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STRONGLY STABLE STATIONARY POINTS FOR A CLASS OF GENERALIZED EQUATIONS*

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Abstract. In this paper we consider a generalized equation that is mainly characterized by a cone-valued mapping. It is well known that optimality conditions for different classes of optimization problems can be formulated as such a generalized equation. Moreover, we generalize Kojima's concept of strong stability and introduce appropriate constraint qualifications. We discuss corresponding properties between strong stability and these constraint qualifications. Finally, we apply these results to the particular class of mathematical programs with complementarity constraints and to that of mathematical programs with abstract constraints.

Key words. generalized equation, strong stability, generalized constraint qualifications, complementarity constraints, abstract constraints

MSC codes. 90C31, 49K40, 90C33, 90C30

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1. Introduction.

In this paper we consider the problem of finding a solution of a given generalized equation:

$$\mathcal{P}^{\Theta}(f,F)\colon\ \text{Find } x\in\mathbb{R}^n \text{ such that } D_xf(x)\in[\Theta(F(x))]^{\mathrm{t}}D_xF(x), \tag{1.1}$$

where $f\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$, $F\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^m)$, and $\Theta\colon\mathbb{R}^m\rightrightarrows\mathbb{R}^m$ is a cone-valued mapping; here, $\mathcal{P}^{\Theta}(f,F)$ can be rewritten as

$$\text{Find } x\in\mathbb{R}^n \text{ such that } D_xf(x)=\theta^{\mathrm{t}}D_xF(x) \text{ for some } \theta\in\Theta(F(x)).$$

Example 1.1. Consider a standard nonlinear program (NLP):

$$\min f(x)\quad \text{s.t.}\quad h_i(x)=0,\, i\in I,\ g_j(x)\geq 0,\, j\in J, \tag{1.2}$$

where I,J are finite index sets and all describing functions belong to $\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$. The problem of finding a stationary point of NLP can be written as in (1.1) by letting

$$F(x)=\begin{pmatrix}h_i(x),\, i\in I\\ g_j(x),\, j\in J\end{pmatrix},$$
$$\Theta(y^h,y^g)=\begin{cases}\mathbb{R}^{|I|}\times(\mathbb{R}^{|J|}_+\cap\{y^g\}^{\perp}) & \text{if } y^h_i=0,\, i\in I,\ y^g_j\geq 0,\, j\in J,\\ \emptyset & \text{otherwise,}\end{cases}$$

where $y^h\in\mathbb{R}^{|I|}$ and $y^g\in\mathbb{R}^{|J|}$.

This generalized equation appears as optimality (KKT-type) conditions for a broad family of optimization problems; see, e.g., [2, 5, 25]. However, we do *not* assume

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that the point under consideration is an optimal solution of a certain optimization problem; our focus lies on stability properties of a given solution of (1.1), which we call a *stationary* point.

When stationarity is motivated by optimality, $\Theta(F(x))$ is often a regular or a limiting normal cone; see, e.g., [27, Chapter 6]. However, considering other cone-valued mappings with (perhaps) larger graphs is also relevant. For instance, in mathematical programs with complementarity constraints, C-stationarity plays a crucial role in the description of topological changes of the feasible sublevel sets via Morse theory, which has applications in global optimization and homotopy methods; see [16].

In this paper, we consider certain constraint qualifications. Roughly speaking, we say that a constraint qualification is stronger than another one if it is always the case that the fulfillment of the former implies that of the latter. In the forthcoming Definition 3.9, $\Theta(F(x))$ is used to define the generalized Mangasarian–Fromovitz constraint qualification (GMFCQ). Here again, an argument could be made in favor of considering at most limiting normal cones: a larger $\Theta(F(x))$ results in a stronger constraint qualification. Note that the tradeoff is worth it when considering, for instance, the *local stability* of feasible sets; see, e.g., [17]. In our view, our general setting regarding GMFCQ and strong stability is a preliminary step for the study of the interplay between different stability concepts.

For a stationary point, we will generalize the concept of *strong stability* which was introduced by Kojima [21] for standard nonlinear programs. This concept refers to the local existence, uniqueness, and continuity of a stationary point for each sufficiently small perturbed problem. Here, the values of perturbations and their derivatives up to second order are taken into consideration. However, they do not necessarily depend on real parameters. In particular, many results about strong stability can be applied whenever only sufficiently small linear and quadratic perturbations are allowed; see, e.g., [21, Corollary 4.3]. We refer to several papers which are related to strong stability [4, 9, 15, 18, 28].

We will also introduce another stability concept, called *weak stability*, which, in general, is weaker than strong stability. However, it turns out that both concepts are equivalent in many cases.

Besides dealing with strong stability, we generalize two constraint qualifications that appeared in the context of mathematical programs with disjunctive constraints: the generalized linear independence constraint qualification (GLICQ) (cf. [23, Definition 3.1]); and the GMFCQ; cf. [5, Definition 4]. For (1.1), it is not necessarily the case that GLICQ still holds after a sufficiently small perturbation. Thus, to strengthen GLICQ, we define the closed linear independence constraint qualification (CLICQ). If CLICQ holds at a point $\bar{x}$, then it also holds in a neighborhood of $\bar{x}$. Furthermore, strong stability and CLICQ imply the existence and uniqueness of a Lagrange vector for any sufficiently small perturbed problem. For NLP, GLICQ and CLICQ are identical, and GLICQ and GMFCQ are the well-known LICQ and MFCQ, respectively.

Another difference between the problems (1.1) and (1.2) is that for the former GMFCQ need not be a necessary condition for strong stability. This situation with corresponding properties is discussed in the forthcoming Theorem 3.25.

The so-obtained general results on strong stability and constraint qualifications for (1.1) will be applied to the important classes of mathematical programs with complementarity constraints (MPCC); see, e.g., [18, 19, 22], and mathematical programs with abstract constraints; see, e.g., [1]. In particular, we will show that for certain stationary points for these classes, GMFCQ is a necessary condition for strong stability. Note that for the class of problems with abstract constraints, GMFCQ and the Robinson Constraint Qualification [24] are equivalent.

Summarizing, the goal of this paper is threefold:

- To extend Kojima's concept of strong stability for the family of generalized equations (1.1).
- To introduce appropriate constraint qualifications and show their relation to strong stability.
- To discuss these results for MPCC, here the concepts of M- and S-stationarity are considered, and for mathematical programs with abstract constraints.

This paper is organized as follows. In section 2 we present basic notation and some auxiliary results. Section 3 starts with a generalization of some well-known terminology related to stationarity. Moreover, the concepts of weak and strong stability as well as appropriate constraints qualifications are presented. Then, section 4 mainly discusses the interplay between both weak and strong stability, and these constraint qualifications, whenever Θ is outer semicontinuous. In sections 5 and 6 we apply the results from sections 3 and 4 to the particular classes of MPCC and mathematical programs with abstract constraint, respectively.

2. Some basic notation and auxiliary results. The first part of this section is mainly taken from [12]. For $w \in \mathbb{R}^n$ let w_i, $i = 1, \dots, n$ denote its components and define the index sets

$$I^0(w) = \{i \in \{1, \dots, n\} \,|\, w_i = 0\},$$
$$I^*(w) = \{i \in \{1, \dots, n\} \,|\, w_i \neq 0\}.$$

Given $\bar{x}, x \in \mathbb{R}^n$, let $\langle \bar{x}, x \rangle$ denote the scalar product of the vectors $\bar{x}$ and x. As usual, here $\|x\|$ stands for the Euclidean norm of x, that is, $\|x\| = \sqrt{\langle x, x \rangle}$. Furthermore, for $\delta > 0$ let

$$B^n(\bar{x}, \delta) = \{x \in \mathbb{R}^n \,|\, \|x - \bar{x}\| \leq \delta\},$$
$$S^n(\bar{x}, \delta) = \{x \in \mathbb{R}^n \,|\, \|x - \bar{x}\| = \delta\}.$$

We abbreviate the sentence "V is a neighborhood of $\bar{x}$" by letting $\mathcal{V}(\bar{x})$ be the *set of all neighborhoods* of $\bar{x}$ and write then the aforementioned statement as "$V \in \mathcal{V}(\bar{x})$."

Let $\mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^m)$ be the space of $k-$times continuously differentiable mappings with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$. For $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ denote the partial derivative of f at $\bar{x} \in \mathbb{R}^n$ with respect to x_i by $\frac{\partial f(\bar{x})}{\partial x_i}$, $i = 1, \dots, n$. In addition, $D_x f(\bar{x})$ stands for its gradient taken as a row vector and $D_x^2 f(\bar{x})$ for its Hessian at $\bar{x}$. Moreover, for $F \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$ let $D_x F(\bar{x}) \in \mathbb{R}^{m \times n}$ be its Jacobian at $\bar{x}$. By $\mathbb{R}^n_+$ we denote the n-dimensional nonnegative orthant.

For applying the concept of strong stability we need a seminorm for functions. Let $V \in \mathcal{V}(\bar{x})$ and $\bar{G} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$. Following [21], denote

(2.1)
$$\|\bar{G}\|^V = \max\left\{ \sup_{x \in V} \max_i \left\{ |\bar{G}_i(x)| \right\}, \sup_{x \in V} \max_{i,j} \left\{ \left| \frac{\partial \bar{G}_i(x)}{\partial x_j} \right| \right\}, \sup_{x \in V} \max_{i,j,k} \left\{ \left| \frac{\partial^2 \bar{G}_i(x)}{\partial x_j \partial x_k} \right| \right\} \right\},$$

where the indices i and j, k are varying in the sets $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. The *set of all neighborhoods* of $\bar{G}$ with respect to the seminorm induced by $V \in \mathcal{V}(\bar{x})$ is denoted by $\mathcal{U}^V(\bar{G})$.

Let $O \subset \mathbb{R}^n$. Throughout this paper, $\operatorname{int} O$ and $\operatorname{cl} O$ denote the *interior* and the *closure* of O, respectively. Moreover, O^{c} and $O^{\perp}$ stand for its *complement* and its *orthogonal complement*, respectively. Furthermore, $\operatorname{aff} O$ and $\operatorname{span} O$ denote the smallest affine set and the smallest subspace containing O; those are the so-called *affine hull* and the *linear hull* of O, respectively. The *relative interior* and the *relative boundary* of O are defined as follows:

$$\operatorname{rel\,int} O = \left\{ x \in O \,\middle|\, \begin{array}{l} \text{there exists } \delta > 0 \text{ with} \\ B^n(x,\delta) \cap \operatorname{aff} O \subset O \end{array} \right\},$$
$$\operatorname{rel\,bd} O = \operatorname{cl} O \setminus \operatorname{rel\,int} O;$$

see, e.g., [26]. For $\lambda \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$ let

$$O + \bar{x} = \{x + \bar{x} \mid x \in O\},$$
$$\lambda O = \{\lambda x \mid x \in O\}.$$

Identifying $\mathbb{R}^n$ with $\mathbb{R}^{n\times 1}$ we define the transpose of O as the set

$$O^{\mathrm{t}} = \{x^{\mathrm{t}} \mid x \in O\},$$

where x^{t} denotes the transpose of the column vector x. For $O^1, O^2 \subset \mathbb{R}^n$ recall that the Minkowski sum is

$$O^1 + O^2 = \{x_1 + x_2 \mid x_1 \in O^1, x_2 \in O^2\}.$$

The set $O^1 - O^2$ is analogously defined.

The set $K \subset \mathbb{R}^m$ is called a *cone* if $\lambda K \subset K$ for each $\lambda > 0$. Note that in this paper it might happen that $0 \notin K$, whereas other authors would not call a set a cone if it does not contain the origin. Regardless, it is always the case that $0 \in \operatorname{cl} K$. The *conic hull* of O is the smallest cone containing O and it is denoted by $\operatorname{cone} O$. The set

$$K^* = \{z \in \mathbb{R}^m \mid \langle z, \theta\rangle \ge 0 \text{ for all } \theta \in K\}$$

is called the *dual cone* of K. The *lineality space* of a convex cone K, denoted by $\operatorname{lin} K$, is the largest subspace contained in K, that is, $\operatorname{lin} K = K \cap (-K)$. If K is also closed, then $\operatorname{lin} K = [K^*]^{\perp}$; see, e.g., [2]. The following two results play a crucial role in section 6.

LEMMA 2.1.
(i) *If $K \subset \mathbb{R}^m$ is a convex cone, then* $\operatorname{rel\,int} K + K = \operatorname{rel\,int} K$.
(ii) *If $C \subset \mathbb{R}^m$ is a convex set, then*

$$\lambda \operatorname{rel\,bd} C + (1-\lambda) \operatorname{rel\,int} C \subset C^{\mathrm{c}}$$

for all $\lambda > 1$.

Proof. Both conditions follow immediately from [26, Theorem 6.1]. ∎

THEOREM 2.2 (see [26, Theorem 11.3]). *Let $C_1, C_2 \subset \mathbb{R}^m$ be nonempty convex sets. The following two conditions are equivalent:*
(i) $\operatorname{rel\,int} C_1 \cap \operatorname{rel\,int} C_2 = \emptyset$.
(ii) *There exist $\omega \in \mathbb{R}^m$ and $\nu \in \mathbb{R}$ such that*

$$\langle \omega, y^1\rangle \le \nu \le \langle \omega, y^2\rangle, \quad y^1 \in C_1, \quad y^2 \in C_2.$$

Moreover, for some $i = 1,2$ and some $\bar{y}^i \in C_i$, it holds that $\langle \omega, \bar{y}^i\rangle \ne \nu$.

Let $A \in \mathbb{R}^{m\times n}$. The rank of A is denoted by rank A and its transpose by A^{t}. The linear subspaces

$$\begin{aligned}\operatorname{im} A &= \{y \in \mathbb{R}^m \,|\, y = Ax,\, x \in \mathbb{R}^n\},\\ \ker A &= \{x \in \mathbb{R}^n \,|\, Ax = 0\}\end{aligned}$$

are the *image* and the *kernel* of A, respectively. These subspaces fulfill the properties

$$\operatorname{im} A = [\ker A^{\mathrm{t}}]^{\perp}, \quad \ker A = [\operatorname{im} A^{\mathrm{t}}]^{\perp};$$

see, e.g., [3, section 2.5]. Let

$$AO = \{Ax \,|\, x \in O\}$$

be the set $O^{\mathrm{t}}A'$ is analogously defined for $A' \in \mathbb{R}^{n\times m}$.

Let $E \subset \mathbb{R}^m$ be a linear subspace. The projection of $y \in \mathbb{R}^m$ onto E is denoted by $\operatorname{proj}_E(y)$. The dimension of E is denoted by $\dim E$.

Next, we present some concepts related to set-valued mappings. Let Z and Z' be two seminormed spaces, and let $\Psi\colon Z \rightrightarrows Z'$ be a set-valued mapping. The *inverse* of Ψ is the set-valued mapping $\Psi^{-1}\colon Z' \rightrightarrows Z$ given by

$$\Psi^{-1}(\psi) = \{z \in Z \,|\, \psi \in \Psi(z)\}.$$

For $V \subset Z$ let

$$\Psi(V) = \bigcup_{z\in V} \Psi(z).$$

The sets $\operatorname{dom}\Psi = \{z \in Z \,|\, \Psi(z) \neq \emptyset\}$ and $\operatorname{im}\Psi = \Psi(Z)$ are the *domain* and the *image* of Ψ, respectively. The set

$$\operatorname{gr}\,\Psi = \{(z, \Psi(z)) \,|\, z \in \operatorname{dom}\Psi\}$$

is called the *graph* of Φ. If the set $\operatorname{gr}\Psi$ is closed, then Φ is said to be a *closed* set-valued mapping. The following definitions are adaptations of those in [27, p. 152]. There, the spaces under consideration are $\mathbb{R}^n$ and $\mathbb{R}^m$, whereas in this paper the definitions are given for two seminormed spaces.

DEFINITION 2.3.

(i) *The outer limit and the inner limit of Ψ at $\bar z$ are given by*

$$\limsup_{z\to\bar z} \Psi(z) = \left\{ \psi \in Z' \;\middle|\; \begin{array}{l} \textit{there exist sequences } z^k \to \bar z \\ \textit{and } \psi^k \to \psi \textit{ with } \psi^k \in \Psi(z^k) \end{array} \right\}$$

and

$$\liminf_{z\to\bar z} \Psi(z) = \left\{ \psi \in Z' \;\middle|\; \begin{array}{l} \textit{for each sequence } z^k \to \bar z \textit{ there} \\ \textit{exists } \psi^k \to \psi \textit{ and } k_0 \in \mathbb{N} \\ \textit{with } \psi^k \in \Psi(z^k),\ k \geq k_0 \end{array} \right\},$$

respectively. The index k used for describing a sequence is always varying over $\mathbb{N}$.

(ii) *Ψ is called outer semicontinuous (osc) at $\bar z \in Z$ if*

$$\limsup_{z\to\bar z} \Psi(z) = \Psi(\bar z),$$

and inner semicontinuous (isc) at $\bar z \in Z$ if

$$\liminf_{z\to\bar z} \Psi(z) = \Psi(\bar z).$$

Moreover, it is continuous at $\bar{z}\in Z$ *if*

$$\liminf_{z\to\bar{z}} \Psi(z) = \limsup_{z\to\bar{z}} \Psi(z) = \Psi(\bar{z}).$$

(iii) Ψ *is called locally bounded at* $\bar{z}$ *if there exists* $V\in\mathcal{V}(\bar{z})$ *such that* $\Psi(V)$ *is bounded.*

In particular, Definition 2.3 implies that

$$\liminf_{z\to\bar{z}} \Psi(z) \subset \Psi(\bar{z}) \subset \limsup_{z\to\bar{z}} \Psi(z);$$

see [27, p. 152]. We end this section by recalling an auxiliary result from topology and presenting two lemmas about $\|\cdot\|^V$.

LEMMA 2.4 (see [14, Lemma 2.2.2a]). *Let* $V^1\subset\mathbb{R}^n$ *be a closed subset, and let* V^2 *be an open neighborhood of* V^1*. Then, there exists a function* $\xi\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ *such that*
(i) $0\le\xi(x)\le 1$ *for all* $x\in\mathbb{R}^n$.
(ii) $\xi(x)=1$ *on some neighborhood of* V^1.
(iii) $\operatorname{supp}\xi\subset V^2$*, where* $\operatorname{supp}\xi = \operatorname{cl}\{x\in\mathbb{R}^n \,|\, \xi(x)\ne 0\}$.

LEMMA 2.5. *Let* $\bar{x}\in\mathbb{R}^n$*,* $f\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$*,* $F\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^m)$*, and* $V\in\mathcal{V}(\bar{x})$*. Then, it holds that*

$$\|f\cdot F\|^V \le 4\|f\|^V\|F\|^V.$$

Proof. The proof is an immediate consequence of the product rule for derivatives. ∎

LEMMA 2.6. *Let* V^1*,* V^2*, and* ξ *be given as in Lemma* 2.4*. Assume that for* $F^k\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^m)$*,* $k\in\mathbb{N}$*, it holds that* $\|F^k-\bar{F}\|^{V^2}\to 0$ *and define*

$$F^{\xi,k}(x) = \xi(x)F^k(x) + (1-\xi(x))\bar{F}(x).$$

Then, the following conditions hold:
(i) $F^{\xi,k}(x) = F^k(x)$ *for all* $x\in V^1$.
(ii) $\|F^{\xi,k}-\bar{F}\|^{V^3}\to 0$ *for all* $V^3\in\mathcal{V}(\bar{x})$ *with* $V^2\subset V^3$.

Proof. (i) The proof easily follows by observing that $\xi(x)=1$ for all $x\in V^1$.
(ii) Note that

$$\|F^{\xi,k}-\bar{F}\|^{V^3} = \|F^{\xi,k}-\bar{F}\|^{V^2} = \|\xi\cdot(F^k-\bar{F})\|^{V^2}.$$

Hence, an application of Lemma 2.5 yields the desired result. ∎

3. Strong stability and constraint qualifications. In this section we consider the problem $P=\mathcal{P}^\Theta(f,F)$ as given in (1.1). Recall that Θ is a cone-valued mapping, which means that the set $\Theta(y)$ is a cone for each $y\in\operatorname{dom}\Theta$. The first part of this section is devoted to the generalization of well-known stationarity concepts mainly related to optimality conditions and also to other aspects of Optimization and Numerical Analysis; see Remark 3.2.

DEFINITION 3.1.
(i) *A point* $\bar{x}\in\operatorname{dom}\Theta\circ F$ *is called a feasible point for* F.
(ii) *A point* $\bar{x}\in\mathbb{R}^n$ *is called a Fritz John point for* P *if*

$$\theta_0 D_x f(\bar{x}) = \theta^{\mathrm{t}} D_x F(\bar{x})$$

for some $(\theta_0, \theta) \in [\mathbb{R}_+ \times \Theta(F(\bar{x}))] \setminus \{0\}$*. The set of Fritz John points for* P *is denoted by* $\Sigma^F(P)$*.*

(iii) *A point* $\bar{x} \in \mathbb{R}^n$ *that solves* P *is called a stationary point for* P*. The set of stationary points for* P *is denoted by* $\Sigma(P)$*.*

Obviously, we have

$$\Sigma(P) \subset \Sigma^F(P) \subset \operatorname{dom} \Theta \circ F. \tag{3.1}$$

Define for $x \in \mathbb{R}^n$ the *set of Lagrange vectors* as

$$\mathcal{L}(P, x) = \left\{\theta \in \Theta(F(x)) \,|\, D_x f(x) = \theta^{\mathrm{t}} D_x F(x)\right\}.$$

In particular, it holds that $x \in \Sigma(P)$ if and only if $\mathcal{L}(P, x) \neq \emptyset$.

Remark 3.2. As mentioned in section 1, the concepts given in Definition 3.1 are mainly related to certain normal cones, whenever optimality conditions are considered. The generality in Definition 3.1 allows the consideration of other cones and stationarity concepts as well, e.g., those related to the topology of the feasible sublevel sets; see [16]. The forthcoming Remark 4.4 presents such a cone.

For the following it is important to mention that $P = \mathcal{P}^\Theta(f, F)$ is more than just a notation. Given Θ as in (1.1), we define the mapping $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \to \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \times \{\Theta\}$ by the expression

$$\mathcal{P}^\Theta(f, F) = (f, F, \Theta).$$

Obviously, the mapping $\mathcal{P}^\Theta$ is a bijection and, thus, it naturally provides the set $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \times \{\Theta\}$ with the same structure as that of $\mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1})$. In particular, given $\bar{x} \in \mathbb{R}^n$, $V \in \mathcal{V}(\bar{x})$, and $P = \mathcal{P}^\Theta(f, F)$, we can define

$$\|P\|^V = \|(f, F)\|^V,$$

where the right-hand side is obtained from (2.1) by choosing $\bar{G} = (f, F)$. Furthermore, for $\delta > 0$ define

$$B^V(\bar{P}, \delta) = \{P \,|\, \|P - \bar{P}\|^V \leq \delta\},$$

and let $\mathcal{W}^V(\bar{P})$ denote the *set of all neighborhoods* of $\bar{P}$.

In the remainder of this section, let $\bar{P} = \mathcal{P}^\Theta(\bar{f}, \bar{F})$ and $\bar{x} \in \mathbb{R}^n$ be the problem and the point under consideration, respectively. Next, we generalize Kojima's definition of a strongly stable stationary point [21].

Definition 3.3. *A point* $\bar{x} \in \Sigma(\bar{P})$ *is called strongly stable if there exists a real number* $\bar{\delta} > 0$ *such that for each* $\delta \in (0, \bar{\delta}]$ *there exists a real number* $\varepsilon > 0$ *such that for every* $P \in B^{B^n(\bar{x}, \bar{\delta})}(\bar{P}, \varepsilon)$ *it holds that*

$$|\Sigma(P) \cap B^n(\bar{x}, \bar{\delta})| = |\Sigma(P) \cap B^n(\bar{x}, \delta)| = 1.$$

The set of strongly stable stationary points for $\bar{P}$ *is denoted by* $\Sigma^S(\bar{P})$*.*

Kojima [21] defined strong stability with respect to a family of functions. Here, we focus on the case where such family is the whole space of twice continuously differentiable functions. In the following we present another stability concept which relaxes Definition 3.3. In the forthcoming Theorem 4.6, we prove that both stability concepts are equivalent whenever Θ is osc.

DEFINITION 3.4. *A point $\bar{x} \in \Sigma(\bar{P})$ is called weakly stable if there exist real numbers $\bar{\delta} > 0$ and $\bar{\varepsilon} > 0$ such that for every $P \in B^{B^n(\bar{x},\bar{\delta})}(\bar{P}, \bar{\varepsilon})$ it holds that*

$$|\Sigma(P) \cap B^n(\bar{x}, \bar{\delta})| = 1.$$

The set of weakly stable stationary points for $\bar{P}$ is denoted by $\Sigma_0^S(\bar{P})$.

Now, we turn our attention to a set-valued mapping that can be osc regardless of strong stability.

DEFINITION 3.5. *Let $V \in \mathcal{V}(\bar{x})$. The set-valued mapping $\widehat{\mathcal{L}}^V : \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^{m+1}) \times \{\Theta\} \rightrightarrows \mathbb{R}^m$ given by*

$$\widehat{\mathcal{L}}^V(P) = \bigcup_{x \in V} \mathcal{L}(P, x)$$

represents the union of Lagrange vectors for all $x \in V$.

LEMMA 3.6. *If $V \in \mathcal{V}(\bar{x})$ is compact and Θ is osc, then $\widehat{\mathcal{L}}^V(\cdot)$ is osc.*

Proof. The proof follows immediately from a continuity argument. ☐

The next example illustrates that strong stability does *not* imply the continuity of $\widehat{\mathcal{L}}^V(\cdot)$.

Example 3.7. Let $n = 1$, $\bar{x} = 0$, and consider the standard nonlinear program

$$\bar{P} : \min x \quad \text{s.t.} \quad x \geq 0, x \geq 0.$$

According to [21, Theorem 7.2], it holds that $0 \in \Sigma^S(\bar{P})$. Fix $V \in \mathcal{V}(0)$ sufficiently small and consider the two sequences of problems given by

$$P^{1,k} : \min x \quad \text{s.t.} \quad x + \frac{1}{k} \geq 0, \quad x \geq 0$$

and

$$P^{2,k} : \min x \quad \text{s.t.} \quad x \geq 0, \quad x + \frac{1}{k} \geq 0.$$

Obviously, it holds that

$$\lim_{k \to \infty} P^{1,k} = \lim_{k \to \infty} P^{2,k} = \bar{P}.$$

However, we have $\widehat{\mathcal{L}}^V(P^{1,k}) = \{(0,1)^{\mathrm{t}}\}$ and $\widehat{\mathcal{L}}^V(P^{2,k}) = \{(1,0)^{\mathrm{t}}\}$, which implies that

$$\liminf_{P \to \bar{P}} \widehat{\mathcal{L}}^V(P) = \emptyset.$$

Thus, strong stability does not imply that $\widehat{\mathcal{L}}^V(\cdot)$ is isc and, thus, continuous at $\bar{P}$.

Next, we generalize the two most well-known constraint qualifications in nonlinear programming.

DEFINITION 3.8. *We say that the GLICQ holds at $\bar{x}$ for $\bar{F}$ if*

$$\ker D_x\bar{F}(\bar{x})^{\mathrm{t}} \cap \operatorname{span}\Theta(\bar{F}(\bar{x})) = \{0\}.$$

DEFINITION 3.9. *We say that the GMFCQ holds at $\bar{x}$ for $\bar{F}$ if*

$$\ker D_x\bar{F}(\bar{x})^{\mathrm{t}} \cap \Theta(\bar{F}(\bar{x})) \subset \{0\}.$$

The set of points for $\bar{P}$ at which GMFCQ does not hold is denoted by $\Sigma_0^F(\bar{P})$.

Remark 3.10. If we consider NLP, which is described in (1.2), then GLICQ and GMFCQ are equivalent to the (standard) LICQ and MFCQ whose definition we recall in the following. We say that LICQ holds at $\bar{x}$ if the vectors

$$D_x h_i(\bar{x}), \quad i \in I, \quad D_x g_j(\bar{x}), \quad j \in I^0(g(\bar{x}))$$

are linearly independent. We say that MFCQ holds at $\bar{x}$ if the following two conditions are fulfilled:

(i) The vectors $D_x h_i(\bar{x})$, $i \in I$ are linearly independent.

(ii) There exists $u \in \mathbb{R}^n$ with $D_x h_i(\bar{x})u = 0$, $i \in I$ and $D_x g_j(\bar{x})u > 0$, $j \in I^0(g(\bar{x}))$.

Remark 3.11. Let $\Theta^1, \Theta^2 \colon \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ be two cone-valued mappings with $\operatorname{gr}\Theta^1 \subset \operatorname{gr}\Theta^2$. Obviously, we have

$$\ker D_x \bar{F}(\bar{x})^{\mathrm{t}} \cap \Theta^1(\bar{F}(\bar{x})) \subset \ker D_x \bar{F}(\bar{x})^{\mathrm{t}} \cap \Theta^2(\bar{F}(\bar{x})).$$

Therefore, according to Definition 3.9, GMFCQ with respect to Θ^2 is stronger than GMFCQ with respect to Θ^1. Stronger constraint qualifications are more restrictive and, thus, cone-valued mappings with smaller graphs are more desirable. However, as mentioned in section 1, the tradeoff of having larger graphs is worth it when considering certain stability properties; see, e.g., [17].

Remark 3.12. If we consider mathematical programs with abstract constraints, as in the forthcoming section 6, then Definition 3.8 is analogous to that of *nondegeneracy* in [2, Definition 4.70]. Moreover, note that Definitions 3.8 and 3.9 generalize [23, Definition 3.1] and [5, Definition 4], respectively.

Obviously, if $\bar{x}$ is not feasible, then GLICQ holds at $\bar{x}$. Moreover, GLICQ implies GMFCQ and also that $|\mathcal{L}(\bar{P},\bar{x})| \leq 1$. By Definition 3.9, we have

$$\Sigma^F(\bar{P}) = \Sigma_0^F(\bar{P}) \cup \Sigma(\bar{P}).$$

LEMMA 3.13. *Assume that for $\bar{\eta} \in \Theta(\bar{F}(\bar{x})) \cap S^m(0,1)$, it holds that $\bar{\eta}^{\mathrm{t}} D_x \bar{F}(\bar{x}) = 0$. Furthermore, for $k \in \mathbb{N}$ define*

$$F^k(x) = \bar{F}(x) + k^{-1}\bar{\eta} D_x \bar{f}(\bar{x})(x - \bar{x}),$$

and let $P^k = \mathcal{P}^{\Theta}(\bar{f}, F^k)$. Then, $k\bar{\eta} \in \mathcal{L}(P^k, \bar{x})$ for all $k \in \mathbb{N}$.

Proof. Fix $k \in \mathbb{N}$. Since $\Theta(\bar{F}(\bar{x}))$ is a cone, we have $k\bar{\eta} \in \Theta(\bar{F}(\bar{x}))$. Moreover, it is $F^k(\bar{x}) = \bar{F}(\bar{x})$ and, therefore, $k\bar{\eta} \in \Theta(F^k(\bar{x}))$. Furthermore, it holds that

$$k\bar{\eta}^{\mathrm{t}} D_x F^k(\bar{x}) = k\bar{\eta}^{\mathrm{t}} D_x \bar{F}(\bar{x}) + k\bar{\eta}^{\mathrm{t}} k^{-1}\bar{\eta} D_x \bar{f}(\bar{x}) = D_x \bar{f}(\bar{x}),$$

which completes the proof. ◻

Remark 3.14. Analogously to [9, Remark 3.1], the perturbation described in Lemma 3.13 can be refined by means of a bump function to get a problem $P^{k,\xi}$ whose describing functions coincide with those of P^k inside a given ball $B^n(\bar{x},\bar{\delta})$ and with those of $\bar{P}$ outside $B^n(\bar{x},2\bar{\delta})$ for certain $\bar{\delta} > 0$. To see this, apply Lemma 2.6 with $V^1 = B^n(\bar{x},\bar{\delta})$ and $V^2 = \operatorname{int} B^n(\bar{x},2\bar{\delta})$ to the functions describing P^k and $\bar{P}$.

PROPOSITION 3.15. *If $\mathcal{L}(\bar{P},\bar{x})$ is a nonempty compact set and $\Theta(\bar{F}(\bar{x}))$ is convex, then GMFCQ holds at $\bar{x}$ for $\bar{F}$.*

Proof. Let $\bar{\theta} \in \mathcal{L}(\bar{P}, \bar{x})$ and suppose contrarily that $\bar{\eta}^{\mathrm{t}} D_x \bar{F}(\bar{x}) = 0$ for some $\bar{\eta} \in \Theta(\bar{F}(\bar{x})) \setminus \{0\}$. The convexity of the cone $\Theta(\bar{F}(\bar{x}))$ yields $\bar{\theta} + t\bar{\eta} \in \mathcal{L}(\bar{P}, \bar{x})$ for all $t \geq 0$, which contradicts the compactness of $\mathcal{L}(\bar{P}, \bar{x})$. ☐

The following stationarity concept plays a crucial role in a necessary condition for the weak stability of stationary points at which GMFCQ does not hold.

DEFINITION 3.16. *A point $\bar{x} \in \mathbb{R}^n$ is called an inner stationary point for $\bar{P}$ if*

$$D_x \bar{f}(\bar{x}) \in \operatorname{int}[\Theta(\bar{F}(\bar{x}))^{\mathrm{t}} D_x \bar{F}(\bar{x})].$$

The set of inner stationary points for $\bar{P}$ is denoted by $\Sigma^{\mathrm{in}}(\bar{P})$.

For standard nonlinear programs, a property analogous to Definition 3.16 is a sufficient condition for a point to be a *local minimizer of order one*; see [31, Remark 3.6].

LEMMA 3.17. *If $\bar{x} \in \Sigma^{\mathrm{in}}(\bar{P})$, then the following conditions hold:*
(i) $D_x \bar{F}(\bar{x}) u \notin \Theta(\bar{F}(\bar{x}))^{\perp}$ *for all $u \in S^n(0,1)$.*
(ii) $\operatorname{im} D_x \bar{F}(\bar{x}) \cap \Theta(\bar{F}(\bar{x}))^{\perp} = \{0\}$.
(iii) $m > n$ *and* $\operatorname{rank} D_x \bar{F}(\bar{x}) = n$, *that is,* $\ker D_x \bar{F}(\bar{x}) = \{0\}$.

Proof. We prove (i). Conditions (ii) and (iii) easily follow from (i). Suppose contrarily that $D_x \bar{F}(\bar{x}) \bar{u} \in \Theta(\bar{F}(\bar{x}))^{\perp}$ for some $\bar{u} \in S^n(0,1)$. Since $\bar{x} \in \Sigma^{\mathrm{in}}(\bar{P})$, for all $\varepsilon > 0$ sufficiently small, there exists $\theta^{\varepsilon} \in \Theta(\bar{F}(\bar{x}))$ such that

$$D_x \bar{f}(\bar{x}) + \varepsilon \bar{u}^{\mathrm{t}} = [\theta^{\varepsilon}]^{\mathrm{t}} D_x \bar{F}(\bar{x}). \tag{3.2}$$

Multiplying by $\bar{u}$, we obtain $\varepsilon = -D_x \bar{f}(\bar{x}) \bar{u}$ which is a contradiction since (3.2) holds for all $\varepsilon > 0$ sufficiently small. ☐

COROLLARY 3.18. *If $\bar{x} \in \Sigma^{\mathrm{in}}(\bar{P})$ and*

$$\operatorname{span} \Theta(\bar{F}(\bar{x})) = \left\{ \theta \in \mathbb{R}^m \,|\, \theta_i \cdot \bar{F}_i(\bar{x}) = 0,\ i \in \{1, \ldots, m\} \right\}, \tag{3.3}$$

then $|I^0(\bar{F}(\bar{x}))| \geq n$ and n of the following vectors:

$$D_x \bar{F}_i(\bar{x}),\ i \in I^0(\bar{F}(\bar{x}))$$

are linearly independent.

Proof. By (3.3), we have

$$\Theta(\bar{F}(\bar{x}))^{\perp} = [\operatorname{span} \Theta(\bar{F}(\bar{x}))]^{\perp} = \left\{ \theta \in \mathbb{R}^m \,|\, I^0(\bar{F}(\bar{x})) \subset I^0(\theta) \right\}.$$

Hence, we have

$$D_x \bar{F}(\bar{x}) u \in \Theta(\bar{F}(\bar{x}))^{\perp}$$

for some $u \in \mathbb{R}^n$ if and only if

$$D_x \bar{F}_i(\bar{x}) u = 0, \quad i \in I^0(\bar{F}(\bar{x})).$$

Thus, by Lemma 3.17 (i), the only possible solution to the previous system of equations is $u = 0$, which yields the desired result. ☐

The next result presents a necessary condition for weak stability.

THEOREM 3.19. *If $\bar{x} \in \Sigma_0^S(\bar{P})$, then there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that*

$$\Sigma(P) \cap V = \Sigma^F(P) \cap V \tag{3.4}$$

and that

$$\Sigma_0^F(P) \cap V \subset \Sigma^{\text{in}}(P) \tag{3.5}$$

for all $P \in W$.

Proof. Take $V = B^n(\bar{x}, \bar{\delta})$ and $W = \operatorname{int} B^{B^n(\bar{x},\bar{\delta})}(\bar{P}, \bar{\varepsilon})$, where the balls are those given in Definition 3.4. Therefore,

$$|\Sigma(P) \cap V| = 1 \tag{3.6}$$

for all $P \in W$. Suppose contrarily that (3.4) does not hold for some $P^0 \in W$. Hence, by (3.1), choose

$$x^0 \in [\Sigma^F(P^0) \setminus \Sigma(P^0)] \cap V.$$

By (3.6), let $x^1 \in \Sigma(P^0) \cap V$. Next, fix $\delta^0 > 0$ such that $x^1 \notin B^n(x^0, 2\delta^0)$. By Definitions 3.1 and 3.9 it follows that GMFCQ does not hold at x^0 for P^0. After applying Remark 3.14 with $\bar{P} = P^0$, $\bar{x} = x^0$, and $\bar{\delta} = \delta^0$, we obtain $P^{k,\xi} \in W$ for $k \in \mathbb{N}$ sufficiently large with

$$|\Sigma(P^{k,\xi}) \cap V| \geq |\{x^0, x^1\}| = 2,$$

which contradicts (3.6).

Next, suppose contrarily that for some $P^0 \in W$, with $P^0 = \mathcal{P}^\Theta(f^0, F^0)$, and some $x^0 \in \Sigma_0^F(P^0) \cap V$, it holds that

$$D_x f^0(x^0) \notin \operatorname{int}[\Theta(F^0(x^0))^{\mathrm{t}} D_x F^0(x^0)].$$

Hence, for some sequence $v^k \in \mathbb{R}^n$ with $v^k \to 0$, we have

$$D_x f^0(x^0) + v^k \notin [\Theta(F^0(x^0))]^{\mathrm{t}} D_x F^0(x^0).$$

Define

$$f^k(x) = f^0(x) + v^k(x - x^0)$$

and $P^k = \mathcal{P}^\Theta(f^k, F^0)$. It is easy to see that

$$x^0 \in \Sigma^F(P^k) \setminus \Sigma(P^k). \tag{3.7}$$

Moreover, for k sufficiently large, we have $P^k \in W$. Consequently, (3.7) yields a contradiction to (3.4), which completes the proof. ∎

LEMMA 3.20. *Assume that for some $V \in \mathcal{V}(\bar{x})$ there exists a sequence F^k with $\|F^k - \bar{F}\|^V \to 0$ and*

$$V \cap \operatorname{dom}(\Theta \circ F^k) = \emptyset. \tag{3.8}$$

Then, $\bar{x} \notin \Sigma^S(\bar{P})$.

Proof. Suppose contrarily that $\bar{x} \in \Sigma^S(\bar{P})$ and fix $B^n(\bar{x}, \bar{\delta})$ as in Definition 3.3. Assume without loss of generality that V is open and that $V \subset B^n(\bar{x}, \bar{\delta})$. Choose $\delta > 0$ with $B^n(\bar{x}, \delta) \subset V$ and apply Lemma 2.6 with $V^1 = B^n(\bar{x}, \delta)$, $V^2 = V$, $V^3 = B^n(\bar{x}, \bar{\delta})$ to the functions F^k and $\bar{F}$. The latter yields $P^k = \mathcal{P}^\Theta(\bar{f}, F^{\xi,k})$ with $\|P^k - \bar{P}\|^{B^n(\bar{x},\bar{\delta})} \to 0$ and

$$\Sigma(P^k) \cap B^n(\bar{x}, \delta) = \emptyset,$$

which contradicts that $\bar{x} \in \Sigma^S(\bar{P})$. ∎

The next lemma is essentially a restatement of a classic idea: having too many constraints implies local unfeasibility for sufficiently small perturbations of $\bar{P}$. Preliminarily, we present two definitions.

Definition 3.21. *The family of sets* $\mathbf{M}$ *is called a subpartition of* $\{1, \ldots, m\}$ *if* $\emptyset \notin \mathbf{M}$ *and the following conditions hold:*

- $M \subset \{1, \ldots, m\}$ *for all* $M \in \mathbf{M}$.
- $M^1 \cap M^2 = \emptyset$ *for all* $M^1, M^2 \in \mathbf{M}$, *with* $M^1 \neq M^2$.

Definition 3.22. *Let* $\mathbf{M}$ *be a subpartition of* $\{1, \ldots, m\}$. *A set* $I \subset \{1, \ldots, m\}$ *is called a choice of indexes from* $\mathbf{M}$ *if there exists a bijective mapping* $\phi\colon I \to \mathbf{M}$ *with* $i \in \phi(i)$ *for all* $i \in I$. *The set of choices of indexes from* $\mathbf{M}$ *is denoted by* $\mathcal{I}(\mathbf{M})$.

Roughly speaking, Definition 3.22 means that each choice of indexes $I \in \mathcal{I}(\mathbf{M})$ is obtained by choosing exactly one index from each $M \in \mathbf{M}$.

Lemma 3.23. *Let* $\mathbf{M}$ *be a subpartition of* $\{1, \ldots, m\}$, *and let* $a^1, \ldots, a^m \in \mathbb{R}^m$ *be linearly independent vectors. If* $|\mathbf{M}| > n$ *and for some* $V' \in \mathcal{V}(\bar{F}(\bar{x}))$ *it holds that*

$$V' \cap \operatorname{dom}\Theta \subset \bigcup_{I \in \mathcal{I}(\mathbf{M})} \{y \in \mathbb{R}^m \,|\, \langle a^i, y\rangle = 0,\, i \in I\}, \tag{3.9}$$

then $\bar{x} \notin \Sigma^S(\bar{P})$.

Proof. Let A be the matrix whose rows are $(a^1)^{\mathrm{t}}, \ldots, (a^m)^{\mathrm{t}}$. Analogously, for $I \in \mathcal{I}(\mathbf{M})$, let A^I be the matrix whose rows are $(a^i)^{\mathrm{t}}$, $i \in I$. Since $|I| = |\mathbf{M}| > n$, by [13, Lemma 1.1, p. 68], for any $I \in \mathcal{I}(\mathbf{M})$ the set $A^I \bar{F}(\mathbb{R}^n)$ has Lebesgue measure zero on $\mathbb{R}^{|\mathbf{M}|}$. In the remainder of the proof, we use the convention that the rows of A^I are indexed by the elements of $\mathbf{M}$ instead of $1, \ldots, |\mathbf{M}|$. Choose a sequence $v^k = (v^k_M)_{M \in \mathbf{M}}$ with $v^k \to 0$ and

$$v^k \notin \bigcup_{I \in \mathcal{I}(\mathbf{M})} A^I \bar{F}(\mathbb{R}^n). \tag{3.10}$$

For $k \in \mathbb{N}$, let the components of $\hat{v}^k \in \mathbb{R}^m$ be given as follows:

$$\hat{v}^k_j = \begin{cases} v^k_M & \text{if } j \in M \text{ for some } M \in \mathbf{M}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for $w^k = A^{-1}\hat{v}^k$, it holds that $w^k \to 0$ and that

$$A^I w^k = A^I A^{-1} \hat{v}^k = v^k. \tag{3.11}$$

Define

$$F^k(x) = \bar{F}(x) - w^k.$$

By (3.10) and (3.11), we obtain

$$0 \notin \bigcup_{I \in \mathcal{I}(\mathbf{M})} A^I F^k(\mathbb{R}^n),$$

and therefore,

$$\bigcup_{I \in \mathcal{I}(\mathbf{M})} \{y \in \mathbb{R}^m \,|\, \langle a^i, y\rangle = 0,\, i \in I\} \cap F^k(\mathbb{R}^n) = \emptyset. \tag{3.12}$$

Now, assume without loss of generality that V' is open and choose $V \in \mathcal{V}(\bar{x})$ with $\mathrm{cl}\, \bar{F}(V) \subset V'$. The latter implies $F^k(V) \subset V'$ for k sufficiently large which, together with (3.9) and (3.12), yields

$$F^k(V) \cap \mathrm{dom}\, \Theta = \emptyset,$$

and therefore, we have

$$V \cap \mathrm{dom}\, (\Theta \circ F^k) = \emptyset.$$

By Lemma 3.20, we get $\bar{x} \notin \Sigma^S(\bar{P})$. ∎

The previous result generalizes [10, Lemma 5.2]. In many applications, A is the identity matrix and the property that defines the class of problems is given by the subpartition. For instance, [10, Lemma 5.2] follows from Lemma 3.23 by taking $\bar{F}_{2l-1}(x) = \bar{r}_l(x)$, $\bar{F}_{2l}(x) = \bar{s}_l(x)$, and $\mathbf{M} = \{\{2l-1, 2l\}, l \in L\}$. The following corollary deals with a case where considering certain A, instead of the identity matrix, is advantageous.

Corollary 3.24. *Let $V' \in \mathcal{V}(\bar{F}(\bar{x}))$. If $\dim[V' \cap \mathrm{dom}\, \Theta]^\perp > n$, then $\bar{x} \notin \Sigma^S(\bar{P})$.*

Proof. Let $m^0 = \dim[V' \cap \mathrm{dom}\, \Theta]^\perp$ and $\mathbf{M} = \{\{1\}, \ldots, \{m^0\}\}$. Choose linearly independent vectors $a^1, \ldots, a^m \in \mathbb{R}^m$ such that the first m^0 vectors form a basis of the space $[V' \cap \mathrm{dom}\, \Theta]^\perp$. Note that

$$V' \cap \mathrm{dom}\, \Theta \subset \{y \in \mathbb{R}^m \,|\, \langle a^i, y\rangle = 0,\, i = 1, \ldots, m^0\}.$$

Thus, by Lemma 3.23, it follows that $\bar{x} \notin \Sigma^S(\bar{P})$. ∎

In general, GMFCQ need not to be a necessary condition of a stationary point. As an example we refer to [11, section 5], where a C-stationary point for a mathematical program with complementarity constraints is considered. However, as stated in the next result, several conditions are necessary for strong stability when GMFCQ does not hold; cf. [10, section 5].

Theorem 3.25. *Assume that $\bar{x} \in \Sigma^S(\bar{P})$ and that GMFCQ does not hold at $\bar{x}$ for $\bar{F}$. Then,*
(i) *$\bar{x} \in \Sigma^{\mathrm{in}}(\bar{P})$.*
(ii) *$\dim[\mathrm{span}\, \Theta(\bar{F}(\bar{x}))] > n$.*
(iii) *$\dim \mathrm{span}\, [V' \cap \mathrm{dom}\, \Theta] \geq m - n$ for all $V' \in \mathcal{V}(\bar{F}(\bar{x}))$.*

Proof. (i) The proof immediately follows from Lemma 3.19.

(ii) Considering the orthogonal space in Lemma 3.17 (ii), we get

$$\ker D_x \bar{F}(\bar{x})^{\mathrm{t}} + \mathrm{span}\, \Theta(\bar{F}(\bar{x})) = \mathbb{R}^m. \tag{3.13}$$

Moreover, since GMFCQ does not hold at $\bar{x}$ for $\bar{F}$, it follows that

$$\ker D_x \bar{F}(\bar{x})^{\mathrm{t}} \cap \mathrm{span}\, \Theta(\bar{F}(\bar{x})) \neq \{0\}. \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$\dim[\operatorname{span}\Theta(\bar{F}(\bar{x}))] > m - \dim[\ker D_x\bar{F}(\bar{x})^{\mathrm{t}}],$$

which, together with Lemma 3.17 (iii), yields (ii).

(iii) Suppose contrarily that

$$\dim\operatorname{span}[V' \cap \operatorname{dom}\Theta] < m - n$$

for some $V' \in \mathcal{V}(\bar{F}(\bar{x}))$. Since

$$\dim\operatorname{span}[V' \cap \operatorname{dom}\Theta] + \dim[V' \cap \operatorname{dom}\Theta]^{\perp} = m,$$

it follows that

$$n < \dim[V' \cap \operatorname{dom}\Theta]^{\perp}.$$

Thus, by Corollary 3.24, we get a contradiction to $\bar{x} \in \Sigma^S(\bar{P})$. ∎

4. Strong stability when the cone-valued mapping is osc. Throughout this section, we assume that Θ is osc, which implies some interesting properties:

- GMFCQ locally holds after sufficiently small perturbations.
- GMFCQ implies the boundedness of the set of Lagrange vectors.
- Weak stability and strong stability are equivalent properties.

In this section we prove the latter three properties and introduce yet another constraint qualification.

LEMMA 4.1. *Assume that GMFCQ holds at $\bar{x}$ for $\bar{F}$. Then, there exist $V \in \mathcal{V}(\bar{x})$ and $U \in \mathcal{U}^V(\bar{F})$ such that for all $x \in V$ and all $F \in U$ the condition GMFCQ holds at x for F.*

Proof. First, we show that there exists $V \in \mathcal{V}(\bar{x})$ such that for all $x \in V$ the condition GMFCQ holds at x for $\bar{F}$. Suppose contrarily that there are sequences $x^k \to \bar{x}$ and $\eta^k \in \Theta(\bar{F}(x^k)) \cap S^m(0,1)$ with

$$[\eta^k]^{\mathrm{t}} D_x\bar{F}(x^k) = 0. \tag{4.1}$$

After perhaps reducing to an appropriate subsequence, assume that $\eta^k \to \bar{\eta}$ with $\bar{\eta} \in S^m(0,1)$. Since Θ is closed, by letting $k \to \infty$ and using (4.1), we obtain $\bar{\eta} \in \Theta(\bar{F}(\bar{x}))$ with $\bar{\eta}^{\mathrm{t}} D_x\bar{F}(\bar{x}) = 0$, which is a contradiction since GMFCQ holds at $\bar{x}$ for $\bar{F}$.

Now, we assume without loss of generality that V is compact and show that there exists $U \in \mathcal{U}^V(\bar{F})$ such that for all $x \in V$ and all $F \in U$ the condition GMFCQ holds at x for F. Suppose contrarily that for some sequences $x^k \in V$, $F^k \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$, and $\eta^k \in \Theta(F^k(x^k)) \cap S^m(0,1)$ it holds that $\|F^k - \bar{F}\|^V \to 0$ and that

$$[\eta^k]^{\mathrm{t}} D_x F^k(x^k) = 0. \tag{4.2}$$

After perhaps reducing to appropriate subsequences, assume that $x^k \to x^0$ for some $x^0 \in V$ and that $\eta^k \to \eta^0$. Since Θ is closed, by letting $k \to \infty$ and using (4.2), we obtain $\theta^0 \in \Theta(\bar{F}(x^0))$ and $[\eta^0]^{\mathrm{t}} D_x\bar{F}(x^0) = 0$, which is a contradiction since GMFCQ holds at x^0 for $\bar{F}$. ∎

Remark 4.2. Note that in the previous proof the existence of $V \in \mathcal{V}(\bar{x})$ and, in a second step, that of $U \in \mathcal{U}^V(\bar{F})$ were shown in an analogous way. For the sake of simplicity, in the remainder of this paper we will often avoid repetition by skipping the proof of the existence of V such that a certain property holds for the mapping or problem under consideration on V. As an example, we could have started the previous proof by saying "Suppose contrarily that there exist sequences $x^k \to \bar{x}$, $F^k \to \bar{F} \ldots$ with (4.2)" and then proceeding with the last three lines of the proof.

THEOREM 4.3. *The following three conditions are equivalent:*

(i) *GMFCQ holds at $\bar{x}$ for $\bar{F}$.*

(ii) *There exist $V \in \mathcal{V}(\bar{x})$, $W \in \mathcal{W}^V(\bar{P})$, and a compact set $Q \subset \mathbb{R}^m$ such that*

$$\mathcal{L}(P,x) \subset Q$$

for all $x \in V$ and all $P \in W$.

(iii) *There exists $V \in \mathcal{V}(\bar{x})$ such that $\widehat{\mathcal{L}}^V$ is locally bounded at $\bar{P}$.*

Proof. (i) $\Rightarrow$ (ii) Suppose contrarily that there exist sequences $x^k \to \bar{x}$, $P^k \to \bar{P}$, and $\theta^k \in \mathcal{L}(P^k, x^k)$ with $\|\theta^k\| \to \infty$. By $\theta^k \in \mathcal{L}(P^k, x^k)$, it follows that $\theta^k \in \Theta(F^k(x^k))$ and that

$$D_x f^k(x^k) = [\theta^k]^{\mathrm{t}} D_x F^k(x^k). \tag{4.3}$$

Assume without loss of generality that

$$\frac{\theta^k}{\|\theta^k\|} \to \bar{\eta}$$

for some $\bar{\eta} \in \Theta(\bar{F}(\bar{x})) \cap S^m(0,1)$. Since $\|\theta^k\| \to \infty$ and $D_x f^k(x^k) \to D_x \bar{f}(\bar{x})$, dividing by $\|\theta^k\|$ and letting $k \to \infty$ in (4.3) yield a contradiction to (i).

(ii) $\Rightarrow$ (i) Suppose contrarily that for some $\bar{\eta} \in \Theta(\bar{F}(\bar{x})) \cap S^m(0,1)$ it holds that $\bar{\eta}^{\mathrm{t}} D_x \bar{F}(\bar{x}) = 0$. For $k \in \mathbb{N}$, let P^k be given as in Lemma 3.13. Note that for k sufficiently large, we have $P^k \in W$. By Lemma 3.13 and (ii), we get

$$k\bar{\eta} \in Q, \tag{4.4}$$

where Q is a compact set. By letting $k \to \infty$ in (4.4), we get a contradiction.

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (ii) Since $\widehat{\mathcal{L}}^V$ is locally bounded at $\bar{P}$, there exist $W \in \mathcal{W}^V(\bar{P})$ and a compact set $Q \subset \mathbb{R}^m$ with

$$\widehat{\mathcal{L}}^V(W) \subset Q. \tag{4.5}$$

By (4.5), for any $x \in V$ and any $P \in W$ it holds that

$$\mathcal{L}(P,x) \subset \widehat{\mathcal{L}}^V(P) \subset Q,$$

which completes the proof. ∎

Remark 4.4. Note that Theorem 4.3 is analogous to that of Gauvin [6] for standard nonlinear programs. However, in general, the set $\mathcal{L}(\bar{P}, \bar{x})$ can be compact even when GMFCQ does not hold at $\bar{x}$. We refer, e.g., to [10, Remark 1], where L is a finite index set, $\bar{r}_l, \bar{s}_l \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, $l \in L$, $\bar{F} = (\bar{r}, \bar{s})$, $\theta = (\rho, \sigma) \in \mathbb{R}^{2|L|}$, $y = (y^r, y^s) \in \mathbb{R}^{2|L|}$, and

$$\Theta(y^r, y^s) = \begin{cases} \Xi_{\mathrm{c}}(y^r, y^s) & \text{if } \min\{y_l^r, y_l^s\} = 0, \, l \in L, \\ \emptyset & \text{otherwise,} \end{cases} \tag{4.6}$$

with

$$\Xi_{\mathrm{c}}(y^r, y^s) = \left\{ (\rho, \sigma) \in \mathbb{R}^{2|L|} \;\middle|\; \begin{array}{rl} y_l^r \cdot \rho_l = y_l^s \cdot \sigma_l = 0, & l \in L, \\ \rho_l \cdot \sigma_l \geq 0, & l \in L \end{array} \right\}.$$

There, the stationarity concept under consideration is the C-stationarity for mathematical programs with complementarity constraints.

LEMMA 4.5. *Assume that for all $V \in \mathcal{V}(\bar{x})$ and all $W \in \mathcal{W}^V(\bar{P})$ there exist $x \in V$ and $P \in W$ with $x \in \Sigma^F(P)$. Then, it holds that $\bar{x} \in \Sigma^F(\bar{P})$.*

Proof. If GMFCQ does not hold at $\bar{x}$ for $\bar{F}$, then $\bar{x} \in \Sigma^F(\bar{P})$. Now, assume that GMFCQ holds at $\bar{x}$ for $\bar{F}$. By Lemma 4.1, Theorem 4.3 (ii) and a continuity argument, we get $\bar{x} \in \Sigma(\bar{P})$. ∎

THEOREM 4.6. *Assume that $\bar{x} \in \Sigma(\bar{P})$. Then, the following three conditions are equivalent:*
- (i) $\bar{x} \in \Sigma_0^S(\bar{P})$.
- (ii) $\bar{x} \in \Sigma^S(\bar{P})$.
- (iii) *There exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that for all $P \in W$ the set $\Sigma(P) \cap V$ contains exactly one element which we denote by $\hat{x}(P)$. The mapping $W \to V$, $P \mapsto \hat{x}(P)$ is continuous.*

Proof. (i) $\Rightarrow$ (iii) Take $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ as in Lemma 3.19. Suppose contrarily that there exist $P^0 \in W$ and a sequence $P^k \in W$ with $P^k \to P^0$ and $\hat{x}(P^k) \not\to \hat{x}(P^0)$. Since V is compact, assume without loss of generality that $\hat{x}(P^k) \to x^1$ with $x^1 \neq \hat{x}(P^0)$. By Lemma 4.5, we get $x^1 \in \Sigma^F(P^0)$. Note that

$$\Sigma(P^0) \cap V = \{\hat{x}(P^0)\} \subsetneq \{\hat{x}(P^0), x^1\} \subset \Sigma^F(P^0) \cap V.$$

Consequently, by Lemma 3.19 and, in particular, (3.4), we get a contradiction.

(iii) $\Rightarrow$ (i) By using an analogous argument as in the proof of Lemma 3.19, a moment of reflection shows that there exists $\bar{\delta} > 0$ such that

$$\Sigma^F(\bar{P}) \cap B^n(\bar{x}, 2\bar{\delta}) = \{\bar{x}\} \tag{4.7}$$

and that $B^n(\bar{x}, 2\bar{\delta}) \subset V$. Now, suppose contrarily that there exists a sequence $P^k = (f^k, F^k)$ with $\|P^k - \bar{P}\|^{B^n(\bar{x},2\bar{\delta})} \to 0$ such that

$$|\Sigma(P^k) \cap B^n(\bar{x}, 2\bar{\delta})| \neq 1. \tag{4.8}$$

By (4.7) and Lemma 4.5, assume without loss of generality that

$$\Sigma(P^k) \cap [B^n(\bar{x}, 2\bar{\delta}) \setminus B^n(\bar{x}, \bar{\delta})] = \emptyset. \tag{4.9}$$

Next, apply Lemma 2.6 with $V^1 = B^n(\bar{x}, \bar{\delta})$, $V^2 = \operatorname{int} B^n(\bar{x}, 2\bar{\delta})$, and $V^3 = V$ to the functions describing P^k and $\bar{P}$. From (4.8) and (4.9), we have

$$|\Sigma(P^{k,\xi}) \cap B^n(\bar{x}, \bar{\delta})| \neq 1. \tag{4.10}$$

Moreover, it holds that $\|P^{k,\xi} - \bar{P}\|^V \to 0$, which, together with (4.10), yields a contradiction to (iii).

(i) $\Rightarrow$ (ii) It follows analogously to the proof of (i) $\Rightarrow$ (iii) by choosing $P^0 = \bar{P}$ and noting that $\hat{x}(P^0) = \bar{x}$.

(ii) $\Rightarrow$ (i) It immediately follows from Definitions 3.3 and 3.4. ∎

Now, we turn our attention to a constraint qualification which is stronger than GLICQ. First, we define the set-valued mapping

$$(\operatorname{cs}\Theta)(y) = \limsup_{y' \to y} \operatorname{span}\Theta(y'),$$

which is osc at each $y \in \operatorname{dom}\Theta$. This notation stands for the closure of the set-valued mapping $\operatorname{span}\Theta(\cdot)$, we refer to [27, p. 155] for more details about the semicontinuity of $\operatorname{cs}\Theta$. For simplicity, in what follows we write $\operatorname{cs}\Theta(y)$ instead of $(\operatorname{cs}\Theta)(y)$.

Definition 4.7. *We say that the closed linear independence constraint qualification (CLICQ) holds at $\bar{x}$ for $\bar{F}$ if*

$$\ker D_x \bar{F}(\bar{x})^{\mathrm{t}} \cap \operatorname{cs}\Theta(\bar{F}(\bar{x})) = \{0\}.$$

Note that CLICQ implies GLICQ because of $\operatorname{span}\Theta(\bar{F}(\bar{x})) \subset \operatorname{cs}\Theta(\bar{F}(\bar{x}))$. Moreover, CLICQ and GLICQ are equivalent whenever the set-valued mapping $\operatorname{span}\Theta(\cdot)$ is osc at $\bar{F}(\bar{x})$. One might expect that, as it is in the case of LICQ for standard nonlinear programs, GLICQ holds in a neighborhood of the point at which GLICQ holds. The following example illustrates that this does not happen in general. The motivation to consider the stronger constraint qualification CLICQ is that it holds after sufficiently small perturbations as it is shown in the next lemma.

Example 4.8. Let $n = 1$, $m = 2$, $\bar{x} = 0$, and consider the problem $\bar{P}$ whose describing functions are

$$\bar{f}(x) = x, \quad \bar{F}_1(x) = x, \quad \bar{F}_2(x) = 1,$$

and its cone-valued mapping is

$$\Theta(y_1, y_2) = \begin{cases} \operatorname{cone}\{(\cos y_1, \sin y_1)^{\mathrm{t}}, (1,0)^{\mathrm{t}}\} & \text{if } y_1 \in [0, \frac{\pi}{2}], \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that

$$\operatorname{span}\Theta(y_1, y_2) = \begin{cases} \mathbb{R} \times \{0\} & \text{if } y_1 = 0, \\ \mathbb{R}^2 & \text{if } y_1 \in (0, \frac{\pi}{2}], \\ (0,0)^{\mathrm{t}} & \text{otherwise,} \end{cases}$$

whereas

$$\operatorname{cs}\Theta(y_1, y_2) = \begin{cases} \mathbb{R}^2 & \text{if } y_1 \in [0, \frac{\pi}{2}], \\ (0,0)^{\mathrm{t}} & \text{otherwise.} \end{cases}$$

Consequently, GLICQ holds at $\bar{x}$ for $\bar{F}$, but CLICQ does not. Neither GLICQ nor CLICQ hold at x for $\bar{F}$ for any $x > 0$ sufficiently small.

Lemma 4.9. *Assume that CLICQ holds at $\bar{x}$ for $\bar{F}$. Then, there exist $V \in \mathcal{V}(\bar{x})$, $W \in \mathcal{W}^V(\bar{P})$ such that CLICQ holds at x for F for all $x \in V$ and all $\mathcal{P}^{\Theta}(f, F) \in W$.*

Proof. The proof follows analogously to the proof of Lemma 4.1. Note that if CLICQ does not hold at x^k for $P^k = \mathcal{P}^{\Theta}(f^k, F^k)$, then

$$[\eta^k]^{\mathrm{t}} D_x F^k(x^k) = 0$$

for some $\eta^k \in [\operatorname{cs}\Theta(F^k(x^k))] \cap S^m(0,1)$, and recall that $\operatorname{cs}\Theta$ is osc at each $y \in \operatorname{dom}\Theta$. ∎

Theorem 4.10. *Assume that CLICQ holds at $\bar{x} \in \Sigma^S(\bar{P})$. Then, there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that the following occur:*

(i) *Condition* (iii) *in Theorem* 4.6 *holds.*

(ii) *For all $P \in W$ the set $\mathcal{L}(P, \hat{x}(P))$ contains exactly one element which we denote by $\hat{\theta}(P)$.*

(iii) *The mapping $W \to \mathbb{R}^m, P \mapsto \hat{\theta}(P)$ is continuous.*

Proof. (i) and (ii). These conditions follow from Lemma 4.9 after perhaps shrinking V and W.

(iii) By (ii), we have $\widehat{\mathcal{L}}^V(P) = \{\hat{\theta}(P)\}$. Hence, by Lemma 3.6, Theorem 4.3 and [27, Corollary 5.20], we get (iii). ∎

THEOREM 4.11. *If $\bar{x} \in \Sigma^S(\bar{P})$, then there exist $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ such that*

$$\Sigma(P) \cap V = \Sigma^F(P) \cap V = \Sigma^S(P) \cap V = \{\hat{x}(P)\}$$

for all $P \in W$, where $\hat{x}(P)$ is defined as in Theorem 4.6.

Proof. By choosing $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ as in the proofs of Lemma 3.4 and Theorem 4.6, we have

$$\Sigma(P) \cap V = \Sigma^F(P) \cap V = \{\hat{x}(P)\} \tag{4.11}$$

for all $P \in W$. For simplicity of notation denote $\operatorname{int} V$ by V and, after perhaps shrinking W, fix $P \in W$ with $\hat{x}(P) \in V$. Note that $V \in \mathcal{V}(\hat{x}(P))$ and $W \in \mathcal{W}^V(P)$. Hence, by Theorem 4.6, we get

$$\hat{x}(P) \in \Sigma^S(P). \tag{4.12}$$

Obviously, $\Sigma^S(P) \subset \Sigma(P)$. Thus, by (4.11) and (4.12), the desired result follows. ∎

For several classes of optimization problems, GMFCQ is a necessary condition for strong stability; we refer, e.g., to NLP [9]. In the following two sections we consider further two classes where GMFCQ turns out to be a necessary condition for strong stability.

5. Application to M-stationary and S-stationary points for MPCC.

In this section we consider the following MPCC:

$$\min f(x) \quad \text{s.t.} \quad x \in M[r,s]$$

with

$$M[r,s] = \{x \in \mathbb{R}^n \,|\, \min\{r_l(x), s_l(x)\} = 0,\, l \in L\},$$

where L is a finite index set and all describing functions $f : \mathbb{R}^n \to \mathbb{R}$ and $r_l, s_l : \mathbb{R}^n \to \mathbb{R}$, $l \in L$, are assumed to be twice continuously differentiable. For the definition as well as properties and applications of MPCC, we refer, e.g., to [20, 22, 29, 32, 33]. For $\bar{x} \in M[r,s]$ we define the following active index sets:

$$\begin{aligned}
\bar{I}_r(\bar{x}) &= \{l \in L \,|\, r_l(\bar{x}) = 0\},\\
\bar{I}_s(\bar{x}) &= \{l \in L \,|\, s_l(\bar{x}) = 0\},\\
I_r(\bar{x}) &= \{l \in L \,|\, r_l(\bar{x}) = 0,\, s_l(\bar{x}) > 0\},\\
I_s(\bar{x}) &= \{l \in L \,|\, r_l(\bar{x}) > 0,\, s_l(\bar{x}) = 0\},\\
I_{rs}(\bar{x}) &= \{l \in L \,|\, r_l(\bar{x}) = 0,\, s_l(\bar{x}) = 0\}.
\end{aligned}$$

In particular, there are several stationarity concepts for MPCC; see, e.g., [20]. We consider two of them: M-stationarity and S-stationarity. In both cases the feasible set is the same. By Lemma 3.23, we can restrict ourselves to the case $|L| \le n$. Next, we recall the definition of an M-stationary point for MPCC.

DEFINITION 5.1. *A point $\bar{x} \in M[r,s]$ is called an M-stationary point for MPCC if there exists $(\rho,\sigma) \in \mathbb{R}^{2|L|}$ such that*

$$\begin{gathered} D_x\,\mathbf{L}^{\mathrm{cc}}(\bar{x},\rho,\sigma) = 0, \\ \rho_l \cdot r_l(\bar{x}) = \sigma_l \cdot s_l(\bar{x}) = 0,\ l \in L, \\ \rho_l > 0,\ \sigma_l > 0 \quad \text{or} \quad \rho_l \cdot \sigma_l = 0,\ l \in L, \end{gathered}$$

where

$$\mathbf{L}^{\mathrm{cc}}(x,\rho,\sigma) = f(x) - \sum_{l \in L} [\rho_l \cdot r_l(x) + \sigma_l \cdot s_l(x)]$$

is the MPCC-Lagrange function.

Having (1.1) in mind, we consider a corresponding generalized equation with $y = (y^r, y^s) \in \mathbb{R}^{2|L|}$ and

$$\Theta(y^r,y^s) = \begin{cases} \Xi_{\mathrm{m}}(y^r,y^s) & \text{if } \min\{y_l^r, y_l^s\} = 0,\, l \in L, \\ \emptyset & \text{otherwise,} \end{cases} \tag{5.1}$$

where

$$\Xi_{\mathrm{m}}(y^r,y^s) = \left\{ (\rho,\sigma) \in \mathbb{R}^{2|L|} \;\middle|\; \begin{array}{rl} y_l^r \cdot \rho_l = y_l^s \cdot \sigma_l = 0, & l \in L, \\ \rho_l > 0, \sigma_l > 0 \text{ or } \rho_l \cdot \sigma_l = 0, & l \in L \end{array} \right\}.$$

For Θ given in (5.1) and $\bar{F} = (\bar{r},\bar{s})$, GMFCQ does not hold at $\bar{x} \in M[\bar{r},\bar{s}]$ for $\bar{F}$ if and only of there exists $(\alpha,\beta) \in S^{2|L|}(0,1)$ with

$$\sum_{l \in L} [\alpha_l \cdot D_x\bar{r}_l(\bar{x}) + \beta_l \cdot D_x\bar{s}_l(\bar{x})] = 0, \tag{5.2}$$

$$\begin{gathered} \alpha_l \cdot \bar{r}_l(\bar{x}) = \beta_l \cdot \bar{s}_l(\bar{x}) = 0,\ l \in L, \\ \alpha_l > 0,\ \beta_l > 0 \quad \text{or} \quad \alpha_l \cdot \beta_l = 0,\ l \in L. \end{gathered} \tag{5.3}$$

The following theorem shows that GMFCQ is a necessary condition for strong stability in this context.

THEOREM 5.2. *Let Θ be given as in (5.1), $\bar{F} = (\bar{r},\bar{s})$ and $\bar{P} = \mathcal{P}^{\Theta}(\bar{f},\bar{F})$. If $\bar{x} \in \Sigma^S(\bar{P})$, then GMFCQ holds at $\bar{x}$ for $\bar{F}$.*

Proof. Suppose contrarily that $\bar{x} \in \Sigma^S(\bar{P})$ and that GMFCQ does not hold at $\bar{x}$ for $(\bar{r},\bar{s})$. Then, there exists $(\alpha,\beta) \in \Theta(\bar{r}(\bar{x}),\bar{s}(\bar{x})) \cap S^{2|L|}(0,1)$ such that (5.2) holds. For $\varepsilon > 0$, define

$$\begin{aligned} r_i^\varepsilon(x) &= \bar{r}_i(x) + \varepsilon, \quad i \in I^0(\alpha) \cap I_{\bar{r}\bar{s}}(\bar{x}), \\ r_i^\varepsilon(x) &= \bar{r}_i(x), \quad i \in L \setminus [I^0(\alpha) \cap I_{\bar{r}\bar{s}}(\bar{x})], \\ s_j^\varepsilon(x) &= \bar{s}_j(x) + \varepsilon, \quad j \in [I^0(\beta) \cap I_{\bar{r}\bar{s}}(\bar{x})] \setminus I^0(\alpha), \\ s_j^\varepsilon(x) &= \bar{s}_j(x), \quad j \in L \setminus [[I^0(\beta) \cap I_{\bar{r}\bar{s}}(\bar{x})] \setminus I^0(\alpha)], \end{aligned}$$

and $P^\varepsilon = \mathcal{P}^{\Theta}(\bar{f}, r^\varepsilon, s^\varepsilon)$. It is easy to see that $(\alpha,\beta) \in \Theta(r^\varepsilon(\bar{x}), s^\varepsilon(\bar{x}))$ and that

$$\sum_{l \in L} [\alpha_l \cdot D_x r_l^\varepsilon(\bar{x}) + \beta_l \cdot D_x s_l^\varepsilon(\bar{x})] = 0, \tag{5.4}$$

that is, $\bar{x} \in \Sigma^F(P^\varepsilon)$. Hence, by Theorem 4.11, it follows that $\bar{x} \in \Sigma^S(P^\varepsilon)$. Note that $I_{r^\varepsilon s^\varepsilon}(\bar{x}) = I^*(\alpha) \cap I^*(\beta)$. Since $|L| \leq n$, by Theorem 3.25 and Corollary 3.18 we have $I_{r^\varepsilon s^\varepsilon}(\bar{x}) \neq \emptyset$. For $k \in \mathbb{N}$ define

$$r_l^k(x) = r_l^\varepsilon(x) - \frac{1}{k}, \quad l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}), \quad r_l^k(x) = r_l^\varepsilon(x), \quad l \in L \setminus I_{r^\varepsilon s^\varepsilon}(\bar{x}),$$
$$s_l^k(x) = s_l^\varepsilon(x) - \frac{1}{k}, \quad l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}), \quad s_l^k(x) = s_l^\varepsilon(x), \quad l \in L \setminus I_{r^\varepsilon s^\varepsilon}(\bar{x}),$$

and $P^k = \mathcal{P}^\Theta(\bar{f}, r^k, s^k)$. By $\bar{x} \in \Sigma^S(P^\varepsilon)$, let $x^k \in \Sigma^S(P^k)$. Since

$$r_l^k(\bar{x}) = -\frac{1}{k} < 0, l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}),$$

it follows that $\bar{x} \neq x^k$. Assume without loss of generality that

$$\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \to u$$

for some $u \in S^n(0,1)$. Moreover, by $x^k \in M[r^k, s^k]$, we obtain

$$r_l^\varepsilon(x^k) \geq 0, \quad l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}), \quad r_l^\varepsilon(x^k) = 0, \quad l \in I_{r^\varepsilon}(\bar{x}),$$
$$s_l^\varepsilon(x^k) \geq 0, \quad l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}), \quad s_l^\varepsilon(x^k) = 0, \quad l \in I_{s^\varepsilon}(\bar{x}).$$

Hence, by using the mean value theorem and letting $k \to +\infty$, we get

$$D_x r_l^\varepsilon(\bar{x})u \geq 0, \quad l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}), \quad D_x r_l^\varepsilon(\bar{x})u = 0, \quad l \in I_{r^\varepsilon}(\bar{x}), \tag{5.5}$$

$$D_x s_l^\varepsilon(\bar{x})u \geq 0, \quad l \in I_{r^\varepsilon s^\varepsilon}(\bar{x}), \quad D_x s_l^\varepsilon(\bar{x})u = 0, \quad l \in I_{s^\varepsilon}(\bar{x}). \tag{5.6}$$

By Corollary 3.18, it follows that there are n linearly independent vectors in the set

$$\left\{D_x r_l^\varepsilon(\bar{x}), l \in \bar{I}_{r^\varepsilon}(\bar{x}), D_x s_l^\varepsilon(\bar{x}), l \in \bar{I}_{s^\varepsilon}(\bar{x})\right\}.$$

Thus, assume without loss of generality that

$$D_x r_{l^0}^\varepsilon(\bar{x})u > 0 \tag{5.7}$$

for some $l^0 \in \bar{I}_{r^\varepsilon s^\varepsilon}(\bar{x})$. By $(\alpha, \beta) \in \Theta(r^\varepsilon(\bar{x}), s^\varepsilon(\bar{x}))$, multiplying by u in (5.4), using (5.5) and (5.6), it follows that

$$\begin{aligned} 0 &= \sum_{l \in L} [\alpha_l \cdot D_x r_l^\varepsilon(\bar{x})u + \beta_l \cdot D_x s_l^\varepsilon(\bar{x})u] \\ &= \sum_{l \in I_{r^\varepsilon s^\varepsilon}(\bar{x})} [\alpha_l \cdot D_x r_l^\varepsilon(\bar{x})u + \beta_l \cdot D_x s_l^\varepsilon(\bar{x})u] \geq \alpha_{l^0} D_x r_{l^0}^\varepsilon(\bar{x})u, \end{aligned}$$

which, together with $\alpha_{l^0} > 0$, yields a contradiction to (5.7). This completes the proof. ∎

Now, we recall the definition of an S-stationary point for MPCC.

Definition 5.3. *A point $\bar{x} \in M[r,s]$ is called an S-stationary point for MPCC if there exists $(\rho,\sigma) \in \mathbb{R}^{2|L|}$ such that*

$$\begin{aligned} D_x \mathbf{L}^{\mathrm{cc}}(\bar{x},\rho,\sigma) &= 0, \\ \rho_l \cdot r_l(\bar{x}) = \sigma_l \cdot s_l(\bar{x}) &= 0, \quad l \in L, \\ \rho_l \geq 0, \quad \sigma_l &\geq 0, \quad l \in I_{rs}(\bar{x}). \end{aligned}$$

Analogously to (5.1), for the S-stationarity we consider a corresponding generalized equation with

$$\Theta(y^r,y^s) = \begin{cases} \Xi_{\mathrm{s}}(y^r,y^s) & \text{if } \min\{y^r_l,y^s_l\} = 0,\ l \in L, \\ \emptyset & \text{otherwise}, \end{cases} \tag{5.8}$$

where

$$\Xi_{\mathrm{s}}(y^r,y^s) = \left\{ (\rho,\sigma) \in \mathbb{R}^{2|L|} \;\middle|\; \begin{array}{rl} y^r_l \cdot \rho_l = y^s_l \cdot \sigma_l = 0, & l \in L, \\ \rho_l \geq 0,\ \sigma_l \geq 0, & l \in I^0(y^r) \cap I^0(y^s) \end{array} \right\}.$$

For Θ given in (5.8) and $\bar{F} = (\bar{r},\bar{s})$, GMFCQ does not hold at $\bar{x} \in M[\bar{r},\bar{s}]$ for $\bar{F}$ if and only if there exists $(\alpha,\beta) \in S^{2|L|}(0,1)$ with (5.2), (5.3), and

$$\alpha_l \geq 0, \quad \beta_l \geq 0, \quad l \in I_{\bar{r}\bar{s}}(\bar{x}).$$

In this context again, GMFCQ is a necessary condition for strong stability. However, since Θ is *not* necessarily closed, the results in section 4 and, particularly, Theorem 4.11, cannot be applied as in the proof of Theorem 5.2. Instead, we use two auxiliary results. For simplicity of notation, in the remainder of this section we assume without loss of generality that $I_{\bar{r}\bar{s}}(\bar{x}) = L$. Moreover, fix $(\alpha,\beta) \in \Theta(\bar{r}(\bar{x}),\bar{s}(\bar{x})) \setminus \{0\}$ such that (5.2) holds, and let

$$I^{\alpha\beta} = I^0(\alpha) \cap I^0(\beta),$$

whenever GMFCQ does not hold at $\bar{x}$ for $(\bar{r},\bar{s})$.

Lemma 5.4. *Assume that GMFCQ does not hold at $\bar{x}$ for $(\bar{r},\bar{s})$. If for some $I \subset I^{\alpha\beta}$ it holds that*

$$\dim \operatorname{span} \left\{ D_x\bar{r}_i(\bar{x}),\ i \in I^*(\alpha) \cup I,\ D_x\bar{s}_j(\bar{x}),\ j \in I^*(\beta) \cup [I^{\alpha\beta} \setminus I] \right\} < n, \tag{5.9}$$

then $\bar{x} \notin \Sigma^S(\bar{P})$.

Proof. Suppose contrarily that $\bar{x} \in \Sigma^S(\bar{P})$. For $\varepsilon > 0$ sufficiently small let

$$\begin{aligned} &r^\varepsilon_i(x) = \bar{r}_i(x),\ i \in I^*(\alpha) \cup I, && r^\varepsilon_i(x) = \bar{r}_i(x) + \varepsilon,\ i \in I^0(\alpha) \setminus I, \\ &s^\varepsilon_j(x) = \bar{s}_j(x),\ j \in I^*(\beta) \cup [I^{\alpha\beta} \setminus I], && s^\varepsilon_j(x) = \bar{s}_j(x) + \varepsilon,\ j \in I^0(\beta) \setminus [I^{\alpha\beta} \setminus I], \end{aligned}$$

and $P^\varepsilon = \mathcal{P}(\bar{f},r^\varepsilon,s^\varepsilon)$. It is easy to see that $\bar{x} \in \Sigma^F_0(P^\varepsilon)$. By Theorem 3.19, we get $\bar{x} \in \Sigma^{\mathrm{in}}(P^\varepsilon)$. Hence, by Corollary 3.18, we obtain a contradiction to (5.9). ∎

Lemma 5.5. *Assume that GMFCQ does not hold at $\bar{x}$ for $(\bar{r},\bar{s})$. Moreover, for $k \in \mathbb{N}$ and $I \subset I^{\alpha\beta}$ let*

$$M^{I,k}[\bar{r},\bar{s}] = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{rl} \min\{\bar{r}_l(x),\bar{s}_l(x)\} = k^{-1}, & l \in L \setminus I^{\alpha\beta}, \\ \bar{r}_i(x) = 0, & i \in I, \\ \bar{s}_j(x) = 0, & j \in I^{\alpha\beta} \setminus I \end{array} \right\}.$$

If for some $I \subset I^{\alpha\beta}$ it holds that

$$\dim \operatorname{span}\left\{D_x \bar r_i(\bar x),\, i \in I^*(\alpha)\cup I,\, D_x \bar s_j(\bar x),\, j \in I^*(\beta)\cup [I^{\alpha\beta}\setminus I]\right\} = n.$$

Then, there exist $V^I \in \mathcal{V}(\bar x)$ and $k^I \in \mathbb{N}$ such that

$$M^{I,k}[\bar r, \bar s] \cap V^I = \emptyset$$

for all $k \geq k^I$.

Proof. The proof follows analogously to the part right after taking sequences in the proof of Theorem 5.2. Note that we mainly use there the feasibility of x^k. ∎

THEOREM 5.6. *Let Θ be given as in* (5.8), *$\bar F = (\bar r, \bar s)$, and $\bar P = \mathcal{P}^{\Theta}(\bar f, \bar F)$. If $\bar x \in \Sigma^S(\bar P)$, then GMFCQ holds at $\bar x$ for $\bar F$.*

Proof. Suppose contrarily that $\bar x \in \Sigma^S(\bar P)$ and that GMFCQ does not hold at $\bar x$ for $\bar F$. By Lemmas 5.4 and 5.5, there exist $V \in \mathcal{V}(\bar x)$ and $k^0 \in \mathbb{N}$ such that

$$\bigcup_{I \in I^{\alpha\beta}} M^{I,k}[\bar r, \bar s] \cap V = \emptyset \tag{5.10}$$

for all $k \geq k^0$. Let

$$\begin{aligned} r_l^k(x) &= \bar r_l(x) - \frac{1}{k}, \quad l \in L \setminus I^{\alpha\beta}, \quad r_l^k(x) = \bar r_l(x), \quad l \in I^{\alpha\beta},\\ s_l^k(x) &= \bar s_l(x) - \frac{1}{k}, \quad l \in L \setminus I^{\alpha\beta}, \quad s_l^k(x) = \bar s_l(x), \quad l \in I^{\alpha\beta}. \end{aligned}$$

Note that

$$M[r^k, s^k] \subset \bigcup_{I \in I^{\alpha\beta}} M^{I,k}[\bar r, \bar s],$$

which, together with (5.10), yields

$$M[r^k, s^k] \cap V = \emptyset.$$

By Lemma 3.20, we get a contradiction to $\bar x \in \Sigma^S(\bar P)$. ∎

We end this section by mentioning the relation between GLICQ and the so-called MPCC-LICQ; see, e.g., [18]. Note that

$$\operatorname{span} \Xi_{\mathrm{s}}(y^r, y^s) = \left\{(\rho, \sigma) \in \mathbb{R}^{2|L|} \,\middle|\, y_l^r \cdot \rho_l = y_l^s \cdot \sigma_l = 0,\, l \in L\right\}$$

and that

$$\operatorname{span} \Xi_{\mathrm{c}}(y^r, y^s) = \operatorname{span} \Xi_{\mathrm{m}}(y^r, y^s) = \operatorname{span} \Xi_{\mathrm{s}}(y^r, y^s).$$

Hence,

$$\operatorname{span} \Theta(y^r, y^s) = \begin{cases} \operatorname{span} \Xi_{\mathrm{s}}(y^r, y^s) & \text{if } \min\{y_l^r, y_l^s\} = 0,\ l \in L,\\ \{0\} & \text{otherwise,} \end{cases}$$

whenever Θ is given as in (4.6), (5.1), or (5.8). Consequently, in these cases the set-valued mapping $\operatorname{span}\Theta$ is closed, and therefore, GLICQ, CLICQ, and MPCC-LICQ are equivalent. Thus, by Theorem 4.10, the mapping of Lagrange vectors $\hat\theta(P)$ is well defined and continuous in a neighborhood of $\bar P$. However, this result has been already established in [18] for C-stationary points and in [8] for M- and S-stationary points.

6. Application to mathematical programs with abstract constraints. In this section we consider the following mathematical program with abstract constraints:

$$\min f(x) \quad \text{s.t.} \quad F(x) \in C, \tag{6.1}$$

where (f, F) are defined as in (1.1) and $C \subset \mathbb{R}^m$ is a closed and convex set; see, e.g., [1]. Recall that the tangent cone and the normal cone of C at $y \in C$ are

$$\begin{aligned} T_C(y) &= \operatorname{cl} \operatorname{cone}(C - y), \\ N_C(y) &= \{\theta \in \mathbb{R}^m \,|\, \langle \theta, z - y \rangle \le 0,\, z \in C\}, \end{aligned} \tag{6.2}$$

respectively, and that $T_C(y) = N_C(y) = \emptyset$, whenever $y \notin C$. It is

$$N_C(y) = -[T_C(y)]^* \tag{6.3}$$

and, in particular, $T_C(y)$ and $N_C(y)$ are closed and convex cones for each $y \in C$.

We focus on the optimality condition given in [2, section 3.1] which can be written as

$$D_x f(\bar{x}) = \theta^{\mathrm{t}} D_x F(\bar{x}), \quad \theta \in -N_C(F(\bar{x})).$$

In particular, we study the strong stability of a stationary point $\bar{x}$ for a problem $\bar{P} = \mathcal{P}^{\Theta}(\bar{f}, \bar{F})$, where $\bar{f}$, $\bar{F}$, C are fixed and $\Theta(y) = -N_C(y)$. Recall that the Robinson Constraint Qualification (RCQ) [24] holds at a feasible point $\bar{x}$ for $\bar{F}$ if

$$0 \in \operatorname{int} \left[\bar{F}(\bar{x}) + D_x \bar{F}(\bar{x}) \mathbb{R}^n - C\right]. \tag{6.4}$$

It is well known that, under the assumptions of this section, RCQ and GMFCQ are equivalent; see [2, Corollary 2.98].

LEMMA 6.1. *Let $K \subset \mathbb{R}^m$ be a closed and convex cone. Assume that $y \in K$, $\theta \in K^*$, and that $\langle y, \theta \rangle = 0$. Then, the following two conditions hold:*
(i) *$y \in \operatorname{rel} \operatorname{bd} K$ or $\theta \in K^{\perp}$.*
(ii) *$\theta \in \operatorname{rel} \operatorname{bd} K^*$ or $y \in [K^*]^{\perp}$.*

Proof. We prove (i). The proof of (ii) runs analogously since $K = K^{**}$. Suppose contrarily that $y \in \operatorname{rel} \operatorname{int} K$ and that $\theta \notin K^{\perp}$. Hence, for $\varepsilon > 0$ sufficiently small, it holds that $y - \varepsilon \operatorname{proj}_{\operatorname{span} K}(\theta) \in K$ and that

$$\langle y - \varepsilon \operatorname{proj}_{\operatorname{span} K}(\theta), \theta \rangle = -\varepsilon \|\operatorname{proj}_{\operatorname{span} K}(\theta)\|^2 < 0,$$

which contradicts that $\theta \in K^*$. ∎

The next lemma will be used in the proof of Theorem 6.5, and it is a generalization of Stiemke's theorem of the alternatives [30].

LEMMA 6.2. *Let $A \in \mathbb{R}^{m \times n}$ and $K \subset \mathbb{R}^m$ be a closed and convex cone. Then, exactly one of the following two conditions hold:*
(i) *There exists $y \in \operatorname{rel} \operatorname{int} K$ with $A^{\mathrm{t}} y = 0$.*
(ii) *There exists $u \in \mathbb{R}^n$ with $Au \in K^* \setminus K^{\perp}$.*

Proof. First, we show that (i) and (ii) cannot hold both. Suppose contrarily that there exist $y \in \operatorname{rel} \operatorname{int} K$ and $u \in \mathbb{R}^n$ with $A^{\mathrm{t}} y = 0$ and $Au \in K^* \setminus K^{\perp}$, respectively. Note that

$$\langle y, Au \rangle = \langle A^{\mathrm{t}} y, u \rangle = 0,$$

which is not possible according to Lemma 6.1 (i). Second, we show that if (i) does not hold, then (ii) holds. If

$$\ker A^{\mathrm{t}} \cap \operatorname{rel\,int} K = \emptyset,$$

then, by Theorem 2.2 and by noting that $\ker A^{\mathrm{t}}$ is a subspace, we obtain $\omega \in \mathbb{R}^m \setminus \{0\}$ with

$$\langle \omega, y^1 \rangle = 0, \quad y^1 \in \ker A^{\mathrm{t}}, \tag{6.5}$$

$$\langle \omega, y^2 \rangle \geq 0, \quad y^2 \in K. \tag{6.6}$$

By (6.5) and (6.6), we get $\omega = Au$ for some $u \in \mathbb{R}^n$ and that $\omega \in K^*$, respectively. Furthermore, according to Theorem 2.2, there exists $\bar{y}^2 \in K$ with $\langle \omega, \bar{y}^2 \rangle \neq 0$. Thus, $\omega \notin K^\perp$, which completes the proof. ∎

Remark 6.3. In Lemma 6.2, if K is a subspace, then (i) is fulfilled with $y = 0$ and (ii) cannot hold because of $K^* \setminus K^\perp = \emptyset$. If we have $K = \mathbb{R}^m_+$, then Lemma 6.2 yields Stiemke's theorem of the alternatives [30].

LEMMA 6.4. *Assume that $\bar{x} \in \Sigma(\bar{P})$ and that GMFCQ does not hold at $\bar{x}$ for $\bar{F}$. If $\bar{F}(\bar{x}) \in \operatorname{rel\,int} C$, then $\bar{x} \notin \Sigma^S(\bar{P})$.*

Proof. By $\bar{F}(\bar{x}) \in \operatorname{rel\,int} C$, choose $V' \in \mathcal{V}(\bar{F}(\bar{x}))$ with

$$V' \cap \operatorname{dom} \Theta = V' \cap C = V' \cap \operatorname{span} C.$$

Moreover, we have $T_C(\bar{F}(\bar{x})) = \operatorname{span} C$. Hence, by (6.3), we obtain

$$\operatorname{span} \Theta(\bar{F}(\bar{x})) = \Theta(\bar{F}(\bar{x})) = [T_C(\bar{F}(\bar{x}))]^* = C^\perp.$$

Thus,

$$\dim \operatorname{span} [V' \cap \operatorname{dom} \Theta] + \dim [\operatorname{span} \Theta(\bar{F}(\bar{x}))] = \dim \operatorname{span} C + \dim C^\perp = m.$$

By Theorem 3.25 (ii) and (iii), we get $\bar{x} \notin \Sigma^S(\bar{P})$. ∎

The next theorem states that GMFCQ is a necessary condition for strong stability.

THEOREM 6.5. *If $\bar{x} \in \Sigma^S(\bar{P})$, then GMFCQ holds at $\bar{x}$ for $\bar{F}$.*

Proof. Fix $V \in \mathcal{V}(\bar{x})$ and $W \in \mathcal{W}^V(\bar{P})$ as in Theorem 4.6 (iii). Suppose contrarily that GMFCQ does not hold at $\bar{x}$ for $\bar{F}$, that is, there exists $\bar{\eta} \in N_C(\bar{F}(\bar{x})) \cap S^m(0,1)$ with

$$\bar{\eta}^{\mathrm{t}} D_x \bar{F}(\bar{x}) = 0.$$

Since $N_C(\bar{F}(\bar{x}))$ is a nonempty convex set, by [26, Theorem 6.2], it follows that $\operatorname{rel\,int} N_C(\bar{F}(\bar{x})) \neq \emptyset$. By Lemma 2.1 (i), fix $\eta^R \in \operatorname{rel\,int} N_C(\bar{F}(\bar{x})) \cap S^m(0,1)$ arbitrarily close or perhaps equal to $\bar{\eta}$. Next, choose a rotation matrix $R \in \mathbb{R}^{m \times m}$ arbitrarily close or perhaps equal to the identity matrix with $\eta^R = R\bar{\eta}$. Now, define

$$F^R(x) = \bar{F}(x) + (RD_x\bar{F}(\bar{x}) - D_x\bar{F}(\bar{x}))(x - \bar{x}),$$

and let $P^R = \mathcal{P}^{-N_C}(\bar{f}, F^R)$. Note that

$$[\eta^R]^{\mathrm{t}} D_x F^R(\bar{x}) = \bar{\eta}^{\mathrm{t}} R^{\mathrm{t}} R D_x \bar{F}(\bar{x}) = \bar{\eta}^{\mathrm{t}} D_x \bar{F}(\bar{x}) = 0 \tag{6.7}$$

and that $F^R(\bar{x}) = \bar{F}(\bar{x})$. Hence, $\bar{x} \in \Sigma^F(P^R)$. By Theorem 4.11, we obtain $\bar{x} \in \Sigma^S(P^R)$. Fix $v \in \operatorname{rel\,int} C$, and for $k \in \mathbb{N}$ let

$$\begin{aligned} \lambda^k &= 1 + \frac{1}{k}, \\ F^k(x) &= \lambda^k F^R(x) + (1-\lambda^k)v, \\ P^k &= \mathcal{P}^{-N_C}(\bar{f}, F^k). \end{aligned}$$

By Lemma 6.4, we have $F^R(\bar{x}) \in \operatorname{rel\,bd} C$. Applying Lemma 2.1 (ii) with $\lambda = \lambda^k$, we obtain

$$F^k(\bar{x}) = \lambda^k F^R(\bar{x}) + (1-\lambda^k)v \notin C,$$

and, hence, $\bar{x} \notin \Sigma(P^k)$. Let $x^k \in \Sigma(P^k)$ for k sufficiently large. By the convexity of C, we get

$$F^R(x^k) = \frac{1}{\lambda^k} F^k(x^k) + \left(1 - \frac{1}{\lambda^k}\right) v \in C,$$

which implies that

$$\frac{F^R(x^k) - F^R(\bar{x})}{\|x^k - \bar{x}\|} \in T_C(F^R(\bar{x})). \tag{6.8}$$

By Theorem 4.6, it follows that $x^k \to \bar{x}$. Assume without loss of generality that

$$\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \to u$$

for some $u \in S^n(0,1)$. By letting $k \to +\infty$ in (6.8), we obtain

$$D_x F^R(\bar{x}) u \in T_C(F^R(\bar{x})) \tag{6.9}$$

and, by (6.3),

$$\Theta(F^R(\bar{x})) = -N_C(F^R(\bar{x})) = [T_C(F^R(\bar{x}))]^*.$$

Therefore, by (6.7), (6.9), and applying Lemma 6.2 with $K = \Theta(F^R(\bar{x}))$, it follows that

$$D_x F^R(\bar{x}) u \in \Theta(F^R(\bar{x}))^\perp. \tag{6.10}$$

By Theorem 3.25 (i) and (6.10), we get a contradiction to $\bar{x} \in \Sigma^S(P^R)$, which completes the proof. ∎

The following lemma will be used in the proof of Theorem 6.8 and it is a generalization of Gordan's theorem of the alternatives [7].

LEMMA 6.6. *Let $A \in \mathbb{R}^{m\times n}$ and $K \subset \mathbb{R}^m$ be a closed and convex cone. Then, exactly one of the following two conditions hold:*

(i) *There exists $y \in K \setminus \operatorname{lin} K$ with $A^{\mathrm{t}} y = 0$.*
(ii) *There exists $u \in \mathbb{R}^n$ with $Au \in \operatorname{rel\,int} K^*$.*

Proof. The proof follows analogously to the proof of Lemma 6.2 by using $\operatorname{lin} K = [K^*]^\perp$ and applying Lemma 6.1 (ii). ∎

Remark 6.7. An application of Lemma 6.6 with $K = \mathbb{R}^m_+$ yields Gordan's theorem of the alternatives [7].

As mentioned above, the constraint qualifications RCQ and GMFCQ are equivalent for the class of problems considered in this section. Several characterizations of RCQ can be found in [2]. The next theorem presents yet another one. Note that an analogous characterization holds in the setting of Banach spaces whenever $\operatorname{int} C \neq \emptyset$; see [2, Lemma 2.99].

THEOREM 6.8. *Let $\bar{x}$ be a feasible point for $\bar{F}$. Then, GMFCQ holds at $\bar{x}$ for $\bar{F}$ if and only if the following both conditions hold:*

(i) $\ker D_x\bar{F}(\bar{x})^{\mathrm{t}} \cap (C - \bar{F}(\bar{x}))^{\perp} = \{0\}$.

(ii) *There exists $u \in \mathbb{R}^n$ with $D_x\bar{F}(\bar{x})u + \bar{F}(\bar{x}) \in \operatorname{rel\,int} C$.*

Proof. First, we find expressions for the sets $\operatorname{lin} K$ and $\operatorname{rel\,int} K^*$ for $K = -N_C(\bar{F}(\bar{x}))$. Note that

$$\operatorname{lin} K = \left\{\theta \in \mathbb{R}^m \,|\, \langle \theta, z - \bar{F}(\bar{x})\rangle \leq 0 \,\forall z \in C\right\} \cap \left\{\theta \in \mathbb{R}^m \,|\, \langle \theta, z - \bar{F}(\bar{x})\rangle \geq 0 \,\forall z \in C\right\},$$

which yields

$$\operatorname{lin} K = (C - \bar{F}(\bar{x}))^{\perp}.$$

Moreover, by (6.2) and (6.3), it holds that

$$\operatorname{rel\,int} K^* = \operatorname{rel\,int} T_C(\bar{F}(\bar{x})) = \operatorname{rel\,int}\,[\operatorname{cl}\operatorname{cone}(C - \bar{F}(\bar{x}))].$$

By consecutively applying [26, Theorem 6.3, Corollaries 6.8.1 and 6.6.2], it follows that

$$\begin{aligned}\operatorname{rel\,int}\,[\operatorname{cl}\operatorname{cone}(C - \bar{F}(\bar{x}))] &= \operatorname{rel\,int}\,[\operatorname{cone}(C - \bar{F}(\bar{x}))]\\ &= \operatorname{cone}\,[\operatorname{rel\,int}(C - \bar{F}(\bar{x}))]\\ &= \operatorname{cone}(\operatorname{rel\,int} C - \bar{F}(\bar{x})).\end{aligned}$$

Thus, we have

$$\operatorname{rel\,int} K^* = \operatorname{cone}(\operatorname{rel\,int} C - \bar{F}(\bar{x})).$$

Second, note that (ii) is equivalent to the existence of $u^0 \in \mathbb{R}^n$ with

$$D_x\bar{F}(\bar{x})u^0 \in \operatorname{cone}(\operatorname{rel\,int} C - \bar{F}(\bar{x})).$$

Let $A = D_x\bar{F}(\bar{x})$. Now, we show that GMFCQ implies (i) and (ii). If GMFCQ holds at $\bar{x}$ for $\bar{F}$, then

$$\ker A^{\mathrm{t}} \cap K = \{0\}, \tag{6.11}$$

which, in particular, implies that

$$\ker A^{\mathrm{t}} \cap \operatorname{lin} K = \{0\}. \tag{6.12}$$

Hence, (i) holds. Moreover, by (6.11) and (6.12), it follows that

$$\ker A^{\mathrm{t}} \cap (K \setminus \operatorname{lin} K) = \emptyset.$$

Therefore, by Lemma 6.6, we obtain (ii). The proof of that (i) and (ii) together imply GMFCQ runs analogously. ∎

Remark 6.9. In Theorem 6.8, if C is a closed and convex cone, the condition (i) can be rewritten as

$$\ker D_x\bar{F}(\bar{x})^{\mathrm{t}} \cap C^{\perp} = \{0\}.$$

That is, $\bar{F}(\bar{x})$ does not play any role in (i). To see this, observe that

$$\operatorname{span} C = \operatorname{span}(C - \bar{F}(\bar{x})).$$

Remark 6.10. For the standard nonlinear program (1.2), the conditions (i) and (ii) in Theorem 6.8 are equivalent to the conditions (i) and (ii) in MFCQ, respectively; see Remark 3.10.

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