Research Article

Bruno Franchi and Irina Markina*

Exceptional families of measures on Carnot groups

https://doi.org/10.1515/agms-2022-0148
received September 17, 2022; accepted January 4, 2023

Abstract: We study the families of measures on Carnot groups that have vanishing p-module, which we call M_p-exceptional families. We found necessary and sufficient Conditions for the family of intrinsic Lipschitz surfaces passing through a common point to be M_p-exceptional for $p \geq 1$. We describe a wide class of M_p-exceptional intrinsic Lipschitz surfaces for $p \in (0, \infty)$.

Keywords: nilpotent Lie group, module of families of measures, Hausdorff measure, intrinsic Lipschitz graph

MSC 2020: Primary 28A78, 53C17, Secondary 31B15, 22E30

1 Introduction and motivation

"Negligible" sets appear customarily in measure theory and stochastic theory, as the sets of measure zero, as well as the sets of vanishing p-capacity when dealing with regularity issues for solutions of partial differential equations, and as thin and polar sets in potential theory. Sets of a family of measures having the so-called M_p-module zero belong to this category of negligible or exceptional subsets of families of measures, see definitions in Section 3.

The notion of a module of a family of curves "or in another terminology extremal length" originated in the theory of complex analytic functions as a conformal invariant [2], and later was widely used for the quasiconformal analysis and extremal problems of functional spaces [3,9,31,34,44,47]. Fuglede in his seminal paper [21] proposed to extend the notion of module from families of curves to families of measures. He characterised the completion, with respect to L^p norm, of some functional classes by using a family of surfaces having vanishing M_p-module. He also described some classes of systems of measures with vanishing module and related it to the potential theory. In spite of the fact that the definition of the module of a family of measures is given for an arbitrary measure space, most of the applications in [21] were done for $\mathbb{R}^n$.

The development of the analysis on metric measure spaces inspired us to look for examples of interesting systems of measures in a more general setting than the Euclidean space. Several studies have been carried out on this context [10,11,27,40,41]. However, most of the preceding works were dealing with families of curves. Our main interest focuses on families of (suitably defined) intrinsic surfaces on Carnot groups [17,20,46].

Carnot groups are connected, simply connected, nilpotent Lie groups and are one of the most popular examples of metric measure spaces. Being endowed with a rich structure of translations and dilations makes the Carnot groups akin to Euclidean spaces. Euclidean spaces are commutative Carnot groups

*** Corresponding author: Irina Markina,** Department of Mathematics, University of Bergen, P.O. Box 7803, Bergen N-5020, Norway, e-mail: irina.markina@uib.no
Bruno Franchi: Department of Mathematics, University of Bologna, Piazza di Porta S. Donato, 5, 40126 Bologna, Italy, e-mail: bruno.franchi@unibo.it

and, more precisely, the only commutative Carnot groups. The simplest but, at the same time, non-trivial instance of non-Abelian Carnot groups is provided by Heisenberg groups H^n.

Carnot groups possess an intrinsic metric, the so-called Carnot-Carathéodory metric (cc-distance), see for instance, [12,16,24]. It is also well-known that non-commutative Carnot groups, endowed with the cc-distance, are not Riemannian manifolds because the cc-distance makes them not locally Lipschitz equivalent to a Riemannian manifold at any scale [39]. Carnot groups are particular instances of the so-called sub-Riemannian manifolds.

Although Carnot groups are analytic manifolds, the study of measures supported on submanifolds (for instance the Hausdorff measures associated with their cc-distance) cannot be reduced to the well-established theory for submanifolds of Euclidean spaces, since it has been clear for a long time that considering Euclidean regular submanifolds, even in Heisenberg groups, may be both too general and too restrictive, see [25] for a striking example related to the second instance. Through this article, we shall rely on the theory of intrinsic submanifolds in Carnot groups that has been recently developed by making use of the notion of *intrinsic graphs*, see e.g. [17,19,20]. A discussion of different alternatives leading to this notion can be found e.g. in [20], together with the main properties of the most relevant instances, the so-called intrinsic Lipschitz graphs. Let us sketch this construction, restricting ourselves to stress the difficulties arising when we want to extend the theory of p-modules from the Euclidean setting to Carnot groups. For deep algebraic reasons, due to the non-commutativity of the group, the most flexible notion of submanifold of a Carnot group is the counterpart of the Euclidean notion of a graph of a function. However, the notion of intrinsic graph is not a straightforward translation of the corresponding Euclidean notion, since Carnot groups not always can be expressed as a direct product of subgroups. Because of that, we argue as follows: an intrinsic graph inside $\mathbb{G}$ is associated with a decomposition of $\mathbb{G}$ as a product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ of two homogeneous *complementary subgroups* $\mathbb{M}$, $\mathbb{H}$, see Section 3.4.1. Then the intrinsic (left) graph of $f : \Omega \to \mathbb{H}$, where Ω is an open subset of $\mathbb{M}$, is the set

$$\operatorname{graph}(f) = \{g \cdot f(g) : g \in \Omega\}.$$

Another deep peculiarity of Carnot groups is the poor structure of the isometry group preserving the grading structure of their Lie algebras. The isometry of a Carnot group is a composition of translations and elements of the group automorphisms, see [26, Theorem1.2]. A Carnot group is said to be rigid if the space of maps preserving the horizontal distribution is finite dimensional. For instance, H-type Lie groups having dimension of the centre greater than two are rigid [33, Theorem, page 705]. A wider class of two-step Carnot groups introduced in [29] contains a non-rigid subclass, see [32, Theorem 3], where some other examples of non-rigid Carnot groups can be found by making use of the Tanaka prolongation approach.

The main results of the present work are formulated in Theorem 3.31, Section 3.5.1, where we show that quite a wide class of families of intrinsic Lipschitz surfaces (sets which are locally intrinsic Lipschitz graphs of the same "metric dimension") has vanishing p-module for $p \in (0, 1)$. We did not reach the full generality as in the Euclidean space due to the complex structure of decompositions of an arbitrary Carnot groups into the product of two homogeneous non-isomorphic subgroups, see, for instance [5,23]. Another result, contained in Theorem 3.32, Section 3.5.2, is the sufficient condition for a family of surfaces passing through a common point to be M_p-exceptional. In order to find a necessary condition we construct a family of intrinsic Lipschitz graphs passing through one point by making use of the orthogonal Grassmannians on some specific two-step Carnot groups, see Section 4. The construction of the orthogonal Grassmannians and the study of measures on them have an independent interest. Examples of exceptional families of measures that are not related to intrinsic Lipschitz graphs are contained in Examples 3.5 and 3.9 in Section 3.3.

We are trying to keep the article as accessible as possible for a wider audience.

2 Carnot groups

In this section, we establish notation and collect the basic notions concerning Carnot groups and their Lie algebras.

2.1 General definition of Carnot groups

A Carnot group $\mathbb{G}$ is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ of the left-invariant vector fields is a graded stratified nilpotent Lie algebra of step l, i.e. the Lie algebra $\mathfrak{g}$ satisfies:

$$\mathfrak{g} = \bigoplus_{k=1}^{l} \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}, \quad \mathfrak{g}_{l+1} = \{0\}.$$

We denote by $N = \sum_{k=1}^{l} \dim(\mathfrak{g}_k)$ the topological dimension of $\mathbb{G}$. The number $Q = \sum_{k=1}^{l} k \dim(\mathfrak{g}_k)$ is called the *homogeneous dimension* of the group $\mathbb{G}$. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp:\mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism.

One can identify the group $\mathbb{G}$ with $\mathbb{R}^N \cong \mathfrak{g}$ by making use of exponential coordinates of the first kind by the following procedure. We fix a basis

$$X_{11}, \dots, X_{1\mathbf{d}_1}, X_{21}, \dots, X_{2\mathbf{d}_2}, \dots, X_{l1}, \dots, X_{l\mathbf{d}_l}, \quad \mathbf{d}_k = \dim(\mathfrak{g}_k), \tag{2.1}$$

of the Lie algebra $\mathfrak{g}$ which is adapted to the stratification. If $g \in \mathbb{G}$ and $V \in \mathfrak{g}$ are such that

$$g = \exp(V) = \exp\left(\sum_{k=1}^{l} \sum_{j=1}^{\mathbf{d}_k} x_{kj} X_{kj} \right),$$

then (with a slight abuse of notations) we associate with the point $g \in \mathbb{G}$ a point $x \in \mathbb{R}^N$ having the following coordinates:

$$g = \left(x_{11}, \dots, x_{1\mathbf{d}_1}, x_{21}, \dots, x_{2\mathbf{d}_2}, \dots, x_{l1}, \dots, x_{l\mathbf{d}_l} \right) = x. \tag{2.2}$$

Thus, the identity $e \in \mathbb{G}$ is identified with the origin in $\mathbb{R}^N$ and the inverse g^{-1} with $-x$.

The stratification $\mathfrak{g} = \oplus_{k=1}^{l} \mathfrak{g}_k$ of $\mathfrak{g}$ induces the one-parameter family $\{\delta_\lambda\}_{\lambda>0}$ of automorphisms of $\mathfrak{g}$, where each $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined as

$$\delta_\lambda(X) := \lambda^k X, \quad \text{for all } X \in \mathfrak{g}_k \quad \text{and} \quad \lambda > 0.$$

The exponential map allows us to transfer these *automorphisms of* $\mathfrak{g}$ to a family of *automorphisms of the Lie group* $\mathbb{G}$: $\delta_\lambda^{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$, so-called intrinsic dilations, defined as

$$\delta_\lambda^{\mathbb{G}} := \exp \circ \delta_\lambda \circ \exp^{-1}, \quad \text{for all } \lambda > 0.$$

We keep denoting by $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ the intrinsic dilations, if no confusion arises.

In coordinates (2.2), the group automorphism $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, $\lambda > 0$, is written as

$$\begin{aligned} \delta_\lambda g &= \delta_\lambda\left(x_{11}, \dots, x_{1\mathbf{d}_1}, x_{21}, \dots, x_{2\mathbf{d}_2}, \dots, x_{l1}, \dots, x_{l\mathbf{d}_l} \right) \\ &= \left(\lambda x_{11}, \dots, \lambda x_{1\mathbf{d}_1}, \lambda^2 x_{21}, \dots, \lambda^2 x_{2\mathbf{d}_2}, \dots, \lambda^l x_{l1}, \dots, \lambda^l x_{l\mathbf{d}_l} \right). \end{aligned} \tag{2.3}$$

The group product on $\mathbb{G}$, written in coordinates (2.2), has the form

$$x \cdot y = x + y + Q(x, y), \quad \text{for all } x, y \in \mathbb{R}^N, \tag{2.4}$$

where $Q = (Q_1, \dots, Q_N) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and each Q_k is a homogeneous polynomial with respect to group dilations, see, for instance [16, Propositions 2.1 and 2.2]. When the grading structure is not important, we will use the one-index notation

$$X_1, \dots, X_N, \quad x_1, \dots, x_N$$

for basis (2.1) of the Lie algebra $\mathfrak{g}$ and for coordinates (2.2) on $\mathbb{G}$. The basis vectors of the Lie algebra $\mathfrak{g}$ viewed as left invariant vector fields X_j, $j = 1, \dots, N$, on $\mathbb{G}$ have polynomial coefficients and take the form in the coordinate frame:

$$X_j = \partial_j + \sum_{i>j}^{N} q_{i,j}(x)\partial_i, \quad \text{for } j = 1, \dots, N, \tag{2.5}$$

where $q_{i,j}(x) = \frac{\partial Q_i}{\partial y_j}(x, y)|_{y=0}$. The vector fields X_j, $j = 1, \dots, \mathbf{d}_1$, are called horizontal and they are homogeneous of degree 1 with respect to the group dilation. Their span at $q \in \mathbb{G}$ is called the horizontal vector space $H_q\mathbb{G} \subset T_q\mathbb{G}$.

2.2 Distance functions

In the present article, we use the following distance functions on a Carnot group $\mathbb{G}$ identified with $\mathbb{R}^N$ through exponential coordinates (2.2).

(D_1) The standard Euclidean distance d_E which is associated with the Euclidean norm $\|x\|_E$: $d_E(x, y) = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2} = \|x - y\|_E$.

However, such a distance is neither left-invariant under group translations nor 1-homogeneous with respect to group dilations. Thus, let us introduce further distances (not Lipschitz equivalent to d_E) enjoying these properties.

Definition 2.1. Let $\mathbb{G}$ be a Carnot group. A *homogeneous norm* $\|\cdot\|$ is a continuous function $\|\cdot\| : \mathbb{G} \to [0, +\infty)$ such that

$$\begin{gathered} \|p\| = 0 \quad \text{if and only if } p = 0; \\ \|p^{-1}\| = \|p\|, \quad \|\delta_\lambda(p)\| = \lambda\|p\| \quad \text{for all } p \in \mathbb{G} \quad \text{and} \quad \lambda > 0; \\ \|p \cdot q\| \le \|p\| + \|q\| \quad \text{for all } p, q \in \mathbb{G}. \end{gathered} \tag{2.6}$$

Remark 2.2. A homogeneous norm $\|\cdot\|$ induces a homogeneous left invariant distance in $\mathbb{G}$ as follows:

$$d(p, q) := d(q^{-1} \cdot p, 0) := \|q^{-1} \cdot p\| \quad \text{for all } p, q \in \mathbb{G}. \tag{2.7}$$

(D2) On any Carnot group $\mathbb{G}$ there exists a homogeneous norm $\|\cdot\|_{\mathbb{G}}$ that is smooth away of the origin, see [43, Page 638]. It induces the distance $d_{\mathbb{G}}(x, y) := \|y^{-1} \cdot x\|_{\mathbb{G}}$ on $\mathbb{G}$.

(D3) We also use the homogeneous norm $\|\cdot\|_H$:

$$\|x\|_H = \max\{\varepsilon_1\|\mathbf{x}_1\|_E, \varepsilon_2\|\mathbf{x}_2\|_E^{1/2}, \dots, \varepsilon_l\|\mathbf{x}_l\|_E^{1/l}\},$$

where $\mathbf{x}_k = (x_{k1}, \dots, x_{k\mathbf{d}_k}) \in \mathfrak{g}_k$, $\|\cdot\|_E$ is the Euclidean norm, making the adapted basis (1) orthonormal. The suitable constants $\varepsilon_1, \dots, \varepsilon_l$ are positive, see [16, Theorem 5.1]. The induced distance is $d_H(x, y) := \|y^{-1} \cdot x\|_H$.

(D4) The Carnot-Carathéodory distance $d_{cc}(x, y)$ which is induced by an Euclidean scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}_1}$ on $\mathfrak{g}_1$, making the horizontal vector fields X_j, $j = 1, \dots, \mathbf{d}_1$, orthonormal [1,24].

The distances defined in (D_2)–(D_4) are Lipschitz equivalent, since they are invariant under the left translation on $\mathbb{G}$ and are homogeneous functions of degree 1 with respect to dilation (2.3). We denote by d_ρ any of the distances mentioned in (D_2)–(D_4). Then the above observation and [12, Corollaries 5.15.1 and 5.15.2] imply:

Proposition 2.3. *Let $\mathbb{G}$ be a Carnot group of step l. Then*

(*i*) *a set $A \subset \mathbb{G}$ is d_ρ-bounded if and only if it is d_E-bounded*;

(*ii*) *for any bounded set* $A \subset \mathbb{G}$ *there is* $C_A > 0$ *such that*

$$C_A^{-1} d_E(x, y) \le d_\rho(x, y) \le C_A d_E(x, y)^{1/l}, \quad x, y \in A;$$

(*iii*) *the topologies induced by* d_ρ *and* d_E *coincide.*

2.3 Measures on Carnot groups

The push forward of the N-dimensional Lebesgue measure $\mathcal{L}^N$ on $\mathbb{R}^N$ to the group $\mathbb{G}$ under the exponential map is the Haar measure $\mathbf{g}_{\mathbb{G}}$ on the group $\mathbb{G}$. Hence, if $E \subset \mathbb{R}^N \cong \mathbb{G}$ is measurable, then $\mathcal{L}^N(x \cdot E) = \mathcal{L}^N(E \cdot x) = \mathcal{L}^N(E)$ for all $x \in \mathbb{G}$. Moreover, if $\lambda > 0$, then $\mathcal{L}^N(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^N(E)$.

Recall the definition of the Hausdorff measure in a metric space (X, ρ). For all sets $E, E_i \subseteq X$, closed balls $B_\rho(x_i, r_i)$, real numbers $\mathbf{m} \in [0, \infty)$, and $\delta > 0$ one writes

$$\mathcal{H}^{\mathbf{m}}_{\rho,\delta}(E) := \inf\left\{\sum_i \operatorname{diam}(E_i)^{\mathbf{m}} : E \subset \bigcup_i E_i, \quad \operatorname{diam}(E_i) \le \delta\right\},$$

$$\mathcal{S}^{\mathbf{m}}_{\rho,\delta}(E) := \inf\left\{\sum_i \operatorname{diam}(E_i)^{\mathbf{m}} : E \subset \bigcup_i B_\rho(x_i, r_i), \operatorname{diam}(B_\rho(x_i, r_i)) \le \delta\right\},$$

where we assume $\operatorname{diam}(E_i)^0 = 1$ for $E_i \ne \emptyset$, and $\mathcal{H}^{\mathbf{m}}_{\rho,\delta}(\emptyset) = \mathcal{S}^{\mathbf{m}}_{\rho,\delta}(\emptyset) = 0$. For all $E \subseteq X$ and $\mathbf{m} \in [0, \infty)$ the $\mathbf{m}$ *-Hausdorff measure* $\mathcal{H}^{\mathbf{m}}_\rho(E)$ and the *spherical* $\mathbf{m}$*-Hausdorff measure* $\mathcal{S}^{\mathbf{m}}_\rho(E)$ are defined, respectively, as

$$\mathcal{H}^{\mathbf{m}}_\rho(E) = \lim_{\delta\to 0} \mathcal{H}^{\mathbf{m}}_{\rho,\delta}(E), \quad \mathcal{S}^{\mathbf{m}}_\rho(E) = \lim_{\delta\to 0} \mathcal{S}^{\mathbf{m}}_{\rho,\delta}(E).$$

Both $\mathcal{H}^{\mathbf{m}}_\rho$ and $\mathcal{S}^{\mathbf{m}}_\rho$ are Borel regular measures.

When $\mathbb{G}$ is a Carnot group considered as a metric space $(\mathbb{G}, d_\rho)$, where d_ρ is one of the distances (D_1)–(D_4), we denote by $\mathcal{H}^{\mathbf{m}}_{d_\rho}$ and $\mathcal{S}^{\mathbf{m}}_{d_\rho}$ the $\mathbf{m}$-dimensional Hausdorff and spherical Hausdorff measures associated with the distance d_ρ, respectively. The measures $\mathcal{H}^{\mathbf{m}}_{d_\rho}$ and $\mathcal{S}^{\mathbf{m}}_{d_{\rho'}}$, where d_ρ and $d_{\rho'}$ are the distance functions of types (D_2)–(D_4), satisfy

$$c\mathcal{H}^{\mathbf{m}}_{d_\rho}(E) \le \mathcal{H}^{\mathbf{m}}_{d_{\rho'}}(E) \le C\mathcal{H}^{\mathbf{m}}_{d_\rho}(E), \quad k\mathcal{H}^{\mathbf{m}}_{d_\rho}(E) \le \mathcal{S}^{\mathbf{m}}_{d_{\rho'}}(E) \le K\mathcal{H}^{\mathbf{m}}_{d_\rho}(E),$$

for some positive constants c, k, C, K and a set $E \subset \mathbb{G}$. The same is true if we interchange $\mathcal{H}^{\mathbf{m}}_{d_\rho}$ and $\mathcal{S}^{\mathbf{m}}_{d_\rho}$. Finally,

$$\tilde{c}\mathcal{H}^{Q}_{d_\rho}(E) \le \mathcal{L}^N(E) \le \tilde{C}\mathcal{H}^{Q}_{d_\rho}(E), \quad E \subset \mathbb{G}, \tag{2.8}$$

for some positive constants $\tilde{c}, \tilde{C}$, the homogeneous dimension Q, and the topological dimension N of the Carnot group.

2.4 *H*-type Lie groups

One of the core examples for the present article will be H-type Lie groups that are particular examples of two-step Carnot groups. Consider a real Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$ with the underlying vector space $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the decomposition is orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, and $\mathfrak{g}_2$ is the centre of the Lie algebra $\mathfrak{g}$. The inner product space $(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_{\mathbb{R}})$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the restriction of the scalar product to the subspace $\mathfrak{g}_2 \subset \mathfrak{g}$, generates the Clifford algebra $\mathrm{Cl}(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_{\mathbb{R}})$. The Clifford algebra $\mathrm{Cl}(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_{\mathbb{R}})$ admits a representation on the vector space $\mathfrak{g}_1$:

$$J : \mathrm{Cl}(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_{\mathbb{R}}) \to \mathrm{End}(\mathfrak{g}_1).$$

We use the notation J_z, $z \in \mathfrak{g}_2$, for the value of the map J restricted to the vector space $\mathfrak{g}_2 \subset \mathrm{Cl}(\mathfrak{g}_2, \langle\cdot,\cdot\rangle_{\mathbb{R}})$. From the definition of the Clifford algebra we have

$$J_z^2 = -\langle z, z\rangle_{\mathbb{R}}\,\mathrm{Id}_{\mathfrak{g}_1}, \quad z \in \mathfrak{g}_2. \tag{2.9}$$

The Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$ is called H-type if

$$\langle J_z u, v\rangle_{\mathbb{R}} = \langle z, [u, v]\rangle_{\mathbb{R}}, \quad z \in \mathfrak{g}_2, \quad u, v \in \mathfrak{g}_1. \tag{2.10}$$

An H-type Lie group is a connected simply connected Lie group whose Lie algebra is an H-type Lie algebra $(\mathfrak{g}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$.

2.4.1 The Heisenberg group

The nth Heisenberg group H^n is diffeomorphic to $\mathbb{R}^{2n+1}$ and is the simplest example of H-type Lie group. Its $(2n + 1)$-dimensional Lie algebra $\mathfrak{h}^n_{\mathbb{R}}$ has one-dimensional centre $\mathfrak{h}_2$. Let $\mathfrak{h}_1$ be the $2n$-dimensional orthogonal complement to $\mathfrak{h}_2$ with respect to an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}^n_{\mathbb{R}}$. We choose an orthonormal basis

$$X_1, \dots, X_n, Y_1, \dots, Y_n \quad \text{for } \mathfrak{h}_1 \quad \text{and} \quad \varepsilon \quad \text{for } \mathfrak{h}_2, \tag{2.11}$$

satisfying the commutation relations

$$[X_j, Y_i] = \delta_{ji}\varepsilon, \quad [X_j, X_i] = [Y_j, Y_i] = 0. \tag{2.12}$$

Then the map J_ε defined in (2.10) satisfies

$$J_\varepsilon^2 = -\mathrm{Id}_{\mathfrak{h}_1}, \quad J_\varepsilon(X_i) = Y_i, \quad J_\varepsilon(Y_i) = -X_i.$$

3 Module of a family of measures

We start explaining the notion of a p-module of a system of measures that Fuglede introduced in his celebrated paper [21]. Let $(X, \mathfrak{M}, m)$ be an abstract measure space with a fixed basic measure $m : \mathfrak{M} \to [0, +\infty]$ defined on a σ-algebra $\mathfrak{M}$ of subsets of X. We denote by $\mathbf{M}$ the system of all measures on X, whose domains of definition contain $\mathfrak{M}$.

With an arbitrary subset $\mathbf{E}$ of the system of measures $\mathbf{M}$ we associate a class of functions that we call *admissible* for $\mathbf{E}$ and denote by $\mathrm{Adm}(\mathbf{E})$. Namely,

$$\mathrm{Adm}(\mathbf{E}) = \left\{ f : X \to \mathbb{R} : f \text{ is } m\text{-measurable}, \quad f \geq 0, \quad \text{and} \quad \int_X f\,\mathrm{d}\mu \geq 1, \quad \text{for all } \mu \in \mathbf{E} \right\}.$$

Definition 3.1. For $0 < p < \infty$, the module $M_p(\mathbf{E})$ of a system of measures $\mathbf{E}$ is defined as

$$M_p(\mathbf{E}) = \inf_{f \in \mathrm{Adm}(\mathbf{E})} \int_X f^p\,\mathrm{d}m,$$

interpreted as $+\infty$ if $\mathrm{Adm}(\mathbf{E}) = \varnothing$.

The reader can find the fundamental properties of the p-module of measures in [21, Chapter 1].

A system of measures $\mathbf{E} \subset \mathbf{M}$ can be associated with the set where the measures are supported [31,45].

(I) Consider, for instance, *a family of rectifiable curves* $\Gamma = \{\gamma : [a_\gamma, b_\gamma] \to \mathbb{R}^n\}$ and the associated system of measures

$$\mathbf{E} = \left\{ \mathrm{Var}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) = \mathcal{H}^1_{d_E}(|\gamma|) : \gamma \in \Gamma \right\}.$$

Here $\mathrm{Var}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)$ is the total variation of the vector-valued Radon measure $\frac{\mathrm{d}\gamma}{\mathrm{d}t}$, which coincides with the Hausdorff measure $\mathcal{H}^1_{d_E}(|\gamma|)$ of the locus $|\gamma|$ of the curve $\gamma \in \Gamma$.

If we consider a subfamily $\tilde{\Gamma} = \{\tilde{\gamma} : [\tilde{a}_\gamma, \tilde{b}_\gamma] \to \mathbb{R}^n\} \subset \Gamma$ of absolutely continuous curves, then the corresponding measures can be calculated by

$$\mathrm{Var}\left(\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t}\right) = \int_{a_{\tilde{\gamma}}}^{b_{\tilde{\gamma}}} \left|\frac{\mathrm{d}\tilde{\gamma}(t)}{\mathrm{d}t}\right| \mathrm{d}t = \int_{|\tilde{\gamma}|} \mathrm{d}\mathcal{H}^1_{d_E}, \quad \tilde{\gamma} \in \tilde{\Gamma}.$$

(II) A *family of locally Lipschitz k-dimensional surfaces* in $\mathbb{R}^n$. Each surface locally is the image of an open set of $\mathbb{R}^k$ under a Lipschitz map f. In this case, a surface measure locally coincides with $\mathrm{d}\sigma = |J(f, t)|\mathrm{d}t$. Here, $J(f, t)$ is the Jacobian of f for the points $t \in \mathbb{R}^k$, where the Jacobian $J(f, t)$ is defined. The corresponding family of measures is supported on these Lipschitz k-dimensional surfaces.

(III) A *family of countable $\mathcal{H}^k_{d_E}$-rectifiable subsets* in $\mathbb{R}^n$, where we understand the rectifiability in the sense of [14]. The system of measures consists of k-dimentional Hausdorff measures $\mathcal{H}^k_{d_E}$, associated with the family of countable $\mathcal{H}^k_{d_E}$-rectifiable sets.

3.1 Exceptional families of measures

A system $\mathbf{E}_0 \subset \mathbf{M}$ is called M_p-exceptional, if $M_p(\mathbf{E}_0) = 0$. A statement concerning measures $\mu \in \mathbf{M}$ is said to hold M_p-almost everywhere if it fails to hold for an M_p-exceptional system $\mathbf{E}_0$. The question that we are interested in is to study M_p-exceptional sets of measures on Carnot groups. Let us remind that a point-set $\mathbf{E}_0 \subset X$ in a measure space (X, m) has vanishing measure: $m(\mathbf{E}_0) = 0$ if and only if there is a function $f \in L^p(X, m)$ such that $f(x) = +\infty$ for all $x \in \mathbf{E}_0$. A generalisation of this fact to a system of measures $\mathbf{E}_0 \subset \mathbf{M}$ is given by Fuglede.

Theorem 3.2. [21, Theorem 2] *A system of measures $\mathbf{E}_0 \subset \mathbf{M}$ is M_p-exceptional if and only if there exists a non-negative function $f \in L^p(X, m)$ such that*

$$\int_X f\,\mathrm{d}\mu = +\infty \quad \text{for every } \mu \in \mathbf{E}_0.$$

Remark 3.3. By Theorem 3.2, it is easy to see that a family of curves in $\mathbb{R}^n$ that are not locally rectifiable is a M_p-exceptional set for $p \geq 1$.

3.2 Exceptional family of curves on Carnot groups

Recall the definition of horizontal subbundle $H\mathbb{G} \subset T\mathbb{G}$ from Section 2.1. We say that a function $f : I \to \mathbb{G}$, $I \subset \mathbb{R}$ is d_ρ-Lipschitz continuous if it is Lipschitz continuous between metric spaces (I, d_E) and $(\mathbb{G}, d_\rho)$.

Definition 3.4. A continuous curve $\gamma : I \to \mathbb{G}, I \subset \mathbb{R}$, is called *horizontal* if it is d_E-Lipschitz continuous and the tangent vector $\dot{\gamma}(t)$ belongs to $H_{\gamma(t)}\mathbb{G}$ for almost all $t \in I$.

We mention a well-known example of an M_p-exceptional family of curves on a Carnot group.

Example 3.5. We define $\mathbf{M} := \{\mathcal{H}^1_{d_\rho} \llcorner \gamma, \gamma \in \Gamma\}$, where

$$\Gamma := \{\gamma : [0, 1] \to \mathbb{G} \quad \text{is a } d_E\text{-Lipschitz continuous curve}\},$$

and

$$\Gamma_H := \{\gamma : [0,1] \to \mathbb{G} \text{ is a horizontal curve}\}, \quad \mathbf{M}_H := \left\{\mathcal{H}^1_{d_\rho} \llcorner \gamma, \quad \gamma \in \Gamma_H\right\}.$$

We claim that $\mathbf{M} \setminus \mathbf{M}_H$ is M_p-exceptional for all $p > 0$. By Theorem 3.2, it is enough to find a nonnegative function $f \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ such that

$$\int_\gamma f \, d\mathcal{H}^1_{d_\rho} = \infty$$

for all $\gamma \in \Gamma \setminus \Gamma_H$. Without loss of generality we can assume that the loci of curves $\gamma \in \Gamma$ are contained in a compact set $K \subset \mathbb{G}$. Then the function

$$f(x) = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K, \end{cases}$$

belongs to $L^p(\mathbb{G}, \mathcal{L}^N)$. Assume now, by contrary, that there is $\gamma \in \Gamma \setminus \Gamma_H$ such that $\int_\gamma f \, d\mathcal{H}^1_{d_\rho} = \mathcal{H}^1_{d_\rho}(\gamma) < \infty$. We will show that $\gamma \in \Gamma_H$, yielding a contradiction. The proof is more or less standard, but we prefer to give complete arguments. By the property of module of a minorised family of curves, see [45, Theorem 6.4] we can assume that the curve γ is injective. Since $\gamma([0,1])$ is a closed and connected set, by [4, Theorem 4.4.8] we can write

$$\gamma([0,1]) = \gamma_0 \bigcup \left(\bigcup_{k=1}^{\infty} \gamma_k([0,1]) \right). \tag{3.1}$$

Here γ_0 is a Borel set such that $\mathcal{H}^1_{d_\rho}(\gamma_0) = 0$ and $\gamma_k : [0,1] \to \gamma([0,1])$ are d_ρ-Lipschitz continuous functions. Note also that

$$d_E(x,y) \le C d_\rho(x,y) \quad \Rightarrow \quad 0 \le \mathcal{H}^1_{d_E}(\gamma) \le \mathcal{H}^1_{d_\rho}(\gamma) \tag{3.2}$$

and that $\gamma^{-1}(\gamma_0)$ has zero Lebesgue measure in $[0,1]$. Indeed (3.2) implies that $\mathcal{H}^1_{d_E}(\gamma_0) = 0$. Thus, we can apply the area formula of [4, Theorem 3.3.1] for $A_0 := \gamma^{-1}(\gamma_0) \subset [0,1]$:

$$\mathcal{L}^1(A_0) = \int_{A_0} \left| \frac{d\gamma(s)}{ds} \right| d\mathcal{L}^1(s) = \int_{\gamma_0} \operatorname{card}\,(\gamma^{-1}(y)) d\mathcal{H}^1_{d_E}(y) = 0. \tag{3.3}$$

Since both $\mathbb{R}$ and $\mathbb{G}$ are Carnot groups, by Pansu-Rademacher theorem, all γ_k's are Pansu differentiable in a set $[0,1] \setminus A_1$ with $\mathcal{L}^1(A_1) = 0$. For any $k \in \mathbb{N}$, let us denote by $d_P\gamma_k(t)$ the Pansu differential at a point $t \in [0,1] \setminus A_1$. Set now $A := A_0 \cup A_1$. Arguing as in the proof of [19, Theorem 3.5 (2)] the Euclidean tangent space to γ at a point $\gamma(t)$, $t \in [0,1] \setminus A$, coincides with $d_P\gamma_k(\tau)(\mathbb{R})$ if k, τ are such that $\gamma(t) = \gamma_k(\tau)$. Since $d_P\gamma_k$ is a group homomorphism between $\mathbb{R}$ and $\mathbb{G}$, it maps $\mathbb{R}$ into the first (horizontal) layer of $\mathbb{G}$, so that $\dot\gamma(t) \in H_{\gamma(t)}\mathbb{G}$, yielding a contradiction.

Remark 3.6. By [4, Theorem 4.2.1] and [4, Remark 4.1.3], we can always assume that if $t \in [0,1] \setminus A$ then $|\dot\gamma(t)| = 1$. Thus, if $\gamma(t) = \gamma_k(\tau)$, and

$$\gamma_k(\tau) = \sum_{j=1}^{\mathbf{d}_1} u_j(\tau) X_{j1}(\gamma_k(\tau)),$$

then $\|u\|_{L^\infty} \le C$, where C is independent of τ. Thus, in the definition of Γ_H we can replace "horizontal" by "admissible," see [1].

3.3 Exceptional families of Radon measures on Carnot groups

Definition 3.7. If μ is a measure on a metric space (X, ρ), and $h > 0$, then the values

$$\Theta_*^h(\mu, x) = \liminf_{r\to 0} \frac{\mu(B_\rho(x, r))}{r^h}, \quad \text{and} \quad \Theta^{h,*}(\mu, x) = \limsup_{r\to 0} \frac{\mu(B_\rho(x, r))}{r^h},$$

are called the lower and upper h-density of μ at the point $x \in X$, respectively. We say that measure μ has h-density $\Theta^h(\mu, x)$ if

$$0 < \Theta_*^h(\mu, x) = \Theta^h(\mu, x) = \Theta^{h,*}(\mu, x) < \infty.$$

Lemma 3.8. *Let a Carnot group $\mathbb{G}$ of topological dimension N and homogeneous dimension Q be endowed with a distance function d_ρ of type (D_2)–(D_4). Let μ be a Radon measure on $\mathbb{G}$. If h-density $\Theta^h(\mu, x)$ is defined for $1 < h < Q$ and μ-almost every $x \in \mathbb{G}$, then $\Theta^h(\mu, \cdot) \in L^p(\mathbb{G}, \mathbf{g}_\mathbb{G})$ for any $p > 0$.*

Proof. Recall the relation $\mathbf{g}_\mathbb{G} \sim \mathcal{L}^N$. We note that the map $x \to \Theta^h(\mu, x)$ is Borel measurable, see e.g. [42, Remark 3.1] and $\Theta^h(\mu, x) > 0$ for μ-a.e. $x \in \mathbb{G}$. Let us show that

$$\mathbf{g}_\mathbb{G}(\{x \in \mathbb{G};\ \Theta^h(\mu, x) > 0\}) = 0. \tag{3.4}$$

Fix $R > 0$ and assume by contradiction that

$$\mathbf{g}_\mathbb{G}(B_{d_\rho}(e, R) \cap \{x \in \mathbb{G};\ \Theta^h(\mu, x) > 0\}) > 0.$$

We have

$$\mathbf{g}_\mathbb{G}(B_{d_\rho}(e, R) \cap \{x \in \mathbb{G};\ \Theta^h(\mu, x) > 0\}) = \lim_{k\to\infty} \mathbf{g}_\mathbb{G}\left(B_{d_\rho}(e, R) \cap \left\{x \in \mathbb{G};\ \Theta^d(\mu, x) > \frac{1}{k}\right\}\right).$$

Denote $E_k = B_{d_\rho}(e, R) \cap \{x \in \mathbb{G};\ \Theta^h(\mu, x) > \frac{1}{k}\}$. Then there exists $k \in \mathbb{N}$ such that $0 < \mathbf{g}_\mathbb{G}(E_k) < \infty$. Thus, by [4, Theorem 2.4.3],

$$\mathcal{H}^h_{d_\rho}(E_k) \le k\omega\ \mu(E_k) \le k\omega\mu\ \left(B_{d_\rho}(e, R)\right) < \infty, \tag{3.5}$$

where ω is a normalisation constant. On the other hand, the equivalence (2.8) implies that $\mathcal{H}^h_{d_\rho}(E_k) = \infty$ contradicting (3.5). Letting $R \to \infty$ we obtain (3.4). This accomplishes the proof. ◻

Example 3.9. Consider the Heisenberg group H^1 endowed with a distance function d_ρ of type (D_2)–(D_4). Let $\mathbf{M}$ be the set of all Radon measures μ on H^1 satisfying

$$\Theta^1(\mu, x) \coloneqq \lim_{r\to 0} \frac{\mu(B_{d_\rho}(x, r))}{r} > 0 \quad \text{for } \mu\text{-a.e.} x \in H^1. \tag{3.6}$$

We let $\mathbf{M}_0 \subset \mathbf{M}$ to be the measures for which there exists a countable family of Lipschitz maps $\Phi_i : A_i \to H^1$, $A_i \subset \mathbb{R}$, such that

$$\mu(H^1 \backslash \bigcup_i \Phi_i(A_i)) = 0.$$

We want to show that $\mathbf{M}\backslash\mathbf{M}_0$ is M_p-exceptional for $p > 0$. By Theorem 3.2 and Lemma 3.8, it is enough to show that

$$\int_{H^1} \Theta^1(\mu, x) \mathrm{d}\mu(x) = \infty \quad \text{if } \mu \in \mathbf{M}\backslash\mathbf{M}_0.$$

Suppose by contradiction that the above integral is finite for a given measure $\mu \in \mathbf{M}$. Then $\Theta^1(\mu, x) < \infty$ for μ-a.e. $x \in H^1$. Then applying [7, Theorem 1.3], which is an analogue of one-dimensional Preiss' theorem for H^1, we obtain that $\mu \in \mathbf{M}_0$. This is a contradiction.

Example 3.10. In this example, we refer to the definition of a tangent measure $\mathrm{Tan}_h(\mu, x)$ in [28, Chapter 14]. Consider a Carnot group $\mathbb{G}$ as a metric space $(\mathbb{G}, d_\rho)$ where d_ρ is one of the distance functions (D_2)–(D_4). For $1 < h < Q$ denote by $\mathbf{M}$ the set of all Radon measures μ on $\mathbb{G}$ such that

(i) $\Theta^h_*(\mu, x) > 0$ for μ-a.e. $x \in \mathbb{G}$;
(ii) $\mathrm{Tan}_h(\mu, x) \subset \{\lambda \mathcal{S}^h_{d_\rho} \llcorner \mathbb{V}(x)\}$, where $\mathbb{V}(x)$ is a *complemented* homogeneous subgroup in $\mathbb{G}$.

We let $\mathbf{M}_0 \subset \mathbf{M}$ to be a family of Radon measures such that there exists a countable family $\{\Gamma_i := \text{graph } (\phi_i);\ i = 1, 2, \dots\}$ of compact intrinsic Lipschitz graphs of metric dimension h, see Section 3.4.1, which are intrinsically differentiable almost everywhere, see [20], and such that

$$\mu(\mathbb{G} \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0.$$

Then we claim that $\mathbf{M}\setminus\mathbf{M}_0$ is M_p-exceptional family of measures. By the methods of Lemma 3.8, one can prove that $\Theta^{h,*}(\mu, \cdot) \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ for any $p > 0$. Thus, it is enough to show that

$$\int_{\mathbb{G}} \Theta^{h,*}(\mu, x) d\mu(x) = \infty \quad \text{if} \quad \mu \in \mathbf{M}\setminus\mathbf{M}_0$$

by Theorem 3.2. Suppose by contradiction that the above integral is finite for a measure $\mu \in \mathbf{M}\setminus\mathbf{M}_0$. Then $\Theta^{h,*}(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{G}$ and therefore [6, Theorem 1.5], see also [8], implies that $\mu \in \mathbf{M}_0$, which is a contradiction.

3.4 Families of surfaces on Carnot groups

In this section, we aim to define families of surfaces that will be of our interest.

3.4.1 Intrinsic Lipschitz surfaces on the Carnot groups

In this section, we recall the definition of *intrinsic Lipschitz graphs*, i.e. graphs of *intrinsic Lipschitz functions*, see [18,20]. Then we give a definition of an intrinsic Lipschitz surface. A subgroup $\mathbb{M}$ of a Carnot group $\mathbb{G}$ is called a homogeneous subgroup if $\mathbb{M}$ is a homogeneous group with respect to the dilation δ_λ defined in (2.3). Let us assume that $\mathbb{G}$ is decomposed into complementary homogeneous subgroups: $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, $\mathbb{M} \cap \mathbb{H} = e$, and let $\mathbf{P}_{\mathbb{M}}$ and $\mathbf{P}_{\mathbb{H}}$ be the canonical projections: $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_{\mathbb{H}} : \mathbb{G} \to \mathbb{H}$ defined by the identity $\mathbf{P}_{\mathbb{M}} q \cdot \mathbf{P}_{\mathbb{H}} q \equiv q$ for $q \in \mathbb{G}$. The projections define intrinsic cones:

$$C_{\mathbb{M},\mathbb{H}}(e, \beta) = \{p \in \mathbb{G} | \|\mathbf{P}_{\mathbb{M}} p\| \le \beta \|\mathbf{P}_{\mathbb{H}} p\|\}, \quad C_{\mathbb{M},\mathbb{H}}(q, \beta) = q \cdot C_{\mathbb{M},\mathbb{H}}(e, \beta),$$

where $\beta > 0$ is called the opening of the cone $C_{\mathbb{M},\mathbb{H}}(q, \beta)$ and q is the vertex.

Definition 3.11. The graph of a function $f : \Omega \to \mathbb{H}$, where Ω is an open set of $\mathbb{M}$, is the set

$$\text{graph } (f) = \{q \cdot f(q) \in \mathbb{G} = \mathbb{M} \cdot \mathbb{H} | \quad q \in \Omega \subset \mathbb{M}\}.$$

A function $f : \Omega \to \mathbb{H}$, $\Omega \subset \mathbb{M}$, is an *intrinsic Lipschitz function* in Ω with the Lipschitz constant $L > 0$ if

$$C_{\mathbb{M},\mathbb{H}}(p, 1/L) \cap \text{graph } (f) = \{p\} \quad \text{for all } p \in \text{graph } (f).$$

An *intrinsic Lipschitz graph* is the graph of an intrinsic Lipschitz function.

Left translations of intrinsic Lipschitz graphs are still intrinsic Lipschitz graphs. Following [20, Lemma 2.12], we set

$$c_0(\mathbb{M}, \mathbb{H}) := \inf\{\|mh\| : \|m\| + \|h\| = 1\}. \tag{3.7}$$

Remark 3.12. We emphasise that there is a subtle difference in the notions of a Lipschitz function between metric spaces and that of *intrinsic Lipschitz function* within a Carnot group. We refer the reader to [20].

Definition 3.13. The *topological dimension* $\mathbf{d_t}$ of a (sub)group is the dimension of its Lie algebra. The *metric dimension* $\mathbf{m}$ of a Borel set $U \subset \mathbb{G}$ is its Hausdorff dimension, with respect to the Hausdorff measure $\mathcal{H}_{d_\rho}$ (or $\mathcal{S}_{d_\rho}$) for a distance function d_ρ of type (D_2)–(D_4). We say that $\mathbb{M}$ is a $(\mathbf{d_t}, \mathbf{m})$-*subgroup of* $\mathbb{G}$ if $\mathbb{M}$ is a homogeneous subgroup of $\mathbb{G}$ with topological dimension $\mathbf{d_t}$ and metric dimension $\mathbf{m}$. We say that graph (f) is intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz grapf if $f : \Omega \to \mathbb{H}$, $\Omega \subset \mathbb{M}$, is an intrinsic Lipschitz function and $\mathbb{M}$ is a $(\mathbf{d_t}, \mathbf{m})$-subgroup of $\mathbb{G}$.

The metric dimension $\mathbf{m}$ of a homogeneous subgroup is an integer usually larger than its topological dimension $\mathbf{d_t}$, see [30] and coincides with the homogeneous dimension, defined in Section 2.1.

Definition 3.14. Suppose $1 \le \mathbf{d_t} \le N - 1$ and $1 \le \mathbf{m} \le Q - 1$. A non-empty subset $S \subset \mathbb{G}$ is called an intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surface (or manifold in the graph representation) in $\mathbb{G}$ if to every point $x \in S$ there correspond an open neighbourhood $U(x) \subset \mathbb{G}$, a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, and an open set $\Omega \subset \mathbb{M}$ such that

- $x \in U$;
- $\mathbb{M}$ is a $(\mathbf{d_t}, \mathbf{m})$-subgroup of $\mathbb{G}$;
- there exists an intrinsic Lipschitz map $f : \Omega \to \mathbb{H}$ such that $S \cap U = \text{graph } (f) \cap U(x)$.

3.4.2 Measures on the intrinsic Lipschitz graphs and surfaces

We suppose that $(\mathbb{G}, \mathcal{B}, \mathcal{L}^N, d_\rho)$ is a Carnot group with the Borel σ-algebra $\mathcal{B}$, the Lebesgue measure $\mathcal{L}^N$, which is identified with the Haar measure $\mathbf{g}_{\mathbb{G}}$, and a distance function d_ρ of types $(D_2 - D_4)$. We assume that $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$. The following result provides the existence of a Borel measure on an intrinsic Lipschitz graph.

Theorem 3.15. [20, Theorem 3.9] *Let S be an intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graph on a Carnot group $(\mathbb{G}, d_\rho)$. Suppose $S = \text{graph } (f)$ is defined by a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ and an intrinsic L-Lipschitz function $f : \Omega \to \mathbb{H}$ in the domain $\Omega \subset \mathbb{M}$. Then there are positive constants c_0, c depending on the decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ such that*

$$\left(\frac{c_0(\mathbb{M} \cdot \mathbb{H})}{1 + L}\right)^{\mathbf{m}} R^{\mathbf{m}} \le \mathcal{S}^{\mathbf{m}}_{d_\rho}\Big(S \cap B_{d_\rho}(x, R)\Big) \le c(\mathbb{M} \cdot \mathbb{H})(1 + L)^{d_m} R^{\mathbf{m}} \tag{3.8}$$

for all points $x \in S$ and $R > 0$. In particular, the Hausdorff dimension of S with respect to d_ρ equals the homogeneous dimension of the group $\mathbb{M}$.

Let $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ be a decomposition of $\mathbb{G}$ into complementary homogeneous subgroups. If $\Omega \subset \mathbb{M}$ is an open set and $f : \Omega \to \mathbb{H}$ is an intrinsic Lipschitz function, then one can define a map $\Phi_f : \Omega \to \mathbb{G}$ by $\Phi_f(m) = m \cdot f(m)$, $m \in \Omega$, which parametrises the intrinsic Lipschitz graph of f. We define two measures

$$\sigma_S(A) = \mathcal{S}^{\mathbf{m}}_d \llcorner \text{graph } (f)(A) = \mathcal{S}^{\mathbf{m}}_d(\text{graph } (f) \cap A) \tag{3.9}$$

and

$$\mu(A) = (((\Phi_f)_\sharp)\mathbf{g}_{\mathbb{M}})(A) = \mathbf{g}_{\mathbb{M}}(\Phi_f^{-1}(A)) = \mathbf{g}_{\mathbb{M}}(\Phi_f^{-1}(\text{graph } (f) \cap A)) \tag{3.10}$$

for any Borel measurable subset $A \subset \mathbb{G}$. Both measures σ_S and μ are concentrated on the set graph $(f) \subset \mathbb{G}$.

Theorem 3.16. *For the measures σ_S and μ defined in* (3.9) *and* (3.10) *there are positive constants C_1 and C_2 such that*

$$C_1\sigma_S(A) \le \mu(A) \le C_2\sigma_S(A) \tag{3.11}$$

for any Borel measurable set $A \subset \mathbb{G}$.

Proof. Note that if $A \subset \mathbb{G}$, then by definition

$$\mu(A) = \mathbf{g}_{\mathbb{M}}(\Phi_f^{-1}(\text{graph }(f) \cap A)) = \mathbf{g}_{\mathbb{M}}(\mathbf{P}_{\mathbb{M}}(\text{graph }(f) \cap A)). \tag{3.12}$$

Let $B_{d_\rho}(x, R) \subset \mathbb{G}$ be a ball centred at $x \in \text{graph }(f)$. Then, by [20, Formula (44)],

$$\mathbf{P}_{\mathbb{M}}\Big(B_{d_\rho}(x, cR)\Big) \subset \mathbf{P}_{\mathbb{M}}\Big(\text{graph }(f) \cap B_{d_\rho}(x, R)\Big) \subset \mathbf{P}_{\mathbb{M}}\Big(B_{d_\rho}(x, R)\Big),$$

where $c = \frac{c_0(\mathbb{M} \cdot \mathbb{H})}{1+L}$. Moreover, it was shown in [20, Lemma 2.20] that

$$\mathbf{g}_{\mathbb{M}}(\mathbf{P}_{\mathbb{M}}(B_{d_\rho}(x, R))) = c_1 R^{\mathbf{m}}, \quad c_1 = \mathbf{g}_{\mathbb{M}}\Big(B_{d_\rho}(e, 1)\Big).$$

It implies

$$c_1 c^{\mathbf{m}} R^{\mathbf{m}} \le \mathbf{g}_{\mathbb{M}}(\mathbf{P}_{\mathbb{M}}(\text{graph }(f) \cap B_{d_\rho}(x, R))) = \mu\Big(B_{d_\rho}(x, R)\Big) \le c_1 R^{\mathbf{m}} \tag{3.13}$$

by (3.12) and the definition of the measure μ. Passing to the upper and lower limits in (3.13) we obtain

$$c_1 c^{\mathbf{m}} \le \liminf_{R \to 0} \frac{\mu(B_{d_\rho}(x, R))}{R^{\mathbf{m}}} \le \limsup_{R \to 0} \frac{\mu(B_{d_\rho}(x, R))}{R^{\mathbf{m}}} \le c_1. \tag{3.14}$$

Therefore, arguing as in [14, Section 2.10.19], we can write (3.13) as

$$\tilde{c}_1 c^{\mathbf{m}} \le \Theta_*^{\mathbf{m}}(\mu, x) \le \Theta^{\mathbf{m},*}(\mu, x) \le \tilde{c}_1 \tag{3.15}$$

for any $x \in \text{graph }(f)$. Again by [14, Section 2.10.19] it follows that

$$\tilde{c}_1 c^{\mathbf{m}} \mathcal{S}_{d_\rho}^{\mathbf{m}}(U) \le \mu(U) \le \tilde{c}_1 2^{\mathbf{m}} \mathcal{S}_{d_\rho}^{\mathbf{m}}(U) \tag{3.16}$$

for any Borel measurable set $U \subset \mathbb{G}$. As a set U we take $U := A \cap \text{graph }(f)$ for a Borel set $A \subset \mathbb{G}$. By (3.16) we obtain

$$C_1 \sigma_S(A) = C_1 \mathcal{S}_{d_\rho}^{\mathbf{m}}(U) \le \mu(U) = \mu(A) \le C_2 \mathcal{S}_{d_\rho}^{\mathbf{m}}(U) = C_2 \sigma_S(A),$$

where $C_1 := \tilde{c}_1 c^{\mathbf{m}}$ and $C_2 := \tilde{c}_1 2^{\mathbf{m}}$. □

Corollary 3.17. *Let $S = \text{graph }(f)$ be an intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graph in a Carnot group $(\mathbb{G}, d_\rho)$, and let $\Phi_f : \Omega \to \mathbb{G}$ be the map defined by $\Phi_f(m) = m \cdot f(m)$, which parametrises S. For the measure σ_S defined above and any Borel non-negative function h on $\mathbb{G}$ one has*

$$C_1 \int_{\mathbb{G}} h(y) \mathrm{d}\sigma_S(y) \le \int_{\Omega} (h \circ \Phi_f)(x) \mathrm{d}\mathbf{g}_{\mathbb{M}}(x) \le C_2 \int_{\mathbb{G}} h(y) \mathrm{d}\sigma_S(y). \tag{3.17}$$

Proof. We recall a result from [28, Theorem 1.19]. Let X and Y be two separable metric spaces, $\Phi : X \to Y$ a Borel map, ν a Borel measure on X, and h is a Borel non-negative function on Y. Then

$$\int_Y h(y) d(\Phi_\sharp \nu)(y) = \int_X (h \circ \Phi)(x) \mathrm{d}\nu(x).$$

We apply the result for the surface locally parametrised by an intrinsic Lipschitz graph of f, taking $X = \Omega \subset \mathbb{M}$, $Y = \mathbb{G}$, $\Phi = \Phi_f$, $\nu = \mathbf{g}_{\mathbb{M}}$. Then (3.17) follows. □

Remark 3.18. We stress that if S is an intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surface, then according to Definition 3.14, the constant $c_0(\mathbb{M}, \mathbb{H})$ in (3.17) depends on S and therefore we can write $c_0(\mathbb{M}, \mathbb{H}) =: c_0(S)$ when we consider this constant on a surface S.

If in Corollary 3.17 we track the definition of the constants C_1, C_2, then we see that they depend (up to geometric constants) only on the decomposition $\mathbb{M} \cdot \mathbb{H}$ and the Lipschitz constant L. So, we can say that C_1, C_2 depend on S and we write $C_1(S)$, $C_2(S)$.

3.4.3 Examples of families of surfaces on Carnot groups

Example 3.19. $(1,1)$-*intrinsic Lipschitz surfaces*. Let $\mathbb{M}_X = \exp(tX)$, $t \in \mathbb{R}$, $X \in \mathfrak{g}_1$ be a one-dimensional commutative subgroup and $\mathbb{H}_X$ a complementary to $\mathbb{M}_X$ subgroup. Let $\phi_X : \mathbb{M}_X \to \mathbb{H}_X$ be a Lipschitz map. The family

$$\Phi = \{\phi_X : \mathbb{M}_X \to \mathbb{H}_X : X \in \mathfrak{g}_1\}$$

is a family of $(1,1)$-intrinsic Lipschitz surfaces, that is a family of horizontal curves.

Example 3.20. *A family of parametrised intrinsic Lipschitz graphs*. Let $\mathbb{M}$ and $\mathbb{H}$ be complementary subgroups and $f : \Omega \to \mathbb{H}$, $\Omega \subset \mathbb{M}$, be an intrinsic Lipschitz function with $S = \text{graph}\,(f)$. We define

$$f_\lambda : \delta_\lambda \Omega \to \mathbb{H} : f_\lambda(m) := \delta_\lambda f(\delta_{1/\lambda} m).$$

Then $S_\lambda = \text{graph}\,(f_\lambda)$ is a family of intrinsic Lipschitz graphs, parametrised by $\lambda > 0$ and it coincides with $\delta_\lambda S$.

We also consider

$$f_q : \Omega_q \to \mathbb{H}, \quad \Omega_q = \{m \in \mathbb{M} : \mathbf{P}_\mathbb{M}(q^{-1}m) \in \Omega\}, \quad q \in E \subseteq \mathbb{G}$$

defined by

$$f_q(m) = (\mathbf{P}_\mathbb{H}(q^{-1}m))^{-1} \cdot f(\mathbf{P}_\mathbb{M}(q^{-1}m)).$$

Then

$$S_q = \text{graph}\,(f_q) = \{(m \cdot f_q(m)) : m \in \Omega_q, q \in E \subset \mathbb{G}, f_q : \Omega_q \to \mathbb{H}\}$$

is a family of intrinsic Lipschitz graphs parametrised by $q \in E \subset \mathbb{G}$ and it coincides with $q \cdot S = L_q(S)$. The details about the properties of S_λ and S_q are given in [20, Theorem 3.2].

Particularly, if $q \in E = \mathbb{H}$, then $\mathbf{P}_\mathbb{M}(q^{-1}m) = m$, $\mathbf{P}_\mathbb{H}(q^{-1}m) = q^{-1}$. It implies $\Omega_q = \Omega$, and the family $f_q(m) = q \cdot f(m)$ is a family of graphs shifted along the subgroup $\mathbb{H}$.

Example 3.21. Let $\mathcal{F} \in \text{Aut}(\mathbb{G})$ be a grading preserving automorphism. Let $f : \mathbb{M} \to \mathbb{H}$ be an (intrinsic) Lipschitz function with $S = \text{graph}\,(f)$. We define $f_\mathcal{F} : \mathcal{F}(\mathbb{M}) \to \mathcal{F}(\mathbb{H})$ by $f_\mathcal{F}(m) = (\mathcal{F} f \mathcal{F}^{-1})(m)$. Then

$$S_\mathcal{F} = \text{graph}\,(f_\mathcal{F}) = \{(m \cdot f_\mathcal{F}(m)) : m \in \mathcal{F}(\mathbb{M})\}$$

is a family of (intrinsic) Lipschitz graphs and it coincides with $\mathcal{F}(S)$. Indeed, let $(m \cdot f(m)) \in S$, then

$$\mathcal{F}(S) \ni \mathcal{F}(m \cdot f(m)) = (\mathcal{F}m \cdot \mathcal{F}f(m)) = (\mathcal{F}m \cdot \mathcal{F}f(\mathcal{F}^{-1}\mathcal{F}m)) = (m' \cdot \mathcal{F}f(\mathcal{F}^{-1}m')) = (m' \cdot f_\mathcal{F}(m')) \in S_\mathcal{F}.$$

If $\mathcal{F} \in \text{Aut}(\mathbb{G})$ preserves the homogeneous norm, then an intrinsic L-Lipschitz graph is transformed to an intrinsic L-Lipschitz graph.

3.5 Exceptional families of intrinsic Lipschitz surfaces

3.5.1 Exceptional families for $0 < p < 1$

Denote by $\Sigma^{(\mathbf{d_t},\mathbf{m})}$ a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surfaces in $\mathbb{G}$. With each surface $S \in \Sigma^{(\mathbf{d_t},\mathbf{m})}$ we associate a measure σ_S as in Section 3.4.2. Let $\Sigma \subset \Sigma^{(\mathbf{d_t},\mathbf{m})}$ be a subfamily and $\mathbf{E}$ the system of measures σ_S, $S \in \Sigma$, associated with Σ. Then $M_p(\Sigma) = M_p(\mathbf{E})$ denotes the p-module of the family of measures $\mathbf{E}$ as well as the family of the surfaces Σ.

Note that in [21, page 187] it was shown that if $0 < p < 1$, then any system of Lipschitz k-dimensional surfaces Σ which intersects the cube

$$\mathrm{Cube}_a = \{x \in \mathbb{R}^n | |x_l| < a, \quad l = 1, \dots, n\}$$

has vanishing p-module for any $a > 0$. It leads to the fact that $M_p(\Sigma) = 0$, $p \in (0, 1)$, by the monotonicity of p-module. We will study an analogue situation on the Carnot groups.

Suppose $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ is a decomposition of $\mathbb{G}$ and denote by $\mathfrak{m}$ and $\mathfrak{g}$ the Lie algebras of the Lie groups $\mathbb{M}$ and $\mathbb{G}$, respectively. Let us fix a weak Malcev basis $\{W_1^{\mathbb{M}}, \dots, W_{\mathbf{d_t}}^{\mathbb{M}}, W_{\mathbf{d_t}+1}, \dots, W_N\}$ for $\mathfrak{g}$ through $\mathfrak{m}$, see [13, Theorem 1.1.13]. We define the following coordinate maps

$$\begin{aligned}
T_{\mathbb{M}} : \mathbb{R}^{\mathbf{d_t}} \to \mathbb{M} : \quad & \zeta = \left(\zeta_1, \dots, \zeta_{\mathbf{d_t}}\right) \mapsto \exp(\zeta_1 W_1^{\mathbb{M}}) \cdot \ldots \cdot \exp\left(\zeta_{\mathbf{d_t}} W_{\mathbf{d_t}}^{\mathbb{M}}\right),\\
T_{\mathbb{G}} : \mathbb{R}^{N-\mathbf{d_t}} \to \mathbb{G} : \quad & s = \left(s_{\mathbf{d_t}+1}, \dots, s_N\right) \mapsto \exp\left(s_{\mathbf{d_t}+1} W_{\mathbf{d_t}+1}\right) \cdot \ldots \cdot \exp(s_N W_N),\\
T : \mathbb{R}^{N-\mathbf{d_t}} \to \mathbb{M} \backslash \mathbb{G} : s = & \left(s_{\mathbf{d_t}+1}, \dots, s_N\right) \mapsto \mathbb{M} \cdot \exp\left(s_{\mathbf{d_t}+1} W_{\mathbf{d_t}+1}\right) \cdot \ldots \cdot \exp(s_N W_N).
\end{aligned}$$

We define the natural projection on the right coset space by

$$\pi : \mathbb{G} \to \mathbb{M} \backslash \mathbb{G} : \quad g \mapsto \mathbb{M} \cdot g. \tag{3.18}$$

The group $\mathbb{G}$ acts on the right on the coset space $\mathbb{M} \backslash \mathbb{G}$ by

$$(\mathbb{M} \backslash \mathbb{G}) \times \mathbb{G} \to \mathbb{M} \backslash \mathbb{G} : (\mathbb{M} \cdot g, \tilde{g}) \mapsto \mathbb{M} \cdot g\tilde{g}.$$

Since $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, the coset space is "parametrised" by the elements of $\mathbb{H}$ in the following sense $\mathbb{M} \cdot g = \mathbb{M} \cdot mh = \mathbb{M} \cdot h$. Moreover, $\mathbb{M} \cdot g\tilde{g} = \mathbb{M} \cdot mh\tilde{m}\tilde{h} = \mathbb{M} \cdot \tilde{\tilde{h}}$, where $\tilde{\tilde{h}}$ is not necessarily $h\tilde{h}$. In the following proposition, we formulate a Fubini-type theorem related to the right quotient space $\mathbb{M} \backslash \mathbb{G}$.

Proposition 3.22. [13, Lemma 1.2.13][37, Theorem 15.24] *The map $T = \pi \circ T_{\mathbb{G}}$ is a diffeomorphism and the push forward measure $\mathbf{g}_{\mathbb{M} \backslash \mathbb{G}} = T_{\#}(\mathcal{L}^{N-\mathbf{d_t}})$ is a right $\mathbb{G}$-invariant measure on the coset space $\mathbb{M} \backslash \mathbb{G}$,* [13, Theorem 1.2.12]. *Moreover, the right invariant measure $\mathbf{g}_{\mathbb{G}}$ on $\mathbb{G}$ and the measure $\mathbf{g}_{\mathbb{M} \backslash \mathbb{G}}$ on $\mathbb{M} \backslash \mathbb{G}$ are related by*

$$\int_{\mathbb{G}} \varkappa(g) d\mathbf{g}_{\mathbb{G}} = \int_{\mathbb{M} \backslash \mathbb{G}} d\mathbf{g}_{\mathbb{M} \backslash \mathbb{G}} \int_{\mathbb{M}} \varkappa(mg) d\mathbf{g}_{\mathbb{M}} \tag{3.19}$$

for any continuous function $\varkappa$ with a compact support.

By making use of the Vitali covering lemma, we consider a countable family of Euclidean balls $\{B(\xi_j, r) \subset \mathbb{R}^{\mathbf{d_t}}, j \in \mathbb{N}\}$ such that

$$\begin{aligned}
&\bullet \{B(\xi_j, r), j \in \mathbb{N}\} \text{ is an open covering of } \mathbb{R}^{\mathbf{d_t}};\\
&\bullet \text{ the balls } B(\xi_j, r/5) \text{ are disjoint}.
\end{aligned} \tag{3.20}$$

In particular, if a finite family of balls $\mathfrak{B} = \{B(\xi_{j_1}, 3r), \dots, B(\xi_{j_K}, 3r)\}$ has nonempty intersection, then $\#\mathfrak{B} \le 30^{\mathbf{d_t}}$. Indeed, suppose for the sake of simplicity that $\mathfrak{B} = \{B(\xi_1, 3r), \dots, B(\xi_K, 3r)\}$ are such that $\zeta_0 \in \cap_{i=1}^K B(\xi_i, 3r)$. By triangle inequality $\cup_{i=1}^K B(\xi_i, 3r) \subset B(\xi_1, 6r)$. Since $\{B(\xi_1, r/5), \dots, B(\xi_K, r/5)\}$ is a disjoint family in $B(\xi_1, 6r)$ we obtain

$$\mathrm{vol(B(0,1))}\frac{r_{\mathbf{t}}^{\mathbf{d}}}{5^{\mathbf{d_t}}} \cdot (\#\mathfrak{B}) = \left| \bigcup_{i=1}^K B(\xi_i, r/5) \right| \le |B(\xi_1, 6r)| = (6r)^{\mathbf{d_t}} \mathrm{vol}(B(0, 1)).$$

Let now $\psi \in C_0^\infty(B(0, 3r))$ be a cutoff function of the ball $B(0, 2r)$ and set

$$\phi(\zeta) := \phi_r(\zeta) = \sum_i 2^{-i} |\xi_i - \zeta|^{-1} \psi(\xi_i - \zeta), \tag{3.21}$$

where ξ_j are centres of the balls in family (3.20).

Lemma 3.23. *For any $r > 0$ and $0 < 1 < p$ we have $\phi_r \in L^p(\mathbb{R}^{\mathbf{d_t}})$.*

Proof. If ζ is fixed, then $\psi(\xi_i - \zeta) > 0$ if and only if $\zeta \in B(\xi_i, 3r)$, which is possible for at most $30^{\mathbf{d_t}}$ values of i. Thus, if $0 < p < 1$, then

$$|\phi_r(\zeta)|^p \leq \left(\sum_i 2^{-i}|\xi_i - \zeta|^{-1}\psi(\xi_i - \zeta)\right)^p \leq C_p \sum_i 2^{-ip}|\xi_i - \zeta|^{-p}\psi^p(\xi_i - \zeta),$$

so that

$$\begin{aligned}\int\limits_{\mathbb{R}^{\mathbf{d_t}}} \phi_r(\zeta)^p &\leq C_p \sum_i 2^{-ip} \int\limits_{\mathbb{R}^{\mathbf{d_t}}} |\xi_i - \zeta|^{-p}\psi^p(\xi_i - \zeta)\mathrm{d}\zeta \\ &\leq C_p \sum_i 2^{-ip} \int\limits_{B(\xi_i,3r)} |\xi_i - \zeta|^{-p}\mathrm{d}\zeta = C_p \sum_i 2^{-ip} \int\limits_{B(0,3r)} |\zeta|^{-p}\mathrm{d}\zeta < \infty. \end{aligned}$$

□

Lemma 3.24. *Let us fix the Vitali covering as in* (3.20) *by balls of a radius r. Then for any $x \in B(\xi_i, r)$ the function ϕ_r defined in* (3.21) *satisfies*

$$\int\limits_{B(x,r)} \phi_r(\zeta)\mathrm{d}\zeta = \infty. \tag{3.22}$$

Proof. We have $B(x, r) \subset B(\xi_i, 2r)$ by the triangle inequality and $x \in B(\xi_i, r)$. Then for $\zeta \in B(x, r)$ we obtain

$$\phi_r(\zeta) \geq 2^{-i}|\xi_i - \zeta|^{-1}\psi(\xi_i - \zeta) = 2^{-i}|\xi_i - \zeta|^{-1}.$$

It implies

$$\int\limits_{B(x,r)} \phi_r(\zeta)\mathrm{d}\zeta \geq \int\limits_{B(x,r)} 2^{-i}|\xi_i - \zeta|^{-1}\mathrm{d}\zeta \geq \int\limits_{B(\xi_i,\tilde{r})} 2^{-i}|\xi_i - \zeta|^{-1}\mathrm{d}\zeta = \infty,$$

for some $\tilde{r} \in (0, r)$. □

A Carnot group $\mathbb{G}$ can admit various decompositions $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ into homogeneous non-isomorphic subgroups of the same topological $\mathbf{d_t}$ and Hausdorff $\mathbf{m}$ dimensions. The full description of decompositions $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ into homogeneous non-isomorphic subgroups is far from being well understood. The difficulty of a classification of decompositions originates in the classification of nilpotent Lie algebras. In dimension 6, there are only finitely many non-isomorphic Carnot algebras, whereas in dimension 7 there are uncountably many non-isomorphic Carnot algebras, see [23]. The latter result allows us to construct 8-dimensional Carnot algebra, which contains uncountably many non-isomorphic Carnot sub-algebras of dimension 7, see [5]. Moreover, this 7-dimensional homogeneous sub-algebras are complemented by 1-dimensional Abelian homogeneous sub-algebras. That is why, we restrict ourself to the following cases. The first one when there are finitely many decompositions $\mathbb{G}_\alpha = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$, $\alpha = 1, 2, \dots, l$ into non-isomorphic pairs $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$. The second case when each pair $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$ belongs to an orbit of the action of grading preserving isometries of $\mathbb{G}$. In this case, we have decomposition into finitely many pairs, such that each pair is a representative in the equivalence class.

It is enough to consider surfaces belonging to an open bounded set $\mathcal{U} \subset \mathbb{G}$, for instance a ball. Let Σ be a system of Lipschitz surfaces with some specific property, which will be specified in theorems below, and let

$$\mathbf{E} \coloneqq \left\{\sigma_S = \mathcal{S}^{\mathbf{m}}_{d_\rho} \llcorner S;\ S \in \Sigma\right\}$$

be the system of the associated measures in $(\mathbb{G}, d_\rho)$. Then we denote by $\Sigma_{\mathcal{U}}$ the system of the Lipschitz surfaces $S \cap \mathcal{U}$, $S \in \Sigma$, and by $\mathbf{E}_{\mathcal{U}}$ the family of associated measures. If we show that $M_p(\mathbf{E}_{\mathcal{U}}) = 0$, then it will imply that the system of measures $\mathbf{E}$ is exceptional by [21, Theorem 3 (b)].

We start from the family of a most simple nature, that is a family of graphs parametrised over a single decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$. Then we consider multiple decompositions and more complicate families of surfaces.

Theorem 3.25. *Let $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ be a decomposition of $\mathbb{G}$, and let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graphs over $\mathbb{M}$ and $\Sigma_{\mathcal{U}}$ the family of $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graphs in a bounded open set $\mathcal{U} \subset \mathbb{G}$. Then the system $\mathbf{E}_{\mathcal{U}}$ is M_p-exceptional for $p \in (0, 1)$.*

Proof. By definition, for any $S \in \Sigma_{\mathcal{U}}$ there exists an intrinsic Lipschitz function $f_S : \Omega_S \to \mathbb{H}$, such that

- Ω_S is an open subset of $\mathbb{M}$;
- $S = \operatorname{graph}(f_S)$, i.e. $S = \Phi_S(\Omega_S)$, where $\Phi_S(m) = m \cdot f_S(m)$, $m \in \Omega_S$;

If $R, N > 0$ we denote by $\Sigma_{\mathcal{U}}(R, N) \subset \Sigma_{\mathcal{U}}$ the family of graphs such that

- the open set $T_{\mathbb{M}}^{-1}\Phi_S^{-1}(\mathcal{U})$ contains an Euclidean ball $B(\zeta_S, R) \subset \mathbb{R}^{\mathbf{d_t}}$;
- the associated measures $\sigma_S = \mathcal{S}^{\mathbf{m}}_{d_\rho} \llcorner S$, according to (3.17), satisfy

$$\int\limits_{\Phi_S^{-1}(\mathcal{U})} (h \circ \Phi_S)(x) \mathrm{d}\mathbf{g}_{\mathbb{M}}(x) \le N \int\limits_{S \cap \mathcal{U}} h(y) \mathrm{d}\sigma_S(y).$$

We fix R and N and denote $\mathbf{E}_{\mathcal{U}}(R, N)$ the family of associated measures to $\Sigma_{\mathcal{U}}(R, N)$. To show that $M_p(\mathbf{E}_{\mathcal{U}}(R, N)) = 0$ it is enough to find $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ such that $\int_{\mathbb{G}} F \mathrm{d}\sigma_S = \infty$ for all $\sigma_S \in \mathbf{E}_{\mathcal{U}}(R, N)$, see Theorem 3.2. We set

$$F = \phi_R \circ T_{\mathbb{M}}^{-1} \circ \Pi_{\mathbb{M}} : \mathbb{G} \to [0, \infty], \tag{3.23}$$

where $\Pi_{\mathbb{M}}$ is the projection over $\mathbb{M}$ associated with $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, as in [20, Formula (28)], and ϕ_R is the function defined in (3.21) for $r = R$.

Step 1: we claim that $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$. Since $\mathcal{U} \subset K \subset \mathbb{G}$ for some compact set K, it is enough to show that $F_K = \chi_K F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$. If we put $\tilde{K} := \pi(K)$, then $\tilde{K} \subset \mathbb{M}/\mathbb{G}$ is compact by the continuity of π and $\chi_K(g) \le \chi_{\tilde{K}}(\pi(g))$, since, if $g \in K$, then $\pi(g) \in \tilde{K}$.

We apply (3.19) with $\varkappa = F_K^p$ and obtain

$$\begin{aligned}
\int\limits_{\mathbb{G}} |F_K|^p(g) \mathrm{d}\mathbf{g}_{\mathbb{G}}(g) &= \int\limits_{\mathbb{M}\setminus\mathbb{G}} \mathrm{d}\mathbf{g}_{\mathbb{M}\setminus\mathbb{G}} \int\limits_{\mathbb{M}} |F_K|^p(mg) \mathrm{d}\mathbf{g}_{\mathbb{M}}(m) \\
&= \int\limits_{\mathbb{M}\setminus\mathbb{G}} \mathrm{d}\mathbf{g}_{\mathbb{M}\setminus\mathbb{G}} \int\limits_{\mathbb{M}} \chi_K(mg) |\phi_R \circ T_{\mathbb{M}}^{-1} \circ \Pi_{\mathbb{M}}(m)|^p \mathrm{d}\mathbf{g}_{\mathbb{M}}(m) \\
&\le \int\limits_{\mathbb{M}\setminus\mathbb{G}} \chi_{\tilde{K}}(\pi(g)) \mathrm{d}\mathbf{g}_{\mathbb{M}\setminus\mathbb{G}} \int\limits_{\mathbb{M}} |\phi_R \circ T_{\mathbb{M}}^{-1}(m)|^p \mathrm{d}\mathbf{g}_{\mathbb{M}}(m) \\
&= c(\tilde{K}) \int\limits_{\mathbb{M}} |\phi_R \circ T_{\mathbb{M}}^{-1}(m)|^p \mathrm{d}\mathbf{g}_{\mathbb{M}}(m) \\
&= c(\tilde{K}) \int\limits_{\mathbb{R}^{\mathbf{d_t}}} |\phi_R|^p(\zeta) \mathrm{d}\mathcal{L}^{\mathbf{d_t}}(\zeta) < \infty.
\end{aligned}$$

Step 2: we claim that $\int_S F \mathrm{d}\sigma_S = \infty$ **for any** $\sigma_S \in \mathbf{E}_{\mathcal{U}}(R, N)$. Assume that $S \in \Sigma_{\mathcal{U}}(R, N)$ is the graph of a Lipschitz function $f_S : \Omega_S \to \mathbb{H}$. We denote by $\Phi_S(m) := m \cdot f_S(m)$, $m \in \Omega_S$, the parametrisation of S associated with f_S. We stress that $(\Pi_{\mathbb{M}} \circ \Phi_S)(m) = m$. By Corollary 3.17 with $N = C_2$, and (3.23) we have:

$$\begin{aligned}
N \int\limits_{S \cap \mathcal{U}} F \mathrm{d}\sigma_S &\ge \int\limits_{\Phi_S^{-1}(\mathcal{U})} F \circ \Phi_S(m) \mathrm{d}\mathbf{g}_{\mathbb{M}} \\
&= \int\limits_{\Phi_S^{-1}(\mathcal{U})} \phi_R \circ T_{\mathbb{M}}^{-1} \circ \Pi_{\mathbb{M}} \circ \Phi_S(m) \mathrm{d}\mathbf{g}_{\mathbb{M}} \\
&= \int\limits_{\Phi_S^{-1}(\mathcal{U})} \phi_R \circ T_{\mathbb{M}}^{-1}(m) \mathrm{d}\mathbf{g}_{\mathbb{M}} \\
&= \int\limits_{T_{\mathbb{M}}^{-1}\Phi_S^{-1}(\mathcal{U})} \phi_R(\zeta) \mathrm{d}\zeta \\
&\ge \int\limits_{B(\zeta_S, R)} \phi_R(\zeta) \mathrm{d}\zeta.
\end{aligned}$$

By the Vitali covering lemma, there exists ξ_i such that $\zeta_S \in B(\xi_i, R)$. Thus, we apply Lemma 3.24 to show that the integral on the right hand diverges.

Step 3: we claim that $M_p(\Sigma_{\mathcal{U}}) = 0$. From Steps 1 and 2 we conclude that $M_p(\Sigma_{\mathcal{U}}(R, N)) = 0$ for any $R, N > 0$. We set $R = \frac{1}{M}$ for $M \in \mathbb{N}$. Then

$$\Sigma_{\mathcal{U}} = \bigcup_{N\in\mathbb{N}} \bigcup_{M\in\mathbb{N}} \Sigma_{\mathcal{U}}\Big(\frac{1}{M}, N\Big).$$

We conclude by [21, Theorem 3 (b)] that $M_p(\Sigma_{\mathcal{U}}) = 0$. □

Corollary 3.26. *Let $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ be a decomposition of $\mathbb{G}$ and let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surfaces, such that locally each surface is represented by an intrinsic Lipschitz graph over $\mathbb{M}$. Let $\Sigma_{\mathcal{U}}$ be the family of Lipschitz surfaces in a bounded open set $\mathcal{U} \subset \mathbb{G}$. Then the system $\mathbf{E}_{\mathcal{U}}$ is M_p-exceptional for $p \in (0, 1)$.*

In the next step, we assume that the family of graphs is parametrised over a decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ that belongs to the orbit of a subgroup $K \subset \mathrm{Iso}(\mathbb{G})$ preserving the decomposition. Here we denote by $\mathrm{Iso}(\mathbb{G})$ the group of grading preserving isometries of $\mathbb{G}$.

Theorem 3.27. *Let $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, and K a subgroup of the group of isometries* $\mathrm{Iso}(\mathbb{G})$ *preserving the decomposition. Let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graphs over the orbit $K(\mathbb{M})$ and $\Sigma_{\mathcal{U}} = \{S \cap \mathcal{U} : S \in \Sigma\}$. Then the system of measures $\mathbf{E}_{\mathcal{U}}$ is M_p-exceptional for $p \in (0, 1)$.*

Proof. By definition, for any $S \in \Sigma_{\mathcal{U}}$ there exists an intrinsic Lipschitz function $\hat{f}_S : \hat{\Omega}_S \to \hat{\mathbb{H}}$, where $\hat{\Omega}_S$ is an open set in the group $\hat{\mathbb{M}}$ such that $\mathbb{M} \cdot \mathbb{H} \in K(\hat{\mathbb{M}} \cdot \hat{\mathbb{H}})$. Then there is an isometric diffeomorphism $k \in K$ such that $\mathbb{M} \cdot \mathbb{H} = k(\hat{\mathbb{M}} \cdot \hat{\mathbb{H}})$. We write

$$f_S = k \circ \hat{f}_S \circ k^{-1} : \Omega_S \to \mathbb{H},$$

where Ω_S is an open subset in $\mathbb{M}$, such that $k(\hat{\Omega}_S) = \Omega_S$. Thus, for any $S \in \Sigma_{\mathcal{U}}$ there exists an intrinsic Lipschitz function $f_S : \Omega_S \to \mathbb{H}$, such that

- Ω_S is an open subset of $\mathbb{M}$;
- $S = \mathrm{graph}\,(f_S)$, i.e. $S = \Phi_S(\Omega_S)$, where

$$\Phi_S(m) = k(\hat{m} \cdot \hat{f}_S(\hat{m})) = k(\hat{m}) \cdot (k \circ \hat{f}_S \circ k^{-1}(k(\hat{m}))), \quad m \in \Omega_S.$$

We define $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ as in (3.23) and argue as in Theorem 3.25. □

Corollary 3.28. *Let $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ and $K \subset \mathrm{Iso}(\mathbb{G})$. Let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surfaces, such that locally each surface is represented by an intrinsic Lipschitz graph over an element of the orbit $K(\mathbb{M})$ as in Theorem* 3.27. *Then $\mathbf{E}_{\mathcal{U}}$ is M_p-exceptional for $p \in (0, 1)$.*

Now we assume that the group $\mathbb{G}$ can be written as $\mathbb{G} = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$ for finitely many $\alpha = 1, 2, \ldots, l$ and $\mathbb{M}_\alpha$ being $(\mathbf{d_t}, \mathbf{m})$-homogeneous non-isomorphic subgroups for all α. Under this assumption we state the following result.

Theorem 3.29. *Let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graphs where each graph is parametrised over one of the decompositions $\mathbb{G} = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$, $\alpha = 1, 2, \ldots, l$. Let $\Sigma_{\mathcal{U}} = S \cap \mathcal{U}$, $S \in \Sigma$, and $\mathcal{U} \subset \mathbb{G}$ be an open set. Then $M_p(\mathbf{E}_{\mathcal{U}}) = 0$ for $p \in (0, 1)$.*

Proof. We use the notation $\mathbb{G}_\alpha = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$, $\alpha = 1, 2, \ldots, l$ and $\mathbb{G}^l = \mathbb{G}_1 \times \ldots \times \mathbb{G}_l$. We define the selection map $\chi_\alpha(g) = m_\alpha h_\alpha$, $m_\alpha \in \mathbb{M}_\alpha$, $h_\alpha \in \mathbb{H}_\alpha$. It can be considered as a composition of the map

$$P : \mathbb{G} \to \mathbb{G}^l = \mathbb{G}_1 \times \ldots \times \mathbb{G}_l$$
$$g \mapsto (m_1 h_1, \ldots, m_l h_l)$$

followed by the projection on α-slot. Then we define

$$F(g) = \sum_{\alpha=1}^{l} \phi_{r_\alpha} \circ T^{-1}_{\mathbb{M}_\alpha} \circ \Pi_{\mathbb{M}_\alpha} \circ \chi_\alpha(g). \tag{3.24}$$

Then as in Theorem 3.25 we show that $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$. Moreover, if $S \in \Sigma_{\mathcal{U}}(R, N)$ is the graph of a Lipschitz function $f_S : (\Omega_\alpha)_S \to \mathbb{H}_\alpha$, for some $\alpha = 1, \ldots, l$, where $(\Omega_\alpha)_S$ is an open set in $\mathbb{M}_\alpha$, then $\Phi_S(m_\alpha) := m_\alpha \cdot f_S(m_\alpha)$ the parametrisation of S associated with f_S. Then

$$N \int_{S \cap \mathcal{U}} F \mathrm{d}\sigma_S \geq \int_{\Phi_S^{-1}(\mathcal{U})} F \circ \Phi_S(m_\alpha) \mathrm{d}\mathbf{g}_{\mathbb{M}_\alpha} \geq \int_{B((\bar{\zeta}_\alpha)_S, r_\alpha)} \phi_{r_\alpha}(\bar{\zeta}_\alpha) \mathrm{d}\bar{\zeta}_\alpha = \infty,$$

as in the proof of Theorem 3.25. □

Corollary 3.30. *Let us assume that $\mathbb{G}$ can be decomposed in finitely many ways $\mathbb{G} = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$, $\alpha = 1, 2, \ldots, l$, with non-isomorphic subgroups $\mathbb{M}_\alpha$.*

Let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surfaces, such that locally each surface is represented by an intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graph over one of the groups $\mathbb{M}_\alpha$ as in Theorem 3.29. Then $M_p(\mathbf{E}_{\mathcal{U}}) = 0$ for $p \in (0, 1)$.

The last result in this section is a combination of Theorems 3.27 and 3.29.

Theorem 3.31. *Let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz surfaces where each graph is parametrised over one of the decompositions $\mathbb{G} = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$, $\alpha = 1, 2, \ldots, l$. Moreover, any term $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$ is an element in the orbit of a subgroup $K_\alpha \subset \mathrm{Iso}(\mathbb{G})$ preserving the decomposition $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$.*

Let $\Sigma_{\mathcal{U}} = S \cap \mathcal{U}$, $S \in \Sigma$, and $\mathcal{U} \subset \mathbb{G}$ be an open set. Then $M_p(\mathbf{E}_{\mathcal{U}}) = 0$ for $p \in (0, 1)$.

Proof. We define function $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ as in (3.24). Moreover, for any point $q \in S \cap \mathcal{U}$ there is a neighbourhood V, such that for $S \cap \mathcal{U} \cap V$ there exists an intrinsic Lipschitz function $\hat{f}_S : (\hat{\Omega}_S)_\alpha \to \hat{\mathbb{H}}_\alpha$, where $(\hat{\Omega}_S)_\alpha \subset \hat{\mathbb{M}}_\alpha$ such that $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha \in K_\alpha(\hat{\mathbb{M}}_\alpha \cdot \hat{\mathbb{H}}_\alpha)$. By choosing an isometry $k_\alpha \in K_\alpha$ we find an intrinsic Lipschitz function $f_S : (\Omega_S)_\alpha \to \mathbb{H}_\alpha$, such that

- $(\Omega_S)_\alpha$ is an open subset of $\mathbb{M}_\alpha$, $k_\alpha((\hat{\Omega}_S)_\alpha) = (\Omega_S)_\alpha$;
- $S \cap \mathcal{U} \cap V = \text{graph}\,(f_S)$, i.e. $S = \Phi_S((\Omega_S)_\alpha)$, where

$$\Phi_S(m) = k_\alpha(\hat{m}) \cdot (k_\alpha \circ \hat{f}_S \circ k_\alpha^{-1}(k_\alpha(\hat{m}))), \quad m \in (\Omega_S)_\alpha, \quad \hat{m} \in (\hat{\Omega}_S)_\alpha.$$

Then we proceed in the same way as in Theorem 3.25 and show that

$$N \int_{S \cap \mathcal{U} \cap V} F \mathrm{d}\sigma_S \geq \int_{B(\zeta_S, R)} \phi_R(\zeta) \mathrm{d}\zeta = \infty.$$ □

3.5.2 Exceptional families for $p \geq 1$

Fuglede proved that the system of k-dimensional Lipschitz surfaces in $\mathbb{R}^n$ which pass through a given point is M_p-exceptional if and only if $kp \leq n$ [21]. Here we show the sufficient part of the analogous statement for Carnot groups.

Theorem 3.32. *Let $\Sigma \subset \Sigma^{(\mathbf{d_t}, \mathbf{m})}$ be a collection of intrinsic $(\mathbf{d_t}, \mathbf{m})$-Lipschitz graphs. Suppose that all the graphs $S \in \Sigma$ contain a common point $g_0 \in \mathbb{G}$. Then for $\mathbf{m}p \leq Q$ we have $M_p(\Sigma) = 0$.*

Proof. Let $\Sigma \subset \Sigma^{(\mathbf{d_t},\mathbf{m})}$ be a collection of intrinsic $(\mathbf{d_t},\mathbf{m})$-Lipschitz graphs containing a common point $g_0 \in \mathbb{G}$. We can assume that $g_0 = e \in \mathbb{G}$ by the translation invariance of measures and the fact that the translation of an intrinsic Lipschitz graph is still an intrinsic Lipschitz graph, see [20, Theorem 3.2]. We need to find a non-negative measurable function $F : \mathbb{G} \to \mathbb{R}$ such that $\int_{\mathbb{G}} F^p \mathrm{d}\mathbf{g}_{\mathbb{G}} < \infty$, but $\int_S F \mathrm{d}\sigma_S = \infty$ for any $S \in \Sigma$.

Let $\|\cdot\|$ be a homogeneous norm on $\mathbb{G}$, for instance one of the types $(D_2)-(D_4)$ and let d_ρ be a metric associated with the norm $\|\cdot\|$. We set

$$F(g) = \begin{cases} \|g\|^{-\mathbf{m}}, & \text{if } \|g\| < 1, \\ 0, & \text{if } \|g\| \geq 1, \end{cases} \quad \mathbf{m}p < Q. \tag{3.25}$$

Then $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ since

$$\int_{B_{d_\rho}(e,1)} |F|^p \mathrm{d}\mathcal{L}^N = \omega \int_0^1 r^{-p\mathbf{m}+Q-1} dr < \infty$$

for $-p\mathbf{m} + Q > 0$, where ω is a suitable constant depending only on $\|\cdot\|$, see, e.g. [15, Proposition 1.15].

Consider intersections $S \cap B_{d_\rho}\left(e, \frac{1}{2^j}\right)$, $j \in \mathbb{N}$. We divide the ball $B_{d_\rho}(e,1)$ into the spherical rings $R_j = B_{d_\rho}\left(e, \frac{1}{2^j}\right) \setminus \overline{B}_{d_\rho}\left(e, \frac{1}{2^{j+1}}\right)$. In each ring R_j we choose a point $p_j \in S \cap \partial B_{d_\rho}\left(e, \frac{3}{2^{j+2}}\right)$, then $B_{d_\rho}\left(p_j, \frac{1}{2^{j+3}}\right) \subset R_j$. We observe that $2^{(j+1)\mathbf{m}} \geq \|g\|^{-\mathbf{m}} > 2^{j\mathbf{m}}$ for $g \in R_j$. Then

$$\begin{aligned} \int_S F \mathrm{d}\sigma_S &\geq \int_{B_{d_\rho}(e,1)\cap S} \|g\|^{-\mathbf{m}} \mathrm{d}\sigma_S = \sum_j \int_{R_j \cap S} \|g\|^{-\mathbf{m}} \mathrm{d}\sigma_S \\ &> \sum_j 2^{j\mathbf{m}} \mathcal{S}^{\mathbf{m}}_{d_\rho}(R_j \cap S) > \sum_j 2^{j\mathbf{m}} \mathcal{S}^{\mathbf{m}}_{d_\rho}\left(B_{d_\rho}(p_j, 2^{-j-3}) \cap S\right) \\ &\geq c_1 2^{-3\mathbf{m}} \sum_j 2^{j\mathbf{m}} 2^{-j\mathbf{m}} = \infty. \end{aligned}$$

If $\mathbf{m}p = Q$, then we need to change the function F to

$$F(g) = \begin{cases} \|g\|^{-\mathbf{m}} \left(\ln \dfrac{2}{\|g\|}\right)^{-\alpha}, & \text{if } \|g\| < 1, \\ 0, & \text{if } \|g\| \geq 1. \end{cases} \tag{3.26}$$

If we choose $\alpha \in \left[\frac{\mathbf{m}}{Q}, 1\right]$, then $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ and $\int_S F(g) \mathrm{d}\sigma_S = \infty$. Indeed,

$$\int_{B_{d_\rho}(0,1)} |F|^p \mathrm{d}\mathbf{g}_{\mathbb{G}} = \omega \int_0^1 r^{-p\mathbf{m}+Q-1} \left(\ln \frac{2}{r}\right)^{-\alpha p} \mathrm{d}r = \omega \int_{\ln 2}^{\infty} t^{-\alpha p} \mathrm{d}t < \infty$$

if $\alpha p > 1$ or equivalently $\alpha > \frac{\mathbf{m}}{Q}$. From the other side,

$$\begin{aligned} \int_S F \mathrm{d}\sigma_S &\geq \int_{R_j \cap S} \|g\|^{-\mathbf{m}} \left(\ln \frac{2}{\|g\|}\right)^{-\alpha} \mathrm{d}\sigma_S \\ &> \sum_j 2^{j\mathbf{m}} (\ln 2^{j+2})^{-\alpha} \mathcal{S}^{\mathbf{m}}_{d_\rho}\left(B_{d_\rho}(p_j, 2^{-j-3}) \cap S\right) \\ &\geq c_1 2^{-3\mathbf{m}} (\ln 2)^{-\alpha} \sum_j (j+2)^{-\alpha} = \infty \end{aligned}$$

for $\alpha < 1$. □

Corollary 3.33. *Let $\Sigma \subset \Sigma^{(\mathbf{d_t},\mathbf{m})}$ be a collection of intrinsic $(\mathbf{d_t},\mathbf{m})$-Lipschitz surfaces. Suppose that all the surfaces $\widehat{S} \in \Sigma$ contain a common point $g_0 \in \mathbb{G}$. Then for $\mathbf{m}p \leq Q$ we have $M_p(\Sigma) = 0$.*

Proof. We assume that $g_0 = e$ and, as in Theorem 3.32, consider the function $F \in L^p(\mathbb{G}, \mathbf{g}_{\mathbb{G}})$ for $\mathbf{m}p \le Q$. To show that $\int_{\widehat{S}} F d\sigma_{\widehat{S}} = \infty$, we fix a surface $\widehat{S} \in \Sigma$ and a neighbourhood U of e, such that $S = \widehat{S} \cap U$ is an intrinsic Lipschitz graph. Then we argue as in Theorem 3.32. □

In the following sections, we study M_p-exceptional sets of intrinsic Lipschitz surfaces passing through a common point when $\mathbf{m}p > Q$. We will make special constructions of intrinsic Lipschitz surfaces that will reveal the situation of being M_p-exceptional. The first step to the construction is the study of an analogue of a Grassmann manifold on special type of Carnot groups.

4 Orthogonal Grassmannians

In this section, we construct orthogonal Grassmannians of Lie subalgebras on specially chosen H-type Lie algebras. We start from the overview of Grassmannians of k-plains of n-dimensional Euclidean space, reminding that they are orbits under the action (modulo the isotropy subgroup) of the isometry group $O(n)$ of the Euclidean space, see Section 4.1. In order to make a proper construction of the Grassmannian of subalgebras we first remind in Section 4.2 the structure of the isometry group Iso of H-type algebras. Then we make construction of orthogonal Grassmannians of subalgebras and study their properties in Section 4.3.

4.1 Overview over the Grassmannians in $\mathbb{K}^n$

4.1.1 Definition of the groups O(n), U(n), and Sp(n)

The orthogonal group $\mathrm{O}(n)$ in $\mathbb{R}^n$, endowed with the standard Euclidean inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, is

$$\mathrm{O}(n) = \{A : \mathbb{R}^n \to \mathbb{R}^n : \langle Av, Av\rangle_{\mathbb{R}} = \langle v, v\rangle_{\mathbb{R}}\}.$$

The unitary group $\mathrm{U}(n)$ acting in $\mathbb{C}^n$, endowed with the standard Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, is defined analogously by

$$\mathrm{U}(n) = \{A : \mathbb{C}^n \to \mathbb{C}^n : \langle Av, Av\rangle_{\mathbb{C}} = \langle v, v\rangle_{\mathbb{C}}\}.$$

Finally, the quaternion unitary group (or compact symplectic group) $\mathrm{Sp}(n)$ acting in right quaternion space $\mathbb{Q}^n$ is defined by

$$\mathrm{Sp}(n) = \{A : \mathbb{Q}^n \to \mathbb{Q}^n : \langle Av, Av\rangle_{\mathbb{Q}} = \langle v, v\rangle_{\mathbb{Q}}\},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{Q}}$ is a quaternion Hermitian product in $\mathbb{Q}^n$, see, for instance [38].

Let us write $\mathbb{K}$ for the division algebras $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$ and $\mathrm{U}(n,\mathbb{K})$ for the groups $\mathrm{O}(n)$, $\mathrm{U}(n)$, and $\mathrm{Sp}(n)$, respectively. We let k be an integer satisfying $0 < k \le n$. The Grassmann manifold or Grassmannian $\mathrm{Gr}(k,\mathbb{K}^n)$ is defined as the set of k-dimensional vector subspaces in $\mathbb{K}^n$:

$$\mathrm{Gr}(k,\mathbb{K}^n) = \{V \text{ is a } k\text{-dimensional vector subspace of } \mathbb{K}^n\}.$$

Note that the vector space $\mathbb{Q}^n$ is defined as the *right vector space* with respect to the right multiplication by scalars from $\mathbb{Q}$. The same agreement is done for the subspaces $V \subset \mathbb{Q}^n$.

The group $\mathrm{U}(n,\mathbb{K})$ acts transitively on the set $\mathrm{Gr}(k,\mathbb{K}^n)$ via

$$A.V = \{Av \in \mathbb{K}^n : v \in V, \quad A \in \mathrm{U}(n,\mathbb{K})\}.$$

Fix a plain $\hat{V} \in \mathrm{Gr}(k,\mathbb{K}^n)$. Let $K_{\hat{V}} = \{A \in \mathrm{U}(n,\mathbb{K}) : A.\ \hat{V} = \hat{V}\}$ be the isotropy group of $\hat{V}$. It follows that the Grassmannian $\mathrm{Gr}(k,\mathbb{K}^n)$ admits the structure of a compact manifold [48] given by

$$\mathrm{Gr}(k,\mathbb{K}^n) = \mathrm{U}(n,\mathbb{K})/K_{\hat{V}},$$

where $K_{\hat{V}}$ is isomorphic to $\mathrm{U}(k,\mathbb{K})\times\mathrm{U}(n-k,\mathbb{K})$. Note that there is a diffeomorphism $\mathrm{Gr}_{n-k}(\mathbb{K}^n)\cong\mathrm{Gr}(k,\mathbb{K}^n)$ mapping any $V\in\mathrm{Gr}_{n-k}(\mathbb{K}^n)$ to its orthogonal complements $V^\perp\in\mathrm{Gr}_k(\mathbb{K}^n)$.

4.1.2 Measure on the Grassmannians

Let us remind the definition of a measure on the Grassmannian $\mathrm{Gr}_k(\mathbb{K}^n)$. The continuous map

$$\psi:\mathrm{U}(n,\mathbb{K})\to\mathrm{Gr}_k(\mathbb{K}^n),$$

which is the composition of the projection map to the quotient and the diffeomorphism giving the manifold structure to $\mathrm{Gr}_k(\mathbb{K}^n)$ is used to push forward a measure from $\mathrm{U}(n,\mathbb{K})$ to $\mathrm{Gr}_k(\mathbb{K}^n)$. We take for granted that the group $\mathrm{U}(n,\mathbb{K})$ carries bi-invariant normalised measure λ:

$$\lambda(AU)=\lambda(UA)=\lambda(U),\quad \lambda(\mathrm{U}(n,\mathbb{K}))=1,$$

for any Borel set $U\subset\mathrm{U}(n,\mathbb{K})$ and any isometry $A\in\mathrm{U}(n,\mathbb{K})$. The measure μ on the Borel sets $\Omega\subset\mathrm{Gr}_k(\mathbb{K}^n)$ is defined as a push forward of λ:

$$\mu(\Omega)=(\psi_\sharp\lambda)(\Omega)=\lambda(\psi^{-1}(\Omega))=\lambda\{A\in\mathrm{U}(n,\mathbb{K}):V=A.\,\hat{V}\in\Omega\}.$$

It can be verified that μ is normalised and it is $\mathrm{U}(n,\mathbb{K})$-invariant. The converse is also true: a normalised $\mathrm{U}(n,\mathbb{K})$-invariant measure on the homogeneous space $\mathrm{Gr}_k(\mathbb{K}^n)$ is a push forward of the normalised Haar measure from $\mathrm{U}(n,\mathbb{K})$, see for instance [28].

Note that an $(n-1)$-dimensional sphere $S(0,R)$ in $\mathbb{R}^n$ can be considered as a particular case of the Grassmannian $\mathrm{Gr}_{n-1}(\mathbb{R}^n)$. If we denote by $K_x(\mathbb{R})$ an isotropy group of a point $x\in S(0,R)$ under the action of $\mathrm{O}(n)$, then the following manifolds are diffeomorphic

$$S(0,R)\sim\mathrm{Gr}_{n-1}(\mathbb{R}^n)\sim\mathrm{O}(n)/K_x(\mathbb{R}).$$

The push forward of the normalised measure λ on $\mathrm{O}(n)$ to $S(0,R)$ coincides with the normalised surface measure on $S(0,R)$, see [28, Theorem 3.7].

4.2 Isometry groups of special H-type Lie algebras

Before we make the construction of orthogonal Grassmannians on some special H-type Lie algebras, we describe the group of isometries of these Lie algebras.

Recall the definition of an H-type Lie algebra $\mathfrak{h}=(\mathfrak{h}_1\otimes\mathfrak{h}_2,[\cdot,\cdot],\langle\cdot,\cdot\rangle_{\mathbb{R}})$ from Section 2.4. The group of isometries $\mathrm{Iso}(\mathfrak{h})$ of H-type Lie algebras were studied in [35,36]. It was shown that

$$\mathrm{Iso}(\mathfrak{h})=\Big\{(A,C)\in\mathrm{O}(\mathfrak{h}_1)\times\mathrm{O}(\mathfrak{h}_2):A^tJ_zA=J_{C^t(z)}\quad\text{for any } z\in\mathfrak{h}_2\Big\}.$$

The group $\mathrm{Iso}(\mathfrak{h})$ is isogenous to the product of the Pin group $\mathrm{Pin}(\mathfrak{h}_2,\langle\cdot,\cdot\rangle_{\mathbb{R}})$ of the Clifford algebra $\mathrm{Cl}(\mathfrak{h}_2,\langle\cdot,\cdot\rangle_{\mathbb{R}})$ and a classical group $\mathbb{A}$. The latter means that there is a surjective morphism

$$\begin{aligned}\phi:\mathrm{Pin}(\mathfrak{h}_2,\langle\cdot,\cdot\rangle_{\mathbb{R}})\times\mathbb{A}&\to\mathrm{Iso}(\mathfrak{h})\subset\mathrm{O}(\mathfrak{h}_1)\times\mathrm{O}(\mathfrak{h}_2)\\ \theta=(\alpha,A)&\mapsto\phi(\theta)=(J_\alpha\circ A,\kappa(\alpha)),\end{aligned}\tag{4.1}$$

with a finite kernel of order 2 or 4. Here

$$\kappa:\mathrm{Pin}(\mathfrak{h}_2,\langle\cdot,\cdot\rangle_{\mathbb{R}})\to\mathrm{O}(\mathfrak{h}_2)\tag{4.2}$$

is the double cover of the orthogonal group defined by

$$\kappa(\alpha)v=\alpha v\alpha^{-1},\quad v\in\mathfrak{h}_2,\quad \alpha\in\mathrm{Pin}(\mathfrak{h}_2,\langle\cdot,\cdot\rangle_{\mathbb{R}}).$$

We will not give the full description of the group of isometries, but rather concentrate on the cases when the restriction $\mathrm{Iso}(\mathfrak{h})|_{\mathfrak{h}_1}$ on $\mathfrak{h}_1$ acts transitively on the spheres $S(0,r)\in\mathfrak{h}_1$ and the vector space $\mathfrak{h}_1$, considered a

Clifford module, is not irreducible. Only in these cases the construction of the Grassmannian on the Lie algebra $\mathfrak{h}$ is not trivial. Thus, we will consider the following H-type Lie algebras:

RH: The Heisenberg algebra $\mathfrak{h}^n_{\mathbb{R}} = (\mathbb{R}^{2n+1}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$, $n > 1$ with $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \cong \mathbb{R}^{2n} \otimes \mathbb{R}$;

CH: The complex Heisenberg algebra $\mathfrak{h}^n_{\mathbb{C}} = (\mathbb{R}^{4n+2}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$, $n > 1$ with $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \cong \mathbb{R}^{4n} \otimes \mathbb{R}^2$;

QH: The quaternion Heisenberg algebra $\mathfrak{h}^n_{\mathbb{Q}} = (\mathbb{R}^{4n+3}, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$, $n > 1$ with $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \cong \mathbb{R}^{4n} \otimes \mathbb{R}^3$.

The commutators in all the cases are defined by (2.10) and the scalar products are the standard Euclidean products making the direct sums orthogonal. The names real, complex, and quaternion are attached to the names of the algebras by the following reasons. For the Lie algebra $\mathfrak{h}^n_{\mathbb{R}}$, the group of real symplectic transformations $\mathrm{Sp}(n,\mathbb{R})$ is a subgroup of automorphisms, leaving the centre $\mathfrak{h}_2$ of the Lie algebra $\mathfrak{h}^n_{\mathbb{R}}$ invariant. The centre $\mathfrak{h}_2 \subset \mathfrak{h}^n_{\mathbb{R}}$ is isomorphic to $\mathbb{R}$. For the Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$, the group of complex symplectic transformations $\mathrm{Sp}(2n,\mathbb{C})$ is a subgroup of the Lie algebra automorphisms, leaving the centre invariant and also $\mathfrak{h}^n_{\mathbb{C}}$ is isomorphic to the complexification of $\mathfrak{h}^n_{\mathbb{R}}$. The centre $\mathfrak{h}_2 \subset \mathfrak{h}^n_{\mathbb{C}}$ is isomorphic to $\mathbb{C}$. For $\mathfrak{h}^n_{\mathbb{Q}}$ the group of quaternion unitary transformations $\mathrm{Sp}(n)$ is a subgroup of the Lie algebra automorphisms, leaving the centre invariant. The centre $\mathfrak{h}_2$ of $\mathfrak{h}^n_{\mathbb{Q}}$ is isomorphic to the space of pure imaginary quaternions.

Now we describe the isometry groups in each case.

4.2.1 The isometry group Iso($\mathfrak{h}_{\mathbb{R}}$)

For the convenience of the forthcoming calculations we slightly change the notation of Section 2.4.1. We write $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and denote

$$X_{11}, X_{12}, \dots, X_{1n}, \quad X_{21}, X_{22}, \dots, X_{2n},$$

the orthonormal basis for $\mathfrak{h}_1$. Let $\varepsilon \in \mathfrak{h}_2 \cong \mathbb{R}$, $\langle\varepsilon,\varepsilon\rangle_{\mathbb{R}} = 1$ be the basis vector of $\mathfrak{h}_2$. The commutation relations are

$$[X_{1l}, X_{2m}] = \delta_{lm}\varepsilon, \quad [X_{1p}, X_{1q}] = [X_{2r}, X_{2s}] = [X_{kl}, \varepsilon] = 0$$

for indices l, m, p, q, r, s running in $\{1, \dots, n\}$, and $k = 1, 2$.

The vector space $\mathfrak{h}_1 \cong \mathbb{R}^{2n}$ carries natural almost complex structure given by the action of J_ε, $\varepsilon \in \mathfrak{h}_2$. Therefore, the space $\mathfrak{h}_1 \cong \mathbb{R}^{2n}$ can be considered as a complex vector space $\mathfrak{h}_1 \cong \mathbb{C}^n$, where the multiplication by a complex number $\alpha \in \mathbb{C}$ is defined by $\alpha v = (a + ib)v = av + bJ_\varepsilon v$, $v \in \mathfrak{h}_1$. The maximal compact subgroup $\mathbb{A} = \mathrm{U}(n) \subset \mathrm{Sp}(2n,\mathbb{R})$ is an isometry on $\mathfrak{h}_1 \cong \mathbb{R}^{2n}$. The corresponding Hermitian form $\langle . | . \rangle_{\mathbb{C}}$ on $\mathfrak{h}_1 \cong \mathbb{C}^n$ is defined by making use of the real scalar product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ by the following

$$\langle u|v\rangle_{\mathbb{C}} = \langle u, v\rangle_{\mathbb{R}} - i\langle J_\varepsilon u, v\rangle_{\mathbb{R}}. \tag{4.3}$$

The imaginary part of the Hermitian product (4.3) defines the symplectic form $\omega : \mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathbb{R}$, that is compatible with the real inner product on $\mathfrak{h}_1$. Namely

$$\omega(u, v) = \langle J_\varepsilon u, v\rangle_{\mathbb{R}} = \langle [u, v], \varepsilon\rangle_{\mathbb{R}}, \tag{4.4}$$

which implies $\omega(J_\varepsilon u, J_\varepsilon v) = \omega(u, v)$, $\quad \omega(u, J_\varepsilon v) = \langle u, v\rangle_{\mathbb{R}}$.

We use the basis described in Section 2.4.1 and give the coordinates to $\mathfrak{h}^n_{\mathbb{R}}$. Consider the product of two spheres on $\mathfrak{h}^n_{\mathbb{R}} \cong \mathbb{R}^{2n} \times \mathbb{R}$ centred at the origin

$$\mathcal{S}_{\mathbb{R}}(0, r_1, r_2) = S^{2n-1}(0, r_1) \times S^0(0, r_2) = \{v \in \mathfrak{h}_1 : \langle v, v\rangle_{\mathbb{R}} = r_1^2\} \times \{z \in \mathfrak{h}_2, \langle z, z\rangle_{\mathbb{R}} = r_2^2\}. \tag{4.5}$$

Lemma 4.1. *The group* $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{R}})$ *acts transitively on* $\mathcal{S}_{\mathbb{R}}(0, r_1, r_2)$.

Proof. The map $\kappa(\varepsilon) \in \mathrm{O}(\mathfrak{h}_2)$, defined in (4.2), acts either as a reflection with respect to the origin of $\mathfrak{h}_2$ or as the identity, since $\mathfrak{h}_2 \cong \mathbb{R}$. Let $(x_1, z), (x_2, -z) \in \mathcal{S}_{\mathbb{R}}(0, r_1, r_2)$. Then we find a map $(J_\varepsilon \circ A, \kappa(\varepsilon)) \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{R}})$ that maps the point (x_1, z) to $(x_3, -z) = (J_\varepsilon \circ Ax_1, \kappa(\varepsilon)z)$, $A \in \mathrm{U}(n)$. Since the subgroup $(\mathrm{U}(n) \times \mathrm{Id}_{\mathfrak{h}_2}) \subset \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{R}})$ acts transitively on $S^{2n-1}(0, r_1)$ we can find the transformation mapping $(x_3, -z)$ to $(x_2, -z)$. ◻

At the end, we recall that the basis of left invariant vector fields on the Heisenberg group $H^n_{\mathbb{R}}$ is

$$X_{1k} = \frac{\partial}{\partial x_{1k}} - \frac{x_{2k}}{2}\frac{\partial}{\partial \varepsilon_1}, \quad X_{2k} = \frac{\partial}{\partial x_{2k}} + \frac{x_{1k}}{2}\frac{\partial}{\partial \varepsilon_2}, \tag{4.6}$$

for $k = 1, \dots, n$.

4.2.2 Isometry groups of H-type Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$

Let $\mathfrak{h}_1 \cong \mathbb{R}^{4n} \subset \mathfrak{h}^n_{\mathbb{C}}$ be the horizontal subspace and $\mathfrak{h}_2 \cong \mathbb{R}^2 \subset \mathfrak{h}^n_{\mathbb{C}}$ be the vertical subspace. Then $\mathrm{Cl}(\mathfrak{h}_2, \langle\cdot,\cdot\rangle_{\mathbb{R}}) \cong \mathrm{Cl}(\mathbb{R}^2)$ contains two elements $\varepsilon_1, \varepsilon_2$ such that

$$\varepsilon_1^2 = \varepsilon_2^2 = -1, \quad \langle\varepsilon_1, \varepsilon_1\rangle_{\mathbb{R}} = \langle\varepsilon_2, \varepsilon_2\rangle_{\mathbb{R}} = 1, \quad \langle\varepsilon_1, \varepsilon_2\rangle_{\mathbb{R}} = 0,$$

and moreover $\varepsilon_3 = \varepsilon_1\varepsilon_2$ satisfies $\varepsilon_3^2 = -1$.

We first consider the case $n = 1$, that is $\mathfrak{h}_1 \cong \mathbb{R}^4 \subset \mathfrak{h}^1_{\mathbb{C}}$. The maps $J_{\varepsilon_i} : \mathfrak{h}_1 \to \mathfrak{h}_1$, $i = 1, 2$, are orthogonal transformations. A convenient orthonormal basis for $\mathfrak{h}_1$ can be constructed by the following. We choose a vector $v \in \mathfrak{h}_1$, such that $\langle v, v\rangle_{\mathbb{R}} = 1$ and define the orthonormal vectors

$$X_1 = v, \quad X_2 = J_{\varepsilon_1}v, \quad X_3 = J_{\varepsilon_2}v, \quad X_4 = J_{\varepsilon_2}J_{\varepsilon_1}v. \tag{4.7}$$

The commutation relations according to (2.10) are

$$[X_1, X_2] = -[X_3, X_4] = \varepsilon_1, \quad [X_1, X_3] = [X_2, X_4] = \varepsilon_2. \tag{4.8}$$

We show now that $\mathfrak{h}^1_{\mathbb{C}}$ is the complexified Lie algebra of $\mathfrak{h}^1_{\mathbb{R}}$. Let $X_1, X_2 = J_\varepsilon X_1$ be an orthonormal basis of $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{R}}$ and $\mathfrak{h}_2 = \mathrm{span}_{\mathbb{R}}\{\varepsilon\}$ is the centre of the Lie algebra $\mathfrak{h}^1_{\mathbb{R}}$. Then the complexification $\mathfrak{h}^{\mathbb{C}}_1 = \mathbb{C} \otimes \mathfrak{h}_1$ of the vector space $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{R}}$ can be described as any of the following direct sums:

$$\begin{aligned}\mathfrak{h}^{\mathbb{C}}_1 &= \mathrm{span}_{\mathbb{R}}\{Z = X_1 + iX_2\} \oplus \mathrm{span}_{\mathbb{R}}\{\bar{Z} = X_1 - iX_2\}\\ &= \mathrm{span}_{\mathbb{R}}\{X_1, X_2\} \oplus \mathrm{span}_{\mathbb{R}}\{iX_1, iX_2\}\\ &= \mathrm{span}_{\mathbb{C}}\{X_1, X_2\}.\end{aligned} \tag{4.9}$$

The Lie bracket on $\mathfrak{h}^{\mathbb{C}}_1$ is a complex linear Lie bracket on $\mathfrak{h}_1$:

$$[X_1, X_2] = -[iX_1, iX_2] = \varepsilon, \quad [X_1, iX_2] = [iX_1, X_2] = i\varepsilon. \tag{4.10}$$

Thus, the centre $\mathfrak{h}^{\mathbb{C}}_2$ of the complexified Heisenberg algebra is given by

$$\mathfrak{h}^{\mathbb{C}}_2 = \mathrm{span}_{\mathbb{C}}\{\varepsilon\} = \mathrm{span}_{\mathbb{R}}\{\varepsilon, i\varepsilon\}. \tag{4.11}$$

Recall that the real Heisenberg algebra has the complex structure $J_\varepsilon : \mathfrak{h}_1 \to \mathfrak{h}_1$ defined by $J_\varepsilon X_1 = X_2$, $J_\varepsilon X_2 = -X_1$. We extend J_ε to $\mathfrak{h}^{\mathbb{C}}_1$ by linearity, meaning that $J_\varepsilon(iu) = iJ_\varepsilon(u)$ for $u \in \mathrm{span}_{\mathbb{R}}\{X_1, X_2\}$. We define another complex structure

$$J_{i\varepsilon} : \mathfrak{h}^{\mathbb{C}}_1 \to \mathfrak{h}^{\mathbb{C}}_1 : \quad J_{i\varepsilon}X_1 = iX_2, \quad J_{i\varepsilon}X_2 = iX_1.$$

It is easy to check that $J_\varepsilon J_{i\varepsilon} = -J_{i\varepsilon}J_\varepsilon$. Note that if we denote

$$\begin{gathered}\varepsilon_1 = \varepsilon, \quad \varepsilon_2 = i\varepsilon,\\ X_1 = X_1, \quad X_2 = J_{\varepsilon_1}X_1, \quad X_3 = iX_2 = J_{\varepsilon_2}X_1, \quad X_4 = iX_1 = J_{\varepsilon_2}J_{\varepsilon_1}X_1,\end{gathered} \tag{4.12}$$

then we recover basis (4.7) and commutation relations (4.8) of $\mathfrak{h}^1_{\mathbb{C}}$.

We show now that the horizontal space $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{C}}$ of the complexified Heisenberg algebra is a complex symplectic space. The real symplectic form ω defined in (4.4) can be extended to the complex symplectic form $\omega^{\mathbb{C}} : \mathfrak{h}_1 \times \mathfrak{h}_1 \to \mathbb{C}$, $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{C}}$, by

$$\omega^{\mathbb{C}}(u + iv, x + iy) = \omega(u, x) + \omega(v, y) + i(\omega(v, x) + \omega(y, u)). \tag{4.13}$$

Here we consider $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{C}}$ as a complex space (4.9). Then $\omega^{\mathbb{C}}$ satisfies

$$\omega^{\mathbb{C}}(z_1, z_2) = -\overline{\omega^{\mathbb{C}}(z_2, z_1)},$$

$$\omega^{\mathbb{C}}(\alpha z_1, z_2) = \alpha\omega^{\mathbb{C}}(z_1, z_2), \quad \omega^{\mathbb{C}}(z_1, \alpha z_2) = \bar{\alpha}\omega^{\mathbb{C}}(z_1, z_2).$$

The symplectic basis $X_1, X_2 = J_{\varepsilon_1}X_1$ over $\mathbb{R}$ for the real symplectic vector space $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{R}}$ is the symplectic basis over $\mathbb{C}$ for the complex symplectic vector space $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{C}}$.

The multidimensional algebra $\mathfrak{h}^n_{\mathbb{C}}$ as a vector space is the Cartesian product

$$\mathfrak{h}^n_{\mathbb{C}} = (\mathfrak{h}^{\mathbb{C}}_1)_1 \times \ldots \times (\mathfrak{h}^{\mathbb{C}}_1)_n \times \mathfrak{h}^{\mathbb{C}}_2, \tag{4.14}$$

where $\mathfrak{h}^{\mathbb{C}}_1$ is defined in (4.9) and $\mathfrak{h}^{\mathbb{C}}_2$ is defined in (4.11). The commutation relations are given by (4.10), if the vectors belong to the same slot $(\mathfrak{h}^{\mathbb{C}}_1)_k$ in the Cartesian product (4.21) and zero otherwise. The real scalar product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is extended to the Cartesian product (4.21) by making the different slots orthogonal. The multidimensional algebra $\mathfrak{h}^n_{\mathbb{C}}$ has real topological dimension $4n + 2$ and the Hausdorff dimension $4n + 4$. The corresponding left invariant vector fields on the complexified Heisenberg group $H^n_{\mathbb{C}}$ are

$$\begin{aligned}
X_{1k} &= \frac{\partial}{\partial x_{1k}} - \frac{x_{2k}}{2}\frac{\partial}{\partial \varepsilon_1} - \frac{x_{3k}}{2}\frac{\partial}{\partial \varepsilon_2}, \\
X_{2k} &= \frac{\partial}{\partial x_{2k}} + \frac{x_{1k}}{2}\frac{\partial}{\partial \varepsilon_1} - \frac{x_{4k}}{2}\frac{\partial}{\partial \varepsilon_2}, \\
X_{3k} &= \frac{\partial}{\partial x_{3k}} + \frac{x_{4k}}{2}\frac{\partial}{\partial \varepsilon_1} + \frac{x_{1k}}{2}\frac{\partial}{\partial \varepsilon_2}, \\
X_{4k} &= \frac{\partial}{\partial x_{4k}} - \frac{x_{3k}}{2}\frac{\partial}{\partial \varepsilon_1} + \frac{x_{2k}}{2}\frac{\partial}{\partial \varepsilon_2},
\end{aligned} \qquad k = 1, \ldots, n. \tag{4.15}$$

The subgroup $\mathbb{A} \cong \mathrm{Sp}(n)$ of the isometry group $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{C}})$ preserves the quaternion Hermitian product $\langle.|.\rangle_{\mathbb{Q}}$ on $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{C}}$, defined by

$$\langle z|w\rangle_{\mathbb{Q}} = \langle z, w\rangle_{\mathbb{R}} - i\langle J_{\varepsilon_1}z, w\rangle_{\mathbb{R}} - j\langle J_{\varepsilon_2}z, w\rangle_{\mathbb{R}} - k\langle J_{\varepsilon_2}J_{\varepsilon_1}z, w\rangle_{\mathbb{R}}. \tag{4.16}$$

The group $\mathbb{A} \cong \mathrm{Sp}(n)$ acts transitively on the unit sphere $S^{4n-1} \subset \mathfrak{h}_1 \cong \mathbb{C}^{2n}$, where $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{C}}$.

4.2.3 Isometry groups of H-type Lie algebra $\mathfrak{h}^n_{\mathbb{Q}}$

Since $\mathbb{R}^3 \cong \mathfrak{h}_2 \subset \mathfrak{h}^1_{\mathbb{Q}}$, there are three linearly independent length one elements $\varepsilon_i \in \mathrm{Cl}(\mathbb{R}^3, \langle\cdot,\cdot\rangle_{\mathbb{R}})$, $i = 1, 2, 3$, satisfying the quaternion relations

$$\varepsilon_1^2 = \varepsilon_3^2 = \varepsilon_3^2 = \varepsilon_1\varepsilon_2\varepsilon_3 = -1. \tag{4.17}$$

We introduce a quaternion structure on $\mathbb{R}^4 \cong \mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{Q}}$ by defining the multiplication by a quaternion number $q = a + ib + jc + kd \in \mathbb{Q}$ as follows:

$$qv = (a + ib + jc + kd)v = av + bJ_{\varepsilon_1}v + cJ_{\varepsilon_2}v + dJ_{\varepsilon_3}v, \quad v \in \mathfrak{h}_1.$$

The quaternion Hermitian product is

$$\langle u|v\rangle_{\mathbb{Q}} = \langle u, v\rangle_{\mathbb{R}} - i\langle J_{\varepsilon_1}u, v\rangle_{\mathbb{R}} - j\langle J_{\varepsilon_2}u, v\rangle_{\mathbb{R}} - k\langle J_{\varepsilon_3}u, v\rangle_{\mathbb{R}}. \tag{4.18}$$

To construct an orthonormal basis we choose $v \in \mathfrak{h}_1$ with $\langle v, v\rangle_{\mathbb{R}} = 1$ and set

$$X_1 = v, \quad X_2 = J_{\varepsilon_1}v, \quad X_3 = J_{\varepsilon_2}v, \quad X_4 = J_{\varepsilon_3}v. \tag{4.19}$$

The commutation relations are

$$[X_1, X_2] = -[X_3, X_4] = \varepsilon_1, \quad [X_1, X_3] = [X_2, X_4] = \varepsilon_2, \quad [X_1, X_4] = -[X_2, X_3] = \varepsilon_3. \tag{4.20}$$

The space $\mathfrak{h}_1 = \mathrm{span}_{\mathbb{R}}\{X_1, X_2, X_3, X_4\} \subset \mathfrak{h}^1_{\mathbb{C}}$ is isomorphic and isometric to the one-dimensional quaternion space endowed by the quaternion Hermitian product (4.18).

We note the relation between $\mathfrak{h}^1_{\mathbb{C}}$ and $\mathfrak{h}^1_{\mathbb{Q}}$. Due to (4.17), the action of J_{ε_3} can be expressed as $J_{\varepsilon_3} = J_{\varepsilon_1} J_{\varepsilon_2}$ and therefore the space $\mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{Q}}$ can be considered as a complex symplectic space with respect to $\omega^{\mathbb{C}}$ in (4.13). In addition to that we add a real symplectic form $\omega_3(u, v) = \langle J_{\varepsilon_3} u, v\rangle_{\mathbb{R}}$.

The multidimensional algebra $\mathfrak{h}^n_{\mathbb{Q}}$ as a vector space is the Cartesian product

$$\mathfrak{h}^n_{\mathbb{Q}} = (\mathfrak{h}_1)_1 \times \ldots \times (\mathfrak{h}_1)_n \times \mathfrak{h}_2. \tag{4.21}$$

The commutation relations are given by (4.20), if the vectors belong to the same slot $(\mathfrak{h}_1)_k$ in the Cartesian product (4.21) and zero otherwise. The multidimensional algebra $\mathfrak{h}^n_{\mathbb{Q}}$ has real topological dimension $4n + 3$ and the Hausdorff dimension $4n + 6$. We note that the Clifford module $(\mathfrak{h}_1)_1 \times \ldots \times (\mathfrak{h}_1)_n$ in this case is assumed to be isotypic, which corresponds to the fact that the product $J_{\varepsilon_1} J_{\varepsilon_2} J_{\varepsilon_3}$ acts as minus identity on every slot $(\mathfrak{h}_1)_k$. The subgroup $\mathbb{A} \cong \mathrm{Sp}(n)$ of the isometry group $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{Q}})$ preserves the quaternion Hermitian product on $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{Q}}$, defined in (4.18). The corresponding basis of left invariant vector fields on the group $H^n_{\mathbb{Q}}$ is

$$\begin{aligned}
X_{1k} &= \frac{\partial}{\partial x_{1k}} - \frac{x_{2k}}{2}\frac{\partial}{\partial \varepsilon_1} - \frac{x_{3k}}{2}\frac{\partial}{\partial \varepsilon_2} - \frac{x_{4k}}{2}\frac{\partial}{\partial \varepsilon_3} \\
X_{2k} &= \frac{\partial}{\partial x_{2k}} + \frac{x_{1k}}{2}\frac{\partial}{\partial \varepsilon_1} - \frac{x_{4k}}{2}\frac{\partial}{\partial \varepsilon_2} + \frac{x_{3k}}{2}\frac{\partial}{\partial \varepsilon_3}, \\
X_{3k} &= \frac{\partial}{\partial x_{3k}} + \frac{x_{4k}}{2}\frac{\partial}{\partial \varepsilon_1} + \frac{x_{1k}}{2}\frac{\partial}{\partial \varepsilon_2} - \frac{x_{2k}}{2}\frac{\partial}{\partial \varepsilon_3}, \\
X_{4k} &= \frac{\partial}{\partial x_{4k}} - \frac{x_{3k}}{2}\frac{\partial}{\partial \varepsilon_1} + \frac{x_{2k}}{2}\frac{\partial}{\partial \varepsilon_2} + \frac{x_{1k}}{2}\frac{\partial}{\partial \varepsilon_3}.
\end{aligned} \qquad k = 1, \ldots, n. \tag{4.22}$$

We use the bases (4.7) and (4.19) to identify $\mathfrak{h}^n_{\mathbb{C}}$ with $\mathbb{R}^{4n} \times \mathbb{R}^2$ and $\mathfrak{h}^n_{\mathbb{Q}}$ with $\mathbb{R}^{4n} \times \mathbb{R}^3$. Define the product of spheres

$$\begin{aligned}
\mathcal{S}_{\mathbb{C}}(0, r_1, r_2) &= \{(x, z) \in \mathfrak{h}^n_{\mathbb{C}} : x \in S^{4n-1}(0, r_1) \subset \mathfrak{h}_1, z \in S^1(0, r_2) \subset \mathfrak{h}_2\} \\
&\cong S^{4n-1}(0, r_1) \times S^1(0, r_2), \quad \mathcal{S}_{\mathbb{C}}(0, r_1, r_2) \subset \mathfrak{h}^n_{\mathbb{C}}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_{\mathbb{Q}}(0, r_1, r_2) &= \{(x, z) \in \mathfrak{h}^n_{\mathbb{Q}} : x \in S^{4n-1}(0, r_1) \subset \mathfrak{h}_1, z \in S^2(0, r_2) \subset \mathfrak{h}_2\} \\
&\cong S^{4n-1}(0, r_1) \times S^2(0, r_2), \quad \mathcal{S}_{\mathbb{Q}}(0, r_1, r_2) \subset \mathfrak{h}^n_{\mathbb{Q}}.
\end{aligned}$$

Lemma 4.2. *The groups of isometries* $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{C}})$ *and* $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{Q}})$ *act transitively on the respective products of spheres* $\mathcal{S}_{\mathbb{C}}(0, r_1, r_2)$ *and* $\mathcal{S}_{\mathbb{Q}}(0, r_1, r_2)$.

4.3 Grassmannians on special *H*-type Lie algebras

We will construct orthogonal Grassmannians on the H-type Lie algebras mentioned in Section 4.2. In these cases, the isometry groups act transitively on the spheres $\mathcal{S}_{\mathbb{K}}(0, r_1, r_2)$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{Q}$. This allows us to define the measure on the Grassmannians. Moreover, the transitive action permits us to realise the Grassmannians as orbit spaces under the action of the isometry groups. Note that the spaces $\mathbb{R}^n$, $\mathbb{C}^n$, and $\mathbb{Q}^n$ are Abelian algebras with respect to the summation. Any k-dimensional vector subspace V is a subalgebra that has $(n-k)$-dimensional orthogonal complement that is also a subalgebra. This property is not trivial for the non-commutative subalgebras and therefore we restrict ourself to the consideration of *orthogonally complemented* Grassmannians.

4.3.1 Orthogonally complemented Grassmannians

A subalgebra $V \subset \mathfrak{h}^n_{\mathbb{K}}$ is called *homogeneous* if it is invariant under the action of dilation (2.3). In the following definition, we use the inner product from the definition of H-type Lie algebra $\mathfrak{h} = (\mathfrak{h}_1 \oplus \mathfrak{h}_2, [\cdot,\cdot], \langle\cdot,\cdot\rangle_{\mathbb{R}})$.

Definition 4.3. We say that a homogeneous subalgebra $V \subset \mathfrak{h}^n_{\mathbb{K}}$ is an orthogonally complemented homogeneous subalgebra of $\mathfrak{h}^n_{\mathbb{K}}$ if the orthogonal complement $V^{\perp}$ is a homogeneous subalgebra of $\mathfrak{h}^n_{\mathbb{K}}$.

Lemma 4.4. *Let $V \subset \mathfrak{h}^n_{\mathbb{K}}$ be an orthogonally complemented homogeneous subalgebra. Then:*

(i) *In the case $\mathfrak{h}^n_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{R}, \mathbb{H}$ we have*
 (i-1) *if $\dim_{\mathbb{R}} V \le n$, then $V \subset \mathfrak{h}_1$ (and hence V is commutative);*
 (i-2) *if $\dim_{\mathbb{R}} V > n$, then $\mathfrak{h}_2 \subset V$.*

(ii) *In the case $\mathfrak{h}^n_{\mathbb{C}}$ we have*
 (ii-1) *if $\dim_{\mathbb{R}} V \le 2n$, then $V \subset \mathfrak{h}_1$ (and hence V is commutative);*
 (ii-2) *if $\dim_{\mathbb{R}} V > 2n$, then $\mathfrak{h}_2 \subset V$.*

Proof. We start from the case $\mathfrak{h}^n_{\mathbb{R}}$. We sketch the construction of the orthogonally complemented homogeneous subalgebras. The horizontal vector space $\mathfrak{h}_1$ is a symplectic vector space with the symplectic form (4.4), see Section 4.2.1. Let $W \subset \mathfrak{h}_1$ be a vector space of dimension $d = \dim_{\mathbb{R}}(W) \ge \frac{1}{2}\dim_{\mathbb{R}}(\mathfrak{h}_1) = n$. It was shown in [19, Lemma 3.26] that there is a vector space $W' \subset \mathfrak{h}_1$ such that $W' \oplus W = \mathfrak{h}_1$ and $\omega(v, w) = 0$ for all $v, w \in W'$. The relation between the symplectic form and the commutation relation in (4.4) shows that W' is a commutative subalgebra of $\mathfrak{h}_1$. We choose an orthonormal basis $\{e_1, \dots, e_{d'}\}$, $d' = \dim(W') \le n$ for W' and extend it to an orthonormal basis

$$\{e_1, \dots, e_{d'}, e_{d'+1}, \dots, e_n, f_1 = J_\varepsilon e_1, \dots, f_n = J_\varepsilon e_n\}$$

for $\mathfrak{h}_1$. We denote

$$V = W', \quad V' = \operatorname{span}\{e_{d'+1}, \dots, e_n, f_1 = J_\varepsilon e_1, \dots, f_n = J_\varepsilon e_n\} \oplus \operatorname{span}\{\varepsilon\}.$$

Then it is easy to see that $V \oplus V'$ are orthogonally complemented subalgebras, satisfying hypothesis of Lemma 4.4. Note that the construction above gives all possible complementary subalgebras by exchanging the role of V and V'.

We turn to $\mathfrak{h}^n_{\mathbb{C}}$ and will consider $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{C}}$ as a complex symplectic space with the complex symplectic form $\omega^{\mathbb{C}}$. We have $\dim_{\mathbb{C}}(\mathfrak{h}_1) = 2n$ and n is even. We can show that for any W such that $\dim_{\mathbb{C}}(W) \ge n$ there is a vector space $W' \subset \mathfrak{h}_1$ satisfying $W' \oplus W = \mathfrak{h}_1$ and $\omega^{\mathbb{C}}(v, w) = 0$ for all $v, w \in W'$. The arguments are the same as at the beginning of the proof, since the arguments do not depend on the choice of the fields $\mathbb{R}$ or $\mathbb{C}$, but only on the construction of the basis.

Thus, by the construction, the vector space W' is a complex isotropic vector space with respect to the complex symplectic form $\omega^{\mathbb{C}}$ with $\dim_{\mathbb{C}} W' \le n$. It contains the isotropic subspace $\tilde{W}'$ with respect to the real symplectic form ω and the complex span of $\tilde{W}'$ coincides with W'. The vector space $\tilde{W}'$ is a commutative real subalgebra and its complexification W' will be a commutative complex subalgebra of the complex Heisenberg algebra $\mathfrak{h}^n_{\mathbb{C}}$. By making use of notation (4.12) we conclude that $\dim_{\mathbb{R}} W' \le 2n$, and W' is a commutative subalgebra of $\mathfrak{h}^1_{\mathbb{C}}$ considered as a real Lie algebra.

We consider two cases: $\dim_{\mathbb{R}} W' = 2k \le n$ and $\dim_{\mathbb{R}} W' = 2k > n$. Let $\dim_{\mathbb{R}} W' = 2k \le n$. We find a real commutative orthonormal basis $\{e_1, \dots, e_{2k}\}$ for W' and extend it to an orthonormal basis $\{e_1, \dots, e_n\}$. Then we denote $V = W'$ and set

$$V' = \operatorname{span}_{\mathbb{R}}\Big\{e_{2k+1}, \dots, e_n, J_{\varepsilon_1} e_i, J_{\varepsilon_2} e_i, J_{\varepsilon_2} J_{\varepsilon_1} e_i; \quad i = 1, \dots, n\Big\} \oplus \operatorname{span}_{\mathbb{R}}\{\varepsilon_1, \varepsilon_2\}.$$

In the case $\dim_{\mathbb{R}} W' = 2k > n$, we choose orthonormal vectors $\{e_1, \dots, e_n\}$ in W' and extend them to the orthonormal basis

$$\Big\{e_1, \dots, e_n, J_{\varepsilon_2} J_{\varepsilon_1} e_1, \dots, J_{\varepsilon_2} J_{\varepsilon_1} e_n\Big\}$$

of the maximal commutative subalgebra of $\mathfrak{h}^1_{\mathbb{C}}$. Without loss of generality we can assume that

$$\Big\{e_1, \dots, e_n, J_{\varepsilon_2} J_{\varepsilon_1} e_1, \dots, J_{\varepsilon_2} J_{\varepsilon_1} e_p\Big\}, \quad n + p = 2k$$

is an orthonormal basis for W'. Now we denote $V = W'$ and set

$$V' = \mathrm{span}_{\mathbb{R}}\left\{J_{\varepsilon_1}e_i, J_{\varepsilon_2}e_i; \quad i=1,\dots,n, \quad J_{\varepsilon_2}J_{\varepsilon_1}e_{p+1},\dots,J_{\varepsilon_2}J_{\varepsilon_1}e_n\right\} \oplus \mathrm{span}_{\mathbb{R}}\{\varepsilon_1,\varepsilon_2\}.$$

We recall that

$$\langle e_i, J_{\varepsilon_1}J_{\varepsilon_2}e_i\rangle_{\mathbb{R}} = -\langle J_{\varepsilon_1}e_i, J_{\varepsilon_2}e_i\rangle_{\mathbb{R}} = \langle \varepsilon_1,\varepsilon_2\rangle_{\mathbb{R}}\,\langle e_i, e_i\rangle_{\mathbb{R}} = 0$$

because of the orthogonality of vectors $\varepsilon_1, \varepsilon_2$. Also $\langle e_i, J_{\varepsilon_1}J_{\varepsilon_2}e_j\rangle_{\mathbb{R}} = 0$ for $i \neq j$ because of the orthogonal decomposition (4.21).

The last case concerns with $\mathfrak{h}^n_{\mathbb{H}}$. We start from lower dimensional subalgebra $\mathfrak{h}^1_{\mathbb{H}}$. We choose a vector $v \in \mathfrak{h}_1 \subset \mathfrak{h}^1_{\mathbb{H}}$ with $\langle v, v\rangle_{\mathbb{R}} = 1$ and define an orthonormal basis of the real Heisenberg algebra

$$X_1 = v, \quad X_2 = J_{\varepsilon_1}v, \quad \varepsilon_1.$$

We use ε_2 and construct the symplectic complex space $\mathfrak{h}_1^{\mathbb{C}}$ as in the previous case. Then the constructed multidimensional complexified Heisenberg algebra satisfies the commutation relations of the first line in (4.20). As in the previous case, we find a space W' such that $\omega^{\mathbb{C}}(u,v)=0$ for all $u, v \in W'$. Since W' is a complex vector space it has even real dimension $\dim_{\mathbb{R}}(W') = 2k \le 2n$ and therefore we can define an additional real symplectic form on W' (considered W' as a real vector space) by

$$\omega_3(u_1,u_2) = \langle J_{\varepsilon_3}u_1, u_2\rangle_{\mathbb{R}}, \quad u_1,u_2 \in W'.$$

Then, we set $V' = W' \cap L(W')$, where $L(W')$ is the Lagrangian subspace of the real symplectic space (W',ω_3), that is a maximal isotropic subspace of (W',ω_3). We obtain $\dim_{\mathbb{R}}(V) = p \le k \le n$ and it is by construction a commutative subspace of $\mathfrak{h}^1_{\mathbb{H}}$. Now we choose an orthonormal basis $\{e_1,\dots,e_p\}$ for V and complement it to an orthonormal basis

$$\{e_1,\dots,e_p, e_{p+1},\dots,e_n\}. \tag{4.23}$$

In the last step, we extend (4.23) to an orthonormal basis of $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{H}}$ by

$$\left\{e_1,\dots,e_n, J_{\varepsilon_l}e_1,\dots,J_{\varepsilon_l}e_n; \quad l=1,2,3\right\}.$$

We have obtained the orthogonally complemented subalgebras

$$V, \quad V' = \mathrm{span}_{\mathbb{R}}\left\{e_{p+1},\dots,e_n, J_{\varepsilon_l}e_1,\dots,J_{\varepsilon_l}e_n; \quad l=1,2,3\right\} \oplus \mathfrak{h}_2$$

satisfying the statement of Lemma 4.4. □

Theorem 4.5. *The group* $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ *acts transitively on the family of orthogonally complemented subalgebras of* $\mathfrak{h}^n_{\mathbb{K}}$*, i.e. if* V *and* V' *are subalgebras of the same dimension that have orthogonal subalgebras* $V^\perp$ *and* $V'^\perp$*, then there exists* $\mathcal{A} \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ *such that*

$$V' = \mathcal{A}(V), \quad V'^\perp = \mathcal{A}(V^\perp).$$

Proof. We start from $\mathfrak{h}^n_{\mathbb{R}}$. Suppose first that $k = \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V' \le n$. Then both V and V' are commutative by Lemma 4.4. Let $\{z_1,\dots,z_k\}$ and $\{z'_1,\dots,z'_k\}$ be orthonormal bases of the corresponding V and V' with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. Since both of the bases belong to the isotropic space of the real symplectic form ω in (4.4), then the bases are orthonormal with respect to the Hermitian scalar product (4.3), that is

$$\langle z_i | z_j\rangle_{\mathbb{C}} = \langle z_i, z_j\rangle_{\mathbb{R}} - i\omega(z_i,z_j) = 0, \tag{4.24}$$

and the same holds for $\{z'_1,\dots,z'_k\}$.

By the Gram-Schmidt procedure for Hermitian scalar products the orthonormal families $\{z_1,\dots,z_k\}$ and $\{z'_1,\dots,z'_k\}$ can be extended to orthonormal bases $\mathcal{Z} = \{z_1,\dots,z_k, z_{k+1},\dots,z_n\}$ and $\mathcal{Z}' = \{z'_1,\dots,z'_k, z'_{k+1},\dots,z'_n\}$ of $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{R}}$. Then $\mathfrak{h}_1$ spanned over $\mathbb{C}$ by $\mathcal{Z}$ and also by $\mathcal{Z}'$ is an n-dimensional complex space. We can find $A \in \mathrm{U}(n)$ such that $A(z_j) = z'_j$. Then $\mathcal{A} = A \times \mathrm{Id}_{\mathfrak{h}_2} \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{R}})$ and $A(V) = V'$. Since

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{Sp}(2n,\mathbb{R}),$$

we conclude that $A \in \mathrm{O}(2n)$ and $\mathcal{A} \in \mathrm{O}(2n+1)$, and therefore $\mathcal{A}(V^{\perp}) = V'^{\perp}$. This completes the proof of the assertion when $k \leq n$.

Suppose now $k > n$ and let V and V' be two orthogonally complemented subalgebras of $\mathfrak{h}^n_{\mathbb{R}}$. The assertion follows by the previous arguments applied to the orthogonal complements $V^{\perp}$ and $V'^{\perp}$.

Consider the Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$. The complex dimension of $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{C}}$ is equal to $2n$. Let V and V' be orthogonally complemented subalgebras of complex dimension $k \leq n$. By the construction in Lemma 4.4 the bases of V or V' will also be orthonormal bases with respect to the quaternion Hermitian product (4.16). Then we extend the bases of V and V' to bases of $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{C}}$ and apply the Gram-Schmidt procedure to make the bases orthonormal with respect to the quaternion Hermitian product. Noting that $\mathrm{Sp}(n) \cong \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$, we obtain that the bases will be orthogonal with respect to the original real scalar product and therefore will preserve the orthogonally complemented subalgebras. We finish the proof as in the previous case.

The last case is the Lie algebra $\mathfrak{h}^n_{\mathbb{Q}}$. In this case, we use the similar arguments noting that an orthonormal basis (with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$) for a commutative subalgebra V will be orthogonal with respect to the quaternion Hermitian form (4.18). Therefore, the basis can be extended to an orthonormal basis for $\mathfrak{h}^n_{\mathbb{Q}}$ with respect to the quaternion Hermitian form (4.18). The group $\mathrm{Sp}(n)$ acts transitively on a set of such kind of extended bases for $\mathfrak{h}^n_{\mathbb{Q}}$, preserving the orthogonality. □

Definition 4.6. The set $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$, $1 \leq k \leq \dim_{\mathbb{R}} \mathfrak{h}^n_{\mathbb{K}}$ of orthogonally complemented homogeneous subalgebras of the same topological dimension k is called the Grassmannian of the Heisenberg algebra $\mathfrak{h}^n_{\mathbb{K}}$.

According to (4.1) we can write $\mathcal{A} \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ as $\mathcal{A} = (\mathcal{U}, \mathcal{V}) \subset \mathrm{O}(\mathfrak{h}_1) \times \mathrm{O}(\mathfrak{h}_2)$. If $V = H \otimes T$, $H \subset \mathfrak{h}_1$, $T \subset \mathfrak{h}_2$, is an orthogonally complemented subalgebra then the action of $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ on $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ is given by

$$\mathcal{A}.\, V = \mathcal{U}.\, H \times \mathcal{V}.\, T. \tag{4.25}$$

Theorem 4.5 immediately implies the corollary.

Corollary 4.7. *The Grassmannian* $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ *is the orbit of the action of the isometry group* $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ *issued from an orthogonally complemented homogeneous k-dimensional subalgebra* $\hat{V} \subset \mathfrak{h}^n_{\mathbb{K}}$.

Example 4.8. Consider the Heisenberg algebra $\mathfrak{h}^2_{\mathbb{R}}$ with the basis (2.11) and the commutation relations (2.12). We write $\hat{V} = \mathrm{span}_{\mathbb{R}}\{X_1, X_2\}$. The space $\hat{V}$ is a commutative subalgebra of $\mathfrak{h}^2_{\mathbb{R}}$ orthogonally complemented by the commutative subalgebra $V = \mathrm{span}_{\mathbb{R}}\{Y_1, Y_2, \varepsilon\}$. The isometry group $\mathrm{Iso}(\mathfrak{h}^2_{\mathbb{R}})$ is given by (4.1) with $\mathbb{A} = \mathrm{U}(2)$. The orbit of the isometry group $\mathrm{Iso}(\mathfrak{h}^2_{\mathbb{R}})$ acting on $\hat{V}$ is the Grassmannian $\mathrm{Gr}(2, \mathfrak{h}^2_{\mathbb{R}})$. The planes in this Grassmannian intersect in one point and it is analogue of the Grassmann manifold of 2-dimensional planes in $\mathbb{R}^4 \cong \mathfrak{h}_1 \subset \mathfrak{h}^2_{\mathbb{R}}$. The orbit of the isometry group $\mathrm{Iso}(\mathfrak{h}^2_{\mathbb{R}})$ acting on the complementary subalgebra V is the Grassmannian $\mathrm{Gr}(3, \mathfrak{h}^2_{\mathbb{R}})$. The planes in $\mathrm{Gr}(3, \mathfrak{h}^2_{\mathbb{R}})$ intersect in the straight line coinciding with the centre of $\mathfrak{h}^2_{\mathbb{R}}$.

In the following example, we show that the complementary subalgebras can both contain the elements of the centre and be both commutative.

Example 4.9. Let us consider the Lie algebra $\mathfrak{h}^1_{\mathbb{C}}$ with the basis (4.7) and the commutation relations (4.8). We set

$$\hat{V} = \mathrm{span}_{\mathbb{R}}\{X_1, X_4, \varepsilon_1\} = \mathrm{span}_{\mathbb{C}}\{X_1\} \oplus \mathrm{span}_{\mathbb{R}}\{\varepsilon_1\}.$$

It is a commutative subalgebra of $\mathfrak{h}_{\mathbb{C}}$ that is orthogonally complemented by the commutative subalgebra $V = \mathrm{span}_{\mathbb{R}}\{X_2, X_3, \varepsilon_2\}$. The isometry group $\mathrm{Iso}(\mathfrak{h}^1_{\mathbb{C}})$ is given by (4.1) with $\mathbb{A} = \mathrm{Sp}(1) = \mathrm{Sp}(2, \mathbb{C}) \cap \mathrm{U}(2)$. The orbit of the isometry group $\mathrm{Iso}(\mathfrak{h}^1_{\mathbb{C}})$ acting on $\hat{V}$ is the Grassmannian $\mathrm{Gr}(3, \mathfrak{h}^1_{\mathbb{C}})$. The planes in $\mathrm{Gr}(3, \mathfrak{h}^1_{\mathbb{C}})$ intersect in one point.

4.3.2 Grassmannians $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ as quotient spaces

Let us fix an orthogonally complemented subalgebra $\hat{V} \subset \mathfrak{h}^n_{\mathbb{K}}$ and write

$$K(\hat{V}) = \{\mathcal{A} \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}}) : \mathcal{A}.\, \hat{V} = \hat{V}\}$$

for the isotropy group of $\hat{V}$. The canonical projection

$$\Pi : \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}}) \to \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})/K(\hat{V}) \tag{4.26}$$

is a continuous map. The action of $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ on $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ is transitive, by Lemma 9. We identify the left cosets from $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})/K(\hat{V})$ with elements in $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ by

$$\begin{aligned} \Gamma : \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})/K(\hat{V}) &\to \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}}) \\ \mathcal{A}.\, K(\hat{V}) &\mapsto V = \mathcal{A}.\, \hat{V}. \end{aligned} \tag{4.27}$$

The map Γ is a diffeomorphism, see [48, Theorem 3.62].

4.3.3 Measure on the Grassmannians $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$

The groups $\mathrm{Pin}(\mathfrak{h}_2, \langle\cdot,\cdot\rangle)$, $\mathfrak{h}_2 \subset \mathfrak{h}^n_{\mathbb{K}}$ and $\mathbb{A}$ from Section 4.2 are compact Lie groups and therefore they carry normalised Haar measures that we will denote by λ_{Pin} and $\lambda_{\mathbb{A}}$, respectively. We also will denote $\lambda = \lambda_{\mathrm{Pin}} \times \lambda_{\mathbb{A}}$ the normalised product measure on the space $\mathbb{P} = \mathrm{Pin}(\mathfrak{h}_2, \langle\cdot,\cdot\rangle) \times \mathbb{A}$. The map $\phi : \mathbb{P} \to \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ from (4.1) mapping $\theta \in \mathbb{P} \to \phi(\theta) = \mathcal{A} = (\mathcal{U}, \mathcal{V}) \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ is continuous surjective map that makes possible to push forward the measure λ to $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$. Then the map $\Psi = \Gamma \circ \Pi \circ \phi$, where Π and Γ are defined in (4.26) and (4.27), respectively, allows us to push forward the normalised Haar measure λ from $\mathbb{P}$ to $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$. We say that a set $\Omega \subset \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ is measurable if $\Psi^{-1}(\Omega) \subset \mathbb{P}$ is measurable with respect to the measure λ. The measure μ on $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ is defined by

$$\mu(\Omega) = (\Psi_\sharp\lambda)(\Omega) = \lambda(\Psi^{-1}(\Omega)) = \lambda\{\theta \in \mathbb{P} : V = \phi(\theta).\, \hat{V} = \mathcal{A}.\, \hat{V} \in \Omega\},$$

for any measurable $\Omega \subset \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$. We express the push forward in the integral form

$$\int\limits_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} f(V)\mathrm{d}\mu(V) = \int\limits_{\mathbb{P}} f(\phi(\theta).\, \hat{V})\mathrm{d}\lambda(\theta) \tag{4.28}$$

for any measurable function f on the Grassmannian.

4.3.4 The groups $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ and the product of spheres

According to Lemmas 4.1 and 4.2, the groups $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ act transitively on the product of two spheres

$$\mathcal{S}_{\mathbb{K}}(0, r_1, r_2) = S^h(0, r_1) \times S^v(0, r_2) \subset \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{h}^n_{\mathbb{K}},$$

with

$$\begin{aligned} S^h(0, r_1) &= \{g = (x, 0) \in \mathfrak{h}^n_{\mathbb{K}} : \|x\|_E = r_1\}, \\ S^v(0, r_2) &= \{g = (0, t) \in \mathfrak{h}^n_{\mathbb{K}} : \|t\|_E = r_2\}. \end{aligned}$$

The group $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ acts on $\mathcal{S}_{\mathbb{K}}(0, r_1, r_2)$ by the following

$$\mathcal{A}.\, (y, w) = (\mathcal{U}y, \mathcal{V}w) \quad \text{for any } \mathcal{A} = (\mathcal{U}, \mathcal{V}) \in \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}}), \quad (y, w) \in \mathcal{S}_{\mathbb{K}}(0, r_1, r_2).$$

We fix $(x, t) \in \mathcal{S}_{\mathbb{K}}(0, r_1, r_2)$ and define the isotropy subgroups

$$\begin{aligned} K^h_{(x,t)} &= \{\mathcal{A} \in \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}}) : \mathcal{A}.\,(x,t) = (\mathcal{U}x, \mathcal{V}t) = (x, \mathcal{V}t)\},\\ K^v_{(x,t)} &= \{\mathcal{A} \in \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}}) : \mathcal{A}.\,(x,t) = (\mathcal{U}x, \mathcal{V}t) = (\mathcal{U}x, t)\}. \end{aligned} \tag{4.29}$$

We can realise both spheres as homogeneous spaces under the action of the respective groups, see [48, Theorem 3.62]. Namely, we write

$$\Pi : \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}}) \to \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}})/(K^h_{(x,t)} \times K^v_{(x,t)}) \cong S^h(0, r_1) \times S^v(0, r_2).$$

We will use the projections

$$\Pi^h : \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}}) \to S^h(0, r_1), \quad \Pi^v : \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}}) \to S^v(0, r_2).$$

The map ϕ from (4.1) is continuous and surjective. It allows us to define the push forward measures

$$\mu^h = (\Pi^h \circ \phi)_\sharp \lambda_{\mathrm{Pin}} \quad \text{and} \quad \mu^v = (\Pi^v \circ \phi)_\sharp \lambda_{\mathbb{A}}.$$

Lemma 4.10. *The measures μ^h and μ^v are normalised measures on the spheres $S^h(0, r_1)$ and $S^v(0, r_2)$, respectively. Moreover,*

$$\int\limits_{S^v(0,r_2)} \mathrm{d}\mu^v(w) \int\limits_{S^h(0,r_1)} f(y, w)\mathrm{d}\mu^h(y) = \int\limits_{\mathbb{P}} f(\phi(\theta).\,(x, t))\mathrm{d}\lambda(\theta)$$

for any measurable function f on $S_{\mathbb{K}}(0, r_1, r_2)$ and the isotropy point (x, t) from (4.29).

Proof. The transitive action of the isometry group on $S^h(0, r_1)$ and $S^v(0, r_2)$ ensures that the measures μ^h and μ^v are uniformly distributed on the respective spheres. Therefore, they are the spherical measures up to constants, see [28].

Let $C^h \subset S^h(0, r_1)$, $C^v \subset S^v(0, r_2)$ be measurable sets and let $C \subset \mathbb{P}$ be its preimage under the map $\Pi \circ \phi = (\Pi^h \circ \phi, \Pi^v \circ \phi)$ from (4.1). We write $\theta = (\alpha, A) \in \operatorname{Pin}(\mathfrak{h}_2, \langle .,. \rangle) \times \mathbb{A}$ and $\phi(\theta) = \phi(\alpha, A) = (\mathcal{U}, \mathcal{V}) \in \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}})$, where $\mathcal{U} = J_\alpha \circ A$ and $\mathcal{V} = \kappa(\alpha)$ then

$$\begin{aligned} \int\limits_C f(\phi(\theta).\,(x,t))\mathrm{d}\lambda(\theta) &= \int\limits_{\{\alpha\in \mathrm{Pin}:(\kappa(\alpha).t)\in C^v\}} \mathrm{d}\lambda_{\mathrm{Pin}}(\alpha) \int\limits_{\{A\in\mathbb{A}:(J_\alpha\circ A.x)\in C^h\}} f(\mathcal{U}.\,x, \mathcal{V}.\,t)\mathrm{d}\lambda_{\mathbb{A}}(A)\\ &= \int\limits_{\{\alpha\in \mathrm{Pin}:(\kappa(\alpha).t)\in C^v\}} \mathrm{d}\lambda_{\mathrm{Pin}(\alpha)} \int\limits_{\{A\in\mathbb{A}:(A.x)\in C^h\}} f(\mathcal{U}.\,x, \mathcal{V}.\,t)\mathrm{d}\lambda_{\mathbb{A}}(A)\\ &= c\int\limits_{C^v} \mathrm{d}\mu^v(w)\int\limits_{C^h} f(y, w)\mathrm{d}\mu^h(y). \end{aligned}$$

In the third line, we used the fact that the group $\mathbb{A}$ is already acts transitively on the spheres $S^h(0, r_1)$ and therefore the push forward measure $\mu^h = (\Pi^h \circ \phi)_\sharp \lambda_{\mathrm{Pin}}$ is up to a constant the Hausdorff spherical measure of $S^h(0, r_1)$ due to the fact that both measures will be uniformly distributed on $S^h(0, r_1)$. ☐

Remark 4.11. Consider the subgroup $\mathbb{A} \times \mathrm{Id}_{\mathfrak{h}_2} \subset \operatorname{Iso}(\mathfrak{h}^n_{\mathbb{K}})$. Since $\mathbb{A} \times \mathrm{Id}_{\mathfrak{h}_2}$ leaves the centre $\mathfrak{h}_2$ invariant we can define the action of $\mathbb{A}$ only on the horizontal slot of coordinates. Since the group $\mathbb{A}$ acts transitively on the spheres $S^h(0, r)$, the sphere $S^h(0, r)$ passing through the point $x \in \mathfrak{h}_1$ with $\|x\|_E = r$ is a homogeneous manifold realised as a quotient of $\mathbb{A}$ by a subgroup fixing x. Then the integral form of the push forward of a measure $\tilde{\lambda}$ from $\mathbb{A}$ is the following

$$\int\limits_{S^h(0,r)} f(y)\mathrm{d}\mu^h(y) = \int\limits_{\mathbb{A}} f(\mathcal{U}x)\mathrm{d}\tilde{\lambda}(\mathcal{U}), \quad x \in S^h(0, r), \tag{4.30}$$

for any measurable function f on $S^h(0, r)$.

5 Integral formula on “special” H-type algebras

5.1 Overview of the formula in $\mathbb{R}^n$

For the Grassmann manifolds in the Euclidean space the following formula is known [22]. Let $V \in \mathrm{Gr}(k, \mathbb{R}^n)$ and

$$F(V) = \int_V f(x)\mathrm{d}\sigma(x),$$

where f is a non-negative measurable function in $\mathbb{R}^n$ and σ is the k-dimensional Lebesgue measure on the plane V. Then

$$\int_{\mathrm{Gr}(k,\mathbb{R}^n)} F(V)\mathrm{d}\mu(V) = \frac{m(S^{k-1}(0,1))}{m(S^{n-1}(0,1))} \int_{\mathbb{R}^n} \|x\|_E^{k-n} f(x)\mathrm{d}x, \tag{5.1}$$

where μ is a normalised invariant under the rotational group measure on the Grassmann manifold, $m(S^{k-1}(0,1))$ is the measure of the unit sphere $S^{k-1}(0,1) \subset \mathbb{R}^k$, $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$, and $\|x\|_E$ is the Euclidean norm of x. Our aim is finding an analogous expression for the three types of the Heisenberg algebras mentioned in Section 4.2.

5.2 Formula for special H-type Lie algebras

We start from the case of orthogonally complemented Grassmannians $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ which elements consist of the commutative subalgebras and do not include elements of the centre. We call them shortly "*horizontal*" *Grassmannians*. In this case, we recover formula (5.1).

5.2.1 Formula for the “horizontal” Grassmannians

We consider both manifolds $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ and $S^h(0, \|x\|) \subset \mathfrak{h}^n_{\mathbb{K}}$ as homogeneous subspaces under the action of the subgroup $\mathbb{A} \times \mathrm{Id} \subset \mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$. Here $\|x\|$ is the Euclidean, or Hermitian norm on $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{K}}$, according to $\mathbb{K}$.

Let $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ be the Grassmannian consisting of orthogonally complemented commutative subalgebras that do not contain elements of the centre. In this case, the topological and homogeneous dimensions of $V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ coincide and we denote them by $\mathbf{m} = \mathbf{d_t} = k$. Let also $\mathcal{L}^k$ denote k-dimensional Lebesgue measure on a generic plain $V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ with $V \subset \mathfrak{h}_1$, $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{K}}$. Set

$$F(V) = \int_V f(y)\mathrm{d}\mathcal{L}^k(y), \quad y \in V \subset \mathfrak{h}_1. \tag{5.2}$$

Here $f : \mathfrak{h}_1 \to \mathbb{R}$ is a non-negative measurable function.

Theorem 5.1. *The formula*

$$\int_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} F(V)\mathrm{d}\mu(V) = \int_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} \mathrm{d}\mu(V) \int_V f(y)\mathrm{d}\mathcal{L}^k(y) = C \int_{\mathfrak{h}_1} \|z\|^{k-m_1} f(z)\mathrm{d}\mathcal{L}^{m_1}(z)$$

holds for any measurable non-negative function $f : \mathfrak{h}_1 \to \mathbb{R}$ and an orthogonally complemented Grassmannian $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ of commutative subalgebras that do not contain elements of the centre of $\mathfrak{h}^n_{\mathbb{K}}$. Here $\mathcal{L}^{m_1}$, $m_1 = \dim_{\mathbb{R}}(\mathfrak{h}_1)$ is the Lebesgue measure on $\mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{K}}$ and $C > 0$ is a constant.

Proof. Note first that $V = A.\,\hat{V}$ for any $V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ and some $A \in \mathbb{A}$. Therefore,

$$F(V) = F(A.\,\hat{V}) = \int\limits_{A.\hat{V}} f(y)\mathrm{d}\mathcal{L}^k(y) = \int\limits_{\hat{V}} f(Ax)\mathrm{d}\mathcal{L}^k(x)$$

$$\begin{aligned}\int\limits_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} F(V)\mathrm{d}\mu(V) &= \int\limits_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} F(A.\,\hat{V})\mathrm{d}\mu(A.\,\hat{V}) \\ &= \int\limits_{\mathbb{A}} F(A.\,\hat{V})\mathrm{d}\lambda(A) \\ &= \int\limits_{\mathbb{A}} \mathrm{d}\lambda(A) \int\limits_{\hat{V}} f(Ax)\mathrm{d}\mathcal{L}^k(x) \\ &= \int\limits_{\hat{V}} \mathrm{d}\mathcal{L}^k(x) \int\limits_{\mathbb{A}} f(Ax)\mathrm{d}\lambda(A)\end{aligned} \tag{5.3}$$

by (4.28). □

Let us consider the last integral, where $x \in \hat{V}$ will also be considered as a point on the sphere $S^h(0, \|x\|) \subset \hat{V} \subset \mathfrak{h}_1$. For that $x \in S^h(0, \|x\|)$ we can consider $S^h(0, \|x\|)$ as a homogeneous manifold under the action of $\mathbb{A}$ with the isotropy group that fixes x. We denote that sphere by $S^{h,x}(0, \|x\|)$, emphasising the fixed point on the sphere. Then we use the push forward μ^h of the normalised measure λ from $\mathbb{A}$ to $S^{h,x}(0, \|x\|)$ and obtain

$$\int\limits_{\mathbb{A}} f(Ax)\mathrm{d}\lambda(A) = \int\limits_{S^{h,x}(0,\|x\|)} f(z)\mathrm{d}\mu^h(z) = \tilde{C} \int\limits_{S^h(0,1)} f(\|x\|\xi)\mathrm{d}\mathcal{S}^{m_1-1}(\xi), \tag{5.4}$$

where $\mathrm{d}\mathcal{S}^{m_1-1}(\xi)$ is the surface measure on the unit sphere $S^h(0, 1)$. In the last step, we used the following calculations

$$\begin{aligned}\int\limits_{S^{h,x}(0,\|x\|)} f(z)\mathrm{d}\mu^h(z) &= c \int\limits_{S^{h,x}(0,\|x\|)} \|x\|^{1-m_1} f(\|x\|\xi)\mathrm{d}S(\|x\|\xi) \\ &= C \int\limits_{S^h(0,1)} \|x\|^{1-m_1}\|x\|^{m_1-1} f(\|x\|\xi)\mathrm{d}\mathcal{S}^{m_1-1}(\xi) \\ &= C \int\limits_{S^h(0,1)} f(\|x\|\xi)\mathrm{d}\mathcal{S}^{m_1-1}(\xi),\end{aligned}$$

where $\mathrm{d}S(\|x\|\xi)$ is the surface measure on the sphere $S^{h,x}(0, \|x\|)$, $x \in \hat{V}$. Substituting integral (5.4) into (5.3), we obtain (for $\rho = \|x\|$)

$$\begin{aligned}\int\limits_{\hat{V}} \mathrm{d}\mathcal{L}^k(x) \int\limits_{S^h(0,1)} f(\|x\|\xi)\mathrm{d}\mathcal{S}^{m_1-1}(\xi) &= \int\limits_0^\infty \rho^{k-1}\mathrm{d}\rho \underbrace{\int\limits_{S^{k-1}(0,1)} \mathrm{d}\mathcal{S}^{k-1}(\zeta)}_{\text{constant}} \int\limits_{S^h(0,1)} f(\|x\|\xi)\mathrm{d}\mathcal{S}^{m_1-1}(\xi) \\ &= \hat{C} \int\limits_0^\infty \rho^{k-1-(m_1-1)}\mathrm{d}\rho \int\limits_{S^h(0,1)} f(\rho\xi)\rho^{m_1-1}\mathrm{d}\mathcal{S}^{m_1-1}(\xi) \\ &= \hat{C} \int\limits_{\mathfrak{h}_1} \|z\|^{k-m_1} f(z)\mathrm{d}\mathcal{L}^{m_1}(z).\end{aligned} \tag{5.5}$$

5.2.2 Formula for the “vertical” Grassmannians

Let $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ be a Grassmannian, where a typical orthogonally complemented subalgebra $V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ contains a non-trivial part of the centre of $\mathfrak{h}^n_{\mathbb{K}}$. Let $\hat{V} = \hat{V}_h \oplus \hat{V}_v$, $\hat{V}_v \neq \{0\}$ be an orthogonally complemented subalgebra, such that $\hat{V}_h \subset \mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{K}}$ and $\hat{V}_v \subset \mathfrak{h}_2 \subset \mathfrak{h}^n_{\mathbb{K}}$. We write $k_v = \dim(\hat{V}_h)$, $k_v = \dim(\hat{V}_v)$ for the topological dimensions of the vector spaces $\hat{V}_h$ and $\hat{V}_v$. Thus, $k = \dim \hat{V} = k_h + k_v$ is the topological dimension of orthogonally complemented subalgebra $\hat{V}$. A generic element $V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ is the image of $\hat{V}$ under the action of $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$. We write $(x, t) \in V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$. Let

$$F(V) = \int_V f(x, t) \mathrm{d}\mathcal{L}^k(x, t),$$

where $f : \mathfrak{h}^n_{\mathbb{K}} \to \mathbb{R}$ is a measurable non-negative function and $\mathcal{L}^k$, $k = \dim_{\mathbb{R}}(V)$ is the Lebesgue measure on V. We denote $m_1 = \dim_{\mathbb{R}}(\mathfrak{h}_1)$, $m_2 = \dim_{\mathbb{R}}(\mathfrak{h}_2)$, the topological dimensions of the horizontal $\mathfrak{h}_1$ and vertical $\mathfrak{h}_2$ layers of $\mathfrak{h}^n_{\mathbb{K}} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Thus, $N = m_1 + m_2$ is the topological dimension of the Lie algebra $\mathfrak{h}^n_{\mathbb{K}}$. Moreover, $\mathcal{L}^{m_1}$ and $\mathcal{L}^{m_2}$ are the respective Lebesgue measures on the vector spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$.

Theorem 5.2. *The formula*

$$\begin{aligned}\int_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} F(V)\mathrm{d}\mu(V) &= \int_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} \mathrm{d}\mu(V) \int_V f(x, t)\mathrm{d}\mathcal{L}^k(x, t) \\ &= C \int_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}} \|x\|^{k_h-m_1} \|t\|^{k_v-m_2} f(x, t)\mathrm{d}\mathcal{L}^{m_1}(x)\mathrm{d}\mathcal{L}^{m_2}(t)\end{aligned}$$

holds for any measurable non-negative function $f : \mathfrak{h}^n_{\mathbb{K}} \to \mathbb{R}$ and an orthogonally complemented Grassmannian $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ of subalgebras that contain a nontrivial element of the centre of $\mathfrak{h}^n_{\mathbb{K}}$. Here $C > 0$ is a constant.

Proof. The isometry group $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ does not act transitively on the spheres with respect to any of the metrics (D_2)–(D_4). This fact does not allow us to obtain a uniformly distributed measure on a sub-Riemannian sphere by pushing forward the measure from the isometry group $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$. Nevertheless, the transitive action of $\mathrm{Iso}(\mathfrak{h}^n_{\mathbb{K}})$ on the product of spheres allows us to prove Lemma 4.10. In the product of the integrals

$$\int_{\mathbb{P}} f(\phi(\theta).\ (x, t))\mathrm{d}\lambda(\theta) = \int_{S^v(0,r_2)} \mathrm{d}\mu^v(w) \int_{S^h(0,r_1)} f(y, w)\mathrm{d}\mu^h(y) \tag{5.6}$$

the measures $\mathrm{d}\mu^v$ and $\mathrm{d}\mu^h$ are the normalised measures on the spheres $S^v(0, r_2)$ and $S^h(0, r_1)$, respectively. We can use successive independent dilations in the vertical and horizontal variables and write the right-hand side of (5.6) as a product of integrals with respect to the Hausdorff measures on the unit spheres

$$C \int_{S^v(0,1)} \mathrm{d}\mathcal{S}^{m_2-1}(\eta) \int_{S^h(0,1)} f(r_2\xi, r_1\eta)\mathrm{d}\mathcal{S}^{m_1-1}(\xi). \tag{5.7}$$

Furthermore, by making use of the polar coordinates in each of the vector spaces $\hat{V}_h$ and $\hat{V}_v$ we obtain

$$\begin{aligned}\int_{\hat{V}} \mathrm{d}\mathcal{L}^{\mathbf{d_t}}(x, t) &= \int_{\hat{V}_h} \mathrm{d}\mathcal{L}^{k_h}(x) \int_{\hat{V}_v} \mathrm{d}\mathcal{L}^{k_v}(t) \\ &= \int_0^\infty \rho^{k_h-1}\mathrm{d}\rho \int_{S^{k_h-1}(0,1)} \mathrm{d}\mathcal{S}^{k_h-1}(\phi) \int_0^\infty r^{k_v-1}\mathrm{d}r \int_{S^{k_v-1}(0,1)} \mathrm{d}\mathcal{S}^{k_v-1}(\psi) \\ &= \tilde{C} \int_0^\infty \rho^{k_h-1}\mathrm{d}\rho \int_0^\infty r^{k_v-1}\mathrm{d}r.\end{aligned}$$

We recall that the measure $\mu(V)$ on $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ is the push forward of the measure $\lambda(\theta)$ from the group $\mathbb{P}$. It allows us to write

$$\begin{aligned}\int\limits_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} F(V)\mathrm{d}\mu(V) &= \int\limits_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} \mathrm{d}\mu(V)\int\limits_{V} f(x,t)\mathrm{d}\mathcal{L}^k(x,t)\\ &= \int\limits_{\hat V}\mathrm{d}\mathcal{L}^k(x,t)\int\limits_{\mathbb{P}} f(\phi(\theta).\,(x,t))\mathrm{d}\lambda(\theta)\\ &= C\int\limits_0^\infty \rho^{k_h-1}\mathrm{d}\rho\int\limits_0^\infty r^{k_v-1}\mathrm{d}r\int\limits_{S^v(0,1)}\mathrm{d}\mathcal{S}^{m_2-1}(\eta)\int\limits_{S^h(0,1)} f(\rho\xi, r\eta)\mathrm{d}\mathcal{S}^{m_1-1}(\xi)\\ &= C\int\limits_0^\infty \rho^{k_h-m_1}\mathrm{d}\rho\int\limits_0^\infty r^{k_v-m_2}\mathrm{d}r\int\limits_{S^v(0,1)}\mathrm{d}\mathcal{S}^{m_2-1}(\eta)\int\limits_{S^h(0,1)} \rho^{m_1-1}r^{m_2-1}f(\rho\xi, r\eta)\mathrm{d}\mathcal{S}^{m_1-1}(\xi)\\ &= \int\limits_{\mathbb{R}^{m_2}\times\mathbb{R}^{m_1}} \|x\|^{k_h-m_1}\|t\|^{k_v-m_2}f(x,t)\mathrm{d}\mathcal{L}^{m_1}(x)\mathrm{d}\mathcal{L}^{m_2}(t).\end{aligned}$$

It finishes the proof. ☐

5.3 Application of the integral formula

Let $\mathbb{G}$ be one of the special Heisenberg-type Lie groups of the topological dimension N and the homogeneous dimension Q with the Lie algebra $\mathfrak{h}^n_{\mathbb{K}}$, such that $\dim_{\mathbb{R}}(\mathfrak{h}_1) = m_1$.

Corollary 5.3. *Let $\Sigma \subset \Sigma^{(\mathbf{d_t},\mathbf{m})}$ be a collection of intrinsic $(\mathbf{d_t},\mathbf{m})$-Lipschitz graphs on $\mathbb{G}$. Suppose that all the graphs $S \in \Sigma$ contain a common point $g_0 \in \mathbb{G}$. Then for $\mathbf{m}p \le Q$, $p > 1$, we have $M_p(\Sigma) = 0$. In the case $\mathbf{d_t} = \mathbf{m}$ if $p\mathbf{d_t} > m_1$, then there is a family Σ of intrinsic Lipschitz graphs such that $M_p(\Sigma) \neq 0$.*

Proof. The proof of the statement that $\mathbf{m}p \le Q$ implies $M_p(\Sigma) = 0$ is the proof of Theorem 3.32. To show the statement in the opposite direction, we consider the Grassmannian $\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})$ consisting of orthogonally complemented commutative subalgebras that do not contain elements of the centre. In this case, the topological and homogeneous dimensions of $V \in \mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})$ coincides, that is $\mathbf{m} = \mathbf{d_t} = k$.

Consider $S = V \cap B^h(0,1)$, where $V \in \mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})$ and $B^h(0,1) \subset \mathfrak{h}_1$ is the Euclidean unit ball and assume by contrary that $M_p(\Sigma) = 0$, where

$$\Sigma = \{S = V \cap B^h(0,1),\ V \in \mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})\}.$$

Let $f \in L^p(\mathfrak{h}_1,\mathcal{L}^{m_1})$, $m_1 = \dim_{\mathbb{R}}(\mathfrak{h}_1)$. Then from Theorem 5.1 and the Hölder inequality for $\frac{1}{p}+\frac{1}{q} = 1$ we obtain

$$\int\limits_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})}\mathrm{d}\mu(V)\int\limits_{V\cap B^h(0,1)} f(y)\mathrm{d}\mathcal{L}^{\mathbf{d_t}}(y) = C\int\limits_{\mathfrak{h}_1\cap B^h(0,1)}\|x\|^{\mathbf{d_t}-m_1}f(x)\mathrm{d}\mathcal{L}^{m_1}(x) \le C\|f\|_{L^p(\mathfrak{h}_1)}\Bigg(\int\limits_0^1 r^{\frac{p\mathbf{d_t}-m_1}{p-1}-1}\Bigg)^{\frac{p-1}{p}}.$$

Then since $p\mathbf{d_t} > m_1$ the last integral is finite. It implies

$$\int\limits_{V\cap B^h(0,1)} f(y)\mathrm{d}\mathcal{L}^{\mathbf{d_t}}(y) < \infty \tag{5.8}$$

for μ-almost all $S = V \cap B^h(0,1) \in \Sigma$, which contradicts the assumption $M_p(\Sigma) = 0$. ☐

For the following corollary, we assume that $\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})$ is a Grassmannian, where a typical orthogonally complemented subalgebra $V \in \mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})$ contains a non-trivial part of the centre of $\mathfrak{h}^n_{\mathbb{K}}$. In this case, necessarily $\mathbf{d_t} < \mathbf{m}$. Let $V = V_h \oplus V_v$ where $V_v \neq \{0\}$, be an orthogonally complemented subalgebra, such that $V_h \subset \mathfrak{h}_1 \subset \mathfrak{h}^n_{\mathbb{K}}$ and $V_v \subset \mathfrak{h}_2 \subset \mathfrak{h}^n_{\mathbb{K}}$. We write $k_h = \dim_{\mathbb{R}}(V_h)$, $k_v = \dim_{\mathbb{R}}(V_v)$, $\mathbf{d_t} = k = k_h + k_v$, for the topological dimensions of the vector spaces V_h and V_v, and $m_1 = \dim_{\mathbb{R}}(\mathfrak{h}_1)$, $m_2 = \dim_{\mathbb{R}}(\mathfrak{h}_2)$, the topological dimensions of the horizontal and the vertical layers of $\mathfrak{h}^n_{\mathbb{K}}$.

Corollary 5.4. *Let $\Sigma \subset \Sigma^{(\mathbf{d_t},\mathbf{m})}$ be a collection of intrinsic $(\mathbf{d_t},\mathbf{m})$-Lipschitz graphs. Suppose that all the graphs $S \in \Sigma$ contain a common point $g_0 \in \mathbb{G}$. Then for $\mathbf{m}p \le Q$, $p > 1$, we have $M_p(\Sigma) = 0$. In the case $\mathbf{d_t} < \mathbf{m}$, if $pk_h > m_1$, $pk_v > m_2$, then there is a family Σ of intrinsic Lipschitz graphs such that $M_p(\Sigma) \neq 0$.*

Proof. We argue as in the proof of Corollary 5.4. Consider $S = V \cap (B^h(0,1) \times B^v(0,1))$, where $V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ and $B^h(0,1) \in \mathfrak{h}_1$, $B^v(0,1) \in \mathfrak{h}_2$ are the Euclidean unit balls. Assume that $M_p(\Sigma) = 0$, where

$$\Sigma = \{S = V \cap (B^h(0,1) \times B^v(0,1)), \quad V \in \mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})\}.$$

Let $f \in L^p(\mathfrak{h}^n_{\mathbb{K}}, \mathcal{L}^N)$, $N = \dim_{\mathbb{R}}(\mathfrak{h}^n_{\mathbb{K}})$. Then from Theorem 5.2 and the Hölder inequality for $\frac{1}{p} + \frac{1}{q} = 1$ we obtain

$$\begin{aligned}
&\int_{\mathrm{Gr}(k,\mathfrak{h}^n_{\mathbb{K}})} \mathrm{d}\mu(V) \int_V f(x,t)\mathrm{d}\mathcal{L}^k(x,t) \\
&= C \int_{\mathbb{R}^{m_2}\cap B^v(0,1)\times\mathbb{R}^{m_1}\cap B^h(0,1)} \|x\|^{k_h-m_1}\|t\|^{k_v-m_2} f(x,t)\mathrm{d}\mathcal{L}^{m_1}(x)\mathrm{d}\mathcal{L}^{m_2}(t) \\
&\le C\|f\|_{L^p(\mathfrak{h}^n_{\mathbb{K}})} \int_{\mathbb{R}^{m_2}\cap B^v(0,1)\times\mathbb{R}^{m_1}\cap B^h(0,1)} \|x\|^{(k_h-m_1)\frac{p}{p-1}}\|t\|^{(k_v-m_2)\frac{p}{p-1}}\mathrm{d}\mathcal{L}^{m_1}(x)\mathrm{d}\mathcal{L}^{m_2}(t) \\
&= \tilde{C}\|f\|_{L^p(\mathfrak{h}^n_{\mathbb{K}})} \int_0^1 r^{\frac{pk_h-m_1}{p-1}-1}\mathrm{d}r \int_0^1 \rho^{\frac{pk_v-m_2}{p-1}-1}\mathrm{d}\rho < \infty.
\end{aligned}$$

Since $pk_h > m_1$ and $pk_v > m_2$, then $\int_V f(x,t)\mathrm{d}\mathcal{L}^k(x,t) < \infty$ for μ-almost all plains $S \in \Sigma$, which contradicts to the assumption $M_p(\Sigma) = 0$. □

Remark 5.5. Two conditions

$$pk_h > m_1 \quad \text{and} \quad pk_v > m_2$$

imply

$$p(k_h + k_v) = p\mathbf{d_t} > m_1 + m_2 = N, \quad p(k_h + 2k_v) = p\mathbf{m} > m_1 + 2m_2 = Q.$$

In general $p\mathbf{m} > Q$, $p > 1$, does not imply both conditions $pk_h > m_1$ and $pk_v > m_2$ despite that the second one for $\mathfrak{h}^n_{\mathbb{R}}$ and $\mathfrak{h}^n_{\mathbb{Q}}$ is always fulfilled for the Grassmannians $\mathrm{Gr}(k, \mathfrak{h}^n_{\mathbb{K}})$ where a typical orthogonally complemented subalgebra V contains a non-trivial part of the centre. In both cases $\mathfrak{h}^n_{\mathbb{R}}$ and $\mathfrak{h}^n_{\mathbb{Q}}$ the subalgebra V necessarily contains the entire centre, and therefore $k_v = m_2$. From the other side we note that the Lipschitz surfaces meet each other not only at one point but at the entire centre.

In the case of $\mathfrak{h}^n_{\mathbb{C}}$ and $V = V_h \oplus V_v$ with $k_h = \dim_{\mathbb{R}}(V_h)$ and $k_v = \dim_{\mathbb{R}}(V_v) = 1$ we obtain that

$$p\mathbf{m} = p(k_h + 2) > Q = m_1 + 4 \;\Rightarrow\; pk_h > m_1 + 4 - 2p \ge m_1 \text{ if } p \le 2$$

but

$$pk_v = p > m_2 = 2 \text{ if } p > 2,$$

Thus, even if the Lipschitz surfaces intersect in one point, our example does not give the answer to the question: is there an example on a Carnot group that is not $\mathbb{R}^n$ where the condition $p\mathbf{m} \le Q$ with $\mathbf{d_t} < \mathbf{m}$ is necessary for the system of Lipschitz surfaces intersecting in one point to be M_p-exceptional?

Funding information: The work of Irina Markina was partially supported by the project Pure Mathematics in Norway, funded by Trond Mohn Foundation and Tromsø Research Foundation.

Conflict of interest: Bruno Franchi is currently a member of the Editorial Advisory Board of Analysis and Geometry in Metric Spaces. The authors declare no other conflict of interest.

References

[1] A. Agrachev, D. Barilari, and U. Boscain, *A comprehensive introduction to sub-Riemannian geometry*, Cambridge Studies in Advanced Mathematics, vol. 181, Cambridge University Press, Cambridge, 2020, From the Hamiltonian viewpoint, with an appendix by Igor Zelenko.

[2] L. Ahlfors and A. Beurling, *Conformal invariants and function-theoretic null-sets*, Acta Math. **83** (1950), 101–129.

[3] H. Aikawa and M. Ohtsuka, *Extremal length of vector measures*, Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 1, 61–88.

[4] L. Ambrosio and P. Tilli, *Topics on analysis in metric spaces*, Oxford Lecture Series in Mathematics and its Applications, vol. 25, Oxford University Press, Oxford, 2004.

[5] G. Antonelli and E. Le Donne, *Pauls rectifiable and purely Pauls unrectifiable smooth hypersurfaces*, Nonlinear Anal. **200** (2020), 111983, 30.

[6] G. Antonelli and A. Merlo, *On rectifiable measures in Carnot groups: existence of density*, J. Geom. Anal. **32** (2022), no. 9, Paper No. 239, 67.

[7] G. Antonelli and A. Merlo, *On rectifiable measures in Carnot groups: Marstrand-Mattila rectifiability criterion*, J. Funct. Anal. **283** (2022), no. 1, Paper No. 109495, 62.

[8] G. Antonelli and A. Merlo, *On rectifiable measures in Carnot groups: representation*, Calc. Var. Partial Differential Equations **61** (2022), no. 1, Paper No. 7, 52.

[9] Z. M. Balogh, K. Fässler, and I. D. Platis, *Modulus of curve families and extremality of spiral-stretch maps*, J. Anal. Math. **113** (2011), 265–291.

[10] Z. M. Balogh, K. Fässler, and I. D. Platis, *Modulus method and radial stretch map in the Heisenberg group*, Ann. Acad. Sci. Fenn. Math. **38** (2013), no. 1, 149–180.

[11] J. Bjorn, *Boundary continuity for quasiminimizers on metric spaces*, Illinois J. Math. **46** (2002), no. 2, 383–403.

[12] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, *Springer Monographs in Mathematics*, Springer, Berlin, 2007.

[13] L. Corwin and F. P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications: Volume 1, Part 1, Basic Theory and Examples*, vol. 18, Cambridge University Press, Cambridge, 1990.

[14] H. Federer, *Die Grundlehren der mathematischen Wissenschaften*, Band 153, Springer-Verlag New York Inc., New York, 1969.

[15] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[16] B. Franchi, R. Serapioni, and F. S. Cassano, *On the structure of finite perimeter sets in step 2 Carnot groups*, J. Geom. Anal. **13** (2003), no. 3, 421–466.

[17] B. Franchi, R. Serapioni, and F. S. Cassano, *Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups*, Comm. Anal. Geom. **11** (2003), no. 5, 909–944.

[18] B. Franchi, R. Serapioni, and F. S. Cassano, *Intrinsic Lipschitz graphs in Heisenberg groups*, J. Nonlinear Convex Anal. **7** (2006), no. 3, 423–441.

[19] B. Franchi, R. Serapioni, and F. S. Cassano, *Regular submanifolds, graphs and area formula in Heisenberg groups*, Adv. Math. **211** (2007), no. 1, 152–203.

[20] B. Franchi and R. P. Serapioni, *Intrinsic Lipschitz graphs within Carnot groups*, J. Geom. Anal. **26** (2016), no. 3, 1946–1994.

[21] B. Fuglede, *Extremal length and functional completion*, Acta Math. **98** (1957), 171–219.

[22] B. Fuglede, *An integral formula*, Math. Scand. **6** (1958), 207–212.

[23] M.-P. Gong, *Classification of Nilpotent Lie Algebras of Dimension 7* (*Over Algebraically Closed Fields and R*), ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)-University of Waterloo (Canada), 1998.

[24] M. Gromov, *Carnot-Carathéodory Spaces Seen from Within, Sub-Riemannian Geometry*, Progress in Mathematics, vol. 144, Birkhäuser, Basel, 1996, pp. 79–323.

[25] B. Kirchheim and F. S. Cassano, *Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), no. 4, 871–896.

[26] V. Kivioja and E. Le Donne, *Isometries of nilpotent metric groups*, J. Éc. Polytech. Math. **4** (2017), 473–482.

[27] I. Markina, *p-module of vector measures in domains with intrinsic metric on Carnot groups*, Tohoku Math. J. (2) **56** (2004), no. 4, 553–569.

[28] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability.

[29] G. Métivier, *Hypoellipticité analytique sur des groupes nilpotents de rang 2*, Duke Math. J. **47** (1980), no. 1, 195–221.

[30] J. Mitchell, *On Carnot-Carathéodory metrics*, J. Differential Geom. **21** (1985), no. 1, 35–45.

[31] M. Ohtsuka, *Extremal Length and Precise Functions*, GAKUTO International Series. Mathematical Sciences and Applications, vol. 19, Gakkōtosho Co., Ltd., Tokyo, 2003, With a preface by Fumi-Yuki Maeda.

[32] A. Ottazzi and B. Warhurst, *Algebraic prolongation and rigidity of Carnot groups*, Monatsh. Math. **162** (2011), no. 2, 179–195.

[33] H. M. Reimann, *Rigidity of H-type groups*, Math. Z. **237** (2001), no. 4, 697–725.

[34] S. Rickman, *Quasiregular Mappings*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* (*3*) [*Results in Mathematics and Related Areas* (*3*)], vol. 26, Springer-Verlag, Berlin, 1993.
[35] C. R. Riehm, *The automorphism group of a composition of quadratic forms*, Trans. Amer. Math. Soc. **269** (1982), no. 2, 403–414.
[36] C. R. Riehm, *Explicit spin representations and Lie algebras of Heisenberg type*, J. London Math. Soc. (2) **29** (1984), no. 1, 49–62.
[37] K. A. Ross and E. Hewitt, *Abstract Harmonic Analysis: Structure of Topological Groups: Integration Theory*. Group representations, 1963.
[38] W. Rossmann, *Lie Groups: An Introduction Through Linear Groups*, vol. 5, Oxford University Press, Oxford, 2006.
[39] S. Semmes, *On the nonexistence of bi-lipschitz parameterizations and geometric problems about ainfty-weights*, Rev. Mat. Iberoamericana **12** (1996), no. 2, 337–410.
[40] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279.
[41] N. Shanmugalingam, *Harmonic functions on metric spaces*, Illinois J. Math. **45** (2001), no. 3, 1021–1050.
[42] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.
[43] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
[44] K. Strebel, *On the existence of extremal Teichmueller mappings*, J. Analyse Math. **30** (1976), 464–480.
[45] J. Väisälä, *Lecture Notes in Mathematics*, vol. 229, Springer-Verlag, Berlin-New York, 1971.
[46] D. Vittone, *Lipschitz surfaces, perimeter and trace theorems for BV functions in Carnot-Carathéodory spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), no. 4, 939–998.
[47] B. V. Šabat, *The modulus method in space*, Soviet Math. Dokl. **1** (1960), 165–168.
[48] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983, Corrected reprint of the 1971 edition.