Journal of Combinatorial Theory, Series B 160 (2023) 206–214

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

journal homepage: www.elsevier.com/locate/jctb

Grid induced minor theorem for graphs of small degree

Tuukka Korhonen [1]

Department of Informatics, University of Bergen, Norway

ARTICLE INFO

Article history:
Received 10 April 2022
Available online 25 January 2023

Keywords:
Treewidth
Induced minor
Grid

ABSTRACT

A graph H is an induced minor of a graph G if H can be obtained from G by vertex deletions and edge contractions. We show that there is a function $f(k,d) = \mathcal{O}(k^{10}+2^{d^5})$ so that if a graph has treewidth at least $f(k,d)$ and maximum degree at most d, then it contains a $k \times k$-grid as an induced minor. This proves the conjecture of Aboulker, Adler, Kim, Sintiari, and Trotignon (2021) [1] that any graph with large treewidth and bounded maximum degree contains a large wall or the line graph of a large wall as an induced subgraph. It also implies that for any fixed planar graph H, there is a subexponential time algorithm for maximum weight independent set on H-induced-minor-free graphs.

1. Introduction

A graph H is a minor of a graph G if H can be obtained as a contraction of a subgraph of G. An induced minor is a minor that is obtained as a contraction of an induced subgraph.

E-mail address: tuukka.korhonen@uib.no.

[1] Supported by the Research Council of Norway via the project BWCA (grant no. 314528).

https://doi.org/10.1016/j.jctb.2023.01.002

The famous grid minor theorem of Robertson and Seymour [19] (see also [9,10]) states that there is a function $f : \mathbb{N} \to \mathbb{N}$ so that any graph with treewidth at least $f(k)$ contains a $k \times k$-grid as a minor. A question with several applications in graph theory and algorithms [1,5,16] is when can "minor" be replaced by "induced minor" in the grid minor theorem. In general this is not possible: complete graphs have unbounded treewidth but do not contain a 2×2-grid as an induced minor. However, one could expect that minors and induced minors behave similarly in sparse graphs. Fomin, Golovach, and Thilikos proved that for any fixed graph H, there is a constant c_H so that any H-minor-free graph with treewidth at least $c_H \cdot k$ contains a $k \times k$-grid as an induced minor [13]. In this paper, we give a grid induced minor theorem for another important class of sparse graphs, in particular for the class of graphs with bounded maximum degree.

Theorem 1. *There is a function $f(k, d) = \mathcal{O}(k^{10} + 2^{d^5})$ so that for any positive integers k, d it holds that if a graph has treewidth at least $f(k, d)$ and maximum degree at most d, then it contains a $k \times k$-grid as an induced minor.*

For graphs with treewidth at least 2^{d^5}, the size of the grid that we obtain is up to a subpolynomial $2^{\mathcal{O}(\log^{5/6} k)}$ factor the same as the size of the grid in the grid minor theorem. In particular, the bound $\mathcal{O}(k^{10} + 2^{d^5})$ on $f(k, d)$ follows from the most recent bound on the grid minor theorem [10] and will improve if the bound on the grid minor theorem is improved.

Together with some routing arguments from [1], Theorem 1 implies that the following conjecture of Aboulker, Adler, Kim, Sintiari, and Trotignon [1] holds.

Corollary 1. *For every $d \in \mathbb{N}$, there is a function $f_d : \mathbb{N} \to \mathbb{N}$ such that every graph with maximum degree at most d and treewidth at least $f_d(k)$ contains a $k \times k$-wall or the line graph of a $k \times k$-wall as an induced subgraph.*

In addition to [1], this conjecture has been explicitly mentioned by Abrishami, Chudnovsky, Dibek, Hajebi, Rzążewski, Spirkl, and Vušković in three articles of their "Induced subgraphs and tree decompositions" series [2–4] and the main results of the first two articles of this series [2,5] are special cases of this conjecture.

Theorem 1 has several direct algorithmic implications. In particular, by known algorithms using treewidth [7,11], it implies that a large number of combinatorial problems can be solved in linear-time on graphs that exclude a planar graph as an induced minor and have bounded maximum degree. Note that in contrast, for any non-planar graph H, for example the maximum independent set problem is NP-complete on subcubic graphs that exclude H as an induced minor [18].

Theorem 1 implies also the following algorithmic result.

Corollary 2. *For every fixed planar graph H, there is a $2^{\mathcal{O}(n/\log^{1/6} n)}$ time algorithm for maximum weight independent set on H-induced-minor-free graphs, where n is the number of vertices of the input graph.*

Algorithms for maximum weight independent set on H-induced-minor-free graphs for specific planar graphs H have recently received attention [12,14,15]. In this area a central open question asked by Dallard, Milanic, and Storgel [12] is if there exists a fixed planar graph H so that maximum weight independent set is NP-complete on H-induced-minor-free graphs, or whether a polynomial time or quasipolynomial time algorithm could be obtained for H-induced-minor-free graphs for all planar graphs H. Theorem 1 can be seen as a step towards the latter direction.

2. Preliminaries

We use log to denote base-2 logarithm. A graph G has a set of vertices $V(G)$ and a set of edges $E(G)$. For a vertex $v \in V(G)$, $N(v)$ denotes the set of its neighbors. For a set of vertices $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. The distance between two vertices in a graph is the minimum number of edges on a path between them, or if they are in different connected components the distance is infinite.

A minor model of a graph H in a graph G is a collection $\{X_v\}_{v \in V(H)}$ of pairwise disjoint vertex sets $X_v \subseteq V(G)$ called *branch sets* so that each induced subgraph $G[X_v]$ is connected, and if there is an edge $uv \in E(H)$, then there is an edge between X_u and X_v in G. A graph G contains H as a minor if and only if there is a minor model of H in G. An induced minor model of H is a minor model with an additional constraint that if $u, v \in V(H)$, $u \neq v$, and $uv \notin E(H)$, then there are no edges between X_u and X_v. A graph G contains H as an induced minor if and only if there is an induced minor model of H in G.

A tree decomposition of a graph G is a pair (T, β), where T is a tree and β is a function $\beta : V(T) \to 2^{V(G)}$ mapping nodes of T to sets of vertices of G called bags so that

1. $V(G) = \bigcup_{i \in V(T)} \beta(i)$,
2. for every $uv \in E(G)$ there exists $i \in V(T)$ with $\{u, v\} \subseteq \beta(i)$, and
3. for every $v \in V(G)$, the nodes $\{i \in V(T) \mid v \in \beta(i)\}$ induce a connected subtree of T.

The width of a tree decomposition is $\max_{i \in V(T)} |\beta(i)| - 1$ and the treewidth of a graph is the minimum width of a tree decomposition of it.

We recall a standard lemma on treewidth.

Lemma 1. *Let G be a graph of treewidth k and let $X_1, \ldots, X_p$ be pairwise disjoint subsets of $V(G)$ so that for each X_i it holds that $|X_i| \leq q$ and $G[X_i]$ is connected. The treewidth of the graph obtained by contracting each set X_i into a single vertex is at least k/q.*

Proof. Let G' be the graph obtained from G by contracting each set X_i into a vertex x_i. Suppose that G' has a tree decomposition of width at most $k/q - 1$. We construct a tree decomposition of G by replacing each vertex x_i in the decomposition by the set X_i. This increases the bag sizes by a factor of q, so we obtain a tree decomposition of G of width at most $k-1$, which is a contradiction. □

3. Proof of Theorem 1

We will first define *sparsifiable graphs* and show that minors in sparsifiable graphs correspond to induced minors. Then we show that if a graph has treewidth k and maximum degree at most $\log^{1/5} k$, then it has an induced subgraph that is sparsifiable and has treewidth $k/2^{\mathcal{O}(\log^{5/6} k)}$.

3.1. Sparsifiable graphs

We say that a vertex v of a graph is *sparsifiable* if it satisfies one of the following conditions:

1. v has degree at most 2,
2. v has degree 3 and all of its neighbors have degree at most 2, or
3. v has degree 3, one of its neighbors has degree at most 2, and the two other neighbors form a triangle with v.

We call a graph sparsifiable if every vertex of it is sparsifiable. We say that a vertex is of type 2 if it satisfies the condition 2 and that a vertex is of type 3 if it satisfies the condition 3 but does not satisfy the condition 2.

Now we show that minors and induced minors are highly related in sparsifiable graphs.

Lemma 2. *Let G be a sparsifiable graph and H a graph of minimum degree at least 3. If G contains H as a minor, then G contains H as an induced minor.*

Proof. Let $\{X_v\}_{v \in V(H)}$ be a minor model of H in G. Say that an edge $ab \in E(G)$ is violating if there exists non-adjacent vertices u, v of H so that $a \in X_u$ and $b \in X_v$. Note that if there are no violating edges, then $\{X_v\}_{v \in V(H)}$ is also an induced minor model of H. Suppose that $\{X_v\}_{v \in V(H)}$ minimizes the number of violating edges among all minor models of H. We will show by contradiction that the number of violating edges must be zero.

Let $ab \in E(G)$ be a violating edge with $a \in X_u$ and $b \in X_v$. Both a and b must have degree at least 2 because otherwise u or v would have degree 1. First, consider the case that ab has a degree 2 endpoint, and by symmetry assume that the degree 2 endpoint is $a \in X_u$. The neighbor of a not in X_v must be in X_u because u has degree at least 3. It also holds that $G[X_u \setminus \{a\}]$ is connected because a has degree 1 in $G[X_u]$. The only other branch set that a is adjacent to than X_u is X_v, but there is no edge between u

and v in H, so we can replace X_u by $X_u \setminus \{a\}$ in the minor model. This decreases the number of violating edges.

The other case is that both a and b have degree 3. Because they are adjacent, they both are of type 3, so they form a triangle together with a vertex c. First, consider the case that $c \in X_v$. Now, a is adjacent to two vertices in X_v and the third vertex it is adjacent to must be in X_u because u has degree at least 3. Therefore, we can replace X_u by $X_u \setminus \{a\}$ in the minor model, which decreases the number of violating edges.

The same argument as above works also if $c \in X_u$ or if c is not in any branch set. The remaining case is that $c \in X_w$ for some $w \in V(H) \setminus \{u, v\}$. If w is not adjacent to u (resp. to v), then removing a from X_u (resp. b from X_v) again decreases the number of violating edges, so we can assume that w is adjacent to both u and v. Because u and v have degree at least 3 and u and v are not adjacent, it holds that the unique vertex in $N(a) \setminus \{b, c\}$ is in X_u and that the unique vertex in $N(b) \setminus \{a, c\}$ is in X_v. We replace X_u by $X_u \setminus \{a\}$, X_v by $X_v \setminus \{b\}$, and X_w by $X_w \cup \{a, b\}$. This removes the violating edge ab, and does not create any new violating edges because w is adjacent to both u and v, and both $N(a)$ and $N(b)$ are contained in $X_u \cup X_v \cup X_w$. □

Lemma 2 cannot be directly used when H is the $k \times k$-grid because its corners have degree 2. However, contracting four edges each incident to a distinct corner of the $k \times k$-grid yields a graph of minimum degree 3 that contains the $k-2 \times k-2$-grid as an induced minor. Therefore Lemma 2 implies that if a sparsifiable graph contains a $k \times k$-grid minor, then it contains a $k-2 \times k-2$-grid induced minor.

3.2. Sparsifying a graph

We will make use of the following theorem proved by Chekuri and Chuzhoy that every graph contains a degree-3 subgraph that approximately preserves its treewidth.

Theorem 2 *([8]). There exists a constant δ so that every graph with treewidth $k \geq 2$ has a subgraph with maximum degree 3 and treewidth at least $k/\log^{\delta} k$.*

For the rest of this section we will use δ to denote the constant δ given by Theorem 2. We note that instead of Theorem 2 we could alternatively use the grid minor theorem, but that would yield a significantly worse dependence on the treewidth in our result.

A distance-5 independent set in a graph G is a set $I \subseteq V(G)$ of vertices so that for any pair $u, v \in I$ of distinct vertices, the distance between u and v in G is at least 5. Next we show how to make all vertices in a distance-5 independent set sparsifiable while approximately preserving treewidth. This will be then used to prove Theorem 1 by observing that the vertices of a graph with maximum degree d can be partitioned into $d^4 + 1$ distance-5 independent sets.

Lemma 3. *Let G be a graph of treewidth k and maximum degree d with $2d^2 \leq k$, and let $I \subseteq V(G)$ be a distance-5 independent set in G. There exists an induced subgraph $G[S]$*

of G so that $G[S]$ has treewidth at least $k/((d^2+1)\log^{\delta} k)$ and every vertex in $I \cap S$ is sparsifiable in $G[S]$.

Proof. For each vertex $v \in I$, let B_v be the set of vertices at distance at most 2 from v (i.e. $B_v = \{v\} \cup N(v) \cup N(N(v))$). The induced subgraphs $G[B_v]$ are connected, and because I is a distance-5 independent set, the sets B_v are disjoint. Let H be the graph obtained by contracting each set B_v into one vertex. Because $|B_v| \leq d^2+1$, Lemma 1 implies that the treewidth of H is at least $k/(d^2+1)$. By Theorem 2, there exists a subgraph H' of H with maximum degree 3 and treewidth at least $k/((d^2+1)\log^{\delta} k)$. We can assume that $V(H') = V(H)$. Let $Q \subseteq V(G)$ be the vertices of G that are not in any B_v. The graph H' is a minor of G, with a minor model whose branch sets are the sets B_v for each $v \in I$ and singleton sets $\{u\}$ for each $u \in Q$.

We will construct a set $S \subseteq V(G)$ so that $Q \subseteq S$, all vertices of $I \cap S$ are sparsifiable in $G[S]$, and H' has a minor model in G whose branch sets are the sets $B_v \cap S$ for all $v \in I$ and singleton sets $\{u\}$ for all $u \in Q$. The graph $G[S]$ therefore will contain H' as a minor and therefore will have treewidth at least $k/((d^2+1)\log^{\delta} k)$.

Consider a vertex $v \in I$. We will construct $B_v \cap S$ so that either $v \notin S$ or v is sparsifiable in $G[S]$. Because H' has maximum degree 3, we can choose a set $T \subseteq B_v$ of at most three terminal vertices that are at distance 2 from v whose connectivity should be preserved in $G[B_v \cap S]$ in order to preserve the minor model of H'. We start by setting $B_v \cap S$ to be the union of the shortest paths from the terminals $t \in T$ to v. Note that the only vertices at distance 2 from v that are on the shortest paths are the terminals T. Then, we say that a terminal $t \in T$ is private to a vertex $u \in N(v) \cap S$ if u is the only vertex in $N(v) \cap S$ adjacent to t. If a vertex $u \in N(v) \cap S$ does not have a private terminal, we remove u from S. Now, every vertex in $N(v) \cap S$ has a private terminal. If $|N(v) \cap S| \leq 2$, the vertex v has degree at most 2 in $G[S]$ and we are done. The remaining case is that $|N(v) \cap S| = 3$ and each vertex in $N(v) \cap S$ has a private terminal, implying that $|T| = 3$ and the edges between $N(v) \cap S$ and T form a matching. If the graph $G[N(v) \cap S]$ is connected, we remove v from S and are done. If the graph $G[N(v) \cap S]$ is not connected, it contains at most one edge. If it contains no edges, the vertices $N(v) \cap S$ have degree 2 in $G[S]$, and therefore v is a type 2 vertex in $G[S]$. If it contains one edge, then v is a type 3 vertex in $G[S]$. □

Next we finish the proof of Theorem 1 by applying Lemma 3 d^4+1 times.

Lemma 4. *If a graph G has treewidth k and maximum degree at most $\log^{1/5} k$, then it contains an induced subgraph that is sparsifiable and has treewidth at least $k/2^{\mathcal{O}(\log^{5/6} k)}$.*

Proof. Let G be a graph with treewidth k and maximum degree d. The vertices of G can be partitioned into d^4+1 distance-5 independent sets $I_1, \ldots, I_{d^4+1}$ by observing that there are at most d^4 vertices at distance at most 4 from any vertex and using a greedy method. We then sequentially apply Lemma 3 with these distance-5 independent sets, in particular letting $G_0 = G$, and then for each i with $1 \leq i \leq d^4+1$ letting G_i be the graph obtained by applying Lemma 3 with G_{i-1} and $I_i \cap V(G_{i-1})$. As taking induced subgraphs only increases distances, $I_i \cap V(G_{i-1})$ is a distance-5 independent set in G_{i-1}. It also holds that once a vertex becomes sparsifiable, it will stay sparsifiable because taking induced subgraphs cannot increase vertex degrees or add new neighbors. Therefore G_{d^4+1} is sparsifiable. The graph G_{d^4+1} has treewidth at least $k/((d^2+1)\log^\delta k)^{d^4+1}$, meaning that when $d^5 \leq \log k$, the decrease in the treewidth is by a factor of at most

$$((d^2+1)\log^\delta k)^{d^4+1} = 2^{\mathcal{O}(d^4(\log(d^2+1)+\log\log^\delta k))} = 2^{\mathcal{O}(d^4 \log\log^\delta k)} = 2^{\mathcal{O}(\log^{5/6} k)}. \quad \square$$

Theorem 1 follows from using Lemma 4 to obtain a sparsifiable induced subgraph with treewidth at least $k/2^{\mathcal{O}(\log^{5/6} k)}$, then applying the grid minor theorem [10] to obtain a $\Omega(k^{1/10}) \times \Omega(k^{1/10})$ grid minor, and then using Lemma 2 to argue that this grid minor, after contracting the corners, is also an induced minor.

4. Proofs of corollaries

We detail how Corollary 1 and Corollary 2 follow from Theorem 1.

Corollary 1. *For every $d \in \mathbb{N}$, there is a function $f_d : \mathbb{N} \to \mathbb{N}$ such that every graph with maximum degree at most d and treewidth at least $f_d(k)$ contains a $k \times k$-wall or the line graph of a $k \times k$-wall as an induced subgraph.*

Proof. The proof of Theorem 1.1 in [1] shows that there is a function $g : \mathbb{N} \to \mathbb{N}$ so that if a graph contains a triangulated $g(k) \times g(k)$-grid as an induced minor, then it contains either a $k \times k$-wall or the line graph of a $k \times k$-wall as an induced subgraph. Then, this corollary follows from Theorem 1 by observing that a $k \times k$-grid contains a triangulated $k/6 \times k/6$-grid as an induced minor and setting $f_d(k) = c \cdot (g(k)^{10} + 2^{d^5})$ for some large enough constant c. $\square$

Corollary 2. *For every fixed planar graph H, there is a $2^{\mathcal{O}(n/\log^{1/6} n)}$ time algorithm for maximum weight independent set on H-induced-minor-free graphs, where n is the number of vertices of the input graph.*

Proof. Let G be the input graph and assume that n is sufficiently large compared to H. While there is a vertex of degree at least $\log^{1/5}(n/\log n)$ in G, we branch from this vertex. This branching tree has size at most

$$n\binom{n}{n/\log^{1/5}(n/\log n)} < n(e \cdot \log^{1/5}(n/\log n))^{n/(\log^{1/5}(n/\log n))} = 2^{\mathcal{O}(n/\log^{1/6} n)}.$$

Then, if all vertices of G have degree less than $\log^{1/5}(n/\log n)$ and G has treewidth more than $cn/\log n$ for some constant c, then by Theorem 1 G contains a $\Omega(n^{1/11}) \times$

$\Omega(n^{1/11})$ grid induced minor, and therefore contains any planar graph H as an induced minor if n is sufficiently large compared to H. Therefore, if all vertices of G have degree less than $\log^{1/5}(n/\log n)$, then G must have treewidth $\mathcal{O}(n/\log n)$, and we can use a parameterized single-exponential time constant-factor approximation of treewidth [17] together with dynamic programming [6] to solve maximum weight independent set in $2^{\mathcal{O}(n/\log n)}$ time. □

Acknowledgments

I thank Daniel Lokshtanov for telling me about the conjecture of Aboulker et al.

References

[1] Pierre Aboulker, Isolde Adler, Eun Jung Kim, Ni Luh Dewi Sintiari, Nicolas Trotignon, On the tree-width of even-hole-free graphs, Eur. J. Comb. 98 (2021) 103394, https://doi.org/10.1016/j.ejc.2021.103394.

[2] Tara Abrishami, Maria Chudnovsky, Cemil Dibek, Sepehr Hajebi, Paweł Rzążewski, Sophie Spirkl, Kristina Vušković, Induced subgraphs and tree decompositions II. Toward walls and their line graphs in graphs of bounded degree, arXiv:2108.01162, 2021.

[3] Tara Abrishami, Maria Chudnovsky, Sepehr Hajebi, Sophie Spirkl, Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth, Adv. Combin. 2022 (2022) 6, https://doi.org/10.19086/aic.2022.6.

[4] Tara Abrishami, Maria Chudnovsky, Sepehr Hajebi, Sophie Spirkl, Induced subgraphs and tree decompositions IV. (Even hole, diamond, pyramid)-free graphs, arXiv:2203.06775, 2022.

[5] Tara Abrishami, Maria Chudnovsky, Kristina Vušković, Induced subgraphs and tree decompositions I. Even-hole-free graphs of bounded degree, J. Comb. Theory, Ser. B 157 (2022) 144–175, https://doi.org/10.1016/j.jctb.2022.05.009.

[6] Stefan Arnborg, Andrzej Proskurowski, Linear time algorithms for NP-hard problems restricted to partial k-trees, Discrete Appl. Math. 23 (1) (1989) 11–24, https://doi.org/10.1016/0166-218X(89)90031-0.

[7] Hans L. Bodlaender, A linear-time algorithm for finding tree-decompositions of small treewidth, SIAM J. Comput. 25 (6) (1996) 1305–1317, https://doi.org/10.1137/S0097539793251219.

[8] Chandra Chekuri, Julia Chuzhoy, Degree-3 treewidth sparsifiers, in: Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, SIAM, 2015, pp. 242–255.

[9] Chandra Chekuri, Julia Chuzhoy, Polynomial bounds for the grid-minor theorem, J. ACM 63 (5) (2016) 40, https://doi.org/10.1145/2820609.

[10] Julia Chuzhoy, Zihan Tan, Towards tight(er) bounds for the excluded grid theorem, J. Comb. Theory, Ser. B 146 (2021) 219–265, https://doi.org/10.1016/j.jctb.2020.09.010.

[11] Bruno Courcelle, The monadic second-order logic of graphs. I. Recognizable sets of finite graphs, Inf. Comput. 85 (1) (1990) 12–75, https://doi.org/10.1016/0890-5401(90)90043-H.

[12] Clément Dallard, Martin Milanic, Kenny Storgel, Tree decompositions with bounded independence number and their algorithmic applications, arXiv:2111.04543, 2021.

[13] Fedor V. Fomin, Petr A. Golovach, Dimitrios M. Thilikos, Contraction obstructions for treewidth, J. Comb. Theory, Ser. B 101 (5) (2011) 302–314, https://doi.org/10.1016/j.jctb.2011.02.008.

[14] Peter Gartland, Daniel Lokshtanov, Independent set on P_k-free graphs in quasi-polynomial time, in: Proceedings of 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, IEEE, 2020, pp. 613–624.

[15] Peter Gartland, Daniel Lokshtanov, Marcin Pilipczuk, Michal Pilipczuk, Pawel Rzazewski, Finding large induced sparse subgraphs in $C_{>t}$-free graphs in quasipolynomial time, in: Proceedings of 53rd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2021, ACM, 2021, pp. 330–341.

[16] Tuukka Korhonen, Lower bounds on dynamic programming for maximum weight independent set, in: Proceedings of the 48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, in: LIPIcs, vol. 198, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021, 87.

[17] Tuukka Korhonen, A single-exponential time 2-approximation algorithm for treewidth, in: Proceedings of 62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, IEEE, 2021, pp. 184–192.
[18] Bojan Mohar, Face covers and the genus problem for apex graphs, J. Comb. Theory, Ser. B 82 (1) (2001) 102–117, https://doi.org/10.1006/jctb.2000.2026.
[19] Neil Robertson, Paul D. Seymour, Graph minors. V. Excluding a planar graph, J. Comb. Theory, Ser. B 41 (1) (1986) 92–114, https://doi.org/10.1016/0095-8956(86)90030-4.