Treewidth Is NP-Complete on Cubic Graphs

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Abstract

In this paper, we show that Treewidth is NP-complete for cubic graphs, thereby improving the result by Bodlaender and Thilikos from 1997 that Treewidth is NP-complete on graphs with maximum degree at most 9. We add a new and simpler proof of the NP-completeness of treewidth, and show that Treewidth remains NP-complete on subcubic induced subgraphs of the infinite 3-dimensional grid.

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1 Introduction

Treewidth is one of the most studied graph parameters, with many applications for both theoretical investigations as well as for applications. The problem of deciding the treewidth of a given graph, and finding corresponding tree decomposition, single-handedly lead to a plethora of studies, including exact algorithms, algorithms for special graph classes, approximations, upper and lower bound heuristics, parameterised algorithms and more. In this paper, we look at the basic problem to decide, for a given graph G and integer k, whether the treewidth of G is at most k.

This problem was shown to be NP-complete in 1987 by Arnborg et al. [1]; their proof also gives NP-completeness on co-bipartite graphs. As the treewidth of a graph (without parallel edges) does not change under subdivision of edges, it easily follows and is well known that TREEWIDTH is NP-complete on bipartite graphs. In 1997, Bodlaender and Thilikos [4] modified the construction of Arnborg et al. [1] and showed that TREEWIDTH remains NP-complete if we restrict the inputs to graphs with maximum degree 9. In this paper, we sharpen this bound of 9 to 3. Our proof uses a simple transformation, whose correctness follows from well-known facts about treewidth and simple insights. We also give a new simple proof of the NP-completeness of TREEWIDTH on arbitrary (and on co-bipartite) graphs. We obtain a number of corollaries of the results, in particular NP-completeness of TREEWIDTH on d-regular graphs for each fixed $d \geq 3$, and for graphs that can be embedded in a 3-dimensional grid.

Our techniques are based on the techniques in [1] and [4] with streamlined and simplified arguments, and some additional new but elementary ideas. As a starting point for the reductions, we use the NP-complete problems CUTWIDTH on cubic graphs and PATHWIDTH; the NP-completeness proofs for these were given by Monien and Sudborough [6] in 1987.

This paper is organised as follows. In Section 2, we give basic definitions and some well-known results on treewidth. In Section 3, we give a new simple proof of the NP-completeness of TREEWIDTH on co-bipartite graphs that uses an elementary transformation from pathwidth. Section 4 gives our main result: NP-completeness for TREEWIDTH on cubic graphs (i.e. graphs with each vertex of degree 3). In Section 5, we derive as consequences some additional NP-completeness results: on d-regular graphs for each fixed d and on graphs that can be embedded in a 3-dimensional grid. Some final remarks are made in Section 6.

2 Definitions and preliminaries

Throughout the paper, we denote the number of vertices of the graph G by n. All graphs considered in this paper are undirected. A graph G is d-regular if each vertex has degree d. We say that a graph G is *cubic* if G is 3-regular. If each vertex of G has degree at most 3, we say that G is *subcubic*. All numbers considered are assumed to be integers, and an interval $[a, b]$ denotes the set of integers $\{a, a+1, a+2, \ldots, b-1, b\}$. Furthermore, for a positive integer a, we denote by $[a]$ the interval $[1, a]$. A graph G is a *minor* of a graph H, if G can be obtained from H by zero or more vertex deletions, edge deletions, and edge contractions. For a graph G and a set of vertices $A \subseteq V(G)$, we write $G + \mathit{clique}(A)$ for the graph obtained by adding an edge between each pair of distinct non-adjacent vertices in A, i.e. by turning A into a clique.

A *tree decomposition* of a graph G is a pair (T, β) such that T is a tree and β is a mapping assigning each node x of T to a *bag* $\beta(x) \subseteq V(G)$, satisfying the following conditions: every vertex of G belongs to some bag, for every edge of G there exists a bag containing both endpoints of the edge, and for every vertex of G, the set of nodes x of T such that $v \in \beta(x)$ induces a connected subtree of T. The *width* of a tree decomposition (T, β) is the maximum, over all nodes x of T, of the value of $|\beta(x)| - 1$. The *treewidth* of a graph G, denoted by $\mathtt{tw}(G)$, is the minimum width of a tree decomposition of G. Path decompositions and pathwidth (denoted by $\mathtt{pw}(G)$) are defined analogously, but with the additional requirement that the tree T is a path.

We use a number of well-known facts about treewidth and tree decompositions.

▶ **Lemma 1** (Folklore). *Let G be a graph, and (T,β) a tree decomposition of width k of G. Then the following statements hold.*

1. *Let W be a clique in G. Then, there is a node x of T with $W \subseteq \beta(x)$.*
2. *Suppose $v, w \in V(G)$, $\{v,w\} \notin E(G)$. If there is a node x of T, with $v, w \in \beta(x)$, then (T,β) is a tree decomposition of width k of the graph obtained by adding the edge $\{v,w\}$ to G.*
3. *Suppose $W \subseteq V(G)$. Then, there is a node x in T such that when we remove $\beta(x)$ and all incident edges from G, then each connected component of G contains at most $n/2$ vertices of W.*
4. *Let y be a leaf of T, with neighbour y'. If $\beta(y) \subseteq \beta(y')$, then removing y with its bag from the tree decomposition (T,β) yields another tree decomposition of G of width at most k.*
5. *If H is a minor of G, then $\mathtt{tw}(H) \leq \mathtt{tw}(G)$, and $\mathtt{pw}(H) \leq \mathtt{pw}(G)$.*

A graph G is *co-bipartite* if $V(G) = A \cup B$ with A a clique and B a clique (that is, the complement of G is bipartite). The following fact is also well known, and follows implicitly from the proofs of Arnborg et al. [1]. For completeness, we give a proof here.

▶ **Lemma 2** (See, e.g. [1]). *Let G be a co-bipartite graph, with $V(G) = A \cup B$ where A and B are cliques. Then:*

1. *$\mathtt{tw}(G) = \mathtt{pw}(G)$.*
2. *G has a path decomposition (P,β) with width equal to $\mathtt{tw}(G)$ such that $A \subseteq \beta(p_1)$ and $B \subseteq \beta(p_r)$, where p_1 and p_r are the two endpoints of P.*

Proof. Suppose (T,β) is a tree decomposition of G of width $\mathtt{tw}(G)$. By Lemma 1(1), there is a node x in T with $A \subseteq \beta(x)$, and a node y in T with $B \subseteq \beta(y)$. Let P be the path from x to y in T.

If T has nodes not in P, then we can apply the following step. Take a leaf z of T, not in P. Let z' be the neighbour of z in T. For each $v \in A \cap \beta(z)$, it holds that $v \in \beta(z')$ as z' is on the path from z to x, and for each $v \in B \cap \beta(z)$, it holds that $v \in \beta(z')$ as z' is on the path from z to y. So, by Lemma 1(4), we can remove z from T and obtain another tree decomposition of G. Repeating this step as long as possible gives the desired result. ◀

The *vertex separation number* of a graph G is denoted by $\mathtt{vsn}(G)$ and defined as the minimum, over all orderings $\sigma = (v_1, \ldots, v_n)$ of the vertex set of G, of the maximum, over all $i \in \{1, \ldots, n\}$, of the number of vertices v_j such that $j > i$ and v_j has a neighbour in $\{v_1, \ldots, v_i\}$. Kinnersley proved the following characterisation of pathwidth.

▶ **Theorem 3** (Kinnersley [5]). *The pathwidth of every graph equals its vertex separation number.*

Treewidth is the following decision problem: Given a graph G and an integer k, is the treewidth of G at most k? The problems Pathwidth and Vertex Separation Number are defined analogously.

In 1987, Arnborg, Corneil, and Proskurowski established NP-completeness of Treewidth in the class of co-bipartite graphs [1]. Ten years later, Bodlaender and Thilikos [4] proved that Treewidth is NP-complete on graphs with maximum degree at most 9. Monien and Sudborough [6] proved that Vertex Separation Number is NP-complete on planar graphs with maximum degree at most 3. Combining this result with Theorem 3 directly shows the following.

▶ **Theorem 4** (Monien and Sudborough [6]). *Pathwidth is NP-complete on planar graphs with maximum degree at most 3.*

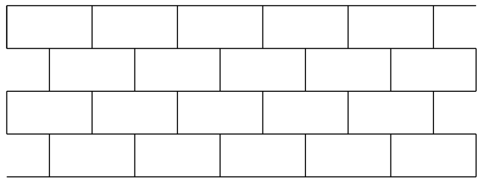

Figure 1 A brick wall with 5 rows and 12 columns.

A well-known type of graphs are the *walls*. A wall with r rows and c columns has $r \times c$ vertices. We refrain from giving a formal definition here, as the concept is clear from Figure 1[1]

It is well known that the pathwidth and treewidth of an n by r grid equal $\min\{n, r\}$, see, e.g. [3, Lemmas 87 and 88]. Since any brick wall is a subgraph of a grid, the upper bound also holds for brick walls, and the standard construction gives the following result.

▶ **Lemma 5** (Folklore). *Let $B_{r,c}$ be a brick wall with r rows and c columns. Then $\mathtt{tw}(B_{r,c}) \leq \mathtt{pw}(B_{r,c}) \leq r$ and there is a path decomposition (P, β) of $B_{r,c}$ of width r with $\beta(p_1)$ the set of vertices on the first column of $B_{r,c}$, and $\beta(p_q)$ the set of vertices on the last column of $B_{r,c}$, where p_1 and p_r are the two endpoints of P.*

A *linear ordering* of a graph G is a bijection $f : V(G) \to \{1, \ldots, n\}$. The *cutwidth* of a linear ordering of G is

$$\max_{i \in [n]} \left| \left\{ \{v, w\} \in E(G) \;\middle|\; f(v) \leq i < f(w) \right\} \right|.$$

The *cutwidth* of a graph G, denoted by $\mathtt{cw}(G)$, is the minimum cutwidth of a linear ordering of G.

The Cutwidth problem asks to decide, for a given graph G and integer k, whether the cutwidth of G is at most k. Monien and Sudborough [6] showed that Cutwidth is NP-complete on graphs of maximum degree three (using the problem name Minimum Cut Linear Arrangement). As their proof does not generate vertices of degree one, and the cutwidth of a graph does not change by subdividing an edge, from their proof, the next result follows.

▶ **Theorem 6** (Monien and Sudborough [6]). *Cutwidth is NP-complete on cubic graphs.*

3 A simple proof for co-bipartite graphs

In this section, we give a new simple proof that Treewidth is NP-complete. Our proof borrows elements from the NP-completeness proof from Arnborg et al. [1], but uses an easy transformation from Pathwidth.

Let G be a graph. We denote by $F(G)$ the graph obtained from G as follows. The vertices of $F(G)$ consist of two copies v and v' for every $v \in V(G)$; we denote by V and V' the sets $V(G)$ and $\{v' \mid v \in V(G)\}$, respectively. Moreover, the graph $F(G)$ contains for every $v \in V(G)$ an edge between v and v', and for every edge $\{u, v\} \in E(G)$, it contains one edge between u and v' and one edge between v and u'. Finally, $F(G)$ contains all edges between every pair of distinct vertices in V and every pair of distinct vertices in V'. Note that each of the sets V and V' are cliques in $F(G)$. In particular, G is co-bipartite. An example is given in Figure 2.

[1] The most common notion of wall does not have the vertices of degree one which we see at the bottom left and top right corner of Figure 1. We keep these degree one vertices, for slightly easier notation.

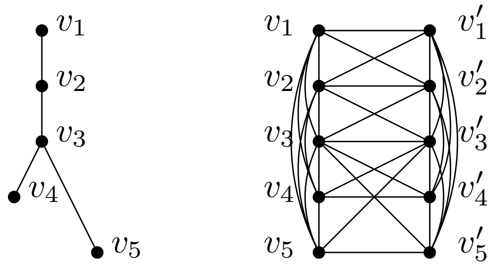

Figure 2 A graph G with $F(G)$.

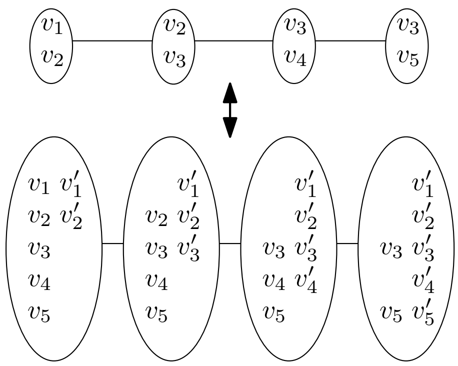

Figure 3 A path decomposition of the graph G from Figure 2 and the corresponding path decomposition of $F(G)$.

▶ **Lemma 7.** *Let G be a graph. Then, $\mathtt{tw}(F(G)) = \mathtt{pw}(F(G)) = n+\mathtt{pw}(G)$, where $n = |V(G)|$.*

Proof. First, we show that $\mathtt{pw}(F(G)) \leq n+\mathtt{pw}(G)$. Let $k = \mathtt{pw}(G)$. Take a path decomposition (P, β) of G of width k, with $P = (p_1, \ldots, p_r)$. Now, let $\gamma(p_i)$ be a set of vertices of $F(G)$ defined as follows:

- For each $v \in V(G)$ such that there is a $j \geq i$ with $v \in \beta(p_j)$, add v to $\gamma(p_i)$.
- For each $v \in V(G)$ such that there is a $j \leq i$ with $v \in \beta(p_j)$, add v' to $\gamma(p_i)$.

An example of this construction, applied to the graphs G and $F(G)$ of Figure 2, is given in Figure 3.

We claim that (P, γ) is a path decomposition of $F(G)$ of width $n + k$. We first verify that (P, γ) is a path decomposition. The first and third conditions of path decompositions are clearly satisfied. Notice that $V \subseteq \gamma(p_1)$, and $V' \subseteq \gamma(p_r)$. So, for each edge in $F(G)$ between two vertices in V, or between two vertices in V', there is a bag in (P, γ) containing the two endpoints of the edge, namely, the bag corresponding to the node p_1 or p_r, respectively. Consider an edge $\{v, v'\}$ for a vertex $v \in V(G)$. There is a node p_v with $v \in \beta(p_v)$, and therefore $v, v' \in \gamma(p_v)$. Consider an edge $\{v, w'\}$ in $F(G)$, corresponding to an edge $\{v, w\} \in E(G)$. There is a node p_{vw} with $v, w \in \beta(p_{vw})$. Now, $v, v', w, w' \in \gamma(p_{vw})$.

To see that the width is $n + k$, consider some bag $\gamma(p_i)$ and a vertex $v \in V(G)$. There are three possible cases:

1. For each j with $v \in \beta(p_j)$, $j > i$. Now, $v \in \gamma(p_i)$; $v' \notin \gamma(p_i)$.
2. For each j with $v \in \beta(p_j)$, $j < i$. Now, $v' \in \gamma(p_i)$; $v \notin \gamma(p_i)$.
3. If the previous two cases do not hold, then there is $j \leq i$ with $v \in \beta(p_j)$, and $j' \geq i$ with $v \in \beta(p_{j'})$. From the definition of path decompositions, it follows that $v \in \beta(p_i)$. From the construction of γ, we have $v, v' \in \gamma(p_i)$.

In each of the cases, we have one vertex more in $\gamma(p_i)$ than in $\beta(p_i)$, so for each node, the size of its γ-bag is exactly n larger than the size of its β-bag. The claim follows.

Now, suppose the treewidth of G equals ℓ. From Lemma 2(2), it follows that we can assume we have a path decomposition (P, γ) of $F(G)$ of width ℓ, with P having successive bags $p_1, p_2, \ldots, p_r$, and with $V \subseteq \gamma(p_1)$ and $V' \subseteq \gamma(p_2)$.

We now define a path decomposition (P, δ) of G, as follows. For each node x on P, set $\delta(x) = \{v \in V \mid v \in \gamma(x) \wedge v' \in \gamma(x)\}$. (Note that this is the reverse of the operation in the first part of the proof; compare with Figure 3.)

We now verify that (P, δ) is indeed a path decomposition of G. For each vertex v, $\{v, v'\}$ is an edge in $F(G)$, so there is a node x_v with $v, v' \in \gamma(x_v)$, hence $v \in \delta(x_v)$. For each edge $\{v, w\} \in E(G)$, the set $\{v, v', w, w'\}$ forms a clique in $F(G)$, so there is a node x_{vw} with $\{v, v', w, w'\} \subseteq \gamma(x_{vw})$ (see Lemma 1(1)). Hence $v, w \in \delta(x_{vw})$. Finally, for each $v \in V(G)$, the set of nodes x with $v \in \delta(x)$ is the intersection of the nodes with $v \in \gamma(x)$ and the nodes with $v' \in \gamma(x)$; the intersection of connected subtrees is connected, so the third condition in the definition of path (tree) decompositions also holds.

Finally, we show that the width of (P, δ) is $\ell - n$. Consider a vertex v, and $i \in [r]$. There must be i_v with $\{v, v'\} \subseteq \gamma(p_{i_v})$. If $i \leq i_v$, then $v \in \gamma(p_i)$; if $i \geq i_v$, then $v' \in \gamma(p_i)$ (using that $v \in \gamma(p_1)$ and $v' \in \gamma(p_r)$). So, we have $\{v, v'\} \cap \gamma(p_i) \neq \emptyset$.

Now, for each node p_i, $i \in [r]$, for each vertex v, we have that $\gamma(p_i)$ contains both vertices from the set $\{v, v'\}$ when $v \in \delta(p_i)$, and $\gamma(p_i)$ contains exactly one vertex from the set $\{v, v'\}$ when $v \notin \delta(p_i)$. So, $|\gamma(p_i)| = |\delta(p_i)| + n$. As this holds for each bag, we have that the width of (P, γ) is exactly n larger than the width of (P, δ). It follows that $\texttt{pw}(G) \leq \texttt{tw}(F(G)) - n \leq \texttt{pw}(F(G)) - n$, which shows the result. ◀

Lemma 7, together with the NP-completeness of Vertex Separation Number [6], and the equivalence between the pathwidth and the vertex separation number (Theorem 3), leads to an alternative simple proof of NP-completeness of Treewidth in the class of co-bipartite graphs.

▶ **Corollary 8.** *Treewidth is NP-complete on co-bipartite graphs.*

One can obtain a proof of the NP-completeness of Treewidth on graphs with maximum degree five by combining the proof above with the technique of replacing a clique with a wall or grid (as in [4] or in the next section). Instead of this, we give in the next section a proof that reduces from Cutwidth and shows NP-completeness of Treewidth on graphs of degree three.

4 Cubic graphs

In this section, we give an NP-completeness proof for Treewidth on cubic graphs. The construction uses a few steps. The first step is a simplified version of the NP-completeness proof from Arnborg et al. [1]; the second step follows the idea of Bodlaender and Thilikos [4] to replace the cliques by grids or walls. After this step, we have a graph with maximum degree 7. In the third step, we replace vertices of degree more than 3 by trees of maximum degree 3, and show that this step does not change the treewidth (it actually can change the pathwidth). The fourth step makes the graph 3-regular by simply contracting over vertices of degree 2.

▶ **Theorem 9.** *Treewidth is NP-complete on regular graphs of degree 3.*

Proof. We use a transformation from CUTWIDTH on 3-regular graphs.

Let G be an n-vertex 3-regular graph and k an integer. Using a sequence of intermediate steps and intermediate graphs G_1, G_2, G_3, we construct a 3-regular graph G_4 with the property that G has cutwidth at most k, if and only if G_4 has treewidth at most $3n + k + 2$.

Step 1: From Cutwidth to Treewidth. The first step is a streamlined version of the proof from Arnborg et al. [1]. For each vertex $v \in V(G)$, we take a set $A_v = \{v^1, v^2, v^3\}$ which has three copies of v.

For each edge $e \in E(G)$, we have a set $B_e = \{e^1, e^2\}$, which consists of two vertices that represent the edge.

Let $A = \bigcup_{v \in V(G)} A_v$, and $B = \bigcup_{e \in E(G)} B_e$. We create G_1 by taking $A \cup B$ as vertex set, turning A into a clique, turning B into a clique, and for each pair v, e with v an endpoint of e, adding edges between all vertices in A_v and all vertices in B_e.

▷ Claim 10. Let G and G_1 be as above. $\mathtt{tw}(G_1) = \mathtt{pw}(G_1) = \mathtt{cw}(G) + 3n + 2$.

Proof. First, assume G has cutwidth k, and let f be a linear ordering of G of cutwidth k, and denote the ith vertex in the linear ordering as $v_i = f^{-1}(i)$.

Build a path decomposition (P, β) with P the path with nodes $p_1, \ldots, p_n$. For $i \in [n]$, set

$$\begin{aligned}\beta(p_i) = &\left\{ v_j^a \;\middle|\; j \geq i \wedge a \in \{1,2,3\} \right\} \\ &\cup \left\{ e^b \;\middle|\; e = \{v_j, v_{j'}\} \in E(G) \wedge \min\{j, j'\} \leq i \wedge b \in [2] \right\}.\end{aligned}$$

That is, we take the representatives of the vertices $v_i, v_{i+1}, \ldots, v_n$, and all vertices that represent an edge with at least one endpoint in $\{v_1, v_2, \ldots, v_i\}$.

We can verify that (P, β) is a path decomposition of G_1. From the construction, it directly follows that $A \subseteq \beta(p_1)$ and $B \subseteq \beta(p_n)$. For the second condition of path decompositions, it remains to look at edges in G_1 with one vertex of the form v_i^a and one vertex of the form e^b. Necessarily, v_i is an endpoint of e, and now we can note that both vertices are in bag $\beta(p_i)$. From the construction, it directly follows that the third condition of path decompositions is fulfilled.

To show that the width of this path decomposition is at most $k + 3n + 2$, we use an accounting system. Consider $\beta(p_i)$. Give each vertex $v \in V(G)$ three credits, except v_i, which gets six credits. Each edge that "crosses the cut", i.e. it belongs to the set $\{\{v, w\} \in E(G) \mid f(v) \leq i < f(w)\}$, gets one credit. All other edges get no credit. We handed out at most $k + 3n + 3$ credits. We now redistribute these credits to the vertices in $\beta(p_i)$. Each vertex v_j, $j \geq i$, gives one credit to each vertex of the form v_j^a, $a \in \{1, 2, 3\}$. For an edge $e = \{v_j, v_{j'}\}$, with $j < i$ and $j' < i$, the vertices e^1 and e^2 get, respectively, a credit from v_j and $v_{j'}$. For an edge $e = \{v_j, v_{j'}\}$, with $j \leq i < j'$, the vertices e^1 and e^2 get, respectively, a credit from v_j and a credit from e. Now, each vertex and edge precisely spends its credit: a vertex v_j with $j < i$ gives one credit to each of its incident edges, v_i gives one credit to each of its copies v_i^1, v_i^2, v_i^3, and one credit to each of its incident edges, and v_j with $j > i$ gives one credit to each of its copies v_j^1, v_j^2, v_j^3. Each vertex in the bag $\beta(p_i)$ gets one credit, so the size of the bag is at most $k + 3n + 3$. As this holds for each bag, the width of the path decomposition is at most $k + 3n + 2$.

Now, assume that we have a tree decomposition (T, γ) of G_1 of width ℓ. By Lemma 1(1), as A and B are cliques, there is a bag p_1 with $A \subseteq \gamma(p_1)$, and a bag p_r with $B \subseteq \gamma(p_r)$. As in the proof of Lemma 2, we can remove all bags not on the path from p_1 and p_r, and still keep a tree decomposition of G_1. So, we can assume we have a path decomposition (P, γ) of width at most ℓ of G_1, where P is a path with successive vertices $p_1, p_2, \ldots, p_r$, and $\gamma(p_1) = A$ and $\gamma(p_r) = B$.

For each $v \in V(G)$, set $g(v)$ to the maximum i such that $\{v^1, v^2, v^3\} \subseteq \beta(p_i)$. (As $\{v^1, v^2, v^3\} \subseteq A \subseteq \beta(p_1)$, $g(v)$ is well defined and in $[r]$.)

Take a linear ordering f of G such that for all $v, w \in V(G)$, $g(v) < g(w) \Rightarrow f(v) < f(w)$. (That is, order the vertices with respect to increasing values of g, and arbitrarily break the ties when vertices have the same value $g(v)$.) We claim that f has cutwidth at most $\ell - 3n - 2$.

Consider a vertex $v \in V(G)$, and suppose $g(v) = i'$. Let e be an edge incident to v. The set $\{v^1, v^2, v^3, e^1, e^2\}$ is a clique in G_1, so there is an i_e with $\{v^1, v^2, v^3, e^1, e^2\} \subseteq \beta(p_{i_e})$. From the definition of path decompositions and the construction of g, we have $i_e \leq i'$. As $\{e^1, e^2\} \subseteq \beta(p_{i_e}) \cap \beta(p_r)$, we have that $\{e^1, e^2\} \subseteq \beta(p_{i'})$.

Now, consider an $i \in [n]$. Let $v = f^{-1}(i)$ be the ith vertex of the ordering and $C = f^{-1}[i]$ be the first i vertices in the linear ordering. Let E^1 be the set of edges with exactly one endpoint in C, and let E^2 be the set of edges with both endpoints in C. Suppose $g(v) = i'$. We now examine which vertices belong to $\beta(p_{i'})$:

- By definition, v^1, v^2, v^3.
- For each $w \in V(G) \setminus C$, there is an $i_w \geq i'$ with $\{w^1, w^2, w^3\} \subseteq \beta(p_{i_w})$, hence w^1, w^2, and w^3 are in $\beta(p_{i'})$. (Use here that these vertices are in $\beta(p_1)$.) The number of such vertices is $3n - 3i$.
- For each edge $e \in E^1 \cup E^2$, from the discussion above it follows that there is an $i_e \leq i'$ with $e^1, e^2 \in \beta(p_{i_e})$, and, as these vertices are in $\beta(p_r)$, we have $\{e^1, e^2\} \subseteq \beta(p_{i'})$.

Thus, the size of $\beta(p_{i'})$ is at least $3n - 3i + 3 + 2 \cdot |E_1| + 2 \cdot |E_2|$. As each vertex in C is incident to exactly three edges, we have $3i = |E_1| + 2 \cdot |E_2|$. Now, $\ell \geq |\beta(p_{i'})| - 1 \geq 3n - 3i + 2 + 2 \cdot |E_1| + 2 \cdot |E_2| = 3n + 2 + |E_1|$. It follows that the size of the cut $\Big|\big\{\{x, y\} \in E(G) \;\big|\; f(x) \leq i < f(y)\big\}\Big| = |E_1| \leq \ell - 3n - 2$. As this holds for each $i \in [n]$, the bound of $\ell - 3n - 2$ on the cutwidth of f follows.

We have thus shown that $\texttt{pw}(G_1) \leq \texttt{cw}(G) + 3n + 2$ and that $\texttt{cw}(G_1) \leq \texttt{tw}(G_1) - 3n - 2$. Together with the inequality $\texttt{tw}(G_1) \leq \texttt{pw}(G_1)$, this proves the claim. $\lhd$

Step 2: The wall construction. In the second step, we use a technique from Bodlaender and Thilikos [4]. We construct a graph G_2 from the graph G_1 by removing the edges between vertices in A and the edges between vertices in B; then, we add a wall with $3n$ rows and $24n$ columns, and add a matching from the vertices in the last column of the wall to the vertices in A. Similarly, we add another wall with $3n$ rows and $24n$ columns, and add a matching from the vertices in the first column of this wall to the vertices in B.

As applying the wall construction to a graph obtained from the first step would be unwieldy, the example in Figure 4 shows the wall construction applied to the graph from the previous section.

$\rhd$ Claim 11. $\texttt{tw}(G_1) = \texttt{pw}(G_1) = \texttt{tw}(G_2) = \texttt{pw}(G_2)$. Moreover, there is a path decomposition of G_2 of optimal width with a node x_A with $A \subseteq \beta(x_A)$ and a node x_B with $B \subseteq \beta(x_B)$.

Proof. Suppose we have a tree decomposition (T, β) of G_2 of optimal width k. By Lemma 1(3), there is a node x such that each connected component of $G_2 \setminus \beta(x)$ contains at most $36n^2$ vertices of the left wall. Note that $\beta(x)$ must contain a vertex of each row from the left wall. Suppose not. Each pair of two successive columns in the wall is connected; there are at least $12n - |\beta(x)|$ disjoint pairs of columns which do not contain a vertex from $\beta(x)$. All vertices on

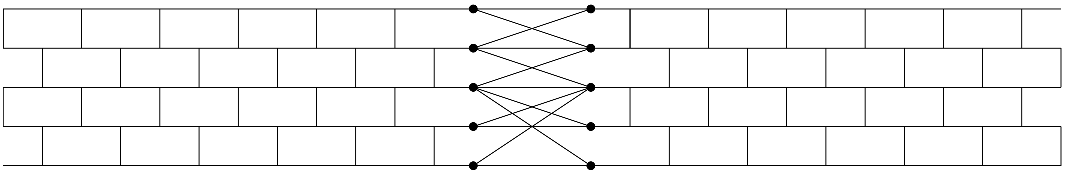

Figure 4 Illustration of the wall construction. Here, it is applied to the graphs from Figure 2, and the number of columns shown is smaller than that in the actual construction.

these columns are connected in $G_2 \setminus \beta(x)$ as they intersect the row without vertices in $\beta(x)$. As the number of vertices in these columns is larger than $36n^2$, since $k \leq |E(G)| = 3n/2$, we have a contradiction.

By Lemma 1(2), (T, β) is also a tree decomposition of the graph obtained from G_2 by adding edges between each pair of vertices in $\beta(x)$. Apply the same step to the right wall. We see that (T, β) is a tree decomposition of width k of a graph that for each pair of rows in the left wall contains an edge between a pair of vertices from these rows, and similarly for the right wall. Now, if we contract each row of the left wall to the neighbouring vertex in A, and contract each row of the right wall to the neighbouring vertex in B, we obtain G_1 as minor: G_1 is a minor of a graph of treewidth k, so has treewidth at most k.

By Lemma 2, $\mathtt{tw}(G_1) = \mathtt{pw}(G_1)$, and there is a path decomposition (P, γ) of G_1 of optimal width ℓ such that $A \subseteq \gamma(p_1)$ and $B \subseteq \gamma(p_q)$, where p_1 and p_q are the endpoints of P.

We can now build a path decomposition of G_2 of the same width ℓ as follows: first, take the successive bags of a path decomposition of the left wall, of width $3n$, where we can end with a bag that contains all vertices of A. Then, we take the bags of (P, γ). Now, we add a path decomposition of the right wall, of width $3n$, that starts with a bag containing all vertices in B. $\lhd$

Step 3: Making the graph subcubic. Note that the maximum degree of a vertex in G_2 is seven. A vertex in A has one neighbour in the wall, and six neighbours in B (the vertex it represents has three incident edges, and each is represented by two vertices). Similarly, a vertex in B has degree seven: again, one neighbour in the wall, and six neighbours in A (each endpoint of the edge it represents is represented by three vertices). Vertices in the walls have degree at most three.

Given G_2, we build a subcubic graph G_3. We do this by replacing each vertex in A and in B by a tree, and replacing edges to vertices in A and B by edges to leaves or the root of these trees.

For vertices v^α in A (with $v \in V(G)$, $\alpha \in [3]$), we take an arbitrary tree with a root of degree 2, all other internal vertices of degree 3, and six leaves. The root (which we denote by the name of the original vertex v^α) is made adjacent to the neighbour of v^α in the wall.

Each vertex $e^\alpha \in B$ (with $e \in E(G)$, $\alpha \in [2]$) is also replaced by a tree with a root of degree 2, all other internal vertices of degree 3, and six leaves, but here we need to use a specific shape of the tree. Suppose e has endpoints v and w. Figure 5 shows this tree. In particular, note that the root is made adjacent to the neighbour of e^α in the wall, and the leaves that go to the subtrees that represent v are grouped together, and the leaves that go to the subtrees that represent w are grouped together.

Each edge between a vertex v^α in A and a vertex $e^{\alpha'}$ in B now becomes an edge from a leaf of the tree representing v^α, to a leaf of the tree representing $e^{\alpha'}$; $\alpha \in [3]$, $\alpha' \in [2]$. The roots of the trees are made adjacent to a vertex in the wall; this is the same vertex as the wall neighbour of the original vertex in G_2.

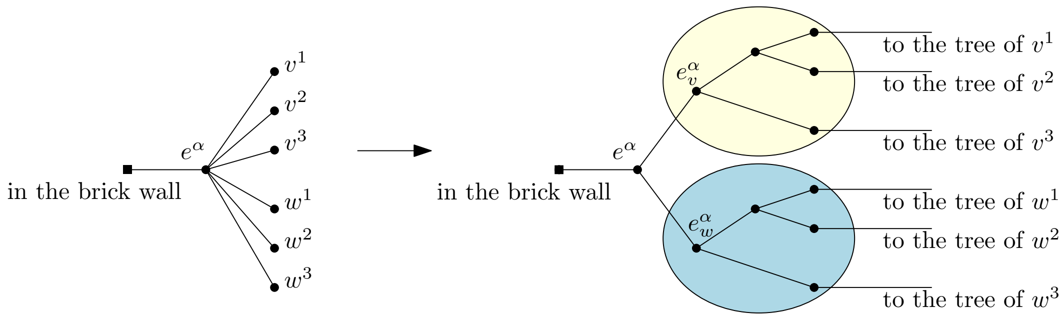

Figure 5 Replacing a vertex e^α from B by a tree; e is here the edge $\{v, w\}$.

▷ Claim 12. Suppose $\mathtt{tw}(G_2) \geq 68$. Then $\mathtt{tw}(G_2) = \mathtt{pw}(G_2) = \mathtt{tw}(G_3)$.

Proof. We have already established that $\mathtt{tw}(G_2) = \mathtt{pw}(G_2)$.

First, note that G_2 is a minor of G_3: we obtain G_2 from G_3 by contracting each of the new trees to its original vertex. By Lemma 1(5), we have $\mathtt{tw}(G_2) \leq \mathtt{tw}(G_3)$.

Suppose we have a path decomposition (P, β) of G_2 of optimal width $\ell = \mathtt{pw}(G_2) = \mathtt{tw}(G_2)$. By Claim 11, we can also assume that there is a bag that contains all vertices in A, and that there is a bag that contains all vertices in B.

For each vertex $v \in V(G)$, we claim that there is a node p_{i_v} with $v^1, v^2, v^3 \in \beta(p_{i_v})$ and $e^1, e^2 \in \beta(p_{i_v})$ for each edge e incident to v. This can be shown as follows. The pair (P, β) is also a path decomposition of the graph $G + \mathit{clique}(A) + \mathit{clique}(B)$, obtained from G_2 by adding edges between each pair of vertices in A, and each pair of vertices in B (since there is a bag containing all vertices of A and a bag containing all vertices of B and by Lemma 1(2).) The claim now follows from Lemma 1(1) by observing that these nine vertices (v^1, v^2, v^3, and e^1, e^2 for each edge incident to v) form a clique in $G + \mathit{clique}(A) + \mathit{clique}(B)$.

Now, we can construct a tree decomposition of G_3 as follows. Take (P, β). Replace each vertex in A and each vertex in B by the root of the tree it represents. For each vertex $v \in V(G)$, we add one additional bag to the tree decomposition; this bag becomes a leaf of the tree decomposition. (Note that after this step, we no longer have a path decomposition.)

Consider a vertex $v \in V(G)$. Take a new node x_v, and make x_v adjacent to p_{i_v} in the tree. Let the bag of x_v contain the following vertices: all vertices in the subtrees that represent v^1, v^2, v^3, for each edge e with v as endpoint the vertices e^1, e^1_v, e^2, e^2_v, and the descendants of e^1_v and e^2_v in the respective subtrees (the vertices in the yellow area in Figure 5, assuming that $e = \{v, w\}$).

Each vertex in A is represented by a binary tree with a root of degree two and six leaves, so by eleven vertices. For each of the three edges incident to v, we have two subtrees of which we take six vertices each, so the total size of this new bag is $3 \cdot 11 + 3 \cdot 2 \cdot 6 = 69$. One easily verifies that we have a tree decomposition of G_3, and as the original bags keep the same size when $\ell \geq 68$, we have a tree decomposition of G of width at most ℓ. ◁

By taking a sufficiently large n (e.g. $n \geq 22$ works), we can assume that $\ell \geq 68$.

Step 4: Making the graph 3-regular. The fourth step is simple. Note that when the treewidth of a graph is at least three, the treewidth does not change when we contract a vertex of degree at most two to a neighbour (see [2]), possibly removing parallel edges. We apply this step as long as possible, and let G_4 be the resulting graph. The graph G_4 is a 3-regular graph, and, when $n \geq 22$, its treewidth equals the treewidth of G_1, which is $\mathtt{cw}(G) + 3n + 2$. As we can construct G_4 in polynomial time, this completes the transformation, and we can conclude that TREEWIDTH is NP-complete on 3-regular graphs. ◀

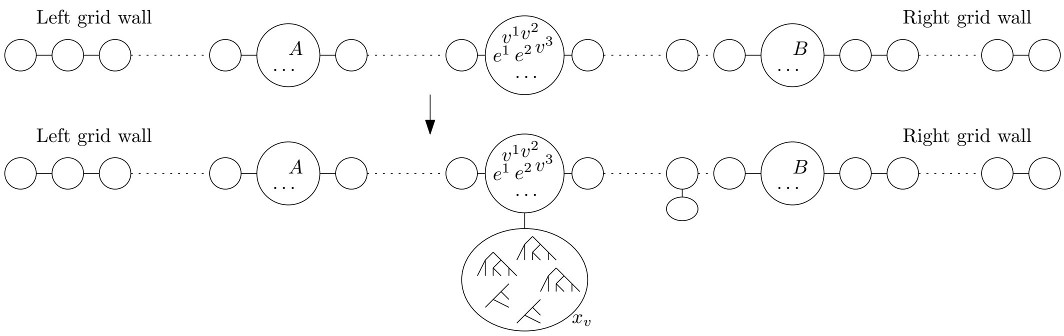

Figure 6 Illustration of the proof. The path decomposition before and after adding the new node x_v.

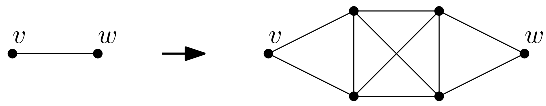

Figure 7 Increasing the degree of two adjacent vertices by one.

5 Special cases

In this section, we give two NP-completeness proofs for Treewidth on special graph classes, which follow from minor modifications of the proof of Theorem 9. We first observe that for any fixed $d \geq 4$, Treewidth is NP-complete on d-regular graphs.

▶ **Proposition 13.** *For each $d \geq 3$, Treewidth is NP-complete on d-regular graphs.*

Proof. The result for $d = 3$ was given as Theorem 9.

A small modification of the proof of Theorem 9 gives the result for 4-*regular* graphs: instead of using a wall, use a grid. At the borders of this grid, we have vertices of degree less than 3. We can avoid these by first contracting vertices of degree 2, and then noting that there is a perfect matching with the vertices of degree 3 at the sides of the grid. Replace each edge in this matching by a small subgraph, as shown in Figure 7. Note that this step increases the degree of v and w by one, while, when the treewidth of G is at least 5, the step will not change the treewidth of the graph.

In the step where we change vertices of degree 7 to vertices of degree 3 by replacing a vertex by a small tree, we instead use a tree with the root having two children, each with three children. These roots are made adjacent to the grid. Now, the roots have degree 3, and we add an arbitrary perfect matching between these root vertices in A, and similarly for B. (Note that in the construction, there is a bag containing all roots for A, and similarly B; these sets have even size.) This gives the result for $d = 4$.

Consider the following gadget. Take a clique with $d + 1$ vertices, and remove one edge, say $\{x, y\}$, from this clique. For a vertex v in a graph G, add an edge from x to v, and an edge from y to v. See Figure 8.

If G has treewidth at least d, then this step increases the degree of v by 2 without changing the treewidth. Now, if d is odd, we can take an instance of the hardness proof on 3-regular graphs, and add to each vertex of that instance $(d - 3)/2$ copies of this gadget. We obtain an equivalent instance that is d-regular. If d is even, we add $(d - 4)/2$ copies of the gadget to an instance of the hardness proof on 4-regular graphs. ◀

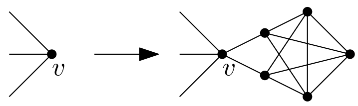

Figure 8 Increasing the degree of a vertex: if $\mathtt{tw}(G) \geq 4$, then the step increases the degree of v from 3 to 5, but does not change the treewidth.

A *d-dimensional grid graph* is a finite induced subgraph of the infinite d-dimensional grid. Observe that d-dimensional grid graphs have degree at most $2d$, and in particular the 3-dimensional grid graphs have degree at most 6. As a consequence of lowering the degree of hard TREEWIDTH instances from 9 to at most 6, we can show that computing the treewidth of 3-dimensional grid graphs is NP-complete. Since we lowered the degree of hard instances down to at most 3, we can even show the following.

▶ **Proposition 14.** *TREEWIDTH is NP-complete on subcubic 3-dimensional grid graphs.*

Proof. The argument is simply that every n-vertex (sub)cubic graph admits a subdivision of polynomial size that is a 3-dimensional grid graph. We give a simple such embedding.

We reduce from TREEWIDTH on cubic graphs, which is NP-hard by Theorem 9. Let G be any cubic graph, $v_0, v_1, \ldots, v_{n-1}$ its vertices, and $e_1, e_2, \ldots, e_{3n/2}$ its edges. We build a subcubic induced subgraph H of the $(6n-1) \times (3n+1) \times 3$ grid that is a subdivision of G. In particular, $\mathtt{tw}(H) = \mathtt{tw}(G)$ and H has $O(n^2)$ vertices and edges, thus we can conclude.

For each $i \in [0, n-1]$, vertex v_i is encoded by the path made by the 5 vertices $(x, 0, 0)$ with $x \in [6i, 6i+4]$. We arbitrarily assign $(6i, 0, 0)$, $(6i+2, 0, 0)$, $(6i+4, 0, 0)$ each with a distinct neighbour of v_i in G, say $v_{i(0)}$, $v_{i(1)}$, $v_{i(2)}$, respectively.

Every edge $e_k = \{v_i, v_j\}$ of G with $i < j$ is encoded in the following way. Let $a, b \in [0, 2]$ be such that $i(a) = j$ and $j(b) = i$. We build a path from $(6i+2a, 0, 0)$ to $(6j+2b, 0, 0)$ with degree-2 vertices, by first adding all the vertices $(6i+2a, y, 0)$ and $(6j+2b, y, 0)$ for $y \in [2k]$, then bridging $(6i+2a, 2k, 0)$ and $(6j+2b, 2k, 0)$ by adding $(6i+2a, 2k, 1)(6i+2a, 2k, 2)(6i+2a+1, 2k, 2)(6i+2a+2, 2k, 2)\ldots(6j+2b-1, 2k, 2)(6j+2b, 2k, 2)(6j+2b, 2k, 1)$.

This finishes the construction of H. All of its vertices have degree 2, except the vertices at $(6i+2, 0, 0)$, which have degree 3. It is easy to see that H is a subdivision of G (where each edge gets subdivided at most $12n+5$ times). ◀

We can easily adapt the previous proof to show hardness for finite subcubic (non-induced) subgraphs of the $\infty \times \infty \times 2$ grid.

6 Conclusions

In this paper, we gave a number of NP-completeness proofs for TREEWIDTH. The first proof is an elementary reduction from PATHWIDTH to TREEWIDTH on co-bipartite graphs; while the hardness result is long known, our new proof has the advantage of being very simple, and presentable in a matter of minutes. Our second main result is the NP-completeness proof for TREEWIDTH on cubic graphs, which improves upon the over 25-years-old bound of degree 9.

We end this paper with a few open problems. A long standing open problem is the complexity of TREEWIDTH on planar graphs. While the famous ratcatcher algorithm solves the related BRANCHWIDTH problem in polynomial time [7], it is still unknown whether TREEWIDTH on planar graphs is polynomial time solvable or whether it is NP-complete. Also, no NP-hardness proofs for TREEWIDTH on graphs of bounded genus, or H-minor free graphs for some fixed H are known. An easier open problem might be the complexity of BRANCHWIDTH for graphs of bounded degree, and we conjecture that BRANCHWIDTH is NP-complete on cubic graphs.

While "our" reductions are simple, the NP-hardness of TREEWIDTH is derived from the NP-hardness of PATHWIDTH or CUTWIDTH. Thus, it would be good to have simple NP-hardness proofs for PATHWIDTH and/or CUTWIDTH, preferably building upon "classic" NP-hard problems like SATISFIABILITY, elementary graph problems like CLIQUE, or BIN PACKING.

The reductions in our hardness proofs increase the parameter by a term linear in n, so shed no light on the parameterised complexity of TREEWIDTH. Hence, it would be interesting to obtain parameterised reductions (i.e. reductions that change k to a value bounded by a function of k), and also aim at lower bounds (e.g. based on the (S)ETH) on the parameterised complexity of TREEWIDTH. It is also not known whether one can obtain a time lower bound of $2^{\Omega(n)}$ for TREEWIDTH.

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