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Who benefits from postponement in multi-period supply channel optimization?☆

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ABSTRACT

Duopolistic price-setting supply channels competing in a bilevel framework have been extensively studied in single-period (static) settings. However, such supply channels typically face uncertain and time-varying demand; and thus require a dynamic analysis. Dynamic channel optimization while addressing uncertain demand has received limited attention due to the highly nested structure of the ensuing equilibrium problems. The level of complexity rises when demand is dependent on current and previous prices. We consider a decentralized (non-cooperative) supply channel whose members, a manufacturer and a retailer, competing in a Stackelberg framework, must address the demand for a perishable commodity within a multi-period discrete-time setting. In the first part of the paper, we propose a constructive theorem providing an explicit solution algorithm to obtain equilibrium states at each period. Next, we prove that the resulting equilibria are *subgame perfect*. In the second part, we allow the retailer (follower) to postpone the supply and pricing decisions until demand uncertainty is resolved in each period. Using subgame perfection of the equilibria, we propose solution algorithms that use the delayed information obtained by the postponement. Our comparative theorems and simulated scenarios indicate that postponement strategies are always beneficial for the follower, and, for a centralized (cooperative) channel. Whereas in a decentralized channel, due to vertical competition, there may be scenarios wherein postponement strategies, i.e., access to extra information, turn out to be detrimental to the manufacturer (leader).

1. Introduction

In the sequential decision-making process known as the Stackelberg game, the equilibrium problem for a competing pair of agents in a duopoly is solved in a bilevel setting. That is, one of the agents in the duopoly is considered the leader (upstream agent) and acts first. The other agent follows accordingly and hence is deemed the follower (downstream agent). Stackelberg games have long been used in modeling decentralized supply channels consisting of two agents, a manufacturer and a retailer addressing uncertain demand. Typically, the manufacturer is considered the leader, and the retailer is the follower (see for example, [1–4]). The objective of each channel member is to maximize their respective profit. The decision variables can be either only the supply quantity (q) or a combination of supply quantity and prices (q, r). In the latter case, the supply channel is referred to as *price-setting*.

The Stackelberg game model has been extensively used in finding equilibrium states in decentralized price-setting supply channels operating in static (single-period) frameworks. However, the time-dependent (multi-period) feedback Stackelberg game in a price-setting channel has received limited attention in the literature due to its complexity and the highly nested structure of the corresponding equilibrium problems. The complexity mainly stems from the nature of the state in a price-setting supply channel optimization problem — the uncertain demand. In a *static* (single-period) supply channel optimization problem modeled as a Stackelberg game, the inverse effect of price on demand can be easily embedded within the objective functions. In contrast, in a multi-period (dynamic) setting, in addition to seeking a Nash equilibrium between themselves, the price-setting decision-makers must keep the balance between maximizing profit in the current period and not stifling future demand (i.e., the yet-unobserved state). The resulting Nash–Stackelberg equilibria, as we will see, become highly nested in time. Moreover, addressing an uncertain demand, the downstream agent faces a *newsvendor problem*; thus, adding another layer of complexity to the equilibrium problem [5,6].

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In our model, a decentralized supply channel is to address the stochastic demand for a perishable commodity within a multi-period time horizon. The channel is composed of two price-setting suppliers, the manufacturer (leader) and the retailer (follower), competing in a Stackelberg framework. The manufacturer sets the wholesale price (w) and receives a supply quantity order from the retailer. The manufacturer is bound to address the retailer's order entirely. The retailer faces the stochastic demand signal. Thus, she has to solve a time-dependent price-setting newsvendor problem. In the retailer's optimization problem, the decision variables are the retail price (r), and the supply quantity (q).

This paper has two main sections. In the first (Sections 3 and 4), both agents are risk-neutral, aiming to maximize their *expected* revenue based on a known demand distribution, a conventional approach cited in several studies (e.g., [2,7,8]). What sets our work apart is the multi-period context with potential time-variance in variables and parameters. Plus, our results apply to almost all continuous demand distributions.[1] The results, outlined in Theorem 4.1 and related corollaries, offer explicit multi-period feedback Stackelberg solutions. These are termed *open-loop* solutions, as they do not account for demand uncertainty feedback (no operational delay permitted).

The paper's second part (Sections 5 and 6) focuses on postponed feedback Stackelberg games in dynamic contexts. Drawing from [9,10], and [11], we allow the retailer to receive a feedback signal from uncertain demand, permitting the delay of decision variables (q or r) until demand uncertainty clarifies. These outcomes are labeled as closed-loop (postponement) solutions. Section 5 delves into order postponement, while Section 6 covers price postponement. Theorem 4.2, confirming that open-loop solutions are subgame perfect, links the paper's two segments on open-loop and closed-loop Nash–Stackelberg equilibria.

Our results indicate that for each postponement strategy, the closed-loop equilibria solutions are always superior to the open-loop results for the retailer (follower). However, there may be scenarios wherein the manufacturer is worse off if a strategy of postponement is utilized. This is due to the imperfections caused by vertical competition within a decentralized channel. Hence, it does not happen in a centralized channel facing the same demand realization (see Corollary 5.6). This finding is comparable to a result demonstrated in [12], where a supplier in a decentralized channel with asymmetric information is better off without receiving the delayed information. For a comparative analysis of the retailer's performance in open-loop and closed-loop solutions, see Theorem 5.1 (covering q-postponement scenarios) and Section 6.3 (covering r-postponement scenarios).

Finally, in Section 7, we provide simulated realizations of the random scenarios described in the previous sections. The purpose of these examples is to provide a comparative illustration of the performance of each channel member with and without implementing postponement. That is, in a variety of scenarios, the same supply chain is to address an uncertain demand with identical mean and volatility functions but under different realizations of the demand's stochastic variable. In each scenario, the expected and real values of each supplier's revenues are compared. For example, certain simulation results suggest that in some instances, acquiring extra information via postponement might adversely impact the manufacturer's revenue, as highlighted in Sections 7.2.2 and 7.3.2.

Diagram of section dependency

See Diagram 1

[1] In the theoretical analysis, the only restriction imposed on the uncertain demand is that its cumulative distribution function must be invertible over its support. See Section 3.

2. Comparative literature analysis: Bridging the research gap

Demand uncertainty poses challenges to supply chains' capacity, production, and pricing decisions. The interplay between the timings of demand realization and operational decisions endows firms with different capabilities. In particular, flexible firms can postpone certain operational decisions until the actual demand curve is observed [9–11,13]. The literature highlights two primary postponement strategies: order and price postponement.

Supply chains, particularly major online retailers like Amazon, Walmart, and eBay, utilize order postponement to harness upcoming demand signals. Customers can create "wish lists" or pre-order items before their release, giving companies insights for determining supplier order volumes. As a result, many retailers delay order placements until receiving more accurate demand information. This strategy extends to large appliances and conference organization [10,11,14]. Additionally, apparel retailers such as Zara in their attempt to address "fast fashion" demands adopt two-stage ordering policies, resonating with our order postponement study [15].

Price postponement, or responsive pricing, enables firms to adjust prices based on factors like location, time, and customer segment to maximize profits. This strategy shields companies from demand uncertainty by setting prices after demand is known, supporting strategic decisions in the face of unpredictable demand [13,16,17]. Its ease of implementation is also a key benefit [13]. Examples include car dealerships' negotiable pricing [17], Amazon's dynamic pricing based on demand and customer history, and retailers signaling potential price changes for upcoming releases [11]. Grocery providers also use this approach [16].

The literature extensively explores postponement strategies in supply chains under uncertain, single-period demand. In a static model, [17] examines such strategies and finds that with price flexibility, suppliers can base capacity decisions on deterministic reasoning despite demand uncertainty. A Nash–Stackelberg solution in a static Stackelberg game is presented by [11], who studies the effects of postponing order quantity (q) or retail price (r) announcements and focuses on a multiplicative uncertain demand structure. Delving into price postponement in a single-period newsvendor model, [18] determines conditions for unimodal profit functions. Meanwhile, [19] evaluates the effects of price postponement on production and pricing under supply risk, suggesting that while postponement offers flexibility, it may not offset potential fixed costs, especially for risk-averse firms. For a succinct review of supply chain postponement strategies, one can refer to [20]. To the best of our knowledge, all of these studies are conducted within the framework of static (single-period) models.

However, in various scenarios, especially in market-penetration cases where new entrants introduce lower initial prices, a multi-period analysis of pricing and demand becomes vital. Single-period solutions often overlook the long-term impacts of pricing on demand and profit. In multi-period frameworks, channel members not only seek equilibrium amongst themselves but also consider the repercussions of present pricing on future demand. This introduces a strategic dimension where immediate profit maximization is weighed against potential future earnings. In particular, only a multi-period model can cover the behavior of *strategic buyers* — those market-savvy customers who may postpone their purchase until they observe a considerable drop in prices [21–23]. See also Example 6 in 7.3.3.

On the other hand, incorporating the interplay between the current pricing decisions and future demand makes the equilibrium problems highly nested in time. The studies in [22,24] presents a *memory-based approach* to decouple the ensuing nested equilibria problems in a decentralized channel. Following a similar approach, in the first part of this paper (Section 4), we significantly enhance and generalize the solution algorithm so that it incorporates postponement strategies as well.

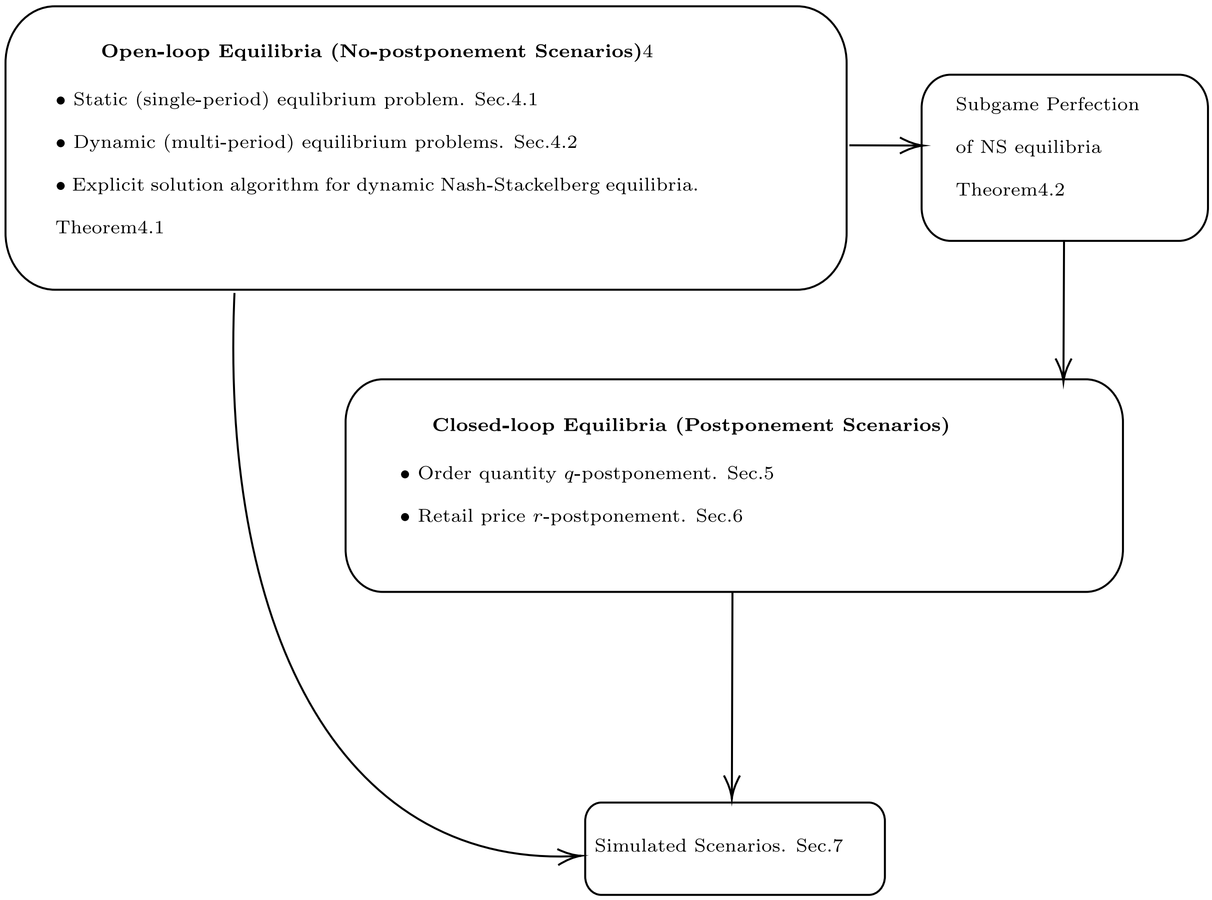


Diagram 1.

Thus, our work differs from other studies in the literature in that, our solution algorithm provides explicit solutions to the equilibrium outcome of a decentralized price-setting supply channel in a dynamic (multi-period) framework, while also implementing price or order postponement strategies. These solutions are applicable to a general setting where the demand is uncertain and can be characterized by an arbitrary distribution. Moreover, many of the earlier referenced works focus solely on a multiplicative uncertain demand structure. These purely multiplicative demand structures, despite their mathematical tractability, pose a significant limitation: they assume that the coefficient of variation of the demand remains constant. We find this assumption to be rather strong and not always justifiable.[2] In contrast to these studies, we consider a more general additive-multiplicative demand structure.

3. Preliminary model description

In a dynamic setting and time-dependent structure, first we propose a general model for stochastic demand at each point in time. Then, in Section 4, embedding this demand structure into various profit-optimization games, we arrive at equilibria solutions for each scenario. We divide the time scope into n discrete intervals referred to as periods. All the model variables and parameters are assumed to remain constant within each period.

In general, we consider demand at each period k to be a function of the entire retail price history, and time.

$$D_k = \tilde{\mu}_k(\mathbf{r}_k) + \tilde{\sigma}_k(\mathbf{r}_k)\,\epsilon_k \tag{1}$$

where r_k is the retail price at k, $\mathbf{r}_k = \{r_1, \dots, r_k\}$ is the set of the entire retail price history up to period k. Moreover, $\tilde{\mu}_k(\cdot)$ and $\tilde{\sigma}_k(\cdot)$, i.e. the mean and standard deviation of demand respectively, are deterministic functions of $\mathbf{r}_k$ and time (period k). Finally, ϵ_k is the stochastic variable at k.

The stochastic variable ϵ_k is normalized such that $\mathrm{E}[\epsilon_k] = 0$ and $\mathrm{Var}[\epsilon_k] = 1$. We also assume that the density function for ϵ_k and its cumulative distribution function, $f_{\epsilon_k}(\cdot)$ and $F_{\epsilon_k}(\cdot)$ respectively, are known over its support $[\underline{\epsilon}_k, \overline{\epsilon}_k]$. Plus, we assume that F_{ϵ_k} is invertible on the support interval and denote the resulting inverse cumulative distribution function (quantile function) by $F^{-1}_{\epsilon_k}(\cdot)$.

In a purely additive model for the uncertain demand [25], the standard deviation of the demand is considered to be constant and in a purely multiplicative model [26], the coefficient of variation of demand is assumed constant. Both assumptions are restrictive and not always justifiable [27]. An additive-multiplicative model, on the other hand, allows us to cover cases with coefficient of variation of demand being affected by the retail price.

3.1. Open-loop and closed-loop equilibria problems

Having outlined our general demand structure in Section 3, we embed it in a class of channel optimization problems where the suppliers of a perishable good face the uncertain demand described earlier. Considering the uncertain demand for the product, at the beginning of each period k, the manufacturer sets the optimal wholesale price w_k, and

[2] In the multiplicative uncertain demand model, $D = \mu(r)\zeta$, where ζ is the stochastic variable with $E(\zeta) = 1$. This becomes a highly specialized case of our model ($D = \mu(r) + \sigma(r)\epsilon$, $E(\epsilon) = 0$), where the demand mean and standard deviation are equal.

the retailer has to find the optimal retail price r_k, and order quantity q_k accordingly. We denote the equilibrium values of the wholesale and retail prices and order quantity by w_k^*, r_k^*, and q_k^* respectively.

In the non-postponement analysis, both the agents are risk-neutral and their optimization problems are based on maximizing their respective expected profits within the n-period time scale. In such scenarios, after w_k^* is announced, the retailer announces her r_k^* and q_k^* without postponement.

Whereas in the order or retail price postponement scenario, the retailer postpones declaration of one of her decision variables (either q_k or r_k) until she has observed demand uncertainty ϵ_k. At each period k, the retailer uses this extra delayed information in order to incorporate the real value of her period (i.e. local-in-time) profit in her optimization problem. In Theorem 5.1, its Corollary 5.4, and in Sections 6.3 and 6.3.1 we discuss how different postponement strategies, allowing for post-observation optimization, will affect the profits for the two decision makers and for the whole channel.

We refer to the post-observation equilibrium variables as $\hat{w}_k$, $\hat{r}_k$, and $\hat{q}_k$. In the subsequent sections, we refer to the non-postponement optimization procedures as the open-loop, or pre-observation analyses. We also use the terms post-observation, closed-loop, and ex-post analysis, interchangeably to refer to the postponement analysis.

4. Pre-observation equilibria: An open-loop model without postponement

At the beginning of each period, the manufacturer offers a wholesale price. Then the retailer sends her order quantity (which may be zero) to the manufacturer and declares her retail price to the market. At the end of the period, if the retailer is left with a surplus of items, which means her order quantity was larger than the actual demand, she will sell them for a salvage price. She may or may not receive a buy-back offer from the manufacturer for the surplus items. Because the commodity is perishable, she will not be able to store the unsold items to be offered to the market in the next periods.

In this section we solve the problem of maximizing the expected profits within the whole timescale encompassing all the periods. Thus, for instance, a pricing strategy that is optimal for a single period problem may be found out to be suboptimal within the multi-period setting. Thereby, the prescribed pricing and order quantity for the manufacturer and the retailer will enable then to make strategic sacrifices in order to boost the demand and rip the highest expected profits within the multi-period timescale. The decision variables to be determined are the wholesale price, retail price, and order quantity in each period, and the objective functions to be maximized are the holistic discounted expected profit for each decision maker.

4.1. The static (single-period) equilibrium problem

The final model in Section 4.5, its equilibrium structure, and our proposed algorithm for its numerical solution presented in Theorem 4.1, will include the general multi-period problem. However, for illustration purposes we start out with a single-period Stackelberg equilibrium problem. Later we expand the scheme to solve the generalized equilibrium problem in a multi-period (dynamic) setting.

Model Variables and Parameters

$w =$ wholesale price per unit, (decision variable)

$r =$ retail price per unit, $r > w$ (decision variable)

$q =$ quantity of items to be supplied to the market, (decision variable)

$D =$ uncertain demand

$c_m =$ manufacturing cost per unit, $c_m < w$ (given parameter)

$c_r =$ retailer's marginal cost per unit, $c_r < r - w$ (given parameter)

$s =$ salvage price per unit

$b =$ buy-back price per unit

$\pi^m =$ manufacturer's profit

$\pi^r =$ retailer's profit

Note that because this is a single-period analysis, we have suppressed the subscripts k. In such a single-period setting the general demand expression in (1) will turn into a specific simplified form described below.

$$D = \mu(r) + \sigma(r)\epsilon \tag{2}$$

In the multi-period analysis, however, all the decision variables and parameters may vary with time. This feature adds up to the level of non-autonomy the model can cover.

In general the single-period equilibrium is obtained by solving the following bilevel maximization problem.

$$\begin{aligned} &\max_w \mathrm{E}[\pi^m \mid q(r)] \\ &s.t. \quad r, q \in \{\operatorname{argmax} \mathrm{E}[\pi^r \mid w]\} \end{aligned} \tag{3}$$

The manufacturer's optimization problem includes that of the retailer (follower). An algorithm for deriving the explicit outcomes of the Nash–Stackelberg equilibrium typically operates sequentially. First, the follower's optimization problem is solved to obtain her best response in an implicit form, i.e., as a function of the leader's decision variable (w). Next, the follower's response is substituted in the leader's optimization problem to find the optimal values for her decision variables. Finally, the leader's optimal decision variable is brought back to the follower's response, this time yielding explicit results [28,29]. This procedure is outlined in (4).

$$\begin{aligned} &\max_q \mathrm{E}[\pi^r(r,w,q)] \quad \text{to obtain} \quad q^*(r,w), \\ &\max_r \mathrm{E}[\pi^r(r,w)] \quad \text{to obtain} \quad r^*(w) \\ &\max_w \mathrm{E}[\pi^m(w)] \quad \text{to obtain} \quad w^* \to r^*, q^* \end{aligned} \tag{4}$$

Note that in (4), optimization procedures are applied on expected values of the players' profits. The profits for the retailer and the manufacturer, denoted by π^r and π^m respectively, are calculated as below.

$$\begin{aligned} \pi^r(r,q,w) &= r\min(D,q) + s(q-D)^+ - c_r q - wq + b(q-D)^+ \\ &= (r-s-b)\min(D,q) + (s+b-c_r-w)q \\ \pi^m &= (w-c_m)q - b(q-D)^+ = (w-c_m-b)q + b\min(D,q), \end{aligned} \tag{5}$$

where $(q-D)^+$ denotes the positive component of $(q-D)$; i.e. $(q-D)^+ = \max(q-D,0)$ indicating that the buy-back and salvage prices are applicable only if there exists a surplus of items.

The general solution to the bilevel optimization problem in (4) is given below. (See Appendix A for the proof.) A term superscripted by an asterisk represents the optimal value for the associated variable.

$$\begin{aligned} q^*(r,w) &= \mu(r) + \sigma(r)\, F_\epsilon^{-1}\left(\frac{r-w-c_r}{r-s-b}\right) \\ \overline{\pi}^r(r,w) &= (r-w-c_r)\,\mu(r) + (r-s-b)\,\sigma(r)\int_{\underline{\epsilon}}^{z} t f_\epsilon(t)dt \\ &\quad \text{where} \quad z(r,w) = F_\epsilon^{-1}\left(\frac{r-w-c_r}{r-s-b}\right) \\ \overline{\pi}^m(w) &= \mu\left(r^*(w)\right)\left(w-c_m\right) + \sigma\left(r^*(w)\right)\Big[z^*(w)\Big(w-c_m-\frac{r^*-w-c_r}{r^*-s-b}\Big) \\ &\quad + b\int_{\underline{\epsilon}}^{z^*} t f_\epsilon(t)dt \Big] \\ &\quad \text{where} \quad z^*(w) = F_\epsilon^{-1}\left(\frac{r^*-w-c_r}{r^*-s-b}\right) \end{aligned} \tag{6}$$

A numerical solution to $\max_w \overline{\pi}^m$ will complete the procedure in (4) and yield the equilibrium values of w^*, r^*, and q^*. In (5), for the sake of generality, we have considered buy-back contracts represented by

b. In a buy-back contract the manufacturer pays the retailer $b < w$ per unit unsold. It should be noted that a buy-back contract does not necessarily mean that the unsold items will be physically sent back to the manufacturer (Chacon 2003). In order to share the risks stemming from market uncertainty and incentivize a larger order quantity, the manufacturer credits the retailer for each unsold item. Obviously $r > b + s$.

4.2. The dynamic (multi-period) equilibrium problems

Having solved the open-loop equilibrium problem in a single-period setting, we now proceed to the general open-loop problem in a multi-period time frame. In a multi-period setting, both the manufacturer and the retailer try to maximize their total expected profit over the whole duration of n periods. We start with analyzing the retailer's optimization problem. The manufacturer will face an structurally identical problem.

$$\max_{\mathbf{r}_k} \overline{\Pi}^r = \sum_{k=1}^{n} \alpha_k \mathrm{E}[\pi_k^r | D_1, \ldots, D_{k-1}] \tag{7}$$

where α_k is the given discount factor at period k, $(\alpha_1 = 1)$.[3]

From the structure of the expected profit at a single-period in (52) and without loss of generality we can conclude that $\mathrm{E}[\pi_k^r]$ is a function of the mean and variance of the demand, which in turn may depend on the entire price history. The dependence of $\mu(\mathbf{r}_k)$ and $\sigma(\mathbf{r}_k)$ on the vector of the whole retail prices in the past makes the optimization problem (7) highly nested.

4.3. A general solution procedure

Using the reasoning method referred to as *backward induction* in dynamic programming, we begin the solution of the multi-variable nested optimization problem by analyzing the final period. It is readily observable that the only profit expression in (7) which depends on r_n is $\mathrm{E}[\pi_n^r]$. Thus maximization of $\overline{\Pi}^r$ with respect to r_n is equivalent to maximization of $\mathrm{E}[\pi_n^r]$ with respect to r_n.

$$\max_{r_n} \overline{\Pi}^r \equiv \max_{r_n} \mathrm{E}[\pi_n^r] \tag{8}$$

Moreover, at period n all of the previous decision variables and demands have become common knowledge. Therefore, given $\mathbf{r}_{n-1}^*$ and $\mathbf{D}_{n-1} = [D_1, \ldots, D_{n-1}]$ and assuming that the mapping $r_n \mapsto \mathrm{E}[\pi_n^r | \mathbf{D}_{n-1}]$ has a global maximum, this global maximum can be expressed as a function of the previous retail prices and demand history.

$$r_n^* = r_n^*(\mathbf{r}_{n-1}, \mathbf{D}_{n-1}) \tag{9}$$

Now the backward induction method proceeds to the period $n-1$ where having r_n^* as expressed in (9) enables us to conclude that maximization of $\overline{\Pi}^r$ with respect to r_{n-1} is equivalent to maximization of $\alpha_{n-1}\mathrm{E}[\pi_{n-1}^r]+ \alpha_n \mathrm{E}[\pi_n^r]$ with respect to r_{n-1}. The resulting r_{n-1}^* will be a function of $(\mathbf{r}_{n-2}^*, \mathbf{D}_{n-2})$. Inserting this new function into (7) and iterating the same procedure backward in time, we obtain the vector $\mathbf{r}_n^*$.

4.4. Generalizing demand's dependence on time and prices

The microeconomic relationship between an elastic demand structure and the current price is classically portrayed as $D_k = \psi(r_k)$, where k denotes the current period. However, not every market behaves in such a simple manner, as strategic buyers base their purchase on the (possibly repetitive) trends of previous prices to which they have become *anchored* [30]. In general, potential buyer's valuation of a commodity and, in turn, their purchase decision may become biased by their comparison of the current price and those of the past. For example, in a specific scenario, a price increase by 20% may reduce the customer base by, for example, 10%. Thus, a general time-dependent model of supply and price optimization should also consider the effect of anchoring to the past prices on current demand. Following [22], we base our time-dependent model of uncertain demand on the simple premise that the probability of an item being sold at time k for the price of r_k depends on the customers' interest, which in its own right, in general, may depend on the past prices,

$$D_k = \psi_k(r_k) \cdot \Phi_k(r_{k-1}, \ldots, r_1) \tag{10}$$

where the functional form Φ represents price *history*. Obtaining such a functional form may fall into the domain of behavioral economics. Obviously, such a general demand model, which considers the effects of anchoring to the past prices, also covers the classical *memoryless* demand case where $\Phi_k = 1$. If the demand functional format remains identical, i.e. $\psi_k(r_k) = \psi(r_k)$, the procedure outlined in Section 4.3 turns into a *repeated game*. In contrast, a fully dynamic game emerges when the functional formats for $\psi_k(r_k)$s vary with time, adding to the level of non-autonomy in the ensuing equilibrium problems. In addition, assuming demand's dependence on past prices, i.e. $\Phi_k = \Phi_k(r_{k-1}, \ldots, r_1) := \Phi_k(\mathbf{r}_{k-1})$, makes the equilibrium problems highly nested.

In Theorems 4.1, 5.1, and 6.1 we propose solution algorithms for the general non-autonomous dynamic games. Obviously, the proposed solution algorithms are significant generalizations which among others, cover the trivial n-periodic repeated games as well as the non-trivial fully non-autonomous memory-based cases.

4.4.1. Memory-based uncertain demand structure

In our expression for memory-based demand, we embed a class of functional forms within the uncertain demand structure such that the demand at each period be not only a function of price at that period, but also carry the effects of pricing policies and the demand in the previous periods. We will refer to these functional forms as memory functions and denote them by $\Phi_k(\mathbf{r}_{k-1})$.

As discussed earlier, the additive-multiplicative structure of demand in (1) enables us to cover general demand expressions with non-constant coefficient of variation. Here, for the sake of greater generality, we consider the coefficient of variation of demand to be a function of the retail price as well.

$$CV_{D_k} = CV_{D_k}(r_k) \tag{11}$$

In this paper we limit our analysis to the case where previous prices scale the level of the current demand.

$$\begin{aligned} &D_k(\mathbf{r}_k) = \Phi_k(\mathbf{r}_{k-1}) d_k(r_k) \\ &\text{where } d_k(r_k) = \mu_k(r_k) + \sigma_k(r_k)\epsilon_k \end{aligned} \tag{12}$$

Comparing (12) with (1) we observe that

$$\begin{aligned} &\widetilde{\mu}_k(\mathbf{r}_k) = \Phi_k(\mathbf{r}_{k-1})\mu_k(r_k) \\ &\widetilde{\sigma}_k(\mathbf{r}_k) = \Phi_k(\mathbf{r}_{k-1})\sigma_k(r_k). \end{aligned} \tag{13}$$

This memory-based structure enables us to explicitly incorporate the impact of previous pricing decisions on the current demand's mean and standard deviation. The memory functions embedded within the uncertain demand $D_k(\mathbf{r}_k)$ must be such that at the $k+1$st period, $\Phi_{k+1}(\mathbf{r}_k)$ retains the information from the entire previous periods' memories while being affected by the last piece of information that has becomes available, i.e. r_k. This feature can be obtained by the following expression.

$$\frac{\Phi_{k+1}}{\Phi_k} = \phi_k(r_k) \tag{14}$$

[3] This allows for time-dependent discounting which in turn allows for different lengths of periods.

We call these $\phi_k(r_k)$s the memory elements. Notice that the possibility of having different functional forms for ϕ_ks in different periods enables our demand structure to cover more non-autonomy. With the memory structure in (14), we will have:

$$\begin{aligned}\Phi_k(\mathbf{r}_{k-1}) &= \prod_{i=1}^{k} \phi_i(r_{i-1}) \\ \Phi_1(\cdot) &= \phi_1(\cdot) = 1\end{aligned} \tag{15}$$

4.5. Embedding the demand structure in the equilibrium problems

The general construction outlined in Section 4.3 is sufficiently explicit to enable solutions of the problem for most choices of functions $\widetilde{\mu}$ and $\widetilde{\sigma}$. However, as discussed in Section 4.2 the resulting bilevel optimization problem in its multi-period setting is so deeply nested that one cannot expect to find an analytical solution. The importance of our memory-based demand scheme lies in the structure it will create when embedded inside the expressions for the channel members' expected profits. At each period k, we denote the local-in-time profit for the retailer and the manufacturer by $\widetilde{\pi}^r_k$ and $\widetilde{\pi}^m_k$, respectively. We state the final results of this section in the following two theorems.

Theorem 4.1. *Let n be the number of periods and assume that the uncertain demand at period k is given by*

$$D_k(\mathbf{r}_k) = \Phi_k(\mathbf{r}_{k-1})\Big(\mu_k(r_k) + \sigma_k(r_k)\epsilon_k\Big) \tag{16}$$

where

$$\Phi_1(\cdot) = \phi_1(\cdot) = 1, \quad \Phi_k(\mathbf{r}_{k-1}) = \prod_{i=1}^{k} \phi_i(r_{i-1})$$

and where ϵ_ks are continuously distributed with $\mathrm{E}[\epsilon_k] = 0$ *and* $\mathrm{Var}[\epsilon_k] = 1$ *for all k. with $f_{\epsilon_k} > 0$ almost everywhere on their supports. If for each k the single-period Stackelberg problem below has an equilibrium at r^*_k and w^*_k*

$$\begin{aligned}\mathfrak{J}^r_k &= \overline{\pi}^r_k + \phi_{k+1}(r_k)\mathcal{F}^r_k \\ \mathfrak{J}^m_k &= \overline{\pi}^m_k + \phi_{k+1}(r_k)\mathcal{F}^m_k\end{aligned} \tag{17}$$

where $\mathcal{F}^r_k$ and $\mathcal{F}^m_k$ are found recursively from:

$$\begin{aligned}&\mathcal{F}^r_n = 0, \quad \mathcal{F}^m_k = 0 \\ &\mathcal{F}^r_{k-1} = \frac{\alpha_{k-1}}{\alpha_k}\mathfrak{J}^r_k(r^*_k), \quad \mathcal{F}^m_{k-1} = \frac{\alpha_{k-1}}{\alpha_k}\mathfrak{J}^m_k(w^*_k), \quad k = n, \dots, 2\end{aligned} \tag{18}$$

and

$$\begin{aligned}\overline{\pi}^r_k &= (r_k - w_k - c_{r_k})\mu_k(r_k) + (r_k - s_k - b_k)\,\sigma_k(r_k)\int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t)dt \\ \overline{\pi}^m_k &= \mu_k\left(r^*_k(w_k)\right)\left(w_k - c_{m_k}\right) \\ &+\sigma_k\left(r^*_k(w_k)\right)\Big[\, z^*_k(w_k)\Big(w_k - c_{m_k} - \frac{r^*_k - w_k - c_{r_k}}{r^*_k - s_k - b_k}\Big) \\ &+b_k\int_{\underline{\epsilon}_k}^{z^*_k} t f_\epsilon(t)dt\,\Big],\ z^*_k(w) = F^{-1}_\epsilon\Big(\frac{r^*_k - w_k - c_{r_k}}{r^*_k - s_k - b_k}\Big)\end{aligned} \tag{19}$$

then the bilevel (Stackelberg) optimization problem

$$\begin{aligned}\overline{\Pi}^r &= \sum_{k=1}^{n}\alpha_k \mathrm{E}[\pi^r_k] = \sum_{k=1}^{n}\alpha_k\Phi_k(\mathbf{r}_{k-1})\overline{\pi}^r_k \\ \overline{\Pi}^m &= \sum_{k=1}^{n}\alpha_k \mathrm{E}[\pi^m_k] = \sum_{k=1}^{n}\alpha_k\Phi_k(\mathbf{r}^*_{k-1})\overline{\pi}^m_k\end{aligned} \tag{20}$$

has an equilibrium at $\mathbf{r}^*_n = [r^*_1, \dots, r^*_n]$ and $\mathbf{w}^*_n = [w^*_1, \dots, w^*_n]$.

The optimal order quantity at k is then calculated as below.

$$q^*_k = \Phi_k(\mathbf{r}^*_{k-1})\left[\mu_k(r^*_k) + \sigma_k(r^*_k)\,F^{-1}_{\epsilon_k}\Big(\frac{r^*_k - w^*_k - c_{r_k}}{r^*_k}\Big)\right] \tag{21}$$

See the proof and discussion in Appendix B.

Remark. Finding the numerical values of w^*_ks allows us follow the procedure outlined in (4) in reverse order and calculate the numerical values of $r^*_k(w^*_k)$s which in turn yield q^*_ks. Next, we prove that the results of Theorem 4.1 are subgame perfect.

Theorem 4.2. *The equilibrium obtained in Theorem 4.1 is subgame perfect. That is, subsets of the equilibrium results covering the time interval between an arbitrary period j and n, i.e. $[r^*_j, \dots, r^*_n]$ and $[w^*_j, \dots, w^*_n]$ and, a fortiori, their resulting $[q^*_j, \dots, q^*_n]$ will also constitute an equilibrium for the corresponding subgame of the original problem covering that time interval:*

$$\begin{aligned}J^r_j &= \alpha_j\Phi_j(\mathbf{r}_{j-1})\overline{\pi}^r_j(r_j) + \dots + \alpha_n\Phi_n(\mathbf{r}_{n-1})\overline{\pi}^r_n(r_n) \\ J^m_j &= \alpha_j\Phi_j(\mathbf{r}^*_{j-1})\overline{\pi}^r_j(w_j) + \dots + \alpha_n\Phi_n(\mathbf{r}^*_{n-1})\overline{\pi}^r_n(w_n)\end{aligned} \tag{22}$$

Proof. See Appendix B.1. □

In Section 6, we will use the subgame perfection of the open-loop equilibrium in the analysis of the closed-loop equilibrium in a price-postponement scenario.

5. Post-observation equilibrium: Postponing the order quantity

In this section, we analyze the closed-loop equilibrium in an order-postponement scenario. At each period, the decision-making process is divided into two phases: before and after the realization of the demand uncertainty. At the beginning of the period, both decision-makers are aware that the order-quantity will be sent to the manufacturer after demand uncertainty has been resolved. At the first step in each period k, the manufacturer and the retailer set their corresponding prices obtained from the open-loop equilibria solutions (given by Theorem 4.1). The retailer, however, postpones her order quantity until after demand randomness at that period has been resolved; i.e. ϵ_k has been observed. In the following sections, a term superscripted by a circumflex represent the optimal value of the associated variable after employing a postponement strategy.

At the second step, when the retailer observes demand uncertainty ϵ_k, it is obvious that in order to optimize her local-in-time profit, i.e. $(r^*_k - w^*_k - c_{r_k})\,\hat{q}_k$, she must pick the highest possible value for $\hat{q}_k$ which will be $D_k = \Phi_k(\mathbf{r}^*_{k-1})\Big(\mu_k(r^*_k) + \sigma_k(r^*_k)\hat{\epsilon}_k\Big)$. With a zero lead time, the optimal order quantity for the retailer is simply equal to the real demand.

$$\hat{q}_k = D_k = \Phi_k(\mathbf{r}^*_{k-1})\left[\mu_k(r^*_k, k) + \sigma_k(r^*_k, k)\,\epsilon_k\right] \tag{23}$$

In Theorem 5.1, we show that the retailer always benefits from an order-postponement strategy. The manufacturer on the other hand, may either benefit from or be adversely impacted by the retailer's deviation from q^*_k depending on whether $\hat{q}_k > q^*_k$ or $\hat{q}_k < q^*_k$ respectively. Moreover, in the beginning of the $k+1$th period the manufacturer faces exactly the same expected profit optimization problem as the corresponding one in the open-loop equilibrium problem. This is because in our price-setting optimization problem, the memory functions depend only on prices. Since in an order-postponement scenario, the retailer changes only the order quantities obtained from the open-loop solutions, her optimal prices r^*_ks remain the same and thus leaving the memory functions unchanged. In other words, a change in order quantity at period k does not affect the *expected* profit of the manufacturer in the future periods because the retailer has not deviated from the r^*_k obtained by open-loop equilibrium solution. Besides the manufacturer has no strategic means to influence the occurrence of $\hat{\epsilon}_k$. Thus she will not deviate from previously calculated w^*_{k+1}. Hence the results $[w^*_k, r^*_k, \hat{q}_k]$ constitute the *ex-post* equilibrium state at k.

$$\Pi^r = \sum_{k=1}^{n}\alpha_k(r^*_k - w^*_k - c_{r_k})\,\hat{q}_k \tag{24}$$

$$\Pi^m = \sum_{k=1}^{n}\alpha_k(w^*_k - c_{m_k})\,\hat{q}_k \tag{25}$$

5.1. Comparison between open-loop and closed-loop profits

We consider two hypothetical scenarios each with a retailer facing the same uncertain demand (ϵ_k). One of the two retailers does not postpone her declaration of order quantity; instead she adheres to the pre-observation optimal order quantity, q^*_k. The other retailer postpones her declaration of optimal order quantity $\hat{q}_k$ until after observation of $\hat{\epsilon}_k$. In this hypothetical scenario, they both face the same $\hat{\epsilon}_k$. We refer to the (real) profit obtained by the non-postponing retailer as $\pi^r_{OL_k}$ (open-loop profit) and to the postponing one's as $\pi^r_{CL_k}$ (closed-loop profit).

Below we show that for the retailers, the closed-loop profit is always greater than or equal to the open-loop profit. Hence, for the retailer, it is always beneficial to postpone her declaration of the order quantity until after she has observed demand uncertainty.

Theorem 5.1. *Between two retailers who will face the same uncertain demand D_k, the profit obtained by the one who postpones her order quantify $\hat{q}_k$ until she observes the demand uncertainty $\hat{\epsilon}_k$ is higher than or equal to that of the retailer who instead of postponing, adheres to the order quantity obtained from the open-loop equilibrium q^*_k, as given by (21).*

Proof. See Appendix C. □

Remark 5.2. Notice that while the order quantities and retail prices prescribed by the open-loop equilibrium guarantee a non-negative *expected* profit for the retailer, the *real* retail profit can become negative in extreme over-supplying cases, i.e when $q^*_k \gg D_k$ (see Remarks 7.1 for Example 2, and the corresponding Fig. 3(a)). In contrast, order quantity postponement guarantees an always-positive profit for the retailer.

Corollary 5.3. *In the hypothetical scenario described in Theorem 5.1, the holistic profit, over the span of n periods, for the postponing retailer is higher than or equal to that of the non-postponing retailer.*

Proof. See Appendix C.1. □

5.1.1. Order postponement and channel profit

Corollary 5.4 (*The Aggregate Channel Profit*)**.** *In a hypothetical scenario with two price-setting decentralized channels, the aggregate channel profit for a channel with an order-postponing retailer is higher than or equal to that of the channel with a non-postponing retailer.*

Proof. See Appendix C.2. □

Remark 5.5. Corollary 5.4 shows that in an order-postponement scenario, despite the fact that the manufacturer may lose potential profits due to postponement (see Example 3 in Section 7.2.2), the channel always benefits from postponement.

Corollary 5.6 (*Centralized Channel (Cooperating Agents)*)**.** *In a hypothetical scenario with two price-setting centralized channels, the channel that postpones supplying the market until after demand uncertainty has been resolved will benefit higher than or equal to a non-postponing channel.*

Proof. See Appendix C.3. □

6. Price postponement

In this section, we analyze another closed-loop variant of the problem, in which the retailer postpones the announcement of retail price until after the demand uncertainty has been resolved. We use essentially the same notations for the model variables and parameters as those in Section 4. We use $\hat{r}_k$, and $\hat{q}_k$ to denote the optimal retail price and order quantity, respectively.

Here, again the two players start from the open-loop equilibrium solutions and obtain $\mathbf{r}^*_n$, $\mathbf{w}^*_n$, and $\mathbf{q}^*_n$. At the beginning of the first period the manufacturer sets w^*_1 and the retailer orders $\hat{q}_1 = q^*_1$. But the retailer postpones the announcement of the retail price $\hat{r}_1$ until after she observes $\hat{\epsilon}_1$. In Sections 6.2 and 6.3.1 we solve the equilibrium problems for each player to obtain the optimal post-observation decision variables at an arbitrary period k.

Furthermore, since in the price-postponement scenario the entire demand is not necessarily addressed by the retailer, for the sake of generality we must also consider a (possibly time-dependent) salvage price for the retailer, and a buy-back contract between the two agents.

6.1. Observing the feedback: Closing the loop

In this scenario, at the beginning of the kth period the manufacturer sets the w^*_k and the retailer orders $\hat{q}_1 = q^*_k$ items. However, the retailer postpones her declaration of the retail price until after she has observed the demand uncertainty $\hat{\epsilon}_k$.

It should be noted that while in the ex-ante analysis of the no-postponement equilibria states, we used the backward induction method, here in the ex-post analysis of price-postponement scenario we use a forward induction approach. Thereby, we incorporate the newly-revealed information in the form of feedback signals into the decision-making process. This is due to the fact that we now change future demand by our postponement.

6.2. Post-observation bilevel optimization

In our analysis of the retail price-postponement scenario, we divide the decision-making process into two steps. First, at the beginning of each period k, both the retailer and the manufacturer solve the expected profit optimization (equilibrium) problem in a Stackelberg framework within the time interval $\{k, \dots, n\}$. The manufacturer then declares the equilibrium wholesale price and the retailer submits her order quantity to the manufacturer. However, the retailer does not declare her retail price to the market. Instead, she postpones doing so until after she observes demand uncertainty.

In the second step, having observed $\hat{\epsilon}_k$, the retailer incorporates this new information and solves the equilibrium problem again while considering the manufacturer's response for the next periods. That is, after observing $\hat{\epsilon}_k$ the retailer tries to find optimal retail prices within $\{k, \dots, n\}$ while being subject to the optimality of the wholesale prices within $\{k+1, \dots, n\}$. The equilibrium solution will provide the retailer with her post-observation optimal retail price vector $[\hat{r}_k, \dots, \hat{r}_n]$. Then she declares the first element of her newly found optimal price vector, $\hat{r}_k$, to the market.

We begin the analysis of the equilibrium problem from the first period and using forward induction reasoning delineate a general optimization procedure for all periods. At the first step in the first period, both the retailer and the manufacturer solve the equilibrium problem aimed at maximizing their own respective expected holistic profit while subject to the optimality of the other player's solution. Thus they obtain the results of the open-loop equilibrium, i.e. $\{\mathbf{r}^*_k, \mathbf{q}^*_k, \mathbf{w}^*_k\}$. Therefore at $k = 1$ the manufacturer proceeds with declaring w^*_1 and the retailer orders q^*_1 items. However, instead of declaring r^*_1 to the market, the retailer waits for the uncertainty of demand, ϵ_1 to be resolved. In the second step and after observing $\hat{\epsilon}_1$, the retailer (and the manufacturer) solve the following equilibrium problem to obtain the optimal retail prices.

$$\max_{\mathbf{r}_n} \Pi^r$$
$$\max_{w_2,\dots,w_n} J_2^m$$
$$\Pi^r = \pi_1^r(r_1, w_1^*, q_1^*) + \cdots + \alpha_k \Phi_k(\mathbf{r}_{k-1}) \overline{\pi}_k^r(r_k, w_k, q_k) + \cdots + \alpha_n \Phi_n(\mathbf{r}_{n-1}) \overline{\pi}_n^r(r_n, w_n, q_n) = \pi_1^r + J_2^r$$
$$J_2^m = \alpha_2 \Phi_2(\hat{r}_1(w_1)) \overline{\pi}_2^m(w_2) + \cdots + \alpha_n \Phi_n(\hat{r}_1(w_1), \dots, \hat{r}_{n-1}(w_{n-1})) \overline{\pi}_n^m(w_n)$$
$$\text{where } \pi_1^r = (r_1 - s_1 - b_1) \min \overbrace{\left(\mu_1(r_1) + \sigma_1(r_1)\hat{\epsilon}_1\right)}^{D_1(r_1)} + (s_1 + b_1 - c_{r_1} - w_1^*) q_1^* \tag{26}$$

Note that the only difference between the retailer's problem expression in (26) and the one in (65) is in the first term, where the expected value of the profit in the first period $\overline{\pi}_1^r$ is replaced by the real profit π_1^r. Thus the retailer, having observed $\hat{\epsilon}_1$, tries to find the vector of optimal retailer prices $\hat{\mathbf{r}}_n$ to optimize the sum of her real profit at the first period π_1^r and the expected (discounted) profits in the future J_2^r.

To solve (26) we use the backward induction reasoning again. Starting with the retailer's problem in the last period n we observe that in order to obtain $\hat{r}_n$ from (26) the retailer has to solve (64) once again. This means that $\hat{r}_n(w_n)$ equals $r_n^*(w_n)$ which was obtained in the pre-observation optimization. In general, going backward in time from period n to 2, the retailer will face the exact same optimization problems as the ones in the pre-observation analysis, i.e. $\hat{r}_k(w_k) = r_k^*(w_k)$, $k \in \{2, \dots, n\}$. However, when the backward induction reaches the first period, it will face the only term in the objective function which is different from the corresponding one in (65), i.e. π_1^r. Thus, in general the optimal $\hat{r}_1$ is different from r_1^*.

$$\max_{r_1} \Pi^r = \pi_1^r(r_1) + \phi_2(r_1) \times \underbrace{\left[\frac{\alpha_2}{\alpha_1}\overline{\pi}_2^r(r_2^*) + \cdots + \frac{\alpha_n}{\alpha_k}\overline{\pi}_n^r(r_n^*) \prod_{i=3}^{n} \phi_i(r_{i-1}^*)\right]}_{\mathcal{F}_1^r \text{ future expected profit, given (obtained from pre-observation analysis)}} \tag{27}$$

Therefore the vector of optimal decision variables for the retailer after observing $\hat{\epsilon}_1$ is $[\hat{r}_1, r_2^*, \dots, r_n^*]$ where $\hat{r}_1$ is obtained from (27) and r_k^*s ($k = 2 \cdots n$) are equal to the ones obtained from the pre-observation optimization problems.

Now we proceed to the manufacturer's part of the equilibrium (26), considering the effect of the new retail pricing scheme on future (time interval $\{2, \dots, n\}$) demand.

$$\max_{w_2,\dots,w_n} J_2^m = \max_{w_2,\dots,w_n} \left[\alpha_2 \Phi_2(\hat{r}_1) \overline{\pi}_2^m(w_2) + \cdots + \alpha_n \Phi_n(\hat{r}_1, r_2^*, \dots, r_{n-1}^*) \overline{\pi}_n^m(w_n)\right]$$
$$\text{where } \hat{r}_1 = \hat{r}_1(w_1^*), r_k^* = r_k^*(w_k) \quad 1 < k \tag{28}$$

where each $\overline{\pi}_k^m$ is calculated from .

Analogously, observing that the term w_n appears only in the profit expression for the final period $\overline{\pi}_n^m$, we start the backward induction process from the nth period.

$$\max_{w_n} J_2^m \equiv \max_{w_n} \overline{\pi}_n^m \tag{29}$$

But this problem has already been solved in the open-loop analysis and it will yield the same optimal decision variable as before, i.e. w_n^*. Going backward in time, in general, at each period $j \in \{2, \dots, n\}$, the manufacturer faces the optimization problem (30). Note that for this arbitrary period j we have $\max_{w_j} J_2^m \equiv \max_{w_j} J_j^m$. This is due to the result of Theorem 4.2 about the subgame perfection of the equilibrium aimed at maximization of the expected profits on time interval between 2 and n.

$$\max_{w_j} J_2^m \equiv \max_{w_j} J_j^m = \max_{w_j} \alpha_k \Phi_j(\tilde{\mathbf{r}}_{j-1}) \overbrace{\left[\overline{\pi}_k^m(w_j) + \phi_{j+1}\left(r_k^*(w_j)\right)\mathcal{F}_j^m\right]}^{\Im_k^m(w_j)}$$
$$\text{where } \tilde{\mathbf{r}}_{j-1} = \{\hat{r}_1(w_1^*), r_2^*(w_2), \dots, r_{j-1}^*(w_{j-1})\},$$
$$\Phi_j(\tilde{\mathbf{r}}_{j-1}) = \phi_2(\hat{r}_1) \prod_{i=3}^{j} \phi_i(r_{i-1}^*) \tag{30}$$

$$\mathcal{F}_j^m = \frac{\alpha_{j+1}}{\alpha_j} \overline{\pi}_{j+1}^m(w_{j+1}) + \cdots + \frac{\alpha_n}{\alpha_j} \overline{\pi}_n^m(w_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)$$
$$\mathcal{F}_n^m = 0 \tag{31}$$

Again we have $\max_{w_j} J_j^m \equiv \max_{w_j} \Im_j^m$. Plus, when solving $\max_{w_j} \Im_j^m$ we observe that the choice of $\hat{r}_1$ does not affect $\mathcal{F}_j^m$. Therefore the results of $\max_{w_j} \Im_j^m$ will be exactly as equal to ones obtained by the open-loop solutions. Comparing (30) with the general expression (32), from the proof of Theorem 4.1 in Appendix B, we can conclude that after the observation of $\hat{\epsilon}_1$ and declaration of $\hat{r}_1$ in the first period, the manufacturer's optimal price vector for the rest of the periods (from 2 to n) does not change.

$$\max_{w_k} J_k^m(\mathbf{w}_k) = \alpha_k \Phi_k(\mathbf{r}_{k-1}^*) \overbrace{\left[\overline{\pi}_k^m(w_k) + \phi_{k+1}\left(r_k^*(w_k)\right)\mathcal{F}_k^m\right]}^{\Im_k^m(w_k)} \tag{32}$$

However, $\phi_2(\hat{r}_1)$ in (30) will scale $\Im_j^m$ differently from $\phi_2(r_1^*)$ in the corresponding open-loop equilibrium. Hence while the same w_j^*s will come out of the two equilibrium problems, the expected values of the total profits will be different due to different memory elements.

After analyzing the two-step solution for the players in the period 1, we try to find a general solution procedure at a period k. The players arrive at period k with the memory function containing the already declared $\hat{\mathbf{r}}_{k-1}$. In the first step they have to solve the following bilevel optimization (Stackelberg equilibrium) problem.

$$\max_{r_k,\dots,r_n} J_k^r$$
$$\max_{w_k,\dots,w_n} J_k^m \tag{33}$$

From Theorem 4.2 we know that the equilibrium aimed at maximization of the expected profits is subgame perfect. Hence, in the first step, each decision maker obtains a subset of her original open-loop equilibrium results; i.e. $[r_k^*, \dots, r_n^*]$ and $[w_k^*, \dots, w_n^*]$. Thus, at the first step in period k, the manufacturer declares w^* and the retailer orders $\hat{q}_k = \Phi_k(\hat{\mathbf{r}}_{k-1}) \left[\mu_k(r_k^*) + \sigma_k(r_k^*) F_{\epsilon_k}^{-1}\left(\frac{r_k^* - w_k^* - c_{r_k}}{r_k^*}\right)\right]$.

At the second step, after the retailer observes $\hat{\epsilon}_k$ the following bilevel equation has to be solved.

$$\max_{r_k,\dots,r_n} \alpha_k \Phi_k(\hat{\mathbf{r}}_{k-1}) \pi_k^r(r_k) + J_{k+1}^r \qquad \text{over } k, \dots, n \tag{34}$$

$$\max_{w_{k+1},\dots,w_n} J_{k+1}^m = \max_{w_{k+1},\dots,w_n} \sum_{i=k+1}^{n} \alpha_i \Phi_i(\tilde{\mathbf{r}}_{i-1}) \overline{\pi}_i^r(w_i) \qquad \text{over } k+1, \dots, n$$
$$\text{where } \tilde{\mathbf{r}}_{i-1} = [\hat{\mathbf{r}}_{k-1}, r_k, \dots, r_{i-1}] \tag{35}$$

Similarly, starting the backward induction from the final period, it is evident that from period n to $k+1$ the retailer will face the exact same optimization problems as the ones in the pre-observation analysis. The only term in the entire objective function which is different from its corresponding term in (65) is π_k^r (the real profit at k which has replaced its own expected value, $\overline{\pi}_k^r$). Therefore the retailer's optimization problem boils down to the following.

$$J_k^r = \alpha_k \overbrace{\Phi_k(\hat{\mathbf{r}}_{k-1})}^{\text{price history}} \times \overbrace{\left(\pi_k^r(r_k) + \phi_{k+1}(r_k) \underbrace{\left[\frac{\alpha_{k+1}}{\alpha_k} \overline{\pi}_{k+1}^r(r_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k} \overline{\pi}_n^r(r_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}) \right]}_{\mathcal{F}_k^r = \text{ expected (future) values, given (obtained from pre-observation problem)}} \right)}^{\mathfrak{J}_k^r(r_k)}$$

$$\max_{r_k} J_k^r \equiv \max_{r_k} \mathfrak{J}_k^r \tag{36}$$

Note that by the time the backward induction process reaches the kth period $\mathcal{F}_k^r$ in (36), i.e. the future expected profit, is already calculated and is treated as a constant. Solving the single-variable optimization problem in (36) yields $\hat{r}_k$ while the rest of the optimal retail prices remain equal to those obtained in the pre-observation (open-loop) optimization problem. Thus at the second step in the kth period, the retailer obtains her optimal decision variables $[\hat{r}_k(w_k^*), r_{k+1}^*(w_{k+1}), \ldots, r_n^*(w_n)]$ as functions of corresponding manufacturing prices.

In order to find the numerical values of $\hat{r}_k(w_k)$ and the rest of the optimal retail prices, the retailer has to solve the manufacturer's problem of an optimal response for the next periods.

$$\max_{w_{k+1},\ldots,w_n} J_{k+1}^m$$
$$J_{k+1}^m = \alpha_{k+1}\Phi_{k+1}(\hat{\mathbf{r}}_k)\overline{\pi}_{k+1}^m + \cdots + \alpha_n \Phi_k(\hat{\mathbf{r}}_k) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)\overline{\pi}_n^m \tag{37}$$
$$= \alpha_{k+1}\Phi_{k+1}(\hat{\mathbf{r}}_k)\Big[\overline{\pi}_{k+1}^m + \cdots + \frac{\alpha_n}{\alpha_{k+1}} \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)\overline{\pi}_n^m\Big]$$

The numerical results for optimal wholesale prices are obtained using the recursive solution procedure delineated below.

$$J_j^m = \alpha_j \Phi_j(\tilde{\mathbf{r}}_{j-1})\Big[\overline{\pi}_j^m(w_j) + \phi_{j+1}(r_j^*)\mathcal{F}_j^m\Big] \quad k+1 \le j \le n$$
$$\tilde{\mathbf{r}}_{j-1} = (\hat{\mathbf{r}}_k, r_{k+1}^* \cdots, r_{j-1}^*) \Rightarrow \Phi_j(\tilde{\mathbf{r}}_{j-1}) = \prod_{i=1}^{k+1} \phi_i(\hat{r}_{i-1}) \prod_{i=k+2}^{j} \phi_i(r_{i-1}^*) \tag{38}$$

$$\mathcal{F}_n^m = 0$$
$$\mathcal{F}_j^m = \frac{\alpha_{j+1}}{\alpha_j} \overline{\pi}_{j+1}^m(w_{j+1}) + \cdots + \frac{\alpha_n}{\alpha_j} \overline{\pi}_n^m(w_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*) \tag{39}$$

Comparing (37) with (76) and (77), and using the result of Theorem 4.2, it is evident that after announcing $\hat{r}_k$ at period k, and when solving manufacturer's optimization problem for the time interval $\{k+1,\ldots,n\}$, the backward induction process will yield the same $\{w_{k+1}^*,\ldots,w_n^*\}$ as those obtained in the open-loop equilibrium problem. To see this, note that in (76), at period k, the decision variable w_{k+1} appears only inside the bracket, while the pricing effect is factored out within the memory function. When the backward induction process reaches the kth period, $\mathcal{F}_k^m$ has already been calculated. This makes $\mathfrak{J}_k^m$ a function of only w_k. However, the manufacturer's expected profit will be different from the results of the open-loop solutions. This is due to the scaling factor $\Phi_k(\hat{\mathbf{r}}_{k-1})$ which in general will be different from $\Phi_k(\mathbf{r}_{k-1}^*)$ in (76). The results of this section are expressed in the following theorem.

Theorem 6.1. *In a retail price postponement scenario where the retailer and the manufacturer face the uncertain demand described in Theorem 4.1, the retailer at each period k postpones the declaration of her price until after observing demand uncertainty ϵ_k.*

Assuming that there exists an equilibrium state $[\mathbf{r}_k^*, \mathbf{w}_k^*]$ for the open-loop problem described in Theorem 4.1, if the following objective function has a global maximum, $\hat{r}_k$,

$$\mathfrak{J}_k^r(r_k) = \pi_k^r(r_k) + \phi_{k+1}(r_k)\mathcal{F}_k^r$$

where $\mathcal{F}_k^r = \frac{\alpha_{k+1}}{\alpha_k} \overline{\pi}_{k+1}^r(r_{k+1}^*) + \cdots + \frac{\alpha_n}{\alpha_k} \overline{\pi}_n^r(r_n^*) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)$

then the closed-loop problem of price postponement has an equilibrium with the following optimal decision variables.

$$\hat{\mathbf{r}}_n = [\hat{r}_1,\ldots,\hat{r}_n]$$
$$\mathbf{w}_n^* = [w_1^*,\ldots,w_n^*]$$
$$\hat{\mathbf{q}}_n = [\hat{q}_1,\ldots,\hat{q}_n]$$

where $\hat{q}_k = \Phi_k(\hat{\mathbf{r}}_{k-1})\Big[\mu_k(r_k^*) + \sigma_k(r_k^*)\, F_{\epsilon_k}^{-1}\Big(\frac{r_k^* - w_k^* - c_{r_k}}{r_k^*}\Big)\Big] = \frac{\Phi_k(\hat{\mathbf{r}}_{k-1})}{\Phi_k(\mathbf{r}_{k-1}^*)} q_k^*$

6.3. Comparison between the open-loop and closed-loop total profits

At the end of period n, the set of post-observation optimal retail prices, $[\hat{\mathbf{r}}_n]$ is the result of the optimization problem $\max_{\mathbf{r}_n} \Pi^r$ considering the real values of π_k^rs. Whereas the set of pre-observation optimal retail prices, $[\mathbf{r}_n^*]$ is the result of optimization $\max_{\mathbf{r}_n} \overline{\Pi}^r$ considering the expected values of the profits at each period $\overline{\pi}_k^r$s. Thus it is trivial that in a hypothetical n-period scenario where two retailers face the same ϵ_k at each period k, the one that postpones the declaration of her prices ($\hat{r}_k$s) until after observation of each ϵ_k gains higher profit compared to the retailer who adheres to sub-optimal r_k^*s. In other words, in a price-postponement scenario, because $\hat{r}_k$s are the results of the real profit optimizations, any other set of decision variables (including the set of r_k^*s) will be sub-optimal. Therefore, $\Pi_{CL}^r \ge \Pi_{OL}^r$ where Π^r is the total discounted real profit gained through n periods. Obviously, this conclusion holds true for centralized channel, as a centralized channel's equilibrium problem is structurally equivalent to a that of a retailer (see the discussion in Corollary 5.6).

6.3.1. Closed-loop optimization for the manufacturer

In general, in the closed-loop optimization scenario, at each period k the retailer enjoys the strategic means to find an optimal $\hat{r}_k$ maximizing the sum of her current profit and expected future profits. Whereas the manufacturer always faces the structurally identical (though differently scaled) expected profit optimization.

At each period k after observing $\hat{\epsilon}_k$, the retailer deviates from the previously obtained equilibrium price r_k^* by declaring $\hat{r}_k$ instead. Due to the structure of the memory functions, this new pricing scheme will affect the future demand and thereby the future earnings for both the retailer and the manufacturer. The retailer's optimization problem as generalized in Theorem 6.1 is tailored such that an optimal $\hat{r}_k$ will maximize the sum of the current real profit and expected future profits. Thus after declaring each $\hat{r}_k$, it is the manufacturer's turn to modify her own optimal pricing scheme for the future considering the effects of the retail prices on future demand and expected earnings.

Comparing to the non-postponement solutions, the retailer always benefits from postponing her retail price. Whereas the manufacturer's may either benefit or lose potential profit compared to the non-postponement case, depending on the structure of demand mean and variance, and different realizations of the uncertain demand. In Section 7.3, we provide simulated examples with price-postponement having different effects on the manufacturer's profit. For a hypothetical scenario wherein the manufacturer is worse off due to price postponement, see Example 5 in Section 7.3.2.

7. Monte Carlo simulations of the model

In this section, we illustrate the theoretical results and implement the solution algorithms discussed in Sections 4, 5, and 6. In the examples analyzed in this section, we use Cobb–Douglas demand functions. In the construction of the example scenarios and in their Monte Carlo

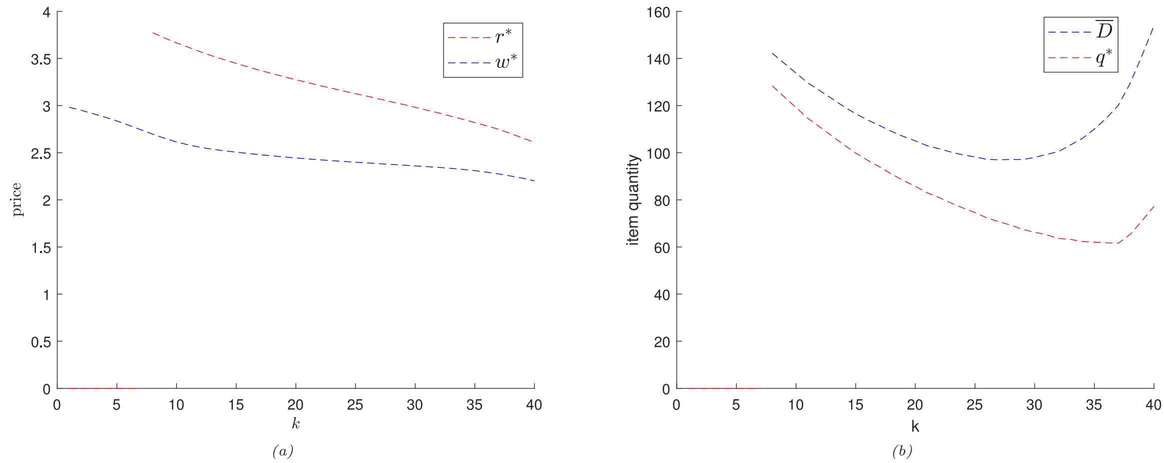


Fig. 1. Retail and Wholesale Prices (a), Supply and Expected Demand (b) at Equilibrium States (Example 1, Section 7.1.1).

simulations, we use a truncated normal distribution for ϵs to ensure that the negative noise terms do not cause the entire demand to become negative.

7.1. The open-loop equilibria solutions

Following the order by which the scenarios were presented, we begin by providing an example of the open-loop equilibria wherein no postponement strategies is employed; optimization takes place based on only the expected values of discounted profits within a number of periods.

7.1.1. Example 1. Boosting the demand through initial free distribution

For the first example, we consider the following scaled demand function.

$$d_k(r_k) = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon$$
$$\text{where } \mu_k(r_k) = \frac{10^4}{r^{3-0.01k}}, \quad \sigma_k(r_k) = 0.5\mu_k(r_k) + 10/r_k \tag{40}$$

Multiplicative memory functions scale the future demand such that an increase in the current retail price decreases the future demand. Thus, the memory function at period $k+1$ which will scale the future demand D_{k+1} is monotonically decreasing with respect to the retail price at all previous periods.

$$\frac{\partial \Phi_{k+1}(r_1,\ldots,r_n)}{\partial r_k} = \frac{\partial \prod_{k=1}^{n} \phi_{k+1}(r_k)}{\partial r_k} < 0, \quad \forall k \in \{1,\ldots,n\} \tag{41}$$

This means that the memory element at $k+1$ must be monotonically decreasing with respect to r_k.

$$\frac{\partial \phi_{k+1}(r_k)}{\partial r_k} < 0 \tag{42}$$

Here, for illustration purpose, we use the following functional structure for memory elements

$$\phi_{k+1}(r_k) = 1 + \gamma_k(\kappa_k - r_k) \tag{43}$$

where $\gamma_k \geq 0$, the memory strength factor at period k, is a given parameter. The given parameter $\kappa_k \geq 0$ can be interpreted as a price cap; i.e., any initial price above κ_k reduces demand, whereas demand is more likely to increase if $r_k < \kappa_k$. If the scaling factor is negative, i.e., $\phi_{k+1}(r_k) \leq 0$, the optimal order q_{k+1} is zero. To avoid this problem, we consider

$$\phi_{k+1}(r_k) = [1 + \gamma_k(\kappa_k - r_k)]^+. \tag{44}$$

For simplicity, we set $\gamma_k = 0.02$, $\kappa_k = 6$, $\alpha_k = 1$, $c_{m_k} = 2$, $c_{r_k} = 0$, $s_k = 1$, and $b_k = 0.1$ for all ks. The number of periods, n, is set to be 40.

The decision variables at the equilibrium state obtained from Theorem 4.1 are given in Fig. 1. Fig. 1a illustrates r_k^* and w_k^* and Fig. 1b shows q_k^* and the expected demand $\overline{D}_k = \Phi_k\,\mu_k$ at each period. We observe that the holistic optimization algorithm prescribes the retailer to set $r^*_{1,\ldots,7} = 0$ and $q^*_{1,\ldots,7} \approx 0$. We also observe the resulting boost in demand mean $\overline{D}$ in Fig. 1b to begin at period 7. The expected profits for the two suppliers are $\overline{\Pi}^r = 957.38$ and $\overline{\Pi}^m = 1151.66$.

A strategy of this type makes good sense economically; it corresponds to a situation in which a small number of items ($q \approx 0$) are given away for free at earlier periods to create increased interest for the product in the next periods. Marketing schemes of this type, referred to as *freemium* business models, are also employed by entrant suppliers who intend to boost the demand in the future by intruding very low prices in the beginning. According to an article published by the Wall Street Journal, the freemium market approach is the most effective strategy for firms seeking to expand their customer base and enhance their value [31]. In 2015, Samsung utilized this strategy to increase its market share by permitting Apple iPhone users to test several of its Galaxy models for a fee of only $1 for 30 days, according to Forbes [32]. Additionally, a Harvard Business Review report states that online start-ups frequently adopt the freemium model [33].

7.2. Order postponement scenarios

In this section, we implement the two-step optimization algorithm delineated in Section 5. In the first step, the retailer and the manufacturer, both aware of a forthcoming order-postponement phase, have to solve the open-loop problem. In order to provide illustrative examples of different scenarios that may happen in the second step, we simulate different realizations of the stochastic variable ϵ_k. We create these ϵ_ks based on a given truncated normal distribution and normalized as discussed in Section 3.

7.2.1. Example 2

In this section, we analyze the order-postponement strategy where the decentralized supply channel faces the following scaled demand function.

$$d_k = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon_k \quad \mu_k = \frac{10^4}{r_k^{2-\beta\frac{n-k}{n}}},\ \sigma_k(r_k) = \frac{1}{2}\mu_k(r_k) + \frac{5000}{r^3}$$
$$\alpha_k = 0.95^{k-1},\ \gamma_k = 0.01,\ \kappa_k = 5, c_{m_k} = 2, c_{r_k} = 0, s_k = b_k = 0, n = 40, \beta = 0.4. \tag{45}$$

We use the same functional structure in (44) for the memory elements in this analysis with $\gamma_k = 0.01$ and $\kappa_k = 5$. Note that because both the suppliers are aware that there will be an order-postponement, there

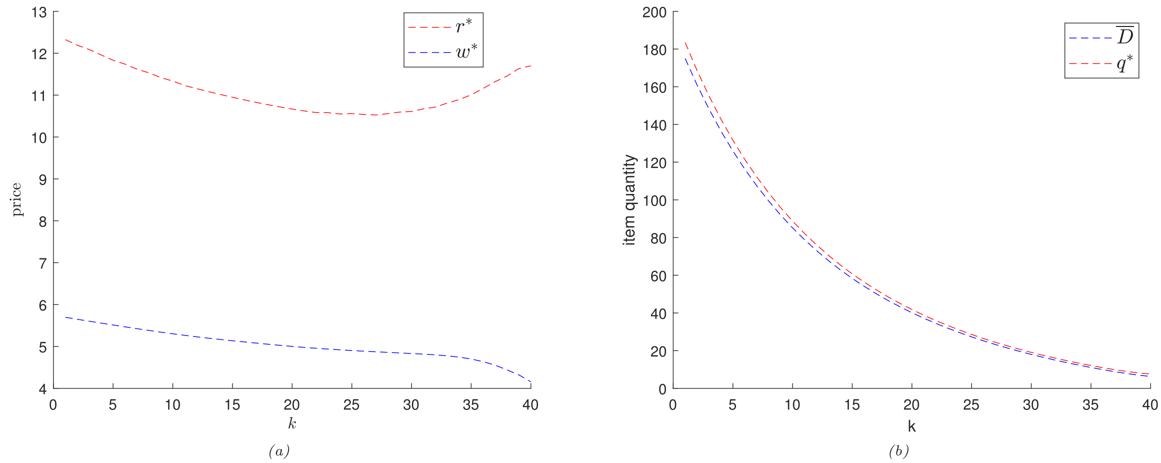


Fig. 2. Retail and Wholesale Prices (a), Supply and Expected Demand (b) at Equilibrium States (Example 2, Section 7.2.1).

is no buy-back feature embedded in their contract; $b_k = 0$. Besides, because, according to (23), the retailer will always address the demand, no salvage price is needed in the model; $s_k = 0$.

At the beginning of the first step, solving the open-loop equilibrium problem, we obtain the following results.

$$\overline{\Pi}^r = 5430.44, \overline{\Pi}^m = 5066.09$$

Fig. 2(a) illustrates the optimal retail and wholesale prices that the suppliers set at the first step, in the beginning of each period. The equilibrium state supply quantities and corresponding expected demands are depicted in Fig. 2(b).

At each period, the retailer postpones her order-quantity until after demand uncertainty at that period is resolved. Fig. 3(c) shows a possible realization of uncertain demand stemmed from a simulated set of ϵ_ks. After observing each ϵ_k, the retailer sets her $\hat{q}_k$ accordingly (see Eq. (23)). The figure also shows the expected values of demand at each period.

The demand realization shown in Fig. 3(c) results in the following profits for the two hypothetical channels described in Theorem 5.1 and its Corollary 5.4. The subscripts CL and OL denote closed-loop (order-postponement) and open-loop (no-postponement) scenarios, and the superscripts r, m, c denote retailer, manufacturer, and (the whole) channel, respectively.

$$\Pi^r_{OL} = 7123.93 < \Pi^r_{CL} = 10110.65 \quad \Pi^m_{OL} = 5066.09 < \Pi^m_{CL} = 5556.08$$

$$\Pi^c_{OL} = 12190.03 < \Pi^c_{CL} = 15666.74$$

The results of this simulated scenario show that facing this set of ϵ_ks, both the retailer and the manufacturer, and a fortiori, the whole channel benefit from order postponement. We also observe that in a no-postponement scenario, the real profit obtained by the manufacturer equals her expected profit: $\Pi^m_{OL} = \overline{\Pi}^m$. This is due to the fact that in the no-postponement scenario with $b_k = 0$, the manufacturer receives the order quantity at the beginning of each period and thus does not share the risk of facing an uncertain demand.[4]

Remark 7.1. Comparing the retailer's open-loop and closed-loop overall profits (in the span of the entire n periods), we observe that she is better off with order-postponement. This observation is compatible with Corollary 5.3. The superiority of the closed-loop equilibrium solutions for the retailer is readily observable in Fig. 3(a), where the open-loop retailer, despite having a positive overall profit, experiences negative revenues at periods $7, 12, 15, 23, 24, 28$, and 33. In contrast, following the closed-loop equilibrium solution ensures a non-negative profit for the retailer at all ks. Immune to the risks stemmed from market volatility, the manufacturer never experiences negative revenues (again, due to setting $b_k = 0$). See Fig. 3(b).

7.2.2. Example 3. Order-postponement detrimental to the manufacturer

Different realizations of demand uncertainty ϵ_k may indeed cause different real profits for the channel members. Iterating the simulation with a different realization of ϵ_ks, in Fig. 4, we illustrate the results of the same channel as the one in Example 2, catering to an ensuing realization of the uncertain demand. Similarly, we consider two channels facing this realization of demand, one with an order-postponing retailer and one with a retailer who adheres to the open-loop solutions. In this simulation, the realized profits for these two hypothetical channels and their individual members are as follows.

$$\Pi^r_{OL} = 6357.22 < \Pi^r_{CL} = 8509.45 \quad \Pi^m_{OL} = 5066.09 > \Pi^m_{CL} = 4648.47$$

$$\Pi^c_{OL} = 11423.32 < \Pi^c_{CL} = 13157.93$$

In this case, the manufacturer does not benefit from order postponement (see Fig. 4(b) for details). Despite her relatively lower profit following the closed-loop solution procedure, the whole channel still benefits from order postponement. This observation is consistent with the result of Corollary 5.4.

7.3. Price postponement scenarios

In this section we provide examples of price postponement scenarios and implement the two-step optimization algorithm discussed in Section 6. Since in a price-postponement scenario the retailer does not necessarily address the entire demand, for the sake of generality, we have to consider non-zero salvage and buy-back prices in the profit optimization expressions. In the examples, following the procedure outlined in Theorem 6.1, at the first step, we provide the open-loop solutions. Based on the open-loop equilibrium results, the channel members determine their initial decision variables, i.e. the wholesale price and the order quantity $w^*_k, \boldsymbol{\Phi}_k(\hat{\mathbf{r}}_{k-1})q^*_k$, respectively. Next, we simulate different scenarios by generating a sequence of ϵ_ks based on a given truncated normal distribution and normalized as stated in Section 3. The retailer then observes this demand uncertainty and finds her optimal retail price $\hat{r}_k$ accordingly.

[4] Note that only through paying a non-zero buy-back price ($b_k \neq 0$) to the retailer, does the manufacturer share the risk stemmed from demand uncertainty with the retailer. In return, a $b_k \neq 0$ encourages the retailer to order more items to the manufacturer.

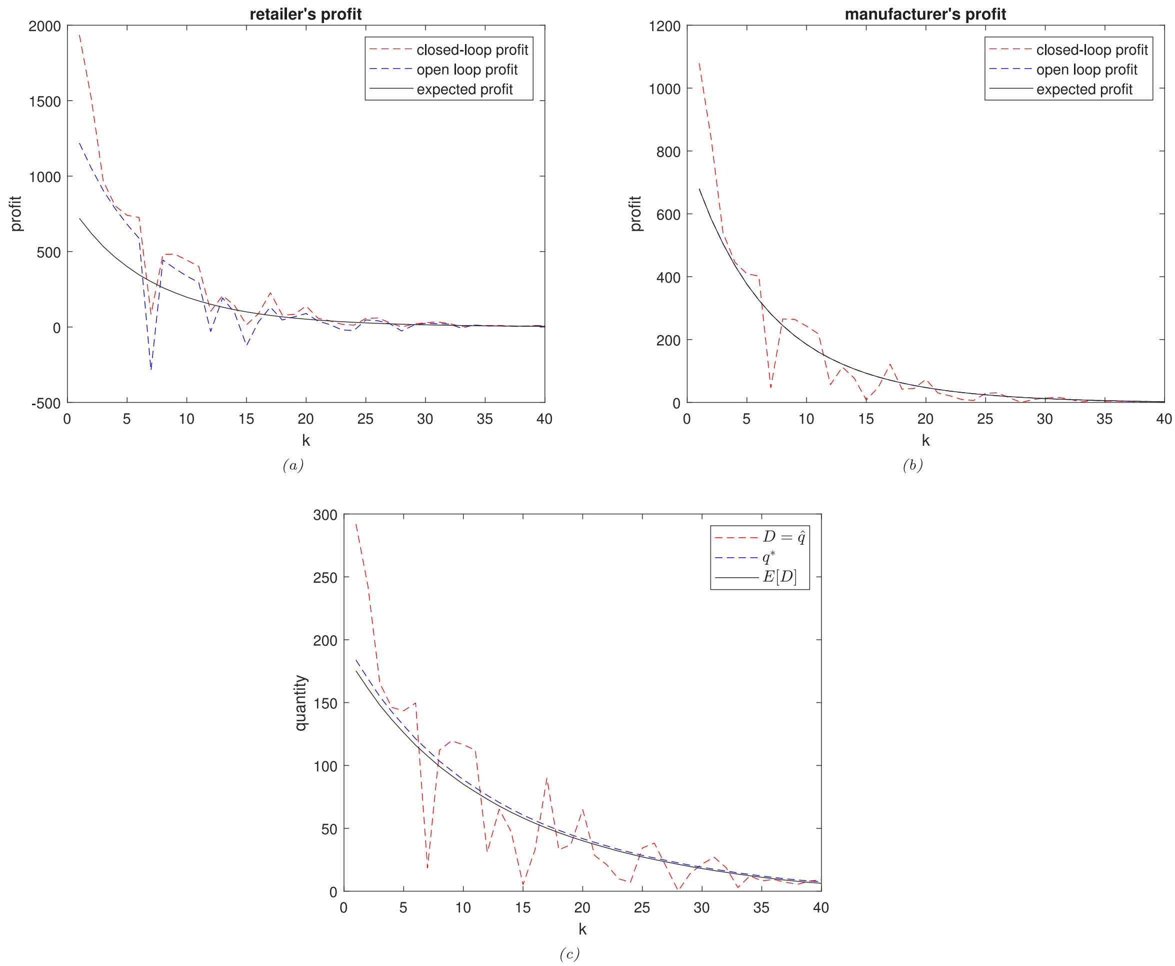


Fig. 3. Open-loop (No postponement) and Closed-loop (Order-Postponement) Equilibrium States (Example 2, Section 7.2.1).

7.3.1. Example 4. Given buy-back prices

We consider the following Cobb–Douglas functions in the demand structure. We also use the same functional structure in (44) for the memory elements.

$$d_k = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon_k, \quad \mu_k(r_k) = \frac{10^4}{r_k^3},\ \sigma_k(r_k) = 0.3\,\mu_k(r_k) + \frac{100}{r_k^2}$$
$$\alpha_k = 0.95^{k-1},\ \gamma_k = 0.01,\ \kappa_k = 4 \tag{46}$$
$$c_{m_k} = 2, c_{r_k} = 0, b_k = 1.5, s_k = 0.1, n = 40$$

The open-loop (no-postponement) solution results for this scenario are given in Figs. 5 and 6. In Fig. 6, a supply policy change happens at period 26 when the optimal supply policy for the retailer changes from under-supplying the market ($q^* < \overline{D}$) to over-supplying ($q^* > \overline{D}$). The corresponding expected values for the profits are as below.

$$\overline{\Pi}^r = 1536.30, \overline{\Pi}^m = 1433.55$$

Now that both the channel members have obtained the open-loop solutions, at each period, the retailer updates her objective function after demand uncertainty is resolved. She then declares her optimal retail price $\hat{r}_k$ to the market.

To compare the channel members' performance with and without price-postponement, analogous to the analysis in the previous section, we consider two hypothetical channels facing the same set of ϵ_ks at each period. In one channel (denoted by the subscript OL), the retailer does not postpone her declaration of the retail price, i.e. she always adheres to open-loop solutions for optimal prices — r_k^*s. In the other channel (denoted by the subscript CL), the retailer postpones her decision on retail price after demand uncertainty is resolved and then declares $\hat{r}_k$ instead (following the procedure delineated in Theorem 6.1).

$$\Pi^r_{OL} = 1784.639 < \Pi^r_{CL} = 2256.92 \quad \Pi^m_{OL} = 1547.84 < \Pi^m_{CL} = 1673.55$$

$$\Pi^c_{OL} = 3332.48 < \Pi^c_{CL} = 3930.472$$

The closed-loop solution results for this realization of ϵ_ks are given in Figs. 7 and 8. In accordance with the results of the discussion in Section 6.3, the total expected profit for the price-postponing retailer (following the closed-loop equilibria solutions) exceeds that of the non-postponing retailer who adheres to the open-loop equilibria solutions. (See Fig. 7.) Fig. 8 illustrates similar results for two hypothetical manufacturers. For this realization of the ϵ_ks, the manufacturer, too, benefits from price-postponement strategy; $\Pi^m_{OL} < \Pi^m_{CL}$.

7.3.2. Example 5. Price-postponement detrimental to the manufacturer

In the previous example, we observed a realization of stochastic demand, catering to which the manufacturer was better off following a price-postponement strategy. This fortunate outcome, however, is not guaranteed to occur for the manufacturer, as discussed in Section 6.3.1. That is, unlike the retailer who is always better off by price postponement, the manufacturer may indeed lose potential revenue following that strategy. In example 5, we iterate Example 4 (with the same supply channel and market) with another set of randomly generated ϵ_ks which turn out to be detrimental to the manufacturer. As it can be seen in Fig. 10, catering to the stochastic demand ensued from these ϵ_ks, the manufacturer would have been better off had the channel adhered to

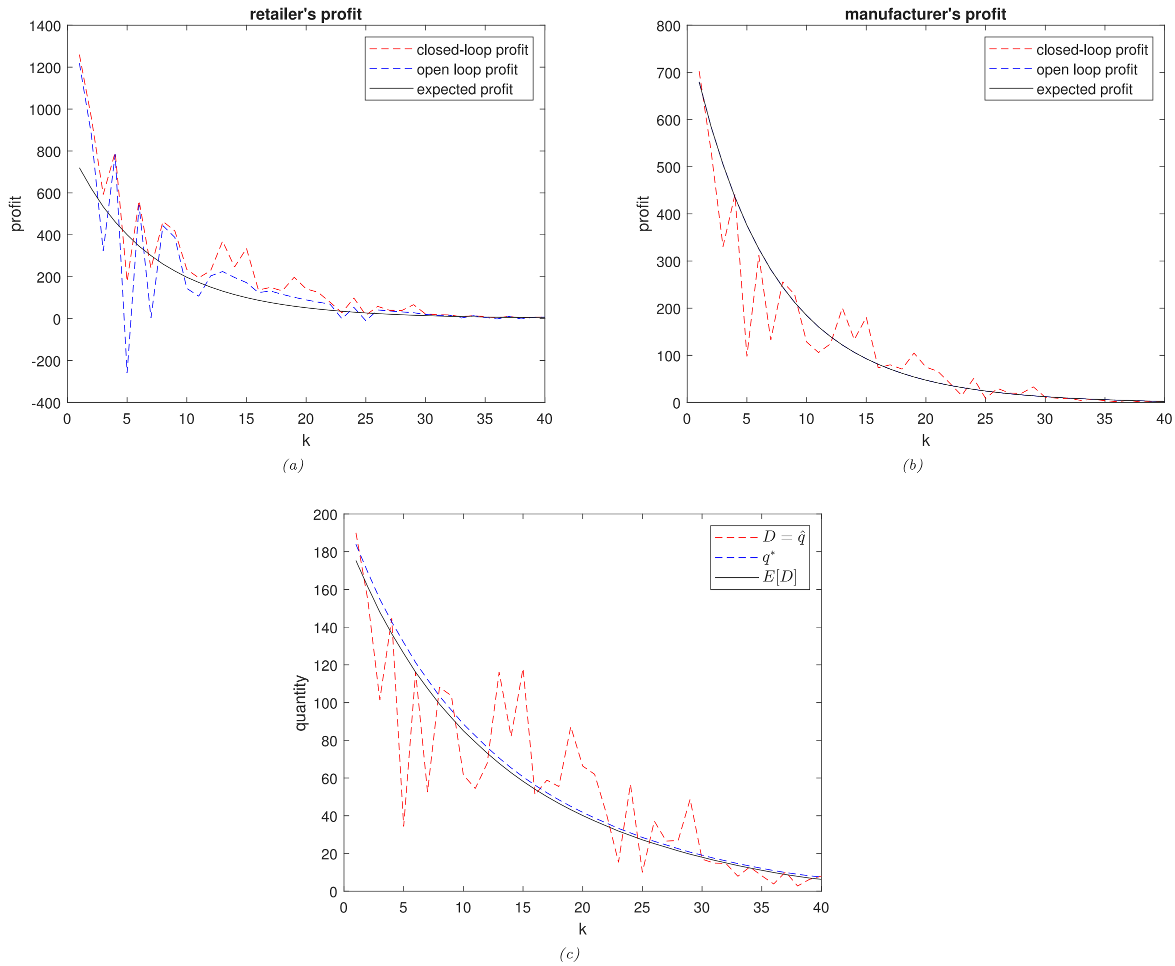


Fig. 4. Example 3: the manufacturer is worse off following order postponement, Section 7.2.2.

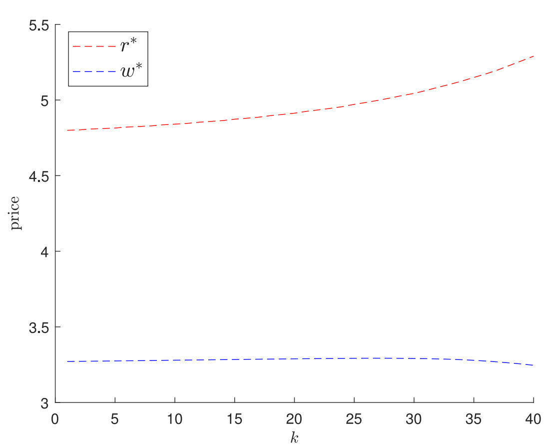


Fig. 5. Open-loop Equilibrium: Prices (Ex.4).

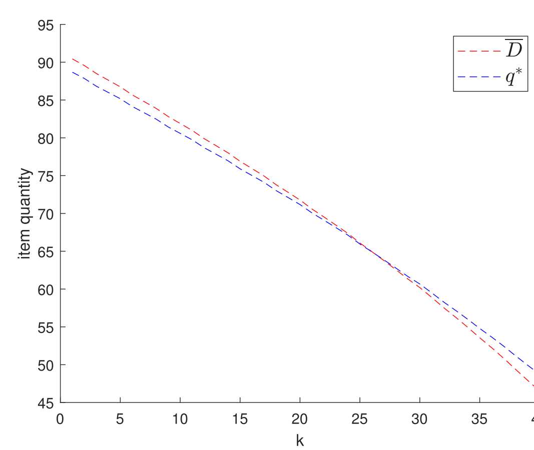


Fig. 6. Open-Loop Equilibrium: Supply & Demand (Ex.4).

the open-loop (non-postponement) solution. The open-loop and closed-loop equilibrium results for the two channel members and the entire channel have been juxtaposed below for comparison.

$\Pi^r_{OL} = 2009.64 < \Pi^r_{CL} = 2456.52 \quad \Pi^m_{OL} = 1651.28 > \Pi^m_{CL} = 1633.07$

$\Pi^c_{OL} = 3660.92 < \Pi^c_{CL} = 4089.59$

It is noteworthy that despite the manufacturer's potential loss due to price-postponement, the entire channel still benefits from postponement. That is, the retailer's extra revenue due to postponement outweighs the manufacturer's potential loss; thus making $\Pi^c_{CL} > \Pi^c_{OL}$. (See the retailer's equilibrium state results in Fig. 9.)

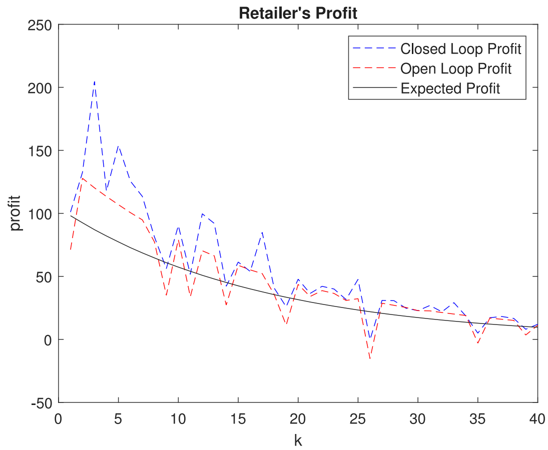


Fig. 7. Closed-loop Equilibrium States for the retailer, Ex.4.

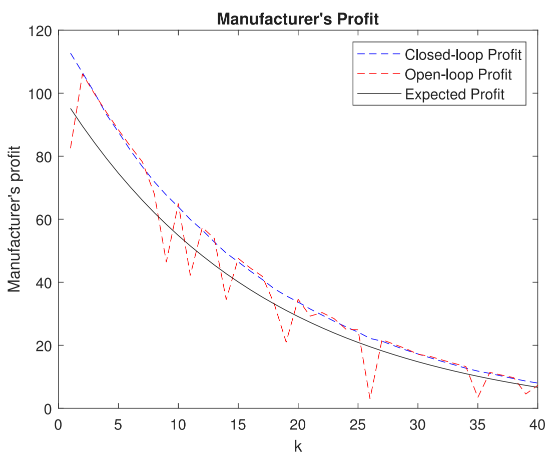


Fig. 8. The manufacturer is better off by price-postponement. Ex.4.

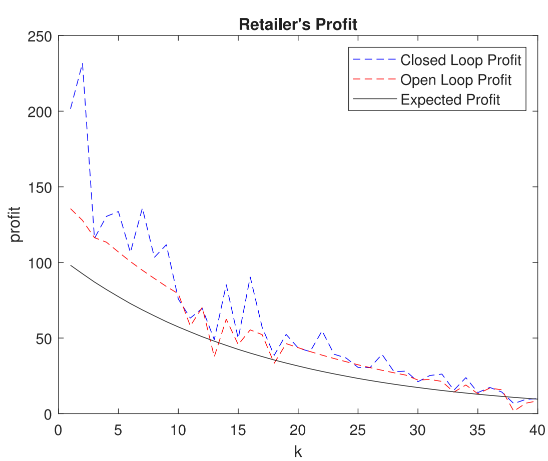


Fig. 9. The retailer benefits from price-postponement. Ex.5.

7.3.3. Example 6. Optimal timing for supply/pricing policy change

In Example 1 (Section 7.1.1), we saw a scenario wherein the equilibrium results given by the open-loop algorithm prescribed abrupt changes in the pricing strategy. The solution algorithm in that example, in addition to finding the optimal pricing and supply policies, determines the optimal duration for the costly, yet demand-boosting strategy of free distribution (See Fig. 1). Similarly, in Example 4 (Section 7.3.1),

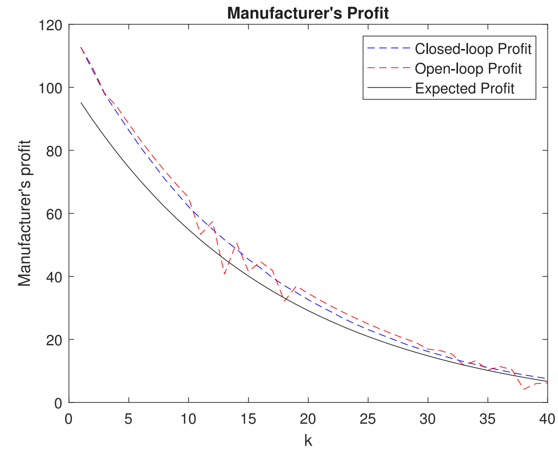


Fig. 10. The manufacturer is worse off by price-postponement. Ex.5.

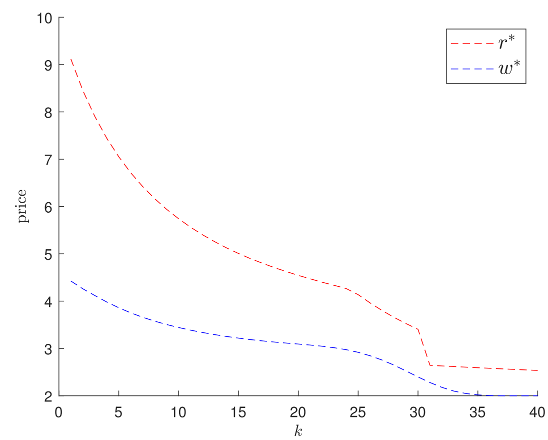


Fig. 11. Open-loop Equilibrium Prices. Ex.6.

a supply policy change was prescribed by the open-loop equilibrium solution (See Fig. 6).

Analogously, the closed-loop equilibrium solution can be employed to apply changes in the pricing and supply strategies at optimal times. In this example, we present a scenario where the model determines the optimal time for an *end-season sale*, when an abrupt drop in prices boosts demand. The optimal time in this case is found to be the 31st period. Pricing strategies of this type are typically employed to cater to the preferences of strategic customers who may postpone their purchase until they see a significant drop in prices. (See Figs. 11, 12, and 13.)

In this example we use the following functional structures and parameters for the uncertain demand.

$$d_k = \mu_k(r_k) + \sigma_k(r_k)\,\epsilon_k \quad \mu_k = \frac{10^4}{r_k^{2+\beta k}},\ \sigma_k(r_k) = 0.5\,\mu_k(r_k) + \frac{5000}{r^3}$$
$$\alpha_k = 0.95^{k-1},\ \beta = 0.1,\ \gamma_k = 0.01,\ \kappa_k = 4, c_{m_k} = 2, c_{r_k} = 0, n = 40 \qquad (47)$$
$$b_k = 1.5, s_k = 0.1, \forall k \in \{1,\dots,n\}.$$

Similarly, first, the two suppliers solve the open-loop equilibrium problems and obtain the following results. The decision variables r_k^*, w_k^*, q_k^* for all periods are given in Figs. 11 and 12.

$$\overline{\Pi}^r = 1428.49,\ \overline{\Pi}^m = 1081.48,\ \overline{\Pi}^c = \overline{\Pi}^r + \overline{\Pi}^m = 2509.97$$

Now, at each period, the retailer postpones her choice of retail price until demand uncertainty is resolved and then declares $\hat{r}_k$. The optimal prices for a specific scenario based on a simulated realization of ϵ_ks are given in Fig. 13(c) (comparing the closed-loop $\hat{r}$s and open-loop r^*s). The corresponding profits for two hypothetical postponing and

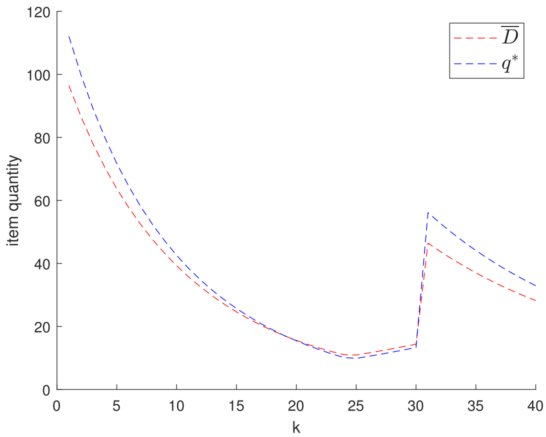


Fig. 12. Open-loop Equilibrium Order Quantities. Ex.6.

non-postponing channels facing the same realization of ϵ_ks are given below. In this scenario, too, both the retailer and the manufacturer benefit from price postponement.

$$\Pi^r_{OL} = 2307.13 < \Pi^r_{CL} = 3012.14 \quad \Pi^m_{OL} = 1289.51 < \Pi^m_{CL} = 1309.06$$

$$\Pi^c_{OL} = 3596.63 < \Pi^c_{CL} = 4321.21$$

8. Concluding remarks

Despite the ubiquity of its potential applications, multi-period supply channel optimization in the face of uncertain, time-varying, and price-dependent demand has received limited attention due to its high level of complexity. The complexity partly stems from the nestedness of the ensuing equilibria problems: in a time-dependent setting, a pricing decision at the present may affect the demand in the future. Therefore, in addition to reaching an equilibrium between themselves, the supply channel members must find a balance between immediate profit and reduced revenue in the future. A static (single-period) analysis inevitably cannot cover the effects of current pricing on future demand. Moreover, the interdependence of the decision variables – supply quantity and prices – with the behavior of strategic buyers adds another level of complexity to the problem.

In this paper, we have developed an analytical method for the dynamic problem of channel optimization in the face of time-varying and uncertain demand. In doing so, we have considered two types of settings referred to as the open-loop and closed-loop scenarios.

In the first part of the paper, covering the open-loop cases, we developed an explicit solution algorithm for the problem of finding Nash–Stackelberg equilibria without postponement strategies. In this part, the agents, facing an uncertain and time-varying demand, are risk-neutral as they try to maximize their expected profits within a given number of periods.

The second part of the paper is dedicated to solving postponed feedback Stackelberg games in multi-period (dynamic) frameworks. Inspired by the (single-period) analysis in [11], we allow the follower in the feedback Stackelberg setting to postpone one of her decisions (either supply quantity or retail price) until after demand uncertainty is resolved at each period. Thus, the results of this section are denoted as closed-loop (postponement) solutions. Analogously, in the second part, we propose solution algorithms that use the extra information obtained due to postponement in devising optimal supply and pricing strategies. To achieve this, we use the proven subgame perfection of the equilibria obtained in the first part. Finally, in a number of comparison theorems and simulated examples, we study the effect of each postponement strategy on individual profits and the overall channel efficiency.

Our findings indicate that postponement strategies are always beneficial for the retailer and for the centralized channel (whose revenue structure is identical to that of a retailer). However, for a decentralized channel, due to vertical competitions, there may be scenarios wherein postponement strategies, i.e., access to extra information, turn out to be detrimental to the manufacturer (the leader in the Stackelberg setting.) Monte Carlo simulations of several archetypal scenarios demonstrate cases wherein postponement strategies are detrimental to the manufacturer while the overall channel may still benefit from postponing operational decisions. As demonstrated by our simulations, the occurrence of such scenarios depends on the structure of the contract between the two supply channel members as well as the realization of the uncertain demand. Our simulations illustrate the performance of each channel member with and without implementing postponement, providing valuable insights into the benefits and trade-offs of postponement strategies in practice. Our analytical solution algorithms are presented in several constructive theorems. Thus, not only do they prove the existence of equilibria in a wide variety of scenarios but provide programmable instructions for simulating those scenarios. As illustrated in the examples, these simulations can be used for further comparative studies.

CRediT authorship contribution statement

Reza Azad Gholami: Conceptualization, Methodology, Writing – original draft, Writing – review & editing, Software, Data curation, Visualization. **Leif Kristoffer Sandal:** Conceptualization, Methodology, Supervision. **Jan Ubøe:** Conceptualization, Methodology, Supervision.

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

Appendix A. A general solution to the bilevel optimization problem presented in (4)

In the procedure outlined below, we calculate the expected values of the profits for the retailer and the manufacturer, $\overline{\pi}^r$ and $\overline{\pi}^m$ respectively in (50) and (54). Each channel member tries to maximize her expected profit by finding the optimal decision variables. Fortunately, the problem of optimizing $\overline{\pi}^r$ with respect to q is convex and has a closed-form solution (52). However, the problem of optimizing $\overline{\pi}^r$ with respect to r and $\overline{\pi}^m$ with respect to w should generally be solved using numerical methods.

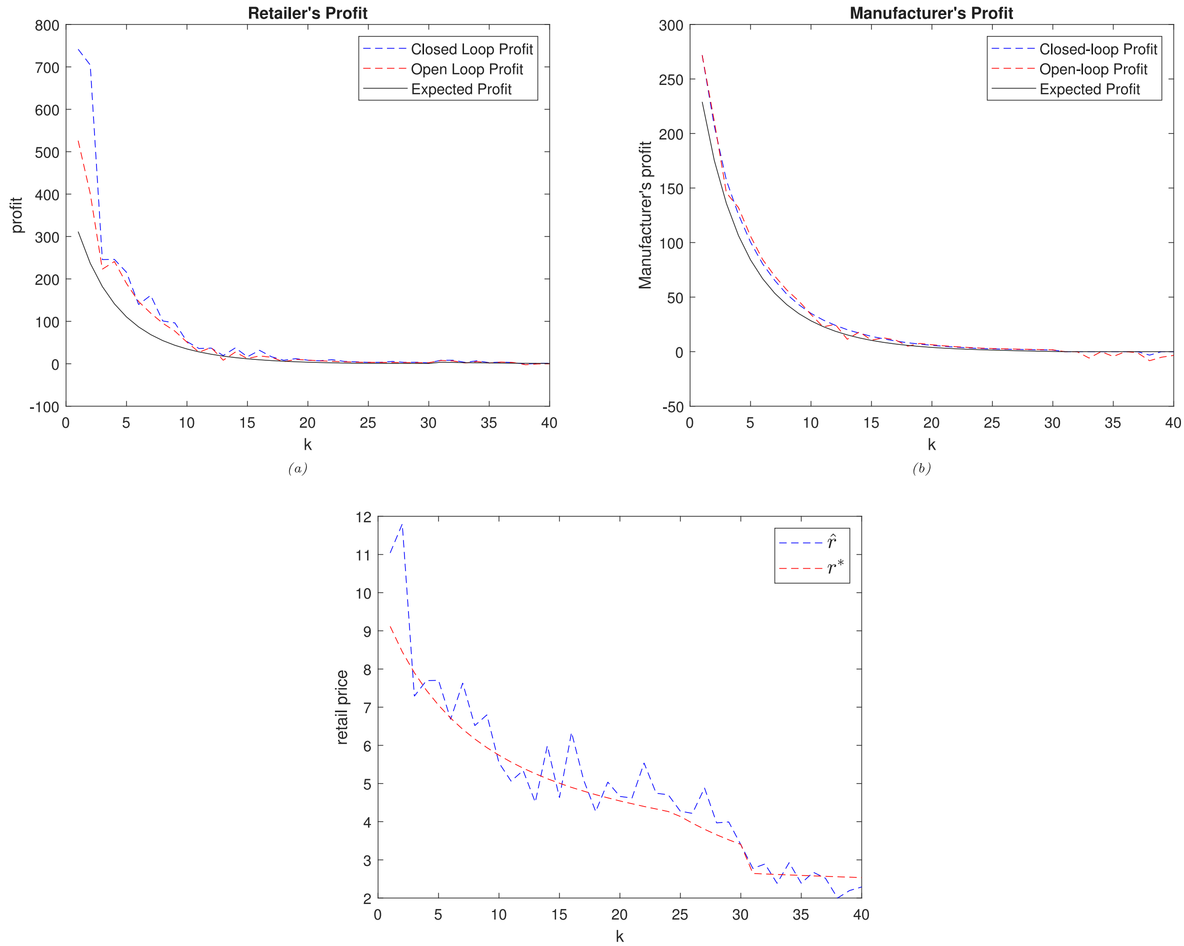
In order to obtain the expected value of the retailer's profit, we need to calculate $\mathrm{E}\big[\min(D,q)\big]$. Given f_ϵ, F_ϵ, and $\underline{\epsilon}$ we define and calculate the expected sales, $\mathcal{S}$, as follows.

$$\begin{aligned} \mathcal{S}(q) &:= \mathrm{E}\big[\min(D,q)\big] = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \min(\mu+\sigma t, q)\, f_\epsilon(t)\, dt \\ &= \int_{\underline{\epsilon}}^{\frac{q-\mu}{\sigma}} (\mu+\sigma t)\, f_\epsilon(t)\, dt + \int_{\frac{q-\mu}{\sigma}}^{\overline{\epsilon}} q\, f_\epsilon(t)\, dt \\ &= q - (q-\mu) F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big) + \sigma \int_{\underline{\epsilon}}^{\frac{q-\mu}{\sigma}} t f_\epsilon(t) dt \end{aligned} \tag{48}$$

$$\frac{\partial \mathcal{S}(q)}{\partial q} = 1 - F_\epsilon\Big(\frac{q-\mu}{\sigma}\Big) \tag{49}$$

From (48), we obtain the expected value of the retailer's profit $\overline{\pi}^r$.

$$\overline{\pi}^r(r,w,q) := \mathrm{E}[\pi^r(r,w,q)] = (r-s-b)\,\mathcal{S}(q) + (b+s-c_r-w)\,q \tag{50}$$



(a) (b)

(c) Closed-loop equilibrium prices ($\hat{r}_k$s) drop after $k = 31$

Fig. 13. Open-loop (No postponement) and Closed-loop (Price-Postponement) Equilibrium States (Example 6, Section 7.3.3).

The optimal order quantity for the retailer, q^* as a function of r and w is now obtained as below.

$$\frac{\partial \overline{\pi}^r}{\partial q} = (r - s - b)\left(1 - F_\epsilon\left(\frac{q-\mu}{\sigma}\right)\right) + (b + s - c_r - w) = 0 \tag{51}$$

From the expressions in (49) and (50) it is readily observable that $\mathrm{E}[\pi^r(r, w, q)]$ is convex with respect to q; therefore, solving (51) yields $q^*(r, w)$ as the argmax of the retailer's expected profit.

$$q^*(r, w) = \mu(r) + \sigma(r)\, F_\epsilon^{-1}\left(\frac{r - w - c_r}{r - s - b}\right) \tag{52}$$

Substituting (52) in (48) and the result in (50), we obtain the following.

$$\overline{\pi}^r(r, w) = (r - w - c_r)\,\mu(r) + (r - s - b)\,\sigma(r)\int_{\underline{\epsilon}}^{z} t f_\epsilon(t)dt \quad \text{where} \quad z(r, w) = F_\epsilon^{-1}\left(\frac{r - w - c_r}{r - s - b}\right) \tag{53}$$

Note that because $\underline{\epsilon} < z < \delta$, the term $\int_{\underline{\epsilon}}^{z} t f_\epsilon(t)dt$ is always negative, which in turn makes $(r - s - b)\sigma(r)\int_{\underline{\epsilon}}^{z} t f_\epsilon(t)dt$ also negative. This means that stochasticity in demand always reduces the expected profit for the retailer.

Following the procedure outlined in (4) a numerical solution to $\max_r \overline{\pi}^r(r, w)$ in (53) yields $r^*(w)$ which is in turn substituted in the expression for the manufacturer's expected profit (55).

$$\pi^m = (w - c_m)q - b(q - D)^+ = (w - c_m - b)q + b\min(D, q) \tag{54}$$

$$\overline{\pi}^m(w) = \mu\left(r^*(w)\right)\left(w - c_m\right) + \sigma\left(r^*(w)\right)\left[\, z^*(w)\left(w - c_m - \frac{r^* - w - c_r}{r^* - s - b}\right) + b\int_{\underline{\epsilon}}^{z^*} t f_\epsilon(t)dt \,\right] \quad \text{where} \quad z^*(w) = F_\epsilon^{-1}\left(\frac{r^* - w - c_r}{r^* - s - b}\right) \tag{55}$$

Appendix B. Theorem 4.1

Proof. The memory-based expression for demand at each period D_k is given by (12). Due to linearity of the expressions for $\widetilde{\pi}_k^r$ and $\widetilde{\pi}_k^m$ with respect to D in the single-period case, for the kth period, the resulting expressions for the order quantity and the expected values of the profits will be as below.

$$D_k(k, \mathbf{r}_k) = \tilde{\mu}_k(\mathbf{r}_k) + \tilde{\sigma}_k(\mathbf{r}_k)\,\epsilon_k = \boldsymbol{\Phi}_k(\mathbf{r}_{k-1})\left[\mu_k(r_k, k) + \sigma_k(r_k, k)\,\epsilon_k\right] \tag{56}$$

$$\begin{aligned}\mathrm{E}[\widetilde{\pi}_k^r] &= (r_k - w_k - c_{r_k})\,\widetilde{\mu}_k(\mathbf{r}_k) + (r_k - s_k - b_k)\,\widetilde{\sigma}_k(\mathbf{r}_k)\int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t)dt \\ &= \overbrace{\left[(r_k - w_k - c_{r_k})\mu_k(r_k) + (r_k - s_k - b_k)\,\sigma_k(r_k)\int_{\underline{\epsilon}_k}^{z_k} t f_\epsilon(t)dt\right]}^{:=\overline{\pi}_k^r} \cdot \boldsymbol{\Phi}_k(\mathbf{r}_{k-1}) \\ &\text{where} \quad z_k(r_k, w_k) = F_\epsilon^{-1}\left(\frac{r_k - w_k - c_{r_k}}{r_k - s_k - b_k}\right)\end{aligned} \tag{57}$$

We refer to $\overline{\pi}_k^r$ as scaled expected profit for the retailer at k. Thus (57) can be simplified as below.

$$\mathrm{E}[\widetilde{\pi}^r_k] = \overline{\pi}^r_k \cdot \boldsymbol{\Phi}_k(\mathbf{r}_{k-1}) \tag{58}$$

The manufacturer's local-in-time expected profit is calculated as below.

$$\begin{aligned}\mathrm{E}[\widetilde{\pi}^m_k] = \Bigg[&\mu_k\left(r^*_k(w_k)\right)\left(w_k - c_{m_k}\right) \\ +\sigma_k\left(r^*_k(w_k)\right)\Bigg(&z^*_k(w_k)\Big(w_k - c_{m_k} - \frac{r^*_k - w_k - c_{r_k}}{r^*_k - s_k - b_k}\Big) \\ &+b_k \int_{\underline{\epsilon}_k}^{z^*_k} t f_\epsilon(t)dt \Bigg)\Bigg] \cdot \boldsymbol{\Phi}_k(\mathbf{r}^*_{k-1}) \\ \text{where} \quad &z^*_k(w) = F^{-1}_\epsilon\Big(\frac{r^*_k - w_k - c_{r_k}}{r^*_k - s_k - b_k}\Big)\end{aligned} \tag{59}$$

Analogous to the single-period case, the numerical value for the optimal order quantity is then obtained from the following expression.

$$q^*_k = \boldsymbol{\Phi}_k(\mathbf{r}^*_{k-1})\left[\mu_k(r^*_k) + \sigma_k(r^*_k)\, F^{-1}_{\epsilon_k}\Big(\frac{r^*_k - w^*_k - c_{r_k}}{r^*_k}\Big)\right] \tag{60}$$

Similarly we refer to the term inside the brackets in as the scaled expected profit for the manufacturer at k and denote it by $\overline{\pi}^r_m$. Whence is simplified as below.

$$\mathrm{E}[\widetilde{\pi}^m_k] = \overline{\pi}^m_k \cdot \boldsymbol{\Phi}_k(\mathbf{r}^*_{k-1}) \tag{61}$$

In general, the argmax of the expected profit in a specific period k for either supplier, i.e. the result of $\max_{r_k,m_k} \mathrm{E}[\widetilde{\pi}^{r,m}_k]$ is not equal to the value of the kth optimal decision variable for that supplier when the objective function is the whole expected profit within the periods 1 to n. In other words, in general,

$$\max_{r_k,m_k} \mathrm{E}[\widetilde{\pi}^{r,m}_k] \not\equiv \max_{r_k,m_k} \overline{\Pi}^{r,m}. \tag{62}$$

We refer to the results of the RHS of (62) as myopic solutions and to those of its LHS as the holistic ones. Our objective is to find the vectors of the latter — those decision variables which, considering the effect of the pricing in the past on current and future demand, manipulate the demand such that they yield highest amounts of expected profits for each decision maker over the whole time interval between 1 and n.

To that end, we begin by analyzing the retailer's optimization problem and re-write the general optimization problem in (7) using the results of (58).

$$\begin{aligned}\max_{\mathbf{r}_n} \overline{\Pi}^r &= \overline{\pi}^r_1(r_1, w_1, q_1) + \cdots + \alpha_k \boldsymbol{\Phi}_k(\mathbf{r}_{k-1})\,\overline{\pi}^r_k(r_k, w_k, q_k) \\ &+ \cdots + \alpha_n \boldsymbol{\Phi}_n(\mathbf{r}_{n-1})\,\overline{\pi}^r_n(r_n, w_n, q_n)\end{aligned} \tag{63}$$

Analogous to the approach adopted in Section 4.3, we observe that the variable r_n appears only in the final discounted profit term — more precisely in $\overline{\pi}^r_n$. Thus following the backward induction process, we begin the optimization from the final period.

$$\max_{r_n} \overline{\Pi}^r(\mathbf{r}_n) \equiv \max_{r_n} \overline{\pi}^r_n(r_n) \tag{64}$$

At each period k we define J^r_k as the discounted expected value of the profit obtained from that period onward, i.e. within the time interval $\{k, \dots, n\}$.

$$J^r_k = \alpha_k \boldsymbol{\Phi}_k(\mathbf{r}_{k-1})\overline{\pi}^r_k(r_k) + \cdots + \alpha_n \boldsymbol{\Phi}_n(\mathbf{r}_{n-1})\overline{\pi}^r_n(r_n) \tag{65}$$

Notice that $J^r_1 = \overline{\Pi}^r$. We also observe that in this structure, beginning from the last period, the variable r_k in Π^r appears for the first time in the expression for J^r_k. Having solved the RHS of (64) we obtain r^*_n and proceed to the previous period $n-1$. Knowing r^*_n means that in the holistic optimization problem (63) the unknown variable r_{n-1} appears only in the two final terms for the expected profit. This is stated below.

$$\begin{aligned}J^r_{n-1}(\mathbf{r}_{n-1}) &= \alpha_{n-1}\boldsymbol{\Phi}_{n-1}(\mathbf{r}_{n-2})\overline{\pi}^r_{n-1}(r_{n-1}) + \alpha_n \boldsymbol{\Phi}_n(\mathbf{r}_{n-1})\overline{\pi}^r_n(r^*_n) \\ &= \boldsymbol{\Phi}_{n-1}(\mathbf{r}_{n-2}) \overbrace{\left[\overline{\pi}^r_{n-1}(r_{n-1}) + \frac{\alpha_n}{\alpha_{n-1}}\phi_n(r_{n-1})\underbrace{\overline{\pi}^r_n(r^*_n)}_{\text{given}}\right]}^{:=\mathfrak{J}^r_{n-1}:\ \text{a function of } r_{n-1} \text{ only}}\end{aligned} \tag{66}$$

The multiplier effect in (66) is the crucial observation in this paper as it demonstrates how the problem of finding the optimal r^*_{n-1} boils down to the following single-variable optimization problem.

$$\max_{r_{n-1}} \overline{\Pi}^r(\mathbf{r}_{n-1}) \equiv \max_{r_{n-1}} J^r_{n-1}(\mathbf{r}_{n-1}) \equiv \max_{r_{n-1}} \mathfrak{J}^r_{n-1}(r_{n-1}) \tag{67}$$

Going backward in time, we can generalize this procedure as shown in (68), given that $\alpha_1 = 1$ and $\boldsymbol{\Phi}_1(\cdot) = 1$.

$$\begin{aligned}&J^r_k := \alpha_k \underbrace{\boldsymbol{\Phi}_k(\mathbf{r}_{k-1})}_{\text{price history}} \\ &\overbrace{\left(\overline{\pi}^r_k(r_k) + \phi_{k+1}(r_k)\underbrace{\left[\frac{\alpha_{k+1}}{\alpha_k}\,\overline{\pi}^r_{k+1}(r^*_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k}\,\overline{\pi}^r_n(r^*_n)\prod_{i=k+2}^{n}\phi_i(r^*_{i-1})\right]}_{:=\mathcal{F}^r_k=\ \text{expected (future) values, given at } k\text{th period}}\right)}^{:=\mathfrak{J}^r_k(r_k)} \\ &\max_{r_k} J^r_k\end{aligned} \tag{68}$$

Below, we define $\mathcal{F}^r_k$ as the scaled expected future profit within $\{k+1, \dots, n\}$, and $\mathfrak{J}^r_k$ as the scaled expected profit within $\{k, \dots, n\}$.

$$\mathcal{F}^r_k := \frac{1}{\alpha_k}\sum_{j=k+1}^{n}\prod_{i=k+2}^{j}\phi_i(r^*_{i-1})\cdot \alpha_j \overline{\pi}^r_j(r^*_j) \tag{69}$$

$$\mathfrak{J}^r_k(r_k) := \overline{\pi}^r_k + \phi_{k+1}(r_k)\mathcal{F}^r_k \tag{70}$$

As it is demonstrated in (68), when the backward induction process reaches the kth period, the scaled profit expected to gain in the future denoted by $\mathcal{F}^r_k$ has been determined and is treated as a constant. We also observe the following relationship between $\mathfrak{J}^r_{k+1}$ and $\mathcal{F}^r_k$.

$$\mathfrak{J}^r_{k+1}(r^*_{k+1}) = \frac{\alpha_k}{\alpha_{k+1}}\mathcal{F}^r_k \quad 1 \le k < n \tag{71}$$

Note that, unlike $\mathcal{F}^r_k$ and $\mathfrak{J}^r_{k+1}$, J^r_{k+1} includes the entire pricing history $\boldsymbol{\Phi}_k(\mathbf{r}_{k-1})$ and hence is not known at k. In fact, J^r_ks are not resolved until the backward induction reaches $k = 1$. The effect of the past represented by $\boldsymbol{\Phi}_k(\mathbf{r}_{k-1})$, though not yet determined by backward induction, is factorized in (68) such that it only scales the expected profit from k onward. Therefore, we will have:

$$\max_{r_k} \overline{\Pi}^r(\mathbf{r}_n) \equiv \max_{r_k} J^r_k(\mathbf{r}_k) \equiv \max_{r_k} \mathfrak{J}^r_k(r_k) \tag{72}$$

Combining (68) and (71) we can summarize the retailer's part of the multi-period bilevel optimization in the following recursive procedure.

$$\begin{aligned}&\mathcal{F}^r_n = 0 \quad \text{no future earning after } n \\ &\max_{r_k} \mathfrak{J}^r_k(r_k) = \max_{r_k}\left[\overline{\pi}^r_k(r_k) + \phi_{k+1}(r_k)\mathcal{F}^r_k\right] \quad k = n, \dots, 1 \text{ (backward)} \ \rightarrow \text{ yields } r^*_k \\ &\mathcal{F}^r_{k-1} = \frac{\alpha_{k-1}}{\alpha_k}\,\mathfrak{J}^r_k(r^*_k) \quad k = n, \dots, 2 \text{ (backward)}\end{aligned} \tag{73}$$

From the procedure outlined in (73) it is readily observable that, in general, the holistic optimal retail prices (r^*_ks) are not the optimizers of individual $\overline{\pi}^r_k$s. The only situation where $r_k = argmax(\overline{\pi}^r_k)$ is when $\phi_{k+1} = C_k$, where C_k is a constant. A scenario in which all the memory elements are constants, will create identical repeated games at different periods.

The same structure is employed to decouple the nested optimization problems of the manufacturer. Notice that as in the single-period case in (4), each r_k^* is obtained as a function of manufacturing price at k, i.e. $r_k^* = r_k^*(w_k)$.

$$\max_{\mathbf{w}_n} \overline{\Pi}^m(\mathbf{w}_n) = \max_{\mathbf{w}_n} \sum_{k=1}^{n} \alpha_k \Phi_k(\mathbf{r}_k^*) \overline{\pi}_k^m(w_k) \tag{74}$$

$$J_k^m(\mathbf{w}_k) = \sum_{i=k}^{n} \alpha_i \Phi_i(\mathbf{r}_i^*) \overline{\pi}_i^m(\mathbf{w}_i) \tag{75}$$

$$\max_{w_k} J_k^m(\mathbf{w}_k) = \alpha_k \Phi_k(\mathbf{r}_{k-1}^*) \overbrace{\left[\overline{\pi}_k^m(w_k) + \phi_{k+1}\left(r_k^*(w_k)\right)\mathcal{F}_k^m\right]}^{\mathfrak{J}_k^m(w_k)} \tag{76}$$

Where $\mathcal{F}_k^m$ in (76) is the scaled expected value of future (time interval within $\{k+1,\dots,n\}$) discounted profit. By the time the backward induction process reaches the kth period, $\mathcal{F}_k^m$ has already been calculated. This makes $\mathfrak{J}_k^m$ a function of only w_k.

$$\mathcal{F}_k^m = \frac{\alpha_{k+1}}{\alpha_k} \overline{\pi}_{k+1}^m(w_{k+1}) + \cdots + \frac{\alpha_n}{\alpha_k} \overline{\pi}_n^m(w_n) \prod_{i=k+2}^{n} \phi_i(r_{i-1}^*)$$
$$\mathcal{F}_n^m = 0 \tag{77}$$

Finally, we can decouple the nested $n-$variable optimization problem into n single variable optimization problems.

$$\max_{w_k} \overline{\Pi}^m(\mathbf{w}_n) \equiv \max_{w_k} J_k^m(\mathbf{w}_k) \equiv \max_{w_k} \mathfrak{J}_k^m(w_k) \tag{78}$$

Analogous to the retailer's case, the manufacturer's part of the multi-period bilevel optimization is outlined in the following recursive procedure.

$$\mathcal{F}_n^m = 0 \quad \text{no future earning after } n$$
$$\max_{w_k} \mathfrak{J}_k^m(w_k) = \max_{w_k}\left[\overline{\pi}_k^r(w_k) + \phi_{k+1}\left(r_k(w_k)\right)\mathcal{F}_k^m\right] \quad k=n,\dots,1 \rightarrow \text{ yields } w_k^* \quad \square$$
$$\mathcal{F}_{k-1}^m = \frac{\alpha_k}{\alpha_{k-1}} \mathfrak{J}_k^m(w_k^*) \quad k=n,\dots,2. \tag{79}$$

B.1. Theorem 4.2

Proof (*By Induction*). We need to prove that if $\{r_j^*,\dots,r_n^*\}$ and $\{w_j^*,\dots,w_n^*\}$ are subsets of the equilibrium results for $[\overline{\Pi}^r, \overline{\Pi}^m, 1:n]$, then they also constitute an equilibrium for $[J_j^r, J_j^m, j:n]$.

Beginning from the final period, we analyze the two agents' equilibrium problem. In the expressions for both J_k^r and $\overline{\Pi}^r$ the variable r_n appears in $\overline{\pi}_n^r(r_n)$ only. The same logic is applicable to the manufacturer's solution procedure.

$$\max_{r_n} J_k^r \equiv \max_{r_n} \overline{\pi}_n^r \equiv \max_{r_n} \Pi^r$$

$$\max_{w_n} J_k^m \equiv \max_{w_n} \overline{\pi}_n^m \equiv \max_{w_n} \Pi^m$$

Thus, at n the conclusion is obvious. The rest of the proof for an arbitrary k, $j<k<n$ has been explained in detail within the discussion resulting in (72) and (78). □

Appendix C. Theorem 5.1

Proof. We have to show that $\pi_{CL_k}^r \geq \pi_{OL_k}^r$.

$$\pi_{CL_k}^r = r_k^* D_k - w_k^* \hat{q}_k \overset{(D_k=\hat{q}_k)}{=} (r_k^* - w_k^*) D_k$$

$$\pi_{OL_k}^r = r_k^* \min\Big(\overbrace{\Phi_k(\mathbf{r}_{k-1}^*)\left[\mu_k(r_k^*) + \sigma_k(r_k^*)\,\hat{\epsilon}_k\right]}^{D_k}, q_k^*\Big) - w_k q_k^*$$
$$= r^* \min(D_k, q_k^*) - w_k^* q_k^*$$

Comparing with a no-postponement scenario, we have to analyze two possible situations.

1 - When the retailer under-orders: $q_k^* < D_k$, $\min(D_k, q_k^*) = q_k^*$
Then $\pi_{OL_k}^r = (r_k^* - w_k^*) q_k^* < (r_k^* - w_k^*) D_k = \pi_{CL_k}^r$

2- When the retailer over-orders: $q_k^* > D_k$, $\min(D_k, q_k^*) = D_k^*$

$$\pi_{OL_k}^r = r_k^* D_k - w_k^* q_k^* < r_k^* D_k - w_k^* D_k = \pi_{CL_k}^r$$

Note that in the proof above, we compared $\hat{q}_k = D_k$ with a general $q_k^* \neq \hat{q}_k$. We did not use the fact that $q_k^* = \mathrm{E}[D_k]$ which stems from the a priori knowledge of the decision-makers about an order-postponement taking place in the second step. The proof thus shows that for a given r_k^*, an order quantity equal to the resulting uncertain demand will outperform any other arbitrary order quantity, including the one prescribed by the open-loop solution. Equality, $\pi_{CL_k}^r = \pi_{OL_k}^r$, happens when the mean of demand is equal to the real demand, $q_k^* = \mathrm{E}[D_k] = D_k$. □

C.1. Corollary 5.3

Proof.

$$\Pi_{CL}^r = \sum_{k=1}^{n} \alpha_k \pi_{CL_k}^r \geq \sum_{k=1}^{n} \alpha_k \pi_{OL_k}^r = \Pi_{OL}^r \quad \square$$

C.2. Corollary 5.4

Proof. We denote the channel profit for the postponing channel at period k by $\pi_{CL_k}^c$ and for the non-postponing channel by $\pi_{OL_k}^c$.

$$\pi_{CL_k}^c = (r_k^* - c_{m_k}) D_k$$

$$\pi_{OL_k}^c = r_k^* \min(D_k, q_k^*) - c_{m_k} q_k^*$$

$$\pi_{CL_k}^c - \pi_{OL_k}^c = (r_k^* - c_{m_k}) D_k - r_k^* \min(D_k, q_k^*) + c_{m_k} q_k^*$$
$$\geq (r_k^* - c_{m_k})\big(D_k - \min(D_k, q_k^*)\big) \geq 0 \quad \square$$

C.3. Corollary 5.6

Proof. It suffices to show that the profit expression for centralized channels is identical to that of a retailer.

$$\pi_k^c = \pi_k^r + \pi_k^m = r \min(D_k, q_k) - c_{m_k} q_k$$

We observe that a centralized channel is equivalent to a retailer (newsvendor) who has to pay only a fixed manufacturing cost c_{m_k} at each period. Thus, the result of Theorem 5.1 is applicable to centralized channels. □

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Further reading

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