold-Kan theorem 2.6.6, the homology groups of the chain complex we get from taking the alternating sum of the face maps are naturally isomorphic to the homotopy groups. Thus, if we for a moment consider A, B as simplicial abelian groups by taking the forgetful functor down to the underlying additive groups. Then by taking the Moore complex of B, to be defined later in 2.6.1, we do indeed have B as a proper resolution of A. The example above will therefore be of importance when we consider the stability of the higher Hochschild homology, which is a stable invariant for free algebras, but fails to be so in general.

2.4 Spectral Sequences

In our study of stability, we will need some knowledge of spectral sequences. In particular, the Greenlees spectral sequence gives the convergence of a sequence fitting very well with the problem at hand. Note that in working with spectral sequences we use E^r to denote the r-th page of the sequence.

Definition 2.4.1 (Filtration). A *filtration* of an object H is a family of subobjects $FH = (F_nH)_{n\in\mathbb{Z}}$, so that $F_nH \subseteq F_{n+1}H$.

Remark 2.4.2. If the object exists in a category with initial object I and $F_nH = I$ for $n \leq 0$, we say that the filtration is *bounded below* and write the filtration $FH = (F_nH)_{n\geq 0}$.

Filtrations are common sources of spectral sequences. One may think of them as approximations of the objects they filter.

Example 2.4.3 (Sets as Filtered Colimits). Any set X can of course be written as the union of its subsets. In particular we can write it as a union of its finite subsets. Explicitly, any set is bounded below by the empty set and we may form a filtration of finite sets by appending one element at a time. The resulting filtration gives us the means of encoding an object X in **Sets** as the colimit of a sequence of objects F_nX

$$\emptyset \subseteq F_0X \subseteq F_1X \subseteq \cdots \subseteq F_nX \subseteq \dots$$

coming from **Fin**, the category of finite sets. Note that the statement $X = \operatorname{colim}_i F_iX$ only makes sense after taking the inclusion into **Sets**.

Example 2.4.4 (Skeleta of Simplicial Sets). Given a simplicial set X define the n-th skeleta $\operatorname{sk}_n X$ as the subcomplex generated by the simplices of degree $\leq n$. Then it is clear that we have a filtration of X as $F_nX = sk_nX$ for $n \geq 0$.

Definition 2.4.5 (Graded Ring). A *graded ring* is a ring R along with a family $(R_n)_{n\geq 0}$ of subgroups of its underlying additive group such that $R = \bigoplus_{n=0}^{\infty} R_n$ and $R_mR_n \subseteq R_{m+n}$ for all $m, n \geq 0$.

Similarly, we say that a ring R is *bigraded* if $R = \bigoplus_{p,q,\in\mathbb{Z}} A_{p,q}$, where each $A_{p,q}$ is an additive subgroup and $A_{p,q}A_{r,s} \subseteq A_{p+r,q+s}$.

Remark 2.4.6. In particular, R_0 is a subring of R since it is closed under multiplication and by the same reasoning every R_n, $n > 0$ is an R_0-module. We say that an algebra is graded if it is graded as a ring.

Example 2.4.7 (Associated Graded Ring). Let R be a graded ring and FR a filtration such that $F_0R = R$. We define its *associated graded ring* $(S_n)_{n\geq 0}$ by

$$S_n = F_nR/F_{n+1}R.$$

For $x \in R_n$ we write the image in S_n as $\bar{x}$. Then the multiplication is defined as $\bar{x}\bar{y} = \overline{xy}$ for $x \in R_n, y \in R_m$.

As we mentioned briefly, the tensor algebra is a graded algebra, and consequently so are the symmetric and exterior algebras as well.

Definition 2.4.8 (Differential Graded Algebra). A *differential graded algebra*, often abbreviated as DG-algebra, is a graded algebra A with a degree -1 linear mapping $d : A \to A$ such that d is a derivation satisfying the Leibniz rule

$$d(a \cdot a') = d(a) \cdot a' + (-1)^{|a|} a \cdot d(a').$$

A *differential bigraded algebra* is similarly a bigraded algebra A along with a total degree -1 map

$$d : \bigoplus_{p+q=n} A_{p,q} \mapsto \bigoplus_{r+s=n-1} A_{r,s}$$

satisfying a similar Leibniz rule

$$d(a \cdot a') = d(a) \cdot a' + (-1)^{p+q} a \cdot d(a').$$

Example 2.4.9 (Hochschild Homology as DG-Algebra). Equipping the Hochschild homology with the shuffle product, which we will familiarize ourselves with later, we get the structure of a graded algebra on $H_*(A)$ for a commutative algebra A. Furthermore, the Hochschild boundary is a graded derivation with respect to this product making $H_*(A)$ a differential graded algebra. See [Lod98, cor. 4.2.7] and the discussion following lemma 2.6.19.

Example 2.4.10. If $(A, d), (B, d')$ is a pair of differential graded algebras, their tensor product $A \otimes B$ is a differential bigraded algebra with differential

$$d_{\otimes}(a \otimes b) = d(a) \otimes b + (-1)^{|a|} a \otimes d'(b).$$

Definition 2.4.11 (Differential Bigraded Module). A *differential bigraded module* E over a ring k, is a collection of k-modules $\{E_{p,q}\}_{p,q\in\mathbb{Z}}$ along with k-linear maps $d : E_{p,q} \to E_{p-r,q-r-1}$ called *differentials* of degree r for some $r \geq 0$ satisfying $d \circ d = 0$.

Definition 2.4.12 (Spectral Sequence of Modules). A *spectral sequence of modules* is then a collection of differential bigraded modules $\{(E^n, d)\}_{n>0}$ where the differentials of E^n is of degree n and for all p, q, n we have $E^{n+1}_{p,q} \cong H_{p,q}(E^n, d)$.

Remark 2.4.13. By $H_{p,q}(E^n, d)$ we mean explicitly the module

$$\ker(d : E^n_{p,q} \to E^n_{p-n,q-n-1}) / \operatorname{Im}(d : E^n_{p+n,q+n-1} \to E^n_{p,q}).$$

Definition 2.4.14 (Spectral Sequence of Algebras). A *spectral sequence of algebras* is a collection of differential graded algebras $\{(E^n, d)\}$ such that if φ_n denotes the product of E^n, then it induces the product of E^{n+1} through the composition

$$\begin{aligned}\varphi_{n+1} : E^{n+1}_{p,q} \otimes E^{n+1}_{p,q} \xrightarrow{\cong} & H_{p,q}(E^n, d) \otimes H_{p,q}(E^n, d) \\ & \xrightarrow{p} H_{p,q}(E^n \otimes E^n, d_{\otimes}) \xrightarrow{H\varphi_n} H_{p,q}(E^n, d) \xrightarrow{\cong} E^{n+1}_{p,q},\end{aligned}$$

where p is given by $[u] \otimes [v] \mapsto [u \otimes v]$.

The reason that spectral sequences are interesting is that they can be used to approximate objects. To this end there are many theorems regarding their convergence, that is if the spectral sequence collapses for some $n = N$ meaning the differentials are 0 for $n \geq N$, then the objects of the sequence stabilize

$$E^N_{p,q} \cong E^{N+1}_{p,q} \cong \dots E^\infty_{p,q}.$$

There are subtleties to convergence of spectral sequences that we will not mention, referring instead to [McC01] for a discussion of them.

The theorem of convergence that we are going to be interested in is the following:

Theorem 2.4.15 (Greenlees Spectral Sequence). *Let $A \to B \to C$ be a cofibre sequence of simplicial commutative algebras augmented over k such that $\pi_0(A) = k$ and B is of upward finite type as an A-module. Then there is a multiplicative first quadrant spectral sequence*

$$E^2_{p,q} = \pi_p(C) \otimes_k \pi_q(A) \implies \pi_{p+q}(B)$$

with differentials

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

By augmented over k we mean that there is a map $A \to k$ such that $k \to A \to k$ is the identity. Note also that B being of upward finite type is implied if $\pi_n(B)$ is finite dimensional for all n.

Proof. See [Gre16, lemma 3.1]. □

A cofibre sequence of algebras $A \to B \to C$ augmented over k means exactly that we have a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ k & \longrightarrow & C \end{array}$$

so that we may state Greenlees spectral sequence as

$$E^2_{p,q} = \pi_p(k \otimes_A B) \otimes_k \pi_q(A) \implies \pi_{p+q}(B).$$

This will be a very useful when we consider the cofibre sequence

$$\mathbb{Q}[t] \xrightarrow{t \mapsto t^2} \mathbb{Q}[t] \to \mathbb{Q}[t]/(t^2)$$

that will be central in our later developments.

2.5 Higher Hochschild Homology

We will construct the higher Hochschild homology iteratively over several steps, starting with defining a prototype of the Loday functor for the skeleton of finite sets that we then extend to be defined for sets and subsequently simplicial sets. A central part of the construction relies on working over a monoidal category.

Definition 2.5.1 (Monoidal Category). A *monoidal category* $\mathcal{C}$ is a category to which there exists

i an associative bifunctor $\bigotimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

ii an object $e \in \mathcal{C}$;

iii a natural isomorphism $\alpha : (- \otimes (- \otimes -)) \to ((- \otimes -) \otimes -)$;

iv a natural isomorphism $\lambda : (e \otimes -) \to (-)$;

v a natural isomorphism $\rho : (- \otimes e) \to (-)$.

such that for $a, b, c, d \in \mathcal{C}$

$$\begin{array}{ccccc} a \otimes (b \otimes (c \otimes d)) & \xrightarrow{\alpha} & (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\alpha} & ((a \otimes b) \otimes c) \otimes d \\ \downarrow{\scriptstyle 1 \otimes \alpha} & & & & \downarrow{\scriptstyle \alpha \otimes 1} \\ a \otimes ((b \otimes c) \otimes d) & & \xrightarrow{\qquad\quad \alpha \qquad\quad} & & (a \otimes (b \otimes c) \otimes d) \end{array}$$

and

$$\begin{array}{ccc} a \otimes (e \otimes c) & \xrightarrow{\alpha} & (a \otimes e) \otimes c \\ & {\scriptstyle 1 \otimes \lambda} \searrow \quad \swarrow {\scriptstyle \rho \otimes 1} & \\ & a \otimes c & \end{array}$$

commutes.

Remark 2.5.2. Any category with finite products is a monoidal category with the product as the monoidal product and terminal object as the monoidal unit. The existence of the latter follows by assumption as the empty product, i.e. it is the limit of the empty diagram $\mathbf{0} \to \mathcal{C}$. The same is of course true for the dual statement.

Definition 2.5.3 (Category of Finite Sets). Denote the category consisting of finite sets and functions between them by **Fin**. Then **Fin** is a full subcategory of **Sets** with skeleton $\overline{\mathbf{Fin}}$ consisting of the ordinal numbers $[n] = \{1, \ldots, n\}$. Note that the simplex category Δ embeds into $\overline{\mathbf{Fin}}$, but has fewer morphisms since the morphisms of $\overline{\mathbf{Fin}}$ are not required to be order preserving.

Suppose we have a category $\mathcal{C}$ with finite coproducts and consequently an initial object. Examples include of course the categories of sets, abelian groups, commutative rational algebras and so on. Then there exists a functor

$$\overline{\mathbf{Fin}} \times \mathcal{C} \to \mathcal{C}, \quad (X, A) \mapsto \coprod_X A,$$

where $X \mapsto \coprod_X A$ is defined uniquely up to isomorphism by preserving coproducts and by the condition that $\coprod_{\{1\}} A = A$.

The examples listed above are indeed also categories with arbitrary colimits. In particular we can extend the functor from finite sets to sets by taking a given set as the filtered colimit of its finite subsets as in example 2.4.3. Explicitly, for a set $X = \operatorname{colim} X_i$, where X is filtered by finite sets $X_i \hookrightarrow X_{i+1}$, $i \geq 0$, we define $\mathbf{Sets} \times \mathcal{C} \to \mathcal{C}$ to be given by

$$(X, A) \mapsto \coprod_X A = \coprod_{\operatorname{colim} X_i} A \cong \operatorname{colim} \coprod_{X_i} A.$$

If we are now given a simplicial set X and an object $A \in \mathcal{C}$, we may take the composition

$$\Delta^{\mathrm{op}} \xrightarrow{X} \mathbf{Sets} \xrightarrow{\coprod_\bullet A} \mathcal{C}$$

to get a simplicial object $\coprod_X A$ in $\mathcal{C}$ defined by $(\coprod_X A)_n = \coprod_{X_n} A$, again unique up to isomorphism. Since maps of simplicial sets are defined degreewise, we may view the above as a functor $\mathbf{sSets} \times \mathcal{C} \to s\mathcal{C}$. As an application of this construction we define:

Definition 2.5.4 (Loday Functor). The *Loday functor* is given by

$$L : \mathbf{sSets} \times \mathbf{kAlg} \to \mathbf{skAlg}$$

is given by

$$L(X, A) = \bigotimes_X A$$

where $\bigotimes_X A$ is the simplicial k-algebra with n-simplices $(\bigotimes_X A)_n = \bigotimes_{X_n} A$. We can augment the Loday functor by an A-algebra M

$$L(X, A, M) = M \otimes_A L(X, A).$$

Remark 2.5.5. Note that we could modify the definition to allow for simplicial k-algebras as well, with a regular k-algebra viewed as a discrete simplicial k-algebra with $A_n = A$ for all n. Then the image would be the diagonal of the bisimplicial k-algebra $\bigotimes_{X_p} A_q$. Note also that the construction of the Loday functor is a special example of a more general construction over symmetric monoidal categories.

Since the forgetful functor **skAlg** $\to$ **sSets** factors through the subcategory of simplicial abelian groups, every object of **skAlg** is fibrant. As such there exists a simplicial model category structure on **skAlg** where a map of k-algebras $A \to B$ is a weak equivalence if and only if it is a weak equivalence of the simplicial sets, in effect after applying the forgetful functor [GJ09, th. 4.1, th. 4.4, lemma 5.1]. This allows us to consider the homotopy groups of simplicial k-algebras, leading to the higher Hochschild homology.

Definition 2.5.6 (Higher Hochschild Homology). Let A be a commutative k-algebra. The *higher Hochschild homology* of A over a simplicial set X is given by

$$L_*(X, A) = \pi_* \left(\bigotimes_X A \right).$$

If the Loday functor is augmented by some A-algebra M, we refer to the higher Hochschild homology

$$L_*(X, A, B) = \pi_* \left(B \otimes_A \bigotimes_X A \right)$$

as homology with coefficients in M.

Remark 2.5.7. Although it might at the moment seem like a misnomer referring to the homotopy groups of the Loday functor as higher Hochschild homology, we will see in the next section that the Loday functor defines a chain complex, its Moore complex, whose homology groups are naturally isomorphic to the higher Hochschild homology as defined above. In the particular case resulting from taking $X = S^1$, the Moore complex of the Loday functor yields the Hochschild complex, hence $L_*(S^1, A) \cong HH_*(A)$. From this point of view the construction generalizes the Hochschild homology of commutative algebras, motivating the usage of the term higher Hochschild homology.

There is also something to be said about taking coefficients in M requiring working with pointed simplicial sets, the mention of which is absent from the development above. However, this is a minor issue since an analogous development with pointed sets above yields the pointed Loday functor. We do remark that the important part of the pointed theory is that the coefficients should correspond to the base point through the identification $M \otimes_A A_{x_0} \cong M$ for the the pointed Loday functor on the pointed simplicial set (X, x_0), where of course $A_{x_0} = A$.

Noting that the higher Hochschild homology L_* can be written as $\pi_* L$ we see that if we fix a commutative k-algebra A, we may in accordance with definition 2.3.1 consider the stability of the resulting functor

$$L(-, A) : \mathbf{sSets} \to \mathbf{skAlg}.$$

We will try to divulge some properties of the higher Hochschild homology, starting off with the important fact that $L(-, A)$ is not in general a stable functor.

Example 2.5.8. It was shown by [Ten16] that for the algebra $A = \mathbb{Q}[t]/(t^2)$ that

$$L_*(T^2, A, \mathbb{Q}) \ncong L_*(S^1 \vee S^1 \vee S^2, A, \mathbb{Q}),$$

even though we know from theorem 2.1.9 that T^2 is weakly equivalent to $S^1 \vee S^1 \vee S^2$ after suspension, giving a counter example to general stability of $L(-, A)$ for arbitrary A.

On the other hand, there are positive results that lead one to earlier believe that higher Hochschild homology could be a stable invariant.

Example 2.5.9 (Higher Hochschild Homology of a Smooth Algebra). In [DT18, example 2.6] it is shown that for a free symmetric algebra, or more generally a smooth algebra A, the Loday functor $L(-, A)$ *is* stable. In example 2.3.7 we showed that it is always possible to find a free simplicial resolution B of an algebra A by doing the trick with the extra degeneracy map from the free-forgetful adjunction. This entails that if A is an algebra for which $L(-, A)$ is not stable, such as in the case of the previous example, there has to arise some issue with respect to stability when going from $L(X, B_n)$ to $L(X, B)$.

Note that it makes sense to talk about $L(X, B)$, where this becomes a bisimplicial algebra with simplicial degree both in the simplicial set and in the simplicial algebra. See the later definition 2.6.9 for reference.

Explicitly, let $A = \mathbb{Q}[t]/(t^2)$ and let $f : B \xrightarrow{\sim} A$ be a free simplicial resolution of A, where f is considered a map of simplicial algebras with A the discrete simplicial algebra $A_n = A$. Then for each $n \geq 0$ we have a diagram of simplicial algebras

$$\begin{array}{ccc} L(T^2, B_n, \mathbb{Q}) & \xrightarrow{\sim} & L(S^1 \vee S^1 \vee S^2, B_n, \mathbb{Q}) \\ \downarrow{\scriptstyle (f_n)_*} & & \downarrow{\scriptstyle (f_n)_*} \\ L(T^2, A, \mathbb{Q}) & & L(S^1 \vee S^1 \vee S^2, A, \mathbb{Q}) \end{array}$$

that does not extend to a weak equivalence on the bottom.

Even though the map of algebras f_n constitute a weak equivalence f, the resulting diagram of bisimplicial algebras

$$\begin{array}{ccc} L(T^2, B, \mathbb{Q}) & \longrightarrow & L(S^1 \vee S^1 \vee S^2, B, \mathbb{Q}) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ L(T^2, A, \mathbb{Q}) & & L(S^1 \vee S^1 \vee S^2, A, \mathbb{Q}) \end{array}$$

cannot extend to square of weak equivalences either, since the counter example tells us that

$$L_*(T^2, A, \mathbb{Q}) \not\cong L_*(S^1 \vee S^1 \vee S^2, A, \mathbb{Q}).$$

Another example of stable behaviour arises when the weak equivalence on the suspension is induced by a map of the unsuspended spaces.

Example 2.5.10. Let $f : X \to Y$ be a simplicial map inducing a weak equivalence

$$\Sigma f : \Sigma X \xrightarrow{\sim} \Sigma Y.$$

The induced map $f_* : L(X, A) \to L(Y, A)$ is then a weak equivalence [DT18, example 2.7]. Hence we can elaborate further on the previous example by concluding that the weak equivalences for fixed B_n cannot be induced by a simplicial map $f : T^2 \to S^1 \vee S^1 \vee S^2$. It is also possible to show that no such weak equivalence f of spaces exists either.

Before moving on we make a remark that some care should be taken when indexing the Loday functor and higher Hocschild homology, where we denote

$$L(X, A, M)_n = M \otimes_A \bigotimes_{X_n} A$$

while

$$L_n(X, A, M) = \pi_n \left(M \otimes_A \bigotimes_{X} A \right).$$

This distinction becomes even more important further on when we consider the Moore complex of $L(X, A, M)$ whose chain groups are given by $L(X, A, M)_n$ while we will see that the resulting homology of the chain complex is given by $L_n(X, A, M)$.

2.6 Classical Results

We will in this section do a short exposition of three classical results: the Dold-Kan correspondence 2.6.6, the Eilenberg-Zilber theorem 2.6.20 and the Künneth formula 2.6.22. Each of them will be used ubiquitously in our following efforts. Starting off with the Dold-Kan correspondence, we will see that the structure encoded in a simplicial abelian group allows us to construct multiple chain complexes in a functorial manner, allowing us to form an equivalence of categories **sAb** and $\mathbf{Ch}_+(\mathbf{Ab})$. This equivalence of categories will prove to behave nicely with respect to the usual topological invariants, homology and homotopy, with the added benefit that we can then use tools from both theories in subsequent calculations.

Definition 2.6.1 (Moore Complex). Let A be a simplicial object in an abelian category $\mathcal{A}$. Define $\partial : A_n \to A_{n-1}$ by $\partial = \sum_{i=0}^{n}(-1)^i d_i$. As noted earlier in remark 2.2.11, using the simplicial identities $d_i d_j = d_{j-1} d_i,\ i < j$ one can verify that $\partial^2 = 0$ and consequently that the sequence

$$\cdots \longrightarrow A_n \xrightarrow{\partial} \cdots \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \xrightarrow{\partial} 0$$

is a chain complex CA, which we call the *Moore complex* of A.

Example 2.6.2 (Singular Homology). To put things under a more familiar guise, we can consider initially a topological space X. We can then recover the singular homology H_* of X by composing the functors

$$\mathbf{Top} \xrightarrow{\text{Sing}} \mathbf{sSets} \xrightarrow{\mathbb{Z}} \mathbf{sAb} \xrightarrow{C} \mathbf{Ch}_+(\mathbf{Ab}) \xrightarrow{H} \mathbf{grAb}.$$

Note that we could analogously describe homology with coefficients in M by the modification

$$M \otimes_{\mathbb{Z}} \mathbb{Z} : \mathbf{sSets} \to \mathbf{sAb}.$$

As we will see promptly, the Moore complex is closely related to the normalized complex:

Definition 2.6.3 (Normalized Chain Complex). Let again A be a simplicial object in an abelian category and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i : A_n \to A_{n-1}) \subset A_n.$$

Using the simplicial identities one can again easily verify that

$$\cdots \longrightarrow NA_n \xrightarrow{(-1)^n d_n} \cdots \xrightarrow{d_2} NA_1 \xrightarrow{-d_1} NA_0 \xrightarrow{d_0} 0$$

is a chain complex, which we denote NA called the *normalized chain complex* of A.

Remark 2.6.4. It is readily seen that the constructions above gives us functors $C, N : s\mathcal{A} \to \mathbf{Ch}_+(\mathcal{A})$ for an abelian category $\mathcal{A}$. The only thing to check is functoriality, but this follows immediately since a map of simplicial sets is a natural transformation of contravariant set valued functors on Δ, meaning they in particular commute with the face maps d_i, and in an abelian category one may use the additional linearity of the corresponding simplicial maps to see that the boundary commutes with simplicial maps as well.

If A is a simplicial abelian group there is a subgroup $DA_n \subset A_n$ generated by the degenerate simplices. The boundary of the Moore complex induces a homomorphism

$$\partial : A_n/DA_n \to A_{n-1}/DA_{n-1}$$

giving us a chain complex that we will denote by CA/DA. It is clear that there are chain maps

$$NA \overset{i}{\hookrightarrow} CA \overset{p}{\to} CA/DA$$

induced by inclusion and projection, respectively.

Theorem 2.6.5. *The composite*

$$NA \overset{pi}{\to} CA/DA$$

is a chain equivalence.

Proof. See [GJ09, theorem III.2.1]. □

There is a procedure outlined in [GJ09, p.147-149], whose development our exposition is based upon, that shows how to construct a certain functor $\Gamma : \mathbf{Ch}_+(\mathbf{Ab}) \to \mathbf{sAb}$. Explicitly, Γ is defined on objects to be

$$\Gamma(C)_n = \bigoplus_{n \twoheadrightarrow k} C_k,$$

where we have written $n \twoheadrightarrow k$ to denote that the sum is taken over the epimorphisms between the ordinal numbers. To define the functor on morphisms one uses that that we can epi-mono factorize any ordinal number map, albeit the definition and proving functoriality is not trivial. Recall that from the discussion preceding definition 2.1.3, that the cosimplicial face and degeneracies generate all ordinal number morphisms. We will however, for the sake of keeping the exposition brief, omit further details regarding this construction and proof of its properties, referring instead to the development of the source above.

As a final note before moving on, we do remark the following interesting fact: Given a chain complex C concentrated at an abelian group G in degree n, the resulting simplicial abelian group $\Gamma(C)$ is the Eilenberg-Mac Lane object $K(G, n)$. This fact can be used in part to show that any simplicial abelian group is homotopy equivalent to a product of Eilenberg-Mac Lane spaces [GJ09, proposition III.2.20.]. This we can compare to the extra degeneracy argument of example 2.3.7, where we constructed a free simplicial resolution B of a discrete simplicial algebra A and found that as simplicial sets we had $\mathcal{U}(B) \to K(\pi_0(\mathcal{U}(A)), 0)$ a homotopy equivalence.

The first classical theorem is then the following:

Theorem 2.6.6 (Dold-Kan Correspondence). *The functors N, Γ form an equivalence of categories*

$$N : \mathbf{sAb} \rightleftarrows \mathbf{Ch}_{+} : \Gamma.$$

Proof. [GJ09, corollary III.2.3]. □

This equvialence of categories is particularly interesting due to the following fact:

Theorem 2.6.7. *The inclusion $i : NA \to CA$ is a chain homotopy equivalence natural with respect to simplicial abelian groups A.*

Proof. [GJ09, Theorem III.2.4.]. □

The Dold-Kan correspondence tells us that from a categorical point of view, chain complexes of abelian groups are similar to simplicial abelian groups. Now, we can define simplicial homology by $H_*(C\mathbb{Z}X)$ for a simplicial set X, where $\mathbb{Z}$ is the free functor

$$\mathbb{Z} : \mathbf{Sets} \to \mathbf{Ab}, \quad S \mapsto \mathbb{Z}\{S\},$$

as we implicitly did in example 2.6.2. Since this coincides with the singular homology of topological spaces through the singular-realization adjunction of example 2.3.5 and both the homology and homotopy of spaces are topological invariants, one may ask how and if the equivalence of categories provided by the Dold-Kan correspondence relates to any similarity of the invariants. The answer to this question is as nice as possible:

Theorem 2.6.8. *There are isomorphisms of abelian groups*

$$\pi_n(A, 0) \cong H_n(NA) \cong H_n(CA)$$

natural in simplicial abelian groups A.

Proof. See [GJ09, Corollary III.2.7]. □

We have now established an equivalence between simplicial abelian groups and non-negative chain complexes, and an equivalence between the homotopy and associated homology groups of a simplicial abelian group. The last equivalence we are going to need is one regarding products, which will be provided by the Eilenberg-Zilber theorem and the Künneth formula.

Definition 2.6.9 (Bisimplicial Sets). A *bisimplicial set A* is a simplicial object in the category of simplicial sets. Equivalently, through the exponential law, it is a functor $A : \Delta^{\text{op}} \times \Delta^{\text{op}} \to \mathbf{Sets}$. We write $A(m, n) = A([m], [n])$ and refer to this as the simplicial set (of bisimplices) of bidegree (m, n), where m is the *horizontal degree* and n the *vertical degree.* In general we can define a bisimplicial object in $\mathcal{C}$ to be a simplicial object in the category of its simplical objects $s\mathcal{C}$, or equivalently, using the exponential law as above, a functor $\Delta^{\text{op}} \times \Delta^{\text{op}} \to \mathcal{C}$. As a shorthand we will write $s^2\mathcal{C}$ for the category of bisimplical objects in $\mathcal{C}$.

It should come as no surprise that the product of a pair of simplicial objects is closely related to the notion of a bisimplicial object. We can make this precise for simplicial sets in the following way:

Definition 2.6.10 (External Product and Diagonal). Given a pair of simplicial sets K, L, we define their *external product* $K\tilde{\times}L$ to be the bisimplicial set

$$K\tilde{\times}L(m, n) = K_m \times L_n.$$

The *diagonal* of a bisimplicial set A is the simplicial set $\text{diag}^*(A)$, where $\text{diag}^*(A)_n = A(n, n)$.

Remark 2.6.11. Note that from this point of view, the regular product of simplicial sets K, L is just the diagonal of their external product $K \times L = \text{diag}^*(K\tilde{\times}L)$, with diag^* being the precomposition with the diagonal $\Delta^{\text{op}} \to \Delta^{\text{op}} \times \Delta^{\text{op}}$. Of course this generalizes to simplicial objects in arbitrary cartesian monoidal categories.

Expanding on the previous remark, we can also consider the *external tensor product* $A \tilde{\otimes} B$ of simplicial abelian groups A, B by taking the pairwise tensor product of the abelian groups $A_m \otimes B_n$ in bidegree (m, n). Explicitly, $A \tilde{\otimes} B$ is given as the composition

$$\Delta^{\text{op}} \times \Delta^{\text{op}} \xrightarrow{A\times B} \mathbf{Ab} \times \mathbf{Ab} \xrightarrow{\otimes} \mathbf{Ab}.$$

Definition 2.6.12 (Moore Bicomplex). Similarly to how we can define the Moore complex of a simplicial abelian group, we can define the *Moore bicomplex* CA of a bisimplicial abelian group A to be the bicomplex with (p, q)-chains $CA_{p,q} = A(p, q)$, and with horizontal boundary

$$\partial_h = \sum_{i=0}^{p} (-1)^i d_i : A(p, q) \to A(p-1, q),$$

and vertical boundary

$$\partial_v = \sum_{j=0}^{q} (-1)^{p+j} d_j : A(p, q) \to A(p, q-1).$$

Remark 2.6.13. We make a careful remark that it is not unusual to take the Moore complex to be a functor $C : \mathbf{s^2Ab} \to \mathbf{Ch}_+\mathbf{Ch}_+(\mathbf{Ab})$, where the latter stands for the category of double complexes in abelian groups. This differs from our definition above in that the squares are then commuting, instead of anti-commuting, but bears resemblance to how one can view a bisimplicial object as a simplicial object in a category of simplicial objects. The reason we have opted for the definition above is more a matter of preference and in part to make it easier to later define the total complex.

We will at some point start suppressing the notational distinction between A and CA in the cases that A is either a simplicial or bisimplicial abelian group and CA the respective associated Moore complex or bicomplex. In the first case, this abuse of notation is justified by the Dold-Kan correspondence above and the second case will follow similarly by our pending results.

Definition 2.6.14 (Total Complex). The *total complex* $\operatorname{Tot} A$ of a bisimplicial abelian group A is the total complex of the associated Moore bicomplex $\operatorname{Tot} CA$. Explicitly it has chain groups given by

$$\operatorname{Tot} C_n A = \bigoplus_{p+q=n} C_{p,q} A$$

with boundary

$$\partial = \partial_h + \partial_v : \operatorname{Tot} C_n A \to \operatorname{Tot} C_{n-1} A.$$

Definition 2.6.15 (Tensor Product of Chain Complexes). Given a pair of chain complexes C, C', we define their tensor product $C \otimes C'$ to be the chain complex with n-chains

$$(C \otimes C')_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

and boundary $\partial : (C \otimes C')_n \to (C \otimes C')_{n-1}$ given by

$$\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y.$$

Remark 2.6.16. Following the previous remark, we see that we have to adjust for the sign in the tensor product of chain complexes while it is taken care of for the total complex by our definition of the bicomplex.

There is some redundancy in our definitions above, namely one that we can summarize as the existence of the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{sAb} \times \mathbf{sAb} & \xrightarrow{\tilde{\otimes}} & s^2\mathbf{Ab} & \xrightarrow{C} & \mathbf{BiCh}_+(\mathbf{Ab}) \\ \downarrow{\scriptstyle C\times C} & & & & \downarrow{\scriptstyle \operatorname{Tot}} \\ \mathbf{Ch}_+(\mathbf{Ab}) \times \mathbf{Ch}_+(\mathbf{Ab}) & & \xrightarrow{\otimes} & & \mathbf{Ch}_+(\mathbf{Ab}) \end{array} \tag{2.1}$$

In effect, this is the observation that $\operatorname{Tot} C(A \tilde{\otimes} B) = CA \otimes CB$, which follows easily from the definitions. We have opted to present both cases individually to make more explicit that these are two independent and equivalent points of view.

The diagram (2.1) also describes one of two standard methods for extracting a chain complex from a simplicial abelian group: Starting with A in $s^2\mathbf{Ab}$ one can take the Moore bicomplex and its associated total complex to end up with a chain complex. The other method is to take the diagonal $\operatorname{diag}^* : s^2\mathbf{Ab} \to \mathbf{sAb}$ mentioned earlier and then take the Moore complex. What we are interested in is the case that the bisimplicial abelian group is the exterior tensor product of a pair of simplicial abelian groups $A \widetilde{\otimes} B$. We then have

$$C \operatorname{diag}^*(A \widetilde{\otimes} B) = C(A \otimes B)$$

and

$$\operatorname{Tot} C(A \widetilde{\otimes} B) = CA \otimes CB,$$

and we may ask whether these chain complexes are equivalent. This will be shown after stating a pair of definition-lemmas:

Lemma 2.6.17 (Alexander-Whitney Map). *Let A, B be simplicial abelian groups and $C : \mathbf{sAb} \to \mathbf{Ch}_+$ be the Moore functor. There is a natural chain map $\mathrm{AW} : C(A \otimes B) \to CA \otimes CB$ given by*

$$\mathrm{AW}(a \otimes b) = \sum_{i=0}^{n} (\bar{d})^{n-i} a \otimes (d_0)^i b$$

called the Alexander-Whitney map, where $a \in A_n, b \in B_n$. The map $\bar{d}$ acting on a k-simplex x is the face map d_k, i.e. $\bar{d}(x) = d_{|x|}(x)$.

The Alexander-Whitney map is also associative up to homotopy in the sense that there exists a natural chain homotopy $(1 \otimes \mathrm{AW})\,\mathrm{AW} \simeq (\mathrm{AW} \otimes 1)\,\mathrm{AW}$, hence it can be iterated respecting the associativity of the tensor product

$$C(A_1 \otimes A_2 \otimes A_3) \xrightarrow{\mathrm{AW}} C(A_1) \otimes C(A_2 \otimes A_3) \xrightarrow{1 \otimes \mathrm{AW}} C(A_1) \otimes C(A_2) \otimes C(A_3).$$

Proof. See [Mac63, theorem VIII.8.8.]. □

Definition 2.6.18 (Shuffle). For a pair of nonnegative integers p, q we define a (p, q)-shuffle (μ, γ) to be a partition of the set of integers $\{0, \dots, p + q - 1\}$ as a pair of disjoint subsets with $\mu_1 < \cdots < \mu_p$ and $\gamma_1 < \cdots < \gamma_q$ so that $\{\mu_1, \dots, \mu_p, \gamma_1, \dots, \gamma_q\}$ defines a permutation of the set. We denote this permutation also by (μ, γ).

Lemma 2.6.19 (The Shuffle Map). *In the opposite direction, we define the Shuffle map* $\mathrm{sh} : CA \otimes CB \to C(A \otimes B)$ *by*

$$\mathrm{sh}(a \otimes b) = \sum_{(\mu,\nu)} \mathrm{sgn}(\mu,\nu) s_{\nu_q} \dots s_{\nu_1}(a) \otimes s_{\mu_p} \dots s_{\mu_1}(b),$$

where $a \in A_p$, $b \in B_q$ *with* $p + q = n$ *and* (μ, ν) *running over all* (p, q)*-shuffles. In addition to being a natural map of complexes this map is also graded commutative.*

Graded commutative means explicitly that with respect to the twisting map $T : A \otimes B \to B \otimes A$ we get

$$T(\mathrm{sh}(a \otimes b)) = (-1)^{|a|\cdot|b|} \,\mathrm{sh}(T(a \otimes b)) = (-1)^{|a|\cdot|b|} \,\mathrm{sh}(b \otimes a)$$

Proof. See [Mac63, theorem VIII.8.8.]. □

It is worth mentioning that for an algebra A the shuffle map induces a product on the Hochschild complex

$$C_pA \otimes C_qA \to C_{p+q}(A \otimes A),$$

where $C_nA = A^{\otimes n}$, and that the Hochschild boundary is a graded derivation of this product [Lod98, proposition 4.2.2.]. If A is commutative then composing with the map induced by the product map $A \times A \to A$ gives us a product on CA making it a differential graded algebra and consequently giving Hochschild homology the structure of a graded commutative algebra [Lod98, pp. 4.2.6–7]. We will return to this later in section 3.2.

For now, we arrive at the second classical result:

Theorem 2.6.20 (Eilenberg-Zilber). *The shuffle map* sh *and Alexander-Whitney map* AW *are chain equivalences*

$$\mathrm{sh} : CA \otimes CB \rightleftarrows C(A \otimes B) : \mathrm{AW}$$

inverse to each other on homology.

Proof. A proof can be found in [Mac63, theorem VIII.8.1] or [May92, theorem 29.3]. Both use the method of acyclic models, which are described in the previous sections of the same sources. □

There is a more generalized version of the theorem that we briefly state for the sake of the exposition:

Theorem 2.6.21 (Generalized Eilenberg-Zilber). *The total complex* Tot A *and the diagonal* $\mathrm{diag}^*(A)$ *of a bisimplicial abelian group* A *are chain homotopy equivalent, naturally with respect to morphisms of bisimplicial abelian groups* A*.*

Proof. See [GJ09, theorem IV.2.4]. □

The last classical result is that there is a class of nice cases where the homology of a product splits as the homology of the factors.

Theorem 2.6.22 (Künneth Formula). *If* k *is a principal ideal domain and* C *a chain complex of free* k*-modules, then there is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes_k H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n-1} \mathrm{Tor}\, k(H_p(C), H_q(C')) \to 0$$

and this sequence splits.

Note that the Künneth formula generalizes the universal coefficient theorem in the sense that it appears as the special case when C' is concentrated on the coefficient group G in dimension 0.

Proof. See [Hat01, theorem 3B.5]. □

In particular when k is a field, and C, C' are chain complexes with coefficients in k, the Künneth formula simplifies since the Tor terms are zero.

Corollary 2.6.23. *If* k *is a field and* C *a chain complex of free* k*-modules, we have an isomorphism*

$$\bigoplus_{p+q=n} H_p(C) \otimes_k H_q(C') \to H_n(C \otimes C')$$

for all n*.*

In combination with the Eilenberg-Zilber theorem, we can for the simplicial homology make the following strong assertion:

Corollary 2.6.24. *Let* X, Y *be a pair of simplicial sets and* k *a field. Then*

$$H_n(X \times Y, k) \cong \bigoplus_{p+q=n} H_p(X, k) \otimes H_q(Y, k).$$

Proof. Note that the chain groups with coefficients in k are given by $C(k \otimes_{\mathbb{Z}} \mathbb{Z}(X))$ as in example 2.6.2. Let $C(X, k)$ denote these chain groups. Then

$$H_n(C(X \times Y, k)) \cong H_n(C(X, k) \otimes C(Y, k)) \cong \bigoplus_{p+q=n} H_p(X, k) \otimes H_q(X, k),$$

where the first isomorphism follows from the Eilenberg-Zilber theorem and the second from the Künneth formula. □

Chapter 3

Calculations With Higher Hochschild Homology

3.1 Stability of Functors and Attaching Maps of Tori

In topological spaces we may of course construct the standard torus $T^2 = S^1 \times S^1$ by gluing together the opposite sides of a sheet and identifying the corners to a single point, as indicated by the figure below:

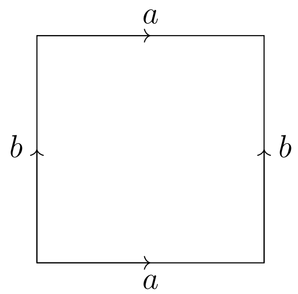


Figure 3.1: Identifying the sides $a \sim a$ and $b \sim b$ in sequence gives us first a cylinder, then a torus. The unfilled square is of course homeomorphic to standard circle S^1. Explicitly one may deformation retract each point radially by normalizing over the unit interval.

Equivalently, we can view the torus as a pushout diagram of topological spaces

$$\begin{array}{ccc} S^1 & \hookrightarrow & D^2 \\ \downarrow c & & \downarrow \\ S^1 \vee S^1 & \hookrightarrow & S^1 \times S^1 \end{array} \tag{3.1}$$

where the map c, the commutator map, is given by twisting S^1 into the figure-8, or explicitly as $aba^{-1}b^{-1}$ if first deformed into the square in figure 3.1. We will also refer to the commutator as the attaching map of the 2-cell $e^2 \subset D^2$ of the standard cellular complex of the torus.

To translate the above from topological spaces to simplicial sets, we need to define a simplicial model for S^1 with sufficient structure to allow us to encode the commutator map. The smallest simplicial set allowing this is the simplicial set given by

$$X = \partial\Delta^1 \times \Delta^1 \coprod_{\partial\Delta^1 \times \partial\Delta^1} \Delta^1 \times \partial\Delta^1, \tag{3.2}$$

which has geometric realization equivalent to the unfilled square in figure 3.1. We define a simplicial map $f : X \to S^1$ sending all nondegenerate simplices in X_1 but $(01, 00)$ to the degeneration of the base point in S^1. Explicitly f is defined by

$$(00, 01), (01, 11), (11, 01) \mapsto (00) \text{ and } (01, 00) \mapsto (01).$$

Notation wise we remark that we will from now on use exclusively S^n to mean the simplicial circle

$$S^n = \Delta^n / \partial\Delta^n$$

and rather refer to its realization whenever we want to treat it as a topological space. We have also used and will continue to use the same notation that we defined in 2.1.4 for the standard simplex and the simplicial sets constructed from it. It is clear that we may deformation retract the sides and top edges of $|X|$ to a point, giving a deformation retraction $|X| \to |S^1|$, showing that the simplicial map f is indeed a weak equivalence.

If we use $\Delta^1 \times \Delta^1$ as our model for D^2, we get a commutative square

$$\begin{array}{ccc} X & \hookrightarrow & \Delta^1 \times \Delta^1 \\ \downarrow \sim & & \downarrow \sim \\ S^1 & \hookrightarrow & D^2 \end{array}$$

where the vertical arrows are weak equivalences and by comparing with (3.1), we then have that the pushout P_1 of the diagram

$$\begin{array}{ccc} X & \longrightarrow & \Delta^1 \times \Delta^1 \\ \downarrow{\scriptstyle c} & & \downarrow \\ S^1 \vee S^1 & \longrightarrow & P_1 \end{array} \tag{3.3}$$

is weakly equivalent to the torus $T^2 = S^1 \times S^1$.

Consider then also the following diagram of simplicial sets

$$\begin{array}{ccc} X & \longrightarrow & \Delta^1 \times \Delta^1 \\ \downarrow & & \downarrow \\ * & \hookrightarrow & P_2 \\ \downarrow & & \downarrow \\ S^1 \vee S^1 & \hookrightarrow & S^1 \vee S^1 \vee P_2 \end{array} \tag{3.4}$$

with the top and bottom square a pushout. We have then that the pushout P_2 is weakly equivalent to

$$* \coprod_{S^1} D^2 = D^2/S^1 = S^2,$$

hence the bottom pushout is weakly equivalent to $S^1 \vee S^1 \vee S^2$. We remark that theorem 2.1.9 asserts that ΣT^2 is weakly equivalent to $\Sigma(S^1 \vee S^1 \vee S^2)$, which is of course why we are relating these simplicial sets with commutative diagrams in our interest of studying stability.

We then consider the implications of applying the Loday functor

$$L(-, A, k) = k \otimes_A \bigotimes_{\bullet} A$$

to the constructions above, for some choice of a commutative k-algebra A.

If the commutator map of (3.3) induces a map

$$c_* : k \otimes_A \bigotimes_X A \to k \otimes_A \bigotimes_{S^1 \vee S^1} A$$

that factors through k up to homotopy, then it is homotopic to the map induced by the composition

$$X \to * \to S^1 \vee S^1$$

in (3.4) since $L(*, A, k) \cong k$. Of course, factoring through k up to homotopy is equivalent to inducing the zero map on all nonzero homotopy groups, so we will refer to this condition informally as the commutator being zero or inducing the zero map.

We know in particular that $L(-, A, k)$ preserves pushouts since it preserves colimits, so applying it to the diagrams (3.3) and (3.4) gives a pair of pushout diagrams in the the category of commutative rational algebras. Indeed, if c_* is homotopic to the composition

$$k \otimes_A \bigotimes_X A \to k \to k \otimes_A \bigotimes_{S^1 \vee S^1} A$$

we have by taking the pushout of the composite square (3.4) and the uniqueness of pushouts that

$$k \otimes_A \bigotimes_{S^1 \times S^1} A \cong k \otimes_A \bigotimes_{S^1 \vee S^1 \vee S^2} A.$$

The above verifies that if the commutator factors through the coefficients, up to homotopy, then we have

$$L_*(T^2, A, k) \cong L_*(S^1 \vee S^1 \vee S^2, A, k). \tag{3.5}$$

However, it was shown by [Ten16] that for A the algebra to be the dual rational numbers

$$A = \mathbb{Q}[t]/(t^2)$$

the isomorphism of (3.5) does not hold. In fact this was used as a counter example to the stability of $L_*(-, A)$. Hence we know that the commutator is more complex than what is allowed for something factoring through the coefficient ring. A calculation with the Greenlees spectral sequence shows that if the algebra in question is the dual rational numbers or more generally just the rational polynomials $\mathbb{Q}[t]$, then the converse is also true. The calculation in question, that we will not display here, can be done by projecting the $S^1 \vee S^1$ to a point in the diagrams above and taking the pushouts both equivalent to S^2, where there is a difference in convergence of the corresponding Greenlees spectral sequences if and only if the commutator is zero. Hence we can detect the unstable behaviour in the commutator, with it factoring through $\mathbb{Q}$ up to homotopy if and only if the stability is observed in the case (3.5) for $A = \mathbb{Q}[t], \mathbb{Q}[t]/(t^2)$.

3.2 The Shuffle Map on Cycles

Before we can direct our full attention to the study of stability through some commutator maps, we will need some knowledge of the chains on which they act. To this end let for the rest of the section A denote the dual numbers $\mathbb{Q}[t]/(t^2)$. We note that $L_*(S^1, A, \mathbb{Q}) = HH_*(A, \mathbb{Q})$ and that we have from example 2.2.20 and the discussion preceding it that $HH_n(A, \mathbb{Q}) = \mathbb{Q}$. We denote the chain groups of the

Hochschild complex by $C_*(A, \mathbb{Q})$ and note also that we have chains $e_n = 1 \otimes t \otimes \cdots \otimes t$ in $C_n(A, \mathbb{Q})$ for all $n \geq 0$.

A simple boundary argument shows that the e_i are indeed the cycles generating the corresponding homology groups. Note that the product map $A \otimes A \to A$ is a homomorphism since A is commutative. Composing the shuffle product map with the product map gives us an inner shuffle product map

$$\mathrm{sh} : C_p(A, \mathbb{Q}) \otimes C_q(A, \mathbb{Q}) \to C_{p+q}(A, \mathbb{Q})$$

defined by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_p) \otimes (a_0' \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}) \mapsto \sum_\sigma \mathrm{sgn}(\sigma)(a_0 a_0' \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(p+q)}),$$

where the sum is taken over all (p, q)-shuffles σ. By counting transpositions, we see that the $(1, q)$-shuffle

$$\sigma_i(a_0 \otimes \cdots \otimes a_{q+1}) = a_0 \otimes a_2 \otimes \cdots \otimes a_{i-1} \otimes a_1 \otimes a_i \otimes \cdots \otimes a_{q+1}$$

has sign $\mathrm{sgn}(\sigma_i) = (-1)^{i-1}$. We remark that the number of $(1, q)$-shuffles σ_i is $q + 1$ so that the sum $\sum_{\sigma_i} \mathrm{sgn}(\sigma_i)$ is given by

$$\sum_{i=1}^{q+1} (-1)^{i-1} = \begin{cases} 1, & q \text{ even} \\ 0, & q \text{ odd} \end{cases}$$

Since $\sigma_i(e_{q+1}) = e_{q+1}$ we have

$$\mathrm{sh}(e_1 \otimes e_q) = \sum_{\sigma_i} \mathrm{sgn}(\sigma_i)\sigma_i(e_{q+1}) = e_{q+1} \sum_{\sigma_i} \mathrm{sgn}(\sigma_i)$$

so that $\mathrm{sh}(e_1 \times e_q)$ is 0 for q odd and e_{q+1} for q even.

Since we can decompose a $(2, q)$-shuffle μ as a $(1, q)$-shuffle σ_i composed with a $(1, i-1)$-shuffle σ_j, we see from the argument above that

$$\mathrm{sgn}(\mu) = \mathrm{sgn}(\sigma_j)\,\mathrm{sgn}(\sigma_i) = (-1)^{j+i}.$$

One way we can think about it is that we transpose a_2 to the i-th position followed by transposing a_1 to the j-th position with $i > j \geq 1$. Following this line of thought we have that $\sum_\mu \mathrm{sgn}(\mu)$ is given by

$$\sum_{i=2}^{q+1} \sum_{j=1}^{i-1} (-1)^{j+i} = \sum_{i=2}^{q+1} (-1)^{i-1} \sum_{j=1}^{i-1} (-1)^{j-1},$$

but since we have by the above that

$$\sum_{j=1}^{i-1}(-1)^{j-1} = \frac{1+(-1)^{i-1}}{2},$$

we thus have

$$\sum_{\mu} \mathrm{sgn}(\mu) = \sum_{i=2}^{q+1}(-1)^{i-1}\frac{1+(-1)^{i-1}}{2} = \sum_{i=2}^{q+1}\frac{(-1)^{i-1}+1}{2}.$$

For the rest of this section we will denote the inner shuffle product map by multiplication. The last thing we want to assert is then that $e_2 \cdot e_{2q} = (q+1)e_{2(q+1)}$. This is evident from looking at the expression for $\sum_\mu \mathrm{sgn}(\mu)$ and noting $2q+1$ is odd, so that the sum equates to $q+1$. We have then inductively that $e_2^n = n!e_{2n}$ and that $e_1 \cdot e_2^n = n!e_{2n+1}$. Since the natural numbers are surely invertible in $\mathbb{Q}$, we have thus proven

Proposition 3.2.1. *The homology groups $L_*(S^1, A, \mathbb{Q})$ are generated by*

$$L_*(S^1, A, \mathbb{Q}) = E(1 \otimes t) \otimes P(1 \otimes t \otimes t). \tag{3.6}$$

where the odd generator $y_1 = 1 \otimes t$ is of degree 1 and the even generator $y_2 = 1 \otimes t \otimes t$ is of degree 2.

Now, consider the following pushout diagram:

$$\begin{array}{ccc} * & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^1 \vee S^1 \end{array} \tag{3.7}$$

Since $\bigotimes_\bullet A$ preserves colimits, we get from (3.7) that

$$\bigotimes_{S^1 \vee S^1} A \cong \bigotimes_{S^1} A \otimes_A \bigotimes_{S^1} A. \tag{3.8}$$

The identity $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$ gives us further that

$$\mathbb{Q} \otimes_A \bigotimes_{S^1} A \otimes_A \bigotimes_{S^1} A \cong (\mathbb{Q} \otimes_A A \bigotimes_{S^1} A) \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes_A A \bigotimes_{S^1} A),$$

hence tensoring (3.8) by $\mathbb{Q} \otimes_A (-)$ gives us

$$\mathbb{Q} \otimes_A \bigotimes_{S^1 \vee S^1} A \cong (\mathbb{Q} \otimes_A A \bigotimes_{S^1} A) \otimes_{\mathbb{Q}} (\mathbb{Q} \otimes_A A \bigotimes_{S^1} A)$$

Now, since $\mathbb{Q}$ is a field, we thus get by the Eilenberg-Zilber theorem followed by the Künneth formula that $L_*(S^1 \vee S^1, A, \mathbb{Q}) \cong L_*(S^1, A, \mathbb{Q}) \otimes L_*(S^1, A, \mathbb{Q})$. This readily gives us the generators of the wedge sum of spheres $S^1 \vee S^1$. As such we have:

Proposition 3.2.2. *The generators of* $L_*(S^1 \vee S^1, A, \mathbb{Q})$ *are given by*

$$\begin{aligned} L_*(S^1 \vee S^1, A, \mathbb{Q}) &\cong E(y_1^h) \otimes_{\mathbb{Q}} P(y_2^h) \otimes_{\mathbb{Q}} E(y_1^v) \otimes_{\mathbb{Q}} P(y_2^v) \\ &\cong E(y_1^h, y_1^v) \otimes_{\mathbb{Q}} P(y_2^h, y_2^v). \end{aligned} \tag{3.9}$$

where use h, v *to denote the generators generators of proposition 3.2.1 corresponding the copies* $S^1 \times *$ *and* $* \times S^1$ *in*

$$S^1 \vee S^1 = S^1 \times * \coprod_{* \times *} * \times S^1.$$

To avoid confusion we write explicitly what we mean by vertical and horizontal factors in the case of y_1. The generator $y_1 = 1 \otimes t$ has the nonunital factor t in position corresponding to the simplex (01). A general chain in $L(S^1 \vee S^1, A, \mathbb{Q})_1$ can be written as $q \otimes a_{(01,00)} \otimes a_{(00,01)}$ and we have explicitly $y_1^h = 1 \otimes t \otimes 1$ and $y_1^v = 1 \otimes 1 \otimes t$.

3.3 The Attaching Map of the Torus

We can now start our analysis of the map

$$c_* : L_*(X, A, \mathbb{Q}) \to L_*(S^1 \vee S^1, A, \mathbb{Q})$$

induced by the commutator map

$$c : X \to S^1 \vee S^1,$$

where X is the simplicial square defined in (3.2). The brunt of the task concerns itself with tracing the generators y_1, y_2 backwards along the isomorphism induced by the weak equivalence $f : X \xrightarrow{\sim} S^1$.

We will denote degeneracies by $s_0(i, j) = (s_0 i, s_0 j)$ and will for clarity list the simplices X in lower degrees, with simplices appearing in the dictionary order we have defined as our convention:

$$X_0 = \{(0,0), (0,1), (1,0), (1,1)\}$$

and

$$X_1 = \{s_0(0,0), (00,01), s_0(0,1), (01,00), (01,11), s_0(1,0), (11,01), s_0(1,1)\}.$$

As such a representative for a homotopy class in $L_1(X, A, \mathbb{Q})$ is of the form

$$q \otimes_A \bigotimes_{x \in X_1} a_x$$

where the identification $\mathbb{Q} \otimes_A A \cong \mathbb{Q}$ takes $q \otimes p(t) \mapsto qp(0)$.

Inspired by the shape of our simplicial set X, we define the following notation for portraying the contents of the product

$$q \otimes_A \bigotimes_{x \in X_1} a_x = \begin{array}{ccc} a_{s_0(0,1)} & a_{(01,11)} & a_{s_0(1,1)} \\ a_{(00,01)} & & a_{(11,01)} \\ qa_{s_0(0,0)} & a_{(01,00)} & a_{s_0(1,0)} \end{array}$$

where $qa_{s(0,0)} \in \mathbb{Q}$. This visualization gives us the advantage that we can take aid in the geometric nature of the boundary maps in the calculations. We advocate that this will make the picture of what is going on easier to understand and at a later point reveal some symmetries of chains.

Now, the reader may readily verify that

$$x_1 = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & & 1 \\ 1 & t & 1 \end{array} + \begin{array}{ccc} 1 & 1 & 1 \\ 1 & & t \\ 1 & 1 & 1 \end{array} - \begin{array}{ccc} 1 & t & 1 \\ 1 & & 1 \\ 1 & 1 & 1 \end{array} - \begin{array}{ccc} 1 & 1 & 1 \\ t & & 1 \\ 1 & 1 & 1 \end{array}$$

is an element of $L_1(X, A, \mathbb{Q})$ such that $f_1(x_1) = y_1$. To see that it is a cycle and demonstrate the usefulness of the visual representation, we do an explicit calculation of the boundary to verify that x_1 is indeed a cycle

$$\begin{aligned} \partial x_1 &= d_0 x_1 - d_1 x_1 \\ &= \left(\begin{matrix} 1 & 1 \\ 1 & t \end{matrix} + \begin{matrix} 1 & t \\ 1 & 1 \end{matrix} - \begin{matrix} 1 & t \\ 1 & 1 \end{matrix} - \begin{matrix} t & 1 \\ 1 & 1 \end{matrix} \right) - \left(\begin{matrix} 1 & 1 \\ t & 1 \end{matrix} + \begin{matrix} 1 & 1 \\ 1 & t \end{matrix} - \begin{matrix} t & 1 \\ 1 & 1 \end{matrix} - \begin{matrix} 1 & 1 \\ t & 1 \end{matrix} \right) \\ &= 0. \end{aligned}$$

Finding the representative x_1 combinatorially by permuting the nonunital factor along the factors of the square is a lot easier than x_2. After all, in the case of x_2 not only do we have to shuffle two factors around at the same time, we also have to do it over a larger set. These difficulties will be evident shortly. Notation wise we remark that for brevity we will write $s_0^2(i,j)$ and for the degenerations of vertices in X_2 and so on. We also denote products of $L_2(X, A, \mathbb{Q})$ analogously to what we did before:

$$q \otimes_A \bigotimes_{x \in X_2} a_x = \begin{array}{cccc} a_{s_0^2(0,1)} & a_{s_0(01,11)} & a_{s_1(01,11)} & a_{s_0^2(1,1)} \\ a_{s_1(00,01)} & & & a_{s_1(11,01)} \\ a_{s_0(00,01)} & & & a_{s_0(11,01)} \\ qa_{s_0^2(0,0)} & a_{s_0(01,00)} & a_{s_1(01,00)} & a_{s_0^2(1,0)} \end{array}$$

Now, the way we found x_1 was of course not by just blindly permuting a t along the factors and guessing the signs of the terms. What we heuristically did and what we will continue to do for x_2 was to start off with the term that we knew had to be in the cycle and then considered what chains we needed to add to make the boundary zero. In effect this amounts to looking at chains with common faces and add them together with appropriate signs so they cancel out after taking the boundary. Geometrically we can think of working with rational coefficients as placing a copy of $\mathbb{Q}$ at the base point, where the multiplication forces any copy of t to be zero. Also since f takes all but one nondegenerate simplex of X to the base point, where we have placed the copy of $\mathbb{Q}$, we do not need to worry that adding other chains of this form to make a cycle changes the image over f_*.

However, as we have hinted at the procedure for finding x_2 is vastly more involved than that of x_1 amounting to a sum of 17 chains that are not so intuitively arranged around the square. As such, to keep things simple, we will opt for not listing the chains in the order that follows from shared faces, but will organize them according to symmetry, which the reader may then utilize for an easier time validating the result. Later on this symmetry will prove to give an idea of another approach to finding such representatives. Explicitly, we have that x_2 is the sum of chains

$$
\begin{aligned}
x_2 = {} & \begin{array}{cccc} 1&1&1&1\\ 1&&&1\\ 1&&&1\\ 1&t&t&1 \end{array} + \begin{array}{cccc} 1&1&1&1\\ 1&&&t\\ 1&&&t\\ 1&1&1&1 \end{array} - \begin{array}{cccc} 1&t&t&1\\ 1&&&1\\ 1&&&1\\ 1&1&1&1 \end{array} - \begin{array}{cccc} 1&1&1&1\\ t&&&1\\ t&&&1\\ 1&1&1&1 \end{array}\\
+ {} & \begin{array}{cccc} 1&t&1&1\\ t&&&1\\ 1&&&1\\ 1&1&1&1 \end{array} + \begin{array}{cccc} 1&1&1&1\\ 1&&&1\\ t&&&1\\ 1&t&1&1 \end{array} - \begin{array}{cccc} 1&1&1&1\\ 1&&&1\\ 1&&&t\\ 1&1&t&1 \end{array}\\
+ {} & \begin{array}{cccc} 1&t&1&1\\ 1&&&1\\ 1&&&1\\ 1&1&t&1 \end{array} + \begin{array}{cccc} 1&1&1&1\\ 1&&&t\\ t&&&1\\ 1&1&1&1 \end{array} - \begin{array}{cccc} 1&1&t&1\\ 1&&&1\\ 1&&&1\\ 1&t&1&1 \end{array} - \begin{array}{cccc} 1&1&1&1\\ t&&&1\\ 1&&&t\\ 1&1&1&1 \end{array}\\
+ {} & \begin{array}{cccc} 1&1&1&1\\ 1&&&t\\ 1&&&1\\ 1&t&1&1 \end{array} - \begin{array}{cccc} 1&1&t&1\\ 1&&&1\\ t&&&1\\ 1&1&1&1 \end{array}\\
+ {} & \begin{array}{cccc} 1&t&1&1\\ 1&&&t\\ 1&&&1\\ 1&1&1&1 \end{array} + \begin{array}{cccc} 1&1&1&1\\ 1&&&1\\ t&&&1\\ 1&1&t&1 \end{array} - \begin{array}{cccc} 1&1&1&1\\ t&&&1\\ 1&&&1\\ 1&t&1&1 \end{array} - \begin{array}{cccc} 1&1&t&1\\ 1&&&1\\ 1&&&t\\ 1&1&1&1 \end{array}
\end{aligned}
$$

Note that the the missing square of symmetry

$$-\begin{array}{cccc}1&1&t&1\\1&&&t\\1&&&1\\1&1&1&1\end{array}$$

from the second line and the sum of the pair

$$+\begin{array}{cccc}1&t&1&1\\1&&&1\\1&&&t\\1&1&1&1\end{array}-\begin{array}{cccc}1&1&1&1\\t&&&1\\1&&&1\\1&1&t&1\end{array}$$

absent from the fourth line are in fact boundaries, which is why we have opted not to include them when representing x_2.

It is not a hard, but a rather tedious task to verify the two properties needed of x_2, namely that $f_*(x_2) = y_2$ and that $\partial(x_2) = 0$. To verify the former note that the first term hits y_2 and that all others are taken to 0 by the argument that we have multiplied $\mathbb{Q}$ at the base point by working with rational coefficients. Although we have of course verified everything and the reader is welcome to do so for themselves, we do not find the process of explicitly writing out and cancelling the 51 terms of the boundary to be very enlightening hence will refrain from doing so here.

Proposition 3.3.1. *The chains x_1, x_2 are cycles generating $L_*(X, A, \mathbb{Q})$ by*

$$L_*(X, A, \mathbb{Q}) \cong E(x_1) \otimes P(x_2).$$

Proof. That x_1 is a cycle we demonstrated explicitly and we have omitted the explicit calculation showing that so is x_2, but we remark that this is immediately verifiable from the sum above. Since f_* is an isomorphism on homotopy groups the result follows from the above along with proposition 3.2.1. □

The next result we want to show is the following:

Proposition 3.3.2. *The commutator $c : X \to S^1 \vee S^1$ induces a map*

$$c_* : L_*(X, A, \mathbb{Q}) \to L_*(S^1 \vee S^1, A, \mathbb{Q})$$

defined by $c_*(x_1) = 0$ and $c_*(x_2) = y_1^h y_1^v$.

Recall from (3.9) that

$$L_*(S^1 \vee S^1, A, \mathbb{Q}) = E(y_1^h, y_1^v) \otimes P(y_2^h, y_2^v),$$

hence the proposition states that $c_*(y_2) \neq 0$ and that from section 3.1 we have that this means that
$$L_*(T^2, A, \mathbb{Q}) \ncong L_*(S^1 \vee S^1 \vee S^2, A, \mathbb{Q}),$$
implying the result originally shown by [Ten16] that $L_*(-, A)$ is not a stable invariant. Before proceeding with the proof we underline that although pleasant to arrive at this result from another end, this is *not* the purpose of this section. What we have done so far will serve two uses for us: First, it will serve as a stepping stone for doing a vastly more involved calculation than what we have seen so far. Second, it gives us an explicit description of the commutator which is exactly what we need to further study how the stability breaks.

Proof. The commutator map $c : X \to S^1 \vee S^1$ is explicitly defined by

$$(01, 00), (01, 11) \mapsto (01, 00)$$

and
$$(00, 01), (11, 01) \mapsto (00, 01),$$
where we identify the figure-8 by

$$S^1 \vee S^1 = S^1 \coprod_* S^1 \cong S^1 \times * \coprod_{* \times *} * \times S^1.$$

Writing an element of $L_1(S^1 \vee S^1, A, \mathbb{Q})$ as

$$qa_{s_0(0,0)} \otimes a_{(01,00)} \otimes a_{(00,01)},$$

we then have that

$$c_*(x_1) = 1 \otimes t \otimes 1 + 1 \otimes 1 \otimes t - 1 \otimes t \otimes 1 - 1 \otimes 1 \otimes t = 0.$$

Similarly, we write an element of $L_2(S^1 \vee S^1, A, \mathbb{Q})$ as

$$qa_{s(0,0)} \otimes a_{(001,000)} \otimes \otimes a_{(011,000)} \otimes a_{(000,001)} \otimes a_{(000,011)}$$

and one can verify that

$$c_*(x_2) = 2 \cdot (1 \otimes t \otimes 1 \otimes 1 \otimes t - 1 \otimes 1 \otimes t \otimes t \otimes 1) + 1 \otimes t \otimes 1 \otimes t \otimes 1.$$

We remark that the last term is the degeneracy

$$s_0(1 \otimes t \otimes t) = 1 \otimes t \otimes 1 \otimes t \otimes 1$$

hence is zero upon taking the homology class, since as we have seen the degenerate elements form an acyclic subcomplex.

Recall that the commutativity of (3.3) implies that the composition

$$X \xrightarrow{c} S^1 \vee S^1 \longrightarrow P_1$$

is nullhomotopic since $\Delta^1 \times \Delta^1$ is contractible, where P_1 is weakly equivalent to T^2. As such the composition $L_*(X, A, \mathbb{Q}) \to L_*(T^2, A, \mathbb{Q})$ is zero and it follows that $c_*(x_2)$ can not be equal to y_2^h, y_2^v since projecting from $T^2 = S^1 \times S^1$ to the component spheres S^1 turns the vertical and horizontal generators back to y_2, which is of course nonzero in $L_*(S^1, A, \mathbb{Q})$. As we will see, this entails that $c_*(x_2) = q y_1^h y_1^v + b$ for some $q \in \mathbb{Q}$ and some boundary b since $c_*(x_2)$ is going to be nonzero.

To show that this is indeed the case, recall from our calculations with the shuffle map i section 3.2 that

$$\mathbb{Q} \otimes_A \bigotimes_{S^1 \vee S^1} A \cong \mathbb{Q} \otimes_A \bigotimes_{S^1} A \otimes \mathbb{Q} \otimes_A \bigotimes_{S^1} A,$$

hence upon taking the homology we have by the Eilenberg-Zilber theorem and the Künneth formula that the shuffle map gives an isomorphism

$$\mathrm{sh} : L_*(S^1, A, \mathbb{Q}) \otimes L_*(S^1, A, \mathbb{Q}) \xrightarrow{\cong} L_*(S^1 \vee S^1, A, \mathbb{Q}).$$

Calculating this isomorphism explicitly

$$\begin{aligned} \mathrm{sh}(y_1^h \otimes y_1^v) &= s_1 y_1^h \otimes s_0 y_1^v - s_0 y_1^h \otimes s_1 y_1^v \\ &= 1 \otimes 1 \otimes t \otimes t \otimes 1 - 1 \otimes t \otimes 1 \otimes 1 \otimes t \end{aligned}$$

we see that $c_2(x_2) = -2 y_1^h y_1^v$ plus some boundary. □

We end the section with a final note that the coefficient of the image of x_2 along the commutator indicates that stability may hold if the algebra in question is the dual numbers over a field of characteristic 2. This is true and mentioned in [LR22], where it is also shown that

$$L_*(S^1 \times S^1, \mathbb{F}_p[t]/(t^2), \mathbb{F}_p) \ncong L_*(S^1 \vee S^1 \vee S^2, \mathbb{F}_p[t]/(t^2), \mathbb{F}_p)$$

for any odd prime p.

3.4 A Higher Dimensional Attaching Map

In the previous section we studied the attaching map of the torus and showed that we can observe the unstable behaviour in the map induced by the commutator. However, the commutator itself is only detected by the fundamental group and is trivial after suspension, raising the question whether the instability of the counter example is related to the connectivity of the space and if working over a higher connected space would resolve the issue.

Of course, from a topological point of view, we have that S^1 is equivalent to the Lie group $U(1)$, that is the unitary group with group operation being multiplication of complex units. Furthermore, we have that $SU(2)$, the special unitary group, is equivalent to S^3 with multiplication given by the group of unit quaternions. We can relate the unitary groups by split exact sequences of Lie groups

$$1 \to SU(n) \to U(n) \to U(1) \to 1,$$

where $U(n) \to U(1)$ is given by the determinant. Thus, in particular $U(2)$ is the semidirect product of S^3 and S^1.

This makes it natural for us to consider $S^3 \times S^1$ in our effort to explore the effect of connectivity. After all, through the Freudenthal suspension theorem [Hat01, p. 4.24], we have that $\pi_1(S^1) \cong \pi_3(S^3) \cong \mathbb{Z}$ and as we have outlined above, the product $S^3 \times S^1$ share similarities with the torus the way the orthogonal group $O(2)$ does with $U(2)$. Further, the question of stability is, as in the case of the torus of section 3.1, contingent upon the commutator inducing a trivial map on the higher Hochschild homology.

We will therefore do an analysis of the commutator map $S^3 \to S^3 \vee S^1$, paralleling that of our previous section, at least initially. However, as we will see, the commutator in the case of S^3 is much more complex than that of S^1, making it so that the simplicial set needed to encode it becomes quite big. This added complexity will result in us having to resort to developing other techniques to be able to arrive at the result, which will show that the unstable behaviour persists even in this higher connected case.

Let for this section X denote the simplicial set

$$X = \Delta^3 \times \partial\Delta^1 \sqcup_{\partial\Delta^3 \times \partial\Delta^1} \partial\Delta^3 \times \Delta^1$$

and recall that $S^3 = \Delta^3/\partial\Delta^3$. Let also $A = \mathbb{Q}[t]/(t^2)$ as before. Define now a simplicial map $f : X \to S^3$ by collapsing every non-degenerate simplex but $(0123, 0000)$ to the base point.

Since $|\Delta^3|$ deformation retracts to a point, there is a deformation retraction of $|X|$ to $\left|\Delta^3 \times \{0\} \sqcup_{\partial\Delta^3 \times \{0\}} C(\partial\Delta^3 \times \{0\})\right|$ given by deforming the top copy $|\Delta^3 \times \{1\}|$ in $|X|$ to a point. Here we mean by $C(\partial\Delta^3 \times \{0\})$ the cone on the boundary $\partial\Delta^3 \times \{0\}$. Of course, the cone deformation retracts to a point as well. Thus by identifying $\Delta^3 \times \{0\} \cong \Delta^3$, we have a pair of deformation retractions whose composite becomes a deformation retraction $|X| \to |S^3|$ upon reparametrization, hence f is a weak equivalence.

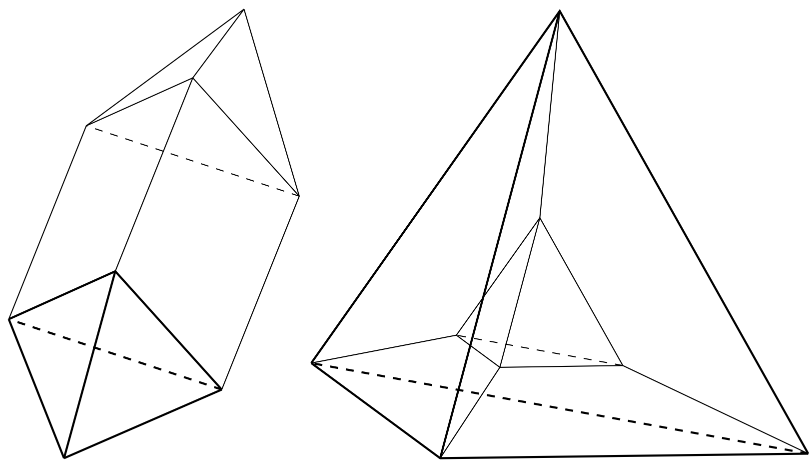


Figure 3.2: To the left, a partial illustration of the simplicial model X. The bottom and top tetrahedron are filled while the interior prism is hollow. There are hollow prisms connecting each pair of faces of the tetrahedrons as demonstrated by the figure to the right. However, the rightmost figure is only apt to illustrate the 2-skeleta of X, although with all faces drawn.

There is a pushout diagram of topological spaces

$$\begin{array}{ccc} S^3 & \hookrightarrow & D^4 \\ \downarrow & & \downarrow \\ S^3 \vee S^1 & \longrightarrow & S^3 \times S^1 \end{array}$$

where the left vertical map is the higher dimensional analogue of the commutator map, which we will define later. In simplicial sets, similar to before, we have

$$\begin{array}{ccc} X & \hookrightarrow & \Delta^3 \times \Delta^1 \\ \downarrow c & & \downarrow \\ S^3 \vee S^1 & \longrightarrow & S^3 \times S^1 \end{array}$$

where $c : X \to S^3 \vee S^1$ is the simplicial higher commutator map. Recall that there are weak equivalences $X \simeq S^3$ and similarly $\Delta^3 \times \Delta^1 \simeq *$, where the latter equivalence can be found by extending the aforementioned deformation retraction of the realization. Since, there is a similar argument as before that the stable invariance is equivalent to the commutator map being zero, we will in what follows study the map induced by the commutator map $c_* : L_*(X, A, \mathbb{Q}) \to L_*(S^3 \vee S^1, A, \mathbb{Q})$, where the algebra in question is still $A = \mathbb{Q}[t]/(t^2)$.

We begin by establishing the following proposition:

Proposition 3.4.1. *The homotopy groups $L_*(S^3, A, \mathbb{Q})$ have one exterior generator of degree 3 and one polynomial generator of degree 4. Explicitly, we have that*

$$L_*(S^3, A, \mathbb{Q}) \cong E(\sigma^2 y_1) \otimes P(\sigma^2 y_2),$$

where

$$\begin{aligned} \sigma^2 y_1 &= 1 \otimes t \otimes t \\ \sigma^2 y_2 &= 1 \otimes t \otimes t \otimes 1 \otimes 1. \end{aligned}$$

where $|\sigma^i y_j| = i + j$.

Proof. Recall from section 3.2 that

$$L_*(S^1, A, \mathbb{Q}) \cong E(y_1) \otimes P(y_2),$$

where $y_1 = 1 \otimes t$ and $y_2 = 1 \otimes t \otimes t$. We have by an application of Greenlees spectral sequence [DT18, lemma 3.4] that

$$L_*(S^3, A, \mathbb{Q}) \cong E(\sigma^2 y_1) \otimes P(\sigma^2 y_2)$$

with $|\sigma^j y_i| = i + j$. Thus the first non-trivial homology group appears in degree 3, where the chain group is given by

$$C_3(S^3, A, \mathbb{Q}) = \mathbb{Q} \otimes_A \bigotimes_{x \in S^3_3} A_x \cong \mathbb{Q} \otimes A_{0123}.$$

By commutativity, all chains in the chain group are cycles. However, there is but one non-degenerate cycle, up to some rational coefficient and boundary of course, namely $1 \otimes t$. Since the normalized chain complex is isomorphic to the chain complex modulo degeneracies, we have from the development of the Dold-Kan correspondence that the degenerate elements cannot generate the homology. Because we know the corresponding homology group to be one dimensional, we may immediately conclude that the non-degenerate cycle must represent $\sigma^2 y_1$.

Similarly, we have the chain group

$$C_4(S^3, A, \mathbb{Q}) = \mathbb{Q} \otimes_A \bigotimes_{x \in S^3_4} A_x \cong \mathbb{Q} \otimes A_{00123} \otimes A_{01123} \otimes A_{01223} \otimes A_{01233}$$

with the corresponding homology group generated by some $\sigma^2 y_2$. A simple calculation with the boundary homomorphism shows that the chain group has three non-boundary cycles

$$\begin{aligned} a &= 1 \otimes t \otimes t \otimes 1 \otimes 1 \\ b &= 1 \otimes 1 \otimes t \otimes t \otimes 1 \\ c &= 1 \otimes 1 \otimes 1 \otimes t \otimes t \end{aligned}$$

subject to the relations

$$[a+b]=[b+c]=[a-c]=0$$

when passing to homology, and as such either represents the generator of the homology group $\sigma^2 y_2$. □

In working with the simplicial set X it will make sense to identify

$$\Delta^3 \times \Delta^1 \cong \mathcal{N}([3]) \times \mathcal{N}([1]) \cong \mathcal{N}([3] \times [1]),$$

where $\mathcal{N}$ denotes the nerve of the posets viewed as categories and pivot into thinking of X as a subset of this nerve. It should then be apparent that an n-simplex of X is, or can at the very least be identified with, a string of composable arrows of length n, with composable arrows given by the partial order

$$(i,j)<(k,l) \text{ if } i<k \text{ or } i=k \text{ and } k<l$$

through the objects

$$\begin{matrix} (0,1) & (1,1) & (2,1) & (3,1) \\ \\ (0,0) & (1,0) & (2,0) & (3,0) \end{matrix}$$

Remark 3.4.2. Note that the following nondegenerate 3-simplices:

$$\begin{aligned} (0123,0001) &= (0,0) \to (1,0) \to (2,0) \to (3,1) \\ (0123,0011) &= (0,0) \to (1,0) \to (2,1) \to (3,1) \\ (0123,0111) &= (0,0) \to (1,1) \to (2,1) \to (3,1), \end{aligned}$$

and nondegenerate 4-simplices:

$$\begin{aligned} (01233,00001) &= (0,0) \to (1,0) \to (2,0) \to (3,0) \to (3,1) \\ (01223,00011) &= (0,0) \to (1,0) \to (2,0) \to (2,1) \to (3,1) \\ (01123,00111) &= (0,0) \to (1,0) \to (1,1) \to (2,1) \to (3,1) \\ (00123,01111) &= (0,0) \to (0,1) \to (1,1) \to (2,1) \to (3,1). \end{aligned}$$

are not in $\mathcal{N}X$. In fact, looking at the corresponding nondegenerates of $\Delta^3 \times \Delta^1$ we see that X is the compliment of the simplices listed above, and of course their degenerates, in $\Delta^3 \times \Delta^1$. Geometrically these constitute the interiors of the hollow prisms connecting the two copies of Δ^3 as shown in figure 3.4. We have listed each of them because they will, perhaps surprisingly, prove useful later on.

For the sake of portraying the geometric aspect of the task and writing the abundance of simplices in a compact manner, we propose to write the nerve as a matrix

$$[a_{(i,j)}] = \begin{bmatrix} a_{(0,1)} & a_{(1,1)} & a_{(2,1)} & a_{(3,1)} \\ a_{(0,0)} & a_{(1,0)} & a_{(2,0)} & a_{(3,0)} \end{bmatrix},$$

where the entries $a_{(i,j)}$ are integers indicating the number of times the string takes the value (i,j). We remark that in keeping with our convention of vertical and horizontal directions, the integers (i,j) differs from the usual matrix position in that they would correspond to the position $(j, 1-i)$. As an example we would write

$$(0123, 0000) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

It has been an underlying idea so far to work with the dual rational numbers with rational coefficients to keep the algebra comparatively simple. We take the point of view that in studying $L(X, A, \mathbb{Q})$ it is the underlying simplicial set that makes the simplicial rational algebra hard to work with. We formalize this notion in the following manner: Define a map

$$\tau : X \wedge A \to L(X, A, \mathbb{Q}) \tag{3.10}$$

given by

$$\tau(x \wedge a) \mapsto \bigotimes_{x' \in X} a_{x'}, \quad a_{x'} = \begin{cases} a, & x' = x \\ 1, & x' \neq x \end{cases}.$$

The smash product

$$X \wedge A = (X \times A)/(X \vee A)$$

is to be considered as pointed sets, in effect the wedge sum is

$$X \vee A = X \sqcup A/(x_0 \sim 0).$$

Fixing a nonzero element $a \in A$ one can identify and take the inclusion

$$X \cong X \times \{a\} \hookrightarrow X \wedge A. \tag{3.11}$$

Writing τ_X^a for the map $X \to L(X, A, \mathbb{Q})$ formed by the composition of (3.10) with (3.11), we get that varying over the simplicial input gives a natural transformation:

Lemma 3.4.3. *For a nonzero $a \in A$, we have that $\tau_\bullet^a$ is a natural transformation*

$$\tau_\bullet^a : \mathrm{id}_{\mathbf{sSets}} \to L(-, A, \mathbb{Q}).$$

Proof. A simplicial map $f : X \to Y$ has to take base point to base point, hence the map $f \sqcup \mathrm{id}_A$ induces a map $X \wedge A \to Y \wedge A$. We need only show that the following square commutes

$$\begin{array}{ccc} X & \xrightarrow{\tau_X^a} & L(X, A, \mathbb{Q}) \\ \downarrow f & & \downarrow f_* \\ Y & \xrightarrow{\tau_Y^a} & L(Y, A, \mathbb{Q}) \end{array}$$

To see this, let $y = f(x)$ and $f(x') = y'$. One may verify that right hand side of

$$f_*\tau_X^a = f_*\tau(x \wedge a) = f_*\left(\bigotimes_{x' \in X} a_{x'}\right) = \bigotimes_{y' \in Y} a_{y'}$$

is equal to the right hand side of

$$\tau_Y^a f(x) = \tau(y \wedge a) = \bigotimes_{y' \in Y} a_{y'}$$

by tracing the simplex corresponding to a. □

Now, since we are interested in the algebra $A = \mathbb{Q}[t]/(t^2)$ which is isomorphic to $\mathbb{Q}\{1, t\}$, we are only going to be interested in $\tau_\bullet^t$ which we will henceforth denote by simply $\tau_\bullet$. We remark that for each simplicial set X, the map τ_X is in each degree an injective map of sets in the sense that

$$x \neq x' \implies \tau_X(x) \neq \tau_X(x')$$

and also these are not multiples of each other with respect to the algebra structures. Since the simplicial $\mathbb{Q}$-algebra maps that we are working with are induced by simplicial set maps, this does mean that we may calculate their effects by dropping down to simplicial sets, using precisely the relation $f_*\tau_X = \tau_Y f$ for $f : X \to Y$ established above. This is helpful since combined with the implication above we get that

$$f(x) = f(x') \iff f_* t_X(x) = f_* t_X(x')$$

allowing us in many cases to avoid working with the much bigger chain group and do as much of our calculations using only the simplicial sets.

We remark that this basic map also helps us in the cases where we have to deal with more than one nonunital factor, since these are the products of images over τ_X. Explicitly, if $\bigotimes_{x \in X_n} a_x$ has nonunital factors indexed by $y \in Y_n \subset X_n$ we have

$$\bigotimes_{x \in X_n} a_x = \prod_{y \in Y_n} t_{X_n}(y),$$

where the product is inherited from the algebra, hence is the usual

$$\bigotimes_{x\in X_n} a_x \cdot \bigotimes_{x\in X_n} b_x = \bigotimes_{x\in X_n} (a_x b_x).$$

We will sometimes write $\tau = \tau_X$ if there is no room for confusion as to which simplicial set we are working over. Note also that $t^2 = 0$ in A hence $\tau(x)^2 = 0$ as well.

Having done somewhat extensive preliminary work, we are now in position to move forward with the task:

Proposition 3.4.4. *The chain* $x_1 = \sum_{i=0}^{13}(-1)^i\tau(\alpha_i)$ *where the simplices* α_i *are given by*

$$\begin{aligned} &\alpha_0 = (0123, 0000), && \alpha_1 = (0122, 0001), && \alpha_2 = (0133, 0001), && \alpha_3 = (0233, 0001),\\ &\alpha_4 = (1233, 0001), && \alpha_5 = (0112, 0011), && \alpha_6 = (0113, 0011), && \alpha_7 = (0223, 0011),\\ &\alpha_8 = (1223, 0011), && \alpha_9 = (0012, 0111), && \alpha_{10} = (0013, 0111), && \alpha_{11} = (0023, 0111),\\ &\alpha_{12} = (1123, 0111), && \alpha_{13} = (0123, 1111), \end{aligned}$$

is a cycle with $f_*(x_1) = \sigma^2 y_1$*.*

Proof. As usual, showing that the image is $\sigma^2 y_1$ follows immediately. In the language we have just defined we can write this a little more explicitly than we did earlier as the observation that

$$\sigma^2 y_1 = \tau_{S^3}(0123) = \tau f(0123, 0000) = f_*\tau_X(\alpha_0).$$

and that by definition of the simplicial map we have

$$f_*(\tau(\alpha_i)) = \tau(f(\alpha_i)) = \tau(0000) = 0$$

for all $i > 0$.

We postpone the proof that x_1 is a cycle to the development below, from which it will follow for free. It is however prudent to remark that one could readily show that $\partial x_1 = 0$ by manual and menial computations, which we only omit due to being made redundant by a nicer development. □

As in the previous section the calculation above can be done by some diligent detective work: By looking at the nondegenerate simplices that share face with α_0 we can deduce which are needed to cancel its residual faces after taking the boundary. This in turn gives us some new terms that we repeat the same procedure for. However, this approach is only viable as long as the number of simplices that share faces remain quite low, and as we are going to want to find x_2 we reach the threshold for where this goes from detective work to more or less guesswork. If in the mood for analogies one could say that the first case can be solved like a puzzle whereas the next we have not only a puzzle with many more pieces, but also a great deal of the new pieces will fit locally while not leading to the correct picture.

The solution we found to this problem is to make use of the interiors of the prisms connecting the two copies of Δ^3, which are the simplices listed explicitly in remark 3.4.2. We propose specifically to consider the inclusion of simplicial sets $\iota : X \hookrightarrow \Delta^3 \times \Delta^1$. We will in what follows also write $C_*(X, A, \mathbb{Q})$ for the chain complex which in degree n is given by $L(X, A, \mathbb{Q})_n$ in order to hopefully avoid any confusion with $L_n(X, A, \mathbb{Q})$, denoting the homology.

The induced chain map $\iota_* : C_*(X, A, \mathbb{Q}) \to C_*(\Delta^3 \times \Delta^1, A, \mathbb{Q})$ gives a commutative diagram

$$\begin{array}{ccc} C_{i+1}(X, A, \mathbb{Q}) & \xrightarrow{\iota_*} & C_{i+1}(\Delta^3 \times \Delta^1, A, \mathbb{Q}) \\ \downarrow \partial & & \downarrow \partial \\ C_i(X, A, \mathbb{Q}) & \xrightarrow{\iota_*} & C_i(\Delta^3 \times \Delta^1, A, \mathbb{Q}) \\ \downarrow \partial & & \downarrow \partial \\ C_{i-1}(X, A, \mathbb{Q}) & \xrightarrow{\iota_*} & C_{i-1}(\Delta^3 \times \Delta^1, A, \mathbb{Q}) \end{array}$$

for $i > 0$.

If we can find a chain $\beta \in C_{i+1}(\Delta^3 \times \Delta^1, A, \mathbb{Q})$ so that

$$\partial\beta \in \operatorname{Im} \iota_*,$$

say

$$\iota_*(\beta') = \partial\beta$$

for some $\beta' \in C_i(X, A, \mathbb{Q})$, then β' is a cycle since by commutativity

$$0 = \partial\partial\beta = \partial\iota_*\beta' = \iota_*\partial\beta'$$

and ι_* is injective.

Essentially this amounts to verifying that all the simplices constituting the boundary of β lies in X. In the framework we have established we can make this precise: Since

$$\iota_* t_X = t_{S^3} \iota,$$

we have that if

$$\partial\beta = \sum_i q_i \prod_j \tau_{\Delta^3 \times \Delta_1}(\iota(x)) = \sum_i q_i \prod_j \iota_* \tau_X(x_j)$$

then β'

$$\beta' = \sum_i q_i \prod_j \tau_X(x_j)$$

is a cycle.

We now finish the proof that x_1 is a cycle by showing that it is, after inclusion, the boundary of some $\alpha \in C_4(\Delta^3 \times \Delta^1, A, \mathbb{Q})$. Specifically, the four 4-simplices

$$a_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},\ a_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},\ a_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},\ a_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

give a chain

$$\alpha = \sum_{i=0}^{3} (-1)^i \tau_{\Delta^3 \times \Delta^1} a_i$$

with $\partial(\alpha) = \iota_*(x_1)$. The boundary of α with no faces cancelled is listed in the appendix for reference. We want to underline here that the four terms of α correspond to a prism operation on the hollow prisms connecting the two copies of Δ^3. As these simplices are stacked on top of each other with alternating signs they cancel the interior faces, in effect the diagonals of the vertical shifts

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

in such a way that the boundary of α is in X. Note that these are indeed the simplices mentioned earlier in remark 3.4.2 whose degeneracies we can think of as generating the compliment of X in $\Delta^3 \times \Delta^1$.

Now, the situation is of course more complicated for finding the cycle $x_2 \in L_2(X, A, \mathbb{Q})$ with $f(x_2) = \sigma^2 y_2$. However, the method outlined above gives us a tool making the previously insurmountable task only a difficult one. Particularly, the four vertical shifts gives us a sense of direction which turns out to be exactly what we need. The added degeneracies will still make the process comparatively hard, but we will spare some of these details and rather outline the procedure to attain the result that may be then verified independently.

Let $B_i \subset X$ be generated by the simplex a_i for $i = 0, 1, 2, 3$. Writing NB_i for the nondegenerate simplices of B_i they are explicitly given as

$$NB_4^i = \{a_i\},\ 0 \le i \le 3.$$

Note that $B^i \cong \Delta^4$ for each i and that $\Delta^3 \times \{0\} \subset B^0 \cap X$. We therefore start with finding a chain β_0 in $L(B_0, A, \mathbb{Q})$ that is such that the part of the boundary that lies in X is taken to $\sigma^2 y_2$. This is a little finicky and it should be stated that we did not find a particularly good method for doing this except trial and error. However, with this part done we need now only find some β^1 so that we cancel all terms in the boundary that correspond to simplices along the vertical shifts. This is what gives a sense of direction and we can use a few symmetric properties akin to what was shown in the previous section, with the caveat that we sometimes needed to add a few terms to make the faces match up orientation wise.

Doing this iteratively we view the top faces of B^i as the bottom B^{i+1} and move the residual faces after taking the boundary from the bottom of B^{i+1} by the means of adding together some chains to a sum β^{i+1}. In this manner we ultimately end up with chains along the sides of each of the B^i, along the top of B^3 and the bottom of B_0. The verification that the corresponding simplices belong to X is then immediate. The alternating sum $\beta = \sum_{i=0}^{3}(-1)^i\beta_i$ resulting from this procedure is really quite big and is listed explicitly in the appendix.

However we claim:

Proposition 3.4.5. *The chain $\beta \in C_5(\Delta^3 \times \Delta^1, A, \mathbb{Q})$ has a boundary in the image of ι_*. Hence $\partial\beta$ can be written as the sum*

$$\partial\beta = \sum_i q_i \iota_*(\tau_X(x_i) \cdot \tau_X(x_i'))$$

and it follows that

$$x_2 = \sum_i q_i \tau_X(x_i) \cdot \tau_X(x_i')$$

in $C_4(X, A, \mathbb{Q})$ is a cycle with $f_*(x_2) = \sigma^2 y_2$.

As before we get a statement regarding the explicit generators of the homology groups.

Corollary 3.4.6. *The cycles x_1, x_2 represents generators of degree 3 and 4 of the higher Hochschild homology*

$$L_*(X, A, \mathbb{Q}) = E(x_1) \otimes P(x_2).$$

We finish the section with a calculation of the images of x_1, x_2 along the commutator $X \to S^3 \vee S^1$. Before we actually define the commutator, recall that we regard $S^3 \vee S^1$ as a subset of $S^3 \times S^1$ given by

$$S^3 \vee S^1 = S^3 \times * \coprod_{* \times *} * \times S^1.$$

Let π_1, π_2 be the vertical and horizontal projections of X, respectively, i.e.

$$\pi_1(a, b) = (a, 0), \;\; \pi_2(a, b) = (0, b)$$

and let q_1 and q_2 be the quotient maps $\Delta^3 \to S^3$ and $\Delta^1 \to S^1$.

Definition 3.4.7. The commutator map $c : X \to S^3 \vee S^1$ is the composition

$$X \xrightarrow{\pi_1 \sqcup \pi_2} \Delta^3 \vee \Delta^1 \xrightarrow{q_1 \sqcup q_2} S^3 \vee S^1,$$

and we denote the induced image by $c_* : L_*(X, A, \mathbb{Q}) \to L_*(S^3 \vee S^1, A, \mathbb{Q})$.

It remains to describe the generators of $S^3 \vee S^1$, which we will now do.

Proposition 3.4.8. *The higher Hochschild homology A over $S^3 \vee S^1$ is given by*

$$L_*(S^3 \vee S^1, A, \mathbb{Q}) \cong E(\sigma^2 y_1^h, y_1^v) \otimes P(\sigma^2 y_2^h, y_2^v),$$

with where the odd generators have degree 3 and 1 and the even degree 4 and 2, respectively.

Proof. From applying the functor $L(-, A, \mathbb{Q})$ to the pushout diagram defining the wedge sum

$$\begin{array}{ccc} * & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ S^3 & \longrightarrow & S^3 \vee S^1 \end{array}$$

we get upon identification

$$\mathbb{Q} \otimes_A \bigotimes_{S^3 \vee S^1} A \cong \left(\mathbb{Q} \otimes_A \bigotimes_{S^3} A \right) \otimes_{\mathbb{Q}} \left(\mathbb{Q} \otimes_A \bigotimes_{S^1} A \right). \tag{3.12}$$

From the Eilenberg-Zilber theorem we know that the the shuffle map

$$\mathrm{sh} : C\left(L(S^3, A, \mathbb{Q})\right) \otimes_{\mathbb{Q}} C\left(L(S^1, A, \mathbb{Q})\right) \to C\left(L(S^3, A, \mathbb{Q}) \otimes_{\mathbb{Q}} L(S^1, A, \mathbb{Q})\right) \tag{3.13}$$

is a chain equivalence. Note that from (3.12) we may identify the right hand side of (3.13) with $C(L(S^3 \vee S^1, A, \mathbb{Q}))$. Now, since $\mathbb{Q}$ is evidently a field, we get from the above and the Künneth formula that

$$L_*(S^3, A, \mathbb{Q}) \otimes_{\mathbb{Q}} L_*(S^1, A, \mathbb{Q}) \cong L_*(S^3 \vee S^1, A, \mathbb{Q}).$$

Of course, we have explicitly found the generators over the components spheres in proposition 3.2.1 and 3.4.1 so we know

$$L_*(S^3 \vee S^1, A, \mathbb{Q}) \cong E(\sigma^2 y_1^h, y_1^v) \otimes P(\sigma^2 y_2^h, y_2^v)$$

where we use h, v to distinguish the horizontal and vertical generators corresponding to our earlier choice of directions for the components of the product to span. □

We can now easily find explicit descriptions of the generators of $L_*(S^3 \vee S^1, A, \mathbb{Q})$. In particular we are interested at generators of $L_4(S^3 \vee S^1, A, \mathbb{Q})$. This group is generated by $\left\{\sigma^2 y_1^h y_1^v, \sigma^2 y_2^h\right\}$. Observe that we can denote the representatives of y_1, y_2 by

$$\tau_{S^1}(01),\ \tau_{S^1}(s_0(01)) \cdot \tau_{S^1}(s_1(01)),$$

respectively, and that the representatives of $\sigma^2 y_1, \sigma^2 y_2$ are likewise in order given by

$$\tau_{S^3}(0123),\ \tau_{S^3}(s_0(0123)) \cdot \tau_{S^3}(s_1(0123)).$$

Note that the vertical and horizontal factors behave exactly as one would expect from their names: Explicitly $y_1^v = \tau_{S^3 \vee S^1}(00, 01)$ with the defining simplex $(00, 01)$ in the subset $* \times S^1$, which is of course equivalent to S^1 by projecting on the second factor turning y_1^v into y_1. Of course $(s_i)_* t_X(x) = t_X(s_i x)$ for any $x \in X$, hence calculating the images of the representatives over the inner shuffle product map is easy:

$$\begin{aligned} \mathrm{sh}(\sigma^2 y_1^h \otimes y_1^v) &= \sum_{(\mu,\gamma)} s_{\gamma_1}(\sigma^2 y_1^h) \cdot s_{\mu_2} s_{\mu_1} s_{\mu_0}(y_1^v) \\ &= \tau(01233, 00000) \cdot \tau(00000, 00001) \\ &\quad - \tau(01223, 00000) \cdot \tau(00000, 00011) \\ &\quad + \tau(01123, 00000) \cdot \tau(00000, 00111) \\ &\quad - \tau(00123, 00000) \cdot \tau(00000, 01111) \end{aligned}$$

Writing the horizontal and vertical factors pairwise in ascending orders, with the horizontal before the vertical ones we can write this shuffle product as

$$\mathrm{sh}(\sigma^2 y_1^h \otimes y_1^v) = 1111tt111 - 111t11t11 + 11t11111t1 - 1t111111t$$

with the two middle terms being boundaries.

Writing the factors of $c_*(x_2)$ in a similar manner we can by a relatively straight forward computation show that

$$c_*(\partial\beta) = -2 \cdot 1111tt111 + 2 \cdot 1t111111t + b,$$

where b are some boundary terms of the image that we have omitted listing. Taking the homology we are again in the situation that

$$c_*(\partial\beta) = -2 \cdot \mathrm{sh}(\sigma^2 y_1^h \otimes y_1^v),$$

hence we conclude that the commutator map is nonzero and it follows by a similar argument as before that

$$L_*(S^3 \times S^1, A, \mathbb{Q}) \not\cong L_*(S^3 \vee S^1 \vee S^4, A, \mathbb{Q}),$$

giving another example of instability of $L_*(-, A, \mathbb{Q})$ and by extension $L_*(-, A)$ for the rational algebra $A = \mathbb{Q}[t]/(t^2)$.

3.5 A Spectral Sequence of the Attaching Map

Having shown that the instability persists even after raising the connectivity of the simplicial set we are working over, we return to the simpler case of $X \xrightarrow{\sim} S^1$, where

$$X = \partial\Delta^1 \times \Delta^1 \coprod_{\partial\Delta^1 \times \partial\Delta^1} \Delta^1 \times \partial\Delta^1.$$

Since we are still in need of explicit calculations to describe what is going on, we will continue working with the comparatively simple algebra $A = \mathbb{Q}[t]/(t^2)$.

We make a quick synopsis before proceeding: In the previous sections we showed explicitly that the attaching map

$$X \to S^1 \vee S^1$$

induces a nonzero map on the homotopy groups

$$L_*(X, A, \mathbb{Q}) \to L_*(S^1 \vee S^1, A, \mathbb{Q}),$$

that we can describe in full detail. As stated in example 2.5.9, we know that $L_*(-, B, \mathbb{Q})$ is a stable invariant whenever B is a smooth commutative algebra, in particular this is true for the rational polynomial algebra $\mathbb{Q}[t]$. In section 3.1 we related this to the condition that the induced morphism $L(X, B, \mathbb{Q}) \to L(S^1 \vee S^1, B, \mathbb{Q})$ must be homotopic to the map factoring through $\mathbb{Q}$.

With this in mind we note that A is isomorphic to the pushout

$$\begin{array}{ccc} \mathbb{Q}[t] & \xrightarrow{t \mapsto s^2} & \mathbb{Q}[s] \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Q}[t]} \mathbb{Q}[s] \end{array} \tag{3.14}$$

where the left vertical map is the homomorphism evaluating t at 0. We remark that the diagram describes A, which is of course not smooth, as the pushout of smooth algebras. We therefore consider the implications of applying the diagram $\left\{ \bigotimes_X(-) \xrightarrow{c_*} \bigotimes_{S^1 \vee S^1}(-) \right\}$ and the properties of the resulting cube

$$\begin{array}{ccccccc} & & \bigotimes_{S^1\vee S^1} \mathbb{Q}[t] & \longrightarrow & & & \bigotimes_{S^1\vee S^1} \mathbb{Q}[s] \\ & \nearrow & & & & \nearrow & \\ \bigotimes_X \mathbb{Q}[t] & & \longrightarrow & & \bigotimes_X \mathbb{Q}[s] & & \downarrow \\ & & \downarrow & & & & \\ \downarrow & & \bigotimes_{S^1\vee S^1} \mathbb{Q} & \longrightarrow & \downarrow & & \bigotimes_{S^1\vee S^1} A \\ & \nearrow & & & & \nearrow & \\ \bigotimes_X \mathbb{Q} & & \longrightarrow & & \bigotimes_X A & & \end{array} \tag{3.15}$$

First of all, we know that each diagonal map induced by the commutator, save the aforementioned map $\bigotimes_X A \to \bigotimes_{S^1 \vee S^1} A$, are homotopic to the zero map. Secondly, we know that $L(-,-)$ preserves colimits in both arguments, so that in particular the front and bottom face resulting from applying $L(X,-)$ and $L(S^1 \vee S^1, -)$ to the pushout (3.14) are pushout squares as well.

In what follows we aim to study the commutator using the the Greenlees spectral sequence on the pair of pushouts in the cube. To do this we need to modify the algebras to have rational coefficients so that $\pi_0(\bigotimes_X \mathbb{Q}[t])$ and $\pi_0(\bigotimes_{S^1 \vee S^1} \mathbb{Q}[t])$ are isomorphic to $\mathbb{Q}$.

Proposition 3.5.1. *Given a simplicial set X, the square*

$$\begin{array}{ccc} \mathbb{Q} \otimes_{\mathbb{Q}[t]} \bigotimes_X \mathbb{Q}[t] & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Q}[s]} \bigotimes_X \mathbb{Q}[s] \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_A \bigotimes_X A \end{array}$$

is a pushout of rational algebras.

Proof. Explicitly $\mathbb{Q}$ viewed as a $\mathbb{Q}[t]$-module has multiplication defined by $(t,q) \mapsto 0$ for $q \in \mathbb{Q}$. We can for a set Y therefore construct the prepushout diagram

$$\begin{array}{ccc} \mathbb{Q} \otimes_{\mathbb{Q}[t]} \bigotimes_Y \mathbb{Q}[t] & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Q}[s]} \bigotimes_Y \mathbb{Q}[s] \\ \downarrow & & \\ \mathbb{Q} & & \end{array} \tag{3.16}$$

where the horizontal map is the identity on $\mathbb{Q}$ and defined by $t \mapsto s^2$ on the remaining factors. The vertical map is similarly the identity on $\mathbb{Q}$ and the evaluation $t \mapsto 0$ on all the other factors.

That the top map is well defined follows from the verification that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Q} \otimes_{\mathbb{Q}[t]} \bigotimes_X \mathbb{Q}[t] & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Q}[s]} \bigotimes_X \mathbb{Q}[s] \\ \uparrow & & \uparrow \\ \mathbb{Q}[t] & \xrightarrow{t \mapsto s^2} & \mathbb{Q}[s] \\ \downarrow & & \downarrow \\ \mathbb{Q} & \xrightarrow{\mathrm{id}_{\mathbb{Q}}} & \mathbb{Q} \end{array}$$

We do an induction over finite subsets of X_n to show that the pushout of (3.16) is $\mathbb{Q} \otimes_A \bigotimes_{X_n} A$ in each degree, proving the result. We remark that for any $\mathbb{Q}$-algebra B and set Y we have that

$$\mathbb{Q} \otimes_B \bigotimes_{Y \coprod \{y\}} B \cong \mathbb{Q} \otimes_B B \otimes_{\mathbb{Q}} \bigotimes_Y B \cong \mathbb{Q} \otimes_{\mathbb{Q}} \bigotimes_Y B \cong \bigotimes_Y B. \tag{3.17}$$

Now, the pushout of the diagram (3.16) where we let Y be a set consisting of two points is thus equivalent to

$$\begin{array}{ccc} \mathbb{Q}[t] & \longrightarrow & \mathbb{Q}[s] \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & A \end{array}$$

More generally, for any finite set $Y_i \subset X_n$ consisting of i elements let $Y = Y_i \cup \{y\}$ where $y \in X_n - Y_i$. We have then by (3.17) that the diagram (3.16) is equivalent to

$$\begin{array}{ccc} \bigotimes_{Y_i} \mathbb{Q}[t] & \longrightarrow & \bigotimes_{Y_i} \mathbb{Q}[s] \\ \downarrow & & \\ \mathbb{Q} & & \end{array}$$

whose pushout is $\bigotimes_{Y_i} A$ by the induction hypothesis (also easily seen directly). As such we have again by (3.17) that

$$\begin{array}{ccc} \mathbb{Q} \otimes_{\mathbb{Q}[t]} \bigotimes_{Y_{i+1}} \mathbb{Q}[t] & \longrightarrow & \mathbb{Q} \otimes_{\mathbb{Q}[s]} \bigotimes_{Y_{i+1}} \mathbb{Q}[s] \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes_A \bigotimes_{Y_{i+1}} A \end{array}$$

is a pushout for $Y_{i+1} = Y$. Since we can write X_n as the filtered colimit $X_n = \operatorname{colim}_i Y_i$ of such finite subsets Y_i and the Loday functor preserves colimits in its first argument, we can extend the pushout to X_n and consequently to X since a pushout of simplicial sets is degreewise a pushout of sets. □

Now, Greenlees spectral sequence 2.4.15 gives us that

$$E^2_{p,q} = L_p\left(X, A, \mathbb{Q}\right) \otimes L_q\left(X, \mathbb{Q}[t], \mathbb{Q}\right) \implies L_{p+q}\left(X, \mathbb{Q}[s], \mathbb{Q}\right) \tag{3.18}$$

and similarly that

$$\begin{aligned} E'^2_{p,q} &= L_p\left(S^1 \vee S^1, A, \mathbb{Q}\right) \otimes L_q\left(S^1 \vee S^1, \mathbb{Q}[t], \mathbb{Q}\right) \\ &\implies L_{p+q}\left(S^1 \vee S^1, \mathbb{Q}[s], \mathbb{Q}\right). \end{aligned} \tag{3.19}$$

Where the attaching map $X \to S^1 \vee S^1$ induces maps $E^2_{p,q} \to E'^2_{p,q}$ as indicated by the cube (3.15). Further we have explicit descriptors of the induced map on the first factor and know the induced map on the second to be zero.

We are therefore in a good position to do some low dimensional calculations on these spectral sequences and the induced maps between them. We begin by considering (3.18). We note that we can for many purposes consider S^1 instead of X to simplify some calculations of the spectral sequence. Although there is a priori no simplicial attaching map for S^1, we have already solved this issue earlier and have total control over the isomorphism $L_*(X, A, \mathbb{Q}) \cong L_*(S^1, A, \mathbb{Q})$ induced by $X \xrightarrow{\sim} S^1$, having an explicit description of the generators.

When working over the simplicial circle, higher Hochschild homology coincides with the regular Hochschild homology of commutative algebras, hence

$$L_*(S^1, B, \mathbb{Q}) = H_*(B, \mathbb{Q})$$

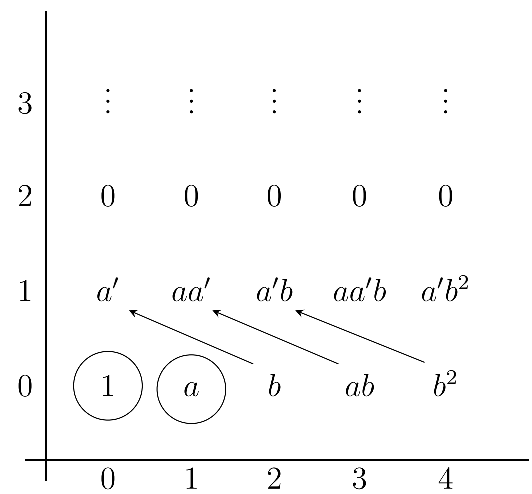
for a $\mathbb{Q}$-algebra B. We know from example 2.2.18 that

$$H_n(\mathbb{Q}[t], \mathbb{Q}) = \begin{cases} \mathbb{Q}\{1\}, & n = 0 \\ \mathbb{Q}\{1 \otimes t\}, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

and from proposition 3.2.1 that

$$H_n(A, \mathbb{Q}) = E(1 \otimes t) \otimes P(1 \otimes t \otimes t).$$

Denoting the generators by $a = 1 \otimes t$ and $b = 1 \otimes t \otimes t$ for the generators of $H_*(A, \mathbb{Q})$ and $a' = 1 \otimes t$ for the generator of $H_1(\mathbb{Q}[t], \mathbb{Q})$, the E^2-page takes the form



where everything but the two lower copies $1, a$ must die. As such, the differentials drawn above have to be isomorphisms with $db \doteq a'$, where we use $\doteq$ to denote equality up to some rational coefficient. Of course, da and da' are zero due to the spectral sequence being concentrated in the first quadrant. Using that the differentials of spectral sequences of algebras are graded derivations with respect to the product, we see that

$$d(ab) = (da)b - a(db) \doteq -aa'$$

and

$$db^2 \doteq 2a'b$$

and so on. In general the differential $d : E^2_{2n,0} \to E^2_{2(n-1),1}$ is given by

$$d(b^n) \doteq a'b^{n-1}$$

and $d : E^2_{2n+1,0} \to E^2_{2n-1,1}$ is given by

$$d(ab^n) \doteq aa'b^{n-1}.$$

We now do a similar calculation of the spectral sequence (3.19). Recall from the argument preceding proposition 3.2.2 that

$$L_*(S^1 \vee S^1, B, \mathbb{Q}) \cong L_*(S^1, B, \mathbb{Q})^{\otimes 2}$$

for a $\mathbb{Q}$-algebra B. Thus akin to our established convention of referring to the first factor as horizontal and the second as vertical, we name $a'^h = 1 \otimes t \otimes 1$ and $a'^v = 1 \otimes 1 \otimes t$ where the product is of the form $a_{(00,00)} \otimes a_{(01,00)} \otimes a_{(00,01)} \in L_*(S^1 \vee S^1, \mathbb{Q}[t], \mathbb{Q})$. We thus have that

$$L_*(S^1 \vee S^1, \mathbb{Q}[t], \mathbb{Q}) \cong E(a'^h, a'^v).$$

Similarly we have from the aforementioned proposition itself that

$$L_*(S^1 \vee S^1, A, \mathbb{Q}) \cong E(a^h, a^v) \otimes P(b^h, b^v)$$

and consequently the E^2 page of the Greenlees spectral sequence (3.19) takes the form

4	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
3	0	0	0	0	0	0
2	$a'^h a'^v$	$E^2_{1,2}$	$E^2_{2,2}$	$E^2_{3,2}$	$E^2_{4,2}$	$E^2_{5,2}$
1	a'^h a'^v	$a'^h a^h$ $a'^h a^v$ $a'^v a^h$ $a'^v a^v$	$E^2_{2,1}$	$E^2_{3,1}$	$E^2_{4,1}$	$E^2_{5,1}$
0	1	a^h a^v	b^h b^v $a^h a^v$	$a^h b^h$ $a^v b^h$ $a^h b^v$ $a^v b^v$	$a^h a^v b^h$ $a^h a^v b^v$ $b^h b^v$	$E^2_{5,0}$
	0	1	2	3	4	5

where we know from projecting to the horizontal and vertical factors that $db^h \doteq a'^h$ and $db^v \doteq a'^v$ and of course $da^h = da^v = 0$. Since d is a graded derivation this gives us all the other differentials, though not in quite as tidy a manner as before.

Now, our previous calculations with the weak equivalence $X \xrightarrow{\sim} S^1$ enables us to describe the induced map of a, b by making appropriate substitutions over the equivalence $L_*(S^1, A, \mathbb{Q}) \cong L_*(X, A, \mathbb{Q})$ and for the latter we have explicit descriptions of the generators over the map induced by the commutator. We will as an abuse of notation for the time being name the generators over X by a, b too. As such we may use the calculated differentials of the spectral sequences to study the commutator map. In particular we can consider the square

$$\begin{array}{ccc} L_2(X, A, \mathbb{Q}) & \xrightarrow{d} & L_1(X, \mathbb{Q}[t], \mathbb{Q}) \\ \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle c_*} \\ L_2(S^1 \vee S^1, A, \mathbb{Q}) & \xrightarrow{d} & L_1(S^1 \vee S^1, \mathbb{Q}[t], \mathbb{Q}) \end{array}$$

and use proposition 3.3.2 showing that $c_*(b) = a^h a^v$ so $dc_* = 0$, but since $\mathbb{Q}[t]$ is smooth we know that the right vertical commutator to be zero, hence $c_* d = 0$

and the square commutes. The calculation of the same proposition also shows that $c_* : L_1(X, A, \mathbb{Q}) \to L_1(X, A, \mathbb{Q})$ is 0.

Due to the shape of the first spectral sequence, we may use the Gysin sequence that we get from splicing the short exact sequences

$$\begin{array}{cccccccccccc}
 & & \downarrow & & & & & & & & & \\
 & & L_{n+1} & & & & & & 0 & & & \\
 & & \downarrow & & & & & & \downarrow & & & \\
0 & \longrightarrow & E^\infty_{n+1,0} & \longrightarrow & E^2_{n+1,0} & \xrightarrow{d} & E^2_{n-1,1} & \longrightarrow & E^\infty_{n-1,1} & \longrightarrow & 0 & \\
 & & \downarrow & & & & & & \downarrow & & & \\
 & & 0 & & & & & & L_n & & & \\
 & & & & & & & & \downarrow & & & \\
 & & & & & & 0 & \longrightarrow & E^\infty_{n,0} & \longrightarrow & E^2_{n,0} & \longrightarrow \\
 & & & & & & & & \downarrow & & & \\
 & & & & & & & & 0 & & &
\end{array}$$

into a long exact sequence

$$\cdots \longrightarrow L_2 \longrightarrow E^2_{2,0} \xrightarrow{d} E^2_{0,1} \longrightarrow L_1 \longrightarrow E^2_{1,0} \longrightarrow 0,$$

where we have above used L_n as the shorthand for $L_n(S^1, \mathbb{Q}[s], \mathbb{Q})$.

Although we can not construct a similar long exact sequence from the second spectral sequence, due to its shape, the same reasoning is valid in low dimensions, i.e. up to $L_2 \to E^2_{2,0}$, hence if we take in account the maps induced by the commutator we have a diagram

$$\begin{array}{ccccccccccc}
L_2(\mathbb{Q}[s]) & \longrightarrow & L_2(A) & \xrightarrow{d} & L_1(\mathbb{Q}[t]) & \longrightarrow & L_1(\mathbb{Q}[s]) & \longrightarrow & L_1(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle c_*} & & \downarrow & & \downarrow & & \downarrow & & \\
L_2(\mathbb{Q}[s]) & \longrightarrow & L_2(A) & \xrightarrow{d} & L_1(\mathbb{Q}[t]) & \longrightarrow & L_1(\mathbb{Q}[s]) & \longrightarrow & L_1(A) & \longrightarrow & 0
\end{array}$$

where we have the top and bottom rows are exact and the vertical maps are induced by the commutator. Note that we have suppressed a lot of redundant notation for the sake of space in the diagram above: The top row are all L_i taken over X and the bottom row are L_i taken over $S^1 \vee S^1$ for $i = 1, 2$. In both rows L_i is also with coefficients in $\mathbb{Q}$.

Observe that all commutator maps save

$$c_* : L_2(X, A, \mathbb{Q}) \to L_2(S^1 \vee S^1, A, \mathbb{Q})$$

are zero due to rational polynomial algebras being smooth. Reading the diagram from right to left, we have also established the commutativity of the first and third square, with the second being commutative trivially.

Even for the nonzero commutator we have as noted that the composition

$$dc_* : L_2(X, A, \mathbb{Q}) \to L_1(S^1 \vee S^1, \mathbb{Q}[t], \mathbb{Q})$$

is zero. We may then do a little bit of diagram chasing using the exactness of the bottom row to find that the map

$$L_2(S^1 \vee S^1, \mathbb{Q}[s], \mathbb{Q}) \to L_2(S^1 \vee S^1, A, \mathbb{Q})$$

maps $a'^h a'^v \mapsto a^h a^v = c_*(b)$. This means that we can as $\mathbb{Q}$-vector spaces make a lift up to homotopy

$$\begin{array}{ccc} & & \mathbb{Q} \otimes_{\mathbb{Q}[s]} \bigotimes_{S^1 \vee S^1} \mathbb{Q}[s] \\ & \overset{f}{\nearrow} & \downarrow \\ \mathbb{Q} \otimes_A \bigotimes_X A & \xrightarrow{c_*} & \mathbb{Q} \otimes_A \bigotimes_{S^1 \vee S^1} A \end{array} \tag{3.20}$$

by defining the lift to be zero on all cycles other than b, and have $f(b) = a'^h a'^v$.

This makes precise exactly how naturality fails for the higher Hochschild homology in this counter example. The square

$$\begin{array}{ccc} L_2(X, \mathbb{Q}[s], \mathbb{Q}) & \longrightarrow & L_2(X, A, \mathbb{Q}) \\ \downarrow & & \downarrow \\ L_2(S^1 \vee S^1, \mathbb{Q}[s], \mathbb{Q}) & \longrightarrow & L_2(S^1 \vee S^1, A, \mathbb{Q}) \end{array}$$

does not commute with the top right composition giving

$$b' \mapsto b \mapsto a^h a^v$$

and the bottom left having $b' \mapsto 0 \mapsto 0$, even though the latter map still maps $a'^h a'^v \mapsto a^h a^v$ and the aforementioned lift exists.

Chapter 4

Appendix

We have deferred writing out some explicit sums to keep the presentation less heavy. Two important ones we have however listed here for reference.

The Boundary of Alpha

The boundary $\partial\alpha$ is the sum given by

$$\begin{aligned}
&+ \tau_{\Delta^3\times\Delta^1}(1233, 0001) - \tau_{\Delta^3\times\Delta^1}(0233, 0001) + \tau_{\Delta^3\times\Delta^1}(0133, 0001) \\
&- \tau_{\Delta^3\times\Delta^1}(0123, 0001) + \tau_{\Delta^3\times\Delta^1}(0123, 0000) - \tau_{\Delta^3\times\Delta^1}(1223, 0011) \\
&+ \tau_{\Delta^3\times\Delta^1}(0223, 0011) - \tau_{\Delta^3\times\Delta^1}(0123, 0011) + \tau_{\Delta^3\times\Delta^1}(0123, 0001) \\
&- \tau_{\Delta^3\times\Delta^1}(0122, 0001) + \tau_{\Delta^3\times\Delta^1}(1123, 0111) - \tau_{\Delta^3\times\Delta^1}(0123, 0111) \\
&+ \tau_{\Delta^3\times\Delta^1}(0123, 0011) - \tau_{\Delta^3\times\Delta^1}(0113, 0011) + \tau_{\Delta^3\times\Delta^1}(0112, 0011) \\
&- \tau_{\Delta^3\times\Delta^1}(0123, 1111) + \tau_{\Delta^3\times\Delta^1}(0123, 0111) - \tau_{\Delta^3\times\Delta^1}(0023, 0111) \\
&+ \tau_{\Delta^3\times\Delta^1}(0013, 0111) - \tau_{\Delta^3\times\Delta^1}(0012, 0111).
\end{aligned}$$

where there are two pairs of terms along the diagonals of the vertical shifts cancelling each other, namely those given by the simplices:

$$(0123, 0001), (0123, 0011), (0123, 0111).$$

The Chain Beta

The chain β, where we have as an abuse suppressed the $\tau_{\Delta_3 \times \Delta^1}$ for brevity, is the sum

$$
\begin{aligned}
\beta = & \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\2&0&3&1\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&1\\2&0&3&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&1\\1&1&1&2\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&0&2\\1&1&1&1\end{smallmatrix}\right] \\
& - \left(\left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\2&0&2&2\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\2&0&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&0&1\\1&1&1&2\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&0&1\\1&1&2&1\end{smallmatrix}\right] \\
& - \left(\left[\begin{smallmatrix}0&0&0&0\\2&1&1&2\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\2&1&1&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&0&1\\1&2&1&1\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&1&1\\0&2&2&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\2&0&3&1\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&1\\2&0&3&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&1\\0&2&2&1\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&2&1\\1&1&1&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&1&2\\1&1&1&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&2&0\\2&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\2&0&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&2&1\\1&1&1&0\end{smallmatrix}\right] \\
& - \left(\left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&1&1\\1&1&2&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\2&1&2&1\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&2&0\\2&1&1&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\2&1&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&1&0\\2&1&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&1&1\\1&2&1&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&1&0\\1&2&2&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\1&2&2&1\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&0&1&1\\2&1&1&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\2&0&3&1\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&1\\2&0&3&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&1\\0&2&2&1\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&1&2&1\\1&1&0&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&1&1\\0&2&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&1&1&2\\1&1&0&0\end{smallmatrix}\right] \\
& - \left(\left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&2&0\\2&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\2&0&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&1&2&1\\1&1&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&1&2&0\\3&0&0&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&1&0&2\\3&0&0&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&2&0\\3&0&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&2\\3&0&1&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&2&1&1\\1&1&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&1&0\\2&1&2&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\2&1&2&1\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&1&2&0\\2&1&0&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&1&0&2\\2&1&0&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&1&1&1\\1&2&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\1&2&2&1\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&1&0\\1&2&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}0&1&1&1\\2&1&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&1&1\\0&2&2&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\5&0&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\2&0&3&1\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&1\\2&0&3&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&1\\0&2&2&1\end{smallmatrix}\right] - \left[\begin{smallmatrix}1&1&2&1\\1&0&0&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}1&1&1&2\\1&0&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&0\\4&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&2&0\\2&0&2&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\2&0&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}1&1&2&1\\1&0&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&2&0\\3&0&1&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0&0&2\\3&0&1&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\3&0&3&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&1&2&0\\3&0&0&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&1&0&2\\3&0&0&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}1&2&1&1\\1&0&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&1&0\\1&2&2&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&0&0\\1&2&2&1\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}1&1&1&1\\2&0&0&0\end{smallmatrix}\right] \\
& + \left(\left[\begin{smallmatrix}0&0&0&0\\2&1&2&1\end{smallmatrix}\right] + \left[\begin{smallmatrix}1&1&2&0\\2&0&0&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}1&1&0&2\\2&0&0&0\end{smallmatrix}\right] - \left[\begin{smallmatrix}0&0&1&0\\2&1&2&0\end{smallmatrix}\right]\right) \cdot \left[\begin{smallmatrix}2&1&1&1\\1&0&0&0\end{smallmatrix}\right]
\end{aligned}
$$

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