Profunctors Between Posets and Alexander Duality

Gunnar Fløystad[1] ![ORCID]

Abstract

We consider profunctors $f : P \longrightarrow Q$ between posets and introduce their *graph* and *ascent*. The profunctors $\mathrm{Pro}(P, Q)$ form themselves a poset, and we consider a partition $\mathcal{I} \sqcup \mathcal{F}$ of this into a down-set $\mathcal{I}$ and up-set $\mathcal{F}$, called a *cut*. To elements of $\mathcal{F}$ we associate their graphs, and to elements of $\mathcal{I}$ we associate their ascents. Our basic results is that this, suitably refined, preserves being a cut: We get a cut in the Boolean lattice of subsets of the underlying set of $Q \times P$. Cuts in finite Booleans lattices correspond precisely to finite simplicial complexes. We apply this in commutative algebra where these give classes of Alexander dual square-free monomial ideals giving the full and natural generalized setting of isotonian ideals and letterplace ideals for posets. We study $\mathrm{Pro}(\mathbb{N}, \mathbb{N})$. Such profunctors identify as order preserving maps $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. For our applications when P and Q are infinite, we also introduce a topology on $\mathrm{Pro}(P, Q)$, in particular on profunctors $\mathrm{Pro}(\mathbb{N}, \mathbb{N})$.

Introduction

This article advocates for general posets P and Q the notion of profunctor $P \longrightarrow Q$ as more effective than the notion of isotone (order preserving) maps $P \to Q$ between posets, especially for applications in algebra. When Q is totally ordered, these notions are practically the same, but when Q is not, profunctors seem to have a clear advantage for developing natural theory.

Let **2** be the two element boolean poset $\{0 < 1\}$. A profunctor $P \longrightarrow Q$ is simply an isotone map $P \times Q^{\mathrm{op}} \to \mathbf{2}$. If P and Q are sets (discrete posets), then this is simply a relation between P and Q. The notion of profunctor may generally be defined between categories or between

✉ Gunnar Fløystad
   nmagf@uib.no

[1]   Matematisk institutt Postboks 7803, Universitetet i Bergen, 5020 Bergen, Norway
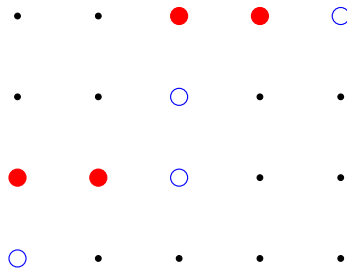
GUNNAR FLØYSTAD

Fig. 1 Profunctor from [5] to [4]

categories enriched in a symmetric monoidal closed category (like **2**), see [2], [3], or, for a recent gentle introduction focusing on applications, [13, Section 4].

The opposite P^{op} of a poset P, has the same elements but order relation reversed. The elements in the distributive lattice $\widehat{P}$ associated to P identifies as pairs (I, F), called *cuts*, where I is a down-set in P and F the complement up-set. There is then a duality between $\widehat{P}$ and $\widehat{P^{\mathrm{op}}}$ sending (I, F) to $(F^{\mathrm{op}}, I^{\mathrm{op}})$. We call two such pairs dual or Alexander dual (as is common in combinatorial commutative algebra).

Denote by $\mathrm{Pro}(P, Q)$ the profunctors $P \longrightarrow\!\!\!\!\!+\, Q$. This is again a a partially ordered set and the opposite of this poset is $\mathrm{Pro}(Q, P)$. The basic notions we introduce associated to a profunctor $f : P \longrightarrow\!\!\!\!\!+\, Q$ between posets are the notions of its *graph* Γf and its *ascent* Λf. These are dual notions in the sense that if $g : Q \longrightarrow\!\!\!\!\!+\, P$ is the dual profunctor, the graph of f equals the ascent of g. Let $U P$ and $U Q$ denote the underlying sets of P and Q. In $\mathrm{Pro}(P, Q)$ let $\mathcal{I}$ be a down-set and $\mathcal{F}$ its complement up-set, so $(\mathcal{I}, \mathcal{F})$ is a cut for $\mathrm{Pro}(P, Q)$. Let $\mathcal{F}_\Lambda$ be the *up-set* in the Boolean lattice of all subsets of $U Q \times U P^{\mathrm{op}}$ generated by the ascents Λf for $f \in \mathcal{F}$. Let $\mathcal{I}_\Gamma$ be the *down-set* in this Boolean lattice generated by the complements of the graphs Γf for $f \in \mathcal{I}$.

Example 0.1 In Fig. 1 the red discs give the graph of a profunctor $[5] \longrightarrow\!\!\!\!\!+\, [4]$. The blue circles give the ascent of this profunctor. The graphs and the ascents are then subsets of $U[4] \times U[5]^{\mathrm{op}}$. The up-set $\mathcal{F}_\Lambda$ is the up-set of the Boolean lattice of subsets of $U[4] \times U[5]^{\mathrm{op}}$ generated by the ascents of $f \in \mathcal{F}$. The down-set $\mathcal{I}_\Gamma$ is the down-set of this Boolean lattice generated by the complements of graphs of $f \in \mathcal{I}$.

Our main theorem states the following.

Theorem 3.7.(Preserving the cut) *Let P and Q be well-founded posets, and $(\mathcal{I}, \mathcal{F})$ a cut for* $\mathrm{Pro}(P, Q)$. *Then $(\mathcal{I}_\Gamma, \mathcal{F}_\Lambda)$ is a cut for the Boolean lattice of subsets of $U Q \times U P^{\mathrm{op}}$.*

Example 0.1 continued. This says that given any subset S of $U[4] \times U[5]^{\mathrm{op}}$, then either S contains an ascent Λf for $f \in \mathcal{F}$, or the complement S^c contains a graph Γf for $f \in \mathcal{I}$. These two cases are also mutually exclusive.

This theorem has alternative formulations in Theorem 3.9 asserting that two up-sets are Alexander dual, with applications to Stanley–Reisner theory, and in Theorem 3.10 asserting that the map sending the ideal $\mathcal{I}$ to the ideal $\mathcal{I}_\Gamma$ respects the duality on profunctors. In Theorem 7.1 we give a version with conditions on P and Q ensuring that Γf and Λf are always finite sets, suitable for applications to monomial ideals, Sect. 8. An open problem is if a more functorial formulation is possible, Problem 3.11.

Although we develop a general theory here, our original motivation came from applications related to commutative algebra.

Applications to Stanley–Reisner theory. When P and Q are finite posets we get general constructions, Sect. 4, of Alexander dual squarefree monomial ideals, generalizing isotonian ideals and letterplace and co-letterplace ideals, [7, 8, 10, 16, 17]. In particular, when Q is a chain these constructions have given very large classes of simplicial balls and spheres, [5, 11].

Applications to order preserving maps $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. The profunctors from $\mathbb{N}$ to $\mathbb{N}$ identify as order preserving maps from $\mathbb{N}$ to the distributive lattice $\widehat{\mathbb{N}}$, and the latter identifies as $\mathbb{N} \cup \{\infty\}$. Profunctors $f : \mathbb{N} \rightarrowtail \mathbb{N}$ are the topic of many our examples. These benefit the reader with a quick access to many of our notions and results: 1.2, 1.4, 2.5, 2.12, 2.13, 3.3, 3.8, 5.10, 6.7, 6.8, 6.9. See also the end of Sect. 8.

Injective order preserving maps $f : \mathbb{N} \to \mathbb{N}$ form the so called increasing monoid, which has gained recent interest. In [20] Nagel and Römer show that ideals in the infinite polynomial ring invariant for the increasing monoid, have an essentially finite Gröbner basis, thereby generalizing previous results for the symmetric group. In [14] Güntürkün and Snowden studies in depth the representation theory of the increasing monoid. Note that the injective order preserving maps $g : \mathbb{N} \to \mathbb{N}$ are in bijection with order preserving maps $f : \mathbb{N} \to \mathbb{N}$ by $g = f + \mathrm{id} - 1$. Order preserving maps $f : \mathbb{N} \to \mathbb{N}$ also occur in the definition of the bicyclic semi-group [6], a basic notion in inverse semi-group theory. Our main application is in [9], relating profunctors $f : \mathbb{N} \rightarrowtail \mathbb{N}$ (i.e. order preserving maps $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$) to the duality theory of strongly stable ideals in the the infinite polynomial ring $k[x_i]_{i \in \mathbb{N}}$.

In order for the substantial parts of our theory, related to graphs and ascents, to work well we must have certain conditions on the posets P and Q. Our weakest condition is that they are *well-founded*. For our applications to polynomial rings, we work in the class of *natural* posets, Sect. 6. These are posets where all anti-chains are finite and for every x in the poset the principal down-set $\downarrow x$ is finite. This is a subclass (closer to natural numbers) of *well partially ordered sets*.

Another feature we introduce is a topology on $\mathrm{Pro}(P, Q)$, Sect. 5, in particular on $\mathrm{Pro}(\mathbb{N}, \mathbb{N})$. This is needed for our applications to commutative algebra. For $\mathrm{Pro}(\mathbb{N}, \mathbb{N})$ a basis for the topology consists of intervals $[f, g]$ where i) the image $f(\mathbb{N})$ is contained in a finite interval, and ii) $g(p) = \infty$ for all but a finite set of p's.

Organization of article:

1. PRELIMINARIES ON POSETS. Notions for posets are recalled, most significantly cuts for posets and the associated distributive lattice. It relates to simplicial complexes, Alexander duality, and Stanley–Reisner rings.
2. PROFUNCTORS BETWEEN POSETS. We introduce these and develop basic theory.
3. THE GRAPH, THE ASCENT AND PRESERVING THE CUT. We give our main theorem on Alexander duality, together with variants.
4. APPLICATIONS TO FINITE POSETS AND ALEXANDER DUALITY. We connect to commutative algebra and get Alexander dual ideals in finite dimensional polynomial rings.
5. TOPOLOGY ON $\mathrm{Pro}(P, Q)$. We define the topology and in particular look at *interior* open down-sets.
6. PROFUNCTORS BETWEEN NATURAL POSETS. We consider natural posets P and Q and investigate the topology in this setting. We show an open down-set is also closed (clopen down-set), if and only if it is finitely generated.

7. NATURAL POSETS AND FINITE TYPE CUTS. We give the version of the main theorem for natural posets, suitable to get Alexander dual ideals in (infinite dimensional) polynomial rings.
8. MONOMIAL IDEALS. When $Q = \mathbb{N}$ we get monomial ideals in the polynomial ring generated by x_p, $p \in P$. We briefly indicate the applications to strongly stable ideals in [9] when $P = Q = \mathbb{N}$.

Note. We let $\mathbb{N} = \{1, 2, 3, \ldots\}$. We only use the ordered structure on this so we could equally well have used $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. Only in the last Sect. 8 do we, in a somewhat different setting, use the commutative monoid structure and then we explicitly write $\mathbb{N}_0$.

1 Preliminaries on Posets

We give basic notions and constructions concerning posets: down-sets, up-sets, dualities, distributive lattices, simplicial complexes. We also recount the algebraic notions of Stanley–Reisner ideal and ring.

1.1 Down- and Up-Sets in *P*

Let P be a partially ordered set. The *opposite poset* P^{op} has the same underlying set as P but with order relation $\leq_{\mathrm{op}}$ where $p \leq_{\mathrm{op}} q$ if $p \geq q$ in P.

A *down-set* I of P is a subset of P closed under taking smaller elements. An *up-set* F in P is a subset of P closed under taking larger elements. If I and F are complements of each other, we call (I, F) a *cut* for P. Since each of I and F determine each other, we sometimes denote this as $(-, F)$ if we focus on F, and similarly with $(I, -)$. Down-sets are sometimes called *order ideals* and up-sets *order filters*, whence the suggestive notation I and F. (The single term *ideal* is usually reserved for order ideals in lattices closed under joins.)

An element $p \in P$ induces the principal up-set $\uparrow p$ consisting of all p' with $p' \geq p$, and a principal down-set $\downarrow p$ consisting of all $p' \leq p$.

Definition 1.1 The *Alexander dual* (or just *dual*) of the cut (I, F) for P is the cut $(F^{\mathrm{op}}, I^{\mathrm{op}})$ for P^{op}. The Alexander dual of the down-set I is the down-set $J = F^{\mathrm{op}}$ of P^{op}, and the Alexander dual of the up-set F is the up-set $G = I^{\mathrm{op}}$.

1.2 The Distributive Lattice

If P and Q are two partially ordered sets, a map $f : P \rightarrow Q$ is *isotone* if it is order-preserving, i.e. $p_1 \leq p_2$ implies $f(p_1) \leq f(p_2)$. We denote by $\mathrm{Hom}(P, Q)$ the set of all isotone (order-preserving) maps $f : P \rightarrow Q$. It is itself a partially ordered set (an internal Hom) by $f \leq g$ if $f(p) \leq g(p)$ for every $p \in P$. The opposite poset $\mathrm{Hom}(P, Q)^{\mathrm{op}}$ naturally identifies as $\mathrm{Hom}(P^{\mathrm{op}}, Q^{\mathrm{op}})$. The category of posets forms a closed symmetric monoidal category and so for any three posets P, Q, R we have:

$$\mathrm{Hom}(P \times Q, R) \cong \mathrm{Hom}(P, \mathrm{Hom}(Q, R)). \tag{1}$$

Denote by **2** the ordered set $\{0 < 1\}$. The (complete) distributive lattice associated to P is $\widehat{P} = \mathrm{Hom}(P^{\mathrm{op}}, \mathbf{2})$. Given an $f \in \widehat{P}$, the elements p in P such that p^{op} maps to $1 \in \mathbf{2}$, constitute a down-set I in P. The complement up-set F in P consists of those $p \in P$ such that p^{op} maps to 0. An element f of $\widehat{P}$ may thus be identified either with:
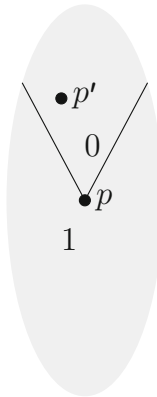
GUNNAR FLØYSTAD

Fig. 2 The poset P and its cut $((\uparrow p)^c, \uparrow p))$, giving $\hat{p}$ in $\hat{P}$

- A down-set I of P,
- An up-set F of P,
- A cut (I, F) for P.

We shall usually identify elements of $\hat{P}$ with the down-sets $\mathbb{D}P$. In categorical terms $\mathbb{D}P$ is the free cocompletion of P. Thus an element $i \in \hat{P}$ is a down-set I of P. However we sometimes will consider the elements of $\hat{P}$ to be cuts (I, F) for P. We speak of a cut (I, F) in $\hat{P}$, or equivalently a cut (I, F) *for* P. The cuts for P are then ordered by

$$(I, F) \leq (J, G) \text{ if and only if } I \subseteq J \text{ or equivalently } F \supseteq G.$$

The distributive lattice $\hat{P}$ has a unique maximal element, denoted ∞. It sends every p^{op} to 1, and corresponds to the cut $(I, F) = (P, \emptyset)$.

Example 1.2

- $P = [n] = \{1 < 2 < \cdots < n\}$ has cuts (I, F) where $I = \{1, 2, \ldots, i - 1\}$ and $F = \{i, \ldots, n\}$ for $i = 1, \ldots, n+1$. Thus $\hat{P} \cong [n+1] = [n] \cup \{\infty\}$ where $\infty = n + 1$.
- Any set A may be considered a discrete poset (only relations are $a \leq a$). Then $\hat{A}$ is the Boolean lattice on A. It consists of subsets $S \subseteq A$. We identify such a subset with the cut (S, S^c). (So for instance the cut $(-, T)$ identifies as the subset T^c in A.)
- If $P = \mathbb{N}$ the natural numbers, then $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Given an element p in P we get a map

$$\hat{p} : P^{\mathrm{op}} \to \mathbf{2},$$

where if $p' \geq p$ we send $p'^{\mathrm{op}} \mapsto 0$ and all other elements of P^{op} to 1. It corresponds to the cut $((\uparrow p)^c, \uparrow p)$ for P (where $()^c$ denotes the complement set).

This gives a distinguished injective poset map

$$P \longrightarrow \hat{P} = \mathrm{Hom}(P^{\mathrm{op}}, \mathbf{2}), \quad p \mapsto \hat{p}, \text{ corresponding to the cut } ((\uparrow p)^c, \uparrow p) \qquad (2)$$

see Fig. 2. Note that the image of the map (2) is in $\hat{P} \backslash \{\infty\}$.

Remark 1.3 A poset may be considered a **2**-category for the symmetric monoidal closed category **2**. Where

$$\mathrm{Hom}_{\mathbf{2}}(p', p) = \begin{cases} 1 & p' \le p \\ 0 & p' \nleq p \end{cases}.$$

The map (2) is not the Yoneda embedding

$$P \to \mathrm{Hom}(P^{\mathrm{op}}, \mathbf{2}), \quad p \mapsto \mathrm{Hom}_{\mathbf{2}}(-, p). \tag{3}$$

Rather the map (2) is derived as follows. One has the Yoneda embedding:

$$P^{\mathrm{op}} \to \mathrm{Hom}(P, \mathbf{2}), \quad p \mapsto \mathrm{Hom}_{\mathbf{2}}(p, -).$$

Taking the opposite of this we get:

$$P \to \mathrm{Hom}(P, \mathbf{2})^{\mathrm{op}}. \tag{4}$$

This is the *co-Yoneda* embedding, [1]. Note that $\mathbf{2}^{\mathrm{op}} \cong \mathbf{2}$ by sending $0^{\mathrm{op}} \mapsto 1$ and $1^{\mathrm{op}} \mapsto 0$. So

$$\mathrm{Hom}(P, \mathbf{2})^{\mathrm{op}} = \mathrm{Hom}(P^{\mathrm{op}}, \mathbf{2}^{\mathrm{op}}) \cong \mathrm{Hom}(P^{\mathrm{op}}, \mathbf{2}) = \widehat{P}. \tag{5}$$

In general however, for instance ordinary categories with **set** instead of **2**, the Yoneda and co-Yoneda embedding map to different categories. Composing (4) above with (5) we get the embedding (2).

Example 1.4 Let $P = [n]$. The map (2), the co-Yoneda embedding, is

$$[n] \longrightarrow \widehat{[n]} = [n+1], \quad i \mapsto i.$$

On the other hand the Yoneda embedding (3)

$$[n] \longrightarrow \widehat{[n]} = [n+1], \quad i \to i+1.$$

In particular the top element $n \mapsto n+1 = \infty$.

In more abstract terms, the Yoneda embedding embeds a chain of length n into the upper part of a chain of length $n+1$, while the co-Yoneda embedding embeds it into the lower part.

1.3 Down- and Up-Sets in $\widehat{P}$

Let $\mathcal{I}$ be a down-set for $\widehat{P}$, and $\mathcal{F}$ the complement up-set of $\widehat{P}$. So $\mathcal{I}$ consists of cuts (I, F) closed under forming cuts with smaller I's, and $\mathcal{F}$ consists of cuts (I, F) closed under forming cuts with larger I's (or equivalently smaller F's). Then $(\mathcal{I}, \mathcal{F})$ is a cut *for* $\widehat{P}$ (note again terminology: (I, F) is a cut *in* $\widehat{P}$). Also $(\mathcal{I}, \mathcal{F})$ is a cut *in* $\widehat{\widehat{P}}$.

Lemma 1.5 *A cut* (I', F') *in* $\widehat{P}$ *is in* $\mathcal{I}$ *iff* $F' \cap I \neq \emptyset$ *for every* (I, F) *in* $\mathcal{F}$.

Proof That (I', F') is in $\mathcal{I}$ means that we cannot find any (I, F) in $\mathcal{F}$ such that $I \subseteq I'$. Alternatively $F' \cap I \neq \emptyset$ for each (I, F) in $\mathcal{F}$. □

For later use we look closer at Alexander duality for $\widehat{P}$ and $\widehat{\widehat{P}}$. A cut (I, F) *for* P, i.e. a cut in $\widehat{P}$, gives a dual cut $(F^{\mathrm{op}}, I^{\mathrm{op}})$ for P^{op}. A cut $(\mathcal{I}, \mathcal{F})$ *for* $\widehat{P}$, i.e. a cut in $\widehat{\widehat{P}}$ gives a dual

cut $(\mathcal{F}^{\mathrm{op}}, \mathcal{I}^{\mathrm{op}})$ for $(\widehat{P})^{\mathrm{op}} = \widehat{P^{\mathrm{op}}}$ which we denote by $P^{\widehat{\mathrm{op}}}$. Thus $(\mathcal{F}^{\mathrm{op}}, \mathcal{I}^{\mathrm{op}})$ is a cut in $P^{\widehat{\widehat{\mathrm{op}}}}$. For $u = (I, F) \in \mathcal{I} \subseteq \widehat{P}$, then $u^{\mathrm{op}} \in \mathcal{I}^{\mathrm{op}}$ is

$$u^{\mathrm{op}} = (I, F)^{\mathrm{op}} = (F^{\mathrm{op}}, I^{\mathrm{op}}) \in P^{\widehat{\mathrm{op}}}.$$

So when we take the dual of the cut $(\mathcal{I}, \mathcal{F})$ for $\widehat{P}$, we not only get a switch $(\mathcal{F}^{\mathrm{op}}, \mathcal{I}^{\mathrm{op}})$ but also the elements $u = (I, F)$ of $\mathcal{I}$ or $\mathcal{F}$ are switched, to $u^{\mathrm{op}} = (F^{\mathrm{op}}, I^{\mathrm{op}})$.

1.4 Finite Type Cuts

The elements of the distributive lattice $\widehat{P}$ identify as cuts (I, F).

Definition 1.6

- $\widehat{P}_{fin}$ is the sublattice of $\widehat{P}$ consisting finite cuts: cuts (I, F) where I is finite.
- $\widehat{P}^{fin}$ is the sublattice of $\widehat{P}$ consisting of cofinite cuts: cuts (I, F) where F is finite.

When P is an infinite discrete poset, an infinite set, such cuts (I, F) of P are a standard example of infinite Boolean algebras, called *finite-cofinite* algebras.

Definition 1.7 A *finite type* $\widehat{P}$-cut is a pair $(\mathcal{I}, \mathcal{F})$ where $\mathcal{F}$ is an up-set for $\widehat{P}_{fin}$ and $\mathcal{I}$ a down-set for $\widehat{P}^{fin}$ such that the following holds.

1. $(I, F) \in \mathcal{I}$ if and only if $F \cap J \neq \emptyset$ for every $(J, G) \in \mathcal{F}$.
2. $(J, G) \in \mathcal{F}$ if and only if $F \cap J \neq \emptyset$ for every $(I, F) \in \mathcal{I}$.

Note that for P finite then 1 and 2 are equivalent. If P is infinite, one of the above will in general not imply the other. The point of having both fulfilled is that $\mathcal{F}$ determines $\mathcal{I}$ by 1, and vice versa by 2. If only 1 holds then $\mathcal{F}$ determines $\mathcal{I}$, but one may not be able to reconstruct $\mathcal{F}$ from $\mathcal{I}$.

In Sect. 7 we construct such finite type cuts for infinite posets.

1.5 Simplicial Complexes and Stanley–Reisner Rings

Let A be a set. A simplicial complex X on A is a set of subsets of A closed under taking smaller subsets, i.e. if $I \in X$ and $J \subseteq I$, then $J \in X$.

The set A may be considered as a poset with the discrete poset structure, i.e. the only comparable elements are $a \leq a$ for $a \in A$. Then $\widehat{A}$ identifies as the Boolean lattice on A (see Example 1.2), consisting of all subsets of A. A cut $(\mathcal{I}, \mathcal{F})$ for $\widehat{A}$ corresponds precisely to a simplicial complex X. The elements I in X give the cuts (I, I^c) in $\mathcal{I}$.

The Alexander dual simplicial complex Y of X consists of all the complements I^c of subsets $I \subseteq A$ such that I is not in X. The Alexander dual cut $(\mathcal{F}^{\mathrm{op}}, \mathcal{I}^{\mathrm{op}})$ for $\widehat{A}^{\mathrm{op}} \cong \widehat{A}$ then corresponds to Y: The cuts $(F^{\mathrm{op}}, I^{\mathrm{op}})$ in $\mathcal{F}^{\mathrm{op}}$ give precisely the elements $F^{\mathrm{op}} = (I^{\mathrm{op}})^c$ in Y.

Denote by $k[x_A]$ the polynomial ring in the variables x_a for $a \in A$. When A is finite, to the simplicial complex X corresponding to the cut $(\mathcal{I}, \mathcal{F})$, we associate a monomial ideal I_X in $k[x_A]$, the Stanley–Reisner ideal of X. It is generated by monomials $x_I = \prod_{i \in I} x_i$ for $(I, F) \in \mathcal{F}$. These are the subset I of A such that I is *not* in the simplicial complex X. The monomials in the Alexander dual Stanley–Reisner ideal I_Y are then precisely those monomials which have non-trivial common divisor with every monomial in I_X, by the characterization of Lemma 1.5.

When A is infinite, we still have a polynomial ring $k[x_A]$. But the construction above does not give meaning if $\mathcal{F}$ is non-empty, since there would be (I, F) in $\mathcal{F}$ with I infinite. However if $(\mathcal{I}, \mathcal{F})$ is a finite type $\widehat{A}-$cut, it gives rise to a Stanley–Reisner ideal I_X of $k[x_A]$, as above.

2 Profunctors Between Posets

We introduce profunctors $P \longrightarrow Q$ between posets. Such a profunctor has a dual $Q \longrightarrow P$ and we investigate this correspondence. For an introduction to profunctors, see [13, Chap.4]. See also [3, Section 7] and [2] where they are called distributors.

2.1 Duality on Profunctors

A *profunctor* $P \longrightarrow Q$ is simply a poset homomorphism $P \to \widehat{Q}$. By the adjunction

$$\mathrm{Hom}(P, \widehat{Q}) = \mathrm{Hom}(P, \mathrm{Hom}(Q^{\mathrm{op}}, \mathbf{2}))$$
$$= \mathrm{Hom}(P \times Q^{\mathrm{op}}, \mathbf{2})$$
$$= \mathrm{Hom}((Q \times P^{\mathrm{op}})^{\mathrm{op}}, \mathbf{2}) = (Q \times P^{\mathrm{op}})\widehat{\ }.$$

Thus a profunctor is equivalently an isotone map $P \times Q^{\mathrm{op}} \to \mathbf{2}$ and this is often taken as the definition. It is also equivalently an element of the distributive lattice $(Q \times P^{\mathrm{op}})\widehat{\ }$, and so corresponds to a cut (I, F) for $Q \times P^{\mathrm{op}}$. (Our convention differs somewhat from [13], since there a profunctor $P \longrightarrow Q$ corresponds to an isotone map $P^{\mathrm{op}} \times Q \to \mathbf{2}$.)

In particular if $Q = B$ and $P = A$ are sets, this is simply a subset of $B \times A^{\mathrm{op}}$ or a relation between the sets A and B. Profunctors are also called distributors or bimodules in the literature. We denote the set of profunctors $P \longrightarrow Q$ as $\mathrm{Pro}(P, Q)(= \mathrm{Hom}(P, \widehat{Q}))$. It is itself a partially ordered set, in fact a distributive lattice.

Remark 2.1 Profunctors can be taken to be the morphisms between two ordered sets. They may be composed and form a category. R.Rosebrugh and R.J.Wood show in [21] that this category is equivalent to the category of *totally algebraic lattices*, those of the form $\widehat{P}$ for some poset P, with supremum-preserving isotone maps. (There the term *ideal* is used for profunctor.)

Order theory may also be done in a more categorical setting, for objects in a topos as in [21], or even more general categories [4]. The result of Rosebrugh and Wood mentioned above also has a more general categorical formulation [22] in terms of a monad on a category where idempotents split.

When (I, F) is the cut of $Q \times P^{\mathrm{op}}$ corresponding to f, the down-set I may be considered as a relation defining the profunctor. We then write qfp when $(q, p^{\mathrm{op}}) \in I$.

Lemma 2.2 *Given a profunctor $f : P \longrightarrow Q$.*

 a. *The following are equivalent: i) $q \in f(p)$ and ii) qfp.*
 b. *The following are equivalent: i) $q \in f(p)^c$, ii) $f(p) \leq \hat{q}$, and iii) $\neg qfp$.*

Proof Let f correspond to $P \to \widehat{Q} = \mathrm{Hom}(Q^{\mathrm{op}}, \mathbf{2})$. That $q \in f(p)$, the latter a down-set, means that $q^{\mathrm{op}} \mapsto 1$. This says $(p, q^{\mathrm{op}}) \mapsto 1$, and so (q, p^{op}) is in the down-set I corresponding to f.

The element $\hat{q}$ corresponds to the cut $((\uparrow q)^c, \uparrow q)$. That $f(p) \leq \widehat{q}$ then means that $f(p) \subseteq (\uparrow q)^c$, or equivalently $q \notin f(p)$.   □

Since $Pro(P, Q)$ identifies as $(Q \times P^{op}\widehat{\,})$, the following is seen to be natural, by taking opposites.

Lemma 2.3 *Let P, Q be posets. There is a a natural isomorphism of posets*

$$Pro(P, Q)^{op} \stackrel{D}{\cong} Pro(Q, P).$$

Proof First

$$\mathrm{Hom}(P, \widehat{Q})^{op} \cong \mathrm{Hom}(P^{op}, (\widehat{Q})^{op}) \cong \mathrm{Hom}(P^{op}, \mathrm{Hom}(Q, \mathbf{2}^{op})).$$

Using that $\mathbf{2}^{op}$ naturally is isomorphic to $\mathbf{2}$, and the adjunction (1) this further becomes

$$\mathrm{Hom}(P^{op}, \mathrm{Hom}(Q, \mathbf{2})) \cong \mathrm{Hom}(P^{op} \times Q, \mathbf{2}) \cong \mathrm{Hom}(Q, \mathrm{Hom}(P^{op}, \mathbf{2}))$$
$$\cong \mathrm{Hom}(Q, \widehat{P}).$$

$\square$

Here is more detail on the duality D.

Lemma 2.4 *Given a profunctor $f : P \longrightarrow Q$ and its dual $g = Df : Q \longrightarrow P$.*

a. *qfp if and only if $\neg\, pgq$,*
b. *$f(p) = \{q \in Q \mid g(q) \leq \widehat{p}\}$.*

In particular $f(p) = \infty$ if and only if $g(q) \leq \widehat{p}$ for every $q \in Q$.

Proof If f corresponds to the cut (I, F) of $Q \times P^{op}$, the dual profunctor g corresponds to the cut (F^{op}, I^{op}) of $P \times Q^{op}$. The statements in a are equivalent to $(q, p^{op}) \in I$.

For part b, the condition $q \in f(p)$ is equivalent to $g(q) \leq \widehat{p}$ by part a and Lemma 2.2. $\square$

Example 2.5 Let $P = Q = \mathbb{N} = \{1, 2, 3, \ldots\}$ and consider a profunctor $f : \mathbb{N} \longrightarrow \mathbb{N}$ (see Example 1.2) which is an isotone map $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. Let its values be

$$2, 2, 4, 5, 5, 7, \ldots.$$

In Fig. 3 the graph of f are marked with red discs ●. We fill in with blue circles ○ to make a connected "snake", starting at position $(1, 1)$. The graph of the dual map $g = Df$ is given by the blue circles by considering the vertical axis as the argument for g. The values of g are

$$1, 3, 3, 4, 6, 6, \ldots.$$

Observe that for a profunctor $f : \mathbb{N} \longrightarrow \mathbb{N}$ (which identifies as an isotone $f : \mathbb{N} \to \widehat{\mathbb{N}}$), then $f(1) \geq 2$ iff the dual map $g = Df$ has $g(1) = 1$. Hence there are no self-dual maps f.

The profunctor f corresponds to the cut (I, F) for $\mathbb{N} \times \mathbb{N}^{op}$ (where $\mathbb{N}$ corresponds to the y-axis and $\mathbb{N}^{op}$ to the (reversed) x-axis) where the up-set F is given by filling in red discs vertically above those in the graph, see Fig. 4, and the ideal I is given by filling in blue circles to the right of those which are present in Fig. 3.
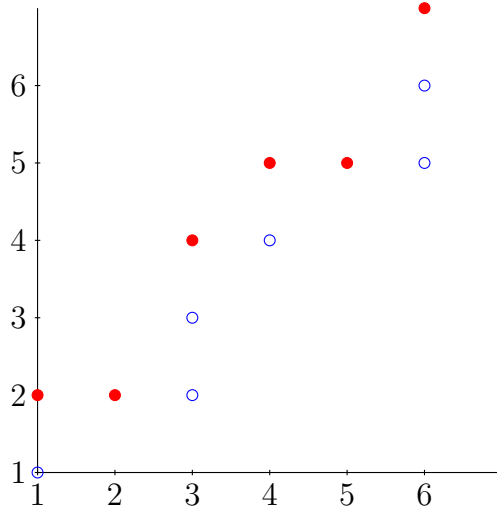
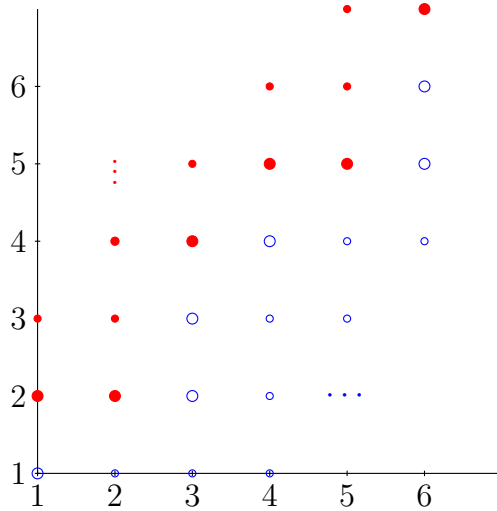**Fig. 3** Graph (red) and ascent (blue) of profunctor f in Example 2.5

Fig. 4 Cut (I, F) (in resp. blue, red) corresponding to the profunctor f in Example 2.5

2.2 Alexander Duality

Definition 2.6 Recall that a profunctor $f : P \longrightarrow Q$ is an isotone $f : P \to \widehat{Q}$.

- For fixed $q \in Q$, the down-set $\mathrm{Pro}_{\leq \widehat{q}}(P, Q)$ is the set of profunctors f such that $f(p) \leq \widehat{q}$ for all p.
- For fixed $p \in P$, the down-set $\mathrm{Pro}^{\mathrm{im}\,(p)<\infty}(P, Q)$ is the set of profunctors f such that $f(p) < \infty$.
- The down-set $\mathrm{Pro}^{<\infty}(P, Q)$ is the set of profunctors f such that $f(p) < \infty$ for every $p \in P$.

Lemma 2.7 *The down-sets $Pro_{\leq \widehat{q}}(P, Q)$ and $Pro^{im\,(q)<\infty}(Q, P)$ are Alexander duals.*

Proof That $f(p) \leq \widehat{q}$ is equivalent to, letting $g = Df$, that $p \in g(q)$. That this holds for all $p \in P$ is equivalent to $g(q) = \infty$. The Alexander dual down-set of $Pro_{\leq \widehat{q}}(P, Q)$ is then those maps g *not* fulfilling $g(q) = \infty$. □

Corollary 2.8 *If Q has a maximal element, then $Pro^{<\infty}(P, Q)$ and $Pro^{<\infty}(Q, P)$ are Alexander dual down-sets.*

Note: By symmetry of the conclusion, this holds under the weaker assumption that P or Q has a maximal element.

Proof If q is the maximal element of Q the down-sets of Lemma 2.7 are respectively $Pro^{<\infty}(P, Q)$ and $Pro^{<\infty}(Q, P)$. □

2.3 Profiles and Co-profiles

Definition 2.9 The *profile* of a profunctor $f : P \to Q$ is the cut (I, F) for P where the *profile up-set F* consists of all $p \in P$ such that $f(p) = \infty$, and the *profile down-set $I = F^c$* is the complement down-set.

The *co-profile* of f is the cut (J, G) for Q where J is the union of all $f(p)$ (considered as a down-set of Q) for $p \in P$, and G is the complement of J.

The following is immediate.

Lemma 2.10 *Let $f : P \longrightarrow\hspace{-0.7em}\rightarrow Q$ be a profunctor.*

    a. Its profile up-set $F = \{p \in P \mid qfp \text{ for every } q \in Q\}$.
    b. Its co-profile up-set $G = \{q \in Q \mid \neg qfp \text{ for every } p \in P\}$.

If $g = Df$ is the dual, the co-profile of f equals the profile of g (by Lemma 2.4a).

We identify the following subsets of $Pro(P, Q)$.

- $Pro^L(P, Q)$ are the profunctors f whose profile down-set I is finite. These maps are called *large*. Then $f(p) = \infty$ for all but a finite number of p's.
- $Pro_S(P, Q)$ are the f whose coprofile down-set J is finite. These maps are called *small*. Then there is a finite down-set bounding the $f(p)$, i.e. all $f(p) \subseteq J$.
- $Pro^u(P, Q)$ are the f which are in neither the above, so both the profile down-set I and co-profile down-set J are infinite.

A consequence of Lemma 2.10 is the following.

Lemma 2.11

    a. The duality D switches $Pro^L(P, Q)$ and $Pro_S(Q, P)$ and maps $Pro^u(P, Q)$ to $Pro^u(Q, P)$.
    b. If P is finite, then $Pro^L(P, Q) = Pro(P, Q)$.
    c. If Q is finite, then $Pro_S(P, Q) = Pro(P, Q)$.

Example 2.12 Consider profunctors $f : \mathbb{N} \longrightarrow\hspace{-0.7em}\rightarrow \mathbb{N}$. By Example 1.2 recall $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Such a map is large if $f(n) = \infty$ for some n. It is small if f is eventually constant, so $f(n) = c$ for all $n \geq n_0$. The maps in $Pro^u(\mathbb{N}, \mathbb{N})$ are the maps $f : \mathbb{N} \to \mathbb{N}$ which are not bounded, so $\lim_{n\to\infty} f(n) = \infty$. We see the naturalness of considering $Pro(\mathbb{N}, \mathbb{N})$ instead of $Hom(\mathbb{N}, \mathbb{N})$: The latter is not self-dual while the former is. $Pro(\mathbb{N}, \mathbb{N})$ has two countable dual "shores" $Pro^L(\mathbb{N}, \mathbb{N})$ and $Pro_S(\mathbb{N}, \mathbb{N})$ and between them an uncountable self-dual "ocean" $Pro^u(\mathbb{N}, \mathbb{N})$.

Remark 2.13 The small profunctors $\mathrm{Pro}_S(\mathbb{N}, \mathbb{N})$, which are simply bounded maps $f : \mathbb{N} \to \mathbb{N}$ are in bijection with the tame (small) increasing monoid of [14] by sending the bounded map f to $f + \mathrm{id} - 1$.

2.4 Adjunctions

Given an isotone map $f : P \to Q$ it induces a pull-back map

$$f^* : \widehat{Q} \to \widehat{P}, \quad (J, G) \mapsto (f^{-1}(J), f^{-1}(G)).$$

This map has a left adjoint

$$f^! : \widehat{P} \to \widehat{Q}, \quad (I, F) \mapsto (f(I)^{\downarrow}, -)$$

where $f(I)^{\downarrow}$ is the smallest down-set in Q containing $f(I)$. There is also a right adjoint of f^*:

$$f^{\mathrm{i}} : \widehat{P} \to \widehat{Q}, \quad (I, F) \mapsto (-, f(F)^{\uparrow})$$

where $f(F)^{\uparrow}$ is the smallest up-set in Q containing $f(F)$. All these maps are functorial in P and Q.

2.4.1 A Variation on the Setting

There is a variation for these maps as follows. There is a forgetful functor $U : \mathbf{Poset} \to \mathbf{Set}$ by mapping a poset to the underlying set. Composing with the natural inclusion $\mathbf{Set} \to \mathbf{Poset}$ we get $U : \mathbf{Poset} \to \mathbf{Poset}$. The inclusion $i : UP \to P$ induces the dual map $i^* : \widehat{P} \to \widehat{(UP)}$.

Suppose we have an isotone map of posets

$$g : UP \to Q.$$

(This instead of an isotone map $f : P \to Q$. Note also that g is really just a map of sets $UP \to UQ$.) We then get composites

$$g_U^{\mathrm{i}} \ (\text{resp. } g_U^!) : \widehat{P} \xrightarrow{\ i^* \ } \widehat{(UP)} \xrightarrow{\ g^{\mathrm{i}} \ (\text{resp. } g^!)\ } \widehat{Q}. \tag{6}$$

Here g_U^{i} sends a cut (I, F) of P to the cut (J, G) of Q where $G = g(F)^{\uparrow}$ is the up-set of Q generated by the $g(p)$ for $p \in F$. Note that the above composite $g^{\mathrm{i}} \circ i^*$ has a left adjoint $i^! \circ g^*$ (although it will not play a role for us).

Also note the following

$$(g_U^!)^{\mathrm{op}} = (g^{\mathrm{op}})_U^{\mathrm{i}} : \widehat{P^{\mathrm{op}}} \to \widehat{Q^{\mathrm{op}}}. \tag{7}$$

3 The Graph, the Ascent, and Preserving the Cut

We define the two significant notions of this article, the graph and ascent of a profunctor $P \longrightarrow Q$, or equivalently the right and left boundaries of the cut (I, F) for $Q \times P^{\mathrm{op}}$ corresponding to this profunctor. Then we state several versions of the main theorem of this article, Theorem 3.7, on preserving cuts.

3.1 The Graph and Ascent

Definition 3.1 Denote by **0** the minimal element in $\mathrm{Pro}(P, Q)$. It sends every $p \mapsto \emptyset$, corresponding to the cut $(\emptyset, Q)$. Denote by **1** the maximal element in $\mathrm{Pro}(P, Q)$. It sends every $p \mapsto Q$, corresponding to the cut $(Q, \emptyset)$.

Recall that for a poset P then UP denotes the underlying set, considered as a discrete poset.

Definition 3.2 Let $f : P \longrightarrow Q$ be a profunctor. Its *ascent* is

$$\Lambda f = \{(q, p) \mid q \in f(p) \text{ but } q \notin f(p') \text{ for } p' < p\} \subseteq UQ \times UP^{\mathrm{op}}. \tag{8}$$

Its *graph* is

$$\Gamma f = \{(q, p) \mid q \text{ minimal in the complement } f(p)^c\} \subseteq UQ \times UP^{\mathrm{op}}. \tag{9}$$

Example 3.3 For a profunctor $f : \mathbb{N} \longrightarrow \mathbb{N}$, see Fig. 3, then Γf is the red discs, and Λf is the blue circles.

Note that Λf and Γf are disjoint. Also note i) $\Lambda \mathbf{0} = \emptyset$ and $\Gamma \mathbf{0} = \min Q \times UP$ where min denotes the minimal elements, ii) $\Lambda \mathbf{1} = UQ \times \min P$ and $\Gamma \mathbf{1} = \emptyset$.

Remark 3.4 Their union $Bf = \Lambda f \cup \Gamma f$, might be called the *boundary* of the cut (I, F). This union will not play a role here, but it does in [5, Section 2].

3.2 Left and Right Boundaries

We have seen that $\mathrm{Pro}(P, Q)$, the profunctors from P to Q, identify as $\widehat{(Q \times P^{\mathrm{op}})}$. So a profunctor $f : P \longrightarrow Q$ corresponds to a cut (I, F) for $Q \times P^{\mathrm{op}}$ where

$$I = \{(q, p) \mid q \in f(p)\}, \quad F = \{(q, p) \mid q \notin f(p)\}. \tag{10}$$

Definition 3.5 If (I, F) is a cut for $Q \times P^{\mathrm{op}}$ its *left and right boundaries* are respectively

$$\Lambda I = \{(q, p) \in I \mid (q, p') \notin I \text{ for } p' < p\} \subseteq UQ \times UP^{\mathrm{op}}$$
$$\Gamma F = \{(q, p) \in F \mid (q', p) \notin F \text{ for } q' < q\} \subseteq UQ \times UP^{\mathrm{op}}$$

We see immediately by (8), (9), and (10), that if f corresponds to (I, F) then $\Lambda f = \Lambda I$ and $\Gamma f = \Gamma F$.

Corollary 3.6 *Given dual profunctors*

$$f : P \longrightarrow Q, \quad g = Df : Q \longrightarrow P.$$

Let (I, F) be the cut for $Q \times P^{op}$ associated to f, and (J, G) the cut for $P \times Q^{op}$ associated to g.

a. $(J, G) = (F^{op}, I^{op})$.
b. $\Gamma G = (\Lambda I)^{op}, \quad \Lambda J = (\Gamma F)^{op}$.
c. $\Gamma g = (\Lambda f)^{op}, \quad \Lambda g = (\Gamma f)^{op}$.

Proof Part a is by Lemma 2.4. Parts b and c are then immediate from a. □

3.3 Extending Λ and Γ to the Next Level

We have looked at cuts (I, F) for $Q \times P^{\text{op}}$. Proceeding to the next level, we look at cuts $(\mathcal{I}, \mathcal{F})$ for $\text{Pro}(P, Q) = \widehat{(Q \times P^{\text{op}})}$. Elements in this latter set are cuts (I, F) for $Q \times P^{\text{op}}$ partially ordered by $(I, F) \leq (I', F')$ if $I \subseteq I'$. Thus the down-set $\mathcal{I}$ consists of cuts (I, F) closed under taking cuts with smaller I's. Similarly the up-set $\mathcal{F}$ is closed under taking larger I's (and so smaller $F = I^c$'s).

We have a map

$$\Lambda : U\text{Pro}(P, Q) \to \widehat{(UQ \times UP^{\text{op}})}, \quad f \mapsto (\Lambda f, -) \tag{11}$$

and a map

$$\Gamma : U\text{Pro}(P, Q) \to \widehat{(UQ \times UP^{\text{op}})}, \quad f \mapsto (-, \Gamma f). \tag{12}$$

By (6) the map Λ induces an isotone map of posets:

$$\Lambda_U^i : \widehat{\text{Pro}(P, Q)} \to \widehat{\widehat{(UQ \times UP^{\text{op}})}}, \quad (\mathcal{I}, \mathcal{F}) \mapsto (-, \Lambda(\mathcal{F})^{\uparrow}).$$

So the image of the cut $(\mathcal{I}, \mathcal{F})$ is the cut in the Boolean lattice $\widehat{(UQ \times UP^{\text{op}})}$ whose up-set is $\Lambda(\mathcal{F})^{\uparrow}$, the up-set generated by all Λf for $f \in \mathcal{F}$.

The map Γ induces an isotone map of posets:

$$\Gamma_U^! : \widehat{\text{Pro}(P, Q)} \to \widehat{\widehat{(UQ \times UP^{\text{op}})}}, \quad (\mathcal{I}, \mathcal{F}) \mapsto (\Gamma(\mathcal{I})^{\downarrow}, -).$$

So the image of the cut $(\mathcal{I}, \mathcal{F})$ is the cut in the Boolean lattice $\widehat{(UQ \times UP^{\text{op}})}$ whose ideal is $\Gamma(\mathcal{I})^{\downarrow}$, the ideal generated by the *complements* (see (12)) of all Γf for $f \in \mathcal{I}$,

3.4 The Main Theorem: Preserving the Cut

In order for the left and right boundaries of a cut (I, F) to give enough information we need to make sure that minimal elements of up-sets of P and Q exists. A poset P is *well-founded* if every subset of P has a minimal element. Equivalently, any descending chain of elements

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq \cdots$$

stabilizes, i.e. for some N we have $p_n = p_N$ for $n \geq N$.

The following theorem is a strong generalization of the results in several articles [7, 8, 10, 16], see Sect. 4 for more on this. The most significant tool in the argument is Zorn's lemma (which is equivalent to the axiom of choice). Note also that $\text{Pro}(P, Q)$ is a complete distributive lattice and so has all joins (colimits) and meets (limits).

Theorem 3.7 *(Preserving the cut) Let P and Q be well-founded posets, and $(\mathcal{I}, \mathcal{F})$ a cut for $\text{Pro}(P, Q)$. Then $(\Gamma(\mathcal{I})^{\downarrow}, \Lambda(\mathcal{F})^{\uparrow})$ is a cut for the Boolean lattice $\widehat{(UQ \times UP^{\text{op}})}$. In other words, the maps $\Gamma_U^! = \Lambda_U^i$.*

Example 3.8 Consider profunctors $\text{Hom}([5], \widehat{[3]})$ where $\widehat{[3]} = [3] \cup \{\infty\}$. Let $\mathcal{I}$ be the ideal consisting of all f not taking the value ∞. The graph of such an f is shown in Fig. 5. Such a path in the rectangle $U[3] \times U[5]^{\text{op}}$ is a *right path*. Let $\mathcal{F}$ be the complement up-set, consisting of all g which take the value ∞ for some argument in $[5]$. The ascent of such a g is also shown in Fig. 5. Such a path in the rectangle $U[3] \times U[5]^{\text{op}}$ is an *up path*. Theorem 3.7 above says that given any subset S of $[5] \times [3]$, exactly one of the following holds: i. S contains an up path, ii. the complement S^c contains a right path. An earlier observation of this is in [12, Lemma 4.4].
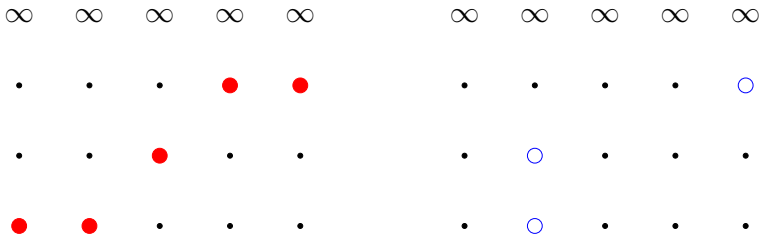
GUNNAR FLØYSTAD

Fig. 5 Profunctors from [5] to [3]. A graph to the left and an (unrelated) ascent to the right

Before proving Theorem 3.7 we state two alternative formulations of this theorem.

Theorem 3.9 *Let P and Q be well-founded posets, and $(\mathcal{I}, \mathcal{F})$ a cut for $Pro(P, Q)$. The up-set $\Lambda^i_U(\mathcal{F})$ for $\widehat{(UQ \times UP^{op})}$ generated by all $(\Lambda f, -)$ for $f \in \mathcal{F}$, and the up-set $(\Gamma^{op})^i_U(\mathcal{I}^{op})$ for $\widehat{(UP \times UQ^{op})}$ generated by all $((\Gamma f)^{op}, -)$ for $f \in \mathcal{I}$, are Alexander dual up-sets. (This is the version applied in Stanley–Reisner theory, giving Alexander dual monomial ideals, see Sect. 4.)*

Proof By the above Theorem 3.7, $(\Gamma(\mathcal{I})^{\downarrow})^{op} = \Gamma^!_U(\mathcal{I})^{op}$ is the Alexander dual up-set of the up-set $\Lambda^i_U(\mathcal{F}) = \Lambda(\mathcal{F})^{\uparrow}$. Then (7) gives $\Gamma^!_U(\mathcal{I}, -)^{op} = (\Gamma^{op})^i_U(-, \mathcal{I}^{op})$. $\qquad \square$

By Corollary 3.6 we have a commutative diagram

$$U\mathrm{Pro}(P, Q) \xrightarrow{\ \Lambda\ } \widehat{(UQ \times UP^{op})}. \tag{13}$$

$$\begin{array}{ccc} & & \\ \downarrow D & & \downarrow D \\ & & \end{array}$$

$$U\mathrm{Pro}(Q, P) \xrightarrow{\ \Gamma\ } \widehat{(UP \times UQ^{op})}$$

In the proof below we write $\Lambda_{P,Q}$ and $\Gamma_{Q,P}$ for these horizontal maps.

Theorem 3.10 *Let P and Q be well-founded posets. The following diagram commutes:*

$$\widehat{Pro(P, Q)} \xrightarrow{\ \Lambda^i_U\ } \widehat{\widehat{(UQ \times UP^{op})}}.$$

$$\begin{array}{ccc} \downarrow D & & \downarrow D \\ \end{array}$$

$$\widehat{Pro(Q, P)} \xrightarrow{\ \Lambda^i_U\ } \widehat{\widehat{(UP \times UQ^{op})}}$$

Proof By the commutative diagram (13), $\Gamma^{op}_{Q,P} = \Lambda_{P,Q}$. Switching P and Q we have $\Gamma^{op}_{P,Q} = \Lambda_{Q,P}$. Hence $(\Gamma^{op}_{P,Q})^i_U$ equals $(\Lambda_{Q,P})^i_U$. By the above Theorem 3.9 $(\Gamma^{op}_{P,Q})^i_U$ is the dual of $(\Lambda_{P,Q})^i_U$ and so $(\Lambda_{Q,P})^i_U$ is the dual of $(\Lambda_{P,Q})^i_U$. $\qquad \square$

Problem 3.11 Theorem 3.7 may be seen as a map

$$\widehat{\mathrm{Pro}(P, Q)} \to \widehat{\mathrm{Pro}(UP, UQ)}.$$

However considering the natural maps $UP \to P$ and $UQ \to Q$ none of the natural functorial ways to get a map as above, gives the map $\Gamma^! = \Lambda^i$. For instance we can not see a natural factorization of the above map through $\widehat{\mathrm{Pro}(UP, Q)}$.

Is there a way to understand or formulate the theorem to get a functorial construction, which works for maps $P' \to P$ and $Q' \to Q$ (or in some other way)?

Proof of Theorem 3.7 We need to show that the down-set $\Gamma(\mathcal{I})^{\downarrow}$ is the complement of the up-set $\Lambda(\mathcal{F})^{\uparrow}$ in the Boolean lattice $(U Q \times U P^{\mathrm{op}} \widehat{\,)}.$ □

Part I. We first show that for any $(I, F) \in \mathcal{I}$ and $(J, G) \in \mathcal{F}$ that

$$\Gamma F \cap \Lambda J \neq \emptyset.$$

By Lemma 1.5 this shows that $\Gamma(\mathcal{I})^{\downarrow}$ and $\Lambda(\mathcal{F})^{\uparrow}$ are disjoint.

If $(I, F) \in \mathcal{I}$ and $(J, G) \in \mathcal{F}$ then $(F \cap J) \subseteq Q \times P^{\mathrm{op}}$ is $\neq \emptyset$ by Lemma 1.5. Let $(q_0, p_0) \in F \cap J$. Define (q_i, p_i) in $F \cap J$ successively as follows. If there is a $q < q_i$ such that (q, p_i) is in F, then let (q_{i+1}, p_{i+1}) be (q, p_i) which is automatically also in J. If there is $p < p_i$ such that (q_i, p) is also in J, then let (q_{i+1}, p_{i+1}) be (q_i, p) which is automatically also in F. We get chains $p_0 \geq p_1 \geq \cdots$ and $q_0 \geq q_1 \geq \cdots$. These must stabilize and for the stable value pair (q, p) we cannot continue the construction and then this pair will be in $\Gamma F \cap \Lambda J$.

Part II. Let $S \subseteq U Q \times U P^{\mathrm{op}}$ be such that $S \cap \Lambda J \neq \emptyset$ for *every* $(J, G) \in \mathcal{F}$. This means that (S^c, S) is not in $\Lambda(\mathcal{F})^{\uparrow}$. We show that $S \supseteq \Gamma F$ for *some* $(I, F) \in \mathcal{I}$ and so (S^c, S) is in $\Gamma(\mathcal{I})^{\downarrow}$. This shows that the union of $\Gamma(\mathcal{I})^{\downarrow}$ and $\Lambda(\mathcal{F})^{\uparrow}$ is all of $U Q \times U P^{\mathrm{op}}$.

Let T consist of $f \in \mathrm{Pro}(P, Q)$ such that $\Lambda f \cap S = \emptyset$. Clearly the minimum **0** of $\mathrm{Pro}(P, Q)$ is in T, so T is non-empty. We use Zorn's lemma to show that T has a maximal element. So let

$$f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots \tag{14}$$

be a chain of functions in T, and f be the colimit (join) of these. We claim that $\Lambda f \cap S = \emptyset$. Let $(q, p) \in \Lambda f$, so $q \in f(p)$ but $q \notin f(p')$ for every $p' < p$. Then $q \in f_i(p)$ for some i and $q \notin f_i(p')$ since $f_i(p') \subseteq f(p')$. Then $(q, p) \in \Lambda f_i$ and so is not in S. Thus f is an upper bound in T for the chain in (14). By Zorn's lemma T has a maximal element f, and take note that f must be in $\mathcal{I}$, by the requirement on S.

Claim 1 $\Gamma f \subseteq S$.

By the second and third lines of Part II, this finishes the proof.

Proof Suppose not, so there is $(q_0, p_0) \in \Gamma f \backslash S$. Recall q_0 is minimal in the complement $(f(p_0))^c$. Let

$$\tilde{f}(p) = \begin{cases} f(p) \cup \{q_0\}, & p \geq p_0, \\ f(p) & \text{otherwise.} \end{cases}$$

This is a profunctor $\tilde{f} : P \longrightarrow Q$. Let us show

$$\Lambda \tilde{f} \subseteq \Lambda f \cup \{(q_0, p_0)\}.$$

That $(q, p) \in \Lambda \tilde{f}$ means that $q \in \tilde{f}(p)$ and $q \notin \tilde{f}(p')$ for $p' < p$.

1. If not $p \geq p_0$ then $\tilde{f}(p') = f(p')$ for all values $p' \leq p$. Therefore (q, p) is in Λf.
2. Let $p \geq p_0$. Then either i. $q \in f(p)$ or ii. $q = q_0$. In case i. $(q, p) \in \Lambda f$. In case ii, where $q = q_0$, if $p > p_0$ then (q_0, p) would not be in $\Lambda \tilde{f}$ since $q \in \tilde{f}(p)$ and $q \in \tilde{f}(p_0)$. We are thus left with $(q, p) = (q_0, p_0)$.

As a consequence we also have $\Lambda \tilde{f} \cap S = \emptyset$. But by definition of S then $\tilde{f} \in T$. This contradicts f being maximal in T. Therefore no such (q_0, p_0) could exist, and $\Gamma f \subseteq S$.   □

4 Applications to Finite Posets and Stanley–Reisner Ideals

In [7] they show that letterplace ideals $L([n], P)$ and co-letterplace ideals $L(P, [n])$ are Alexander dual ideals, and this was taken considerably further in [10].

Here we define such square free ideals in the full general setting for finite posets P and Q and cuts in $\mathrm{Pro}(P, Q)$. We show how the result of [7] above is a consequence. In the following for ease of notation we denote a polynomial ring $k[x_{UQ \times UP^{op}}]$ as $k[x_{Q \times P^{op}}]$.

Definition 4.1 Let P and Q be *finite* posets. From the cut $(\mathcal{I}, \mathcal{F})$ for $\mathrm{Pro}(P, Q)$, we get the cut $(\Gamma(\mathcal{I})^{\downarrow}, \Lambda(\mathcal{F})^{\uparrow})$ for $(UQ \times UP^{op})\widehat{\ }$.

The Λ-ideal. Since $UQ \times UP^{op}$ is simply a set, this gives a Stanley–Reisner ideal in $k[x_{Q \times P^{op}}]$. We denote it $L_{\Lambda}(\mathcal{F}; P, Q)$, or simply $L_{\Lambda}(\mathcal{F})$. As f varies in $\mathcal{F}$, it is generated by the Λf (or rather the squarefree monomials $\prod_{(q,p) \in \Lambda f} x_{q,p}$).

The Γ-ideal. The dual cut $(\Lambda(\mathcal{F})^{\uparrow op}, \Gamma(\mathcal{I})^{\downarrow op})$ for $(UP \times UQ^{op})\widehat{\ }$, also corresponds to a Stanley–Reisner ideal in $k[x_{P \times Q^{op}}]$. We denote this ideal as $L_{\Gamma}(\mathcal{I}; P, Q)$, or simply $L_{\Gamma}(\mathcal{I})$. As f varies in $\mathcal{I}$, it is generated by the $(\Gamma f)^{op}$ (or rather the squarefree monomials $\prod_{(p,q) \in (\Gamma f)^{op}} x_{p,q}$).

By Theorem 3.9 above, the Λ-ideal and Γ-ideal are Alexander dual ideals.

Remark 4.2 Let $Q = [n]$, the chain on n elements, and $\mathcal{I} \subseteq \mathrm{Pro}^{<\infty}(P, [n])$. The Λ-ideals $L_{\Lambda}(\mathcal{F}; P, [n])$ are the letterplace ideals of [10] and are shown to be Cohen-Macaulay ideals. The Alexander dual Γ-ideals $L_{\Gamma}(\mathcal{I}; P, [n])$ are the co-letterplace ideals in loc.cit. and thus have linear resolutions, [19, Thm.5.56]. The above definition and Theorem 3.9 may thus be seen as a full generalization of the setting of [10].

Remark 4.3 With the same setting as in the above remark, the Λ-ideals $L_{\Lambda}(\mathcal{F}; P, [n])$ define simplicial balls by the Stanley–Reisner correspondence, [5]. Furthermore there is a very simple description of the Stanley–Reisner ideal of their boundaries, which are simplicial spheres. This gives the construction of an enormous amount of simplicial spheres, due to the freedom in choosing P, $[n]$ and $\mathcal{I}$.

Corollary 2.8 has the following consequence, see also Example 3.8.

Corollary 4.4 *If P or Q has a maximal element, the ideals $L_{\Gamma}(\mathrm{Pro}^{<\infty}(P, Q))$ and $L_{\Gamma}(\mathrm{Pro}^{<\infty}(Q, P))$ are Alexander dual ideals.*

Definition 4.5 Given an isotone map $f : P \to Q$, let the graph

$$\Gamma_i f = \{(f(p), p^{op}) \mid p \in P\} \subseteq UQ \times UP^{op}.$$

This gives a map

$$\Gamma_i : \mathrm{Hom}(P, Q) \to (UQ \times UP^{op})\widehat{\ }, \quad f \mapsto (-, \Gamma_i f),$$

and so

$$\Gamma_i^{op} : \mathrm{Hom}(P, Q)^{op} \to (UP \times UQ^{op})\widehat{\ }, \quad f^{op} \mapsto ((\Gamma_i f)^{op}, -),$$

The *isotonian* ideal $L(P, Q)$ is the ideal in $k[x_{P \times Q^{op}}]$ generated by the monomials $\prod_{(p,q) \in (\Gamma_i f)^{op}} x_{(p,q)}$ as f varies in $\mathrm{Hom}(P, Q)$. If P is the chain $[n]$, it is called a letterplace ideal, and if $Q = [n]$, it is a co-letterplace ideal, [10].

The canonical map $Q \to \widehat{Q}$ induces an isotone map

$$\alpha : \mathrm{Hom}(P, Q) \to \mathrm{Hom}(P, \widehat{Q}) = \mathrm{Pro}(P, Q), \quad g \stackrel{\alpha}{\mapsto} (p \mapsto \widehat{g(p)} = (-, \uparrow g(p))), \quad (15)$$

and by Sect. 2.4

$$\alpha^! : \widehat{\mathrm{Hom}(P, Q)} \to \widehat{\mathrm{Pro}(P, Q)}. \quad (16)$$

We get commutative diagrams:

$$
\begin{array}{ccc}
U\mathrm{Hom}(P,Q)^{\mathrm{op}} \xrightarrow{\alpha^{\mathrm{op}}} U\mathrm{Pro}(P,Q)^{\mathrm{op}} \, , & & \mathrm{Hom}(P,Q)^{\widehat{\mathrm{op}}} \xrightarrow{(\alpha^{\mathrm{op}})^{\mathrm{i}}} \mathrm{Pro}(P,Q)^{\widehat{\mathrm{op}}} \\
\Gamma_i^{\mathrm{op}} \searrow \quad \downarrow \Gamma^{\mathrm{op}} & & (\Gamma_{i,U}^{\mathrm{op}})^{\mathrm{i}} \searrow \quad \downarrow (\Gamma_U^{\mathrm{op}})^{\mathrm{i}} = (\Gamma_U^!)^{\mathrm{op}} \\
\widehat{(UP \times UQ^{\mathrm{op}})} & & \widehat{(UP \times UQ^{\mathrm{op}})}
\end{array}
\quad (17)
$$

Note that the isotonian ideal $L(P, Q)$ is the squarefree monomial ideal associated to the image by $(\Gamma_i^{\mathrm{op}})^{\mathrm{i}}$ of the cut $(\emptyset, \mathrm{Hom}(P, Q)^{\mathrm{op}})$ for $\mathrm{Hom}(P, Q)^{\widehat{\mathrm{op}}}$.

The poset P is a *forest* if for every two incomparable p_1 and p_2 in P, there is no $p \in P$ such that $p \leq p_1$ and $p \leq p_2$. The connected components of the Hasse diagram of P are then trees with the roots on top.

Lemma 4.6 *If P is a forest, then:*

a. *For the map in* (16), *$\alpha^!(\widehat{\mathrm{Hom}(P, Q)})$ is the down-set $\mathrm{Pro}^{<\infty}(P, Q)$.*
b. *In the diagram* (17),

$$(\Gamma_{i,U}^{\mathrm{op}})^{\mathrm{i}}((\emptyset, \mathrm{Hom}(P, Q)^{op})) = (\Gamma_U^{\mathrm{op}})^{\mathrm{i}}((-, \mathrm{Pro}^{<\infty}(P, Q)^{op})).$$

Hence $L(P, Q) = L_{\Gamma}(\mathrm{Pro}^{<\infty}(P, Q))$.

Proof By the right diagram of (17), part a above implies part b. So we do part a. We show that if f is a profunctor in $\mathrm{Pro}^{<\infty}(P, Q)$, then there exists a $g : P \to Q$ such that $\Gamma f \supseteq \Gamma_i g$. This implies that $\alpha(g) \geq f$ giving part a. It also implies part b.

1. If p is maximal in P then let q be a minimal element in the non-empty set $f(p)^c$. Define then $g(p) = q$.
2. Suppose p is not maximal. Suppose $g(p')$ is defined for $p' > p$, such that $(g(p'), p'^{\mathrm{op}}) \in \Gamma f$. Let p' be the unique cover of p (since P is a forest). Then $f(p) \subseteq f(p')$ and so $f(p)^c \supseteq f(p')^c$. Given $g(p') \in f(p')^c$. Then there will exist a minimal q in $f(p)^c$ with $q \leq g(p')$. Then define $g(p) = q$. This gives the map g. $\qquad \square$

We specialize to $Q = [n]$ and have the consequence:

Corollary 4.7 *The letterplace ideal $L([n], P)$ and the co-letterplace ideal $L(P, [n])$ are Alexander dual ideals.*

Proof The ideal $L(P, [n])$ is $L_{\Gamma}(\mathrm{Pro}^{<\infty}(P, [n]))$ since $\widehat{[n]} = [n] \cup \{\infty\}$. Its Alexander dual is the ideal $L_{\Gamma}(\mathrm{Pro}^{<\infty}([n], P)$ by Corollary 4.4. By the above lemma this identifies as $L([n], P)$. $\qquad \square$

In [15] they fully characterize for which P and Q the isotonian ideals $L(P, Q)$ and $L(Q, P)$ are Alexander dual.

5 Topology on Pro(*P, Q*)

We want to get a setting where P and Q may be infinite but the ascents Λf and graphs Γf are finite. Then we get Λ- and Γ-ideals in infinite-dimensional polynomial rings. This can be achieved if we put a suitable topology on $\mathrm{Hom}(P, Q)$.

5.1 Defining the Topology

We define a topology on $\mathrm{Pro}(P, Q)$ by defining a basis of open subsets.

Definition 5.1 Given a large profunctor $\overline{f}$ in $\mathrm{Pro}^L(P, Q)$ and a small profunctor $\underline{f}$ in $\mathrm{Pro}_S(P, Q)$. The set of profunctors f such that $\underline{f} \leq f \leq \overline{f}$ is denoted $U(\underline{f}, \overline{f})$.

The following is immediate.

Lemma 5.2 *Given two such sets $U_1 = U(\underline{f}_1, \overline{f}_1)$ and $U_2 = U(\underline{f}_2, \overline{f}_2)$, let $\underline{f}$ and $\overline{f}$ be the profunctors defined by the join and meet*

$$\underline{f} = \underline{f}_1 \vee \underline{f}_2, \quad \overline{f} = \overline{f}_1 \wedge \overline{f}_2.$$

Then

$$U_1 \cap U_2 = U(\underline{f}, \overline{f}).$$

As a consequence the sets $U(\underline{f}, \overline{f})$ form the basis of a topology on $\mathrm{Pro}(P, Q)$. Note that if P and Q are finite we get the discrete topology on $\mathrm{Pro}(P, Q)$.

Observation 5.3 a. There is a one-one correspondence between open down-sets in $\mathrm{Pro}(P, Q)$ and down-sets in $\mathrm{Pro}^L(P, Q)$.

b. There is a one-one correspondence between open up-sets in $\mathrm{Pro}(P, Q)$ and up-sets in $\mathrm{Pro}_S(P, Q)$.

Note also by Lemma 2.3 that the open set $U(\underline{f}, \overline{f})$ is mapped by the dual D to the open set $U(D\overline{f}, D\underline{f})$. Hence we get:

Lemma 5.4 *The duality map D of Lemma 2.3 is a homeomorphism of topological spaces.*

5.2 Generalities on Topology

In any topological space X there is distinguished type of open subsets: those that are the interiors of closed subsets, called *regular open sets*. In fact, let $\mathrm{op}\, X$ be the poset of open subsets of X, and $\mathrm{cl}X$ the poset of closed subsets of X. There is a Galois correspondence:

$$\mathrm{op}\, X \underset{()°}{\overset{\overline{()}}{\rightleftarrows}} \mathrm{cl}X,$$

$$U \mapsto \overline{U} = \text{ closure of } U, \quad C \mapsto C° = \text{interior of } C$$

The fix points in $\mathrm{op}\, X$ for the Galois connection, i.e. the fix points in $\mathrm{op}\, X$ for the composition of closure $^-$ and interior $°$, are the regular open sets. We denote these as $\mathrm{reg}\, X$, the open subsets which are interiors of closed subsets.

Lemma 5.5 *The map on open subsets $U \mapsto \overline{U}^c$ where c denotes complement, gives an involution* reg $X \xrightarrow{i}$ reg X. *Furthermore* $\overline{U}^c = (U^c)^\circ$.

Proof Let $V = \overline{U}^c$. We will show that $\overline{V}^c = U$ by proving inclusion either way.

1. We get $V \subseteq U^c$ and since the latter is closed, $\overline{V} \subseteq U^c$ and so $\overline{V}^c \supseteq U$.
2. We also have $\overline{U}^c \subseteq V$ and so $\overline{U} \supseteq \overline{V}^c$. Since U is the interior of $\overline{U}$ we get $U \supseteq \overline{V}^c$.
3. We now show that V is regular. Let W be an open subset of $\overline{V}$. Then $W^c \supseteq \overline{V}^c = U$, so $W^c \supseteq \overline{U} = V^c$. Thus $W \subseteq V$. Hence V is the largest open subset of $\overline{V}$, and so is the interior.
4. Since $U = \overline{V}^c$ we get $U^c = \overline{V}$, and so

$$(U^c)^\circ = \overline{V}^\circ = V = \overline{U}^c.$$

$\square$

5.3 Open and Closed Down-Sets

Lemma 5.6 *Let $\mathcal{I}$ be a down-set in $Pro(P, Q)$. The closure $\overline{\mathcal{I}}$ and the interior $(\mathcal{I})^\circ$ are also down-sets. By duality the analog also holds for up-sets.*

Proof Let $F \in \overline{\mathcal{I}}$ and let $G \leq F$. We will show G is also in $\overline{\mathcal{I}}$. If not there is an open $U(b, c)$ disjoint from $\mathcal{I}$ with $G \in U(b, c)$, so $b \leq G \leq c$.

Then $F \geq b$ with b small and not in $\mathcal{I}$. Then $F \in U(b, \mathbf{1})$ with this basis open set disjoint from $\mathcal{I}$ since b is not in $\mathcal{I}$ and $\mathcal{I}$ is a down-set. But then F is not in the closure of $\mathcal{I}$, contrary to assumption.

As for the interior, if $\mathcal{F}$ is the complement up-set of $\mathcal{I}$, then by a similar argument $\overline{\mathcal{F}}$ is a up-set, and so the complement $(\overline{\mathcal{F}})^c = \mathcal{I}^\circ$ is a down-set. $\square$

Lemma 5.7 *Let $\mathcal{I}$ be an open down-set $\mathcal{I}$. Then $\mathcal{I}$ is regular if and only if $\overline{\mathcal{I}} \backslash \mathcal{I}$ does not contain large maps.*

Similarly an open up-set $\mathcal{F}$ is regular if and only if $\overline{\mathcal{F}} \backslash \mathcal{F}$ does not contain small maps.

Proof We only show the first statement. Suppose $\mathcal{I}$ is regular. Let f in $\overline{\mathcal{I}} \backslash \mathcal{I}$ be large. The closure $\overline{\mathcal{I}}$ is a down-set. It contains f and so the open set $U(\mathbf{0}, f)$, which must then be in the interior of $\overline{\mathcal{I}}$ which is $\mathcal{I}$. This contradicts f being in the gap. Similarly one can argue that an f in the gap cannot be bounded, by considering the up-set $\mathcal{F}$.

If $\mathcal{I}$ is not regular, there is some open $U = U(f, \overline{f}) \subseteq \overline{\mathcal{I}}$ with $U \not\subseteq \mathcal{I}$. But then $\overline{f} \in \overline{\mathcal{I}} \backslash \mathcal{I}$, and so $\overline{\mathcal{I}} \backslash \mathcal{I}$ contains $\overline{f}$ which is large. $\square$

Definition 5.8 A *Dedekind cut* in the poset $\mathrm{Pro}(P, Q)$ is a pair $[\mathcal{I}, \mathcal{F}]$ where $\mathcal{I}$ is an regular open down-set of $\mathrm{Pro}(P, Q)$ and $\mathcal{F}$ is its image by the involution i (so $\mathcal{F}$ is an regular open up-set). The *gap* of the Dedekind cut is

$$\mathcal{G} = \mathrm{Pro}(P, Q) \backslash (\mathcal{I} \cup \mathcal{F}).$$

Corollary 5.9 *Let $\mathcal{I}$ be an open down-set and $\mathcal{F}$ a disjoint open up-set. Then $\mathcal{I}$ and $\mathcal{F}$ form a Dedekind cut if and only if the gap $Pro(P, Q) \backslash (\mathcal{I} \cup \mathcal{F})$ between them, does not contain large or small maps.*

Remark 5.10 For the rational numbers $\mathbb{Q}$ the distributive lattice $\widehat{\mathbb{Q}}$ consists of $\pm\infty$, the irrational numbers, and the rational numbers doubled, since a rational number q gives two cuts:

$$(\langle -\infty, q \rangle, [q, +\infty)), \quad (\langle -\infty, q], (q, +\infty)).$$

However if we have the natural topology on $\mathbb{Q}$, Dedekind cuts give the real numbers together with $\pm\infty$.

6 Profunctors Between Natural Posets

We introduce the class of natural posets as a suitable generalization of the poset of natural numbers. We then have good criteria for when open down-sets of $\mathrm{Pro}(P, Q)$ are closed or regular. We show, Theorem 6.10, that an open down-set is clopen (closed and open) if and only if it has a finite number of maximal elements. The purpose of this section is to get in Sect. 7 a version of Theorem 3.7 which applies to construct Alexander dual ideals in infinite-dimensional polynomial rings. Our main application is to profunctors $\mathbb{N} \longrightarrow \mathbb{N}$, see [9], and briefly indicated in the last Sect. 8.

6.1 Well Partially Ordered Sets

A poset P is *well partially ordered* if the following two conditions holds:

i. Any descending chain in P stabilizes.
ii. Any antichain in P is finite.

The following characterization of well partially ordered sets is classical, see for instance [18].

Proposition 6.1 *The following are equivalent for a poset P.*

(1) P is well partially ordered.
(2) Any up-set of P is finitely generated.
(3) The ascending chain condition holds for up-sets in P, equivalently the descending chain condition holds for $\widehat{P}$.

Lemma 6.2 *Let P and Q be well partially ordered sets, and $f : P \longrightarrow Q$ an isotone map.*

a. If f is large, then Γf is finite.
b. If f is small, then Λf is finite.

Proof a. Let f have profile (I, F). The projection of Γf onto P will be contained in I since the complement $f(p)^c$ is empty for $p \in F$. Furthermore for $p \in I$ the up-set $f(p)^c$ has only a finite set of minimal elements by the above proposition.

b. Let f have co-profile (J, G). The projection of Λf onto Q is contained in J. If $q \in J$, the set of $p \in P$ such that $q \in f(p)$ is a up-set in P and so has a finite set of minimal elements. Hence Λf is finite. □

6.2 Natural Posets

The class of well partially ordered sets is very large, it includes all well-ordered sets. For our purposes we need an extra condition so that our posets are more like natural numbers.

Definition 6.3 A poset P is *down finite* if for each p in P, the principal down-set $\downarrow p$ is finite. A *natural* poset (in analogy with natural numbers) is a poset which is well partially ordered and down finite.

Lemma 6.4 *Suppose every antichain in P is finite. Then P is natural iff there exists a chain of finite down-sets in P*

$$I^0 \subseteq I^1 \subseteq \cdots \subseteq I^n \subseteq \cdots$$

such that $\cup_{i \geq n} I^n$ equals P.

Proof The if statement is clear. Suppose then P is natural. For $p \in P$ the principal ideal $\downarrow p$ is finite. Let the *height* of p, denoted $\ell(p)$, be the length of the longest chain in $\downarrow p$. Then let I^r be the set of elements of P of height $\leq r$. Then $I^r \backslash I^{r-1}$ is the set of elements of height exactly r. These form an anti-chain in P and so a finite set. Hence each I^r is finite. Clearly the union of the I^r is all of P. □

Lemma 6.5 *Let Q be down finite. Then for any large f in $\mathrm{Pro}(P, Q)$ the open set $U(\mathbf{0}, f)$ is also closed.*

Proof Let f have profile (I, F). Consider the set X of all pairs (p, x) where $p \in I$ and x is minimal in $f(p)^c$. For each pair $(p, x) \in X$, let g_{px} be the smallest isotone map $g : P \to \widehat{Q}$ such that $g(p)$ is the principal down-set $\downarrow x$. Then g_{px} is a small map since $\downarrow x$ is finite. Furthermore if h is an isotone map such that h is not $\leq f$, then for at least one pair $(p, x) \in X$ we must have $x \in h(p)$ and so $h \geq g_{px}$. Then the complement

$$U(\mathbf{0}, f)^c = \cup_{(p,x) \in X} U(g_{px}, \mathbf{1}).$$

□

6.3 Criteria for Down-Sets Being Regular Open or for Being Clopen

Let

$$f_1 \leq f_2 \leq f_2 \leq \cdots$$

be a weakly increasing set of maps in $\mathrm{Pro}(P, Q)$, write $\mathrm{colim} f_r$ for their join. If the f_r's are a decreasing sequence we write $\lim f_r$ for their meet.

Posets in our application will typically be natural posets, like finite posets and the natural numbers $\mathbb{N}$. We assume in this subsection that our posets are natural. The following seems to provide the best way to check whether an open down-set of isotone maps is regular.

Proposition 6.6 *Let P and Q be natural posets and $\mathcal{I}$ an* open *down-set in $\mathrm{Pro}(P, Q)$.*

   a. *$\mathcal{I}$ is also closed if and only if it contains the colimit of any increasing sequence of isotone maps in $\mathcal{I}$.*
   b. *$\mathcal{I}$ is regular if and only if it contains any* large *colimit of an increasing sequence of isotone maps in $\mathcal{I}$.*

   There are analogous statements for open up-sets and decreasing sequences of maps in $\mathcal{F}$.

Proof a1. Suppose $\mathcal{I}$ is closed. Let $\{f_i\}$ be a weakly increasing sequence of isotone maps in $\mathcal{I}$, and $f = \mathrm{colim} f_i$. If f is *not* in $\mathcal{I}$, there is an open subset $U(b, c)$ in the complement $\mathcal{I}^c$ containing f.

Since $b(p)$ is finite and $\lim f_i(p) = f(p)$ which contains $b(p)$, there is a number $N(p)$ such that $f_i(p)$ contains $b(p)$ for $i \geq N(p)$.

Let T be the projection of Λb on P. Since b is small, Λb is finite and so is T. Let N be the maximum of $N(p)$ for $p \in T$. We show below that $f_i \geq b$ for $i \geq N$. But then since $f_i \in \mathcal{I}$ and $\mathcal{I}$ is a down-set, this gives $b \in \mathcal{I}$, a contradiction since $U(b, c)$ and $\mathcal{I}$ are disjoint. Thus f must be in $\mathcal{I}$.

To show that $f_i \geq b$, if the contrary were true, let $q \in b(p) \backslash f_i(p)$ for some p and let $p' \leq p$ be minimal in P with $q \in b(p')$. Then:

i. $(q, p') \in \Lambda b$ and so $f_i(p') \supseteq b(p')$ since $i \geq N \geq N(p')$.
ii. Since $p' \leq p$ then $f_i(p') \subseteq f_i(p)$ and so $q \notin f_i(p')$.

But these give a contradiction, and so we must have $f_i \geq b$.

a2. Suppose any co-limit of an increasing sequence in $\mathcal{I}$, is in $\mathcal{I}$. Suppose $\mathcal{I}$ is *not* closed. Then there is $f \in \mathcal{I}^c$ such that any open subset of f intersects $\mathcal{I}$. Now there exists the following:

(i) A sequence of finite posets in P

$$I^1 \subseteq I^2 \subseteq \cdots$$

such that $\cup_i I^i = P$.

(ii) For each $p \in P$ we can find a sequence of finite posets of Q

$$J_p^1 \subseteq J_p^2 \subseteq \cdots$$

such that $\cup_j J_p^j = f(p)$.

Given k, define the isotone maps f_k by:

$$f_k(p) = \cup_{p' \leq p} J_{p'}^k \text{ when } p \in I^k, \quad f_k(p) = \cup_{p' \in I^k, p' \leq p} f_k(p') \text{ when } p \notin I^k.$$

Then

- f_k is isotone and small,
- $i \leq j$ implies $f_i \leq f_j$,
- The open subset $U(f_i, \mathbf{1})$ contains f and so intersects $\mathcal{I}$. Then $f_i \in \mathcal{I}$ since $\mathcal{I}$ is a down-set,
- $\operatorname{colim} f_i = f$ and so $f \in \mathcal{I}$.

But the latter contradicts $f \notin \mathcal{I}$.

b1. Let $\mathcal{I}$ be regular and $\{f_i\}$ an increasing sequence in $\mathcal{I}$, whose colimit f is large. By part a the colimit of f_i is in the closure $\overline{\overline{\mathcal{I}}}$. Since $\mathcal{I}$ is regular $\overline{\overline{\mathcal{I}}} \backslash \mathcal{I}$ does not contain large maps by Lemma 5.7. Hence $f \in \mathcal{I}$.

b2. Suppose $\mathcal{I}$ contains large limits of increasing sequences in $\mathcal{I}$. Let f be a large map in the closure $\overline{\overline{\mathcal{I}}}$. By the construction in a2 there is a sequence $\{f_i\}$ of small maps such that $f = \operatorname{colim} f_i$. Each open subset $U(f_i, f)$ intersects $\mathcal{I}$ and so f_i is in $\mathcal{I}$. But then f is in $\mathcal{I}$. Then by Lemma 5.7, $\mathcal{I}$ is regular. □

Example 6.7 Let the open down-set $\mathcal{I}$ of $\text{Pro}(\mathbb{N}, \mathbb{N})$ be generated by the large maps $f_1, f_2, f_3, \cdots$ given by

$$f_r(n) = \begin{cases} r & n \leq r \\ \infty & n > r \end{cases}$$

Then $\operatorname{colim}_r f_r$ is the maximal isotone map $\mathbf{1}$, which is not in $\mathcal{I}$. Hence $\mathcal{I}$ is not regular. Also, its closure is all of $\text{Pro}(\mathbb{N}, \mathbb{N})$.

Example 6.8 Let the open up-set $\mathcal{F}$ in $Pro(\mathbb{N}, \mathbb{N})$ be generated by the small maps $g_1, g_2, \cdots$ given by

$$g_r(n) = \begin{cases} 1 & n < r \\ 2 & n \geq r \end{cases}.$$

The limit $\lim_r g_r$ is the minimal isotone map $\mathbf{0}$, which is not in $\mathcal{F}$. Hence $\mathcal{F}$ is not regular. Also, its closure is all of $Pro(\mathbb{N}, \mathbb{N})$.

Example 6.9 Let the profunctor $h : \mathbb{N} \to \mathbb{N}$ be given by $h(i) = i$. Let $\mathcal{I}$ be the down-set generated by the large maps h^r for $r \geq 1$, and $\mathcal{F}$ be the up-set generated by the small maps h_r for $r \geq 0$, where

$$h^r(i) = \begin{cases} i & i \leq r \\ r & i = r + 1 \\ \infty & i \geq r + 2 \end{cases}, \quad h_r(i) = \begin{cases} i & i \leq r \\ r + 2 & i \geq r + 1 \end{cases}.$$

Then $[\mathcal{I}, \mathcal{F}]$ is a Dedekind cut and the gap consists of the function h.

In the argument that follows now we use the following construction: Let $I \subseteq J \subseteq P$ be down-sets and $f : I \to \widehat{Q}$ an isotone map. Then there exists a unique *minimal* extension $f^J : J \to \widehat{Q}$ such that the restriction $(f^J)_{|I} = f$. For each $p \in J$ we let

$$f^J(p) = \vee_{p' \in I, p' \leq p} f(p').$$

Note that if f is small and I is finite, the extension f^J is also small.

The following characterizes precisely when the gap is empty.

Theorem 6.10 *Let P and Q be natural posets. A down-set $\mathcal{I}$ of $Pro(P, Q)$ is clopen (closed and open) if and only if $\mathcal{I}$ is a finite union of basis open subsets $U(\mathbf{0}, f)$. Alternatively formulated, an open down-set $\mathcal{I}$ is clopen if and only if it is finitely generated.*

Proof The open subsets $U(\mathbf{0}, f)$ are closed, by Proposition 6.6. Hence any finite union of them is clopen. □

Suppose now $\mathcal{I}$ is clopen. We will show it has only a finite number of minimal generators. Assume the opposite, that there are infinitely many generators of $\mathcal{I}$, none of which are comparable.

1. Take a filtration of finite posets

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_r \subseteq \cdots$$

such that the union of the I_r's is P. Let T_1 be the set of isotone maps $F_1 : I_1 \to \widehat{Q}$ such that:

- For every $f : I_1 \to \widehat{Q}$ with $f \leq F_1$ and the $f(p)$ *finite* down-sets in Q for $p \in I_1$, there are *infinitely many* of the minimal generators g of $\mathcal{I}$ such that the restriction $g_{|I_1} \geq f$. Clearly T_1 is non-empty: we may take F_1 to be the constant map with value the empty set.

2. Let us show that we can use Zorn's lemma on T_1. If

$$F_1^1 \subseteq F_1^2 \subseteq \cdots \subseteq F_1^r \subseteq \cdots$$

is a chain in T_1, let $F_1 = \text{colim} F_1^r$. We claim F_1 is in T_1. Let $f \leq F_1$ with f having finite values. For each p, for some integer $N(p)$ we have, since $f(p)$ is finite, that $f(p) \subseteq$

$F_1^{N(p)}(p)$. Let N be the maximum of the $N(p)$ for $p \in I_1$. Then $f \leq F_1^N$ and so there are infinitely many minimal generators g of $\mathcal{I}$ with $g_{|I_1} \geq f$. This shows that Zorn's lemma applies. Furthermore, there is an increasing sequence $\{f\}$ of such functions whose limit is F_1. Extending these f's to P we get an increasing sequence $\{f^P\}$ whose limit is the extension F_1^P. Each f^P is in $\mathcal{I}$ since there are minimal generators g of $\mathcal{I}$ with $g_{|I_1} \geq f$. Since $\mathcal{I}$ is closed, by Proposition 6.6 we have $F_1^P \in \mathcal{I}$, a fact we use in 5. below.

3. Let F_1 be maximal in T_1. We now show the following refined statement:

Claim 2 For each finite $f \leq F_1$ there are infinitely many minimal generators g such that $f \leq g_{|I_1} \leq F_1$.

Proof Let $p_1, p_2, \ldots, p_m$ be the elements of I_1 in an order such that $p_i < p_j$ implies $i < j$. (This is usually called a linear extension of I_1.) Suppose we have verified that for each $f \leq F_1$ there are infinitely many g's such $f \leq g_{|I_1}$ and with $f(p_i) \leq g(p_i) \leq F_1(p_i)$ for $i < s$. Let X_s be the (finite) set of minimal elements of the complement $F_1(p_s)^c$. If not $g(p_s) \subseteq F_1(p_s)$ then $g(p_s)$ contains some $x \in X_s$. Define the isotone $F_{1x} : I_1 \to \widehat{Q}$ by

$$F_{1x}(p) = \begin{cases} F_1(p), & \text{not } p \geq p_s \\ F_1(p) \cup \{x\} & p \geq p_s \end{cases}$$

Since F_1 is maximal, there is some $f_x \leq F_{1x}$ where f_x has finite values, such that there are not infinitely many g's with $f_x \leq g_{|I_1}$. Let f'_x be the meet $f_x \wedge F_1$. Then let f' be the join $f \vee (\vee_{x \in X_s} f'_x)$, an isotone with finite values. Note that $f' \leq F_1$. Then there are infinitely many g's such that: i) $f' \leq g$ and ii) $f'(p_i) \leq g(p_i) \leq F_1(p_i)$ for $i < s$. But only a finite number of g's are larger than f_x for each x in the finite set X_s. So taking away a finite number of the g's, none of the infinitely many rest will have $g(p_s)$ containing any $x \in X_s$. Hence for these $g(p_s) \subseteq F_1(p_s)$. □

4. Suppose we now have constructed $F_j : I_j \to \widehat{Q}$ for $j = 1, \cdots, r$ such that $F_{j|I_i} = F_i$ for each $i \leq j$ and the F_j are such that for each finite $f : I_j \to \widehat{Q}$ with $f \leq F_j$ there are infinitely many generators g such that $f \leq g_{|I_j} \leq F_j$, and F_j is maximal such. We will extend F_r to a function $F_{r+1} : I_{r+1} \to \widehat{Q}$. Let T_{r+1} be the set of isotone $F : I_{r+1} \to \widehat{Q}$ such that $F_{|I_r} = F_r$ and for each finite $f \leq F$ there are infinitely many g's with $g_{|I_r} \geq f$ and $F_r \geq g_{|I_r} \geq f$. The set T_{r+1} is non-empty since it is easy to see that the extension $F_r^{I_{r+1}}$ fulfills the condition of being in T_{r+1} since F_r does so for T_r. As above T_{r+1} fulfills the condition for Zorn's lemma and so T_{r+1} has a maximal element F_{r+1}. As above we may show that for each finite $f \leq F_{r+1}$ there are infinitely many minimal generators g of $\mathcal{I}$ such that $F_{r+1} \geq g_{|I_{r+1}} \geq f$

5. Consider now the sequence of extensions

$$F_1^P \leq F_2^P \leq \cdots \leq F_r^P \leq \cdots .$$

Let $F = \text{colim} F_r^P$. Since each $F_r^P \in \mathcal{I}$ (by the same reason as at the end of part 2) and $\mathcal{I}$ is closed we get $F \in \mathcal{I}$. Since $\mathcal{I}$ is open there exists then a large $G \geq F$ in $\mathcal{I}$. Let I be the finite profile down-set of G. Due to I being finite we must have $I_r \supseteq I$ for sufficiently large r. But there are infinitely many minimal generators g such that $g_{|I_r} \leq F_r$ which again is $\leq G_{|I_r}$. But since $G(p) = \infty$ for p in the complement I_r^c, we would have $G \geq g$. This is a contradiction since the g's are (infinitely many) minimal generators of $\mathcal{I}$.

The upshot is that there cannot be infinitely many generators of $\mathcal{I}$. □

Problem 6.11 What subsets of $\text{Pro}^u(\mathbb{N}, \mathbb{N})$ can be gaps?

7 Natural Posets and Finite Type Cuts

Here we give the variant of Theorem 3.7 such that the Λ- and Γ-ideals are finitely generated ideals in a polynomial ring. This works when P and Q are natural posets, which we assume in this section.

By Lemma 6.2 the map Λ of (11) restricts to a map:

$$\Lambda : U\mathrm{Pro}_S(P, Q) \to (UQ \times UP^{\mathrm{op}})\widehat{}_{\mathrm{fin}}, \quad f \mapsto (\Lambda f, -).$$

Similarly the map Γ of (12) restricts to a map:

$$\Gamma : U\mathrm{Pro}^L(P, Q) \to (UQ \times UP^{\mathrm{op}})\widehat{}^{\mathrm{fin}}, \quad f \mapsto (-, \Gamma f).$$

For a cut $(\mathcal{I}, \mathcal{F})$ for $\mathrm{Pro}(P, Q)\widehat{}$ let $\mathcal{F}_S$ denote the small maps in $\mathcal{F}$, and $\mathcal{I}^L$ the large maps in $\mathcal{I}$.

By (6) the map Λ induces an isotone map:

$$\Lambda_U^{\mathrm{i}} : \mathrm{Pro}_S(P, Q)\widehat{} \to (UQ \times UP^{\mathrm{op}})\widehat{\widehat{}}_{\mathrm{fin}}, \quad (-, \mathcal{F}_S) \mapsto (-, \Lambda(\mathcal{F}_S)^\uparrow).$$

The map Γ induces an isotone map:

$$\Gamma_U^{!} : \mathrm{Pro}^L(P, Q)\widehat{} \to (UQ \times UP^{\mathrm{op}})\widehat{\widehat{}}^{\mathrm{fin}}, \quad (\mathcal{I}^L, -) \mapsto (\Gamma(\mathcal{I}^L)^\downarrow, -).$$

We have the following variation of Theorem 3.7.

Theorem 7.1 *Let P and Q be natural posets and $[\mathcal{I}, \mathcal{F}]$ a Dedekind cut for $\mathrm{Pro}(P, Q)$. Then $(\Gamma(\mathcal{I}^L)^\downarrow, \Lambda(\mathcal{F}_S)^\uparrow)$ is a finite type cut for the Boolean lattice $(UQ \times UP^{op})\widehat{}$.*

Proof We must show 1 and 2 in Definition 1.7. We do 1 as 2 is analogous. □

Part I. We show the implication $\Rightarrow$ of 1 in Definition 1.7. This amounts to show for any $(I, F) \in \mathcal{I}^L$ and $(J, G) \in \mathcal{F}_S$ that

$$\Gamma F \cap \Lambda J \neq \emptyset.$$

The argument for this is the same as in Part I for Theorem 3.7.

Part II. We show the implication $\Leftarrow$ of 1 in Definition 1.7. Let $S \subseteq UQ \times UP^{\mathrm{op}}$ be a *finite* set such that $S \cap \Lambda J \neq \emptyset$ for *every* $(J, G) \in \mathcal{F}_S$. We show that $S \supseteq \Gamma F$ for *some* $(I, F) \in \mathcal{I}^L$.

Let $I_0 \subseteq P$ and $J_0 \subseteq Q$ be finite down-sets such that the projections of S are contained respectively in I_0 and in J_0. (We know they are finite due to P and Q being down-finite.) For a large f in $\mathrm{Pro}(P, Q)$ denote by $\Lambda_0 f$ the subset of Λf lying above I_0. Let T consist of the large f such that:

- $f(p) = \begin{cases} \infty, & p \notin I_0 \\ \subseteq J_0, & p \in I_0 \end{cases}$.
- $\Lambda_0 f \cap S = \emptyset$.

Clearly the minimum element of those f fulfilling the first point, also fulfills the second, so T is non-empty. We use Zorn's lemma to show that T has a maximal element. So let

$$f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$$

be a chain of maps in T and let $f = \mathrm{colim}\, f_r$. We claim that $\Lambda_0 f \cap S = \emptyset$. So suppose $(q, p) \in \Lambda f$, so $q \in f(p)$ but $q \notin f(p')$ for every $p' < p$. Then $q \in f_i(p)$ for some i and $q \notin f_i(p')$ since $f_i(p') \subseteq f(p')$. Then $(q, p) \in \Lambda f_i$ and so is not in S. Thus f is an upper bound in T for the chain. By Zorn's lemma T has a maximal element, which we denote f.

Claim 3 f is in $\mathcal{I}$.

Proof The map f is not in the gap between $\mathcal{I}$ and $\mathcal{F}$ since it is large. So if f is not in $\mathcal{I}$ it must be in $\mathcal{F}$. Since $\mathcal{F}$ is open, for $B \supseteq J_0$ large enough f_B given by

$$f_B(p) = \begin{cases} f(p), & p \in I_0 \\ B, & p \notin I_0 \end{cases},$$

is in $\mathcal{F}$. But $\Lambda_0 f_B \subseteq \Lambda_0 f$, and so $\Lambda f_B \cap S = \Lambda_0 f_B \cap S$ is the empty set, which is not so by the requirement on S. Hence f is not in $\mathcal{F}$ and not in the gap, so it must be in $\mathcal{I}$. □

Claim 4 $\Gamma f \subseteq S$. This finishes the proof by the second line of Part II.

Proof Suppose not, so there is $(q_0, p_0) \in \Gamma f \backslash S$. Recall q_0 is minimal in the complement $f(p_0)^c$. Let

$$\tilde{f}(p) = \begin{cases} f(p) \cup \{q_0\}, & p \geq p_0 \\ f(p), & \text{otherwise.} \end{cases}$$

We now claim that:

$$\Lambda_0 \tilde{f} \subseteq \Lambda_0 f \cup \{(q_0, p_0)\}.$$

That $(q, p) \in \Lambda_0 \tilde{f}$ means that $p \in I_0$ and $q \in \tilde{f}(p)$, and $q \notin \tilde{f}(p')$ for $p' < p$.

1. If not $p \geq p_0$ then $\tilde{f}(p') = f(p')$ for all values $p' \leq p$. Therefore (q, p) is in $\Lambda_0 f$.
2. Let $p \geq p_0$. Then either i. $q \in f(p)$ or ii. $q = q_0$. In case i. $(q, p) \in \Lambda f$. In case ii. where $q = q_0$, if $p > p_0$ then (q_0, p) would not be in $\Lambda \tilde{f}$ since $q_0 \in \tilde{f}(p)$ and $q_0 \in \tilde{f}(p_0)$. We are thus left with $(q, p) = (q_0, p_0)$.

The upshot is that $\tilde{f}$ fulfills the requirements to be in T. But this contradicts f being maximal in T. Hence no such (q_0, p_0) can exist, and so $\Gamma f \subseteq S$. □

We may now define the Λ- and Γ-ideals for (infinite) natural posets P and Q. These ideals then live in infinite-dimensional polynomial rings. These ideals will be square-free monomial ideals.

Definition 7.2 Let $[\mathcal{I}, \mathcal{F}]$ be a Dedekind cut for $\mathrm{Pro}(P, Q)$. Let $L_\Lambda(\mathcal{F})$ the ideal in $k[x_{Q \times P^{\mathrm{op}}}]$ generated by the monomials $\prod_{(q,p) \in I} x_{q,p}$ for $(I, -) \in \Lambda(\mathcal{F}_S)$. Let $L_\Gamma(\mathcal{I})$ be the ideal in $k[x_{P \times Q^{\mathrm{op}}}]$ generated by the monomials $\prod_{(p,q) \in F^{\mathrm{op}}} x_{p,q}$ for $(F^{\mathrm{op}}, -) \in \Gamma(\mathcal{I}^L)^{\mathrm{op}}$.

By the theorem above, these ideals are Alexander dual ideals, meaning that the monomials in $L_\Lambda(\mathcal{F})$ are precisely those monomials with non-trivial common divisor with every monomial in $L_\Gamma(\mathcal{I})$, and vice versa.

8 Monomial Ideals

For the case $Q = \mathbb{N}$ we show that open down-sets in $\mathrm{Pro}(P, \mathbb{N})$ induce monomial ideals in the polynomial ring $k[x_P]$. In particular when $P = \mathbb{N}$, these monomial ideals are precisely the class of strongly stable ideals in the infinite-dimensional polynomial ring $k[x_\mathbb{N}]$. The results in Sect. 5 then enables defining a duality on a distinguished class of strongly stable ideals, see [9].

Given a map of sets $A \xrightarrow{\alpha} B$. Let $\mathrm{Hom}_{\mathrm{fin}}(A, \mathbb{N}_0)$ be the maps $A \xrightarrow{u} \mathbb{N}_0$ such that the support $\{a \mid u(a) > 0\}$ is a finite set. Then α induces a map

$$\mathrm{Hom}_{\mathrm{fin}}(A, \mathbb{N}_0) \xrightarrow{\tilde{\alpha}} \mathrm{Hom}_{\mathrm{fin}}(B, \mathbb{N}_0)$$

sending a map u to a map $v : B \to \mathbb{N}_0$ where $v(b) = \sum_{\alpha(a)=b} u(a)$.

Consider the projection $UP \times UQ^{\mathrm{op}} \xrightarrow{\pi_1} UP$, and inclusion $\mathbf{2} \xrightarrow{i} \mathbb{N}_0$ sending $0 \mapsto 0$ and $1 \mapsto 1$. We get a composite

$$\mathrm{Hom}_{\mathrm{fin}}(UP \times UQ^{\mathrm{op}}, \mathbf{2}) \xrightarrow{\tilde{i}} \mathrm{Hom}_{\mathrm{fin}}(UP \times UQ^{\mathrm{op}}, \mathbb{N}_0) \xrightarrow{\tilde{\pi}_1} \mathrm{Hom}_{\mathrm{fin}}(UP, \mathbb{N}_0).$$

Given a map $\tau : UP \times UQ^{\mathrm{op}} \to \mathbf{2}$ corresponding to a finite subset $T \subseteq UP \times UQ^{\mathrm{op}}$ (the arguments for which τ is non-zero), then $\tilde{\pi}_1 \circ \tilde{i}(\tau)$ is the map sending p to the cardinality of the set of pairs $(p, q) \in T$ with first coordinate p.

Note that the set $\mathrm{Hom}_{\mathrm{fin}}(UP, \mathbb{N}_0)$ identifies as the set of monomials $\mathrm{Mon}(x_P)$ in the variables x_p for $p \in P$.

Lemma 8.1 *Let $Q = \mathbb{N}$. The composite $\lambda = \tilde{\pi}_1 \circ \tilde{i} \circ \Lambda$:*

$$\lambda : U Pro_S(P, \mathbb{N}) \xrightarrow{\Lambda} Hom_{fin}(UP \times U\mathbb{N}^{op}, \mathbf{2}) \xrightarrow{\tilde{\pi}_1 \circ \tilde{i}} Hom_{fin}(UP, \mathbb{N}_0)$$

is an isomorphism.

Proof This is Proposition 4.3 in [8]. □

We may also consider the composite $\gamma = \tilde{\pi}_2 \circ \tilde{i} \circ \Gamma$:

$$\gamma : U\mathrm{Pro}^L(\mathbb{N}, P) \xrightarrow{\Gamma} \mathrm{Hom}_{\mathrm{fin}}(U\mathbb{N} \times UP^{\mathrm{op}}, \mathbf{2}) \xrightarrow{\tilde{\pi}_2 \circ \tilde{i}} \mathrm{Hom}_{\mathrm{fin}}(UP^{\mathrm{op}}, \mathbb{N}_0),$$

which is likewise an isomorphism.

Lemma 8.2 *There is a commutative diagram:*

Note that the vertical map sends a cut (I, F) to a cut (F^{op}, I^{op}).

As in (6) this extends to a commutative diagram:

The up-sets in $\text{Hom}_{\text{fin}}(UP, \mathbb{N}_0)$ identifies as the monomial ideals in $k[x_P]$. Recall that open up-sets $\mathcal{F}$ in $\text{Pro}(P, \mathbb{N})$ are in bijection with up-sets in $\text{Pro}_S(P, \mathbb{N})$. So we get an upset $\mathcal{F}_S$ in the latter. By λ^i we get a monomial ideal in $k[x_P]$.

Application. When $P = \mathbb{N}$ the monomial ideals one gets are precisely the strongly stable ideals in $k[x_\mathbb{N}]$. There is a one-one correspondence between open up-sets in $\text{Pro}(\mathbb{N}, \mathbb{N})$ and strongly stable ideals in $k[x_\mathbb{N}]$, see [9].

Definition 8.3 Given a Dedekind cut $[\mathcal{I}, \mathcal{F}]$ for $\text{Pro}(\mathbb{N}, \mathbb{N})$. We get an up-set $\lambda^i(\mathcal{F}_S)$ in $\text{Hom}_{\text{fin}}(\mathbb{N}, \mathbb{N}_0)$ which corresponds to a monomial ideal I_λ of $k[x_\mathbb{N}]$. Similarly we get a up-set $(\gamma^{\text{op}})^i(\mathcal{I}^{L,\text{op}})$ in $\text{Hom}_{\text{fin}}(\mathbb{N}, \mathbb{N}_0)$ which corresponds to a monomial ideal I_γ of $k[x_\mathbb{N}]$. The ideals I_λ and I_γ are *dual strongly stable ideals*.

Note that this definition is reasonable. Given I_λ we can reconstruct $\mathcal{F}_S$ and thereby $\mathcal{F}$. Since $[\mathcal{I}, \mathcal{F}]$ is a Dedekind cut, we can construct $\mathcal{I}$ from $\mathcal{F}$ and thereby determine I_γ. And of course we may go the opposite way, from I_γ we may determine I_λ. Note also that not every strongly stable ideal I has a dual. Only the strongly stable ideals such that the corresponding open down-set $\mathcal{I}$ is *regular*, have duals. Versions of this duality for finite-dimensional polynomial rings were first given in [8, Section 7] and independently by [23].

Data Availability   There are now associated data to this manuscript.

Declarations

Conflict of interest   The authors have no competing interests to declare that are relevant to the content of this article.

References

1. Baez, J.C.: Isbell duality, Notices of the American Mathematical Society **70**(1) 140–141 (2023)
2. Bénabou, J.: Distributors at work, Lecture notes of a course at TU Darmstadt written by Thomas Streicher **11** (2000). www2.mathematik.tudarmstadt.de/streicher/FIBR/DiWo.pdf
3. Borceux, F.: Handbook of Categorical Algebra: volume 1, Basic Category Theory, vol. 1. Cambridge University Press, Cambridge (1994)
4. Carboni, Aurelio, Street, Ross: Order ideals in categories. Pac. J. Math. **124**(2), 275–288 (1986)
5. D'Alì, Alessio, Fløystad, Gunnar, Nematbakhsh, Amin: Resolutions of co-letterplace ideals and generalizations of Bier spheres. Trans. Am. Math. Soc. **371**(12), 8733–8753 (2019)
6. Eberhart, Carl, Selden, John: On the closure of the bicyclic semigroup. Trans. Am. Math. Soc. **144**, 115–126 (1969)
7. Ene, Viviana, Herzog, Jürgen., Mohammadi, Fatemeh: Monomial ideals and toric rings of Hibi type arising from a finite poset. Eur. J. Comb. **32**(3), 404–421 (2011)

8. Fløystad, Gunnar: Poset ideals of P-partitions and generalized letterplace and determinantal ideals. Acta Math. Vietnam. **44**(1), 213–241 (2019)
9. Fløystad, G.: Shift modules, strongly stable ideals, and their dualities. arXiv:2105.14604, 1–39 (2021)
10. Fløystad, Gunnar, Møller, B., Herzog, J.: Letterplace and co-letterplace ideals of posets. J. Pure Appl. Algebra **221**(5), 1218–1241 (2017)
11. Fløystad, G., Mafi, A.: Polarizations of artin monomial ideals define triangulated balls. arXiv:2212.09528, (2022)
12. Fløystad, G., Vatne, J.E.: (Bi-) Cohen–Macaulay simplicial complexes and their associated coherent sheaves. Commun. Algebra **33**(9), 3121–3136 (2005)
13. Fong, B., Spivak, D.I.: An Invitation to Applied Category Theory: Seven Sketches in Compositionality. Cambridge University Press, Cambridge (2019)
14. Güntürkün, S., Snowden, A.: The representation theory of the increasing monoid. arXiv:1812.10242 (2018)
15. Herzog, J., Asloob Qureshi, A., Shikama, A.: Alexander duality for monomial ideals associated with isotone maps between posets. J. Algebra Its Appl. **15**(05), 1650089 (2016)
16. Juhnke-Kubitzke, M., Katthän, L., Madani, S.S.: Algebraic properties of ideals of poset homomorphisms. J. Algebraic Comb. **44**(3), 757–784 (2016)
17. Juhnke-Kubitzke, M., Madani, S.S.: Ideals Associated to Poset Homomorphisms: A Survey, Homological and Computational Methods in Commutative Algebra, pp. 129–140. Springer, Berlin (2017)
18. Kruskal, J.B.: The theory of well-quasi-ordering: a frequently discovered concept. J. Comb. Theory Ser. A **13**(3), 297–305 (1972)
19. Miller, E., Sturmfels, B.: Combinatorial Commutative Algebra, vol. 227. Springer, Berlin (2004)
20. Nagel, Uwe, Römer, Tim: Equivariant Hilbert series in non-noetherian polynomial rings. J. Algebra **486**, 204–245 (2017)
21. Rosebrugh, R., Wood, R.J.: Constructive complete distributivity iv. Appl. Categ. Sruct. **2**(2), 119–144 (1994)
22. Rosebrugh, R., Wood, R.J.: Split structures. Theory Appl. Categ. **13**(12), 172–183 (2004)
23. Shibata, Kosuke, Yanagawa, Kohji: Alexander duality for the alternative polarizations of strongly stable ideals. Commun. Algebra **48**(7), 3011–3030 (2020)