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The highly nonlinear shallow water equation: local well-posedness, wave breaking data and non-existence of sech^2 solutions

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Abstract
In the context of the initial data and an amplitude parameter ε, we establish a local existence result for a highly nonlinear shallow water equation on the real line. This result holds in the space H^k as long as $k > 5/2$. Additionally, we illustrate that the threshold time for the occurrence of wave breaking in the surging type is on the order of ε^{-1}, while plunging breakers do not manifest. Lastly, in accordance with ODE theory, it is demonstrated that there are no exact solitary wave solutions in the form of $sech$ and $sech^2$.

Keywords Shallow water equations · Local existence · Wave breaking

Mathematics Subject Classification 35Q35 · 35B40 · 35G25 · 35Q53

1 Introduction

This paper delves into the dynamics of the velocity field evolution in a fluid layer, operating under several key assumptions: the fluid is ideal, incompressible, irrotational (zero vorticity), and subject solely to the influence of gravity [10, 18].

Employing an asymptotic regime, wherein we seek approximate models and solutions, is a conventional strategy for simplifying complex problems. Johnson's formal asymptotic procedures [9] have notably yielded effective approximations to governing water wave equations, especially concerning specific geophysical parameters. Constantin and Lannes [2] subsequently substantiated the relevance of main asymptotical models for shallow water-wave propagation, extending to more nonlinear generaliza-

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tions such as the KdV equations linked to the Camassa-Holm equation [4] and the Degasperis–Procesi equations [5].

The connection between the typical wavelength λ and wave amplitude a with water depth h_0 is encapsulated by the shallowness (dispersion) parameter $\delta = h_0/\lambda$ and the nonlinearity (amplitude) parameter $\varepsilon = a/h_0$, which naturally emerge during the non-dimensionalization process.

Our focus lies in the "shallow water regime for waves of large amplitude" under the scaling

$$\delta \ll 1, \qquad \varepsilon \sim O(\sqrt{\delta}).$$

Building on Johnson's methodology [9, 10], Quirchmayr in [20] derived a highly non-linear one-dimensional equation describing the unknown horizontal velocity $u(t, x)$ of surface waves propagating in one direction at any position $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. This equation is expressed as:

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x - \frac{1}{18}\delta^2(4u_{xxx} + 7u_{xxt}) \\ -\frac{1}{6}\varepsilon\delta^2(uu_{xxx} + 2u_x u_{xx}) + \frac{1}{96}\varepsilon^2\delta^2(398uu_x u_{xx} + 45u^2 u_{xxx} + 154u_x^3) = 0. \tag{1.1}$$

Unlike the Camassa-Holm regime $[\delta^2 \ll 1, \varepsilon \sim O(\delta^{1/4})]$, here the nonlinear effects are more pronounced. Notably, the right-hand side of (1.1) is of order $O(\varepsilon^3\delta^2, \delta^3)$, and the dependence on $\varepsilon^2\delta^2$ is evident on the left-hand side.

Previous works have laid a solid theoretical foundation for this equation. Its local well-posedness has been extensively discussed; in [22], the local well-posedness of the corresponding Cauchy problem with initial data in $H^s(\mathbb{R})$ for $s > 3/2$ was established using Kato's theory. In [7, 24], the authors enhanced the local existence of solutions to (1.1) within the Besov space setting $B^2_{p,q}(\mathbb{R})$ where $p, q \in [0, +\infty]$, and $s > \max(3/2, (p+1)/p$, also providing some blow-up criteria. Recently, the well-posedness for space-periodic solutions is established in $H^s(\mathbb{R}/\mathbb{Z})$ for $s > 3/2$ in [19]. On the other side, Geyer and Quirchmayr in [8] classified all (weak) traveling wave solutions of (1.1) in $H^1_{loc}(\mathbb{R})$, while in [12], it has been proven that all symmetric wave solutions are traveling waves.

1.1 Statement of the results

For the remainder of this paper, we will represent μ as δ^2. In Sect. 2, we mainly tackle the Cauchy problem associated to the shallow water asymptotic scalar equation (1.1)

$$\begin{cases} u_t + u_x + \frac{3}{2}\varepsilon u u_x - \frac{1}{18}\mu(4u_{xxx} + 7u_{xxt}) - \frac{1}{6}\varepsilon\mu(uu_{xxx} + 2u_x u_{xx}) \\ \qquad\qquad + \frac{1}{96}\varepsilon^2\mu(398uu_x u_{xx} + 45u^2 u_{xxx} + 154u_x^3) = 0, \\ (t, x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u_{|t=0} = u_0 = u(0, x) \qquad x \in \mathbb{R}. \end{cases}$$

This section is dedicated to the linear analysis of our problem, followed by a thorough construction of solutions to the nonlinear system. We employ the Picard iteration method, culminating in the long-time existence result within $X^s \simeq H^{s+1}(\mathbb{R})$ (refer to Definition 1) and ε^{-1} time scale, asserting this result as long as $s > 3/2$.

Theorem 1 *[Local existence] Fix $s > \frac{3}{2}$ and an initial data $u_0 \in X^s$. Then, there exists $T = T(|u_0|_{X^s}) > 0$ and a unique solution to (1.1) bounded in $C([0, \frac{T}{\varepsilon}]; X^s(\mathbb{R})) \cap C^1([0, \frac{T}{\varepsilon}]; X^{s-1}(\mathbb{R}))$ such that for any $0 \leq \varepsilon t \leq T$, the following solution size estimates holds*

$$|u(t, \cdot)|_{X^s} \leq C_0^{HN}(|u_0|_{X^s}), \qquad \textit{and} \qquad |\partial_t u(t, \cdot)|_{X^{s-1}} \lesssim \varepsilon |u_0|_{X^s}. \quad (1.2)$$

where $C_0 = C_0^{HN}(|u_0|_{X^s}) > 0$ is a constant depending only on $|u_0|_{X^s}$.

In Sect. 3, addressing the solution of (1.1), it is established that the threshold time for the onset of wave breaking in the surging type (where the slope grows to $+\infty$) is greater than or equal to an instant of order ε^{-1}. Notably, plunging breakers (where the slope decays to $-\infty$) are not observed. In Sect. 4, following ODE theory, it is deduced that there are no exact solitary wave solutions in the form of *sech* and $sech^2$ for the given equation.

2 Long-term well-posedness of (1.1)

In order to motivate the introduction of the energy norm (see Definition 1), it is instructive to look at the linearized system around a reference state $\underline{u}$:

$$\begin{cases} \mathcal{L}(\underline{u}, \partial)u = 0, \\ u_{|t=0} = u^0. \end{cases} \qquad (2.1)$$

Consequently, this requires to write the Eq. (1.1) in a quasi-linear form such that

$$\mathcal{L}(u, \partial)u = 0, \qquad (2.2)$$

where

$$\mathcal{L}(v, \partial)\cdot = \Big(1 - \frac{7}{18}\mu\partial_x^2\Big)\partial_t \cdot + \partial_x \cdot + \frac{3}{2}\varepsilon v \partial_x \cdot - \frac{2}{9}\mu\partial_x^3 \cdot$$

$$- \varepsilon\mu\Big(\frac{1}{6}v\partial_x^3 + \frac{1}{3}v_x\partial_x^2\Big)\cdot + \varepsilon^2\mu\Big(\frac{199}{48}vv_x\partial_x^2 + \frac{15}{32}v^2\partial_x^3 + \frac{77}{48}v_x^2\partial_x\Big)\cdot$$

In a normal manner, for any $s \geq 0$, we define the energy of the linearized system (2.1) as:

$$E^s(u)^2 = \Big(\Lambda^s u, \Big(1 - \frac{7}{18}\mu\partial_x^2\Big)\Lambda^s u\Big) : = |u|_{H^s}^2 + \frac{7}{18}\mu|u_x|_{H^s}^2. \tag{2.3}$$

Here and throughout the rest of the paper, $H^s(\mathbb{R})$, for any real constant s, is the Sobolev space of all tempered distributions f with the norm $|f|_{H^s(\mathbb{R})} = |\Lambda^s f|_{L^2(\mathbb{R})} < \infty$, where Λ is the pseudo-differential operator $\Lambda^s = (1 - \partial_x)^{s/2}$.

Now, under usual conditions, it is therefore commonly to define the energy space to our problem as follows.

Definition 1 [Energy space] For all $s \geq 0$, we denote by $X^s = H^{s+1}(\mathbb{R})$ endowed with the norm:

$$|u|_{X^s}^2 := |u|_{H^s}^2 + \mu|u_x|_{H^s}^2. \tag{2.4}$$

Remark 1 [Control of E^s energy by X^s-norm] Equivalence across the above energy definitions between $|\cdot|_{X^s}$ and $E^s(u)$ stems directly such that

$$E^s(u) \leq |u|_{X^s} \leq \frac{3\sqrt{14}}{7}E^s(u). \tag{2.5}$$

2.1 Linear analysis

This section targets mainly the mathematical analysis of the linearized system (2.1). As a conclusion, in subsection 2.2 we deduce the proof of our main result (Theorem 1), that is the well-posedness of (2.2) on time scales of order ε^{-1}.

Proposition 1 *Fix any $s > \frac{3}{2}$ and assume that $\underline{u} \in X^s(\mathbb{R})$. Then for any initial data $u_0 \in X^s$ there exists $T > 0$ and a unique solution u to (2.1) bounded in* $C([0, \frac{T}{\varepsilon}]; X^s(\mathbb{R})) \cap C^1([0, \frac{T}{\varepsilon}]; X^{s-1}(\mathbb{R}))$ *such that for all* $0 \leq \varepsilon t \leq T$, *it holds*

$$|u(t)|_{X^s} \lesssim |u_0|_{X^s} \qquad \textit{and} \qquad |u_t(t)|_{X^{s-1}} \lesssim \varepsilon C\big(|\underline{u}|_{X^s}, |u|_{X^s}\big). \tag{2.6}$$

Proof By regularizing the operator $\mathcal{L}$ with a sequence of *Friedriches mollifiers* defined by $J_\delta = (1-\delta\partial_x^2)^{-1/2}$ with $(\delta > 0)$, the constructed unique solution for the linearized problem (2.1) exists by a Cauchy-Lipschtiz theorem (see for instance [21]). Indeed, once the X^s energy estimate (2.6) is in hands, the Cauchy-Lipschitz technique requires only similar estimate. For this reason, we do not give details on the implementation strategy as the bulk of the work is to derive a prior energy estimate. We shall focus however on proving the key step (2.6).

Let us consider any $\lambda \in \mathbb{R}$ to be fixed later. Differentiating the component $\frac{1}{2}e^{-\varepsilon\lambda t}E^s(u)$ with respect to time, one gets, using (2.3), and the equation (2.1),

$$\begin{aligned}\frac{1}{2}e^{\varepsilon\lambda t}\partial_t\big(e^{-\varepsilon\lambda t}E^s(u)^2\big) &= -\frac{\varepsilon\lambda}{2}E^s(u)^2 + \left(\Lambda^s\left(1-\frac{7}{18}\mu\partial_x^2\right)u_t, \Lambda^s u\right)\\ &= -\frac{\varepsilon\lambda}{2}E^s(u)^2 + \frac{3}{2}\varepsilon\big(\Lambda^s(\underline{u}\partial_x u), \Lambda^s u\big) + \frac{1}{6}\varepsilon\mu\big(\Lambda^s(\underline{u}\partial_x^3 u), \Lambda^s u\big)\\ &\quad + \frac{1}{3}\varepsilon\mu\big(\Lambda^s(\underline{u_x}\partial_x^2 u), \Lambda^s u\big) - \frac{199}{48}\varepsilon^2\mu\big(\Lambda^s(\underline{uu_x}\partial_x^2 u), \Lambda^s u\big)\\ &\quad - \frac{15}{32}\varepsilon^2\mu\big(\Lambda^s(\underline{u}^2\partial_x^3 u), \Lambda^s u\big) - \frac{77}{48}\varepsilon^2\mu\big(\Lambda^s(\underline{u}_x^2\partial_x u), \Lambda^s u\big)\ ,\end{aligned}$$

where we used the fact that ∂_x and ∂_t commutes with Λ^s and by integrating by parts that $\big(\Lambda^s(-u_x + \frac{2}{9}\mu u_{xxx}), \Lambda^s u\big) = 0$.

Now, since for all skew-symmetric differential polynomial P (that is, $P^* = -P$), and all h smooth enough, one has

$$(\Lambda^s(hPu), \Lambda^s u) = ([\Lambda^s, h]Pu, \Lambda^s u) - \frac{1}{2}([P, h]\Lambda^s u, \Lambda^s u),$$

we deduce (applying this identity with $P = \partial_x$, $P = \partial_x^2$ and $P = \partial_x^3$),

$$\begin{aligned}\frac{1}{2}e^{\varepsilon\lambda t}\partial_t\big(e^{-\varepsilon\lambda t}E^s(u)^2\big) &= -\frac{\varepsilon\lambda}{2}E^s(u)^2 + \frac{3}{2}\varepsilon\big([\Lambda^s, \underline{u}]u_x, \Lambda^s u\big)\\ &- \frac{3}{4}\varepsilon\mu\big([\partial_x, \underline{u}]\Lambda^s u, \Lambda^s u\big) + \frac{1}{6}\varepsilon\mu\big([\Lambda^s, \underline{u}]u_{xxx}, \Lambda^s u\big)\\ &- \frac{1}{12}\varepsilon\mu\big([\partial_x^3, \underline{u}]\Lambda^s u, \Lambda^s u\big) + \frac{1}{3}\varepsilon\mu\big([\Lambda^s, \underline{u_x}]u_{xx}, \Lambda^s u\big) - \frac{1}{6}\varepsilon\mu\big([\partial_x^2, \underline{u}_x]\Lambda^s u, \Lambda^s u\big)\\ &- \frac{199}{48}\varepsilon^2\mu\big([\Lambda^s, \underline{uu}_x]u_{xx}, \Lambda^s u\big) + \frac{199}{96}\varepsilon^2\mu\big([\partial_x^2, \underline{uu}_x]\Lambda^s u, \Lambda^s u\big)\\ &- \frac{15}{32}\varepsilon^2\mu\big([\Lambda^s, \underline{u}^2]u_{xxx}, \Lambda^s u\big) + \frac{15}{64}\varepsilon^2\mu\big([\partial_x^3, \underline{u}^2]\Lambda^s u, \Lambda^s u\big)\\ &- \frac{77}{48}\varepsilon^2\mu\big([\Lambda^s, \underline{u}_x^2]u_x, \Lambda^s u\big) + \frac{77}{96}\varepsilon^2\mu\big([\partial_x, \underline{u}_x^2]\Lambda^s u, \Lambda^s u\big)\\ &= -\frac{\varepsilon\lambda}{2}E^s(u)^2 + A_1 + A_2 + \cdots + A_{12}\ . \end{aligned} \tag{2.7}$$

The key point is to bound from above each term at the right-hand side of the above equation in terms of $E^s(U)^2$ and $E^s(\underline{U})$, then for a specific choice of λ and by Grönwall's inequality the prior energy estimate (2.6) follows.

In the sequel we shall use intensively the estimates introduced by Kato-Ponce [11] and recently improved by Lannes [17], in particular, for any $s > 3/2$, and $f \in H^s(\mathbb{R})$, $g \in H^{s-1}(\mathbb{R})$, the commutator estimate below holds

$$\big|[\Lambda^s, f]g\big|_2 \lesssim |\partial_x f|_{H^{s-1}}|g|_{H^{s-1}}. \tag{2.8}$$

Moreover, for any $f, g \in H^s(\mathbb{R})$, $s > 3/2$, the classical product estimate (see [1, 11, 17]) below holds

$$|fg|_{H^s} \lesssim |f|_{H^s}|g|_{H^s}. \tag{2.9}$$

Also, we shall use the continuous embedding $H^s(\mathbb{R}) \subset W^{1,\infty}(\mathbb{R})$ for $s > 3/2$.

$\underline{Estimation of\, A_3 + A_9}$: Remark that for all m, n smooth enough, the following commutator identity holds

$$[\Lambda^s, m]\partial_x n = \partial_x[\Lambda^s, m]n - [\Lambda^s, \partial_x m]n.$$

Therefore, using (2.8), (2.9), Cauchy–Schwartz inequality and by integration by parts, it holds that

$$\begin{aligned} A_3 + A_9 &= -\frac{1}{6}\varepsilon\mu\big([\Lambda^s, \underline{u}]u_{xx}, \Lambda^s u_x\big) - \frac{1}{6}\varepsilon\mu\big([\Lambda^s, \underline{u}_x]u_{xx}, \Lambda^s u\big) \\ &\quad + \frac{15}{32}\varepsilon^2\mu\big([\Lambda^s, \underline{u}^2]u_{xx}, \Lambda^s u_x\big) + \frac{15}{16}\varepsilon^2\mu\big([\Lambda^s, \underline{u}\underline{u}_x]u_{xx}, \Lambda^s u\big) \\ &\lesssim \varepsilon C\big(E^s(\underline{u})\big)E^s(u)^2 \ . \end{aligned} \tag{2.10}$$

$\underline{Estimation of\, A_5 + A_7 + A_{11}}$: Directly by the Cauchy-Schwartz inequality and estimates (2.8), (2.9), it holds that

$$A_5 + A_7 + A_{11} \lesssim \varepsilon C\big(E^s(\underline{u})\big)E^s(u)^2. \tag{2.11}$$

$\underline{Estimation of\, A_4 + A_6 + A_{12}}$: By definition we have $[\partial_x^{i=1,2,3}, f]g = g\partial_x^{i=1,2,3}(f)$. Therefore, directly by the Cauchy-Schwartz inequality and integration by parts, it holds that

$$\begin{aligned} A_4 + A_6 + A_{12} &= \frac{1}{2}\varepsilon\mu\big(\underline{u}_{xx}\Lambda^s u, \Lambda^s u_x\big) - \frac{77}{48}\varepsilon^2\mu\big(\underline{u}_x^2\Lambda^s u, \Lambda^s u_x\big) \\ &\lesssim \varepsilon C\big(E^s(\underline{u})\big)E^s(u)^2. \end{aligned} \tag{2.12}$$

$\underline{Estimation of\, A_8}$: By definition we have $[\partial_x^2, fg]h = h\partial_x^2(fg) + 2h_x\partial_x(fg)$. Therefore, by integration by parts one may deduce that $A_8 = 0$.

$\underline{Estimation of\, A_{10}}$: By definition we have $[\partial_x^3, fg]h = h\partial_x^3(fg) + 3h_x\partial_x^2(fg) + 3h_{xx}\partial_x(fg)$. Therefore, directly by the Cauchy-Schwartz inequality, (2.9) and by integration by parts, it holds that

$$\begin{aligned} A_{10} &= -\frac{15}{16}\varepsilon^2\mu\big(\partial_x(\underline{u}\underline{u}_x)\Lambda^s u, \Lambda^s u_x\big) - \frac{45}{64}\varepsilon^2\mu\big(\partial_x(\underline{u}^2)\Lambda^s u_x, \Lambda^s u_x\big) \\ &\lesssim \varepsilon^2 C\big(E^s(\underline{u})\big)E^s(u)^2. \end{aligned} \tag{2.13}$$

Similarly as above, we have $A_1 + A_2 \lesssim \varepsilon C\big(E^s(\underline{u})\big)E^s(u)^2$. Gathering the estimates (2.10)–(2.13) in (2.7), we get

$$\frac{1}{2}e^{\varepsilon\lambda t}\partial_t\big(e^{-\varepsilon\lambda t}E^s(u)^2\big) \lesssim \varepsilon C\big(E^s(\underline{u}) - \frac{\lambda}{2}\big)E^s(u)^2.$$

Now, for all $0 \leq \varepsilon t \leq T$, we take $\tilde{\lambda} = \lambda \geq 2C(E^s(\underline{u})) > 0$ so that the differential inequality below holds:

$$\frac{d}{dt}E^s(u) \lesssim \frac{1}{2}\tilde{\lambda}\varepsilon E^s(u).$$

As a result, by Gronwall's inequality the desired energy estimate (2.6) holds.

Now to establish the other derivative energy estimate of (2.6), we use the linearized system (2.1). Indeed, by definition (2.3) and proceeding as for the estimate in (2.7), it holds that

$$\begin{aligned}
E^{s-1}(u_t)^2 &= \left(\Lambda^{s-1}u_t, \left(1 - \frac{7}{18}\mu\partial_x^2\right)\Lambda^{s-1}u_t\right) \\
&= \frac{1}{6}\varepsilon\mu\big([\Lambda^{s-1}, \underline{u}]u_{xxx}, \Lambda^{s-1}u_t\big) - \frac{1}{12}\varepsilon\mu\big([\partial_x^3, \underline{u}]\Lambda^{s-1}u, \Lambda^{s-1}u_t\big) \\
&+ \frac{1}{3}\varepsilon\mu\big([\Lambda^{s-1}, \underline{u_x}]u_{xx}, \Lambda^{s-1}u_t\big) - \frac{1}{6}\varepsilon\mu\big([\partial_x^2, \underline{u}_x]\Lambda^{s-1}u, \Lambda^{s-1}u_t\big) \\
&- \frac{199}{48}\varepsilon^2\mu\big([\Lambda^{s-1}, \underline{uu}_x]u_{xx}, \Lambda^{s-1}u_t\big) + \frac{199}{96}\varepsilon^2\mu\big([\partial_x^2, \underline{uu}_x]\Lambda^{s-1}u, \Lambda^{s-1}u_t\big) \\
&- \frac{15}{32}\varepsilon^2\mu\big([\Lambda^{s-1}, \underline{u}^2]u_{xxx}, \Lambda^{s-1}u_t\big) + \frac{15}{64}\varepsilon^2\mu\big([\partial_x^3, \underline{u}^2]\Lambda^{s-1}u, \Lambda^{s-1}u_t\big) \\
&- \frac{77}{48}\varepsilon^2\mu\big([\Lambda^{s-1}, \underline{u}_x^2]u_x, \Lambda^{s-1}u_t\big) + \frac{77}{96}\varepsilon^2\mu\big([\partial_x, \underline{u}_x^2]\Lambda^{s-1}u, \Lambda^{s-1}u_t\big) \\
&\lesssim \varepsilon C\big(|\underline{u}|_{X^s}, |u|_{X^s}\big)E^{s-1}(u_t)\ .
\end{aligned}$$

Hence the desired estimate. □

2.2 Proof of Theorem 1

The proof here is a standard argument used for hyperbolic systems (see Chapter III B.1 of [1] for the general case). For sake of completeness we will present the main procedure of the proof. We consider first a sequence of nonlinear problems of (2.2) through the induction relation

$$\forall\, n \in \mathbb{N}\ , \qquad \begin{cases} \mathcal{L}\big(u^n, \partial\big)u^{n+1} = 0 \\ u^{n+1}_{|_{t=0}} = u^0 \qquad \text{with} \qquad u^0 = u_0. \end{cases} \tag{2.14}$$

Once Proposition 1 in hands and in combination with additional standard arguments, the convergence of solution $\big(u^n\big)_{n\geq 0}$ is established. We recall that such an iterative scheme is applicable on various models/equations in one or two space dimension once the linear analysis is accomplished (see for instance [13, 15, 16]).

2.2.1 Bounding $\left(u^n\right)_{n\geq 0}$ in $X^{s>3/2}$

Combining the assumption of Theorem 1 with Proposition 1 applied to system (2.14), one may deduce by induction on n that $u^{n+1} \in C([0, \frac{T}{\varepsilon}]; X^s) \cap C^1([0, \frac{T}{\varepsilon}]; X^{s-1})$ such that for any $0 \leq \varepsilon t \leq T$, it holds

$$|u^{n+1}(t)|_{X^s} \lesssim |u_0|_{X^s} \qquad \text{and} \qquad |\partial_t u^{n+1}(t)|_{X^{s-1}} \lesssim \varepsilon |u_0|_{X^s}. \tag{2.15}$$

2.2.2 Convergence of $\left(u^n\right)_{n\geq 0}$ in larger space X^0

Denote by $u^n = u_0 + \sum_{i=0}^{n-1} v^i$ where $\left(v^n\right)_{n\geq 0}$ is the subtraction of two consecutive approximate solutions $u^{n+1} - u^n = v^n$. Eventually, $\left(v^n\right)_{n\geq 0}$ satisfies the system below

$$\forall\, n \in \mathbb{N}\,, \qquad \begin{cases} \mathcal{L}\big(u^n, \partial\big) v^n = -\Big(\mathcal{R}\big(u^n, \partial\big) - \mathcal{R}\big(u^{n-1}, \partial\big)\Big) u^n, \\ (v^n)_{|t=0} = 0, \end{cases} \tag{2.16}$$

with $\mathcal{R}\big(u^i, \partial\big) = \mathcal{L}\big(u^i, \partial\big) - (1 - \frac{7}{18}\mu\partial_x^2) - \partial_x + \frac{2}{9}\mu\partial_x^3$. We refer to appendix A for the control of sequence $\left(v^n\right)_{n\geq 0}$ in the small norm X^0. As a result, for any $0 \leq \varepsilon t \leq T$, it holds that

$$|v^n(t)|_{X^0} \lesssim \frac{|u_0|_{X^s}\big(\varepsilon |u_0|_{X^s} e^{|u_0|_{X^s} T}\big)^n}{n!}.$$

Hence, with the help of the ratio test for convergence of a series, one may deduce that sequence $u^n = u_0 + \sum_{i=0}^{n-1} v^i$ converges in $C([0, \frac{T}{\varepsilon}]; X^0)$ to u.

2.2.3 End of the proof

Section 2.2.1 implies that there exists a weakly convergent subsequence $\left(u^{n_k}\right)_{n_k\geq 0}$ in $C([0, \frac{T}{\varepsilon}]; X^s)$ to $\tilde{u} \in X^s$. Subsection 2.2.2 and since the limit in the sense of distribution is unique, then $\tilde{u} = u \in C([0, \frac{T}{\varepsilon}]; X^s)$ with $s > 3/2$. Therefore, u^{n_k} converges in $C([0, \frac{T}{\varepsilon}]; X^s)$ to u and as a result the limit u of the iterative scheme (2.14) is a unique solution of (2.2) that satisfies the energy estimates (1.2).

3 Wave breaking

The well-posedness result of Theorem 1 asserts that there is a time $T_* < +\infty$ at which some norm of the solution of (1.1) becomes unbounded, the latter phenomenon is known as (finite-time) blow-up. Consequently, a fundamental question in the theory of nonlinear partial differential equations is when and how a singularity can form.

When the solution itself becomes unbounded in a finite time, a simple type of singularity occurs. On the other side, in models of water waves, wave breaking occurs when the solution (representing the wave) remains bounded but its slope becomes infinite as the blow-up time approaches. Eventually, the following definition states when the wave is said to be broken:

Definition 2 [Wave breaking] We say that there is wave breaking for the Eq. (1.1), if there exists a time $0 < T_* < +\infty$ and solutions $u(t, x)$ to (1.1) such that

$$u \in L^\infty([0, T_*] \times \mathbb{R}) \qquad \text{and} \qquad \lim_{t \to T_*} \|\partial_x u(t, \cdot)\|_{L^\infty(\mathbb{R})} = +\infty.$$

3.1 Wave breaking criterion

Our first result describes the precise blow-up pattern for the Eq. (1.1).

Proposition 2 *Fix any $\varepsilon, \mu \in (0, 1)$ and consider any initial data $u_0 = u(0, \cdot) \in H^{s+1}(\mathbb{R})$ with $s > 3/2$. If the maximal existence time $T_{\max} > 0$ of the solution of (1.1) is finite, $T_{\max} < +\infty$, then the corresponding solution $u \in C([0, T_{\max}); H^{s+1}(\mathbb{R})) \cap C^1([0, T_{\max}); H^{s-1}(\mathbb{R}))$ blows up in finite time if and only if*

$$\lim_{t \to T_{\max}} \|\partial_x u(t, \cdot)\|_{L^\infty(\mathbb{R})} = +\infty. \tag{3.1}$$

Proof Applying Theorem 1 and a simple density argument, we only need to show that the theorem holds for some $s \geq 2$. Here, we assume $s = 2$ to prove the above proposition. In view of Theorem 1, given $u_0 \in H^3(\mathbb{R})$, the maximal existence time of the associated solution $u(t)$ is finite if and only if $u(t)$ H^3-blows up in finite time. Thus if (3.1) holds for some time $T > 0$, then the maximal time is finite. We omit the rest of the proof as it follows same lines as the proof of Theorem 1.2 in reference [24]. □

3.2 Wave breaking data

Our subsequent objective is to establish that, even with the introduction of heightened nonlinearity effects in our equation (1.1), wave breaking of the surging type (i.e., where the slope grows to $+\infty$) transpires within a finite time, greater than or equal to an instant of order ε^{-1}. In contrast, plunging breakers (i.e., where the slope decays to $-\infty$) are not observed.

We will prove this by analyzing the equation that describes the evolution of the slope wave (the differentiation of (1.1) with respect to the spacial variable)

$$\begin{aligned}&(1 - \frac{7}{18}\mu\partial_x^2)u_{tx} + u_{xx} + \frac{3}{4}\varepsilon\partial_x^2(u^2) - \frac{2}{9}\mu u_{xxxx} - \frac{1}{6}\varepsilon\mu\partial_x^2(uu_{xx})\\ &-\frac{1}{12}\varepsilon\mu\partial_x^2(u_x^2) + \frac{1}{96}\varepsilon^2\mu\partial_x\big(398uu_xu_{xx} + 45u^2u_{xxx} + 154u_x^3\big) = 0.\end{aligned} \tag{3.2}$$

For further use, set by $p(x) = \dfrac{3}{\sqrt{14\mu}} \exp\Big(-\dfrac{6}{\sqrt{14\mu}}|x|\Big)$ the convolution kernel function whose Fourier transform reads $\widehat{p}(\omega) = (1 + \frac{7}{18}\mu\omega^2)^{-1}$. Denoting by $*$ the convolution with respect to the spatial variable $x \in \mathbb{R}$, we have $(1 - \frac{7}{18}\mu\partial_x^2)^{-1} f = p * f$ and $p * (f - \frac{7}{18}\mu f_{xx}) = f$ for all $f \in L^2(\mathbb{R})$. Moreover, it is not hard to check that

$$\|p\|_{L^\infty(\mathbb{R})} = \sqrt{\frac{3}{14\mu}}, \qquad \|p\|_{L^1(\mathbb{R})} = 1, \qquad \|p\|_{L^2(\mathbb{R})} = \sqrt{\frac{3}{28}\sqrt{\frac{14}{\mu}}} \le \frac{16}{25}\mu^{-1/4}, \tag{3.3}$$

and

$$\begin{aligned} \|p_x\|_{L^\infty(\mathbb{R})} &= \frac{9}{7\mu}, \qquad \|p_x\|_{L^1(\mathbb{R})} = \frac{3}{7}\sqrt{\frac{14}{\mu}}, \qquad \|p_x\|_{L^2(\mathbb{R})} = \sqrt{\frac{27\sqrt{14}}{98}}\mu^{-3/4} \\ &\le \frac{51}{50}\mu^{-3/4}. \end{aligned} \tag{3.4}$$

Applying the operator $(1 - \frac{7}{18}\mu\,\partial_x^2)^{-1}$ to the time evolution slope wave equation (3.2) and using the identity

$$\partial_x^2\, p * f = p_x * \zeta_x = \frac{18}{7\mu}\, p * f - \frac{18}{7\mu} f, \qquad f \in L^2(\mathbb{R}), \tag{3.5}$$

one may write equation (3.2) in the weak non-local form for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ as

$$\begin{aligned} & u_{tx} + \frac{3}{7} p_x * u_x + \frac{3}{2}\varepsilon p_x * u u_x + \frac{4}{7} u_{xx} - \frac{3}{7}\varepsilon p * u u_{xx} \\ & = -\frac{3}{7}\varepsilon u u_{xx} + \frac{3}{14}\varepsilon p * u_x^2 - \frac{3}{14}\varepsilon u_x^2 - \frac{199}{48}\varepsilon^2\mu p_x * u u_x u_{xx} - \frac{15}{32}\varepsilon^2\mu p_x * u^2 u_{xxx} \\ & \quad - \frac{77}{48}\varepsilon^2\mu p_x * u_x^3\,. \end{aligned} \tag{3.6}$$

It is also found that the solution of (1.1) cannot break up to the maximal time of existence of solution.

Proposition 3 *Let the assumption of Proposition 2 be satisfied. Then for almost everywhere on* $[0, T_{\max})$*, it holds that*

$$\inf_{\mathbb{R}} u_x(t,\cdot) \ge -\|\partial_x u_0\|_{L^\infty(\mathbb{R})} - \frac{1}{3\sqrt{\varepsilon}}\sqrt{42K_0}, \tag{3.7}$$

where

$$K_0 = \frac{7}{4}\mu^{-7/4} C_0 + \frac{49}{2}\varepsilon\mu^{-11/4} C_0^2 + \frac{22}{5}\varepsilon\mu^{-9/4} C_0^2 + \frac{8}{5}\varepsilon\mu^{-5/2} C_0^2$$

$$+271\varepsilon^2\mu^{-11/4}C_0^3 + 31\varepsilon^2\mu^{-11/4}C_0^3 + 132\varepsilon^2\mu^{-3}C_0^3,$$

with $C_0 = C_0^{NH}(\|u_0\|_{H^3(\mathbb{R})}) > 0$ *is the constant that appears in Theorem 1 depending only on* $\|u_0\|_{H^3(\mathbb{R})}$.

Proof First, remark that combining the energy size estimate (1.2), we shall use intensively the below estimate

$$\|u\|_{H^3(\mathbb{R})} \le 4\mu^{-1}C_0,$$

where $C_0 = C_0^{NH}(\|u_0\|_{H^3(\mathbb{R})}) > 0$ is the constant that appears in Theorem 1 depending only on $\|u_0\|_{H^3(\mathbb{R})}$. Therefore, using Young's inequality, the imbedding $L^\infty(\mathbb{R}) \subset H^1(\mathbb{R})$, and the estimates (3.3)-(3.4), we obtain that

$$\begin{aligned}
\frac{3}{7}\|p_x * u_x\|_{L^\infty(\mathbb{R})} &\le \|p_x\|_{L^2}\|u_x\|_{L^2} \le \frac{7}{4}\mu^{-7/4}C_0,\\
\frac{3}{2}\varepsilon\|p_x * uu_x\|_{L^\infty(\mathbb{R})} &\le \frac{3}{2}\varepsilon\|p_x\|_{L^2}\|u\|_{L^\infty}\|u_x\|_{L^2} \le \frac{49}{2}\varepsilon\mu^{-11/4}C_0^2,\\
\frac{3}{7}\varepsilon\|p * uu_{xx}\|_{L^\infty(\mathbb{R})} &\le \frac{3}{7}\varepsilon\|p\|_{L^2}\|u\|_{L^\infty}\|u_{xx}\|_{L^2} \le \frac{22}{5}\varepsilon\mu^{-9/4}C_0^2,\\
\frac{3}{14}\varepsilon\|p * u_x^2\|_{L^\infty(\mathbb{R})} &\le \frac{3}{14}\varepsilon\|p\|_{L^\infty}\|u_x^2\|_{L^1} \le \frac{3}{14}\varepsilon\|p\|_{L^\infty}\|u_x\|_{L^2}^2 \le \frac{8}{5}\varepsilon\mu^{-5/2}C_0^2,\\
\frac{199}{48}\varepsilon^2\mu\|p_x * uu_xu_{xx}\|_{L^\infty(\mathbb{R})} &\le \frac{199}{48}\varepsilon^2\mu\|p_x\|_{L^2}\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^2} \le 271\varepsilon^2\mu^{-11/4}C_0^3,\\
\frac{15}{32}\varepsilon^2\mu\|p_x * u^2u_{xxx}\|_{L^\infty(\mathbb{R})} &\le \frac{15}{32}\varepsilon^2\mu\|p_x\|_{L^2}\|u\|_{L^\infty}^2\|u_{xxx}\|_{L^2} \le 31\varepsilon^2\mu^{-11/4}C_0^3,\\
\frac{77}{48}\varepsilon^2\mu\|p_x * u_x^3\|_{L^\infty(\mathbb{R})} &\le \frac{77}{48}\varepsilon^2\mu\|p_x\|_{L^\infty}\|u_x^3\|_{L^1}\\
&\le \frac{77}{48}\varepsilon^2\mu\|p_x\|_{L^\infty}\|u_x\|_{L^\infty}\|u_x\|_{L^2}^2 \le 132\varepsilon^2\mu^{-3}C_0^3,
\end{aligned}$$

At any fixed time $t \in [0, T_{\max})$, (3.6) is an equality in the space of continuous function $C([0, T_{\max}), L^2(\mathbb{R}))$. So let us evaluate both side of equality (3.6) at a point $x = \xi(t) \in \mathbb{R}$, the local minima) of the slope wave. In fact, as $u_x(t, \cdot) \in H^2(\mathbb{R})$ we see that it vanishes at $\pm\infty$ in which the existence of $\xi(t)$ is guarantee. In other words, there exists at least one point $\xi(t)$ such that $n(t)$ is defined as follows

$$n(t) = \inf_{x\in\mathbb{R}} \{u_x(t, x)\} = u_x(t, \xi(t)). \tag{3.8}$$

We may assume that $n(t) < 0$ for all $t \in \mathbb{R}_+$. In fact, when $n(t) \ge 0$ then $u(t, \cdot)$ is non-decreasing function on $\mathbb{R}$ and therefore $u(t, \cdot) = 0$.

For the degree of smoothness of the solution $u(t, \cdot) \in C^1([0, T_{\max}), H^2(\mathbb{R}))$ given by Theorem 1, we know that by the mean value theorem $n(t)$ is locally Lipschitz and therefore Rademacher's theorem [6] implies that $n(\cdot)$ is almost everywhere differentiable on $[0, T_{\max})$ such that

$$\frac{d}{dt}n(t) = u_{tx}(t, \xi(t)) \quad \text{for a.e.} \quad t \in [0, T_{\max}), \tag{3.9}$$

where $\xi(t)$ is any point where $n(t)$ is the minimum of $u_x(t, \xi(t))$ (we refer to Theorem 2.1 of [3] for similar detailed proof). Now, since $u_{xx}(t, \xi(t)) = 0$, from (3.9), (3.6), and the previous estimates we derive, the following differential inequality for the locally Lipschitz function $n(t)$ for almost everywhere on $[0, T_{\max})$

$$|n'(t) + \frac{3}{14}\varepsilon n^2(t)| \leq K_0, \tag{3.10}$$

At this stage, inspired from similar argument in Ref [23], for any $x \in \mathbb{R}$, let us consider the C^1-differential function in $[0, T_{\max})$ defined by

$$g(t) = -n(t) - \|\partial_x u_0\|_{L^\infty(\mathbb{R})} - \frac{1}{3\sqrt{\varepsilon}}\sqrt{42K_0}. \tag{3.11}$$

Clearly, $g(0) < 0$. The proof is completed if $g(t) \leq 0$ almost everywhere on $[0, T_{\max})$. Indeed, suppose that there exists $t_0 \in [0, T_{\max})$ such that $g(t_0) > 0$. Furthermore, let us introduce the time

$$\tilde{t} = \min\{t > t_0 \text{ such that } g(t) = 0\}. \tag{3.12}$$

Clearly, by (3.11)-(3.12) we have that $g(\tilde{t}) = 0$ and $0 \geq g'(\tilde{t}) = -n'(\tilde{t})$. On the other hand, in view of (3.10), it holds that

$$n'(\tilde{t}) \leq -\frac{3}{14}\varepsilon n(\tilde{t})^2 + K_0 = -\frac{3}{14}\varepsilon\left(-\|\partial_x u_0\|_{L^\infty(\mathbb{R})} - \frac{1}{3\sqrt{\varepsilon}}\sqrt{42K_0}\right)^2 + K_0 < 0.$$

Consequently, by contradiction the proof is completed. □

We can also prove the following wave-breaking data.

Proposition 4 *Let the assumption of Proposition 2 be satisfied. Moreover, if the initial wave profile $u_0 \in H^3(\mathbb{R})$ satisfies*

$$\left\|\partial_x u_0\right\|^2_{L^\infty(\mathbb{R})} \geq \frac{7}{2}\varepsilon^{-1}K_0\ , \tag{3.13}$$

where K_0 is the same constant that appears in Proposition 3. Then, for the solution of (1.1), the threshold time to wave breaking is greater than or equal to an instant of order ε^{-1}.

Proof An analogous result clearly yields the existence of at least one point $\underline{\xi}(t)$ such that $m(t)$ is defined as follows

$$m(t) = \sup_{x\in\mathbb{R}}\left\{u_x(t,x)\right\} = u_x(t, \underline{\xi}(t)). \tag{3.14}$$

As in Proposition 3, the following differential inequality for the locally Lipschitz function $m(t)$ for almost everywhere on $[0, T_{\max})$

$$|m'(t) + \frac{3}{14}\varepsilon m^2(t)| \leq K_0. \tag{3.15}$$

We infer that up to the maximal existence time $T_{\max} > 0$ of the solution $u(t)$ of (1.1) the function $m(t)$ must be increasing (i.e. $m(0) < m(t)$), and, moreover using (3.13), we have

$$0 < m'(t) \leq \frac{1}{14}\,\varepsilon\, m^2(t) \quad \text{for a.e. on } [0, T_{\max})$$

Dividing by $m^2(t) \geq m^2(0) > 0$ and integrating on $(0, t)$, we get

$$0 < \frac{1}{m(0)} - \frac{1}{m(t)} \leq \frac{1}{14}\,\varepsilon\, t, \qquad t \in [0, T_{\max})$$

Therefore $\lim_{t\uparrow T_{\max}} m(t) = +\infty$ and $T_{\max} \geq \dfrac{14}{\varepsilon\, m(0)}$. This completes the proof. □

4 Non-existence of exact *sech* and *sech*^{2} solitary wave solutions

The goal of this section is to demonstrate that the model (1.1) does not have any exact *sech* or *sech*^{2} solitary wave solutions based on ideas from [14]. We start with the derivation of the ODE for traveling wave solutions. Let us denote by $\xi = x + x_0 - ct$ with x_0 and c being constants and recall that $\mu = \delta^2$, we seek traveling wave solutions to (1.1) of the form

$$u(t, x) = u(\xi) = u(x + x_0 - ct). \tag{4.1}$$

We assume that $\lim\limits_{|\xi|\to\pm\infty} (u, u', u'') = (0, 0, 0)$ and $c \in \mathbb{R}$ the velocity of the traveling wave. Plugging the above Ansatz into system (1.1), such solutions should satisfy

$$\begin{aligned} -cu' + \frac{7c}{18}\mu u''' + u' + \frac{3}{4}\varepsilon(u^2)' - \frac{2}{9}\mu u''' - \frac{1}{6}\varepsilon\mu(uu'')' \\ -\frac{1}{12}\varepsilon\mu(u'^2)' + \frac{1}{96}\varepsilon^2\mu(45u^2u'' + 154uu'^2)' = 0. \end{aligned} \tag{4.2}$$

Integrating (4.2) in ξ, we therefore get the following second order ODE

$$\begin{aligned} (1 - c)u + \frac{3}{4}\varepsilon u^2 - \frac{1}{12}\varepsilon\mu u'^2 + \frac{77}{48}\varepsilon^2\mu u u'^2 + \left(\frac{7c}{18}\mu - \frac{2}{9}\mu - \frac{1}{6}\varepsilon\mu\right) u'' \\ +\frac{15}{32}\varepsilon^2\mu u^2 u'' = 0. \end{aligned} \tag{4.3}$$

At this stage, by means of the following scaling

$$\xi \to \tau = \mu^{-1/2}\xi, \qquad u \to \varepsilon^{-1}u,$$

the second-order ODE associated to (4.3) in τ reads

$$(1-c)u + \frac{3}{4}u^2 - \frac{1}{12}u'^2 + \frac{77}{48}uu'^2 + \left(\frac{7c}{18} - \frac{2}{9} - \frac{1}{6}\right)u'' + \frac{15}{32}u^2u'' = 0\,, \quad (4.4)$$

where the primes here indicate derivatives with respect to τ and the transformation

$$u(t,x) = \frac{1}{\varepsilon}u\Big(\frac{1}{\delta}(x + x_0 - ct)\Big).$$

As a result, we establish that finding any $sech^2$ solution of (1.1) suffices to find a solution of the ordinary differential equation (4.4). In the below Theorem we show that such solutions does not exist.

Theorem 2 *There is no exact solitary-wave solution u to (1.1) characterized by* $A \in \mathbb{R}^*$ *its maximum amplitude,* $\lambda \in \mathbb{R}^*$ *the wave-spread, and* $c \in \mathbb{R}$ *its phase velocity, under the form*

$$u(\tau) = A\ sech^2(\lambda\tau), \quad (4.5)$$

where $\tau = \delta^{-1}(x + x_0 - ct)$*,* x_0 *an arbitrary constant and* $(t,x) \in \mathbb{R}^2$*. Likewise, no exact solitary-wave solution v to (1.1)*

$$v(\tau) = A\ sech(\lambda\tau). \quad (4.6)$$

Proof Assuming that $u(\tau)$ of (4.5) is a solution of the second-order ordinary differential equation (4.4). By definition, it's not hard to find the following differential identities

$$(u')^2 = \gamma u^2 + \beta u^3 \qquad \text{and} \qquad u'' = \gamma u + \frac{3}{2}\beta u^2,$$

where $\gamma = 4\lambda^2$ and $\beta = -4\lambda^2/A$. Substituting the above identities in (4.4), the left-hand side becomes a bi-quadratic polynomial in u such as

$$\frac{443}{192}\beta u^4 + \left(\frac{199}{96}\gamma - \frac{1}{3}\beta\right)u^3 + \left(\frac{3}{4} - \frac{1}{6}\gamma + \frac{7c}{12}\beta - \frac{1}{3}\beta\right)u^2$$
$$+ \left(1 - c + \frac{7c}{18}\gamma - \frac{2}{9}\gamma\right)u = 0.$$

In this case, for (4.5) to be a nontrivial solution, all the coefficients have to be zero, which is not relevant. Similarly, using the following differential identities

$$(v')^2 = \gamma v^2 + \beta v^4 \qquad \text{and} \qquad v'' = \gamma v + 2\beta v^3,$$

one can deduce that (4.6) cannot be a solution. □

Remark 2 Similar conclusion applies for the less nonlinear version of (1.1), i.e. the equation of order $O(\varepsilon^2\delta^2, \delta^3)$.

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Declarations

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A Control in X^0-norm

By definition (2.3), equation (2.16) and integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial_t}\Big(E^0(v^n)^2\Big) &= \Big((1-\frac{7}{18}\mu\partial_x^2)\partial_t v^n, v^n\Big)\\
&= -\frac{3}{2}\varepsilon(u^n v_x^n, v^n) + \frac{1}{6}\varepsilon\mu\big(u^n v_{xxx}^n, v^n\big) + \frac{1}{3}\varepsilon\mu\big(u_x^n v_{xx}^n, v^n\big)\\
&\quad - \frac{199}{48}\varepsilon^2\mu\big(u^n u_x^n v_{xx}^n, v^n\big)\\
&\quad - \frac{15}{32}\varepsilon^2\mu\big((u^n)^2 v_{xxx}^n, v^n\big) - \frac{77}{48}\varepsilon^2\mu\big((u_x^n)^2 v_x^n, v^n\big)\\
&\quad - \Big(\big(\mathcal{R}(u^n,\partial) - \mathcal{R}(u^{n-1},\partial)\big)u^n, v^n\Big)\\
&= B_1 + B_2 + \cdots + B_7\,. \qquad \text{(A.1)}
\end{aligned}$$

where we recall the explicit expression of $\mathcal{R}$ given by

$$\begin{aligned}
\mathcal{R}\big(u^i,\partial\big) &= \frac{3}{2}\varepsilon u^i\partial_x - \frac{1}{6}\varepsilon\mu u^i\partial_x^3 - \frac{1}{3}\varepsilon\mu u_x^i\partial_x^2\\
&\quad + \frac{199}{48}\varepsilon^2\mu u^i u_x^i\partial_x^2 + \frac{15}{32}\varepsilon^2\mu(u^i)^2\partial_x^3 + \frac{77}{48}\varepsilon^2\mu(u_x^i)^2\partial_x. \qquad \text{(A.2)}
\end{aligned}$$

We focus now on bounding the right-hand side components in terms of the energy norms. For controlling B_1, by integration by parts, Cauchy-Schwartz inequality, the energy estimate (2.15), with $s > 3/2$ and any $0 \le \varepsilon t \le T$, it holds that

$$B_2 = \frac{1}{6}\varepsilon\mu\big(u^n_{xx}v^n_x, v^n\big) + \frac{1}{4}\varepsilon\mu\big(u^n_x v^n_x, v^n_x\big) \lesssim \varepsilon|u^n|_{X^s}|v^n|^2_{X^0} \lesssim \varepsilon|u_0|_{X^s}|v^n|^2_{X^0}.$$

For controlling B_5, by integration by parts, Cauchy-Schwartz inequality, the energy estimate (2.15), with $s > 3/2$ and any $0 \le \varepsilon t \le T$, it holds that

$$\begin{aligned} B_5 = &-\frac{15}{16}\varepsilon^2\mu\big(uu^n_{xx}v^n_x, v^n\big) - \frac{15}{16}\varepsilon^2\mu\big((u^n_x)^2 v^n_x, v^n\big) \\ &-\frac{45}{32}\varepsilon^2\mu\big(uu^n_x v^n_x, v^n_x\big) \lesssim \varepsilon^2|u^n|_{X^s}|v^n|^2_{X^0} \lesssim \varepsilon^2|u_0|_{X^s}|v^n|^2_{X^0}. \end{aligned}$$

Similarly, it holds that $B_1 + B_3 + B_4 + B_6 \lesssim \varepsilon|u_0|_{X^s}|v^n|^2_{X^0}$. Now, using the explicit expression (A.2), we have

$$\begin{aligned} B_7 = &\frac{1}{6}\varepsilon\mu\big(u^n_{xxx}v^{n-1}, v^n\big) + \frac{1}{3}\varepsilon\mu\big(u^n_{xx}v^{n-1}_x, v^n\big) \\ &- \frac{199}{48}\varepsilon^2\mu\big((u^n_x v^{n-1} - u^{n-1}v^{n-1}_x)u^n_{xx}, v^n\big) \\ &- \frac{15}{32}\varepsilon^2\mu\big((u^n v^{n-1} - u^{n-1}v^{n-1})u^n_{xxx}, v^n\big) \\ &- \frac{77}{48}\varepsilon^2\mu\big((u^n_x v^{n-1}_x - u^{n-1}_x v^{n-1}_x)u^n_x, v^n\big) \\ \lesssim &\ \varepsilon|u_0|_{X^s}|v^{n-1}|_{X^0}|v^n|_{X^0}\ , \end{aligned}$$

where we used integration by parts (only where third-order derivatives are applied to one element) combined with (2.15) and Cauchy-Schwartz inequality with $s > 3/2$ and and any $0 \le \varepsilon t \le T$. Gathering the information provided by the above estimates in (A.1) combined with (2.5), it holds that

$$\frac{d}{dt}|v^n|_{X^0} \lesssim \varepsilon|u_0|_{X^s}(|v^n|_{X^0} + |v^{n-1}|_{X^0})$$

Multiplying the above inequality by $exp\big(-\varepsilon|u_0|_{X^s}t\big)$ and integrating on $(0, t)$ for any $0 \le \varepsilon t \le T$, yields

$$\begin{aligned} |v^n(t)|_{X^0} &\lesssim \varepsilon|u_0|_{X^s}\int_0^t e^{\varepsilon|u_0|_{X^s}(t-t_1)}|v^{n-1}(t_1)|_{X^0}\,dt_1 \\ &\lesssim \frac{\big(\varepsilon|u_0|_{X^s}e^{|u_0|_{X^s}T}\big)^n}{n!}\sup_{t_n\in[0,T/\varepsilon]}|u^1(t_n) - u_0(t_n)|_{X^0}. \end{aligned}$$

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