The Hardy Uncertainty Principle and Unique Continuation for the Schrödinger Equation

Torunn Stavland Jensen

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Departement of Mathematics, University of Bergen

Abstract

The Hardy Uncertainty Principle states that if both a function f and its Fourier transform $\hat{f}$ decay faster than the Gaussian function with a specific weight, then $f \equiv 0$. This result can be reformulated for solutions of the free Schrödinger equation, which implies a unique continuation result for this equation. In a series of work [5, 6, 7, 8] Escauriaza, Kenig, Ponce and Vega extended this result to the Schrödinger equation with potential and to the nonlinear Schrödinger equation, by the use of Carleman estimates. More precisely, the authors proved that if u is a solution to the Schrödinger equation with potential, which at two times has Gaussian decay, and given the right conditions on the potential, then $u \equiv 0$.

The formal arguments of the proof, relying on Carleman estimates, are based on calculus and convexity arguments. However, these computations are not straightforward to justify rigorously. In particular, we need to justify that $\|e^{\phi}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all time $0 \leqslant t \leqslant 1$, for a suitable weight function $\phi = \phi(x,t)$. This is not always true, even though u is in $L^2(\mathbb{R}^n)$ for all $0 \leqslant t \leqslant 1$, and $\|e^{\phi}u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\phi}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite. In this thesis, we will study the proof of the main result in [6], which provides a rigorous strategy to justify the use of the Carleman estimates.

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Contents

Introduction

The Schrödinger equation

In the early 1900s, it was discovered by experiments that electrons act as waves. The Schrödinger equation was developed by the Austrian physicist Erwin Schrödinger in 1926 to describe the time evolution of the wave function $u = u(x,t)$ of the electrons and other particles. The equation is given by

$$i\hbar\partial_t u(x,t) = -\frac{\hbar^2}{2m}\Delta u(x,t) + V(x,t)u(x,t),$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$ is the Laplacian operator and $V(x,t)$ is a physical potential which depends on the particle, like a magnetic field, or gravitational field, $\hbar$ denotes the Planck constant and m the mass of the particle. In particular, $|u(x,t)|^2$ describes a probability density function related to the position x at time t. For more details see for example [13].

If we renormalize the equation we get

$$\partial_t u = i(\Delta u + V(x,t)u). \tag{0.1}$$

In the case where $V(x,t) = 0$, the equation becomes

$$\partial_t u = i\Delta u,$$

which is known as the free Schrödinger equation. This is the case of a free particle, where the potential energy does not vary.

The equation

$$\partial_t u = i(\Delta u + F(u,\bar{u})), \tag{0.2}$$

is called the nonlinear Schrödinger equation (NLS). This equation has applications in several areas of physics, such as fiber optics [1], fluid dynamics [17], and quantum field theory [16]. In the case $F(u,\bar{u}) = |u|^2u$ the equation is called the cubic nonlinear Schrödinger equation.

Uncertainty Principles

A well-known principle in quantum physics is the Heisenberg uncertainty principle. It states that it is not possible to measure both the position and momentum of a particle simultaneously. It was established by W. Heisenberg in 1927. A short time after E. H. Kennard and H. Weyl gave a mathematical formulation of the principle (see for example [10]). It states that for $f \in L^2(\mathbb{R}^n)$, and any $x_0, \xi_0 \in \mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} |(x-x_0)f(x)|^2 dx\right)\left(\int_{\mathbb{R}^n} |(\xi-\xi_0)\hat{f}(\xi)|^2 d\xi\right) \geqslant \frac{n}{4}\|f\|^4_{L^2(\mathbb{R}^n)}.$$

Since $\widehat{e^{-a|x|^2}} = \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{\pi^2}{a}|\xi|^2}$ the equality holds only in the case where f is a specific multiple of the Gaussian function. Uncertainty principles in general tell us that a function f and its Fourier transform $\hat{f}$ cannot both be sharply localized. There are other types of uncertainty principles where the Gaussian function also plays an important role. In 1933 the English mathematician G. H. Hardy formulated precisely in [14] that both the function f and its Fourier transform, $\hat{f}$, cannot decay too fast. In fact, if they both decay faster than the Gaussian with a specific weight, then $f \equiv 0$. The principle is mathematically stated as follows.

If $f(x) = O(e^{-|x|^2/\beta^2})$, $\hat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $\alpha\beta < 4$, then $f \equiv 0$. Also, if $\alpha\beta = 4$ then f is a constant multiple of $e^{-|x|^2/\beta^2}$.

Short History on Unique Continuation and Carleman Estimates

Unique continuation for solutions of partial differential equations is about which properties the solutions have to satisfy in order to be zero in the whole domain. It arises from the study of harmonic functions, which are functions u satisfying the equation

$$\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^n,$$

where Ω is an open connected subset of $\mathbb{R}^n$. Suppose that u is harmonic and vanishes of infinite order at a given point $x \in \Omega$. Since harmonic functions are real analytic, $u \equiv 0$ in Ω. This property is called the "strong unique continuation property". In comparison, "the weak unique continuation property"is when $u|_B = 0$, where B is an open subset in Ω, implies that $u \equiv 0$ in Ω. It is clear that the strong unique continuation implies the weak unique continuation.

Harmonic functions are the simplest example of solutions of an elliptic PDE. A well-known result by Hadamard in the early 1900's, is that for any second-order elliptic PDE with real analytic coefficients, the solution will be real analytic. This implies that every such solution will satisfy the strong unique continuation property. However, if the coefficients are not analytic, this method cannot be applied anymore. T. Carleman introduced in 1939 [2] a new way of proving uniqueness results for elliptic PDEs with non-analytic coefficients. These methods are based on weighted L^2 estimates and require less regularity on the operators than having analytic coefficients. These techniques relying on the Carleman estimates have been very successful with several important applications to elliptic and parabolic PDEs. See for example [15], [18] and the references therein.

Unique Continuation and Hardy's Uncertainty Principle for the Schrödinger Equation

Another question that has been central in the work on unique continuation is how fast a solution can decay before it vanishes identically. Recall that Hardy's uncertainty principle said that if both the function f and the Fourier transform $\hat{f}$ decay too fast, then $f \equiv 0$. This result can be reformulated for the free Schrödinger equation.

Consider now the solution u of the free Schrödinger equation, given by

$$u(x,t) = e^{it\Delta}u_0 = \frac{e^{i|x|^2/4t}}{(2it)^{n/2}}(e^{i|\cdot|^2/4t}u_0)\widehat{}\,(x/2t),$$

so that we can relate the solution at any time t with the Fourier transform of the initial data u_0. By this relation, we can apply the Hardy uncertainty principle to the function $f = e^{i|x|^2/4}u_0$ and deduce the following unique continuation result:

If u is a solution of the free Schrödinger equation, $u(x,0) = O(e^{-|x|^2/\beta^2})$, $u(x,1) = O(e^{-|x|^2/\alpha^2})$ and $\alpha\beta < 4$, then $u \equiv 0$.

Because of the application of Hardy's uncertainty principle to the free Schrödinger equation, it is natural to wonder whether this principle also applies to solutions of the Schrödinger equation with a potential, and of the nonlinear Schrödinger equation. The original proof of the Hardy Uncertainty Principle is based on complex analysis techniques such as the Phragmén-Lindelöf Theorem. However, to be able to extend this result to the Schrödinger equation with potential and to the nonlinear Schrödinger equation, we need a proof that does not depend on analyticity.

In a series of works [5, 6, 7, 8] Escauriaza, Kenig, Ponce and Vega showed unique continuation results for the Schrödinger equation with potential and the nonlinear Schrödinger equation. In particular, they proved the following in [6].

Theorem 1 (EKPV)**.** Let u $\in C([0,1], L^2(\mathbb{R}^n))$ be a solution of the Schrödinger equation

$$\partial_t u = i(\Delta u + V(x,t)u)$$

in $\mathbb{R}^n \times [0,1]$, where V is bounded, and either $V(x,t) = V_1(x) + V_2(x,t)$ with V_1 real valued and

$$\sup_{[0,1]} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}}V_2(t)\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

or

$$\lim_{R\to\infty}\int_0^1 \|V(t)\|_{L^\infty(\mathbb{R}^n\setminus B_R)}dt = 0.$$

Then, if there exist constants $\alpha, \beta > 0$ such that $\alpha\beta < 2$ and $\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite, then $u \equiv 0$.

Remark. This result is an extension of the Hardy Uncertainty Principle for the free Schrödinger equation to the Schrödinger equation with potential. The condition on the coefficients, $\alpha\beta < 2$ was not sharp in [6], and a bit weaker than the one from Hardy's uncertainty principle. However, in [7] the result was improved to be as sharp as in the free case, with the condition $\alpha\beta \leqslant 4$.

As a consequence, we can apply the theorem to the NLS and deduce the following result.

Theorem 2 (EKPV)**.** Let u_1 and u_2 be $(C[0,1], H^k(\mathbb{R}^n))$ solutions of (0.2) with $k \in \mathbb{Z}^+,\ k > n/2,\ F : \mathbb{C}^2 \to \mathbb{C},\ F \in C^k$ and $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$. If there are positive constants α and β with $\alpha\beta < 2$ such that $\|e^{\frac{|x|^2}{\beta^2}}(u_1(0) - u_2(0))\|_{L^2(\mathbb{R}^n)}$, and $\|e^{\frac{|x|^2}{\alpha^2}}(u_1(1) - u_2(1))\|_{L^2(\mathbb{R}^n)}$ are finite. Then $u_1 \equiv u_2$.

The proof of Theorem 1 relies heavily on the Carleman estimates. The ideas of these methods are simple and are based on calculus and convexity arguments. The main problem is that these computations are only formal, and it is not so easy to justify them rigorously.

The goal of this Master thesis is to study the proof of Theorem 1 in [6] in detail. We explain the main steps of the proof below.

Outline of the proof of Theorem 1.

Step 1: the conformal/Appell transformation. The first step of the proof is to reduce the problem to the case where the parameters α and β are equal. This can be done with the conformal/Appell transformation. Instead of assuming that $\|e^{\frac{|x|^2}{\beta^2}}u_0\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite for $\alpha\beta < 2$, we can assume that $\|e^{\gamma|x|^2}u_0\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite for some $\gamma > \frac{1}{2}$.

Step 2: heuristic argument. Assume that u is a solution of the equation (0.1). We define $f = e^{\phi}u$ for some weight function $\phi = \phi_R$, to be chosen later, depending on a large parameter R. Moreover, we let $H(t) = \|f(t)\|^2_{L^2(\mathbb{R}^n)}$. We can then show that f satisfies the equation

$$\partial_t f = (\mathcal{S} + \mathcal{A})f,$$

for a symmetric operator $\mathcal{S}$ and a skew-symmetric operator $\mathcal{A}$, both depending on the weight function ϕ. Ideally, we would like to prove a log-convexity inequality for the function H, by computing $\frac{d^2}{dt^2}\log H(t)$. In particular, we want to choose ϕ such that

$$\frac{d^2}{dt^2}\log H(t) \geqslant -h(R,\gamma),$$

where h is a non-negative function depending on γ and ϕ_R. Then, after some computations,

$$\|u(1/2)\|_{L^2(B_{R\epsilon/4})} \leqslant H(0)^{1/2}H(1)^{1/2}e^{-\tilde{h}(R,\gamma)},\ \text{where } \tilde{h}(R,\gamma) \longrightarrow +\infty \text{ when } R \to \infty \text{ and } \gamma > \frac{1}{2}.$$

If we let $R \to \infty$, and $\gamma > 1/2$, the left hand side goes to $\|u(1/2)\|_{L^2(\mathbb{R}^n)}$, while the right hand side goes to 0 since $H(0)^{1/2} = \|e^{\gamma|x|^2}u_0\|_{L^2(\mathbb{R}^n)}$ and $H(1)^{1/2} = \|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite. This implies that $u(1/2) = 0$, and by the well-posedness theory for the Schrödinger equation, $u \equiv 0$.

We will begin by giving the full details for this formal argument for the case $V = 0$ in the second chapter.

However, to be able to rigorously justify this argument, we need to know that $\|e^{\phi}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all time $t \in [0,1]$ and for a suitable weight ϕ. This is not obvious in general, even though we know it is finite at two times 0 and 1. In [6], the authors therefore chose to follow a different path to prove the main result. Their argument still relies on the Carleman methods, but makes it easier to justify that $\|e^{\phi}u(t)\|_{L^2(\mathbb{R}^n)} < \infty$ for all times and some specific weight functions. Most of the work is therefore dedicated to proving this result rigorously.

In particular, we will show the following result from [6].

Theorem. Let u be a solution of the Schrödinger equation with potential (0.1) such that for some $\gamma \in \mathbb{R}$

$$\|e^{\gamma|x|^2}u_0\|_{L^2(\mathbb{R}^n)} \text{ and } \|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)} < \infty. \tag{0.3}$$

If V is a bounded potential such that $V(x,t) = V_1(x) + V_2(x,t)$, with V_1 real and

$$\sup_{t\in[0,1]} \|e^{\gamma|x|^2}V_2(t)\|_{L^\infty(\mathbb{R}^n)} < \infty$$

then $\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$ is logarithmically convex in $[0,1]$ and there is a constant $N(\gamma)$ such that for all $t \in [0,1]$

$$\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} \leqslant N\left(\|e^{\gamma|x|^2}u(0)\|^{1-t}_{L^2(\mathbb{R}^n)}\|e^{\gamma|x|^2}u(1)\|^{t}_{L^2(\mathbb{R}^n)}\right), \tag{0.4}$$

and

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant N\left(\|e^{\gamma|x|^2}u(0)\|_{L^1(\mathbb{R}^n)} + \|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}\right). \tag{0.5}$$

In particular,

$$\sup_{t\in[0,1]} \|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n)} < \infty$$

and

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla \tilde{u}\|_{L^2(\mathbb{R}^n\times[0,1])} < \infty$$

for all time $0 < t < 1$, which will be fundamental in the proof of Theorem 1.

By going back with the Appell transform, the result can be generalized to the case when $\alpha \neq \beta$.

Theorem 3 (EKPV)**.** Assume that $u \in C([0,1], L^2(\mathbb{R}^n))$ satisfies

$$\partial_t u = i(\Delta u + V(x,t)u) \quad \text{in } \mathbb{R}^n \times [0,1], \tag{0.6}$$

$V(x,t) = V_1(x) + V_2(x,t)$, where V_1 is real-valued, $\|V_1\|_{L^\infty(\mathbb{R}^n)} \leqslant M_1$ and $\sup_{[0,1]} \|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < \infty$. If there exists positive numbers α, β such that

$$\|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} < \infty \text{ and } \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)} < \infty,$$

then $\|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}} u(t)\|_{L^2(\mathbb{R}^n)}^{\alpha t+(1-t)\beta}$ is logarithmically convex in $[0,1]$ and there is a constant $N = N(\alpha, \beta)$ such that

$$\|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}} u(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{N(M_1+M_2+M_1^2+M_2^2)} \|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)}^{\frac{\beta(1-t)}{\alpha t+\beta(1-t)}} \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)}^{\frac{\alpha t}{\alpha t+\beta(1-t)}} \tag{0.7}$$

for all $t \in [0,1]$ and where $M_2 = \sup_{[0,1]} \|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} e^{2\sup_{[0,1]} \|Im V_2(t)\|_{L^\infty(\mathbb{R}^n)}}$. Moreover,

$$\begin{aligned} &\|\sqrt{t(1-t)} e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}} \nabla u\|_{L^2(\mathbb{R}^n)\times[0,1])} \\ &\qquad \leqslant N e^{N(M_1+M_2+M_1^2+M_2^2)} \left[\|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)} \right]. \end{aligned} \tag{0.8}$$

Remark. In Theorem 1, it was given two different choices of conditions on the potential V. The condition on the potential in Theorem 3 corresponds to the first condition in Theorem 1. In [6] the authors also proved a corresponding result to Theorem 3, but by using the other condition on the potential. See in particular Theorem 4 in Chapter 3.

Step 3: parabolic regularization, energy estimate. The main strategy to prove Theorem 3 is to perform a parabolic regularization on (0.1). We work on the equation

$$\partial_t u = (A + iB)(\Delta u + V(x,t)u + F(x,t)), \tag{0.9}$$

and prove similar results to Theorem 3 for $A > 0$.

We start by proving an energy estimate which shows that for a specific weight function $\phi(x,t) = a(t)|x|^2$, where $a(t) = \frac{\gamma A}{A+4\gamma(A^2+B^2)t}$,

$$e^{-M_T} \|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)T}} u(T)\|_{L^2(\mathbb{R}^n)} \tag{0.10}$$

$$\leqslant \|e^{\gamma|x|^2} u(0)\|_{L^2(\mathbb{R}^n)} + \sqrt{A^2+B^2} \|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)t}} F(t)\|_{L^1([0,T],L^2(\mathbb{R}^n))}. \tag{0.11}$$

Remark. For this result, we only need the weighted L^2 norm to be finite at one time($t = 0$ here). However, the weight we propagate for $t \geqslant 0$ is smaller than $e^{\gamma A|x|^2}$ and decreases with time.

We rigorously justify this argument by showing that

$$\|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)T}}u(T)\|_{L^2(\mathbb{R}^n)} < \infty.$$

through the use of a regularization argument with a cutoff function on the weight $a(t)|x|^2$.

Step 4: parabolic regularization, Carleman estimate. The next step is to use the energy estimate (0.10) to prove that for a solution u of (0.9), such that (0.3) is satisfied, we have for any time $0 \leqslant t \leqslant 1$,

$$\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} \leqslant N\|e^{\gamma|x|^2}u(0)\|_{L^2(\mathbb{R}^n)}^{1-t}\|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}^{t}. \tag{0.12}$$

This is a classical Carleman argument, and the main problem is to justify rigorously that $\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for $0 < t < 1$. The idea will be to slightly modify the weight $|x|^2$ by introducing

$$\phi_a(x) = \begin{cases} |x|^2 & |x| < 1 \\ \frac{2|x|^{2-a}-a}{2-a} & |x| \geqslant 1, \end{cases}$$

and define $\phi_{a,\rho} = \phi_a * \theta_\rho$, for a radial mollifier θ_ρ, such that at infinity, $e^{\gamma\phi_{a,\rho}}u$ does not grow faster than $e^{a(t)|x|^2}u$, where $a(t)$ is the weight in the energy estimate (0.10). Then we show that $\|e^{\phi_{a,\rho}}u(t)\|_{L^2(\mathbb{R}^n)} < \infty$, and we can rigorously justify the Carleman argument for this weight. Finally, we conclude the proof of (0.12) by letting a and ρ to 0.

We also need a similar result for $e^{\gamma|x|^2}\nabla u$. In particular, we show that

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant N[(1+M_1)\sup_{[0,1]}\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} + \sup_{[0,1]}\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}]. \tag{0.13}$$

In [6] the authors did not include a rigorous justification for this argument. This was not obvious to us. To be able to justify that $\|e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])}$ is finite, we needed to modify the proof of the energy estimate, so that for the same $a(t)$, and for all $0 \leqslant T \leqslant 1$

$$\|e^{a(T)|x|^2}u(T)\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla(e^{a(t)|x|^2}u)\|_{L^2(\mathbb{R}^n\times[0,T])}^2 + \|2a(t)|x|e^{a(t)|x|^2}u\|_{L^2(\mathbb{R}^n\times[0,T])}^2 < \infty.$$

Then, the rigorous justification of (0.13) follows by arguing as above in the justification of (0.12), relying on the same arguments as those used to prove the Carleman estimate (0.12).

Step 5: proof of Theorem 3. By the previous steps, we have proven the equivalent of estimates (0.4) and (0.5) for the regularized equation (0.9) with $A > 0$. To deduce the results for $A = 0$, we can consider the solution u_ϵ of the equation

$$\partial_t u = (\epsilon + i)(\Delta u + V(x,t)),$$

for $\epsilon > 0$. The previous results hold for u_ϵ. By using semigroup theory, we show that this solution converges to a solution u of the original Schrödinger equation when $\epsilon \to 0$.

Step 6: proof of Theorem 1. We start by proving a Carleman estimate for compactly supported functions in both space and time. In particular, for

$$\phi = \mu|x + Rt(1-t)e_1|^2 + (1+\epsilon)R^2 t(1-t)/16\mu, \epsilon > 0, \mu > 0, R > 0 \text{ and } g \in C_0^\infty(\mathbb{R}^{n+1}),$$

$$R\sqrt{\frac{\epsilon}{8\mu}}\|e^\phi g\|_{L^2(\mathbb{R}^{n+1})} \leqslant \|e^\phi(\partial_t - i\Delta)g\|_{L^2(\mathbb{R}^{n+1})}.$$

Then, we introduce the function $g(x,t) = \theta_M(x)\eta_R(t)u(x,t)$, where θ_M and η_R are compactly supported cutoff functions in space and time respectively, and satisfy $g = u$ in an open ball in $\mathbb{R}^{n+1}$. By applying the Carleman estimate, it follows that

$$\begin{aligned} R\|e^\phi g\|_{L^2(\mathbb{R}^n\times[0,1])} &\leqslant N_\epsilon R e^{\gamma/\epsilon} \sup_{t\in[0,1]} \|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n)} \\ &\quad + N_\epsilon \frac{1}{M} e^{\gamma R^2/\epsilon}\|e^{\gamma|x|^2}(|u| + |\nabla u|)\|_{L^2(\mathbb{R}^n\times[\frac{1}{2R},1-\frac{1}{2R}])}. \end{aligned}$$

Note that the quantity $\|e^{\gamma|x|^2}(|u| + |\nabla u|)\|_{L^2(\mathbb{R}^n\times[\frac{1}{2R},1-\frac{1}{2R}])}$ is finite thanks to Step 5, so by letting $M \to \infty$, the last term on the right-hand side goes to 0.

In $B_{\epsilon(1-\epsilon)^2\frac{R}{4}} \times [\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}]$, we can bound ϕ from below, such that

$$\phi(x,t) \geqslant \frac{R^2}{64}(4\mu^2(1-\epsilon)^6 - (1+\epsilon)^3) > 0,$$

which will imply that for some constants $N_1 = N_1(\epsilon, \gamma, u)$ and $N_2 = N_2(\epsilon, \gamma)$m

$$\|u\|_{L^2(B_{\epsilon(1-\epsilon)^2\frac{R}{4}}\times[\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}])} \leqslant N_1 e^{-N_2R^2}.$$

Integrating in time, and using the fact that for all $t \geqslant 0$,

$$N^{-1}\|u(0)\|^2_{L^2(\mathbb{R}^n)} \leqslant \|u(t)\|^2_{L^2(\mathbb{R}^n)} \leqslant N\|u(0)\|^2_{L^2(\mathbb{R}^n)},$$

show that

$$\|u(0)\|_{L^2(\mathbb{R}^n)} \leqslant N_{\gamma,\epsilon,V}e^{-C_{\gamma,\epsilon}R^2} + e^{-\gamma R^2/16}N_{\gamma,\epsilon,V} \longrightarrow 0, \text{ as } R \to \infty,$$

so that $u_0 = 0$, and hence $u \equiv 0$.

Step 7: what can go wrong if we do not justify rigorously the computations. To show how important it was to rigorously justify the computations, we exhibit an example presented in [6] of a formal Carleman argument for the free Schrödinger equation with an explicit weight function, which leads to a false statement.

Step 8: application to the NLS. By treating the nonlinear term $F(u, \bar{u})$ in (0.2) as a potential, we can show that the unique continuation result also holds for the NLS. For the sake of simplicity, we will show the result in the case of the cubic NLS, but the proof of the general case is similar.

The main goal of this thesis is to understand and explain the proofs of Theorem 1 and Theorem 3 in [6]. We have detailed all the computations and provided more details on several steps in the proof. In particular, to justify that $\|e^{\gamma|x|^2}\nabla u(t)\|_{L^2(\mathbb{R}^n)} < \infty$, we needed to slightly change the proof in the energy estimate (see Lemma 3.2).

Structure of the Thesis

In the first chapter, we state some basic preliminary results that will be useful in the rest of the thesis. We start Chapter 2 with a discussion of the Hardy uncertainty principle and its application to solutions of the free Schrödinger equation. We prove this result formally using the argument explained in Step 2. In Chapter 3, we use the parabolic regularization, the energy- and Carleman estimates to prove Theorem 3. Chapter 4 is devoted to the proof of Theorem 1. In Chapter 5, we give an example of what can go wrong when the computations are not rigorously justified, while in Chapter 6 prove Theorem 2 in the case of the cubic NLS. Finally, in the appendices, we do most of the long, technical computations. In Appendix A, we give more details on parabolic regularization, and in Appendix B, we justify computations to prove that the weighted L^2 norms are finite. Appendix C is devoted to semigroup theory: we recall and explain several fundamental results used in the thesis. In Appendix D we discuss solutions of an ODE appearing in the example in Chapter 5.

1 Preliminaries and Notation

1.1 Notation

- N or C will denote arbitrary positive constants, which can change from line to line. Sometimes we write N_γ, $C(\gamma,\epsilon)$, etc. for some parameters γ,ϵ, to specify that the constants may depend on the specific parameters. If the constant matters, we will define it properly.

- $L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, denotes the usual Lebesgue space, with norm

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}.$$

- $L^\infty(\mathbb{R}^n)$ denotes the space of essentially bounded functions with norm

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \inf\{C > 0 : |f(x)| < C \text{ for almost every } x \in \mathbb{R}^n\}.$$

- $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ denotes the usual Sobolev spaces. See also Definition 1.3.

- $< f, g >_H$ denotes the scalar product in the respective Hilbert space H.

- We define the Fourier transform

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx.$$

We now recall some basic results that will be important throughout the thesis. For more details, see for example [9], [11] and [19].

1.2 Convergence Theorems

The Monotone Convergence Theorem. Let $(f_n)_n$ be a sequence of non-negative measurable functions on X such that $f_n(x) \leqslant f_{n+1}(x)$ for all $n \geqslant 1$, $x \in X$ which converges pointwise to a function f. Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

The Dominated Convergence Theorem Let $(f_n)_n$ b a sequence of measurable functions on X and f measurable on X such that

i) $f_n \to f$ a.e. in X

ii) There exists a function $g \in L^1(X)$ not depending on n such that $|f_n| \leqslant g$ a.e. in X, for all $n \geqslant 1$, then

$$\lim_{n\to\infty}\int f_n d\mu = \int f d\mu.$$

Fatou's Lemma Let $(f_n)_n$ be a sequence of non-negative measurable functions on X. Then

i) $\displaystyle\int \liminf f_n d\mu \leqslant \liminf \int f_n d\mu$

ii) If $f_n \to f a.e.,$ then $\displaystyle\int f d\mu \leqslant \liminf \int f_n d\mu$

1.3 Some Important Inequalities

Young's Inequality with ϵ
For $a, b \geqslant 0, \epsilon > 0,$

$$ab < \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$$

Young's Inequality for Convolution
For $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ and for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}.$$

Grönwall's Lemma [12] Let y, ϕ and ψ be nonnegative, continuous functions on the line segment $[a, b]$. If $\forall\ t \in [a, b]$

$$y(t) \leqslant \phi(t) + \int_a^t \psi(s)y(s)ds,$$

then $\forall\ t \in [a, b]$

$$y(t) \leqslant \phi(t) + \int_a^t \phi(s)\psi(s)e^{\int_s^t \psi(u)du}ds. \tag{1.1}$$

1.4 Mollifiers

Definition 1.1. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqslant \theta \leqslant 1$ and $\int_{\mathbb{R}^n}\theta dx = 1$. For $\rho > 1$ define

$$\theta_\rho(x) = \rho^{-n}\theta(\frac{x}{\rho}).$$

If

$$\lim_{\rho\to 0}\theta_\rho(x) = \delta(x),$$

where $\delta(x)$ is the Dirac delta-distribution, we call θ a mollifier.

Definition 1.2. If $f \in L^1_{loc}(\mathbb{R}^n)$ we define

$$f_\rho(x) := f * \theta_\rho(x) : \int_{\mathbb{R}^n} f(y)\theta_\rho(x-y)dy.$$

Theorem 1.1.

(*i*) $f_\rho \in C^\infty(\mathbb{R}^n)$,

(*ii*) $f_\rho \to f$ *a.e.* as $\epsilon \to 0$,

(*iii*) *for* $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, then $\lim_{\rho \to 0} \|f_\rho - f\|_{L^p(\mathbb{R}^n)} = 0$,

(*iv*) *if* $f \in C(\mathbb{R}^n)$ *then* $f_\rho \to f$ uniformly on compact sets.

1.5 Sobolev Spaces

Definition 1.3. For $s \in \mathbb{R}$ we denote $H^s(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2}\hat{f} \in L^2(\mathbb{R}^n)\}$, and $\|f\|_{H^s(\mathbb{R}^n)} = \|(1 + |\xi|^2)^{s/2}\hat{f}\|_{L^2(\mathbb{R}^n)}$.

Observe that for $t < s$ we always have $H^s(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$. In particular for $s \geqslant 0$, $H^s(\mathbb{R}^n) \hookrightarrow H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.

Theorem 1.2. If s is a positive integer, then $H^s(\mathbb{R}^n)$ coincides with the space of functions $f \in L^2(\mathbb{R}^n)$ whose derivatives(in distribution sense) $\partial_x^\alpha f$ belong to $L^2(\mathbb{R}^n)$ for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \leqslant k$. In this case, the norms $\|f\|_{H^s(\mathbb{R}^n)}$ and $\sum_{|\alpha| \leqslant k} \|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)}$ are equivalent.

Theorem 1.3. (Sobolev Embedding Theorem) If $s > n/2 + k$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C^k_\infty(\mathbb{R}^n)$, the space of functions with k derivatives vanishing at infinity. In other words there exist a constant $c_s > 0$ such that $\|f\|_{C^k(\mathbb{R}^n)} \leqslant c_s \|f\|_{H^s(\mathbb{R}^n)}$.

Theorem 1.4. If $s > n/2$ then $H^s(\mathbb{R}^n)$ is an algebra with respect to the product of functions, so that if $f, g \in H^s(\mathbb{R}^n)$, then $fg \in H^s(\mathbb{R}^n)$, with

$$\|fg\|_{H^s(\mathbb{R}^n)} \leqslant C(n)\|f\|_{H^s(\mathbb{R}^n)}\|g\|_{H^s(\mathbb{R}^n)}.$$

1.6 Solutions of the Schrödinger Equation

We start by recalling some basic properties for the free Schrödinger equation(for more details, see for example [19]). Consider the initial value problem

$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t) \text{ in } [0,T] \times \mathbb{R}^n \\ u(x,0) = u_0. \end{cases} \tag{1.2}$$

Let f be a function and $\hat{f}$ its Fourier transform. Taking the Fourier transform of (1.2) the equation becomes

$$\begin{cases} \partial_t \hat{u}(\xi,t) = -i|\xi|^2 \hat{u}(\xi,t) \\ \hat{u}(x,0) = \hat{u_0}, \end{cases} \tag{1.3}$$

which is an ODE we can solve explicitly. Indeed, we get that

$$\hat{u}(\xi,t) = e^{-i|\xi|^2 t}\hat{u_0}(x).$$

By taking the inverse Fourier transform we get that

$$u(x,t) = (e^{-i|\xi|^2 t}\hat{u_0}(x))\check{} = \frac{e^{i|\cdot|^2/4t}}{(4\pi i t)^{n/2}} * u_0(x) := e^{it\Delta}u_0,$$

where $\{e^{i\Delta t}\}$ is a unitary group in L^2. (see appendix C). In particular, the following holds.

Proposition 1.1. For all $t \in \mathbb{R}$

(*i*) For all $t \in \mathbb{R}$ $e^{it\Delta} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is an isometry, i.e. for $u_0 \in L^2(\mathbb{R}^n)$ $\|e^{it\Delta}u_0\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}$.

(*ii*) $e^{it\Delta}e^{it'\Delta} = e^{i(t+t')\Delta}$, and $(e^{it\Delta})^{-1} = e^{-it\Delta}$.

(*iii*) $e^{i0\Delta} = 1$.

(*iv*) If $u_0 \in L^2(\mathbb{R}^n)$, then $e^{it\Delta}u_0 \in C(\mathbb{R}, L^2(\mathbb{R}^n))$.

Remark. Since $e^{it\Delta}$ is a unitary group, we can always translate the solution from starting at $t = 0$ to any $t = t_0$. In particular, we can write the solution $u(x,t) = e^{i(t+t_0)\Delta}u(t_0)$. This means that if $u(t_0) = 0$ for some time t_0, $u \equiv 0$.

Let us now consider the Schrödinger equation with a potential

$$\begin{cases} \partial_t u = i(\Delta u + V(x,t)u) \\ u(x,0) = u_0. \end{cases}$$

In all of this work, we will consider the case where V is a bounded potential. In the case $V = 0$, the solution will be $u = e^{i\Delta t}u_0$, where $e^{i\Delta t}$ is a semigroup (see Appendix C). For the case $V(x,t) = V(x)$ is real, we have that $i(\Delta + V(x))$ generates a semigroup, and we have a well-defined solution $u(x,t) = e^{i(\Delta+V(x))t}u_0$.

If the potential depends both on x and t, we can also justify the solution with semigroup theory by using the Duhamel formula. All of this is explained in more detail in Appendix C.

We now present an energy estimate for solutions of the Schrödinger equation, which we will refer to several times in the thesis.

Lemma 1.1. Suppose $u \in C([0,1], L^2(\mathbb{R}^n))$ satisfies

$$\begin{cases} \partial_t u = i(\Delta u + V(x,t)u) \quad \text{in } \mathbb{R}^n \times [0,1] \\ u(x,0) = u_0, \end{cases}$$

then for $N = e^{\sup_{[0,1]} \|ImV(t)\|_{L^\infty(\mathbb{R}^n)}}$,

$$N^{-1}\|u(0)\|_{L^2(\mathbb{R}^n)} \leqslant \|u(t)\|_{L^2(\mathbb{R}^n)} \leqslant N\|u(0)\|_{L^2(\mathbb{R}^n)}. \tag{1.4}$$

Proof. We start with a formal argument.

$$\begin{aligned}
\partial_t \|u\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \partial_t (u\bar{u}) dx \\
&= 2Re \int_{\mathbb{R}^n} \partial_t u(\bar{u}) dx \\
&= 2Re\ i \int_{\mathbb{R}^n} \Delta u \bar{u} dx + 2Re\ i \int_{\mathbb{R}^n} V(x,t) u\bar{u} dx \\
&= -2Im \int_{\mathbb{R}^n} \Delta u \bar{u} dx - 2Im \int_{\mathbb{R}^n} V(x,t)|u|^2 dx.
\end{aligned}$$

We do the two parts separately. Integration by parts formally shows that

$$-2Im \int_{\mathbb{R}^n} \Delta u \bar{u} dx = 2Im \int_{\mathbb{R}^n} \nabla u \cdot \nabla \bar{u} dx = 2Im \int_{\mathbb{R}^n} |\nabla u|^2 = 0.$$

For the second part,

$$-2Im \int_{\mathbb{R}^n} V|u|^2 dx \leqslant 2\|ImV\|_{L^\infty(\mathbb{R}^n)} \|u\|^2_{L^2(\mathbb{R}^n)}.$$

Hence,

$$\partial_t \|u(t)\|^2_{L^2(\mathbb{R}^n)} \leqslant 2\|ImV\|_{L^\infty(\mathbb{R}^n)} \|u(t)\|^2_{L^2(\mathbb{R}^n)},$$

and

$$\partial_t \|u(t)\|_{L^2(\mathbb{R}^n)} \leqslant \sup_{[0,1]} \|ImV\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}.$$

Moreover,

$$\partial_t (\|u(t)\|_{L^2} e^{-\sup_{[0,1]} \|ImV\|_{L^\infty} t}) \leqslant 0$$

and

$$\begin{aligned}
\|u(t)\|_{L^2(\mathbb{R}^n)} &\leqslant \|u(0)\|_{L^2(\mathbb{R}^n)} e^{\sup_{t\in[0,1]} \|ImV\|_{L^\infty(\mathbb{R}^n)} t} \\
&\leqslant \|u(0)\|_{L^2(\mathbb{R}^n)} e^{\sup_{t\in[0,1]} \|ImV\|_{L^\infty(\mathbb{R}^n)}} \\
&= N\|u(0)\|_{L^2(\mathbb{R}^n)}, \qquad (1.5)
\end{aligned}$$

which proves the second inequality in the lemma. Now we will use this to prove the first inequality. Fix $t \in [0,1]$, then for all $s \in [0,t]$,

$$\begin{cases} \partial_s u = i(\Delta u + Vu) \\ u|_t = u(t). \end{cases}$$

Define $\tilde{u}(x,\tau) = u(x, t-s)$,

$$\begin{cases} \partial_\tau \tilde{u} = -i(\Delta \tilde{u} + V\tilde{u}). \\ \tilde{u}(x,0) = u(x,t). \end{cases}$$

We can now apply (1.5) to $\tilde{u}$, since the energy method will be the same, except for a minus sign. The first part with Δu will still disappear, for the second term with the potential V, the minus sign disappears in the L^∞-norm. Hence,

$$\|\tilde{u}(\tau)|_{L^2(\mathbb{R}^n)} \leqslant N\|\tilde{u}(0)\|_{L^2(\mathbb{R}^n)} \quad \text{for all } s \in [0,t],$$

$$\|u(t-s)|_{L^2(\mathbb{R}^n)} \leqslant N\|u(t)\|_{L^2(\mathbb{R}^n)} \quad \text{for all } s \in [0,t].$$

In particular, for $s = t$ we get

$$N^{-1}\|u(0)|_{L^2(\mathbb{R}^n)} \leqslant \|u(t)\|_{L^2(\mathbb{R}^n)}. \tag{1.6}$$

Combining (1.5) and (1.6), we get

$$N^{-1}\|u(0)\|_{L^2(\mathbb{R}^n)} \leqslant \|u(t)\|_{L^2} \leqslant N\|u(0)\|_{L^2(\mathbb{R}^n)}. \tag{1.7}$$

For $u_0 \in H^s(\mathbb{R}^n)$, $s > n/2+2$, we can rigorously justify the argument. By a density argument we prove it rigorously for $u_0 \in L^2(\mathbb{R}^n)$. Suppose that $u_0 \in L^2(\mathbb{R}^n)$. Then there is a sequence $\{u_0^k\} \in H^s(\mathbb{R}^n)$ such that $u_0^k \longrightarrow u_0$ in $L^2(\mathbb{R}^n)$ and $u^k \longrightarrow u$ uniformly in $L^2(\mathbb{R}^n)$ for all t. Indeed, since by the Duhamel formula

$$u(x,t) = e^{i\Delta t}u_0 + i\int_0^t e^{i(t-s)\Delta}V(s)u(s)ds,$$

it follows that

$$\begin{aligned}\|(u^k - u)(t)\|_{L^2(\mathbb{R}^n)} &\leqslant \|e^{i\Delta t}(u_0^k - u_0)\|_{L^2(\mathbb{R}^n)} + \|\int_0^t e^{i(t-s)\Delta}V(s)(u^k-u)(s)ds\|_{L^2(\mathbb{R}^n)} \\ &\leqslant \|(u_0^k - u_0)\|_{L^2(\mathbb{R}^n)} + \int_0^t \|V(s)(u^k-u)(s)\|_{L^2(\mathbb{R}^n)}ds.\end{aligned}$$

Applying Grönwall's Lemma, Lemma 1.1, we deduce that

$$\begin{aligned}\|(u^k - u)(t)\|_{L^2(\mathbb{R}^n)} &\leqslant \|u_0^k - u_0\|_{L^2(\mathbb{R}^n)}\left((1 + \int_0^t \|V(s)\|_{L^\infty(\mathbb{R}^n)}e^{\int_0^t \|V(u)\|_L^\infty(\mathbb{R}^n)du}ds\right) \\ &\leqslant \|u_0^k - u_0\|_{L^2(\mathbb{R}^n)}\left((1 + \int_0^1 \|V(s)\|_{L^\infty(\mathbb{R}^n)}e^{\int_0^1 \|V(u)\|_L^\infty(\mathbb{R}^n)du}ds\right) \\ &= N\|u_0^k - u_0\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

which goes to 0 as $k \to \infty$, and where N only depends on $\|V\|_{L^\infty(\mathbb{R}^n\times[0,1])}$

From (1.7) we have that

$$\|u^k\|_{L^2(\mathbb{R}^n)} \leqslant N\|u_0^k\|_{L^2(\mathbb{R}^n)},$$

and by letting $k \to \infty$ the result follows. □

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Remark.

(i) By arguing as above, we can obtain the same result also for any $\tau \in [0,1]$. In particular, for all $t \in [\tau, 1]$

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \leqslant N\|u(\tau)\|_{L^2(\mathbb{R}^n)}.$$

(ii) If $u_0 = 0$, then $u \equiv 0$.

2 Hardy's Uncertainty Principle and the Schrödinger Equation

2.1 Application to the Free Schrödinger Equation

In this section, we will discuss the Hardy uncertainty principle and its relations to the Schrödinger equation.

Let f be a function and

$$\hat{f}(\xi) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx$$

its Fourier transform. The Hardy Uncertainty Principle states the following.

Theorem 2.1. [Hardy's Uncertainty Principle] If $f(x) = O(e^{-|x|^2/\beta^2}), \hat{f}(\xi) = O(e^{-4|\xi|^2/\alpha^2})$ and $\alpha\beta < 4$, then $f \equiv 0$. Also, if $\alpha\beta = 4$ then f is a constant multiple of $e^{-|x|^2/\beta^2}$.

In the 1980's Cowling and Price proved a corresponding L^2 result in one dimension of the Hardy Uncertainty Principle [3].

Theorem 2.2. If $\|e^{|x|^2/\beta^2} f\|_{L^2(\mathbb{R})}$ and $\|e^{4|\xi|^2/\alpha^2} \hat{f}\|_{L^2(\mathbb{R})}$ are both finite and $\alpha\beta < 4$, then $f \equiv 0$.

The extension of this result to n dimensions has also been deduced using the Radon transform, see [23]. As we discussed in the introduction, this result has a natural application to the free Schrödinger equation. In particular, consider the solution, u, to the free Schrödinger equation

$$\begin{cases} \partial_t u = i\Delta u \\ u(x,0) = u_0. \end{cases} \tag{2.1}$$

The solution u can be written as

$$u(x,t) = e^{it\Delta}u_0(x) = (e^{-i|\xi|^2 t}\hat{u_0})^{\check{}} = \frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * u_0(x).$$

Writing out the convolution, we deduce that

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \frac{e^{i|x-y|^2/4t}}{(4\pi it)^{n/2}} u_0(y)dy \\ &= \frac{e^{i|x|^2/4t}}{(4\pi it)^{n/2}} \int_{\mathbb{R}^n} e^{-2ix\cdot y/4t} e^{i|y|^2/4t} u_0(y)dy \\ &= \frac{e^{i|x|^2/4t}}{(2it)^{n/2}} (e^{i|\cdot|^2/4t}u_0)\hat{\ }(x/2t), \end{aligned}$$

so that for any t we can write the solution in terms of the Fourier transform of the initial data u_0. In particular, for $t = 1$, we have

$$u(x,1) = \frac{e^{i|x|^2/4}}{(2i)^{n/2}}(e^{i|\cdot|^2/4}u_0)\widehat{\ }(x/2) = \frac{e^{i|x|^2/4}}{(2i)^{n/2}}\hat{f}(x/2) \tag{2.2}$$

where $f(x) = (e^{i|\cdot|^2/4}u_0)(x)$. Observe that

$$\|e^{\frac{|x|^2}{\beta^2}}f\|_{L^2(\mathbb{R}^n)} = \sqrt{\int_{\mathbb{R}^n}|e^{\frac{|x|^2}{\beta^2}}u_0|^2dx} = \|e^{\frac{|x|^2}{\beta^2}}u_0\|_{L^2(\mathbb{R}^n)},$$

and by (2.2)

$$\|e^{\frac{4|\xi|^2}{\alpha^2}}\hat{f}\|_{L^2(\mathbb{R}^n)} = \sqrt{\int_{\mathbb{R}^n}|e^{\frac{4|\xi|^2}{\alpha^2}}(e^{i|\cdot|^2/4}u_0)\widehat{\ }(\xi)|^2d\xi} = \sqrt{\int_{\mathbb{R}^n)}|e^{\frac{|\xi|^2}{\alpha^2}}u(\xi,1)|^2d\xi} = \|e^{\frac{|\xi|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}.$$

This, combined with the n-dimensional extension of Theorem 2.2, leads to the following result:

Theorem 2.3. Let u be a solution of the free Schrödinger equation (2.1). Suppose that $\|e^{|x|^2/\beta^2}u_0\|_{L^2(\mathbb{R}^n)}$ and $\|e^{|x|^2/\alpha^2}u(1)\|_{L^2(\mathbb{R}^n)}$ are both finite. If $\alpha\beta < 4$ then $u \equiv 0$.

This result tells us that if a solution u of (2.1) at two times decays faster than the Gaussian with a specific weight, then $u \equiv 0$. In particular, Hardy's uncertainty principle implies a unique continuation result for the free Schrödinger equation.

As we discussed in the introduction, this result was extended to the Schrödinger equation with potential and to the NLS in [6], and the proof is based on Carleman estimates. We state the result again.

Theorem 1 (EKPV)**.** Let u $\in C([0,1], L^2(\mathbb{R}^n))$ be a solution of the Schrödinger equation

$$\partial_t u = i(\Delta u + V(x,t)u)$$

in $\mathbb{R}^n \times [0,1]$, where V is bounded, and either $V(x,t) = V_1(x) + V_2(x,t)$ with V_1 real valued and

$$\sup_{[0,1]}\|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}}V_2(t)\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

or

$$\lim_{R\to\infty}\int_0^1\|V(t)\|_{L^\infty(\mathbb{R}^n\setminus B_R)}dt = 0.$$

Then, if there exist constants $\alpha, \beta > 0$ such that $\alpha\beta < 2$ and $\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite, then $u \equiv 0$.

We will start by proving this result formally, in the case where $V = 0$. This result corresponds to the Hardy uncertainty principle for the free Schrödinger equation and shows the formal idea of the Carleman estimates. However, these computations are only formal, and we will see later in the thesis that it is not straightforward to justify them rigorously.

Before we start the proof we will present and prove the conformal/Appell transform, which will help us reduce the proof to a simpler case.

Lemma 2.1. The Conformal/Appell Transformation
Let

$$\partial_s u = (A+iB)(\Delta u + V(y,s)u + F(y,s)) \qquad \text{in } \mathbb{R}^n \times [0,1].$$

If $A+iB \neq 0,\ \alpha, \beta > 0,\ \gamma \in \mathbb{R}$, and let

$$\tilde{u}(x,t) = \Big(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\Big)^{n/2} u\Big(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\Big) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}}.$$

Then $\tilde{u}$ satisfies

$$\partial_t \tilde{u} = (A+iB)(\Delta \tilde{u} + \tilde{V}(x,t)\tilde{u} + \tilde{F}(x,t)) \qquad \text{in} \mathbb{R}^n \times [0,1],$$

where

$$\tilde{V}(x,t) = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} V\Big(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\Big)$$

$$\tilde{F}(x,t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\right)^{n/2+2} F\Big(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\Big) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}}$$

Moreover,

$$\|e^{\gamma|x|^2}\tilde{F}(t)\|_{L^2(\mathbb{R}^n)} = \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} \|e^{[\frac{\gamma\alpha\beta}{(\alpha s+\beta(1-s))^2} + \frac{(\alpha-\beta)A}{4(A^2+B^2)(\alpha s+\beta(1-s))}]|x|^2} F(s)\|_{L^2(\mathbb{R}^n)}, \tag{2.3}$$

and

$$\|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)} = \|e^{[\frac{\gamma\alpha\beta}{(\alpha s+\beta(1-s))^2} + \frac{(\alpha-\beta)A}{4(A^2+B^2)(\alpha s+\beta(1-s))}]|x|^2} u(s)\|_{L^2(\mathbb{R}^n)} \tag{2.4}$$

for $s = \frac{\beta t}{\alpha(1-t)+\beta t}$.

Proof. Suppose u satisfies

$$\partial_s u - (A+iB)\Delta u = (A+iB)H(y,s)), \tag{2.5}$$

where $H(y,s) = V(y,s)u + F(y,s)$. For $y = \sqrt{r}x$, $s = rt+\tau$, define $u_1(x,t) = u(\sqrt{r}x, rt+\tau)$.

$$\partial_t u_1 = (A+iB)(\Delta u_1 + rH(\sqrt{r}x, rt+\tau)) \tag{2.6}$$

For $y = \frac{x}{t}$ and $s = \frac{1}{t}$, define $u_2(x,t) = t^{-n/2}u(\frac{x}{t}, \frac{1}{t})e^{\frac{|x|^2}{4(A+iB)t}}$. We will show that u_2 satisfies

$$\partial_t u_2 = -(A+iB)(\Delta u_2 + t^{-n/2-2}H\Big(\frac{x}{t}, \frac{1}{t}\Big))e^{\frac{|x|^2}{4(A+iB)t}}. \tag{2.7}$$

Using the Leibniz rule, it follows that

$$\begin{aligned}\partial_t u_2 &= -\frac{n}{2}t^{-1}u_2 + \nabla_y u_2 \partial_t y + \nabla_s u_2 \partial_t s + u_2 \frac{|x|^2}{4(A+iB)}(\frac{-1}{t^2}) \\ &= \left(\frac{-n}{2}t^{-1} - \frac{|x|^2}{4(A+iB)t^2}\right)u_2 - \nabla_y u_2 \frac{x}{t^2} - \nabla_s u_2 \frac{1}{t^2},\end{aligned}$$

where by $\nabla_y u_2$ and $\nabla_s u_2$, we actually mean $t^{-n/2}e^{\frac{|x|^2}{4(A+iB)t}}\nabla_y u_2$ and $t^{-n/2}e^{\frac{|x|^2}{4(A+iB)t}}\partial_s u_2$ respectively. Moreover,

$$\partial_{x_j} u_2 = \frac{1}{t}\partial_{y_j} u_2 + \frac{2x_j}{4(A+iB)t}u_2$$

where similarly $\partial_{y_j} u_2$ " $=$ " $t^{-n/2}e^{\frac{|x|^2}{4(A+iB)t}}\partial_{y_j} u$.

$$\begin{aligned}\partial_{x_j}^2 u_2 &= \partial_{x_j}(\frac{1}{t}\partial_{y_j} u_2) + \partial_{x_j}(\frac{2x_j}{4(A+iB)t}u_2) \\ &= \frac{1}{t^2}\partial_{y_j}^2 u_2 + \partial_{y_j} u_2 \frac{2x_j}{4(A+iB)t^2} + \frac{2u_2}{4(A+iB)t} + \frac{2x_j}{4(A+iB)t}\left(\frac{1}{t}\partial_{y_j} u_2 + \frac{2x_j}{4(A+iB)t}u_2\right) \\ &= \frac{1}{t^2}\partial_{y_j}^2 u_2 + \partial_{y_j} u_2 \frac{2x_j}{4(A+iB)t^2} + \frac{2u_2}{4(A+iB)t} + \frac{2x_j \partial_{y_j} u_2}{4(A+iB)t^2} + \frac{4x_j^2}{(4(A+iB)t)^2} \\ &= \frac{1}{t^2}\partial_{y_j}^2 u_2 + \frac{x_j \partial_{y_j} u_2}{(A+iB)t^2} + \left(\frac{1}{2(A+iB)t} + + \frac{x_j^2}{4(A+iB)^2t^2}\right)u_2.\end{aligned}$$

Then

$$\Delta u_2 = \sum_{j=1}^n \partial_{x_j} u_2 = \frac{1}{t^2}\Delta_y u_2 + \frac{1}{(A+iB)t^2}x \cdot \nabla_y u_2 + \frac{n}{2}\frac{1}{(A+iB)t}u_2 + \frac{|x|^2}{4(A+iB)^2t^2}u_2,$$

so it follows that

$$\begin{aligned}\partial_t u_2 + (A+iB)\Delta u_2 &= -\frac{1}{t^2}(\nabla_s u_2 - \Delta_y u_2) \\ &= -t^{-2}\left(t^{-n/2}e^{\frac{|x|^2}{4(A+iB)t}}(\partial_s u - \Delta u)\right) \\ &= t^{-n/2-2}H(\frac{x}{t}, \frac{1}{t})e^{\frac{|x|^2}{4(A+iB)t}}.\end{aligned}$$

Let us assume that $\alpha > \beta$. By the change of variables $r \mapsto \frac{\alpha\beta}{\alpha-\beta}$ and $\tau \mapsto -\frac{\beta}{\alpha-\beta}$ (2.6) implies that $\tilde{u} = u(\sqrt{\frac{\alpha\beta}{\alpha-\beta}}x, \frac{\alpha\beta}{\alpha-\beta}t - \frac{\beta}{\alpha-\beta})$ satisfies the equation

$$\partial_t \tilde{u} = (A+iB)(\Delta \tilde{u} + \frac{\alpha\beta}{\alpha-\beta}H(\sqrt{\frac{\alpha\beta}{\alpha-\beta}}x, \frac{\alpha\beta}{\alpha-\beta}t - \frac{\beta}{\alpha-\beta})).$$

Furthermore, letting $t \mapsto (\alpha - t)$ and combining (2.7) and (2.6), the function

$$\frac{1}{(\alpha-t)^{n/2}} u(\frac{\sqrt{\alpha\beta} x}{\sqrt{\alpha-\beta}(\alpha-t)}, \frac{\alpha\beta}{(\alpha-\beta)(\alpha-t)} - \frac{\beta}{\alpha-\beta}) e^{\frac{|x|^2}{4(A+iB)(\alpha-t)}}$$

will satisfy (2.5) but with right hand side

$$\frac{\alpha\beta}{(\alpha-\beta)(\alpha-t)^{n/2+2}} H\left(\frac{\sqrt{\alpha\beta}}{\sqrt{\alpha-\beta}(\alpha-t)} x, \frac{\alpha\beta}{(\alpha-\beta)(\alpha-t)} - \frac{\beta}{\alpha-\beta}\right) e^{\frac{|x|^2}{4(A+iB)(\alpha-t)}}.$$

We let now $(x,t) \mapsto (\sqrt{\alpha-\beta} x, (\alpha-\beta)t)$. It follows that

$$\frac{1}{(\alpha(1-t)+\beta t)^{n/2}} u\left(\frac{\sqrt{\alpha\beta} x}{(\alpha(1-t)+\beta t}, \frac{\alpha\beta}{(\alpha-\beta)(\alpha(1-t)+\beta t)} - \frac{\beta}{\alpha-\beta})\right) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}} \tag{2.8}$$

satisfies (2.5) with right-hand side

$$\frac{\alpha\beta(\alpha-\beta)}{(\alpha-\beta)(\alpha(1-t)+\beta t)^{n/2+2}} e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t+\beta t)}} \times H\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)} x, \frac{\alpha\beta}{(\alpha-\beta)(\alpha(1-t)+\beta t)} - \frac{\beta}{\alpha-\beta}\right). \tag{2.9}$$

A simple computation shows that

$$\frac{\alpha\beta}{(\alpha-\beta)(\alpha(1-t)+\beta t)} - \frac{\beta}{\alpha-\beta} = \frac{\beta t}{\alpha(1-t)+\beta t},$$

so we can write (2.9) as

$$\frac{\alpha\beta}{(\alpha(1-t)+\beta t)^{n/2+2}} H\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)} x, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}}.$$

Finally, multiplying both (2.8) and (2.9) with $(\sqrt{\alpha\beta})^{n/2}$ we get that

$$\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)}\right)^{n/2} u\left(\frac{\sqrt{\alpha\beta} x}{(\alpha(1-t)+\beta t}, \frac{\alpha\beta}{(\alpha-\beta)(\alpha(1-t)+\beta t)} - \frac{\beta}{\alpha-\beta})\right) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}} \tag{2.10}$$

satisfies (2.5) with right hand side

$$\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)}\right)^{n/2+2} H\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)} x, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}}.$$

Since $H(y,s) = V(y,s)u + F(y,s)$ the result follows. Indeed,

$$\begin{aligned} Vu &= \left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)}\right)^{n/2+2} V\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)} x, \frac{\beta t}{\alpha(1-t)+\beta t}\right) \\ &\quad \times u\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)} x, \frac{\beta t}{\alpha(1-t)+\beta t}\right) (e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}} \\ &= \frac{\alpha\beta}{(\alpha(1-t)+\beta t)^2} V\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)} x, \frac{\beta t}{\alpha(1-t)+\beta t}\right) \tilde{u}(x,t) \\ &= \tilde{V}(x,t)\tilde{u}. \end{aligned}$$

Similarly, we get that

$$F(y,s) = \left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)}\right)^{n/2+2} F\left(\frac{\sqrt{\alpha\beta}}{(\alpha(1-t)+\beta t)}x, \frac{\beta t}{\alpha(1-t)+\beta t}\right) e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}}$$
$$= \tilde{F}(x,t).$$

The case $\alpha < \beta$ follows by reversing the time with $s' = 1-s, t' = 1-t$. For the final part, let $y = \frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}$, $s = \frac{\beta t}{\alpha(1-t)+\beta t}$. Observe also that $\alpha s + \beta(1-s) = \frac{\alpha\beta}{\alpha(1-t)+\beta t}$. Then it follows that

$$\begin{aligned}
&\|e^{\gamma|x|^2}\tilde{u}(t)\|^2_{L^2(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} e^{2\gamma|x|^2}\Big(\frac{\sqrt{\alpha\beta}}{\alpha(1-t)+\beta t}\Big)^n \left|u\Big(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t)+\beta t}, \frac{\beta t}{\alpha(1-t)+\beta t}\Big)e^{\frac{(\alpha-\beta)|x|^2}{4(A+iB)(\alpha(1-t)+\beta t)}}\right|^2 dx \\
&= \int_{\mathbb{R}^n} e^{2\gamma|y|^2\frac{(\alpha(1-t)+\beta t)^2}{\alpha\beta}} e^{(\alpha-\beta)|y|^2 A\frac{(\alpha(1-t)+\beta t)^2}{\alpha\beta(\alpha(1-t)+\beta t)4(A^2+B^2)}}|u(y,s)|^2 dy \\
&= \int_{\mathbb{R}^n} e^{2|y|^2\Big(\frac{\alpha\beta\gamma}{(\alpha s+\beta(1-s))^2}+\frac{(\alpha-\beta)A}{4(A^2+B^2)(\alpha s+\beta(1-s))}\Big)}|u(y,s)|^2 ds \\
&= \|e^{|y|^2\Big(\frac{\alpha\beta\gamma}{(\alpha s+\beta(1-s))^2}+\frac{(\alpha-\beta)A}{4(A^2+B^2)(\alpha s+\beta(1-s))}\Big)}u(s)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The argument for $\|e^{\gamma|x|^2}\tilde{F}(t)\|_{L^2(\mathbb{R}^n)}$ is similar. □

Remark. In particular, this transformation lets us reduce the problem from having two different parameters α, β to only having one parameter γ.

2.2 Proof of Theorem 2.3 in the case $\alpha\beta < 2$

Proof. We split the proof into 4 steps.

Step 1: *Reducing the problem to* $\alpha = \beta = \gamma$. We will show that by Lemma 2.1 it suffices to prove the theorem in the case where $\alpha = \beta = \gamma$. Define $\tilde{u}$ as in the conformal/Appell transformation. For $\gamma \in \mathbb{R}$ it follows by (2.4) that

$$\|e^{\gamma|x|^2}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)} = \|e^{\gamma\frac{\alpha}{\beta}|x|^2}u(0)\|_{L^2(\mathbb{R}^n)}$$

$$\|e^{\gamma|x|^2}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)} = \|e^{\gamma\frac{\beta}{\alpha}|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}.$$

In particular, if we let $\gamma = \frac{1}{\alpha\beta}$, then

$$\|e^{\gamma|x|^2}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)} = \|e^{\frac{1}{\beta^2}|x|^2}u(0)\|_{L^2(\mathbb{R}^n)},$$
$$\|e^{\gamma|x|^2}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)} = \|e^{\frac{1}{\alpha^2}|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}.$$

By these relations, we see that if $\|e^{\gamma|x|^2}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\gamma|x|^2}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)}$ are finite for $\alpha\beta < 2$, and if this implies that $u = 0$, then it is equivalent to show that for $\gamma > \frac{1}{2}$, we have $\tilde{u} = 0$.

Hence, we can assume $\alpha = \beta$.

Step 2: *Carleman estimate for the free Schrödinger eqution.*
Define $f = e^{\phi}u$, where ϕ is a real valued function, depending on both x and t, to be chosen later. We also define $H(t) := \|f(t)\|^2_{L^2(\mathbb{R}^n)}$.

Claim 2.1. For a well-chosen ϕ, we can show that $\log H(t)'' \geqslant \frac{-R^2}{4\gamma}$. In other words, this function is "almost"logarithmically convex. In particular, we choose $\phi = \gamma|x + Rt(1-t)e_1|^2$ for $\gamma, R > 0$.

Assuming the claim, consider the function $F(t) := e^{\frac{-R^2 t(1-t)}{8\gamma}} H(t)$. Observe that

$$\begin{aligned}
\log F(t)'' &= (-\frac{R^2}{8\gamma}t(1-t))'' + \log H(t)'' \\
&= 2\frac{R^2}{8\gamma} + \log H(t)'' \\
&\geqslant \frac{R^2}{4\gamma} - \frac{R^2}{4\gamma} \\
&\geqslant 0.
\end{aligned}$$

Hence, $F(t)$ is logarithmically convex.

Step 3: *Finish the proof assuming the claim.* Since $F(t)$ is logarithmically convex, it follows that for $t_1, t_2 \in [0,1]$ and $\lambda \in (0,1)$

$$\log(F(\lambda t_1 + (1-\lambda)t_2) \leqslant \lambda \log F(t_1) + (1-\lambda)F(t_2.)$$

In particular, for $\lambda = \frac{1}{2},\ t_1 = 0,\ t_2 = 1$,

$$e^{-\frac{R^2}{32\gamma}} H(1/2) = F(\frac{1}{2}) \leqslant F(0)^{1/2}F(1)^{1/2} = H(0)^{1/2}H(1)^{1/2}.$$

Then it follows

$$e^{-\frac{R^2}{32\gamma}} \int_{\mathbb{R}^n} e^{2\gamma|x+\frac{R}{4}e_1|^2} u(x,1/2)dx \leqslant \Big(\int_{\mathbb{R}^n} e^{2\gamma|x|^2}|u_0|^2 dx\Big)^{1/2} \Big(\int_{\mathbb{R}^n} e^{2\gamma|x|^2}|u(x,1)|^2 dx\Big)^{1/2}.$$

Now, if $|x| \leqslant \frac{\epsilon R}{4}$, then it follows that $|x + \frac{R}{4}e_1|^2 \geqslant \frac{R}{4}(1-\epsilon)$. Hence,

$$\int_{B(0,\epsilon R/4)} |u(x,1/2)|^2 e^{2\gamma(\frac{R}{4}(1-\epsilon))^2} dx \leqslant \int_{B(0,\epsilon R/4)} |u(x,1/2)|^2 e^{2\gamma|x+\frac{R}{4}e_1|^2} dx. \tag{2.11}$$

So that

$$\begin{aligned}
\int_{B(0,\epsilon R/4)} |u(x,1/2)|^2 dx &\leqslant e^{\frac{R^2}{32\gamma}} e^{-\frac{2\gamma R^2(1-\epsilon)^2}{16}} \|e^{\gamma|x|^2}u_0\|_{L^2(\mathbb{R}^n)} \|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)} \\
&= e^{\frac{R^2}{32\gamma}(1-4\gamma^2(1-\epsilon)^2)} \|e^{\gamma|x|^2}u_0\|_{L^2(\mathbb{R}^n)} \|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

For $\epsilon > 0$ small enough and

$$1 - 4\gamma^2 < 0, \quad \Longleftrightarrow \gamma > \frac{1}{2}$$

we get that the exponent on the right-hand side is negative, so by letting $R \to \infty$ it follows that $u(x, 1/2) = 0$ in $L^2(\mathbb{R}^n)$, so $u \equiv 0$ in $L^2(\mathbb{R}^n)$. Remark that one of the reasons the function ϕ was chosen as it was, was that we needed the condition that $\phi(x, 0) = \phi(x, 1) = \gamma|x|^2$. If not, we would not necessarily get that the right-hand side goes to 0 as $R \to \infty$.

Step 4: *Proof of Claim 2.1.* Observe that

$$\begin{aligned}\partial_t f &= \partial_t \phi f + \partial_t u e^{\phi} \\ &= \partial_t \phi f + i e^{\phi} \Delta u \\ &= \partial_t \phi f + i e^{\phi} \Delta (e^{-\phi} f)\end{aligned}$$

$$\begin{aligned}e^{\phi} \nabla (e^{-\phi} f) &= e^{\phi}(-\nabla(\phi) e^{-\phi} f + \nabla f e^{-\phi}) \\ &= -\nabla \phi f + \nabla f\end{aligned}$$

$$\begin{aligned}e^{\phi} \Delta (e^{-\phi} f) &= e^{\phi} \nabla \cdot (\nabla e^{-\phi} f) \\ &= e^{\phi} \nabla e^{-\phi}) \cdot (e^{\phi} \nabla e^{-\phi}) f \\ &= (-\nabla \phi + \nabla) \cdot (-\nabla \phi + \nabla) f \\ &= (|\nabla \phi|^2 - \Delta \phi - 2\nabla \phi \cdot \nabla + \Delta) f.\end{aligned}$$

So f satisfies the IVP

$$\begin{cases} \partial_t f = \partial_t \phi f + i(\Delta f - 2\nabla \phi \cdot \nabla f + |\nabla \phi|^2 f - \Delta \phi f) \\ f(x, 0) = e^{\phi(x,0)} u_0. \end{cases}$$

We want to divide the operators into symmetric and skew-symmetric parts with respect to the L^2-inner product, to make the computations simpler. We write

$$\partial_t f = (\mathcal{S} + \mathcal{A}) f,$$

where the symmetric part is

$$\mathcal{S} = \partial_t \phi - i(2\nabla \phi \nabla + \Delta \phi) \tag{2.12}$$

and the skew-symmetric part is

$$\mathcal{A} = i(\Delta + |\nabla \phi|^2). \tag{2.13}$$

In particular, let w_1, w_2 be two functions. By a formal integration by parts, and since ϕ is real,

$$\begin{aligned}
\langle \mathcal{S}w_1, w_2\rangle_{L^2(\mathbb{R}^n)} &= \int \partial_t\phi w_1\overline{w_2} - \int 2i\nabla\phi\nabla w_1\overline{w_2} - \int i\Delta\phi w_1\overline{w_2} \\
&= \int w_1\overline{\partial_t\phi w_2} + \int 2iw_1\nabla\cdot(\nabla\phi\overline{w_2}) - \int iw_1\overline{\Delta\phi w_2} \\
&= \int w_1\overline{\partial_t\phi w_2} - \int w_1\overline{2i\nabla\phi\nabla w_2} - \int w_1\overline{i\Delta\phi w_2} \\
&= \langle w_1, \mathcal{S}w_2\rangle_{L^2(\mathbb{R}^n)}
\end{aligned}$$

Similarly, we can show that

$$\langle \mathcal{A}w_1, w_2\rangle_{L^2(\mathbb{R}^n)} = -\langle w_1, \mathcal{A}w_2\rangle_{L^2(\mathbb{R}^n)}.$$

We first compute the derivatives of $H(t)$ and $\log H(t)$.

$$\begin{aligned}
H'(t) &= \langle \partial_t f, f\rangle_{L^2(\mathbb{R}^n)} + \langle f, \partial_t f\rangle_{L^2(\mathbb{R}^n)} \\
&= \langle (\mathcal{S}+\mathcal{A})f, f\rangle_{L^2(\mathbb{R}^n)} + \langle f, (\mathcal{S}+\mathcal{A})f\rangle_{L^2(\mathbb{R}^n)} \\
&= 2\langle Sf, f\rangle_{L^2(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
H''(t) &= 2\langle \partial_t(\mathcal{S}f), f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \mathcal{S}f, \partial_t f\rangle_{L^2(\mathbb{R}^n)} \\
&= 2\langle \partial_t\mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \mathcal{S}\partial_t f, f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \mathcal{S}f, \partial_t f\rangle_{L^2(\mathbb{R}^n)} \\
&= 2\langle \partial_t\mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \mathcal{S}(\mathcal{S}+\mathcal{A})f, f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \mathcal{S}f, (\mathcal{S}+\mathcal{A})f\rangle_{L^2(\mathbb{R}^n)} \\
&= 2\langle \partial_t\mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} + 4\langle \mathcal{S}f, \mathcal{S}f\rangle_{L^2(\mathbb{R}^n)} + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)}
\end{aligned}$$

$$\begin{aligned}
\log H(t)'' &= \left(\frac{H'}{H}\right)' \\
&= \frac{H''H}{H^2} - \frac{H'^2}{H^2} \\
&= \frac{2\langle (\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}])f, f\rangle_{L^2(\mathbb{R}^n)}}{\langle f, f\rangle_{L^2(\mathbb{R}^n)}} + \frac{4\langle \mathcal{S}f, \mathcal{S}f\rangle_{L^2(\mathbb{R}^n)}}{\langle f, f\rangle_{L^2(\mathbb{R}^n)}} - \frac{4\langle \mathcal{S}f, f\rangle^2_{L^2(\mathbb{R}^n)}}{\langle f, f\rangle^2_{L^2(\mathbb{R}^n)}}.
\end{aligned}$$

Observe that by the Cauchy-Schwarz inequality:

$$\frac{4\langle \mathcal{S}f, f\rangle^2_{L^2(\mathbb{R}^n)}}{\langle f, f\rangle^2_{L^2(\mathbb{R}^n)}} \leqslant \frac{4\langle \mathcal{S}f, \mathcal{S}f\rangle_{L^2(\mathbb{R}^n)}}{\langle f, f\rangle_{L^2(\mathbb{R}^n)}},$$

so that the last two terms together are positive. That means that when we want to find a lower bound for $\log H(t)''$, we only need to consider

$$2\frac{\langle (\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}])f, f\rangle_{L^2(\mathbb{R}^n)}}{\langle f, f\rangle_{L^2(\mathbb{R}^n)}}.$$

The next step is to compute the inner product and try to see how we choose ϕ.

$$
\begin{aligned}
\mathcal{S}\mathcal{A}(f) &= \partial_t\phi(i(\Delta f + |\nabla\phi|^2 f)) + 2\nabla\phi\cdot\nabla(\Delta f + |\nabla\phi|^2 f) + \Delta\phi(\Delta f + |\nabla\phi|^2 f)\\
&= i\partial_t\phi\Delta f + i\partial_t\phi|\nabla\phi|^2 f + 2\nabla\phi\cdot\nabla(\Delta f) + 2\nabla\phi\cdot\nabla(|\nabla\phi|^2)f + 2\nabla\phi|\nabla\phi|^2\cdot\nabla f\\
&+ \Delta\phi\Delta f + \Delta\phi|\nabla\phi|^2 f\\
&= i\partial_t\phi\Delta f + i\partial_t\phi|\nabla\phi|^2 f + 2\nabla\phi\cdot\nabla(\Delta f) + 4\nabla\phi\cdot D^2\phi(\nabla\phi) + 2\nabla\phi|\nabla\phi|^2\cdot\nabla f\\
&+ \Delta\phi\Delta f + \Delta\phi|\nabla\phi|^2 f
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}\mathcal{S}(f) &= i(\Delta + |\nabla\phi|^2)(\partial_t\phi f) + (\Delta + |\nabla\phi|^2)(2\nabla\phi\cdot\nabla f) + (\Delta + |\nabla\phi|^2)(\Delta\phi f)\\
&= i\Delta(\partial_t\phi f) + i|\nabla\phi|^2\partial_t\phi f + 2\Delta(\nabla\phi\cdot\nabla f) + 2|\nabla\phi|^2\nabla\phi\cdot\nabla f + \Delta(\Delta\phi f) + |\nabla\phi|^2\Delta\phi f\\
&= i(\Delta\partial_t\phi)f + i\partial_t\phi\Delta f + 2i\nabla(\partial_t\phi)\cdot\nabla f + i|\nabla\phi|^2\partial_t\phi f + 2(2D^2\phi\cdot D^2 f + \nabla(\Delta\phi)\cdot\nabla f\\
&+ \nabla(\Delta f)\cdot\nabla\phi) + 2|\nabla\phi|^2\nabla\phi\cdot\nabla f + \Delta^2\phi f + 2\nabla(\Delta\phi)\nabla f + \Delta\phi\Delta f + |\nabla\phi|^2\Delta\phi f.
\end{aligned}
$$

where we have used that

$$\nabla(\nabla\phi\cdot\nabla f) = D^2\phi(\nabla f) + D^2 f(\nabla\phi),$$

and

$$\Delta(\nabla\phi\cdot\nabla f) = \nabla\cdot(D^2\phi(\nabla f) + D^2 f(\nabla\phi)) = 2D^2\phi\cdot D^2 f + \nabla(\Delta\phi)\cdot\nabla f + \nabla(\Delta f)\cdot\nabla\phi.$$

It therefore follows that

$$
\begin{aligned}
[\mathcal{S},\mathcal{A}] &= 4\nabla\phi\cdot D^2\phi(\nabla\phi)f - i\Delta(\partial_t\phi)f - 2i\nabla(\partial_t\phi)\cdot\nabla f\\
&- 4D^2\phi\cdot D^2 f - 4\nabla(\Delta\phi)\cdot\nabla f - \Delta^2\phi f.
\end{aligned}
\tag{2.14}
$$

Let us now compute $\partial_t\mathcal{S}(f)$.

$$\partial_t\mathcal{S}(f) = (\partial_t^2\phi - i(\partial_t(2\nabla\phi\cdot\nabla + \Delta\phi)))f = (\partial_t^2\phi - 2i\nabla(\partial_t\phi)\cdot\nabla - i\Delta(\partial_t\phi))f. \tag{2.15}$$

By combining (2.14) and (2.15) we get

$$
\begin{aligned}
(\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}])(f) &= \partial_t^2\phi f - 2i\nabla(\partial_t\phi)\cdot\nabla f - i\Delta(\partial_t\phi)f\\
&+ 4\nabla\phi\cdot D^2\phi(\nabla\phi)f - i\Delta(\partial_t\phi)f - 2i\nabla(\partial_t\phi)\cdot\nabla f\\
&- 4D^2\phi\cdot D^2 f - 4\nabla(\Delta\phi)\cdot\nabla f - \Delta^2\phi f\\
&= \partial_t^2\phi f - 4i\nabla(\partial_t\phi)\cdot\nabla f - 2i\Delta(\partial_t\phi)f + 4\nabla\phi\cdot D^2\phi(\nabla\phi)f\\
&- 4\nabla\cdot(D^2\phi(\nabla f)) - \Delta^2\phi f.
\end{aligned}
\tag{2.16}
$$

First, we see what happens if $\phi = \gamma|x|^2$, not depending on t. Then it obviously satisfies $\phi(x,0) = \phi(x,1) = \gamma|x|^2$. Then all the derivatives with respect to t, and all derivatives of higher order than two, will vanish, so we are only left with

$$(\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}])(f) = 4\nabla\phi\cdot D^2\phi(\nabla\phi)f - 4\nabla\cdot(D^2\phi(\nabla f)) = 32\gamma^3|x|^2 f - 8\gamma\Delta f.$$

Then it follows that

$$\begin{aligned}\langle(\partial_t \mathcal{S}+[\mathcal{S},\mathcal{A}])f,f\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} 32|x|^2|f|^2dx + \int_{\mathbb{R}^n} 8\Delta f\overline{f}dx \\ &= \int_{\mathbb{R}^n} 32|x|^2|f|^2 + 8|\nabla f|^2 dx \geqslant 0\end{aligned}$$

and H will be log-convex. The problem with choosing this weight is that we cannot bound it from below which we needed to do in (2.11) to finish the proof. If we let $\phi = \gamma|x+Rt(1-t)e_1|^2$, where γ and R are some positive, real constants, then we can show that

$$\frac{\langle(\partial_t \mathcal{S}+[\mathcal{S},\mathcal{A}])f,f\rangle_{L^2(\mathbb{R}^n)}}{\langle f,f\rangle_{L^2(\mathbb{R}^n)}} \geqslant \frac{-R^2}{8\gamma}.$$

This choice of ϕ is somehow the second simplest choice we can guess where ϕ still satisfies $\phi(x,0) = \phi(x,1) = \gamma|x|^2$, but where it also depends on t. We start with computing the partial derivatives of ϕ.

$$\begin{aligned}\partial_{x_1}\phi &= 2\gamma(x_1+Rt(1-t)) \\ \partial_{x_j}\phi &= 2\gamma x_j, \quad j\neq 1 \\ \partial_{x_j}^2\phi &= 2\gamma \\ \nabla\phi &= 2\gamma(x+Rt(1-t)e_1) \\ \Delta\phi &= 2\gamma n \\ D^2\phi &= 2\gamma I \\ \partial_t\phi &= 2\gamma R(x_1+Rt(1-t))(1-2t) \\ \partial_t^2\phi &= 2\gamma R[R(1-2t)^2 - 2(x_1+Rt(1-t))] \\ \nabla(\partial_t\phi) &= 2\gamma R(1-2t)e_1 \\ \Delta(\partial_t\phi) &= 0.\end{aligned}$$

Then (2.16) becomes

$$\begin{aligned}(\partial_t \mathcal{S}+[\mathcal{S},\mathcal{A}])(f) &= 2\gamma R^2(1-2t)^2 f - 4\gamma R(x_1+Rt(1-t))f \\ &\quad - 8i\gamma R(1-2t)\partial_{x_1}f + 32\gamma^3|x+Rt(1-t)e_1|^2 f - 8\gamma\Delta f\end{aligned}$$

and the inner product

$$\begin{aligned}\langle(\partial_t \mathcal{S}+[\mathcal{S},\mathcal{A}])f,f\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} 2\gamma R^2(1-2t)^2|f|^2dx - \int_{\mathbb{R}^n} 4\gamma R(x_1+Rt(1-t)|f|^2dx \\ &\quad - \int_{\mathbb{R}^n} 8i\gamma R(1-2t)\partial_{x_1}f\overline{f}dx + \int_{\mathbb{R}^n} 32\gamma^3|x+Rt(1-t)e_1|^2|f|^2dx \\ &\quad + \int_{\mathbb{R}^n} 8\gamma|\nabla f|^2dx.\end{aligned}$$

To make the notation easier, let us write

$$\begin{aligned}\langle(\partial_t\mathcal{S}+[\mathcal{S},\mathcal{A}])f,f\rangle_{L^2(\mathbb{R}^n)} &= (1)+(2)+(3)+(4)+(5)\\ &\geqslant (1)-|(2)|-|(3)|+(4)+(5).\end{aligned}$$

Observe that all terms are real, but (2) and (3) are the bad terms which are not positive. Let us deal with (2) first, by using Cauchy-Schwarz and Young's inequalities.

$$\begin{aligned}|(2)| &= \int_{\mathbb{R}^n} 4uR(x_1+Rt(1-t))|f|^2dx\\ &\leqslant 4\gamma R\|(x_1+Rt(1-t))f\|_{L^2(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}\\ &\leqslant 4\gamma R\frac{\epsilon^2}{2}\|(x_1+Rt(1-t))f\|_{L^2(\mathbb{R}^n)}+4\gamma R\frac{1}{2\epsilon^2}\|f\|_{L^2(\mathbb{R}^n)}\\ &\leqslant 4\gamma R\frac{\epsilon^2}{2}\|(x+Rt(1-t)e_1)f\|^2_{L^2(\mathbb{R}^n)}+4\gamma R\frac{1}{2\epsilon^2}\|f\|^2_{L^2(\mathbb{R}^n)}\\ &= 4\gamma R\frac{\epsilon^2}{2}\int_{\mathbb{R}^n}|x+Rt(1-t)e_1|^2|f|^2dx+4\gamma R\frac{1}{2\epsilon^2}\|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}\tag{2.17}$$

By choosing $\epsilon^2=\frac{16\gamma^2}{R}$, we can use (4) to cancel the first term, and we are left with

$$(4)-|(2)|\geqslant -\frac{R^2}{8\gamma}\|f\|^2_{L^2(\mathbb{R}^n)}.\tag{2.18}$$

For (3)

$$\begin{aligned}|(3)| &= \int_{\mathbb{R}^n} 8\gamma R(1-2t)\partial_{x_1}f\bar{f}dx\\ &\leqslant 8\gamma R\|\partial_{x_1}f\|_{L^2(\mathbb{R}^n)}\|(1-2t)f\|_{L^2(\mathbb{R}^n)}\\ &\leqslant 8\gamma R\frac{\epsilon^2}{2}\|\partial_{x_1}f\|^2_{L^2(\mathbb{R}^n)}+8\gamma R\frac{1}{2\epsilon^2}\|(1-2t)f\|^2_{L^2(\mathbb{R}^n)}\\ &\leqslant 8\gamma R\frac{\epsilon^2}{2}\|\nabla f\|^2_{L^2(\mathbb{R}^n)}+8\gamma R\frac{1}{2\epsilon^2}\|(1-2t)f\|^2_{L^2(\mathbb{R}^n)}\\ &= 8\gamma R\frac{\epsilon^2}{2}\int_{\mathbb{R}^n}|\nabla f|^2dx+8\gamma R\frac{1}{2\epsilon^2}\int_{\mathbb{R}^n}|(1-2t)f|^2dx. \end{aligned}\tag{2.19}$$

Now, let $\epsilon^2=\frac{2}{R}$. Then

$$(1)+(5)-|(3)|=0.\tag{2.20}$$

We are left with

$$\langle(\partial_t\mathcal{S}+[\mathcal{S},\mathcal{A}])f,f\rangle_{L^2(\mathbb{R}^n)}\geqslant -\frac{R^2}{8\gamma}\|f\|^2_{L^2(\mathbb{R}^n)},\tag{2.21}$$

so we have shown that

$$\log H(t)''\geqslant 2\frac{\langle(\partial_t\mathcal{S}+[\mathcal{S},\mathcal{A}]f,f\rangle_{L^2(\mathbb{R}^n)}}{\langle f,f\rangle_{L^2(\mathbb{R}^n)}}\geqslant\frac{-R^2}{4\gamma},$$

which concludes the proof of the claim. □

This shows the formal argument that we ideally would have liked to extend to the case with a non-zero potential. However, to justify the argument rigorously is not easy. We therefore follow a slightly different path, and the first step is to perform a parabolic regularization and work on the equation

$$\partial_t u = (A + iB)(\Delta u + V(x,t)u + F(x,t)),$$

for $A > 0$. We will prove energy and Carleman estimates for solutions to this equation.

3 Important Estimates and Proof of Theorem 3

In this chapter, the main goal is to prove Theorem 3. To do this, we start by proving estimates for the regularized equation (0.9), for $A > 0$.

3.1 Energy Estimate

Lemma 3.1. Assume $u \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ satisfies

$$\partial_t u = (A+iB)(\Delta u + V(x,t)u + F(x,t)) \qquad in\ \mathbb{R}^n \times [0,1]$$

$A > 0,\ B \in \mathbb{R}$. Then

$$e^{-M_T}\|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)T}}u(T)\|_{L^2} \leqslant \|e^{\gamma|x|^2}u(0)\|_{L^2} + \sqrt{A^2+B^2}\|e^{\frac{\gamma A|x|^2}{A+4\gamma(A^2+B^2)t}}F(t)\|_{L^1([0,T],L^2(\mathbb{R}^n))},$$

when $\gamma \geqslant 0$, $0 \leqslant T \leqslant 1$ and $M_T = \|A(\mathrm{Re}\ V)^+ - B\ \mathrm{Im}\ V\|_{L^1([0,T],L^\infty(\mathbb{R}^n))}$.

Remark. In Lemma A.1 in Appendix A, we discuss the existence of solutions $u \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ when $u_0 \in L^2(\mathbb{R}^n)$.

Proof. Let $f = e^\phi u$, where ϕ is a real-valued function to be chosen later. Then

$$\begin{aligned}\partial_t f &= \partial_t\phi e^\phi u + e^\phi\partial_t u\\ &= \partial_t\phi f + (A+iB)e^\phi\Delta(e^{-\phi}f) + Vf + Fe^\phi\\ &= \partial_t\phi f + (A+iB)(|\nabla\phi|^2 - \Delta\phi - 2\nabla\phi\cdot\nabla + \Delta)f + Vf + Fe^\phi\\ &= (\mathcal{S}+\mathcal{A})f + Fe^\phi,\end{aligned}$$

where

$$\begin{aligned}\mathcal{S} &= A(\Delta + |\nabla\phi|^2) - iB(2\nabla\phi\cdot\nabla + \Delta\phi) + (\partial_t\phi + A\,\mathrm{Re}\,V - B\,\mathrm{Im}\,V)\\ \mathcal{A} &= iB(\Delta + |\nabla\phi|^2) - A(2\nabla\phi\cdot\nabla + \Delta\phi) + i(B\ \mathrm{Re}\,V + A\,\mathrm{Im}\,V).\end{aligned}$$

We have that

$$\partial_t\|f\|^2_{L^2(\mathbb{R}^n)} = 2\mathrm{Re}\ \langle Sf, f\rangle_{L^2(\mathbb{R}^n)} + 2\mathrm{Re}\ \langle (A+iB)e^\phi F, f\rangle_{L^2(\mathbb{R}^n)}.$$

It follows by integration by parts that

$$\begin{aligned}\mathrm{Re}\,\langle Sf, f\rangle_{L^2(\mathbb{R}^n)} &= \mathrm{Re}\int_{\mathbb{R}^n} A(\Delta f + |\nabla\phi|^2 f)\bar{f}dx - \mathrm{Re}\,iB\int_{\mathbb{R}^n}(2\nabla\phi\cdot\nabla f + \Delta\phi f)\bar{f}dx\\ &\quad + \mathrm{Re}\int_{\mathbb{R}^n}(\partial_t\phi + A\,\mathrm{Re}\,V - B\,\mathrm{Im}\,V)|f|^2dx\\ &= -\int_{\mathbb{R}^n} A|\nabla f|^2dx + \int_{\mathbb{R}^n}(A|\nabla\phi|^2 + \partial_t\phi)|f|^2dx + 2B\,\mathrm{Im}\int_{\mathbb{R}^n}\nabla\phi\cdot\nabla f\bar{f}dx\\ &\quad + \int_{\mathbb{R}^n}(A\mathrm{Re}\,V - B\mathrm{Im}\,V)|f|^2dx\\ &= (1) + (2) + (3) + (4).\end{aligned}$$

We want to find an upper bound for $\partial_t \|f\|^2_{L^2(\mathbb{R}^n)}$ so we will apply similar methods as we did in the proof of Theorem 2.3, with Cauchy-Schwarz and Young's inequalities. Observe that

$$\begin{aligned}
|(3)| &\leqslant 2B \int_{\mathbb{R}^n} |\nabla \phi \cdot \nabla f \bar{f}| dx \\
&\leqslant 2B \|\nabla f\|_{L^2(\mathbb{R}^n)} \|\bar{f} \nabla \phi\|_{L^2(\mathbb{R}^n)} \\
&\leqslant B\epsilon \int_{\mathbb{R}^n} |\nabla f|^2 dx + \frac{B}{\epsilon} \int_{\mathbb{R}^n} |\bar{f} \nabla \phi|^2 dx.
\end{aligned}$$

By letting $\epsilon = \frac{A}{B}$, it follows

$$|(3)| \leqslant A \int_{\mathbb{R}^n} |\nabla f|^2 dx + \frac{B^2}{A} \int_{\mathbb{R}^n} |f|^2 |\nabla \phi|^2 dx. \tag{3.1}$$

Hence, the first term in (3.1) will be canceled by (1), and

$$\mathrm{Re}\langle Sf, f\rangle_{L^2(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}^n} (A + \frac{B^2}{A}) |\nabla \phi|^2 |f|^2 + \partial_t \phi |f|^2 dx + \|A \mathrm{Re}\, V^+ - B\, \mathrm{Im} V\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}.$$

If we require that

$$(A + \frac{B^2}{A}) |\nabla \phi|^2 + \partial_t \phi \leqslant 0, \tag{3.2}$$

then

$$\mathrm{Re}\langle Sf, f\rangle_{L^2(\mathbb{R}^n)} \leqslant \|A \mathrm{Re}\, V^+ - B\, \mathrm{Im} V\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.3}$$

Also, it follows by Cauchy-Schwarz that

$$\begin{aligned}
\mathrm{Re}\, \langle (A + iB) e^{\phi} F, f\rangle_{L^2(\mathbb{R}^n)} &\leqslant \int_{\mathbb{R}^n} \sqrt{(A^2 + B^2)} |e^{\phi} F \bar{f}| dx \\
&\leqslant \sqrt{A^2 + B^2} \|e^{\phi} F\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4),

$$\partial_t \|f\|^2_{L^2(\mathbb{R}^n)} \leqslant 2 \|A \mathrm{Re}\, V^+ - B\, \mathrm{Im} V\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)} + 2\sqrt{A^2 + B^2} \|e^{\phi} F\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)},$$

which implies that

$$\partial_t \|f\|_{L^2(\mathbb{R}^n)} \leqslant \|A\, \mathrm{Re}\, V(t)^+ - B\, \mathrm{Im}\, V(t)\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} + \sqrt{A^2 + B^2} \|e^{\phi} F\|_{L^2(\mathbb{R}^n)},$$

so by Grönwall's differential inequality,

$$\begin{aligned}
\|f\|_{L^2(\mathbb{R}^n)} e^{-M_T} - \|f(0)\|_{L^2(\mathbb{R}^n} &\leqslant \sqrt{A^2 + B^2} \int_0^T e^{-M_t} \|e^{\phi} F\|_{L^2(\mathbb{R}^n)^2} dt \\
&\leqslant \sqrt{A^2 + B^2} \|e^{\phi} F\|_{L^1([0,1], L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.5}$$

Let us now assume $\phi(x,t) = a(t)\Phi(x)$, where we in the end want $\Phi(x) = |x|^2$. (3.2) is satisfied when $a(t)$ satisfies the initial value problem

$$\begin{cases} a'(t) &= -4(A + \frac{B^2}{A})a(t)^2 \\ a(0) &= \gamma, \end{cases}$$

for some $\gamma > 0$, with solution

$$a(t) = \frac{\gamma A}{4(A^2 + B^2)t\gamma + A}. \tag{3.6}$$

To justify the formal computation above, we will do a regularization argument with a cutoff function on the weight ϕ. Define

$$\Phi_R(x) = \begin{cases} |x|^2, & |x| \leqslant R \\ R^2, & |x| > R. \end{cases}$$

Then let $\theta_\rho(x) = \rho^{-n}\theta(\rho^{-1}x)$, where θ is a radial mollifier, and define

$$\begin{aligned} \phi_{\rho,R}(x,t) &= a(t)\theta_\rho * \Phi_R(x) \\ f_{\rho,R}(x,t) &= e^{\phi_{\rho,R}(x,t)}u(x,t). \end{aligned}$$

Since $u \in L^2(\mathbb{R}^n)$ for all $0 \leqslant t \leqslant 1$,

$$\begin{aligned} \|f_{\rho,R}(t)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |e^{a(t)\theta_\rho * \Phi_R(x)}u(x,t)|^2 dx \\ &\leqslant \int_{\mathbb{R}^n} e^{2\gamma R^2}|u(x,t)|^2 dx \\ &\leqslant e^{2\gamma R^2}\|u(t)\|^2_{L^2(\mathbb{R}^n)} < \infty \end{aligned}$$

so, $f_{\rho,R} \in L^2(\mathbb{R}^n)$ for $0 \leqslant t \leqslant 1$. Replacing f with $f_{\rho,R}$ in (3.5) implies that

$$\begin{aligned} &\|f_{\rho,R}(T)\|_{L^2(\mathbb{R}^n)}e^{-M_T} - \|f_{\rho,R}(0)\|_{L^2(\mathbb{R}^n)} \\ &\leqslant \sqrt{A^2 + B^2}\int_0^T e^{-\int_0^t \|A \operatorname{Re} V(s)^+ - B \operatorname{Im} V(s)\|_{L^\infty(\mathbb{R}^n)} ds}\|e^\phi F\|_{L^2(\mathbb{R}^n)} dt \\ &\leqslant \sqrt{A^2 + B^2}\int_0^T \|e^\phi F\|_{L^2(\mathbb{R}^n)} dt, \end{aligned}$$

so that

$$\begin{aligned} &\|e^{\frac{\gamma A}{4(A^2+B^2)T\gamma + A}\theta_\rho * \Phi_R(x)}u(T)\|_{L^2(\mathbb{R}^n)}e^{-M_T} \\ &\qquad \leqslant \|e^{\gamma\theta_\rho * \Phi_R(x)}u(0)\|_{L^2(\mathbb{R}^n)} + \sqrt{A^2 + B^2}\|e^{\frac{\gamma A}{4(A^2+B^2)t\gamma}\theta_\rho * \Phi_R}F\|_{L^1([0,T],L^2(\mathbb{R}^n))}. \end{aligned}$$

By first letting $\rho \to 0$, using The Dominated Convergence Theorem and the properties of θ_ρ being a radial mollifier, then letting $R \to \infty$, using the monotone convergence theorem, we deduce that

$$\|e^{\frac{\gamma A|x|^2}{4(A^2+B^2)T\gamma+A}} u(T)\|_{L^2(\mathbb{R}^n)} e^{-M_T} \leqslant \|e^{\gamma|x|^2} u(0)\|_{L^2(\mathbb{R}^n)} + \sqrt{A^2+B^2} \|e^{\frac{\gamma A|x|^2}{4(A^2+B^2)t\gamma}} F\|_{L^1([0,T],L^2(\mathbb{R}^n))},$$

which concludes the proof. □

Remark.
(i) As we discussed in the introduction it is not always true that the norm $\|e^{\phi}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all time, even though $u \in C([0,1], L^2(\mathbb{R}^n))$. What this lemma tells us, is that for a specific weight function $\phi = a(t)|x|^2 \leqslant \gamma|x|^2$, we can justify that $\|e^{\phi}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all t. This will be important for us later in Lemma 3.4 when we justify that $\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all time $t \in [0,1]$.

(ii) By modifying this argument a little bit we can also prove that $\int_0^T \|\nabla f\|^2_{L^2(\mathbb{R}^n)} dt$ and $\int_0^T \|\nabla \phi f\|^2_{L^2(\mathbb{R}^n)} dt$ are finite for all $T \in [0,1]$. This will in particular be important to us in the proof of Lemma 3.5. This result is not shown in [6], but for us, it was not obvious how to rigorously justify the arguments without it. We only state the result here, and save the proof for the appendix, see Appendix A.

Lemma 3.2. Let $u \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ satisfy

$$\partial_t u = (A+iB)(\Delta u + V(x,t)u + F(x,t)) \quad \text{in} \ \ \mathbb{R}^n \times [0,1], \tag{3.7}$$

$A > 0,\ B \in \mathbb{R}$. Then

$$\begin{aligned} &\|e^{a(T)|x|^2}u(T)\|^2_{L^2(\mathbb{R}^n)} + A\|\nabla(e^{a(t)|x|^2}u)\|^2_{L^2(\mathbb{R}^n\times[0,T])} + 2A\|2a(t)|x|e^{a(t)|x|^2}u\|^2_{L^2(\mathbb{R}^n\times[0,T])} \\ &\qquad \leqslant e^{M_V+\sqrt{A^2+B^2}}\|e^{\gamma|x|^2}u(0)\|_{L^2(\mathbb{R}^n)} + \sqrt{A^2+B^2}e^{M_V+\sqrt{A^2+B^2}}\|e^{a(t)|x|^2}F(t)\|^2_{L^2(\mathbb{R}^n\times[0,T])}, \end{aligned}$$

for $a(t) = \frac{\gamma A}{A+8(A^2+B^2)\gamma t}$, $\gamma \geqslant 0$, $T \in [0,1]$ and $M_V = \sup_{t\in[0,1]} \|A(\text{Re } V)^+ - B \text{ Im } V\|_{L^\infty(\mathbb{R}^n)}$.

Corollary 3.1. Under the same conditions as in Lemma 3.2 and if $\|e^{\gamma|x|^2}u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\frac{\gamma A|x|^2}{A+8\gamma(A^2+B^2)t}}F(t)\|_{L^2(\mathbb{R}^n\times[0,T])}$ are finite, then

$$\|e^{a(T)|x|^2}u(T)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla(e^{a(t)|x|^2}u)\|^2_{L^2(\mathbb{R}^n\times[0,T])} + \|2a(t)|x|e^{a(t)|x|^2}u\|^2_{L^2(\mathbb{R}^n\times[0,T])} < \infty$$

for $a(t) = \frac{\gamma A}{A+8(A^2+B^2)\gamma t}$, $\gamma \geqslant 0$, $T \in [0,1]$.

3.2 Carleman Estimates

Lemma 3.3. Suppose that $\mathcal{S}$ is a symmetric operator, $\mathcal{A}$ is a skew-symmetric operator, both can depend on a time variable, G is a positive function, $f(x,t)$ is a reasonable function

$$H(t) = \langle f, f\rangle_{L^2(\mathbb{R}^n)}, \qquad D(t) = \langle \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}, \qquad N(t) = \frac{D(t)}{H(t)}.$$

Then

$$\begin{aligned}\partial_t^2 H &= 2\partial_t \mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \partial_t \mathcal{S} f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} \\ &\quad + \|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)} - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.8}$$

and

$$N'(t) \geqslant \frac{\langle \partial_t \mathcal{S} f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)}}{H} - \frac{\|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}}{2H}. \tag{3.9}$$

Moreover, if

$$|\partial_t f - \mathcal{A}f - \mathcal{S}f| \leqslant M_1|f| + G \quad \text{in } \mathbb{R}^n \times [0,1], \quad \partial_t \mathcal{S} + [\mathcal{S},\mathcal{A}] \geqslant -M_0, \tag{3.10}$$

and

$$M_2 = \sup_{[0,1]} \left\| \frac{G(t)}{f(t)} \right\|_{L^2(\mathbb{R}^n)} < \infty,$$

then $\log H(t)$ is convex in $[0,1]$ and there is a universal constant N s.t

$$H(t) \leqslant e^{N(M_0+M_1+M_2+M_1^2+M_2^2)} H(0)^{1-t} H(1)^t \qquad \text{when } 0 \leqslant t \leqslant 1. \tag{3.11}$$

Remark. By a "reasonable"function f, we mean that we can justify all of the computations in the proof.

Proof. Define $H(t) = \langle f, f\rangle_{L^2(\mathbb{R}^n)}$. Then

$$\begin{aligned}H'(t) &= 2\mathrm{Re}\langle \partial_t f, f\rangle_{L^2(\mathbb{R}^n)} \\ &= 2\mathrm{Re}\left(\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)} + \langle \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} + \langle \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)}\right) \\ &= 2\mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)} + 2D(t).\end{aligned} \tag{3.12}$$

We can also write

$$H'(t) = \mathrm{Re}\langle \partial_t f + \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} + \mathrm{Re}\langle \partial_t f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}. \tag{3.13}$$

$$D(t) = \langle \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} = \frac{1}{2}\mathrm{Re}\langle \partial_t f + \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} - \frac{1}{2}\mathrm{Re}\langle \partial_t f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}. \tag{3.14}$$

By multiplying (3.14) and (3.13) we have that

$$H'(t)D(t) = \frac{1}{2}\left(\mathrm{Re}\langle \partial_t f + \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}\right)^2 - \frac{1}{2}\left(\mathrm{Re}\langle \partial_t f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}\right)^2.$$

Since the real part of the skew-symmetric operator is 0, it follows

$$H'(t)D(t) = \frac{1}{2}\left(\mathrm{Re}\langle \partial_t f - \mathcal{A}f + \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}\right)^2 - \frac{1}{2}\left(\mathrm{Re}\langle \partial_t f - \mathcal{A}f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)}\right)^2. \quad (3.15)$$

Now,

$$\begin{aligned} D'(t) &= \langle \partial_t \mathcal{S}f + \mathcal{S}\partial_t f, f\rangle_{L^2(\mathbb{R}^n)} + \langle \mathcal{S}f, \partial_t f\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \partial_t \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)} + 2\mathrm{Re}\langle \partial_t f, \mathcal{S}f\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} - \langle [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} + 2\mathrm{Re}\langle \partial_t f, \mathcal{S}f\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \partial_t \mathcal{S} + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} + 2\mathrm{Re}\langle \partial_t f - \mathcal{A}f, \mathcal{S}f\rangle_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The polarization identity, i.e.

$$\mathrm{Re}\langle x, y\rangle = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big),$$

yields that

$$D'(t) = \langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)} - \frac{1}{2}\|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}. \quad (3.16)$$

Hence, from (3.12) and (3.16) we get that

$$\begin{aligned} H''(t) &= 2\partial_t \mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)} + 2D'(t) \\ &= 2\partial_t \mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)} + 2\langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} \\ &+ \|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)} - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}, \end{aligned}$$

which proves (3.8). For (3.9) we have that

$$\begin{aligned} N'(t) &= \frac{D'(t)H(t) - H'(t)D(t)}{H(t)^2} \\ &= \frac{\langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)}}{H} + \frac{\|\partial_t f - \mathcal{S}f + \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}}{2H} - \frac{\|\partial_t f - \mathcal{S}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}}{2H} \\ &- \frac{(\mathrm{Re}\langle \partial_t f - \mathcal{A}f + \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)})^2}{2H^2} + \frac{(\mathrm{Re}\langle \partial_t f - \mathcal{A}f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)})^2}{2H^2} \\ &= \frac{\langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)}}{H} \\ &+ \frac{\|\partial_t f - \mathcal{A}f + \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n)} - (\mathrm{Re}\langle \partial_t f - \mathcal{A}f + \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)})^2}{2H^2} \\ &+ \frac{(\mathrm{Re}\langle \partial_t f - \mathcal{A}f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)})^2 - \|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n)}}{2H^2} \\ &\geqslant \frac{\langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)}}{H} - \frac{\|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}}{2H}. \end{aligned}$$

where we on the last inequality used Cauchy-Schwarz inequality and that

$$(\mathrm{Re}\langle \partial_t f - \mathcal{A}f - \mathcal{S}f, f\rangle_{L^2(\mathbb{R}^n)})^2 \geqslant 0.$$

Now, if (3.10) holds, then

$$N'(t) \geqslant -(M_0 + M_1^2 + M_2^2).$$

Then, by (3.12) for $\phi(t) = 2\mathrm{Re}\langle \partial_t f - \mathcal{S}f - \mathcal{A}f\rangle_{L^2(\mathbb{R}^n)}$ and $\Phi(t) = \int^t \phi(s)ds$,

$$\partial_t(\log H(t) + \Phi(t)) = 2N(t),$$

where since $\phi \leqslant M_1 + M_2$, it implies $\Phi \leqslant M_1 + M_2$ and $\Phi = O(1)$ on $[0,1]$. Similarly, on $[0,1]$,

$$\partial_t^2(\log H(t) + O(1)) \geqslant, 0 \tag{3.17}$$

where now $|O(1)| \leqslant N(M_0 + M_1 + M_2 + M_1^2 + M_2^2))$ in $[0,1]$. By using (3.17), we get that for $0 \leqslant s \leqslant t \leqslant \tau \leqslant 1$,

$$\partial_s(\log H(s) + O(1)) \leqslant \partial_\tau(\log H(\tau) + O(1)).$$

By integrating two times, first from 0 to t, and then from t to 1, we get

$$(1-t)\log\frac{H(t)}{H(0)} \leqslant t\log\frac{H(1)}{H(t)} + O(1),$$

and thus,

$$H(t) \leqslant O(1)H(1)^t H(0)^{1-t} \leqslant e^{N(M_0+M_1+M_2+M_1^2+M_2^2)} H(1)^t H(0)^{1-t},$$

for $0 \leqslant t \leqslant 1$. □

Lemma 3.4. Assume that $u \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n)$ satisfies

$$\partial_t u = (A + iB)(\Delta u + V(x,t)u + F(x,t)) \quad \text{in } \mathbb{R}^n \times [0,1], \tag{3.18}$$

where $A > 0,\ B \in \mathbb{R}$, V is complex-valued, $\gamma > 0$, and $\sup_{[0,1]} \|V(t)\|_{L^\infty(\mathbb{R}^n)} \leqslant M_1$. Set

$$M_2 = \sup_{[0,1]} \frac{\|e^{\gamma|x|^2}F(t)\|_{L^2(\mathbb{R}^n)}}{\|u(t)\|_{L^2(\mathbb{R}^n)}},$$

and assume that $\|e^{\gamma|x|^2}u(1)\|_{L^2(\mathbb{R}^n)}, \|e^{\gamma|x|^2}u(0)\|_{L^2(\mathbb{R}^n)}$ and M_2 are finite. Then $\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$ is logarithmically convex in $[0,1]$, and there is a universal constant N such that

$$\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{N[(A^2+B^2)(M_1^2+M_2^2)+\sqrt{A^2+B^2}(M_1+M_1)]}\|e^{\gamma|x|^2}u(0)\|^{1-t}_{L^2(\mathbb{R}^n)}\|e^{\gamma|x|^2}u(1)\|^t_{L^2(\mathbb{R}^n)}. \tag{3.19}$$

for $0 \leqslant t \leqslant 1$.

Proof. Let $f = e^{\gamma\phi}u$, where $\phi = \phi(x,t)$ is to be chosen later. As in the previous computations, we can show that f satisfies the equation

$$\partial_t f = \mathcal{S}f + \mathcal{A}f + (A+iB)(Vf + e^{\gamma\phi}F),$$

where

$$\begin{aligned}
\mathcal{S} &= A(\Delta + \gamma^2|\nabla\phi|^2) - iB\gamma(2\nabla\phi\cdot\nabla + \Delta\phi) + \gamma\partial_t\phi = AS_1 - iB\gamma S_2 + \gamma\partial_t\phi \\
\mathcal{A} &= iB(\Delta + \gamma^2|\nabla\phi|^2) - A\gamma(2\nabla\phi\cdot\nabla + \Delta\phi) = iBS_1 - A\gamma S_2.
\end{aligned}$$

Let us now compute the commutator.

$$[\mathcal{S},\mathcal{A}] = -\gamma(A^2+B^2)(S_1S_2 - S_2S_1) + iB\gamma(\partial_t\phi S_1 - S_1\partial_t\phi) + A\gamma^2(S_2\partial_t\phi - \partial_t\phi S_2).$$

By the calculations we already did for the free case, we have that

$$-\gamma(A^2+B^2)(S_1S_2 - S_2S_1) = \gamma(A^2+B^2)[\gamma^2 4\nabla\phi\cdot D^2\phi(\nabla\phi) - 4\nabla\cdot(D^2\phi\nabla) - \Delta^2\phi]$$

$$iB\gamma(\partial_t\phi S_1 - S_1\partial_t\phi) = -iB\gamma[2\nabla(\partial_t\phi)\cdot\nabla + \Delta(\partial_t\phi)]$$

$$\begin{aligned}
A\gamma^2(S_2\partial_t\phi - \partial_t\phi S_2) &= A\gamma^2[2\nabla\phi\cdot\nabla + \Delta\phi)(\partial_t\phi) - \partial_t\phi(2\nabla\phi\cdot\nabla + \Delta\phi)] \\
&= A\gamma^2[2\nabla\phi\cdot\nabla(\partial_t\phi) + 2\partial_t\phi\nabla\phi\cdot\nabla + \Delta\phi\partial_t\phi - 2\partial_t\phi\nabla\phi\cdot\nabla - \Delta\phi\partial_t\phi] \\
&= 2A\gamma^2\nabla\phi\cdot\nabla(\partial_t\phi).
\end{aligned}$$

Also,

$$\begin{aligned}
\partial_t\mathcal{S} &= \partial_t(AS_1 - iB\gamma S_2 + \gamma\partial_t\phi) \\
&= A\partial_t(\Delta + \gamma^2|\nabla\phi|^2) - iB\gamma\partial_t(2\nabla\phi\cdot\nabla + \nabla\phi) + \gamma\partial_t^2\phi \\
&= 2A\gamma^2\nabla\phi\cdot\nabla(\partial_t\phi) - iB\gamma(2\nabla(\partial_t\phi)\cdot\nabla + \Delta(\partial_t\phi)) + \gamma\partial_t^2\phi,
\end{aligned}$$

so that

$$\begin{aligned}
\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}] = \gamma\partial_t^2\phi &+ \gamma(A^2+B^2)[4\gamma^2 D^2\phi(\nabla\phi)\cdot\nabla\phi - 4\nabla\cdot(D^2\phi(\nabla) - \Delta^2\phi] \\
&+ 4A\gamma^2[\nabla\phi\cdot\nabla(\partial_t\phi)] - 2iB\gamma[2\nabla(\partial_t\phi)\cdot\nabla + \Delta(\partial_t\phi)]. \qquad (3.20)
\end{aligned}$$

Again as in the free case, if $\phi(x,t) = |x|^2$, it follows that

$$\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}] = \gamma(A^2+B^2)[32\gamma^2|x|^2 - 8\Delta],$$

and if we can justify the integration by parts,

$$\langle\partial_t\mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle = \gamma(A^2+B^2)\int_{\mathbb{R}^n} 32\gamma^2|x|^2|f|^2 + 8|\nabla f|^2 dx, \qquad (3.21)$$

so $\partial_t \mathcal{S} + [\mathcal{S}, \mathcal{A}] \geqslant 0$. We want to use Lemma 3.3, to prove the result. We have

$$|\partial_t f - \mathcal{S}f - \mathcal{A}f| \leqslant \sqrt{A^2+B^2}(|Vf| + |e^{\gamma\phi}F|) \leqslant \sqrt{A^2+B^2}(M_1|f| + e^{\gamma\phi}|F|) = \sqrt{A^2+B^2}(M_1|f| + G), \tag{3.22}$$

so if all calculations can be justified, we can use the lemma to say that $\|e^{\gamma|x|^2}u\|^2_{L^2(\mathbb{R}^n)}$ is logarithmically convex. Moreover, if $\tilde{M_1} = \sqrt{A^2+B^2}M_1$,

$$\tilde{M_2} = \sup_{t\in[0,1]} \sqrt{A^2+B^2}\frac{\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}}{\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}}$$

it follows that

$$\|e^{\gamma|x|^2}u(t)\|^2_{L^2(\mathbb{R}^n)} \leqslant e^{N(\tilde{M_1}+\tilde{M_2}+\tilde{M_1}^2+\tilde{M_2}^2)}\|e^{\gamma|x|^2}u(0)\|^{2(1-t)}_{L^2(\mathbb{R}^n)}\|e^{\gamma|x|^2}u(1)\|^{2t}_{L^2(\mathbb{R}^n)} \tag{3.23}$$

$$\leqslant e^{N(M_1+M_2+M_1^2+M_2^2)}\|e^{\gamma|x|^2}u(0)\|^{2(1-t)}_{L^2(\mathbb{R}^n)}\|e^{\gamma|x|^2}u(1)\|^{2t}_{L^2(\mathbb{R}^n)} \tag{3.24}$$

Where $M_1 = \sup_{t\in[0,1]}\|V(t)\|_{L^\infty(\mathbb{R}^n)}$ and $M_2 = \sup_{t\in[0,1]}\frac{\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}}{\|u(t)\|_{L^2(\mathbb{R}^n)}}$. The result follows after taking the square root on both sides.

For this argument to be rigorously justified, we need to know that $\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all $t \in [0,1]$. The idea will be to modify the weight in such a way that we can use the Lemma 3.1 to justify that it will be finite. See the appendix, in particular Section B.1, for the detailed computations.

We now show a similar estimate to deduce that $\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} < \infty$ for $t \in (0,1)$. □

Lemma 3.5. Assume u, A, B, V, M_1, and γ are as in Lemma 3.4. Then

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n\times[0,1])}$$

$$\leqslant N[(1+M_1)\sup_{[0,1]}\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)} + \sup_{[0,1]}\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}]. \tag{3.25}$$

Proof. We start with the formal proof. Let $f = e^{\gamma|x|^2}u$. Assuming all calculations in Lemma 3.3 are justified for f, we start by multiplying inequality (3.8) with $t(1-t)$ and integrate from 0 to 1. The left side of (3.8) becomes

$$\int_0^1 \partial_t^2 H(t)t(1-t)dt = -\int_0^1 (1-2t)\partial_t H(t)dt$$

$$= -[(1-2t)H(t)|_0^1] - 2\int_0^1 H(t)dt$$

$$= H(0) + H(1) - 2\int_0^1 H(t)dt$$

$$\leqslant H(0) + H(1). \tag{3.26}$$

For the first part of the right-hand side of (3.8), we get

$$2\int_0^1 t(1-t)\partial_t Re\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)}dt = -2\int_0^1 (1-2t)Re\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)}dt. \tag{3.27}$$

It then follows from (3.8), (3.26) and (3.27) that

$$\begin{aligned}
2\int_0^1 & t(1-t)\langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} \\
&\leqslant H(1) + H(0) + 2\int_0^1 (1-2t)Re\langle \partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)}dt \\
&\quad + \int_0^1 t(1-t)\|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}dt.
\end{aligned} \tag{3.28}$$

From (3.21),

$$\begin{aligned}
\langle \partial_t \mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)}dt &= 8N\int_{\mathbb{R}^n} (|\nabla f|^2 + 4\gamma^2|x|^2|f|^2 dxdt \\
&= N\left(2\int_{\mathbb{R}^n} |\nabla f|^2 + 4\gamma^2|x|^2|f|^2dx + 2\int_{\mathbb{R}^n} |\nabla f|^2 + 4\gamma^2|x|^2|f|^2dx\right) \\
&\geqslant N\left(2\int_{\mathbb{R}^n} |\nabla f|^2 + 4\gamma^2|x|^2|f|^2dx + 8\gamma^2\int_{\mathbb{R}^n} |x|^2|f|^2dx\right)
\end{aligned} \tag{3.29}$$

We wan to find a lower bound on $2\int_{\mathbb{R}^n} |\nabla f|^2 + 4\gamma^2|x|^2|f|^2dx$. We have that

$$\begin{aligned}
\nabla f &= e^{\gamma|x|^2}(2\gamma xu + \nabla u) \\
|\nabla f|^2 &= e^{2\gamma|x|^2}(4\gamma|x|^2|u|^2 + |\nabla u|^2 + 2\gamma xu\cdot\nabla\bar{u} + 2\gamma\nabla u\cdot x\bar{u}).
\end{aligned}$$

Integration by parts shows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla f|^2dx \\
&= \int_{\mathbb{R}^n} e^{2\gamma|x|^2}(4\gamma^2|x|^2|u|^2 + |\nabla u|^2)dx + 2\gamma\int_{\mathbb{R}^n} e^{2\gamma|x|^2}ux\cdot\nabla\bar{u}dx + 2\gamma\int_{\mathbb{R}^n} e^{2\gamma|x|^2}\nabla u\cdot x\bar{u}dx \\
&= \int_{\mathbb{R}^n} e^{2\gamma|x|^2}(4\gamma^2|x|^2|u|^2 + |\nabla u|^2)dx - 2\gamma\int_{\mathbb{R}^n} e^{2\gamma|x|^2}\nabla u\bar{u}\cdot xdx - 2\gamma\int_{\mathbb{R}^n} e^{2\gamma|x|^2}|u|^2\nabla\cdot xdx \\
&\quad - 2\gamma\int_{\mathbb{R}^n} 4\gamma e^{2\gamma|x^2}|u|^2|x|^2dx + 2\gamma\int_{\mathbb{R}^n} e^{2\gamma|x|^2}\bar{u}\nabla u\cdot xdx \\
&= \int_{\mathbb{R}^n} e^{\gamma|x|^2}|\nabla u|^2)dx - 2\gamma n\int_{\mathbb{R}^n} e^{2\gamma|x|^2}|u|^2dx - 4\gamma^2\int_{\mathbb{R}^n} |f|^2|x|^2dx,
\end{aligned}$$

where $n = \nabla\cdot x$. This shows

$$\int_{\mathbb{R}^n} |\nabla f|^2 + 4\gamma^2|x|^2|f|^2dx = \int_{\mathbb{R}^n} e^{2\gamma|x|^2}(|\nabla u|^2 - 2n\gamma|u|^2)dx. \tag{3.30}$$

Also, by using Cauchy-Schwarz and Young's inequalities,

$$\begin{aligned}4\gamma\int_{\mathbb{R}^n}(\nabla f\cdot x)\bar{f}dx &\leqslant \int_{\mathbb{R}^n}2|\nabla f|2\gamma|x||f|dx\\ &\leqslant \frac{2}{\epsilon}\left(\int_{\mathbb{R}^n}|\nabla f|^2dx\right)^{1/2}2\epsilon\gamma\left(\int_{\mathbb{R}^n}|x|^2|f|^2dx\right)^{1/2}\\ &\leqslant \frac{2}{\epsilon^2}\int_{\mathbb{R}^n}|\nabla f|^2dx+2\epsilon^2\gamma^2\int_{\mathbb{R}^n}|x|^2|f|^2dx,\end{aligned}$$

In particular, for $\epsilon=\sqrt{2}$ it follows

$$\int_{\mathbb{R}^n}4\gamma|\nabla f||x||f|dx\leqslant\int_{\mathbb{R}^n}|\nabla f|^2dx+4\gamma^2\int_{\mathbb{R}^n}|x|^2|f|^2dx, \tag{3.31}$$

On the other hand, integrating by parts shows

$$2\gamma\int_{\mathbb{R}^n}(\nabla f\cdot x)\bar{f}dx=-2\gamma\int_{\mathbb{R}^n}\nabla\cdot x|f|^2dx-2\gamma\int_{\mathbb{R}^n}fx\cdot\nabla\bar{f},$$

so that

$$2\gamma n\int_{\mathbb{R}^n}|f|^2dx=-2\gamma\left(\int_{\mathbb{R}^n}fx\cdot\nabla\bar{f}dx+\int_{\mathbb{R}^n}\nabla f\cdot x\bar{f}dx\right)\leqslant 4\gamma\int_{\mathbb{R}^n}|f||x||\nabla f|dx. \tag{3.32}$$

Then (3.32) and (3.31) implies that

$$\int_{\mathbb{R}^n}(|\nabla f|^2+4\gamma^2|x|^2|f|^2)dx\geqslant 2\gamma n\int_{\mathbb{R}^n}|f|^2dx \tag{3.33}$$

Adding (3.30) and (3.33), it follows that

$$2\int_{\mathbb{R}^n}(|\nabla f|^2+4\gamma^2|x|^2|f|^2)dx\geqslant\int_{\mathbb{R}^n}e^{2\gamma|x|^2}|\nabla u|^2dx$$

Returning back to (3.29) we see that

$$\begin{aligned}&\int_0^1 t(1-t)\langle\partial_t\mathcal{S}f+[\mathcal{S},\mathcal{A}]f,f\rangle_{L^2(\mathbb{R}^n)}dt\\ &\qquad\geqslant N\left(\int_0^1 t(1-t)\int_{\mathbb{R}^n}e^{2\gamma|x|^2}|\nabla u|^2dxdt+\int_0^1 t(1-t)|x|^2e^{2\gamma|x|^2}|u|^2dxdt\right)\\ &\qquad= N\left(\|\sqrt{t(1-t)}e^{\gamma|x|^2}|\nabla u|\|^2_{L^2(\mathbb{R}^n\times[0,1])}+\|\sqrt{t(1-t)}|x|e^{\gamma|x|^2}u\|^2_{L^2(\mathbb{R}^n\times[0,1])}\right).\end{aligned}$$

Hence, by (3.28)

$$\begin{aligned}&\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|^2_{L^2(\mathbb{R}^n\times[0,1])}+\|\sqrt{t(1-t)}|x|e^{\gamma|x|^2}u\|^2_{L^2(\mathbb{R}^n\times[0,1])}\\ &\qquad\leqslant N\int_0^1 t(1-t)\langle\partial_t\mathcal{S}f+[\mathcal{S},\mathcal{A}]f,f\rangle_{L^2(\mathbb{R}^n)}dt\\ &\qquad\leqslant N\Big((H(1)+H(0)+2\int_0^1(1-2t)Re\langle\partial_t f-\mathcal{S}f-\mathcal{A}f,f\rangle_{L^2(\mathbb{R}^n)}dt\\ &\qquad\quad+\int_0^1 t(1-t)(\|\partial_t f-\mathcal{A}f-\mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}dt\Big).\end{aligned}$$

Now we want to find an upper bound for the right-hand side. Using (3.22) we get that

$$\begin{aligned}\int_0^1 t(1-t)\|\partial_t f - \mathcal{A}f - \mathcal{S}f\|^2_{L^2(\mathbb{R}^n)}dt &\leqslant \sup_{t\in[0,1]}\int_{\mathbb{R}^n}|\partial_t f - \mathcal{A}f - \mathcal{S}f|^2dx\\ &\leqslant N\sup_{t\in[0,1]}\int_{\mathbb{R}^n}(M_1|f| + e^{\gamma|x|^2}|F|)^2dx\\ &\leqslant N\sup_{t\in[0,1]}\|M_1|f| + e^{\gamma|x|^2}F\|^2_{L^2(\mathbb{R}^n)}\\ &\leqslant N\left(\sup_{t\in[0,1]}\|M_1f\|^2_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|e^{\gamma|x|^2}F\|^2_{L^2(\mathbb{R}^n)}\right)\\ &\leqslant N\left(\sup_{t\in[0,1]}\|M_1f\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}\right)^2\end{aligned}$$

Again by using (3.22) it follows

$$\begin{aligned}2\int_0^1(1-2t)Re\langle\partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)} &\leqslant N\sup_{t\in[0,1]}|\langle\partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)}|\\ &\leqslant N\sup_{t\in[0,1]}\int_{\mathbb{R}^n}|\partial_t f - \mathcal{S}f - \mathcal{A}f||f|dx\\ &\leqslant N\sup_{t\in[0,1]}\int_{\mathbb{R}^n}(M_1|f| + e^{\gamma|x|^2}|F|)|f|dx.\end{aligned}$$

Moreover, Young's inequality implies

$$\begin{aligned}&2\int_0^1(1-2t)Re\langle\partial_t f - \mathcal{S}f - \mathcal{A}f, f\rangle_{L^2(\mathbb{R}^n)}\\ &\leqslant N\sup_{t\in[0,1]}\left(\int_{\mathbb{R}^n}\frac{1}{2}(M_1|f| + e^{\gamma|x|^2}|F|)^2 + \frac{1}{2}|f|^2dx\right)\\ &\leqslant N\sup_{t\in[0,1]}\left(\|M_1f + e^{\gamma|x|^2}F\|^2_{L^2(\mathbb{R}^n)} + \|f\|^2_{L^2(\mathbb{R}^n}\right)\\ &\leqslant N\sup_{t\in[0,1]}\left(\|M_1f + e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}\right)^2\\ &\leqslant N\sup_{t\in[0,1]}\left(\|M_1f\|_{L^2(\mathbb{R}^n)} + \|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}\right)^2\end{aligned}$$

Finally, since $H(0) + H(1) \leqslant 2\sup_{t\in[0,1]} \|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$,

$$\begin{aligned}\Big(\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n\times[0,1])}\Big)^2 \\ \leqslant N\Big(\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|^2_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma|x|^2}u\|^2_{L^2(\mathbb{R}^n\times[0,1])}\Big) \\ \leqslant N\left(\sup_{t\in[0,1]}\|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}\right)^2,\end{aligned}$$

which formally proves the result. To prove the result rigorously, we will do an argument similar to the one in the justification of Lemma 3.4. Even though we have proven in Lemma 3.4 that $\|e^{\gamma|x|^2}u(t)\|_{L^2(\mathbb{R}^n)}$ is finite for all $0 \leqslant t \leqslant 1$, it is still not clear that $\|\nabla e^{\gamma|x|^2}u(t)\|^2_{L^2(\mathbb{R}^n\times[0,1])}$ is finite, which we will need to make this argument rigorous. The idea is to modify the weight as we did in Lemma 3.4 and use Lemma 3.2 to justify that the modified weight will be finite. We then proceed by doing the computations we did above, which now can be justified rigorously, for the modified weight. The details can be found in Appendix B.2.

3.3 Proof of Theorem 3

By the previous two lemmas, we have now proven the equivalent estimates in Theorem 3 for the regularized version of the equation, for $A > 0$. To prove Theorem 3, we also need to deduce the results for $A = 0$. Let us first recall the theorem before we prove the result.

Theorem 3 (EKPV)**.** Assume that $u \in C([0,1], L^2(\mathbb{R}^n))$ satisfies

$$\partial_t u = i(\Delta u + V(x,t)u) \quad \text{in } \mathbb{R}^n \times [0,1], \tag{0.6}$$

$V(x,t) = V_1(x) + V_2(x,t)$, where V_1 is real-valued, $\|V_1\|_{L^\infty(\mathbb{R}^n)} \leqslant M_1$ and $\sup_{[0,1]} \|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}V_2(t)\|_{L^\infty(\mathbb{R}^n)} < \infty$. If there exists positive numbers α, β such that

$$\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)} < \infty \text{ and } \|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)} < \infty,$$

then $\|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}u(t)\|^{\alpha t+(1-t)\beta}_{L^2(\mathbb{R}^n)}$ is logarithmically convex in $[0,1]$ and there is a constant $N = N(\alpha,\beta)$ such that

$$\|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}u(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{N(M_1+M_2+M_1^2+M_2^2)}\|e^{\frac{|x|^2}{\beta^2}}u(0)\|^{\frac{\beta(1-t)}{\alpha t+\beta(1-t)}}_{L^2(\mathbb{R}^n)}\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|^{\frac{\alpha t}{\alpha t+\beta(1-t)}}_{L^2(\mathbb{R}^n)} \tag{0.7}$$

for all $t \in [0,1]$ and where $M_2 = \sup_{[0,1]}\|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}V_2(t)\|_{L^\infty(\mathbb{R}^n)}e^{2\sup_{[0,1]}\|ImV_2(t)\|_{L^\infty(\mathbb{R}^n)}}$. Moreover,

$$\|\sqrt{t(1-t)}e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}\nabla u\|_{L^2(\mathbb{R}^n)\times[0,1])}$$
$$\leqslant Ne^{N(M_1+M_2+M_1^2+M_2^2)}\left[\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}\right]. \tag{0.8}$$

Proof. The main idea of the proof will be to work on the regularized equation,

$$\partial_t v = (\epsilon + i)(\Delta v + V(x,t)v),$$

for $\epsilon > 0$, apply Lemma 3.4 and Lemma 3.5 to this solution, and finally use semigroup theory to obtain the result for $\epsilon = 0$. All details we use regarding semigroup theory can be found in Appendix C.

We can assume $\alpha < \beta$. If $\alpha = \beta$ we can do the previous case for $\alpha = \beta + \delta$, and then let δ to zero. If $\alpha > \beta$, then we can let $\bar{u} = u(1-t)$.

Define the operator $H := \Delta + V_1(x)$ and consider the mild solution $v = e^{t(A+iB)H}u_0 \in C([0,1], L^2(\mathbb{R}^n))$ of

$$\begin{cases} \partial_t v = (A+iB)(\Delta v + V_1(x)v) & in\ \mathbb{R}^n \times [0,1] \\ v(0) = u_0, \end{cases}$$

for $A \geqslant 0$. Since $u \in C([0,1], L^2(\mathbb{R}^n))$ is a solution of (0.6) , it satisfies the equation

$$\partial_t u = iHu + iV_2(x,t)u,$$

so that by the Duhamel formula,

$$u(x,t) = e^{itH}u_0 + i\int_0^t e^{i(t-s)H}(V_2(s)u(s))ds \quad in\ \mathbb{R}^n \times [0,1]. \tag{3.34}$$

Let $\epsilon \in (0,1]$. We then define

$$\begin{aligned} F_\epsilon(t) &= \frac{i}{\epsilon + i}e^{\epsilon tH}(V_2(t)u(t)), \\ u_\epsilon(t) &= e^{(\epsilon+i)tH}u_0 + (\epsilon+i)\int_0^t e^{(\epsilon+i)(t-s)H}F_\epsilon(s)ds. \end{aligned}$$

Then u_ϵ is a solution of the equation

$$\begin{cases} \partial_t u_\epsilon = (\epsilon+i)(Hu_\epsilon + F_\epsilon) & in\ \mathbb{R}^n \times [0,1] \\ u_\epsilon(0) = u(0). \end{cases} \tag{3.35}$$

and from Lemma A.1 we have $u_\epsilon \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$. Moreover, since H generates an analytic semigroup,

$$e^{(z_1+z_2)H} = e^{(z_2+z_1)H} = e^{z_1 H} e^{z_2 H}. \tag{3.36}$$

It follows that for $t \in [0,1]$

$$\begin{aligned} u_\epsilon(t) &= e^{\epsilon t H} e^{itH} u(0) + (\epsilon + i) \int_0^t e^{(\epsilon+i)(t-s)H} \frac{i}{(\epsilon+i)} e^{\epsilon s H} V_2 u(s) ds \\ &= e^{\epsilon t H} e^{itH} u(0) + e^{\epsilon t H} i \int_0^t e^{i(t-s)H} V_2 u(s) ds \\ &= e^{\epsilon t H} u(t), \end{aligned} \tag{3.37}$$

and in particular, for $t = 1$,

$$u_\epsilon(1) = e^{\epsilon H} u(1).$$

We now want to apply Lemma 3.1. Let us define the function $u_\epsilon^*(t) := e^{\epsilon t H} u(1)$. Observe that

$$\begin{cases} \partial_t u_\epsilon^* &= \epsilon(\Delta u_\epsilon^* + V_1 u_\epsilon^*) \\ u_\epsilon^*(0) &= u(1), \end{cases}$$

so by Lemma 3.1 with $A + iB = \epsilon, F = 0, \gamma = \frac{1}{\beta^2}, T = 1$, and $M_T = \|\epsilon V_1\|_{L^1(0,1), L^\infty(\mathbb{R}^n)} \leqslant \epsilon \|V_1\|_{L^\infty(\mathbb{R}^n)}$, we get the estimate

$$\|e^{\frac{|x|^2}{\beta^2+4\epsilon}} u_\epsilon(1)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty}} \|e^{\frac{|x|^2}{\beta^2}} u(1)\|_{L^2(\mathbb{R}^n)}. \tag{3.38}$$

Since $u_\epsilon(0) = u(0)$, we also have

$$\|e^{\frac{|x|^2}{\alpha^2}} u_\epsilon(0)\|_{L^2(\mathbb{R}^n)} = \|e^{\frac{|x|^2}{\alpha^2}} u(0)\|_{L^2(\mathbb{R}^n)}. \tag{3.39}$$

We use a similar argument to (3.38), but now to the function $F_\epsilon^*(s) := \frac{i}{\epsilon+i} e^{\epsilon s H} V_2 u(t)$. We have

$$\begin{cases} \partial_s F_\epsilon^* &= \epsilon(\Delta F_\epsilon^* + V_1 F_\epsilon^*) \\ F_\epsilon^*(0) &= V_2 u(t) \\ F_\epsilon^*(t) &= F_\epsilon(t). \end{cases}$$

Then by applying Lemma 3.1 with $A + iB = \epsilon, F = 0, \gamma = \frac{1}{(\alpha t + \beta(1-t))^2}, T = t$ it follows that for all $0 \leqslant t \leqslant 1$,

$$\|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2+4\epsilon t}} F_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty(\mathbb{R}^n)}} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} \|u(t)\|_{L^2(\mathbb{R}^n)}. \tag{3.40}$$

One last application of the lemma to the functions $F_\epsilon^*(s)$ and $u_\epsilon^{**}(s) := e^{\epsilon s H} u(t)$, $A + iB = \epsilon, F = 0, \gamma = 0$ shows that

$$\|F_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty(\mathbb{R}^n)}} \|V_2(t)\|_{L^\infty(\mathbb{R}^n)} \|u(t)\|_{L^2(\mathbb{R}^n)}, \tag{3.41}$$

$$\|u_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty(\mathbb{R}^n)}} \|u(t)\|_{L^2(\mathbb{R}^n)}. \tag{3.42}$$

Define $\alpha_\epsilon = \alpha + 2\epsilon$ and $\beta_\epsilon = \beta + 2\epsilon$, then $\beta_\epsilon^2 = \beta^2 + 4\epsilon + 4\epsilon^2 \geqslant \beta^2 + 4\epsilon$, so that together with (3.38), we get that

$$\|e^{\frac{|x|^2}{\beta_\epsilon^2}} u_\epsilon(1)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty}} \|e^{\frac{|x|^2}{\beta^2}} u(1)\|_{L^2(\mathbb{R}^n)}. \tag{3.43}$$

The same argument for (3.39) and (3.40) shows that

$$\|e^{\frac{|x|^2}{\alpha_\epsilon^2}} u_\epsilon(0)\|_{L^2(\mathbb{R}^n)} \leqslant \|e^{\frac{|x|^2}{\alpha^2}} u(0)\|_{L^2(\mathbb{R}^n)} \tag{3.44}$$

and

$$\|e^{\frac{|x|^2}{(\alpha_\epsilon t+\beta_\epsilon(1-t))^2}} F_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty(\mathbb{R}^n}} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} \|u(t)\|_{L^2(\mathbb{R}^n)}. \tag{3.45}$$

We now want to apply the Appell transformation in Lemma 2.1, to reduce to the case where $\alpha = \beta$. Let $\gamma_\epsilon = \frac{1}{\alpha_\epsilon\beta_\epsilon}$, and

$$\tilde{u}_\epsilon(x,t) = \Big(\frac{\sqrt{\alpha_\epsilon\beta_\epsilon}}{\alpha_\epsilon(1-t)+\beta_\epsilon t}\Big)^{n/2} u_\epsilon\Big(\frac{\sqrt{\alpha_\epsilon\beta_\epsilon}x}{\alpha_\epsilon(1-t)+\beta_\epsilon t}, \frac{\beta_\epsilon t}{\alpha_\epsilon(1-t)+\beta_\epsilon t}\Big) e^{\frac{(\alpha_\epsilon-\beta_\epsilon)|x|^2}{4(\epsilon+i)(\alpha_\epsilon(1-t)+\beta_\epsilon t)}}$$

Since $u_\epsilon \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$, and since $\alpha < \beta$, we have that $\alpha_\epsilon < \beta_\epsilon$ so that $\tilde{u}_\epsilon \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ as well. By Lemma 2.1, $\tilde{u}_\epsilon$ satisfies

$$\partial_t \tilde{u}_\epsilon = (\epsilon+i)(\Delta\tilde{u}_\epsilon + \tilde{V_1}^\epsilon(x,t)\tilde{u}_\epsilon + \tilde{F}_\epsilon(x,t)) \quad \text{ in } \mathbb{R}^n \times [0,1], \tag{3.46}$$

where $\tilde{V_1}^\epsilon$ is real valued, and

$$\tilde{V}_1^\epsilon(x,t) = \frac{\alpha_\epsilon\beta_\epsilon}{(\alpha_\epsilon(1-t)+\beta_\epsilon t)^2} V_1\Big(\frac{\sqrt{\alpha_\epsilon\beta_\epsilon}x}{\alpha_\epsilon(1-t)+\beta_\epsilon t}\Big),$$

$$\tilde{F}_\epsilon(x,t) = \frac{\sqrt{\alpha_\epsilon\beta_\epsilon}}{\alpha_\epsilon(1-t)+\beta_\epsilon t}^{n/2+2} F_\epsilon\Big(\frac{\sqrt{\alpha_\epsilon\beta_\epsilon}x}{\alpha_\epsilon(1-t)+\beta_\epsilon t}, \frac{\beta_\epsilon t}{\alpha_\epsilon(1-t)+\beta_\epsilon t}\Big) e^{\frac{(\alpha_\epsilon-\beta_\epsilon)|x|^2}{4(\epsilon+i)(\alpha_\epsilon(1-t)+\beta_\epsilon t)}}.$$

Moreover,

$$\sup_{[0,1]} \|\tilde{V}_1^\epsilon(t)\|_{L^\infty} \leqslant \frac{\beta_\epsilon}{\alpha_\epsilon} M_1 \leqslant \frac{\beta}{\alpha} M_1, \tag{3.47}$$

$$\|\tilde{F}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant \frac{\beta}{\alpha} \|F_\epsilon(s)\|_{L^2(\mathbb{R}^n)}, \tag{3.48}$$

and from Lemma 2.1, and since $\alpha < \beta$,

$$\begin{aligned}\|e^{\gamma_\epsilon|x|^2}\tilde{F}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} &= \frac{\alpha_\epsilon\beta_\epsilon}{(\alpha_\epsilon(1-t)+\beta_\epsilon t)^2} \|e^{[\frac{\gamma_\epsilon\alpha_\epsilon\beta_\epsilon}{(\alpha_\epsilon s+\beta_\epsilon(1-s))^2} + \frac{(\alpha_\epsilon-\beta_\epsilon)\epsilon}{4(\epsilon^2+i^2)(\alpha_\epsilon s+\beta_\epsilon(1-s))}]|x|^2} F_\epsilon(s)\|_{L^2(\mathbb{R}^n)} \\ &\leqslant \frac{\beta}{\alpha} \|e^{\frac{|x|^2}{(\alpha_\epsilon s+\beta_\epsilon(1-s))^2}} F_\epsilon(s)\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.49}$$

$$\|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} = \|e^{[\frac{1}{(\alpha_\epsilon s+\beta_\epsilon(1-s))^2}+\frac{(\alpha_\epsilon-\beta_\epsilon)\epsilon}{4(\epsilon^2+i^2)(\alpha_\epsilon s+\beta_\epsilon(1-s))}]|x|^2}u_\epsilon(s)\|_{L^2(\mathbb{R}^n)}, \tag{3.50}$$

and

$$\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant \|u_\epsilon(s)\|_{L^2(\mathbb{R}^n)} \tag{3.51}$$

for $s = \frac{\beta t}{\alpha(1-t)+\beta t}$. Observe that $t = 0 \implies s = 0$ and $t = 1 \implies s = 1$. In particular, for $t = 0$, (3.50) implies that

$$\|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(0)\|_{L^2(\mathbb{R}^n)} = \|e^{[\frac{1}{\beta_\epsilon^2}+\frac{(\alpha_\epsilon-\beta_\epsilon)\epsilon}{4(\epsilon^2+i^2)\beta_\epsilon}]|x|^2}u_\epsilon(0)\|_{L^2(\mathbb{R}^n)} \leqslant \|e^{\frac{|x|^2}{\beta^2}}u_\epsilon(0)\|_{L^2(\mathbb{R}^n)}, \tag{3.52}$$

and for $t = 1$ that

$$\|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(1)\|_{L^2(\mathbb{R}^n)} \leqslant \|e^{[\frac{|x|2}{\alpha_\epsilon^2}}u_\epsilon(1)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V_1\|_{L^\infty}}\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}, \tag{3.53}$$

where we also used (3.43). Moreover,

$$\begin{aligned}
\partial_t\|\tilde{u}_\epsilon(t)\|^2_{L^2(\mathbb{R}^n)} &= 2Re\int \partial_t\tilde{u}_\epsilon\overline{\tilde{u}_\epsilon}dx \\
&= 2Re\int(\epsilon+i)(\Delta\tilde{u}_\epsilon)\overline{\tilde{u}_\epsilon}dx + 2Re\int(\epsilon+i)\tilde{V_1}^\epsilon(x,t)|\tilde{u}_\epsilon|^2dx + 2Re\int(\epsilon+i)\tilde{F}_\epsilon\overline{\tilde{u}_\epsilon}dx \\
&\leqslant -2Re\int(\epsilon+i)|\nabla\tilde{u}_\epsilon|^2dx + 2\epsilon\|\tilde{V_1^\epsilon}\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}_\epsilon\|^2_{L^2(\mathbb{R}^n)} + 2Re\int(\epsilon+i)\tilde{F}_\epsilon\overline{\tilde{u}_\epsilon}dx \\
&\leqslant -2\epsilon\int|\nabla\tilde{u}_\epsilon|^2 + 2\epsilon\|\tilde{V_1^\epsilon}\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}_\epsilon\|^2_{L^2(\mathbb{R}^n)} + 2\int 2|\tilde{F}_\epsilon\overline{\tilde{u}_\epsilon}|dx \\
&\leqslant 2\epsilon\|\tilde{V_1^\epsilon}\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}_\epsilon\|^2_{L^2(\mathbb{R}^n)} + 4\|\tilde{F}_\epsilon\|_{L^2(\mathbb{R}^n)}\|\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Moreover, together with (3.47) and (3.48),

$$\begin{aligned}
\partial_t\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} &\leqslant \epsilon\|\tilde{V_1^\epsilon}\|_{L^\infty(\mathbb{R}^n)}\|\tilde{u}_\epsilon\|_{L^2} + 2\|\tilde{F}_\epsilon\|_{L^2(\mathbb{R}^n)} \\
&\leqslant \epsilon\frac{\beta}{\alpha}M_1\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + 2\frac{\beta}{\alpha}\|F_\epsilon(s)\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Lemma 1.1 and (3.41) implies now that

$$\begin{aligned}
\partial_t\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant\ & \epsilon\frac{\beta}{\alpha}M_1\|\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n)} + 2\frac{\beta}{\alpha}e^{\epsilon M_1}\|V_2(t)\|_{L^\infty}\|u(t)\| \\
&\leqslant \epsilon\frac{\beta}{\alpha}M_1\|\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n)} + 2\frac{\beta}{\alpha}e^{\epsilon M_1}\|V_2(t)\|_{L^\infty(\mathbb{R}^n)}N_1\|u(0)\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

We make a uniformly distributed partition of $[0,1]$, $0 = t_0, t_1, ..., t_k = 1$, k to be chosen later. Let $t_{i-1} \leqslant t \leqslant t_i,\ 0 \leqslant i \leqslant k$. Then for $N_1 = e^{\sup_{t\in[0,1]}\|ImV_2(t)\|_{L^\infty(\mathbb{R}^n)}}$,

$$\partial_t(\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}e^{-\epsilon\frac{\beta}{\alpha}M_1 t}) \leqslant 2\frac{\beta}{\alpha}e^{\epsilon M_1}\|V_2(t)\|_{L^\infty(\mathbb{R}^n)}N_1\|u(0)\|_{L^2(\mathbb{R}^n)}e^{-\epsilon\frac{\beta}{\alpha}M_1 t}$$

for $t_{i-1} \leqslant t \leqslant t_i$, and integrating from t to t_i

$$\begin{aligned}
&\|\tilde{u}_\epsilon(t_i)\|_{L^2(\mathbb{R}^n)} e^{-\epsilon\frac{\beta}{\alpha}M_1 t_i} \\
&\leqslant \|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} e^{-\epsilon\frac{\beta}{\alpha}M_1 t} + \int_t^{t_i} 2\frac{\beta}{\alpha} e^{\epsilon M_1(1-s\frac{\beta}{\alpha})} \|V_2(s)\|_{L^\infty(\mathbb{R}^n)} N_1 \|u(0)\|_{L^2(\mathbb{R}^n)} ds.
\end{aligned}$$

$$\begin{aligned}
&\|\tilde{u}_\epsilon(t_i)\|_{L^2(\mathbb{R}^n)} \\
&\leqslant e^{\epsilon\frac{\beta}{\alpha}M_1(t_i-t)} \|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + e^{\epsilon\frac{\beta}{\alpha}M_1 t_i} 2\frac{\beta}{\alpha} N_1 \|u(0)\|_{L^2(\mathbb{R}^n)} \sup_{t\in[0,1]} \|V_2\|_{L^\infty(\mathbb{R}^n)} (t_i - t) e^{\epsilon M_1}.
\end{aligned}$$

Since $t_{i-1} \leqslant t \leqslant t_i$ and $t_i - t < 1$,

$$\|\tilde{u}_\epsilon(t_i)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\frac{\beta}{\alpha}M_1} \|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + N_2(t_i - t_{i-1}) \|u(0)\|_{L^2(\mathbb{R}^n)}, \tag{3.54}$$

where

$$N_2 = 2\frac{\beta}{\alpha} e^{\epsilon M_1(\frac{\beta}{\alpha}+1)} + N_1 \sup_{t\in[0,1]} \|V_2\|_{L^\infty(\mathbb{R}^n)}.$$

Now we choose k such that $N_2 \max_i(t_i - t_{i-1}) \leqslant \frac{1}{4N_1}$. Moreover since

$$\|u_\epsilon(t)\|_{L^2(\mathbb{R}^n)} - \|u(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{\epsilon\|V\|_{L^\infty}} \|u(t)\|_{L^2(\mathbb{R}^n)} - \|u(t)\|_{L^2(\mathbb{R}^n)} \longrightarrow 0$$

when $\epsilon \to 0$, (3.51) implies that

$$\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} - \|u(s)\|_{L^2(\mathbb{R}^n)} \leqslant \|u_\epsilon(s)\|_{L^2(\mathbb{R}^n)} - \|u(s)\|_{L^2(\mathbb{R}^n)} \longrightarrow 0, \quad \text{when } \epsilon \text{ goes to zero,}$$

so $\lim_{\epsilon\to 0^+} \|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} = \|u(s)\|_{L^2(\mathbb{R}^n)}$. This, combined with Lemma 1.1, implies that $\exists\ \epsilon_0$ such that for all $0 < \epsilon < \epsilon_0$,

$$\|\tilde{u}_\epsilon(t_i)\|_{L^2(\mathbb{R}^n)} \geqslant \frac{1}{2}\|u(t_i)\|_{L^2(\mathbb{R}^n)} \geqslant \frac{1}{2N_1}\|u(0)\|_{L^2(\mathbb{R}^n)}. \tag{3.55}$$

Therefore, from (3.54) and the two inequalities above,

$$e^{\epsilon\frac{\beta}{\alpha}M_1} \|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \geqslant \frac{1}{2N_1}\|u(0)\|_{L^2(\mathbb{R}^n)} - \frac{1}{4N_1}\|u(0)\|_{L^2(\mathbb{R}^n)} = \frac{1}{4N_1}\|u(0)\|_{L^2(\mathbb{R}^n)}. \tag{3.56}$$

If $\epsilon = \min\{\epsilon_0, \frac{\alpha \log 2}{\beta M_1}\}$, we get that

$$\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \geqslant \frac{1}{8N_1}\|u(0)\|_{L^2(\mathbb{R}^n)} \tag{3.57}$$

This, together with (3.49), (3.45) and Lemma 1.1, gives us that

$$\begin{aligned}
\sup_{t\in[0,1]} \frac{\|e^{\gamma_\epsilon|x|^2}\tilde{F}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}}{\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}} &\leqslant \sup_{t\in[0,1]} \frac{8N_1\beta}{\|u(0)\|_{L^2(\mathbb{R}^n)}\alpha} e^{\epsilon M_1} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} \|u(t)\|_{L^2(\mathbb{R}^n)} \\
&\leqslant \frac{8N_1^2\beta}{\alpha} e^{\epsilon M_1} \sup_{t\in[0,1]} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} \\
&= \frac{8\beta}{\alpha} M_2(\epsilon), \qquad (3.58)
\end{aligned}$$

where

$$M_2(\epsilon) = N_1^2 e^{\epsilon M_1} \sup_{t\in[0,1]} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)},$$

which is finite when $\epsilon \to 0$ if $\sup_{t\in[0,1]} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} < \infty$, and this allows us to apply 3.4 to $\tilde{u}_\epsilon$. Indeed, since

$$\sup_{t\in[0,1]} \|\tilde{V}_1^\epsilon(t)\|_{L^\infty(\mathbb{R}n)} \leqslant \frac{\beta}{\alpha} M_1,$$

$$\sup_{t\in[0,1]} \frac{\|e^{\gamma_\epsilon |x|^2} \tilde{F}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}}{\|\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}} \leqslant \frac{8\beta}{\alpha} M_2(\epsilon),$$

from (3.52) and (3.53) we have both $\|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(1)\|_{L^2(\mathbb{R}^n)}$ finite, so by Lemma 3.4, $\|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}$ is logarithmically convex in $[0,1]$ and $\exists$ a constant N so that

$$\begin{aligned}
&\|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \\
&\leqslant e^{N[(\epsilon^2+1)(\frac{\beta^2}{\alpha^2}M_1^2+64\frac{\beta^2}{\alpha^2}M_2(\epsilon)^2)+\sqrt{\epsilon^2+1}(\frac{\beta}{\alpha}M_1+8\frac{\beta}{\alpha}M_2(\epsilon))]} \|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)}^{1-t} \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)}^{t} \\
&\leqslant e^{N\frac{\beta^2}{\alpha^2}[M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon)]} \|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)}^{1-t} \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)}^{t}. \qquad (3.59)
\end{aligned}$$

From Lemma 3.5, (3.59) and the bounds on $\tilde{F}_\epsilon$ we also get that

$$\begin{aligned}
&\|\sqrt{t(1-t)} e^{\gamma_\epsilon |x|^2} \nabla \tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x| e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} \\
&\leqslant N\left((1+\frac{\beta}{\alpha}M_1) \sup_{t\in[0,1]} \|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]} \|e^{\gamma_\epsilon |x|^2}\tilde{F}_\epsilon\|_{L^2(\mathbb{R}^n)} \right) \\
&\leqslant N\left(\sup_{t\in[0,1]} \|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]} \frac{\beta}{\alpha} e^{\epsilon M_1} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} N_1 \|u(0)\|_{L^2(\mathbb{R}^n)} \right) \\
&\leqslant N\left(\sup_{t\in[0,1]} \|e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]} \frac{\beta}{\alpha} e^{\epsilon M_1} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} N_1 \|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} \right) \\
&\leqslant N\left(e^{N[M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon)]} \|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)}^{1-t} \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)}^{t} + C\|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} \right),
\end{aligned}$$

where $C = \sup_{t\in[0,1]} \frac{\beta}{\alpha} e^{\epsilon M_1} \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}} V_2(t)\|_{L^\infty(\mathbb{R}^n)} N_1$. Now, by Young's inequality,

$$\begin{aligned}
&\|\sqrt{t(1-t)} e^{\gamma_\epsilon |x|^2} \nabla \tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x| e^{\gamma_\epsilon |x|^2}\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} \\
&\qquad \leqslant N[e^{N[M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon)]} \|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)}) \\
&\qquad + C(\|e^{\frac{|x|^2}{\beta^2}} u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)})].
\end{aligned}$$

Moreover, there exists some constant N such that $C \leqslant e^{N[M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon)]}$, so it follows that

$$\|\sqrt{t(1-t)}e^{\gamma_\epsilon|x|^2}\nabla\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])}$$
$$\leqslant Ne^{N[M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon)]}(\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}).$$

The result follows now by the relations between u and $\tilde{u}$, and by letting $\epsilon \to 0$. We justify this in Appendix B.3. □

Corollary 3.2. Let u be as in Theorem 3 and $\tilde{u}$ defined through the Appell transform(Lemma 2.1), with $\gamma = \frac{1}{\alpha\beta}$. Then under the same assumptions as in Theorem 3, $R > 0$ s.t. $[\frac{1}{2R}, 1-\frac{1}{2R}] \subset [0,1]$

$$\|e^{\gamma|x|^2}(|\tilde{u}| + |\nabla\tilde{u}|)\|_{L^2([\frac{1}{2R},1-\frac{1}{2R}]\times\mathbb{R}^n)} < \infty$$

Proof. For $t \in [\frac{1}{2R}, 1-\frac{1}{2R}]$, $\sqrt{t(1-t)} \geqslant \frac{1}{2R}$, so that by Theorem 3

$$\frac{1}{2R}\|e^{\gamma|x|^2}\nabla\tilde{u}\|_{L^2([\frac{1}{2R},1-\frac{1}{2R}]\times\mathbb{R}^n))} < \infty.$$

Hence,

$$\|e^{\gamma|x|^2}(|\tilde{u}| + |\nabla\tilde{u}|)\|_{L^2([\frac{1}{2R},1-\frac{1}{2R}]\times\mathbb{R}^n)} \leqslant \sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)} + \|e^{\gamma|x|^2}\nabla\tilde{u}\|_{L^2([\frac{1}{2R},1-\frac{1}{2R}]\times\mathbb{R}^n))} < \infty.$$

□

Recall that in Theorem 1, we have two different conditions on the potential V. We will now prove a similar result to Theorem 3 but in the case where $\lim_{R\to\infty}\|V\|_{L^|([0,1],L^\infty(\mathbb{R}^n\setminus B_R))} = 0$. As we said, the result in Theorem 3 will be fundamental to prove the main theorem. The result we obtain from the next theorem is the same, but with a different hypothesis on the potential. To prove this theorem we will admit one result from [5].

Lemma 3.6. There are N and $\epsilon_0 > 0$ such that the following holds: If $\lambda \in \mathbb{R}^n$, V is a complex-valued potential, $\|V\|_{L^1([0,1],L^\infty(\mathbb{R}^n))} \leqslant \epsilon_0$ and $u \in C[0,1], L^2(\mathbb{R}^n))$ satisfies

$$\partial_t u = i(\Delta u + V(x,t))u + F(x,t) \quad \text{in } \mathbb{R}^n \times [0,1]$$

then

$$\sup_{t\in[0,1]}\|e^{\lambda\cdot x}u(t)\|_{L^2(\mathbb{R}^n)} \leqslant N\left[\|e^{\lambda\cdot x}u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\lambda\cdot x}u(1)\|_{L^2(\mathbb{R}^n)} + \|e^{\gamma\cdot x}F(t)\|_{L^1([0,1],L^2(\mathbb{R}^n))}\right].$$

Theorem 4 (EKPV)**.** Assume that $u \in C([0,1], L^2(\mathbb{R}^n))$ satisfies

$$\partial_t u = i(\Delta u + V(x,t)u) \quad \text{in } \mathbb{R}^n \times [0,1],$$

where V is in $L^\infty(\mathbb{R}^n\times[0,1])$, $\lim_{R\to\infty}\|V\|_{L^1([0,1],L^\infty(\mathbb{R}^n)\setminus B_R))}=0$, α and β are positive constants such that $\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}$ and $\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}$ are finite. Then there is $N=N(\alpha,\beta)$ such that

$$\sup_{t\in[0,1]}\|e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}u(t)\|_{L^2(\mathbb{R}^n)}+\|\sqrt{t(1-t)}e^{\frac{|x|^2}{(\alpha t+(1-t)\beta)^2}}\nabla u\|_{L^2(\mathbb{R}^n\times[0,1])}$$
$$\leqslant Ne^{N\sup_{t\in[0,1]}\|V\|_{L^\infty(\mathbb{R}^n)}}\left[\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}+\sup_{t\in[0,1]}\|u(t)\|_{L^2(\mathbb{R}^n)}\right].$$

Proof. Define $\tilde{u}$ and $\tilde{V}$ through the Appell transformation in Lemma 2.1 with $\gamma=\frac{1}{\alpha\beta}$. Then

$$\partial_t\tilde{u}=i(\Delta\tilde{u}+\tilde{V}u). \tag{3.60}$$

Moreover,

$$\sup_{t\in[0,1]}\|\tilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}\leqslant\max\left\{\frac{\alpha}{\beta},\frac{\beta}{\alpha}\right\}\sup_{t\in[0,1]}\|V(t)\|_{L^\infty},$$

and

$$\lim_{R\to\infty}\|\tilde{V}\|_{L^1[0,1],L^\infty(\mathbb{R}^n\setminus B_R))}=0.$$

Now, let R be large enough so that

$$\|\tilde{V}\|_{L^1[0,1],L^\infty(\mathbb{R}^n\setminus B_R))}\leqslant\epsilon_0,$$

and define $\tilde{V}_R(x,t)=\mathbb{1}_{\mathbb{R}^n\setminus B_R}\tilde{V}(x,t)$, $\tilde{F}_R(x,t)=\mathbb{1}_{B_R}\tilde{V}(x,t)\tilde{u}$. Then

$$\partial_t\tilde{u}=i(\Delta\tilde{u}+\tilde{V}_R(x,t)\tilde{u})+\tilde{F}_R(x,t),$$

and we can apply Lemma 3.6. It follows that

$$\sup_{t\in[0,1]}\|e^{\lambda\cdot x}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}$$
$$\leqslant N\left[\|e^{\lambda\cdot x}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\lambda\cdot x}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)}+\|e^{\lambda\cdot x}\tilde{F}_R(t)\|_{L^1([0,1],L^2(\mathbb{R}^n))}\right]$$
$$\leqslant N\left[\|e^{\lambda\cdot x}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\lambda\cdot x}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)}+e^{|\lambda|R}\sup_{t\in[0,1]}\|\tilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}\sup_{t\in[0,1]}\|u(t)\|_{L^2(\mathbb{R}^n)}\right]. \tag{3.61}$$

Now we need to go from exponential weight to Gaussian weight. Since (3.61) hods for all $\lambda\in\mathbb{R}^n$, we can replace λ with $\lambda\sqrt{\gamma}$. Squaring both sides and multiplying with $e^{-|\lambda|^2/2}$ implies that

$$\sup_{t\in[0,1]}\int_{\mathbb{R}^n}e^{2\lambda\sqrt{\gamma}\cdot x-|\lambda|^2/2}|\tilde{u}(x,t)|^2dx$$
$$\leqslant N\Big[\int_{\mathbb{R}^n}e^{2\lambda\sqrt{\gamma}\cdot x-|\lambda|^2/2}|\tilde{u}(x,0)|^2dx+\int_{\mathbb{R}^n}e^{2\lambda\sqrt{\gamma}\cdot x-|\lambda|^2/2}|\tilde{u}(x,1)|^2dx$$
$$+e^{2|\lambda|\sqrt{\gamma}R-\lambda/2}\sup_{t\in[0,1]}\|\tilde{V}\|^2_{L^\infty(\mathbb{R}^n)}\sup_{t\in[0,1]}\|\tilde{u}(t)\|^2_{L^2(\mathbb{R}^n)}\Big],$$

Next we integrate with respect to λ over $\mathbb{R}^n$, and by using the identity

$$\int_{\mathbb{R}^n} e^{2\sqrt{\gamma}\lambda\cdot x-|\lambda|^2/2}d\lambda = (2\pi)^{n/2}e^{2\gamma|x|^2},$$

we deduce that

$$\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}$$
$$\leqslant N\Big[\|e^{\gamma|x|^2}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\gamma|x|^2}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)}+e^{2\gamma R^2}\sup_{t\in[0,1]}\|\tilde{V}\|_{L^\infty(\mathbb{R}^n)}\sup_{t\in[0,1]}\|\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}\Big].$$

Going back with the Appel transform, using the identities

$$\|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)} = \|e^{\frac{|x|^2}{(\alpha s+\beta(1-s))^2}}u(s)\|_{L^2(\mathbb{R}^n)},\quad \|\tilde{u}(t)\|_{L^2(\mathbb{R}^n)} = \|u(s)\|_{L^2(\mathbb{R}^n)}$$

completes the first part of the result. In particular, we have proven that

$$\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}$$
$$\leqslant Ne^{N\sup_{t\in[0,1]}\|V(t)\|_{L^\infty(\mathbb{R}^n)}}\Big[\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}+\sup_{t\in[0,1]}\|u(t)\|_{L^2(\mathbb{R}^n)}\Big]. \quad (3.62)$$

Now we need to prove the same bound for

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla\tilde{u}\|_{L^2(\mathbb{R}^n\times[0,1])}.$$

Here we again need to do a parabolic regularization, similar to what we did in the proof of Theorem 3 since we want to apply Lemma 3.5, which only holds in the case $A>0$. Most of the computations will be very similar to the proof of Theorem 3, so we will not do everything as detailed as we did in the previous theorem. The difference is that since we do not split up the potential as $V(x,t)=V_1(x)+V_2(x,t)$, we work with the semigroup $e^{t\Delta}$ instead of e^{tH}.

Since $\tilde{u}$ satisfies (3.60), $\tilde{u}(x,t)=e^{it\Delta}\tilde{u}_0+i\int_0^t e^{i(t-s)\Delta}(\tilde{V}(s)\tilde{u}(s))ds$. Let $\epsilon\in(0,1)$ and define

$$\tilde{F}_\epsilon(t)=\frac{i}{\epsilon+i}e^{\epsilon t\Delta}(\tilde{V}(t)\tilde{u}(t)), \quad (3.63)$$

$$\tilde{u}_\epsilon(t)=e^{(\epsilon+i)\Delta}\tilde{u}_0+i\int_0^t e^{(\epsilon+i)(t-s)\Delta}(\tilde{V}(s)\tilde{u}(s))ds. \quad (3.64)$$

The above relations, and since Δ generates an analytic semigroup, shows that

$$\tilde{u}_\epsilon(t)=e^{\epsilon t\Delta}\tilde{u}(t). \quad (3.65)$$

We want to apply Lemma 3.1, and as we did in Theorem 3, we define a new function $\tilde{u}^*_\epsilon(s)=e^{\epsilon s\Delta}\tilde{u}(t)$. Then

$$\begin{cases}\partial_s\tilde{u}^*_\epsilon=\epsilon\Delta\tilde{u}^*_\epsilon\\ \tilde{u}^*_\epsilon(x,0)=\tilde{u}(t)\\ \tilde{u}^*_\epsilon(t)=\tilde{u}_\epsilon(t).\end{cases}$$

Applying Lemma 3.1 with $A + iB = \epsilon$, $T = t$, $F = V = 0$ we get that for all $t \in [0,1]$,

$$\|e^{\frac{\epsilon\gamma|x|^2}{\epsilon+4\gamma\epsilon^2 t}}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant \|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}.$$

Since $t \in [0,1]$, $\frac{\epsilon\gamma|x|^2}{\epsilon+4\gamma\epsilon^2 t} \geqslant \frac{\epsilon\gamma|x|^2}{\epsilon+4\gamma\epsilon^2}$, and by letting $\gamma_\epsilon = \frac{\gamma}{1+4\gamma\epsilon}$, it follows that

$$\sup_{t\in[0,1]} \|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant \sup_{t\in[0,1]} \|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}. \tag{3.66}$$

Similarly, we define $\tilde{F}^*_\epsilon(s) = \frac{i}{\epsilon+i}e^{\epsilon s\Delta}(\tilde{V}(t)\tilde{u}(t))$, and apply Lemma 3.1 with $A + iB = \epsilon$ and $T = t$, $F = V = 0$. Then

$$\begin{aligned}\sup_{t\in[0,1]} \|e^{\gamma_\epsilon|x|^2}\tilde{F}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} &\leqslant \sup_{t\in[0,1]} \|e^{\gamma|x|^2}\tilde{V}(t)\tilde{u}(t)\|_{L^2(\mathbb{R}^n)} \\ &\leqslant e^{\sup_{t\in[0,1]}\|\tilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}} \sup_{t\in[0,1]} \|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}. \end{aligned}\tag{3.67}$$

Now we apply Lemma 3.5 to $\tilde{u}_\epsilon$ to deduce that

$$\begin{aligned}&\|\sqrt{t(1-t)}e^{\gamma_\epsilon|x|^2}\nabla\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} \\ &\qquad \leqslant N\left[\sup_{t\in[0,1]} \|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]} \|e^{\gamma_\epsilon|x|^2}\tilde{F}_\epsilon\|_{L^2(\mathbb{R}^n)}\right].\end{aligned}$$

Furthermore, (3.66), (3.67), (3.62) and the relations between $\tilde{u}$ and u imply

$$\begin{aligned}&\|\sqrt{t(1-t)}e^{\gamma_\epsilon|x|^2}\nabla\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}|x|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon\|_{L^2(\mathbb{R}^n)} \\ &\leqslant N\left[\sup_{t\in[0,1]} \|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)} + e^{\sup_{t\in[0,1]}\|\tilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}} \sup_{t\in[0,1]} \|e^{\gamma|x|^2}\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}\right] \\ &\leqslant Ne^{N\sup_{t\in[0,1]}\|V(t)\|_{L^\infty(\mathbb{R}^n)}}\left[\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|u(t)\|_{L^2(\mathbb{R}^n)}\right]. \end{aligned}\tag{3.68}$$

The final result follows by letting $\epsilon \to 0$ The limit process can be rigorously justified in the spirit of the argument we used for Theorem 3 in Appendix B.3. □

4 Proof of Theorem 1

In this section, we prove Theorem 1. To prove the result, we will need a Carleman estimate. This estimate is for functions g, which are compactly supported in both space and time, so there will be no problem with justifying that $\|e^{\phi}g\|_{L^2(\mathbb{R}^{n+1})} < \infty$.

Lemma 4.1. The inequality

$$\begin{aligned}&R\sqrt{\frac{\epsilon}{8\mu}}\|e^{\mu|x+Rt(1-t)e_1|^2-(1+\epsilon)R^2t(1-t)/16\mu}g\|_{L^2(\mathbb{R}^{n+1})}\\ &\leqslant \|e^{\mu|x+Rt(1-t)e_1|^2-(1+\epsilon)R^2t(1-t)/16\mu}(\partial_t - i\Delta)g\|_{L^2(\mathbb{R}^{n+1})}\end{aligned}$$

holds when $\epsilon > 0, \mu > 0, R > 0$ and $g \in C_0^{\infty}(\mathbb{R}^{n+1})$.

Proof. Let $f = e^{\phi}g$, where $\phi = \mu|x + Rt(1-t)e_1|^2 - \frac{(1+\epsilon)R^2t(1-t)}{16\mu}$. We consider the operator $e^{\phi}(\partial_t - i\Delta)e^{-\phi}$, and want to split it into symmetric and skew-symmetric parts on the form,

$$e^{\phi}(\partial_t - i\Delta)e^{-\phi}f = \partial_t f - \mathcal{S}f - \mathcal{A}f.$$

We have

$$\begin{aligned}e^{\phi}(\partial_t - i\Delta)e^{-\phi}f &= -\partial_t\phi f + \partial_t f - ie^{\phi}\Delta(e^{-\phi})f\\ &= -\partial_t\phi f + \partial_t f - i(|\nabla\phi|^2 - \Delta\phi - 2\nabla\phi\cdot\nabla + \Delta)f\\ &= \partial_t f - \mathcal{S}f - \mathcal{A}f,\end{aligned}$$

where $\mathcal{S} = -i(2\nabla\phi\cdot\nabla + \Delta\phi) + \partial_t\phi$ and $\mathcal{A} = i(\Delta + |\nabla\phi|^2)$. This is exactly as in Lemma 3.4 with $A = 0, B = 1$, and $\gamma = 1$, so by (3.20) we have

$$\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}] = \partial_t^2\phi + 4\ D^2\phi(\nabla\phi)\cdot\nabla\phi - 4\nabla\cdot(D^2\phi(\nabla) - \Delta^2\phi + -2i[2\nabla(\partial_t\phi)\cdot\nabla + \Delta(\partial_t\phi)].$$

For $\phi = \mu|x + Rt(1-t)e_1|^2 - \frac{(1+\epsilon)R^2t(1-t)}{16\mu}$, we can use almost the same computations as in Theorem 2.3 for the free Schrödinger equation for all derivatives, and we get

$$\begin{aligned}\partial_t\mathcal{S} + [\mathcal{S},\mathcal{A}] =&2\mu R^2(1-2t)^2 - 4\mu R(x_1 + Rt(1-t)) + \frac{(1+\epsilon)R^2}{8\mu} + 32\mu^3|x + Rt(1-t)e_1|^2\\ &- 8\mu\Delta - 8i\mu R(1-2t)\partial_{x_1}.\end{aligned}$$

The $L^2(\mathbb{R}^n)$ inner product will be

$$\begin{aligned}\langle\partial_t\mathcal{S}f + [\mathcal{S},\mathcal{A}]f, f\rangle_{L^2(\mathbb{R}^n)} &= \int 2\mu R^2(1-2t)^2|f|^2dx - \int 4\mu R(x_1 + Rt(1-t))|f|^2dx\\ &- \int 8i\mu R(1-2t)\partial_{x_1}f\bar{f} + \int 32\mu^3|x + Rt(1-t)e_1|^2|f|^2dx\\ &+ 8\int|\nabla f|^2dx + \frac{(1+\epsilon)}{8\mu}R^2\int|f|^2dx\\ &= (1) + (2) + (3) + (4) + (5) + (6).\end{aligned}$$

Observe that all the terms are equal to the computation we did in Chapter 2, except for (6). Therefore we can do exactly as we did in (2.17) and (2.19), and get that

$$\langle \partial_t \mathcal{S} f + [\mathcal{S}, \mathcal{A}] f, f \rangle_{L^2(\mathbb{R}^n)} \geqslant -\frac{R^2}{8\mu} \|f\|^2_{L^2(\mathbb{R}^n)} + \frac{(1+\epsilon)}{8\mu} R^2 \|f\|^2_{L^2(\mathbb{R}^n)} = \frac{\epsilon R^2}{8\mu} \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{4.1}$$

To conclude the proof we claim that

$$\|\partial_t f - \mathcal{S} f - \mathcal{A} f\|^2_{L^2(\mathbb{R}^{n+1})} \geqslant \int \langle \partial_t \mathcal{S} f + [\mathcal{S}, \mathcal{A}] f, f \rangle_{L^2(\mathbb{R}^n)} dt. \tag{4.2}$$

Then by (4.1) it follows that

$$\|\partial_t f - \mathcal{S} f - \mathcal{A} f\|_{L^2(\mathbb{R}^{n+1})} \geqslant \frac{\epsilon R^2}{8\mu} \int \|f\|^2_{L^2(\mathbb{R}^n)} dt = \frac{\epsilon R^2}{8\mu} \|f\|^2_{L^2(\mathbb{R}^{n+1})},$$

and since $e^{\phi}(\partial_t - i\Delta) e^{-\phi} f = \partial_t f - \mathcal{S} f - \mathcal{A} f$,

$$R\sqrt{\frac{\epsilon}{8\mu}} \|e^{\mu|x+Rt(1-t)e_1|^2 - \frac{(1+\epsilon)R^2 t(1-t)}{16\mu}} g\|_{L^2(\mathbb{R}^{n+1})} \leqslant \|e^{\mu|x+Rt(1-t)e_1|^2 - \frac{(1+\epsilon)R^2 t(1-t)}{16\mu}} (\partial_t - i\Delta) g\|_{L^2(\mathbb{R}^{n+1})}.$$

The claim follows since

$$\begin{aligned}
\|\partial_t f - \mathcal{S} f - \mathcal{A} f\|^2_{L^2(\mathbb{R}^{n+1})} &= \|\partial_t f - \mathcal{A} f\|^2_{L^2(\mathbb{R}^{n+1})} + \|\mathcal{S} f\|^2_{L^2(\mathbb{R}^{n+1})} - 2Re\langle \mathcal{S} f, \partial_t f - \mathcal{A} f\rangle_{L^2(\mathbb{R}^{n+1})} \\
&\geqslant -\int\int \mathcal{S} f (\overline{\partial_t f - \mathcal{A} f}) dx dt - \int\int (\partial_t f - \mathcal{A} f) \overline{\mathcal{S} f} dx dt \\
&= \int\int (\partial_t - \mathcal{A}) \mathcal{S} f \bar{f} dx dt - \int\int \mathcal{S} (\partial_t - \mathcal{A}) f \bar{f} dx dt \\
&= \int\int ((\partial_t \mathcal{S}) f + \mathcal{S} \partial_t f - \mathcal{A} \mathcal{S} f - \mathcal{S} \partial_t f + \mathcal{S} \mathcal{A} f) \bar{f} dx dt \\
&= \int\int (\partial_t \mathcal{S} + [\mathcal{S}, \mathcal{A}]) f \bar{f} dx dt \\
&= \int \langle \partial_t \mathcal{S} + [\mathcal{S}, \mathcal{A}] f, f \rangle_{L^2(\mathbb{R}^n)} dt.
\end{aligned}$$

This concludes the proof. □

Now we are ready to prove Theorem 1.

Proof. Let u be described as in the Theorem 1. Let $\tilde{u}, \tilde{V}$ be defined through the Appell transformation in Lemma 2.1, where $A + iB = i$. We have that

$$\partial_t \tilde{u} = i(\Delta \tilde{u} + \tilde{V}(x,t) \tilde{u}) \quad in\ \mathbb{R}^n \times [0,1],$$

and for $\gamma = \frac{1}{\alpha\beta}$, $\gamma > \frac{1}{2}$, $\|e^{\gamma|x|^2} \tilde{u}(0)\|_{L^2(\mathbb{R}^n)}, \|e^{\gamma|x|^2} \tilde{u}(1)\|_{L^2(\mathbb{R}^n)}$ are both finite. Let $R > 0$, and let μ and $\epsilon > 0$ small enough satisfy

$$\frac{(1+\epsilon)^{3/2}}{2(1-\epsilon)^3} < \mu \leqslant \frac{\gamma}{1+\epsilon}, \tag{4.3}$$

and

$$\frac{1-\epsilon}{2} \geqslant \frac{1}{R}, \quad \frac{1+\epsilon}{2} \leqslant 1 - \frac{1}{R}. \tag{4.4}$$

Remark. $\frac{(1+\epsilon)^{5/2}}{(1-\epsilon)^3}$ will be close to 1 if ϵ is small enough, and since $\gamma > \frac{1}{2}$, there exists μ such that (4.3) is satisfied.

We want to use Lemma 4.1, so we need to define a function g with compact support. Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\theta(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 2 \end{cases}$$

and for $M \geqslant R$, $\theta_M(x) = \theta(\frac{x}{M})$,

$$\theta_M(x) = \begin{cases} 1, & |x| \leqslant M \\ 0, & |x| > 2M. \end{cases} \tag{4.5}$$

Let $\eta_1(t) \in C_0^\infty(\mathbb{R})$ be such that

$$\eta_1(t) = \begin{cases} 1, & t \geqslant 1 \\ 0, & t < \frac{1}{2}, \end{cases}$$

$$\eta_{1,R}(t) = \eta_1(Rt) = \begin{cases} 1, & t \geqslant \frac{1}{R} \\ 0, & t < \frac{1}{2R}. \end{cases}$$

Now let $\eta_2(t) = \eta_1(1-t)$, and

$$\eta_{2,R}(t) = \eta_2(Rt) = \begin{cases} 1, & t \leqslant 1 - \frac{1}{R} \\ 0, & t \geqslant 1 - \frac{1}{2R}. \end{cases}$$

Then we define $\eta_R(t) = \eta_{1,R}(t)\eta_{2,R}(t)$. It follows

$$\eta_R(t) = \begin{cases} 1 & t \in [\frac{1}{R}, 1 - \frac{1}{R}] \\ 0 & t \in [0, \frac{1}{2R}] \cup [1 - \frac{1}{2R}, 1]. \end{cases} \tag{4.6}$$

We compute the derivatives

$$\begin{aligned} \nabla\theta_M(x) &= \frac{1}{M}\nabla\theta(\frac{x}{M}), \\ \Delta\theta_M &= \frac{1}{M^2}\Delta\theta(\frac{x}{M}), \\ \eta_R'(t) &= R\eta'(Rt), \end{aligned}$$

so that

$$\begin{aligned}
\|\nabla\theta_M(x)\|_{L^\infty(\mathbb{R}^n)} &\leqslant \frac{N}{M}\\
\|\Delta\theta_M(x)\|_{L^\infty(\mathbb{R}^n)} &\leqslant \frac{N}{M^2}\\
\|\eta_R'(t)\|_{L^\infty([0,1])} &\leqslant NR
\end{aligned}$$

for some constant N.
Finally, we define $g(x,t) = \tilde{u}(x,t)\theta_M(x)\eta_R(t)$.

It follows from a direct computation that

$$\partial_t g - i(\Delta g + \tilde{V}g) = \theta_M\eta_R'\tilde{u} - i(2\nabla\theta_M\cdot\nabla\tilde{u} + \tilde{u}\Delta\theta_M)\eta_R. \tag{4.7}$$

Now observe that for the first term on the right-hand side of (4.7)

$$supp(\theta_M\eta_R'\tilde{u}) \subset \{(x,t) : |x| < 2M,\ t \in [\frac{1}{2R}, \frac{1}{R}] \cup [1-\frac{1}{R}, 1-\frac{1}{2R}]\},$$

and on this region we have, using Young's inequality and (4.3),

$$\begin{aligned}
\mu|x + Rt(1-t)e_1|^2 &\leqslant \mu(|x|^2 + 2Rt(1-t)|x| + R^2t^2(1-t)^2)\\
&\leqslant \mu(|x|^2 + 2|x| + 1)\\
&\leqslant \mu(|x|^2 + \epsilon|x|^2 + \frac{1}{\epsilon} + 1)\\
&= \mu(|x|^2(1+\epsilon) + \mu(1+\frac{1}{\epsilon})\\
&\leqslant \gamma|x|^2 + \frac{\gamma}{\epsilon}.
\end{aligned} \tag{4.8}$$

For the second term on the right-hand side of (4.7) we have

$$supp((2\nabla\theta_M\cdot\nabla\tilde{u} + \tilde{u}\Delta\theta_M)\eta_R) \subset \{(x,t) : M \leqslant |x| \leqslant 2M,\ t \in (\frac{1}{2R}, 1-\frac{1}{2R})\},$$

so that

$$\begin{aligned}
\mu|x + Rt(1-t)e_1|^2 &\leqslant \mu(|x|^2 + 2R|x|t(1-t) + R^2t^2(1-t)^2)\\
&\leqslant \mu(|x|^2 + 2R|x| + R^2)\\
&\leqslant \mu(|x|^2 + \epsilon|x|^2 + \frac{R^2}{\epsilon} + R^2)\\
&\leqslant \mu|x|^2(1+\epsilon) + R^2(\frac{1}{\epsilon} + 1)\\
&\leqslant \gamma|x|^2 + \gamma\frac{R^2}{\epsilon}.
\end{aligned} \tag{4.9}$$

Since g has compact support in $\mathbb{R}^n \times [0,1]$ we apply the Carleman estimate in Lemma 4.1. Let

$$\phi(x,t) = \mu|x + Rt(1-t)e_1|^2 - \frac{(1+\epsilon)R^2 t(1-t)}{16\mu}.$$

Using the lemma, the bounds for $\nabla\theta_M, \Delta\theta_M$ and η_R' together with (4.8) and (4.9) we get

$$\begin{aligned} R\|e^{\phi}g\|_{L^2(\mathbb{R}^n\times[0,1])} &\leqslant N_\epsilon \|\tilde{V}\|_{L^\infty(\mathbb{R}^n\times[0,1])}\|e^{\phi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \\ &\quad + N_\epsilon R\|e^{\gamma|x|^2+\gamma/\epsilon}\tilde{u}\|_{L^2(\mathbb{R}^n\times[0,1])} \\ &\quad + N_\epsilon \frac{1}{M}\|e^{\gamma|x|^2+\gamma R^2/\epsilon}(|\tilde{u}| + |\nabla\tilde{u}|)\|_{L^2(\mathbb{R}^n\times[0,1])} \\ &\leqslant N_\epsilon \|\tilde{V}\|_{L^\infty(\mathbb{R}^n\times[0,1])}\|e^{\phi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \\ &\quad + N_\epsilon R e^{\gamma/\epsilon}\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)} \\ &\quad + N_\epsilon \frac{1}{M}e^{\gamma R^2/\epsilon}\|e^{\gamma|x|^2}(|\tilde{u}| + |\nabla\tilde{u}|)\|_{L^2(\mathbb{R}^n\times[\frac{1}{2R},1-\frac{1}{2R}])}. \end{aligned} \tag{4.10}$$

For $R \geqslant 2N_\epsilon\|\tilde{V}\|_{L^\infty(\mathbb{R}^n\times[0,1])}$,

$$\begin{aligned} R\|e^{\phi}g\|_{L^2(\mathbb{R}^n\times[0,1])} &\leqslant 2N_\epsilon R e^{\gamma/\epsilon}\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)} \\ &\quad + 2N_\epsilon \frac{1}{M}e^{\gamma R^2/\epsilon}\|e^{\gamma|x|^2}(|\tilde{u}| + |\nabla\tilde{u}|)\|_{L^2(\mathbb{R}^n\times[\frac{1}{2R},1-\frac{1}{2R}])}. \end{aligned} \tag{4.11}$$

From Corollary 3.2 we have that

$$\|e^{\gamma|x|^2}(|\tilde{u}| + |\nabla\tilde{u}|)\|_{L^2(\mathbb{R}^n\times[\frac{1}{2R},1-\frac{1}{2R}])} < \infty. \tag{4.12}$$

By letting M to $+\infty$, the last term on the right-hand side of (4.11) goes to zero, and we are only left with

$$R\|e^{\phi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \leqslant 2N_\epsilon R e^{\gamma/\epsilon}\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)}. \tag{4.13}$$

In $B_{\epsilon(1-\epsilon)^2\frac{R}{4}} \times [\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}]$, we have $|x| < \epsilon(1-\epsilon)^2\frac{R}{4} < R \leqslant M$ and $t \in [\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}] \subset [\frac{1}{R}, 1-\frac{1}{R}]$, which implies that in $B_{\epsilon(1-\epsilon)^2\frac{R}{4}} \times [\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}], g = \tilde{u}$. Moreover,

$$\begin{aligned} \phi(x,t) &\geqslant \mu(Rt(1-t) - |x|)^2 - \frac{(1+\epsilon)R^2t(1-t)}{16\mu} \\ &\geqslant \mu\left(\frac{R(1-\epsilon)^2}{4} - \frac{R\epsilon(1-\epsilon)^2}{4}\right)^2 - \frac{R^2(1+\epsilon)(1+\epsilon)^2}{64\mu} \\ &= \frac{\mu}{16}(1-\epsilon)^6R^2 - \frac{(1+\epsilon)^3R^2}{64\mu} \\ &= \frac{R^2}{64}(4\mu^2(1-\epsilon)^6 - (1+\epsilon)^3) > 0, \end{aligned} \tag{4.14}$$

where we used (4.3) in the last inequality. By using (4.14), and since $\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)}$ is finite, it follows that

$$\begin{aligned} R\|e^{\frac{R^2}{64}(4\mu^2(1-\epsilon)^6-(1+\epsilon)^3)}g\|_{L^2(B_{\epsilon(1-\epsilon)^2\frac{R}{4}}\times[\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}])} &\leqslant R\|e^{\phi}g\|_{L^2(\mathbb{R}^n\times[0,1])} \\ &\leqslant 2N_\epsilon Re^{\gamma/\epsilon}\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)} \\ &\leqslant RN_{\gamma,\epsilon}, \end{aligned}$$

or equivalently, since $g=\tilde{u}$ in $B_{\epsilon(1-\epsilon)^2\frac{R}{4}}\times[\frac{1-\epsilon}{2},\frac{1+\epsilon}{2}]$,

$$Re^{C_{\gamma,\epsilon}R^2}\|\tilde{u}\|_{L^2(B_{\epsilon(1-\epsilon)^2\frac{R}{4}}\times[\frac{(1-\epsilon)}{2},\frac{(1+\epsilon)}{2}])}\leqslant RN_{\gamma,\epsilon}. \tag{4.15}$$

We also have that

$$\|\tilde{u}(t)\|^2_{L^2(\mathbb{R}^n)}\leqslant\int_{B_{\frac{R}{4}}}|\tilde{u}(t)|^2dx+e^{-\gamma\frac{R^2}{16}}\int_{|x|>\frac{R}{4}}|\tilde{u}(t)|e^{\gamma|x|^2}dx\leqslant\|\tilde{u}(t)\|^2_{L^2(B_{\frac{R}{4}})}+e^{-\gamma\frac{R^2}{16}}N_\gamma^2, \tag{4.16}$$

and from Lemma 1.1

$$\frac{1}{N_V}\|\tilde{u}(0)\|_{L^2(\mathbb{R}^n)}\leqslant\|\tilde{u}(t)\|_{L^2(\mathbb{R}^n)}\leqslant N_V\|\tilde{u}(0)\|_{L^2(\mathbb{R}^n)},\ \text{for all } t\in[0,1],\ N_V=e^{\sup_{[0,1]}\|Im\tilde{V}(t)\|_{L^\infty(\mathbb{R}^n)}}. \tag{4.17}$$

Combining the two inequalities above,

$$\frac{1}{N_V^2}\|\tilde{u}(0)\|^2_{L^2(\mathbb{R}^n)}\leqslant\|\tilde{u}(t)\|^2_{L^2(B_{\frac{R}{4}})}+e^{-\gamma R^2/16}N_\gamma^2,$$

Then, integrating in time from $\frac{1-\epsilon}{2}$ to $\frac{1+\epsilon}{2}$ and using (4.15),

$$\begin{aligned} \epsilon\frac{1}{N_V^2}\|\tilde{u}(0)\|^2_{L^2(\mathbb{R}^n)} &\leqslant\|\tilde{u}\|^2_{L^2(B_{\frac{R}{4}}\times[(1-\epsilon)/2,(1+\epsilon)/2]}+e^{-\gamma R^2/16}N^2_{\gamma,\epsilon} \\ &\leqslant N_{\gamma,\epsilon}e^{-C_{\gamma,\epsilon}R^2}+e^{-\gamma R^2/16}N_{\gamma,\epsilon}. \end{aligned}$$

so that finally, by using that $(a+b)^\lambda\leqslant C_\lambda(a^\lambda+b^\lambda)$, $\forall\ a,b,\lambda>0$,

$$\|\tilde{u}(0)\|_{L^2(\mathbb{R}^n)}\leqslant N_{\gamma,\epsilon,V}e^{-C_{\gamma,\epsilon}R^2}+e^{-\gamma R^2/16}N_{\gamma,\epsilon,V}\longrightarrow 0,\ \text{as } R\to\infty.$$

which shows that $\tilde{u}=0$. By going back with the Appell transformation we can conclude the proof.

Remark. Since u is a $C([0,1],L^2(\mathbb{R}^n))$ solution, we can not guarantee that the function $g=\theta_M\eta_R\tilde{u}$ is regular enough. To fix the problem, let $\tilde{u}_\rho=\tilde{u}*h_\rho$, where h is a radial mollifier. Then $g_\rho=\theta_M\eta_R\tilde{u}_\rho\in C_0^\infty(\mathbb{R}^n\times[0,1])$.

Claim 4.1. If

$$\|e^{\gamma|x|^2}\tilde{u}(0)\|_{L^2(\mathbb{R}^n)} < \infty \text{ and } \|e^{\gamma|x|^2}\tilde{u}(1)\|_{L^2(\mathbb{R}^n)} < \infty,$$

then

$$\|e^{\gamma|x|^2}\tilde{u}_\rho(0)\|_{L^2(\mathbb{R}^n)} < \infty \text{ and } \|e^{\gamma|x|^2}\tilde{u}_\rho(1)\|_{L^2(\mathbb{R}^n)} < \infty.$$

Assuming the claim, we can apply the proof to $\tilde{u}_\rho$, to deduce that $\tilde{u}_\rho = 0$, and hence $\tilde{u} = 0$.

Proof of Claim: Let $\delta > 0$. By Young's inequality

$$|x|^2 = |x-y+y|^2 \leqslant |x-y|^2 + 2|x-y||y| + |y|^2 \leqslant (1+\delta)|x-y|^2 + N(\delta)|y|^2,$$

so that

$$\begin{aligned}\|e^{\frac{\gamma}{1+\delta}|x|^2}\tilde{u}_\rho(0)\|_{L^2(\mathbb{R}^n)} &= \left\|e^{\frac{\gamma}{1+\delta}|x|^2}\int_{\mathbb{R}^n}\tilde{u}_0(x-y)h_\rho(y)dy\right\|_{L^2(\mathbb{R}^n)} \\ &\leqslant \left\|\int_{\mathbb{R}^n} e^{\frac{\gamma}{1+\delta}(1+\delta)|x-y|^2}\tilde{u}_0(x-y)e^{\frac{N(\delta)}{1+\delta}\gamma|y|^2}h_\rho(y)dy\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By Young's inequality for convolution, it follows that

$$\|e^{\frac{\gamma}{1+\delta}|x|^2}\tilde{u}_\rho(0)\|_{L^2(\mathbb{R}^n)} \leqslant \|e^{\gamma|x|^2}u_0\|_{L^2(\mathbb{R}^n)}\|e^{\frac{N(\delta)}{1+\delta}|y|^2}h_\rho(y)\|_{L^1(\mathbb{R}^n)},$$

where $\|e^{\frac{N(\delta)}{1+\delta}|y|^2}h_\rho(y)\|_{L^1(\mathbb{R}^n)} < \infty$ by the right choice of h. Since this holds for all $\delta > 0$, we can apply the monotone convergence theorem and deduce the result for $\delta = 0$. By the same argument we can deduce the same result for $\|e^{\gamma|x|^2}\tilde{u}_\rho(1)\|_{L^2(\mathbb{R}^n)}$. □

5 Counterexample of a Formal Carleman Argument

In this chapter, we give an example, presented in [6], of a formal Carleman argument for which the corresponding inequalities lead to a false statement.

Let u be the solution of the free Schrödinger equation (2.1) $\mathbb{R} \times [-1, 1]$, and define $f = e^{a(t)|x|^2} u$, for some function $a(t)$ to be chosen later. We also define $H(t) = \|f(t)\|^2_{L^2(\mathbb{R}^n)}$.

From (2.12) and (2.13) in Chapter 2, we deduce that

$$\partial_t f = \mathcal{S} f + \mathcal{A} f,$$

where $\mathcal{S} = a' x^2 - 4ia(x\partial_x + 1/2)$ and $\mathcal{A} = i(\partial_x^2 + 4a^2x^2)$. Moreover, from (2.16) we get that

$$\partial_t \mathcal{S} + [\mathcal{S}, \mathcal{A}] = a'' x^2 - 8ixa'\partial_x - 4ia' + 32x^2a^3 - 8a\partial_x^2 = \frac{2a'}{a}\mathcal{S} - 8a\partial_x^2 + x^2(a'' + 32a^3 - \frac{2a'^2}{a}).$$

We require now that a satisfies

$$a'' + 32a^3 - \frac{2a'^2}{a} = 0 \quad \text{in } [-1, 1],$$

and we assume the following claim. See Appendix D for details.

Claim 5.1. If a is a solution of the second-order nonlinear ODE

$$\begin{cases} 32a^3 + a'' - \frac{2a'^2}{a} = 0 \\ a(0) = 1, \quad a'(0) = 0, \end{cases}$$

then $a(t) > 0$, even, and $\lim_{R\to\infty} Ra(R) = 0$.

It follows that

$$\langle (\partial_t \mathcal{S} + [\mathcal{S}, \mathcal{A}]) f, f \rangle_{L^2(\mathbb{R})} \geqslant \frac{2a'}{a} \langle \mathcal{S} f, f \rangle_{L^2(\mathbb{R})}.$$

Moreover,

$$\begin{aligned} \partial_t^2 \log H(t) &\geqslant 2 \frac{\langle (\partial_t \mathcal{S} + [\mathcal{S}, \mathcal{A}]) f, f \rangle_{L^2(\mathbb{R})}}{\langle f, f \rangle_{L^2(\mathbb{R})}} \\ &\geqslant 4 \frac{a'}{a} \frac{\langle \mathcal{S} f, f \rangle_{L^2(\mathbb{R})}}{\langle f, f \rangle_{L^2(\mathbb{R})}} \\ &= 2 \frac{a'}{a} \frac{H'}{H} \\ &= 2 \frac{a'}{a} \partial_t (\log H(t)). \end{aligned}$$

From this, we deduce that

$$\partial_t (a^{-2} \partial_t (\log H(t))) \geqslant 0 \quad \text{in } [-1, 1],$$

which yields that for $-1 \leqslant s \leqslant 0 \leqslant \tau \leqslant 1$,

$$a^2(\tau)\partial_s \log H(s) \leqslant a^2(s)\partial_\tau \log H(\tau).$$

Integrating the above inequality from $-1 \leqslant s \leqslant 0$ and then from $0 \leqslant \tau \leqslant 1$, and using that a is an even function, we get that

$$H(0) \leqslant H(-1)^{1/2} H(1)^{1/2}. \tag{5.1}$$

Define the function $a_R(t) = Ra(Rt)$. Then a_R also satisfies the ODE, so by (5.1)

$$\|e^{Rx^2} u(0)\|_{L^2(\mathbb{R})} \leqslant \|e^{Ra(R)x^2} u(-1)\|_{L^2(\mathbb{R})} \|e^{Ra(R)x^2} u(1)\|_{L^2(\mathbb{R})}.$$

If we let $R \to \infty$ in the above inequality, the right-hand side stays finite if we assume that $u \in C([-1,1], L^2(\mathbb{R}))$, while the left-hand side goes to infinity, unless $u \equiv 0$. This would imply that all solutions of the free Schrödinger equation are zero, but we can find an initial data that contradicts this fact.

So what went wrong in this example? Observe first that we did not assume anything on the weighted norms $\|e^{\gamma|x|^2} u(-1)\|_{L^2(\mathbb{R})}$ and $\|e^{\gamma|x|^2} u(1)\|_{L^2(\mathbb{R})}$. If we add this assumption, we have seen from Hardy's uncertainty principle that $u \equiv 0$. Since we did not assume this condition to hold, we can find examples of initial data not satisfying this, which makes the statement false. For example, if $u_0 = e^{-|x|^2}$, then

$$\|e^{\frac{|x|^2}{\beta^2}} u_0\|_{L^2(\mathbb{R})} = \|e^{|x|^2(\frac{1}{\beta^2}-1)}\|_{L^2(\mathbb{R})},$$

which is finite if and only if

$$\beta > 1.$$

Moreover,

$$u(x,t) = (4it-1)^{-1/2} e^{-\frac{-|x|^2}{4it+1}},$$

so that

$$u(x,1) = Ce^{\frac{-4i|x|^2}{17}} e^{\frac{|x|^2}{17}},$$

which means that $\|e^{\frac{|x|^2}{\alpha}^2} u(x,1)\|_{L^2(\mathbb{R})}$ can never be finite for any $\alpha > 0$, and Hardy's uncertainty principle can never be applied.

The reason why this argument breaks down is that because for the specific weight function $e^{a(t)|x|^2}$, the weighted L^2 norm $\|e^{a(t)|x|^2} u(t)\|_{L^2(\mathbb{R})}$ is not finite for $0 < t < 1$.

6 Application to the Non-Linear Schrödinger Equation

Consider now the non-linear Schrödinger equation

$$\begin{cases} \partial_t u = i(\Delta u + F(u,\bar{u})) \quad \text{in } \mathbb{R}^n \times [0,1] \\ u(x,0) = u_0. \end{cases} \tag{6.1}$$

Suppose that u is a solution to (6.1) such that for $\alpha\beta < 2$,

$$\|e^{\frac{|x|^2}{\beta^2}} u_0\|_{L^2(\mathbb{R}^n)} \text{ and } \|e^{\frac{|x|^2}{\alpha^2}} u(1)\|_{L^2(\mathbb{R}^n)} \tag{6.2}$$

are finite. Then

$$\partial_t u = i(\Delta u + V(x,t)u)$$

where $V(x,t) = \frac{F(u,\bar{u})}{u}$. If V satisfies either

$$\lim_{R\to\infty} \|V\|_{L^1([0,1],L^\infty(\mathbb{R}^n \setminus B_R))}) = 0 \tag{6.3}$$

or

$$\sup_{t\in[0,1]} \|e^{\frac{|x|^2}{(\alpha t + (1-t)\beta)^2}} V(t)\|_{L^\infty(\mathbb{R}^n)} < \infty, \tag{6.4}$$

we can conclude by Theorem 1 that $u \equiv 0$. In particular, consider the cubic NLS

$$\partial_t u = i(\Delta u + |u|^2 u) \quad \text{in } \mathbb{R}^n \times [0,1]$$

and let u be a $C([0,1], H^k(\mathbb{R}^n))$ for $k \in \mathbb{Z}^+$ $k > n/2$ solution (see [19] for existence of solutions there) such that (6.2) is satisfied. Then

$$V(x,t) = \frac{|u|^2 u}{u} = |u|^2.$$

Since $k > n/2$ we have by the Sobolev embedding theorem, and since H^k is an algebra, that

$$\begin{aligned} \|V\|_{L^\infty(\mathbb{R}^n\setminus B_R)} &= \||u|^2\|_{L^\infty(\mathbb{R}^n\setminus B_R)} \\ &\leqslant \|u\|^2_{H^k(\mathbb{R}^n\setminus B_R)}, \end{aligned}$$

so that

$$\lim_{R\to\infty} \|V\|_{L^1([0,1],L^\infty(\mathbb{R}^n\setminus B_R))} \leqslant N \lim_{R\to\infty} \int_0^1 \int_{\mathbb{R}^n} \sum_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n\setminus B_R)} = 0$$

by applying the dominated convergence theorem. This means that the potential V satisfies the condition (6.3), so that by applying Theorem 1, $u \equiv 0$.

Remark. The result can also be generalized to the case where u_1 and u_2 are two solutions of the cubic NLS. In particular, this is just a general case of the following Theorem from [6].

Theorem 2 (EKPV)**.** Let u_1 and u_2 be $(C[0,1], H^k(\mathbb{R}^n))$ solutions of (0.2) with $k \in \mathbb{Z}^+,\ k > n/2,\ F : \mathbb{C}^2 \to \mathbb{C},\ F \in C^k$ and $F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0$. If there are positive constants α and β with $\alpha\beta < 2$ such that $\|e^{\frac{|x|^2}{\beta^2}}(u_1(0) - u_2(0))\|_{L^2(\mathbb{R}^n)}$, and $\|e^{\frac{|x|^2}{\alpha^2}}(u_1(1) - u_2(1))\|_{L^2(\mathbb{R}^n)}$ are finite. Then $u_1 \equiv u_2$.

A Parabolic Regularization

A.1 A Classical Energy Estimate

One of the main steps in the proof was to do a parabolic regularization. In this section, we go into more detail on the parabolic regularization to the Schrödinger equation and prove some useful estimates.

We consider u to be the solution to the problem

$$\begin{cases} \partial_t u = (A + iB)(\Delta u + V(x,t)u + F(x,t)) \\ u(x,0) = u_0 \end{cases}, \tag{A.1}$$

where $A > 0, B \in \mathbb{R}$. Observe that if $u_0 \in L^2(\mathbb{R}^n)$, then by the semigroup theory,(see Appendix C) there exists a solution $u \in C([0,1], L^2(\mathbb{R}^n))$. However, we have more regularity on the solution.

Lemma A.1. Let u satisfy the equation

$$\begin{cases} \partial_t u = (A + iB)(\Delta u + V(x,t)u + F(x,t)) \quad \textit{in } \mathbb{R}^n \times [0,1] \\ u(x,0) = u_0 \end{cases}$$

where $A > 0,\ B \in \mathbb{R}$. Then for all $t \in [0,1], M_V = 2\sup_{t\in[0,1]} \|AReV - BImV\|_{L^\infty(\mathbb{R}^n)}$ and $M_{AB} = \sqrt{A^2 + B^2}$ we have

$$\|u(t)\|^2_{L^2(\mathbb{R}^n)} + 2A\int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^n)} ds \leqslant e^{M_V + M_{AB}} \|u_0\|_{L^2(\mathbb{R}^n)} + M_{AB} e^{M_V + M_{AB}} \|F\|^2_{L^2(\mathbb{R}^n \times [0,t])}.$$

In particular, if $u_0 \in L^2(\mathbb{R}^n)$, $F \in L^2(\mathbb{R}^n \times [0,1])$ and V is bounded then

$$u \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n)).$$

Proof. Formally,

$$\begin{aligned} \partial_t \|u\|^2_{L^2(\mathbb{R}^n)} &= 2Re\int_{\mathbb{R}^n} \partial_t u\bar{u}dx \\ &= 2Re(A + iB)\left(\int_{\mathbb{R}^n} \Delta u\bar{u} + V(x,t)|u|^2 + F(x,t)\bar{u}dx\right) \\ &= -2ARe\int_{\mathbb{R}^n} |\nabla u|^2 dx + 2Re(A + iB)\int_{\mathbb{R}^n} V(x,t)|u|^2 + F(x,t)\bar{u}dx \\ &\leqslant -2A\int_{\mathbb{R}^n)} |\nabla u|^2 dx + 2\sup_{t\in[0,1]} \|AReV - BImV\|_{L^\infty(\mathbb{R}^n)} \|u\|^2_{L^2(\mathbb{R}^n)} \\ &\quad + 2\sqrt{A^2 + B^2}\|F\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

By Young's inequality, it follows that

$$\partial_t \|u\|^2_{L^2(\mathbb{R}^n)} \leqslant -2A \int_{\mathbb{R}^n)} |\nabla u|^2 dx + (M_V + M_{AB})\|u\|^2_{L^2(\mathbb{R}^n)} + M_{AB}\|F\|^2_{L^2(\mathbb{R}^n)}.$$

Integrating from 0 to t, we get that

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leqslant \|u(0)\|_{L^2(\mathbb{R}^n)} - 2A\int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^n)} ds + M_{AB}\int_0^t \|F\|^2_{L^2(\mathbb{R}^n)} ds$$
$$+ (M_V + M_{AB})\int_0^t \|u\|^2_{L^2(\mathbb{R}^n)} ds.$$

Applying Grönwall's lemma (1.1) with

$$\phi(t) = \|u(0)\|_{L^2(\mathbb{R}^n)} - 2A\int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^n)} ds + M_{AB}\int_0^t \|F\|^2_{L^2(\mathbb{R}^n)} ds,$$

and $\psi(t) = M_V + M_{AB}$, we deduce that

$$\|u\|^2_{L^2(\mathbb{R}^n)} \leqslant \phi(t) + (M_V + M_{AB})\int_0^t \phi(s) e^{(M_V - M_{AB}(t-s)} ds.$$

Since

$$\begin{aligned}
&(M_V + M_{AB})\int_0^t \phi(s) e^{(M_V+M_{AB})(t-s)} ds \\
&= (M_V + M_{AB})\Big(\int_0^t \|u(0)\|^2_{L^2(\mathbb{R}^n)} e^{(M_V+M_{AB})(t-s)} ds + \int_0^t M_{AB}\int_0^s \|F\|^2_{L^2(\mathbb{R}^n)} d\tau e^{(M_V+M_{AB})(t-s)} ds \\
&\quad - 2A\int_0^t\int_0^s \|\nabla u\|_{L^2(\mathbb{R}^n)} d\tau e^{(M_V+M_{AB})(t-s)} ds\Big) \\
&\leqslant \|f(0)\|^2_{L^2(\mathbb{R}^n)}(e^{(M_V+M_{AB})t} - 1) + M_{AB}\int_0^t \|F\|^2_{L^2(\mathbb{R}^n)} d\tau (e^{(M_V+M_{AB})t} - 1) \\
&\quad - 2A\int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^n)} d\tau (e^{(M_V+M_{AB})t} - 1),
\end{aligned}$$

we deduce that

$$\|u\|^2_{L^2(\mathbb{R}^n)} + 2A\int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \leqslant e^{M_V+M_{AB}}\|u(0)\|^2_{L^2(\mathbb{R}^n)} + M_{AB}e^{M_V+M_{AB}}\|F\|^2_{L^2(\mathbb{R}^n\times[0,t])}.$$

This proves the result formally for $u_0 \in L^2(\mathbb{R}^n)$ and rigorously for $u_0 \in H^s(\mathbb{R}^n)$, $s > n/2+2$, by the Sobolev Embedding Theorem. If $u_0 \in L^2(\mathbb{R}^n)$, there exists a sequence $\{u_0^k\} \in H^s(\mathbb{R}^n)$ such that

$$u_0^k \longrightarrow u_0 \text{ in } L^2(\mathbb{R}^n)$$

and by the formal argument, $u^k \in L^\infty([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$ and

$$\|u^k\|_{L^\infty([0,1],L^2(\mathbb{R}^n))} + \|u^k\|_{L^2([0,1],H^1(\mathbb{R}^n))} < \infty.$$

Then by the Banach-Alaoglu Theorem $\exists$ a subsequence $\{u^{kj}\}$ of $\{u^k\}$ and $\tilde{u}$ such that

$$u^{kj} \longrightarrow \tilde{u} \text{ weak * in } L^\infty([0,1], L^2(\mathbb{R}^n))$$

and

$$u^{kj} \rightharpoonup \tilde{u} \text{ in } L^2([0,1], H^1(\mathbb{R}^n)).$$

On the other hand, we have by the semigroup theory that $u \in C([0,1], L^2(\mathbb{R}^n))$, and that, see in particular the argument we used for convergence in Lemma 1.1,

$$u^k \longrightarrow u \text{ in } C([0,1], L^2(\mathbb{R}^n)),$$

and therefore also weakly * in $L^\infty([0,1], L^2(\mathbb{R}^n))$. By uniqueness of the weak * limit, $\tilde{u} = u$, and hence $u \in C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$. □

Remark. Now it makes sense to consider solutions u of (A.1) to be in $C([0,1], L^2(\mathbb{R}^n)) \cap L^2([0,1], H^1(\mathbb{R}^n))$, which we in particular do in Lemma 3.1, when $u_0 \in L^2(\mathbb{R}^n)$.

A.2 Proof of Lemma 3.2

Proof. We proceed exactly as in Lemma 3.1 to deduce that

$$\begin{aligned}
\operatorname{Re}\langle Sf, f\rangle_{L^2(\mathbb{R}^n)} = &- \int_{\mathbb{R}^n} A|\nabla f|^2 dx + \int_{\mathbb{R}^n} (A|\nabla \phi|^2 + \partial_t \phi)|f|^2 dx + 2B \operatorname{Im} \int_{\mathbb{R}^n} \nabla \phi \cdot \nabla f \bar{f} dx \\
&+ \int_{\mathbb{R}^n} (A\operatorname{Re} V - B\operatorname{Im} V)|f|^2 dx \\
&= (1) + (2) + (3) + (4).
\end{aligned}$$

Using Young with $\epsilon = \frac{A}{2B}$ instead of $\frac{A}{B}$ we get that

$$|(3)| \leqslant \frac{A}{2} \int_{\mathbb{R}^n} |\nabla f|^2 dx + \frac{2B^2}{A} \int_{\mathbb{R}^n} |\nabla \phi|^2 |f|^2 dx.$$

Moreover, we are also interested in keeping $-A \int_{\mathbb{R}^n} |\nabla \phi|^2 |f|^2$, so we write

$$\begin{aligned}
\operatorname{Re}\langle Sf, f\rangle_{L^2(\mathbb{R}^n)} \leqslant &\frac{-A}{2} \int_{\mathbb{R}^n} |\nabla f|^2 dx - A \int_{\mathbb{R}^n)} |\nabla \phi|^2 |f|^2 dx \\
&+ \int_{\mathbb{R}^n} \left[\left(\frac{2B^2}{A} + 2A \right) |\nabla \phi|^2 + \partial_t \phi \right] |f|^2 dx \\
&+ \|AReV - BImV\|_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

If we choose $a(t) = \frac{\gamma A}{8(A^2+B^2)\gamma t + A}$, and $\phi(x,t) = a(t)|x|^2$, then

$$(\frac{2B^2}{A} + 2A)|\nabla\phi|^2 + \partial_t\phi = 0, \tag{A.2}$$

so proceeding with the argument just as in Lemma 3.1, we will deduce that

$$\begin{aligned}\partial_t\|f\|^2_{L^2(\mathbb{R}^n)} \leqslant{}& 2\sup_{t\in[0,1]}\|AReV^+ - BImV\|_{L^\infty(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n} + 2\sqrt{A^2+B^2}\|e^\phi F\|_{L^2(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}\\ &- A\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - 2A\|\nabla(\phi)f\|^2_{L^2(\mathbb{R}^n)}\\ ={}& M_V\|f\|^2_{L^2(\mathbb{R}^n)} + 2M_{AB}\|e^\phi F\|_{L^2(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)} - A\|\nabla f\|^2_{L^2(\mathbb{R}^n)} - 2A\|\nabla(\phi)f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Integrating from 0 to t, and using Young's inequality, implies

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^n)} \leqslant{}&\|f(0)\|^2_{L^2(\mathbb{R}^n)} - A\int_0^t\|\nabla f\|^2_{L^2(\mathbb{R}^n)}ds - 2A\int_0^t\|\nabla\phi f\|^2_{L^2(\mathbb{R}^n)}ds + M_{AB}\int_0^t\|e^\phi F\|^2_{L^2(\mathbb{R}^n)}ds\\ &+ (M_V + M_{AB})\int_0^t\|f\|^2_{L^2(\mathbb{R}^n)}ds.\end{aligned}$$

Applying Grönwall's lemma (1.1), we get

$$\|f\|^2_{L^2(\mathbb{R}^n)} \leqslant \psi(t) + \int_0^t\psi(s)(M_V + M_{AB})e^{(M_V+M_{AB})(t-s)}ds,$$

where

$$\psi(t) = -A\int_0^t\|\nabla f\|^2_{L^2(\mathbb{R}^n)}ds - 2A\int_0^t\|\nabla(\phi)f\|^2_{L^2(\mathbb{R}^n)}ds + \|f(0)\|_{L^2(\mathbb{R}^n)} + M_{AB}\int_0^t\|e^\phi F\|^2_{L^2(\mathbb{R}^n)}ds.$$

Moreover, we have that

$$\begin{aligned}&(M_V + M_{AB})\int_0^t\psi(s)e^{(M_V+M_{AB})(t-s)}ds\\ &= (M_V + M_{AB})\Big(\int_0^t\|f(0)\|^2_{L^2(\mathbb{R}^n)}e^{(M_V+M_{AB})(t-s)}ds\\ &\quad + \int_0^t M_{AB}\int_0^s\|e^\phi F\|^2_{L^2(\mathbb{R}^n)}d\tau e^{(M_V+M_{AB})(t-s)}ds - A\int_0^t\int_0^s\|\nabla f\|_{L^2(\mathbb{R}^n)}d\tau e^{(M_V+M_{AB})(t-s)}ds\\ &\quad - 2A\int_0^t\int_0^s\|\nabla\phi f\|_{L^2(\mathbb{R}^n)}d\tau e^{(M_V+M_{AB})(t-s)}ds\Big)\\ &\leqslant \|f(0)\|^2_{L^2(\mathbb{R}^n)}(e^{(M_V+M_{AB})t} - 1) + M_{AB}\int_0^t\|e^\phi F\|^2_{L^2(\mathbb{R}^n)}d\tau(e^{(M_V+M_{AB})t} - 1)\\ &\quad - A\int_0^t\|\nabla f\|^2_{L^2(\mathbb{R}^n)}d\tau(e^{(M_V+M_{AB})t} - 1) - 2A\int_0^t\|\nabla\phi f\|^2_{L^2(\mathbb{R}^n)}d\tau(e^{(M_V+M_{AB})t} - 1).\end{aligned}$$

Thus,

$$\|f\|^2_{L^2(\mathbb{R}^n)} \leqslant e^{(M_V+M_{AB})t}\Big(- A\int_0^t \|\nabla f\|_{L^2(\mathbb{R}^n)}ds - 2A\int_0^t \|\nabla\phi f\|_{L^2(\mathbb{R}^n)}ds + \|f(0)\|_{L^2(\mathbb{R}^n)} + M_{AB}\int_0^t \|e^{\phi}F\|_{L^2(\mathbb{R}^n)}ds\Big),$$

which implies

$$\|f\|^2_{L^2(\mathbb{R}^n)} + A\|\nabla f\|^2_{L^2(\mathbb{R}^n\times[0,t])} + 2A\|\nabla\phi f\|^2_{L^2(\mathbb{R}^n\times[0,t])} \leqslant \|f(0)\|_{L^2(\mathbb{R}^n)}e^{(M_V+M_{AB})} + M_{AB}e^{(M_V+M_{AB})}\|e^{\phi}F\|_{L^2(\mathbb{R}^n\times[0,t])},$$

when $f(x,t) = e^{\frac{\gamma A|x|^2}{8(A^2+B^2)\gamma t+A}}u(x,t)$, $M_V = \sup_{t\in[0,1]}\|AReV^+ - BImV\|_{L^\infty(\mathbb{R}^n)}$ and $M_{AB} = \sqrt{A^2+B^2}$. The formal argument can be made rigorous by the same argument as in Lemma 3.1. □

B Justification of Computations

B.1 Lemma 3.4

We will now prove the argument in Lemma 3.4 rigorously. Let $\rho \in (0,1), a \in (0,1/2)$ we define a new function

$$\phi_a(x) = \begin{cases} |x|^2, & |x| < 1 \\ \frac{2|x|^{2-a}-a}{2-a}, & |x| \geqslant 1, \end{cases}$$

and let $\phi_{a,\rho} = \phi_a * \theta_\rho$, where $\theta \in C_0^\infty(\mathbb{R}^n)$ is a radial mollifier. Then we define $f_{a,\rho} = e^{\gamma\phi_{a,\rho}}u$. We will replace f with $f_{a,\rho}$, so we need to compute the derivatives and $\partial_t \mathcal{S}_{a,\rho} f_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] f_{a,\rho}$. Observe that at infinity $\phi_{a,\rho}$ does not grow faster then $|x|^{2-a}$, so we can use Lemma 3.1 to justify that $\|e^{\gamma\phi_{a,\rho}}u(t)\|_{L^2(\mathbb{R}^n)}$ will be finite for all time $0 \leqslant t \leqslant 1$. We see that ϕ_a is $C^1(\mathbb{R}^n)$, but not $C^2(\mathbb{R}^n)$. However, taking the second derivative still makes sense pointwise but is not continuous. Indeed, have that

$$\nabla\phi_a = \begin{cases} 2x, & |x| < 1 \\ 2x|x|^{-a} & |x| \geqslant 1 \end{cases}$$

$$\Delta\phi_a = \begin{cases} 2n, & |x| < 1 \\ 2(n-a)|x|^{-a} & |x| \geqslant 1 \end{cases}$$

$\Delta\phi_a$ is not continuous for $|x| = 1$, and we compute $\partial_j \Delta\phi_a$ in the distributional sense. We have that

$$\partial_j \Delta\phi_a(x) = -2a(n-a)x_j|x|^{-a-2}\mathbb{1}_{|x|\geqslant 1} - 2ax_j d\sigma_1, \tag{B.1}$$

where $d\sigma_1$ is the surface measure on ∂B_1. Moreover, ϕ_a is a convex function. Indeed, since ϕ_a is a radial function, and

$$\phi_a' = \begin{cases} 2r, & r < 1 \\ 2r^{1-a}, & r \geqslant 1 \end{cases}$$

$$\phi_a'' = \begin{cases} 2, & r < 1 \\ 2(1-a)r^{-a}, & r \geqslant 1 \end{cases}$$

we see that ϕ_a is convex as a radial function. We claim that this implies that the Hessian-matrix $D^2\phi_a$ is positive definite. We do the computation for $n = 2$, but the general case is similar. For $|x| < 1$ the calculation is trivial, so we only consider the case $|x| \geqslant 1$.

$$D^2\phi_a = \phi_a'' \begin{bmatrix} \frac{x_1^2}{r^2} & \frac{x_1x_2}{r^2} \\ \frac{x_1x_2}{r^2} & \frac{x_2^2}{r^2} \end{bmatrix} + \phi_a' \begin{bmatrix} \frac{1}{r} - \frac{x_1^2}{r^3} & -\frac{x_1x_2}{r^3} \\ -\frac{x_1x_2}{r^3} & \frac{1}{r} - \frac{x_2^2}{r^3} \end{bmatrix} := D_1 + D_2 \tag{B.2}$$

For a vector $(h,k) \in \mathbb{R}^2$ we compute $(h,k)D_1(h,k)^T$ and $(h,k)D^2(h,k)^T$.

$$(h,k)D_1(h,k)^T = \phi_a''(\frac{x_1^2}{r^2}h^2 + 2\frac{x_1x_2}{r^2}hk + \frac{x_2^2}{r^2}k^2) = \phi_a''(\frac{x_1h}{r} + \frac{x_2k}{r})^2, \tag{B.3}$$

$$(h,k)D_2(h,k)^T = \frac{\phi_a'}{r}(1-\frac{x_1^2}{r^2})h^2 - \frac{2x_1x_2}{r^2}hk + (1-\frac{x_2^2}{r^2})k^2 = \frac{\phi_a'}{r}\left(h^2+k^2-(\frac{x_1}{r}h+\frac{x_2}{r}k)^2\right) \tag{B.4}$$

Adding (B.3) and (B.4) we deduce that

$$\begin{aligned}
(h,k)D^2\phi_a(h,k)^T &= 2(1-a)r^{-a}\left(\frac{x_1h}{r}+\frac{x_2k}{r}\right)^2 + 2r^{-a}\left(h^2+k^2-\left(\frac{x_1h}{r}+\frac{x_2k}{r}\right)^2\right)\\
&= -2ar^{-a}\left(\frac{x_1h}{r}+\frac{x_2k}{r}\right)^2 + 2r^{-a}(h^2+k^2)\\
&\geqslant -4ar^{-a}\left(\frac{(x_1h)^2}{r^2}+\frac{(x_2k)^2}{r^2}\right) + 2r^{-a}(h^2+k^2)\\
&\geqslant -4ar^{-a}(h^2+k^2) + 2r^{-a}(h^2+k^2)\\
&= 2r^{-a}(1-2a)(h^2+k^2) \geqslant 0
\end{aligned}$$

for $a < \frac{1}{2}$, and we deduce that ϕ_a is convex. However, since we want to work with $\phi_{a,\rho}$, we want to show that taking the convolution does not change this property. We claim that $D^2(\phi_a * \theta_\rho) = D^2\phi_a * \theta_\rho$. Indeed, for n=2, we have that

$$\begin{aligned}
D^2(\phi_a*\theta_\rho) &= \begin{bmatrix} \frac{\partial^2\phi_a}{\partial x_1^2}*\theta_\rho & \frac{\partial^2\phi_a}{\partial x_1x_2}*\theta_\rho \\ \frac{\partial^2\phi_a}{\partial x_2x_1}*\theta_\rho & \frac{\partial^2\phi_a}{\partial x_2^2}*\theta_\rho \end{bmatrix}\\
&= \begin{bmatrix} \int\frac{\partial^2\phi_a}{\partial x_1^2}(s)\theta_\rho(t-s)ds & \int\frac{\partial^2\phi_a}{\partial x_1x_2}(s)\theta_\rho(t-s)ds \\ \int\frac{\partial^2\phi_a}{\partial x_2x_1}(s)\theta_\rho(t-s)ds & \int\frac{\partial^2\phi_a}{\partial x_2^2}(s)\theta_\rho(t-s)ds \end{bmatrix}\\
&= \int\theta_\rho(t-s)\begin{bmatrix} \frac{\partial^2\phi_a}{\partial x_1^2}(s) & \frac{\partial^2\phi_a}{\partial x_1x_2}(s) \\ \frac{\partial^2\phi_a}{\partial x_2x_1(s)} & \frac{\partial^2\phi_a}{\partial x_2^2}(s) \end{bmatrix}ds\\
&= \int\theta_\rho(t-s)D^2\phi(s)ds\\
&= \theta_\rho * D^2\phi_a
\end{aligned}$$

Since $\theta_\rho \geqslant 0$, it follows that $D^2(\phi_{a,\rho}) \geqslant 0$. Now we need to compute $\partial_t\mathcal{S}_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}]$ for $\phi_{a\,\rho}$. Recall that

$$\begin{aligned}
&\partial_t\mathcal{S}_{a,\rho} + [\mathcal{S}_{a,\rho},\mathcal{A}_{a,\rho}]\\
&= \gamma\partial_t^2\phi_{a,\rho} + \gamma(A^2+B^2)[4\gamma^2D^2\phi_{a,\rho}(\nabla\phi_{a,\rho})\cdot\nabla\phi_{a,\rho} - 4\nabla\cdot(D^2\phi_{a,\rho}(\nabla)) - \Delta^2\phi_{a,\rho}]\\
&\quad + 4A\gamma^2[\nabla\phi_{a,\rho}\cdot\nabla(\partial_t\phi_{a,\rho})] - 2iB\gamma[2\nabla(\partial_t\phi_{a,\rho})\cdot\nabla + \Delta(\partial_t\phi_{a,\rho})].
\end{aligned}$$

Since the weight does not depend on t, we can reduce it to

$$\partial_t\mathcal{S}_{a,\rho} + [\mathcal{S}_{a,\rho},\mathcal{A}_{a,\rho}] = \gamma(A^2+B^2)[4\gamma^2D^2\phi_{a,\rho}(\nabla\phi_{a,\rho})\cdot\nabla\phi_{a,\rho} - 4\nabla\cdot(D^2\phi_{a,\rho}(\nabla)) - \Delta^2\phi_{a,\rho}].$$

From (B.1) we deduce that

$$\begin{aligned}\Delta^2\phi_{a,\rho} &= \sum_{j=1}^n \partial_j\Delta\phi_a * \partial_j\theta_\rho \\ &= \sum_{j=1}^n -2a(n-a)x_j|x|^{-a-2}\mathbb{1}_{|x|\geqslant 1} * \partial_j\theta_\rho - 2ax_j d\sigma_1 * \partial_j\theta_\rho,\end{aligned}$$

and by using Young's inequality for convolutions

$$\begin{aligned}\|\Delta^2\phi_{a,\rho}\|_{L^\infty(\mathbb{R}^n)} &\leqslant \sum_{j=1}^n 2a(n-a)\|x_j|x|^{-a-2}\mathbb{1}_{|x|\geqslant 1} * \partial_j\theta_\rho\|_{L^\infty(\mathbb{R}^n)} + 2a\|x_j d\sigma_1 * \partial_j\theta_\rho\|_{L^\infty(\mathbb{R}^n)} \\ &\leqslant \sum_{j=1}^n 2a(n-a)\||x|^{-1}\mathbb{1}_{|x|\geqslant 1}\|_{L^1(\mathbb{R}^n)}\|\partial_j\theta_\rho\|_{L^\infty(\mathbb{R}^n)} + 2a\|x_j d\sigma_1 * \partial_j\theta_\rho\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

The first part will be bounded by $C(n,\rho)a$. We want to find a bound for the second part. Let ψ be a test function and let $\tilde{\theta}(x) = \theta(-x)$. Then

$$\begin{aligned}\langle x_j d\sigma_1 * \partial_j\theta_\rho, \psi\rangle &= \langle d\sigma, \partial_j\tilde{\theta}_\rho * (x_j\psi)\rangle \\ &= \int_{\partial B_1} \partial_j\tilde{\theta}_\rho * (x_j\psi) d\sigma(x) \\ &= \int_{\partial B_1}\left(\int_{\mathbb{R}^n}\partial_j\tilde{\theta}_\rho(x-y)y_j\psi(y)dy\right)d\sigma(x) \\ &= \int_{\mathbb{R}^n}\psi(y)\left(\int_{\partial B_1} y_j\partial_j\tilde{\theta}_\rho(x-y)d\sigma(x)\right)dy \\ &= \langle\Theta,\psi\rangle\end{aligned}$$

so that $x_j d\sigma_1 * \partial_j\theta_\rho = \Theta = \int_{\partial B_1} y_j\partial_j\tilde{\theta}_\rho(x-y)d\sigma(x)$ as a distribution. Moreover,

$$\Theta(y) = -\int_{\partial B_1}(x_j - y_j)\partial_j\tilde{\theta}_\rho(x-y)d\sigma(x) + \int_{\partial B_1} x_j\partial_j\tilde{\theta}_\rho(x-y)d\sigma(x),$$

and

$$\begin{aligned}\|\Theta\|_{L^\infty(\mathbb{R}^n)} &\leqslant |\partial B_1|\left(\|y_j\partial_j\tilde{\theta}_\rho(y)\|_{L^\infty(\mathbb{R}^n)} + \|\partial_j\tilde{\theta}_\rho\|_{L^\infty(\mathbb{R}^n)}\right) \\ &\leqslant C(n,\rho).\end{aligned}$$

Combining the two parts we get the bound

$$\|\Delta^2\phi_{a,\rho}\|_{L^\infty(\mathbb{R}^n)} \leqslant C(n,\rho)a. \tag{B.5}$$

The above inequality, and the fact that $D^2\phi_{a,\rho} \geqslant 0$ will imply that

$$\begin{aligned}\langle(\partial_t \mathcal{S}_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}])f_{a,\rho}, f_{a,\rho}\rangle =& N_1\langle D^2\phi_{a,\rho}\nabla\phi_{a,\rho}\cdot\nabla\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle \\ &- N_2\langle\nabla\cdot(D^2\phi_{a,\rho}\nabla f_{a,\rho}), f_{a,\rho}\rangle \\ &- N_3\langle\Delta^2\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle \\ &= N_1\langle\nabla\phi_{a,\rho}^T D^2\phi_{a,\rho}\nabla\phi_{a\rho} f_{a,\rho}, f_{a,\rho}\rangle + N_2\langle D^2\phi_{a,\rho}\nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle \\ &- N_3\langle\Delta^2\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle. \end{aligned} \tag{B.6}$$

The first term of (B.6) is non-negative since $\phi_{a,\rho}$ is convex. For the second term, we have

$$\begin{aligned}\langle D^2\phi_{a,\rho}\nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} D^2\phi_{a,\rho}\nabla f_{a,\rho}\cdot\overline{\nabla f_{a,\rho}} \\ &= \int_{\mathbb{R}^n} \nabla f_{a,\rho}^T D^2\phi_{a,\rho}\overline{\nabla f_{a,\rho}}\end{aligned}$$

which also is non-negative since ϕ_a, ρ is a convex function. For the last term of (B.6) we use (B.5) to obtain that

$$\langle\Delta^2\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \leqslant C(n,\rho)a\langle f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)}.$$

Thus we are left with

$$\partial_t\mathcal{S}_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] \geqslant -C(n,\rho)a = -M_0(a,\rho), \tag{B.7}$$

where $M_0(a,\rho) \to 0$ as $a \to 0$. Moreover we have that $\partial_t f_{a\rho} = \mathcal{S}_{a,\rho} f_{a,\rho} + \mathcal{A}_{a,\rho} f_{a,\rho} + (A + iB)(V f_{a,\rho} + e^{\gamma\phi_{a,\rho}}F)$, so that

$$|\partial_t f_{a\rho} - \mathcal{S}_{a,\rho} f_{a,\rho} - \mathcal{A}_{a,\rho} f_{a,\rho}| \leqslant \sqrt{A^2+B^2}(M_1|f_{a,\rho}| + e^{\gamma\phi_{a,\rho}}F). \tag{B.8}$$

Moreover,

$$\begin{aligned}\phi_{a,\rho}(x) &\leqslant |x|^2 * \theta_\rho \\ &\leqslant \int_{\mathbb{R}^n} |x-y|^2\theta_\rho(y)dy.\end{aligned}$$

Since for all $\delta > 0$, $|x-y| \leqslant (1+\delta)|x|^2 + C(\delta)|y|^2$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} |x-y|^2\theta_\rho(y)dy &\leqslant (1+\delta)|x|^2\int_{\mathbb{R}^n}\theta_\rho(y)dy + \int_{\mathbb{R}^n} C(\delta)|y|^2\theta_\rho(y)dy \\ &\leqslant (1+\delta)|x|^2 + \rho^2 C(\delta)\int_{\mathbb{R}^n}|y|^2\theta(y)dy \\ &\leqslant (1+\delta)|x|^2 + \rho^2 C(\delta,n).\end{aligned}$$

By letting $\delta \to 0$, it follows that

$$\phi_{a,\rho} \leqslant |x|^2 + \rho^2 C(n). \tag{B.9}$$

Then we define

$$M_2(a,\rho) = \sup_{t\in[0,1]} \frac{\|e^{\gamma\phi_{a,\rho}}F\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^2(\mathbb{R}^n)}} \leqslant e^{\rho^2 C(n)} M_2.$$

We observe that $e^{\rho^2 C(n)} M_2 \to M_2$ when $\rho \to 0$. Now we can use Lemma 3.3 to see that the function $H_{a,\rho} = \|f_{a,\rho}\|^2_{L^2(\mathbb{R}^n)}$ is logarithmically convex in [0,1]and that

$$H_{a\,\rho}(t) \leqslant e^{N(M_0(a,\rho)+M_1+M_2(a,\rho)+M_1^2+M_2(a,\rho)^2)} H_{a,\rho}(0)^{1-t} H_{a,\rho}(1)^t$$

so that

$$\|e^{\gamma\phi_{a,\rho}}u(t)\|^2_{L^2(\mathbb{R}^n)} \leqslant e^{N(C(n,\rho)a+M_1+\rho^2C(n)M_2+M_1^2+M_2(a,\rho)^2)} \|e^{\gamma\phi_{a,\rho}}u(0)\|^{2(1-t)}_{L^2(\mathbb{R}^n)} \|e^{\gamma\phi_{a,\rho}}u(1)\|^{2t}_{L^2(\mathbb{R}^n)}. \tag{B.10}$$

Finally, we obtain the result, by first letting $a \to 0$, then $\rho \to 0$. In particular, since ϕ_a is a monotone increasing function, and converges to $|x|^2$ pointwise as $a \to 0$, we can use the Monotone convergence theorem to justify that

$$\|e^{\gamma|x|^2*\theta_\rho}u(t)\|^2_{L^2(\mathbb{R}^n)} \leqslant e^{N(M_1+M_2(\rho)+M_1^2+M_2(\rho)^2)} \|e^{\gamma|x|^2*\theta_\rho}u(0)\|^{2(1-t)}_{L^2(\mathbb{R}^n)} \|e^{\gamma|x|^2*\theta_\rho}u(1)\|^{2t}_{L^2(\mathbb{R}^n)}. \tag{B.11}$$

Now we claim that $||\cdot|^2 * \theta_\rho(x) - |x|^2| \leqslant C_1(n)\rho^2 + C_2(n)\rho|x|$. Indeed,

$$\begin{aligned} ||\cdot|^2 * \theta_\rho(x) - |x|^2| &\leqslant \int_{\mathbb{R}^n} \Big||x-y|^2 - |x|^2\Big| \theta_\rho(y)dy \\ &\leqslant \int_{\mathbb{R}^n} (|x|+|y|)^2 - |x|^2)\theta_\rho(y)dy \\ &= \int_{\mathbb{R}^n} \rho^2|y|^2 + 2\rho|x||y|\theta(y)dy \\ &= \rho^2 \int_{\mathbb{R}^n} |y|^2\theta(y)dy + 2\rho|x| \int_{\mathbb{R}^n} |y|^2\theta(y)dy \\ &\leqslant C_1(n)\rho^2 + C_2(n)\rho|x|. \end{aligned}$$

Then, by also using (B.9), we deduce that

$$\begin{aligned} &e^{-C_1(n)\rho^2}\|e^{-C_2(n)\rho|x|}e^{\gamma|x|^2}u\|^2_{L^2(\mathbb{R}^n)} \\ &\leqslant \|e^{\gamma|x|^2*\theta_\rho}u(t)\|^2_{L^2(\mathbb{R}^n)} \\ &\leqslant e^{N(M_1+M_2(\rho)+M_1^2+M_2(\rho)^2)} \|e^{\gamma|x|^2*\theta_\rho}u(0)\|^{2(1-t)}_{L^2(\mathbb{R}^n)} \|e^{\gamma|x|^2*\theta_\rho}u(1)\|^{2t}_{L^2(\mathbb{R}^n)} \\ &\leqslant e^{N(M_1+M_2(\rho)+M_1^2+M_2(\rho)^2)} e^{C(n)\rho^2}\|e^{\gamma|x|^2}u(0)\|^{2(1-t)}_{L^2(\mathbb{R}^n)} e^{C(n)\rho^2}\|e^{\gamma|x|^2}u(1)\|^{2t}_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The final result now follows by letting $\rho \to 0$ and using the Monotone Convergence Theorem on the left-hand side. □

B.2 Lemma 3.5

We recall that for $a \in (0, \frac{1}{2})$ and $\rho \in (0,1)$

$$\phi_a(x) = \begin{cases} |x|^2, & |x| < 1 \\ \frac{2|x|^{2-a} - a}{2-a}, & |x| \geqslant 1, \end{cases}$$

and $\phi_{a,\rho} = \phi_a * \theta_\rho$, where θ is radial mollifier. We define $f_{a,\rho} = e^{\gamma \phi_{a,\rho}} u$. By Lemma 3.2 we can justify that both $\|f_{a,\rho}\|_{L^2(\mathbb{R}^n)}$ and $\|\nabla f_{a,\rho}\|_{L^2(\mathbb{R}^n \times [0,1])}$ will be finite for all $t \in [0,1]$. We proceed as for the formal computation and deduce that

$$\begin{aligned} 2\int_0^1 & t(1-t)\langle \partial_t \mathcal{S} f_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\ &\leqslant H(1) + H(0) + 2\int_0^1 (1-2t) Re\langle \partial_t f_{a,\rho} - \mathcal{S}_{a,\rho} f_{a,\rho} - \mathcal{A}_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} dt \\ &+ \int_0^1 t(1-t)\|\partial_t f_{a,\rho} - \mathcal{A}_{a,\rho} f_{a,\rho} - \mathcal{S}_{a,\rho} f_{a,\rho}\|^2_{L^2(\mathbb{R}^n)} dt. \end{aligned}$$

We first want to find a lower bound for $\langle \partial_t \mathcal{S} f_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)}$. From (B.6),

$$\begin{aligned} &\langle \partial_t \mathcal{S} f_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\ &\geqslant N\Big(\langle \nabla \phi_{a,\rho}^T D^2 \phi_{a,\rho} \nabla \phi_{a\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + \langle D^2 \phi_{a,\rho} \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \qquad \text{(B.12)} \\ &- \quad \langle \Delta^2 \phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)}\Big). \qquad \text{(B.13)} \end{aligned}$$

For the second term of (B.12), we have that

$$\begin{aligned} &\langle D^2 \phi_{a,\rho} \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle (D^2\phi_a - 2I) * \theta_\rho \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + \langle (2I * \theta_\rho) \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle (D^2\phi_a - 2I) * \theta_\rho \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + 2\|\nabla f_{a,\rho}\|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Moreover, by using the expressions for the derivatives of ϕ_a and (B.2),

$$\begin{aligned} D^2\phi_a - 2I &= (\phi_a'' - 2)\begin{bmatrix} \frac{x_1^2}{r^2} & \frac{x_1 x_2}{r^2} \\ \frac{x_1 x_2}{r^2} & \frac{x_2^2}{r^2} \end{bmatrix} + (\phi_a' - 2r)\begin{bmatrix} \frac{1}{r} - \frac{x_1^2}{r^3} & -\frac{x_1 x_2}{r^3} \\ -\frac{x_1 x_2}{r^3} & \frac{1}{r} - \frac{x_2^2}{r^3} \end{bmatrix} \\ &= \mathbb{1}_{||x| \geqslant 1}(2(1-a)|x|^{-a} - 2) D_1 + \mathbb{1}_{|x| \geqslant 1}(2|x|^{-a} - 2)|x| D_2 := C_1(a,x). \end{aligned}$$

Observe that

$$D_1 = \begin{bmatrix} \frac{x_1^2}{r^2} & \frac{x_1 x_2}{r^2} \\ \frac{x_1 x_2}{r^2} & \frac{x_2^2}{r^2} \end{bmatrix} \leqslant \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1, \end{bmatrix},$$

and

$$rD_2 = r\begin{bmatrix} \frac{1}{r} - \frac{x_1^2}{r^3} & -\frac{x_1x_2}{r^3} \\ -\frac{x_1x_2}{r^3} & \frac{1}{r} - \frac{x_2^2}{r^3} \end{bmatrix} \leqslant \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix},$$

so that D_1 and rD_2 are bounded matrices, and $C_1(a,x) \to 0$ pointwise when $a \to 0$. It follows that

$$\langle D^2\phi_{a,\rho}\nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} = \langle C_1(a,x) * \theta_\rho \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + 2\|\nabla f_{a,\rho}\|_{L^2(\mathbb{R}^n)}.$$

For the first term of (B.12) we do similar and write

$$\begin{aligned} &\langle \nabla\phi_{a,\rho}^T D^2\phi_{a,\rho}\nabla\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\ &= \langle \nabla\phi_{a,\rho}^T (D^2\phi_a - 2I) * \theta_\rho \nabla\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + 2\|\nabla\phi_{a,\rho} f\|_{L^2(\mathbb{R}^n)}^2 \\ &= \langle \nabla\phi_{a,\rho}^T (C_1(a,x) * \theta_\rho)\nabla\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + 2\|\nabla\phi_{a,\rho} f_{a,\rho}\|_{L^2(\mathbb{R}^n)}^2, \end{aligned}$$

so that

$$\begin{aligned} &\langle \partial_t \mathcal{S} f_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\ &\geqslant \langle C_1(a,x) * \theta_\rho \nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + 2\|\nabla f_{a,\rho}\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + \langle \nabla\phi_{a,\rho}^T (C_1(a,x) * \theta_\rho)\nabla\phi_{a,\rho} f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + 2\|\nabla\phi_{a,\rho} f_{a,\rho}\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - C(n,\rho) a \|f_{a,\rho}\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Furthermore, since

$$\nabla f_{a,\rho} = e^{\phi_{a,\rho}}(\nabla\phi_{a,\rho} u + \nabla u),$$

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f_{a,\rho}|^2 dx = &\int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}(|\nabla\phi_{a,\rho}|^2|u|^2 + |\nabla u|^2)dx + \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}(\nabla\phi_{a,\rho} u \cdot \nabla\bar{u})dx \\ &+ \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}(\nabla u \cdot \nabla\phi_{a,\rho}\bar{u})dx. \end{aligned}$$

Integrating by parts shows that

$$\begin{aligned} \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}\nabla\phi_{a,\rho} u \cdot \nabla\bar{u} dx = &-\int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}\nabla\phi_{a,\rho} \cdot \nabla u\bar{u} dx - \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}\nabla\cdot(\nabla\phi_{a,\rho})|u|^2 dx \\ &- \int_{\mathbb{R}^n} 2|\nabla\phi_{a,\rho}|^2 e^{2\phi_{a,\rho}}|u|^2 dx, \end{aligned}$$

so that

$$\int_{\mathbb{R}^n} |\nabla f_{a,\rho}|^2 + |\nabla\phi_{a,\rho}|^2|f_{a,\rho}|^2 dx = \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}}|\nabla u|^2 dx - \int_{\mathbb{R}^n} |f_{a,\rho}|^2 \nabla\cdot(\nabla\phi_{a,\rho}) dx. \tag{B.14}$$

In addition, integration by parts, Cauchy-Schwarz and Young's inequalities again show that

$$\int_{\mathbb{R}^n} |f_{a,\rho}|^2 \nabla \cdot (\nabla \phi_{a,\rho}) dx \leqslant 2 \int_{\mathbb{R}^n} |\nabla f_{a,\rho}||f_{a,\rho}||\nabla \phi_{a,\rho}| dx$$
$$\leqslant \int_{\mathbb{R}^n} |\nabla f_{a,\rho}|^2 dx + \int_{\mathbb{R}^n} |f_{a,\rho}|^2 |\nabla \phi_{a,\rho}|^2 dx. \tag{B.15}$$

Combining (B.14) and (B.15),

$$2 \int_{\mathbb{R}^n} |\nabla f_{a,\rho}|^2 + |f_{a,\rho}|^2 |\nabla \phi_{a,\rho}|^2 dx \geqslant \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}} |\nabla u|^2 dx.$$

Thus,

$$\begin{aligned}
&\int_0^1 \langle \partial_t \mathcal{S}_{a,\rho} + [\mathcal{S}_{a,\rho}, \mathcal{A}_{a,\rho}] f_{a,\rho}, f_{a,\rho} \rangle_{L^2(\mathbb{R}^n)} t(1-t) dt \\
&\geqslant 2 \int_0^1 \int_{\mathbb{R}^n} |\nabla f_{a,\rho}|^2 + |\nabla \phi_{a,\rho}|^2 |f_{a,\rho}|^2 t(1-t) dx dt \\
&\quad + \int_0^1 \langle C_1(a,x) * \theta_\rho \nabla f_{a,\rho}, \nabla f_{a,\rho} \rangle_{L^2(\mathbb{R}^n)} + \langle \phi_{a,\rho}^T (C_1(a,x) * \theta_\rho) \nabla \phi_{a,\rho} f_{a,\rho}, f_{a,\rho} \rangle_{L^2(\mathbb{R}^n)} dt \\
&\quad - \int_0^1 C(n,\rho) a \| f_{a,\rho} \|_{L^2(\mathbb{R}^n)} dt \\
&\geqslant \frac{1}{2} \int_0^1 \int_{\mathbb{R}^n} t(1-t) e^{2\phi_{a,\rho}} |\nabla u|^2 dx dt + \int_0^1 \int_{\mathbb{R}^n} |\nabla \phi_{a,\rho}|^2 |f_{a,\rho}|^2 t(1-t) dx dt \\
&\quad + \int_0^1 \langle C_1(a,x) * \theta_\rho \nabla f_{a,\rho}, \nabla f_{a,\rho} \rangle_{L^2(\mathbb{R}^n)} + \langle \phi_{a,\rho}^T (C_1(a,x) * \theta_\rho) \nabla \phi_{a,\rho} f_{a,\rho}, f_{a,\rho} \rangle_{L^2(\mathbb{R}^n)} dt \\
&\quad - \int_0^1 C(n,\rho) a \| f_{a,\rho} \|_{L^2(\mathbb{R}^n)} dt.
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
&\int_0^t t(1-t) \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}} |\nabla u|^2 dx dt + \int_0^t t(1-t) \int_{\mathbb{R}^n} e^{2\phi_{a,\rho}} |u|^2 |\nabla \phi_{a,\rho}|^2 dx dt \\
&\leqslant H(1) + H(0) + 2 \int_0^1 (1-2t) Re \langle \partial_t f_{a,\rho} - \mathcal{S}_{a,\rho} f_{a,\rho} - \mathcal{A}_{a,\rho} f_{a,\rho}, f_{a,\rho} \rangle_{L^2(\mathbb{R}^n)} dt \\
&\quad + \int_0^1 t(1-t) \| \partial_t f_{a,\rho} - \mathcal{A}_{a,\rho} f_{a,\rho} - \mathcal{S}_{a,\rho} f_{a,\rho} \|_{L^2(\mathbb{R}^n)}^2 dt.
\end{aligned}$$

Using (B.8) and the same arguments as in the formal computation, we deduce that

$$\begin{aligned}
&\|\sqrt{t(1-t)}e^{\phi_{a,\rho}}\nabla u\|^2_{L^2(\mathbb{R}^n\times[0,1])} + \|\sqrt{t(1-t)}e^{\phi_{a,\rho}}u\nabla\phi_{a,\rho}\|^2_{L^2(\mathbb{R}^n\times[0,1])} \\
&\leqslant N\left(\sup_{t\in[0,1]}\|e^{\phi_{a,\rho}}u\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|e^{\phi_{a,\rho}}F\|_{L^2(\mathbb{R}^n)}\right) \\
&\quad - \int_0^1 \langle C_1(a,x)*\theta_\rho\nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} + \langle\phi^T_{a,\rho}(C_1(a,x)*\theta_\rho)\nabla\phi_{a,\rho}f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)}dt \\
&\quad + \int_0^1 C(n,\rho)a\|f_{a,\rho}\|_{L^2(\mathbb{R}^n)}dt. \qquad\qquad (B.16)
\end{aligned}$$

The result will follow by passing to the limit as $a,\rho\to 0$. We start with the left-hand side of (B.16). Since both $\phi_{a,\rho}$ and $\nabla\phi_{a,\rho}$ are monotone increasing functions when $a\to 0$, and converges to $|x|^2$ and $2x$ respectively, we can use the Monotone Convergence Theorem and let $a\to 0$. Moreover, since $||x|^2*\theta\rho-|x|^2|\leqslant C_1(n)\rho^2+C_2(n)\rho|x|$, we deduce exactly

$$\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla u\|^2_{L^2([0,1]\times\mathbb{R}^n)} + \|\sqrt{t(1-t)}e^{\gamma|x|^2}xu\|^2_{L^2(\mathbb{R}^n)}$$

on the left-hand side when we proceed by monotone convergence theorem as in the rigorous argument in the previous lemma when letting $\rho\to 0$.

We move on to the right-hand side of (B.16). Since $|x|^2*\theta_\rho\leqslant C(n)\rho^2+|x|^2$ and $\phi_{a,\rho}$ is a monotone increasing function as $a\to 0$, we can use a similar argument as we used in the limit process in the justification of Lemma 3.4 to see that

$$\sup_{t\in[0,1]}\|e^{\phi_{a,\rho}}u\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|e^{\phi_{a,\rho}}F\|_{L^2(\mathbb{R}^n)} \longrightarrow \sup_{t\in[0,1]}\|e^{\gamma|x|^2}u\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,1]}\|e^{\gamma|x|^2}F\|_{L^2(\mathbb{R}^n)}$$

when $a,\rho\to 0$. For the two next terms, observe that

$$\begin{aligned}
&\langle C_1(a,x)*\theta_\rho\nabla f_{a,\rho}, \nabla f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)} \\
&= \int_{|x|\geqslant 1}\big((2(1-a)|x|^{-a}-2)D_1 + (2|x|^{-a}-2)|x|D_2\big)*\theta_\rho|\nabla f_{a,\rho}|^2dx \\
&\leqslant \int_{|x|\geqslant 1}\int_{|x|\geqslant 1}(2(1-a)|y|^{-a}-2)D_1 + (2|y|^{-a}-2)|y|D_2\theta_\rho(x-y)dy|\nabla f_{a,\rho}|^2dx
\end{aligned}$$

Since $a\in(0,\frac{1}{2})$,

$$\int_{|x|\geqslant 1}(2(1-a)|y|^{-a}-2)D_1 + (2|y|^{-a}-2)|y|D_2\theta_\rho(x-y)dy\leqslant N,$$

and thus,

$$\begin{aligned}
&\big((2(1-a)|x|^{-a}-2)D_1 + (2|x|^{-a}-2)|x|D_2\big)*\theta_\rho|\nabla f_{a,\rho}|^2 \\
&\leqslant N|\nabla f_{a,\rho}|^2 \\
&\leqslant |\nabla e^{\frac{\gamma A|x|^2}{8(A^2+B^2)\gamma t+A}}u(x,t)|^2.
\end{aligned}$$

By Lemma 3.2 $|\nabla e^{\frac{\gamma A|x|^2}{4(A^2+2B^2)\gamma t+A}}u(x,t)|^2 \in L^2([0,1]\times\mathbb{R}^n)$, so we can use the dominated convergence theorem to see that this term goes to 0 when $a \to 0$. A similar argument can be used for $\langle \phi_{a,\rho}^T(C_1(a,x)*\theta_\rho)\nabla\phi_{a,\rho}f_{a,\rho}, f_{a,\rho}\rangle_{L^2(\mathbb{R}^n)}$. Indeed,

$$\begin{aligned}\nabla\phi_{a,\rho}^T(C_1(a,x)*\theta_\rho)\nabla\phi_{a,\rho}f_{a,\rho}, f_{a,\rho} &\leqslant N|\nabla\phi_{a,\rho}|^2|f_{a,\rho}|^2\\ &\leqslant N|\nabla\left(\frac{\gamma A|x|^2}{8(A^2+B^2)\gamma t+A}\right)e^{\frac{\gamma A|x|^2}{8(A^2+B^2)\gamma t+A}}u(x,t)|^2,\end{aligned}$$

which also by Lemma 3.2 is in $L^2([0,1]\times\mathbb{R}^n)$, and we can use the Dominated Convergence Theorem.

Finally, and now rigorously justified, we can let $a \to 0$ and then $\rho \to 0$, and we deduce our final result.

B.3 Limits in Theorem 3

We have shown that

$$\|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{N(M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon))}\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}^{1-t}\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}^{t}, \tag{B.17}$$

$$\begin{aligned}&\|\sqrt{t(1-t)}e^{\gamma_\epsilon|x|^2}\nabla\tilde{u}_\epsilon\|_{L^2([0,1]\times\mathbb{R}^n)} + \|\sqrt{t(1-t)}|x|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon\|_{L^2([0,1]\times\mathbb{R}^n)}\\ &\qquad\leqslant Ne^{N(M_1^2+M_2(\epsilon)^2+M_1+M_2(\epsilon))}\left(\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)} + \|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}\right),\end{aligned} \tag{B.18}$$

and formally the result follows by letting ϵ to 0. We want to justify it rigorously and start with (B.17).

1). Since $u_\epsilon(t) \longrightarrow u(x,t)$ in $L^2(\mathbb{R}^n)$, there exists a subsequence $\{\epsilon_k\}$, $\epsilon_k \nearrow 0$, such that $\lim_{k\to\infty} u_{\epsilon_k}(x,t) = u(x,t)$ almost everywhere.

2). By (3.50)

$$\begin{aligned}\|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon(t)\|_{L^2(\mathbb{R}^n)}^2 &= \|e^{\left[\frac{1}{(\alpha_\epsilon s+\beta_\epsilon(1-s))^2}+\frac{(\alpha_\epsilon-\beta_\epsilon)\epsilon}{4(\epsilon^2+i^2)(\alpha_\epsilon s+\beta_\epsilon(1-s))}\right]|x|^2}u_\epsilon(s)\|_{L^2(\mathbb{R}^n)}^2\\ &= \int_{\mathbb{R}^n}|e^{\left[\frac{1}{(\alpha_\epsilon s+\beta_\epsilon(1-s))^2}+\frac{(\alpha_\epsilon-\beta_\epsilon)\epsilon}{4(\epsilon^2+i^2)(\alpha_\epsilon s+\beta_\epsilon(1-s))}\right]|x|^2}u_\epsilon(s)|^2dx,\end{aligned}$$

3. Passing to the limit inside the integral we see that,

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\lim_{\epsilon_k\to 0}|e^{[\frac{1}{(\alpha_\epsilon s+\beta_{\epsilon_k}(1-s))^2}+\frac{(\alpha_{\epsilon_k}-\beta_{\epsilon_k})\epsilon_k}{4(\epsilon_k{}^2+i^2)(\alpha_{\epsilon_k}s+\beta_{\epsilon_k}(1-s))}]|x|^2}u_\epsilon(s)|^2dx\right)^{1/2}\\ &= \left(\int_{\mathbb{R}^n}|e^{[\frac{1}{(\alpha_{\epsilon_k}s+\beta(1-s))^2}|x|^2}u(s))|^2dx\right)^{1/2}\\ &= \|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}}u(t)\|_{L^2(\mathbb{R}^n)}^2.\end{aligned}$$

Then, by Fatou's lemma and (3.59):

$$
\begin{aligned}
\|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}}u(t)\|_{L^2(\mathbb{R}^n)} &= \Big(\int_{\mathbb{R}^n}\lim_{\epsilon_k\to 0}|e^{\gamma_{\epsilon_k}|x|^2}\tilde{u}_{\epsilon_k}(t)|^2dx\Big)^{1/2}\\
&\leqslant \liminf_{\epsilon_k\to 0}\|e^{\gamma_{\epsilon_k}|x|^2}\tilde{u}_{\epsilon}(t)\|^2_{L^2(\mathbb{R}^n)}\\
&\leqslant \liminf_{\epsilon_k\to 0}e^{N[M_1^2+M_2(\epsilon_k)^2+M_1+M_2(\epsilon_k)]}\|e^{\frac{|x|^2}{\beta^2}}u(0)\|^{1-t}_{L^2(\mathbb{R}^n)}\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|^{t}_{L^2(\mathbb{R}^n)}.
\end{aligned}
$$

Since the only part depending on ϵ on the right hand side is $M_2(\epsilon)$ and $M_2(\epsilon)\to M_2$ when $\epsilon\to 0$, we get the result

$$
\|e^{\frac{|x|^2}{(\alpha t+\beta(1-t))^2}}u(t)\|_{L^2(\mathbb{R}^n)} \leqslant e^{N[M_1^2+M_2^2+M_1+M_2]}\|e^{\frac{|x|^2}{\beta^2}}u(0)\|^{1-t}_{L^2(\mathbb{R}^n)}\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|^{t}_{L^2(\mathbb{R}^n)}. \tag{B.19}
$$

For (B.18) we could have used a similar argument, but it was not clear how we could justify that $\nabla\tilde{u}_\epsilon \longrightarrow \nabla u$. If we had chosen the initial data $u_0\in H^1(\mathbb{R}^n)$ it would be clear from the semigroup theory. However, since we considered $u\in C([0,1],L^2(\mathbb{R}^n))$, we need a different argument.

From (B.18) it follows that $\tilde{u}_\epsilon$ is a bounded sequence in the weighted space $L^2(([0,1],t(1-t)dt):H^1(\mathbb{R}^n,e^{\gamma_\epsilon|x|^2}dx))$. We want to use the Banach-Alaoglu Theorem to obtain a weakly convergent subsequence of $\tilde{u}_\epsilon$ in this space. However, since we will take the limit when $\epsilon\to 0$, we want to make the weight independent of ϵ. Recall that $\gamma_\epsilon=\frac{1}{(\alpha+2\epsilon)(\beta+2\epsilon)}$. For all $\delta>0$ there exists a ϵ_δ such that for $0<\epsilon<\epsilon_\delta$, we can make $\gamma-\delta<\gamma_\epsilon<\gamma$, so that

$$
\|\sqrt{t(1-t)}e^{(\gamma-\delta)|x|^2}\nabla\tilde{u}_\epsilon\|_{L^2([0,1]\times\mathbb{R}^n)}<\infty.
$$

Then there exist a subsequence $\{\tilde{u}_{\epsilon_k}\}$ of $\{\tilde{u}_\epsilon\}$ such that

$$
\tilde{u}_{\epsilon_k}\rightharpoonup\tilde{u}\text{ in }L^2(([0,1],t(1-t)):H^1(\mathbb{R}^n,e^{(\gamma-\delta)|x|^2}dx)),
$$

and

$$
\begin{aligned}
&\|\sqrt{t(1-t)}e^{(\gamma-\delta)|x|^2}\nabla\tilde{u}\|_{L^2([0,1]\times\mathbb{R}^n)}\\
&\qquad\leqslant \liminf_{\epsilon_k\to 0}\|\sqrt{t(1-t)}e^{(\gamma-\delta)|x|^2}\nabla\tilde{u}_{\epsilon_k}\|_{L^2([0,1]\times\mathbb{R}^n)}\\
&\qquad\leqslant \liminf_{\epsilon_k\to 0}Ne^{N(M_1^2+M_2(\epsilon_k)^2+M_1+M_2(\epsilon_k))}\left(\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}\right)\\
&\qquad= Ne^{N(M_1^2+M_2^2+M_1+M_2)}\left(\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}\right)<\infty.
\end{aligned}
$$

Now, since $e^{\gamma-\delta}|x|^2\nearrow e^{\gamma}|x|^2$ when $\delta\to 0$, we can conclude by the Monotone Convergence Theorem and let $\delta\to 0$. Finally we obtain that

$$
\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla\tilde{u}\|_{L^2([0,1]\times\mathbb{R}^n)}\leqslant Ne^{N(M_1^2+M_2^2+M_1+M_2)}\left(\|e^{\frac{|x|^2}{\beta^2}}u(0)\|_{L^2(\mathbb{R}^n)}+\|e^{\frac{|x|^2}{\alpha^2}}u(1)\|_{L^2(\mathbb{R}^n)}\right).
$$

Remark. The reason why we did not include

$$\|\sqrt{t(1-t)}|x|e^{\gamma_\epsilon|x|^2}\tilde{u}_\epsilon\|_{L^2([0,1]\times\mathbb{R}^n)}$$

in the argument is because it will not be important for us when we apply the result in the proof of Theorem 1. However, that

$$\sup_{t\in[0,1]}\|e^{\gamma|x|^2}\tilde{u}\|_{L^2(\mathbb{R}^n)}+\|\sqrt{t(1-t)}e^{\gamma|x|^2}\nabla\tilde{u}\|_{L^2([0,1]\times\mathbb{R}^n)}<\infty$$

will be fundamental to prove the main result.

C Semigroups of Linear Operators

This section is meant as a supplement to Theorem 3 and the theory of semigroups. We give the most important definitions and recall the results, without proofs, we need to prove that the operator $L = (A + iB)(\Delta + V(x))$ generates a C_0 semigroup for $A \geqslant 0, B \in \mathbb{R}$ when V is a bounded potential. Moreover, we discuss some applications to both the inhomogeneous and the homogeneous Schrödinger equation and justify the existence of solutions for these equations. For references, see for example [20], [4] or [22]

C.1 Operator Theory

We assume that A is a densely defined operator on a Hilbert space $\mathcal{H}$, i.e. $D(A)$ is dense in $\mathcal{H}$.

Definition C.1.

$$A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

is symmetric if

$$\langle A\phi, \psi \rangle_{\mathcal{H}} = \langle \phi, A\psi \rangle_{\mathcal{H}}$$

for all $\phi, \psi \in D(A)$. We say that A is skew-symmetric, or anti-symmetric, if

$$\langle A\phi, \psi \rangle_{\mathcal{H}} = -\langle \phi, A\psi \rangle_{\mathcal{H}}$$

for all $\phi, \psi \in D(A)$.

Definition C.2. Let A be a densely defined operator. We define the adjoint of A, A^*:

$$\begin{cases} D(A^*) = \{\eta \in \mathcal{H} : \ \exists \ \psi \in \mathcal{H} \ s.t. \ \langle A\phi, \eta \rangle_{\mathcal{H}} = \langle \phi, \psi \rangle_{\mathcal{H}} \\ A^*\eta = \psi \end{cases}$$

Definition C.3. We say that A is self-adjoint if

$$A^* = A, \text{that means } D(A) = D(A^*) \text{ and } A\phi = A^*\phi$$

for all $\phi \in D(A)$. We say that A is skew-adjoint if $A^* = -A$ or that iA is self-adjoint.

Definition C.4. Let $A : D(A) \subset X \longrightarrow X$ be a closed operator. The resolvent set is the set

$$\rho(A) := \left\{ z \in \mathbb{C} : (A - z)^{-1} \text{ is injective and surjective} \right\}.$$

The spectrum of A is the set

$$\Sigma(A) = \mathbb{C} \backslash \rho(A).$$

C.2 Semigroups and the Homogeneous IVP

Definition C.5. Let X be a Banach space. A one-parameter family $S(t)$, $0 \leqslant t < \infty$ of bounded linear operators from X to X is a strongly continuous semigroup, or C_0 semigroup if

(i) $S(0) = I$

(ii) $S(t+s) = S(t)S(s)$

(iii) $S(t)\phi \longrightarrow S(t_0)\phi$ as $t \to t_0$ $\forall\ \phi \in X$.

Definition C.6. Let $S(t)$ be a C_0 semigroup. The infinitesimal generator of S is the operator $L : D(L) \to X$ where

$$\begin{cases} D(L) : \{\phi \in X : \lim_{h\to 0^+} \left(\frac{S(h)-Id}{h}\right)\phi \text{ exists in X}\} \\ L\phi = \lim_{h\to 0^+} \left(\frac{S(h)-Id}{h}\right)\phi \end{cases}$$

Proposition C.1. Let L be the generator of the semigroup $S(t)$. Let ϕ in $D(L)$. Then

(i) $S(t)\phi \in D(L)$, $\forall t \geqslant 0$

(ii) $LS(t)\phi = S(t)L\phi$

(iii) : $t \mapsto S(t)\phi$ is differentiable and $\dfrac{d}{dt}S(t)\phi = LS(t)\phi$.

Observe that by this proposition it makes sense to denote the semigroup $S(t) = e^{Lt}$, and it satisfies the natural properties of the exponential. Moreover, $S(t)\phi$ is a solution to the initial value problem

$$\begin{cases} u'(t) - Lu = 0 \\ u(0) = \phi. \end{cases}$$

Definition C.7. A one parameter family $S(t)$, $-\infty < t < \infty$ of bounded linear operators on a Banach space X is C_0 group if

(i) $S(0) = I$

(ii) $S(t+s) = S(t)S(s)$

(iii) $S(t)\phi \longrightarrow S(t_0)\phi$ as $t \to t_0$ $\forall\ \phi \in X$.

Moreover, if $\|S(t)\phi\| = \|\phi\|$ $\forall\ \phi \in X$, we say that $S(t)$ is a unitary group of operators.

Remark.
(i) For $t \geqslant 0$ a C_0 group is also a C_0 semigroup. We define the infinitesimal generator, L of a group similarly as for a semigroup, but the limit as $h \to 0$ has to be from both sides, not only as $h \to 0^+$.

(ii) Moreover, since $S(t)S(-t) = S(0) = I$, we have a well-defined inverse for a C_0 group. We denote $S(t)^{-1} = S(-t)$. It is now clear that from Proposition 1.1 that $e^{it\Delta}$ is a unitary group.

(iii) For the heat operator $e^{t\Delta}$ we do not have a group, since it is not well-defined for $t \leqslant 0$. More generally, this is the case for the operator $e^{(A+iB)t\Delta}$, when $A > 0$. Since we mainly want to prove results for this operator, we will focus on results for semigroups, rather than only for groups.

Theorem C.1. (Hille-Yosida Theorem) Let X be a Banach Space. Let $L : D(L) \subset X \to X$ be a closed, densely defined linear operator. Then L is the infinitesimal generator of a strongly continuous semigroup $S(t)$ satisfying $\|S(t)\| \leqslant Me^{\beta t}$ if and only if

$(H1)$ $(\beta, \infty) \subset \rho(L),\ \beta \geqslant 0$
$(H2)$ $||R_L(\lambda)^n\| = \|(\lambda I - L)^{-n}\| \leqslant M(\lambda - \beta)^{-n}$ for $\lambda > \beta$, for all $n = 1, 2, ...$

A consequence of this theorem is the following.

Theorem C.2. Let X be a Banach space. Let $L : D(L) \subset X \to X$ be a closed, densely defined operator that satisfies the conditions $(H1)$ and $(H2)$, then $\forall\ u_0 \in D(L)\ \exists!\ u \in C^1([0, \infty), X) \cap C([0, \infty), D(A))$ such that

$$\begin{cases} u'(t) - Lu = 0 \text{ in } X, \quad \forall\ t > 0 \\ u(0) = u_0 \end{cases} \tag{C.1}$$

In particular, $u(t) = S(t)\phi = e^{Lt}u_0$.

Remark. From this result we can only justify the existence of a unique solution in the case where $u_0 \in D(L)$. For the Schrödinger equation this is when $u_0 \in H^2(\mathbb{R}^n)$. In this thesis we usually work with initial data only in $L^2(\mathbb{R}^n)$, which is not enough to have the existence of a unique solution on this form. In this case, we consider a generalized solution to the problem. One way of formally defining the generalized solution is the following way from [20]:

Definition C.8. A continuous function u on $[0, \infty)$ is a generalized solution of (C.1) if there are $x_n \in D(L)$ such that $x_n \longrightarrow_{n\to\infty} u_0$ and $e^{Lt}x_n \longrightarrow u(t)$ uniformly on bounded intervals.

With this definition it makes sense to consider a generalized solution for all $u_0 \in X$ as $u = e^{Lt}u_0$. Since e^{Lt} is a semigroup, we will have that $u \in C([0, T], X)$. We denote $u(t) = e^{Lt}u_0$ as a mild solution of (C.1) for $u_0 \in X$.

We now want to apply these results to the operator $L = (A+iB)(\Delta+V(x)) = (A+iB)H$, where $V(x)$ is a real valued, bounded potential. If we can show that L generates a C_0 semigroup, $u(t) = e^{(A+iB)H}u_0$ is a solution to the initial value problem

$$\begin{cases} \partial_t u - Lu = 0 \\ u(0) = u_0. \end{cases} \tag{C.2}$$

We first consider the case $V = 0$, so that $L = (A + iB)\Delta$, $A \geqslant 0$, $B \in \mathbb{R}$. Taking the Fourier transform of (C.2), we get that for some constant C

$$\begin{cases} \partial_t \hat{u}(\xi, t) = C(A + iB)|\xi|^2 \hat{u}(\xi, t) \\ \hat{u}(0) = \hat{u_0}, \end{cases}$$

and the solution becomes $u(x, t) = \left(e^{-C(A+iB)|\xi|^2 t} \hat{u_0}\right)^{\vee} = e^{(A+iB)t\Delta} u_0$. It is now straightforward to verify that $e^{(A+iB)t\Delta}$ is a C_0 semigroup by Definition C.5 and that $L = (A + iB)\Delta$ is the infinitesimal generator of the semigroup. Let us now consider the case

$$L = (A + iB)(\Delta + V(x)),$$

where V is a real, bounded potential. We have the following theorem on perturbations by a bounded linear operator, see Theorem 3.1 in [20].

Theorem C.3. Let X be a Banach space and let L be the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \leqslant Me^{\omega t}$. If V is a bounded linear operator on X then $L + V$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X satisfying $\|S(t)\| \leqslant Me^{(\omega + M\|V\|)t}$.

The proof of Theorem C.3 relies on the Hille-Yosida Theorem. By this result, it follows that $L = (A + iB)(\Delta + V)$ generates a C_0 semigroup

$$S(t) = e^{(A+iB)(\Delta+V)t},$$

such that

$$\|S(t)\| < e^{A\|V\|_{L^\infty} t}.$$

Indeed, since $\|e^{(A+iB)t\Delta)}\phi\|_{L^2} \leqslant \|\phi\|_{L^2}$, we have that $\|e^{(A+iB)t\Delta)}\| \leqslant 1$, and since $\|V(x)\phi\|_{L^2} \leqslant \|V\|_{L^\infty}\|\phi\|_{L^2}$, it follows that $\|S(t)\| < e^{A\|V\|_{L^\infty} t}$.

Remark.
(i) This result is valid for all $A \geqslant 0, B \in \mathbb{R}$. In particular, the operators $\Delta + V(x)$ and $i(\Delta + V(x))$ also generates C_0 semigroups.

(ii) There are other ways in which we could have proven that the operator $(A + iB)H$ generates a semigroup. For example, we could have deduced from Stones Theorem [20] that since $\Delta + V(x)$ is indeed self-adjoint, iH generates a unitary group. However, we could not have applied this result to $(A+iB)H$, since $A > 0$. Even though the method with perturbation of bounded operators justifies that we have a solution to the initial value problem, it was not clear that it justifies one of the properties we need, which is that $e^{(z_1+z_2)H} = e^{z_1 H} e^{z_2 H}$ for two complex numbers z_1 and z_2 with non-negative real part. This is why we in the next section choose to follow a different approach, which is to prove that H generates an analytic semigroup. By verifying this, we get the property we will need for free.

C.3 Analytic Semigroups

In this section we will follow chapter 2 in [4]. We omit proofs and details.

Definition C.9. Let $\Sigma_\delta = \{z \in \mathbb{C} : |argz| < \delta\}$. A family of linear operators on a Banach space X $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}}$ is called an analytic semigroup of angle $\delta \in (0, \pi/2]$ if

(i) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ $\forall$ $z_1, z_2 \in \Sigma_\delta$,

(ii) the map $: z \mapsto T(z)$ is analytic in Σ_δ,

(iii) $\lim_{z\to 0} T(z)\phi = \phi$ $\forall \phi \in X$. $z \in \Sigma_{\delta'}$, $0 < \delta' < \delta$.

If moreover $\|T(z)\|$ is bounded in $\Sigma_{\delta'}$ we call $\{T(z)\}$ a bounded analytic semigroup.

Observe that if $T(z)$ is an analytic semigroup of some angle δ, then if we restrict z to the non-negative real axis, it is also a C_0 semigroup.

Theorem C.4. For an operator L on a Banach space X the following are equivalent.

(i) L generates a bounded analytic semigroup $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}}$ on X,

(ii) L generates a bounded strongly continuous semigroup on X and there exists a constant $C > 0$ such that$\|R_L(r + is)\| \leqslant \dfrac{C}{|s|}$ for all $r > 0$ and $0 \neq s \in \mathbb{R}$.

Without proof and further details one can deduce from this theorem and the Spectral Theorem, see [4] for details, that a self-adjoint operator that is bounded from above, i.e. that there exists $\omega \in \mathbb{R}$ such that $\langle L\phi, \phi\rangle \leqslant \omega\|\phi\|^2$ $\forall \phi \in D(L)$, generates an analytic semigroup of angle $\pi/2$.

Let us again consider the operator $H = \Delta + V(x)$, where $V(x)$ is a real bounded potential, $D(H) : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$.

Lemma C.1. The operator H is self-adjoint and bounded from above.

To prove the lemma we present some useful theorems. The proofs can be found in [21].

Theorem C.5. Let $\mathcal{H}$ be a Hilbert space. If A is symmetric, then the following are equivalent.

(i) A is self-adjoint,

(ii) A is closed and $ker(A^* \pm i) = \{0\}$

(iii) $R(A \pm i) = \mathcal{H}$.

Theorem C.6. (Kato-Rellich)
Let $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be self-adjoint, and $B : D(B) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be symmetric. Suppose $D(A) \subseteq D(B)$ and that there exists $\alpha, \beta > 0$ such that

$$\|B\phi\|_{\mathcal{H}} \leqslant \alpha\|\phi\|_{\mathcal{H}} + \beta\|A\phi\|_{\mathcal{H}}, \quad \forall\ \phi \in D(A). \tag{C.3}$$

If moreover $\beta_0 = \inf\{\beta$ such that $(C.3)$ holds$\} < 1$, then $(A+B) : D(A) \longrightarrow H$ is self-adjoint.

We now prove the lemma.

Proof. **Step 1: Δ is self-adjoint.**
An application of integration by parts shows that the Laplacian operator is symmetric. To verify that it is self-adjoint we can show that $R(\Delta \pm i) = L^2(\mathbb{R}^n)$. This follows directly since the spectrum $\Sigma(\Delta)$ is real.

Step 2: H is self-adjoint.
We can now use the Kato-Rellich Theorem to see that when we perturb Δ with a bounded, real potential $V(x)$ we still get a self-adjoint operator. Since $V(x)$ is real, it is symmetric. Moreover,

$$\|V\phi\|_{L^2(\mathbb{R}^n)} \leqslant \|V\|_{L^\infty(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)} + 0\|\Delta\phi\|_{L^2(\mathbb{R}^n)} \leqslant \|V\|_{L^\infty(\mathbb{R}^n)} + \beta\|\Delta\phi\|_{L^2(\mathbb{R}^n)},$$

for some $\beta < 1$. We can apply the Kato-Rellich Theorem to deduce that $H = \Delta + V(x)$ is self-adjoint.

Step 3: H is bounded from above.
We observe that

$$\begin{aligned}
\langle H\phi, \phi\rangle_{L^2(\mathbb{R}^n)} &= \langle \Delta\phi, \phi\rangle_{L^2(\mathbb{R}^n)} + \langle V(x)\phi, \phi\rangle_{L^2(\mathbb{R}^n)} \\
&= \langle -|\xi|^2\hat{\phi}, \hat{\phi}\rangle_{L^2(\mathbb{R}^n)} + \langle V(x)\phi, \phi\rangle_{L^2(\mathbb{R}^n)} \\
&\leqslant -|\xi|^2\|\phi\|^2_{L^2(\mathbb{R}^n)} + \|V\|_{L^\infty(\mathbb{R}^n)}\|\phi\|^2_{L^2(\mathbb{R}^n)} \\
&\leqslant \|V\|_{L^\infty(\mathbb{R}^n)}\|\phi\|^2_{L^2(\mathbb{R}^n)},
\end{aligned}$$

which concludes the proof of the lemma. □

Since H is self-adjoint and bounded from above, we deduce that it generates an analytic semigroup $T(z) = e^{zH}$ of angle $\pi/2$. In particular this means that $\forall\ z_1, z_2$ with non-negative real part, we have $e^{(z_1+z_2)H} = e^{z_1 H}e^{z_2 H}$.

C.4 The Inhomogeneous IVP

Consider now the problem

$$\begin{cases} \partial_t u - Lu = f & \text{for } t \in [0, T] \\ u(0) = u_0, \end{cases} \tag{C.4}$$

where L generates a C_0 semigroup e^{Lt}.

Theorem C.7. Let u be a solution of the initial value problem (C.4). Then u is given by

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}f(s)ds.$$

Observe that if $f \in L^1([0,T],X)$, $u_0 \in X$, then $e^{Lt}u_0 + \int_0^t e^{L(t-s)}f(s)ds \in C([0,T],X)$. Even if it is not differentiable, and is not a solution of (C.4) in the classical sense, it still makes sense to consider a solution on this form. We say that if $f \in L^1([0,T],X)$, and $u_0 \in X$, then $u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)}f(s)ds$ is a mild solution of (C.4). If we make stronger assumptions on f and u_0, we see that the mild solution is indeed a classical solution.

Theorem C.8. Let $u_0 \in D(A), f \in C([0,T],X)$ and suppose that $f \in W^{1,1}([0,T],X)$ or $f \in L^1([0,T],D(A))$, then the mild solution is a classical solution.

To sum up everything, let us go back to the specific problem in the proof of Theorem 3. where $(A+iB)H = (A+iB)(\Delta + V_1(x))$ generates a C_0-semigroup for $A \geqslant 0, B \in \mathbb{R}$. Since $u \in C([0,1],L^2(\mathbb{R}^n))$ satisfies

$$\partial_t u = i(\Delta u + (V_1(x) + V_2(x,t))u) = iHu + iV_2(x,t)u$$

with initial data u_0, we have by Theorem C.7 that

$$u(t) = e^{itH}u_0 + i\int_0^t e^{i(t-s)H}(V_2(s)u(s))ds.$$

Then we define

$$u_\epsilon(t) = e^{(\epsilon+i)tH}u_0 + (\epsilon+i)\int_0^t e^{(\epsilon+i)(t-s)H}F_\epsilon(s)ds.$$

This will be a mild solution to the problem

$$\begin{cases} \partial_t u_\epsilon = (\epsilon+i)(Hu_\epsilon + F_\epsilon(t)) \\ u_\epsilon(0) = u_0. \end{cases}$$

Since $u_0 \in L^2(\mathbb{R}^n)$ and

$$\|F_\epsilon\|_{L^1([0,1],L^2(\mathbb{R}^n)} \leqslant \sup_{t\in[0,1]} e^{\epsilon\|V_1\|_{L^\infty(\mathbb{R}^n)}}\|V_2(t)\|_{L^\infty(\mathbb{R}^n)}\|u(t)\|_{L^2(\mathbb{R}^n)} < \infty,$$

$u_\epsilon(t) \in C([0,1],L^2(\mathbb{R}^n))$. However, since $u_0 \notin H^2(\mathbb{R}^n) = D(H)$ we will not get any more regularity of u_ϵ from Theorem C.8. On the other hand, since

$$\partial_t u_\epsilon = (\epsilon+i)(\Delta u_\epsilon + V_1(x)u_\epsilon + F_\epsilon(x,t))$$

we can use Lemma A.1 to obtain more regularity. Indeed, we have $u_\epsilon \in L^\infty([0,1],L^2(\mathbb{R}^n) \cap L^2([0,1],H^1(\mathbb{R}^n))$.

D Problem with an ODE

Consider the initial value problem

$$\begin{cases} a''(t) + 32a^3 - \frac{2(a')^2}{a} = 0 \\ a(0) = 1, \quad a'(0) = 0. \end{cases} \tag{D.1}$$

This is a nonlinear, autonomous second-order ode, which appears naturally in the Carleman estimate in Section 5. We want to prove the following properties for a solution of (D.1).

Lemma D.1. Let a be a solution of the IVP (D.1). Then $a(t)$ is positive, even and $\lim_{t\to\infty} ta(t) = 0$.

Proof. Suppose that a is a solution, and define $\tilde{a}(t) = a(-t)$. Then

$$\tilde{a}'' + 32\tilde{a}^3 - \frac{2(\tilde{a}')^2}{\tilde{a}} = a''(-t) + 32a(-t)^3 - \frac{2(-a'(-t))^2}{a(-t)} = 0,$$

so that $\tilde{a}$ also is a solution.

Observe that the function $F(a, a') = -32a^3 + \frac{2(a')^2}{a}$ is C^1 away from 0, so that by the Cauchy-Lipschitz theorem, we can prove that there exist a local solution a such that $a(0) = 1$. Since $a(0) = \tilde{a}(0)$, we must have $a(t) = a(-t)$, so that a is even.

Now we want to prove existence of a global solution, for $t \in \mathbb{R}$, and that this solution is positive. Let

$$T^* = \sup\{T > 0 : \text{ there exist a solution } a \text{ on } [0, T] \text{ and } a(t) \in (0, 1] \text{ for all } 0 \leqslant t \leqslant T\}.$$

T^* is well defined because of the local existence as explained above, and the fact that $a'(0) = 0$ and $a''(0) < 0$. Assume by contradiction that $T^* < \infty$. It means that either (i) $\lim_{t\to T^*} a(t) = 0$ or (ii) $\lim_{t\to T^*} a(t) > 1$. We will show that neither of these cases can be true.

Define the function

$$f(t) = \frac{a'(t)}{(2a(t))^2}.$$

It follows that $f'(t) = -8a$. If $a(t) \in (0, 1]$, then $f'(t) < 0$, so f is decreasing, and since $f(0) = 0$, $f(t) \leqslant 0$, which implies that also $a'(t) \leqslant 0$, and $a(t)$ is decreasing. This holds for all $t \in [0, T^*)$, so that $\lim_{t\to T^*} a(t) < 1$, and (ii) cannot be true.

Suppose now that $\lim_{t\to T^*} a(t) = 0$. Observe that

$$\int_0^t -8a(s)ds = \int_0^t f'(s)ds = f(t) - f(0) = f(t).$$

Since $a(t) \longrightarrow 0$ when $t \to T^*$,

$$-\int_0^t 8a(s)ds \longrightarrow -c \text{ when } t \to T^*,$$

for $0 < c < \infty$. Thus,

$$f(t) = \frac{a'(t)}{(2a(t))^2} = \frac{1}{4}\left(\frac{-1}{a(t)}\right)' \in [-c, 0], \quad \forall\, t \in [0, T^*).$$

It follows that

$$\int_0^t -cds \leqslant \frac{1}{4}\int_0^t \left(\frac{-1}{a(s)}\right)' ds \leqslant 0,$$

or equivalently that

$$4 \leqslant \frac{1}{a(t)} \leqslant 1 + 4ct \iff \frac{1}{1+4ct} \leqslant a(t) \leqslant \frac{1}{4}, \quad \forall t \in [0, T^*),$$

which contradicts that $\lim_{t\to T^*} a(t) = 0$.

Thus, T has to be infinite, and we have a global, even solution $a(t)$, such that $a(t) \in (0, 1]$ for all $t \in \mathbb{R}$.

Next we show that $\lim_{t\to\infty} a(t) = 0$. We already know that a is decreasing and positive, so we assume by contradiction that $\lim_{t\to\infty} a(t) = c_1 > 0$. It then follows that

$$f'(t) = -8a < -8c_1,$$

for all $t \geqslant 0$, and integrating this inequality shows that

$$f(t) < -8c_1 t.$$

Integrating one more time, we deduce that

$$\frac{1}{4}\left(-\frac{1}{a(t)} + 1\right) < -4c_1 t^2.$$

Then

$$a(t) < \frac{1}{16c_1 t^2 + 1},$$

which shows that $\lim_{t\to\infty} a(t) < \lim_{t\to\infty} \frac{1}{16c_1t^2+1} = 0$, which contradicts the hypothesis.

Now we are only left to prove that $\lim_{t\to\infty} ta(t) = 0$, or that $a(t) = o(\frac{1}{t})$. We did not see how to complete the proof, so instead we have included numerical plots of $a(t)$ and $ta(t)$, which shows that the claimed properties are likely to hold. See Figures 1-3, where we have coded in Python.

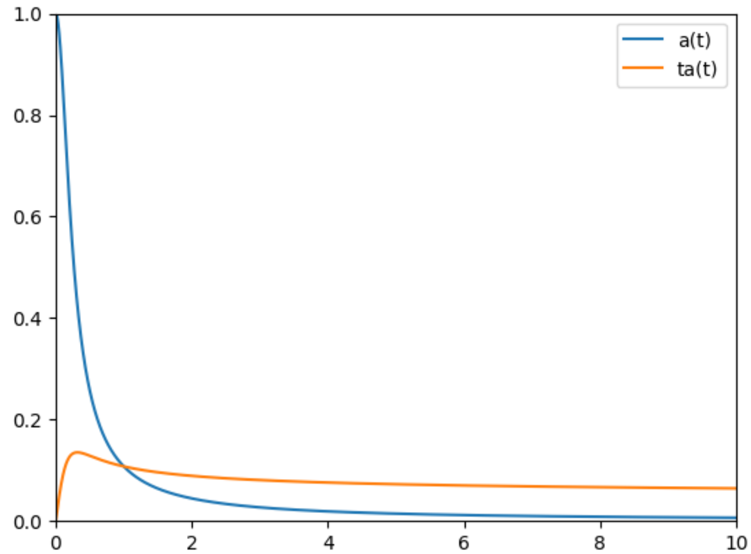

Figure 1: A numerical plot of $a(t)$ and $ta(t)$. We see that the properties we proved for $a(t)$ hold, and it is also likely from the plot that $ta(t)$ stays finite.

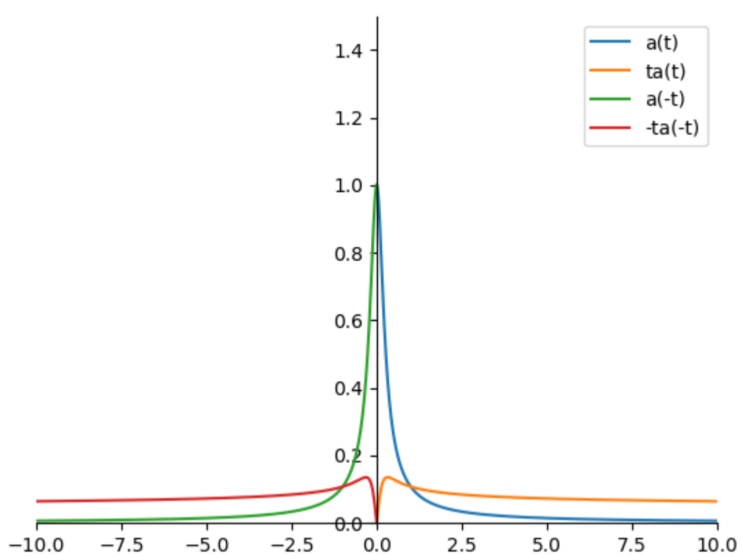

Figure 2: We can see that $a(t)$ is an even function and that the properties we claim for $ta(t)$ are also likely to hold when $t \to -\infty$.

Remark. (i) Observe that in both Figure 1 and Figure 2 we have only plotted the solutions for $t \leqslant 10$. If we plot it over a larger time interval, it is hard to see the behavior of the solution, and we would only see a straight, horizontal line close to 0.

(ii) Even though it is likely from Figure 1 that $ta(t)$ stays finite when $t \to \infty$, it is not obvious that it goes to 0. Therefore, we have also included a plot of $ta(t)$ compared to $\frac{1}{\ln t}$, since we know that $\lim_{t\to\infty} \frac{1}{\ln t} = 0$.

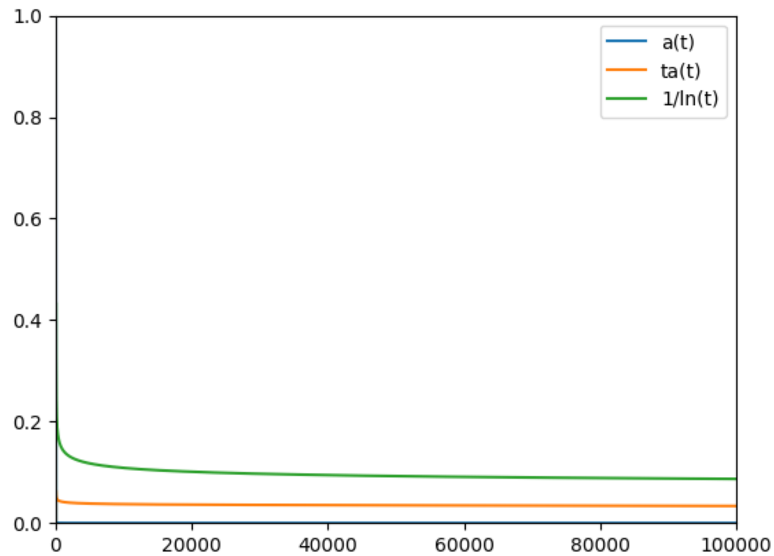

Figure 3: Numerical plot of$ta(t)$, $a(t)$ and $\frac{1}{\ln t}$.

□

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