COMBING A LINKAGE IN AN ANNULUS*

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Abstract. A *linkage* in a graph G of *size* k is a subgraph L of G whose connected components are k paths. The *pattern* of a linkage of size k is the set of k pairs formed by the endpoints of these paths. A consequence of the *Unique Linkage Theorem* is the following: there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if a plane graph G contains a sequence $\mathcal{C}$ of at least $f(k)$ nested cycles and a linkage of size at most k whose pattern vertices lay *outside* the outer cycle of $\mathcal{C}$, then G contains a linkage with the same pattern avoiding the inner cycle of $\mathcal{C}$. In this paper we prove the following variant of this result: Assume that all the cycles in $\mathcal{C}$ are "orthogonally" traversed by a linkage P and L is a linkage whose pattern vertices may lay either outside the outer cycle or inside the inner cycle of $\mathcal{C} := [C_1, \ldots, C_p, \ldots, C_{2p-1}]$. We prove that there are two functions $g, f : \mathbb{N} \to \mathbb{N}$, such that if L has size at most k, P has size at least $f(k)$, and $|\mathcal{C}| \geq g(k)$, then there is a linkage with the same pattern as L that is "internally combed" by P, in the sense that $L \cap C_p \subseteq P \cap C_p$. This result applies to any graph that is partially embedded on a disk (where $\mathcal{C}$ is also embedded). In fact, we prove this result in the most general version where the linkage L is s-scattered: every two vertices of distinct paths are within distance bigger than s. We deduce several variants of this result in the cases where $s = 0$ and $s > 0$. These variants permit the application of the unique linkage theorem on several path routing problems on embedded graphs.

Key words. Linkage, Treewidth, Irrelevant vertex technique

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1. Introduction.

1.1. The Disjoint Paths Problem. One of the most central problems in algorithmic graph theory and combinatorial optimization is the Disjoint Paths Problem. An instance of the Disjoint Paths Problem (in short DPP) is a graph G and a collection $\mathcal{T} = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of pairs of terminals. Given such an instance $(G, \mathcal{T})$, the problem asks whether there are k vertex-disjoint paths $P_1, \ldots, P_k$ in G, where, for every $i \in \{1, \ldots, k\}$, P_i is a path between s_i and t_i. This problem, as well as its directed, edge-disjoint, and half-integral variants, has been extensively studied (see, for example, [41, 42, 6, 61, 50, 14, 53]) and has numerous applications in network routing, transportation, and VLSI design. From the viewpoint of computational complexity, this problem is known to be NP-complete [26], even on planar graphs [45] (see [64, 48, 43] for NP-completeness of other variants of the problem). However, for fixed values of k, Robertson and Seymour, in Volume XIII of their seminal Graph Minors series [54], proved that the problem can be solved in time $\mathcal{O}(n^3)$. Moreover, the problem is solvable in linear time when the input graph is planar [51, 52], or embeddable on a surface of fixed Euler genus [40, 51].

1.2. The irrelevant vertex technique. In order to design their polynomial time algorithm for DPP, Robertson and Seymour [54] introduced the celebrated *irrelevant vertex technique*. This technique focuses on structural characteristics of the input that may permit the detection of a vertex of the graph whose removal does

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not change the answer to the instance. More formally, Robertson and Seymour [54] proved that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, if a given an instance $(G, \mathcal{T})$, where G has treewidth bigger than $f(k)$, then it is possible, in linear time, to find in G a (non-terminal) vertex x that is *irrelevant* in the sense that $(G, \mathcal{T})$ is a yes-instance of DPP if and only if $(G \setminus x, \mathcal{T})$ is. The guiding idea behind the application of this technique is, as long as the treewidth of the graph is bigger than $f(k)$, to detect irrelevant vertices and remove them from the graph, in order to obtain an equivalent instance of bounded treewidth. Then, one can design a dynamic programming algorithm to solve the bounded treewidth instance.

In fact, the main combinatorial condition that allows to characterize a vertex (or some part of the graph) to be irrelevant is the existence of sufficiently many "insulation layers" surrounding the potentially irrelevant part of the graph. These insulation layers permit "rerouting" any possible solution "away" from the insulated part, which then can be safely deleted. To prove this rerouting argument is a quite technical task and the proofs span Volumes XXI and XII of the Graph Minors series [55, 56]. This result, known as the Unique Linkage Theorem, has been further studied and improved – see [38, 2, 47]. Furthermore, in [32], Kawarabayashi, Kobayashi, and Reed proved that the insulation layers can be found in linear time, implying an (improved) quadratic time algorithm for DPP.

Adapting the above arguments for other problems that involve the identification of paths or collections of paths in graphs is a challenging task that has attracted a lot of attention from researchers, elevating the *irrelevant vertex technique* to a standard algorithmic paradigm for solving such problems [10, 51, 23, 44, 35, 46, 40, 20, 30, 27, 36, 34, 16, 13, 22, 63, 33, 24, 25, 12, 21, 37, 1, 5, 28, 11, 4, 57, 58, 59, 60, 18, 15, 3, 8, 9, 19].

Linkages. In [54], Robertson and Seymour defined the notion of *linkages*. A *linkage* L of a graph is a collection of pairwise vertex-disjoint paths, called the *paths* of L. The *size* of L is the number of paths of L. The endpoints of the paths of a linkage are called *terminals*. The *pattern* of a linkage L is the set of pairs of endpoints of the paths in L. Two linkages of a graph are *equivalent* if they have the same pattern. Using this terminology, to declare a vertex $v \in V(G)$ *irrelevant* for DPP, one has to prove that for every linkage between the terminals in $\mathcal{T}$, there is a linkage L' of G that is equivalent to L and does not contain the vertex v.

Insulation of vertices. The combinatorial structure in [54] that allows the "rerouting" of linkages away from a vertex is a sequence of "insulation layers". In plane graphs, this is interpreted as a sequence of *nested* cycles $\mathcal{C} = [C_1, \ldots, C_p]$, each C_i "cropping" an open disk D_i such that, for every $i \in \{1, \ldots, p-1\}$, D_i is contained in D_{i+1}. The cycle C_1 is called the *inner* cycle of $\mathcal{C}$ and the cycle C_p is called the *outer* cycle of $\mathcal{C}$. Under this setting, the consequence of the Unique Linkage Theorem that we are interested in is the following:

PROPOSITION 1.1. *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, if a plane graph G contains a sequence of at least $f(k)$ nested cycles $\mathcal{C} = [C_1, \ldots, C_p]$, then for every linkage L of size at most k whose terminals are outside D_p, there is an equivalent linkage L' that avoids D_1.*

1.3. Our results. In this paper, we prove a variant of Proposition 1.1, that allows to handle linkages in a graph in a more "disciplined" way. This variant permits the application of the Unique Linkage Theorem on several routing problems on embedded graphs and has already served as the combinatorial base of [4, 57, 58, 59, 60, 18, 8, 19]. In order to present it, we introduce some additional definitions.

Railed annuli. In our result, we demand a "richer" structure that this of nested cycles. In fact, we assume that we are given a sequence of nested cycles $\mathcal{C}$ that is "orthogonally" traversed by a linkage P, meaning that the intersection of every path of P with every cycle of $\mathcal{C}$ is a (possibly trivial) path. We call this graph a *railed annulus*, we denote it by $\mathcal{A} = (\mathcal{C}, P)$, and we refer to P as the *rails* of $\mathcal{A}$. Note that one can ask every path of P to have just *non-empty intersection* with every cycle of $\mathcal{C}$ and use planarity in order to obtain a linkage P' that traverses $\mathcal{C}$ orthogonally.

Given a plane graph G and a railed annulus $\mathcal{A} = (\mathcal{C}, P)$ of G, where $\mathcal{C} = [C_1, \ldots, C_{2p-1}]$ for some $p \in \mathbb{N}$, we say that a linkage L of G is *combed* in $\mathcal{A}$, if L "crosses" C_p through the rails of $\mathcal{A}$, i.e., $L \cap C_p \subseteq P \cap C_p$.

THEOREM 1.2 (Informal). *There are two functions $g, f : \mathbb{N} \to \mathbb{N}$, such that if G is a plane graph, $\mathcal{A} = (\mathcal{C}, P)$ is a railed annulus of G where $|\mathcal{C}| \geq f(k)$ and P has size at least $g(k)$, and L is a linkage of G of size at most k whose terminals are either outside the outer cycle of $\mathcal{C}$ or inside the inner cycle of $\mathcal{C}$, then there is a linkage L' equivalent to L that is combed in $\mathcal{A}$.*

In fact, we prove Theorem 1.2 in a more general setting. The planarity condition is only necessary for the part of the graph bounded by the inner and outer cycle of $\mathcal{C}$. Therefore, it suffices to demand for G to be *partially annulus-embedded*, which intuitively means that, given a closed annulus Δ (i.e., a set homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2\}$), there is a subgraph K of G embedded in Δ and there is neither an edge in G from the interior of Δ to the part of G outside Δ, nor an edge connecting the two "parts" of G "cropped out" from Δ. Analogously, we define Δ-embedded railed annuli $\mathcal{A} = (\mathcal{C}, P)$, where Δ is the closed annulus bounded by the inner and the outer cycle of $\mathcal{C}$.

THEOREM 1.3 (Informal). *There are two functions $g, f : \mathbb{N} \to \mathbb{N}$, such that if Δ is a closed annulus, G is a partially Δ-embedded graph, $\mathcal{A} = (\mathcal{C}, P)$ is a Δ-embedded railed annulus of G where $|\mathcal{C}| \geq f(k)$ and P has size at least $g(k)$, and L is a linkage of G of size at most k whose terminals are outside Δ, then there is a linkage L' equivalent to L that is combed in $\mathcal{A}$.*

Linkage reducible graph classes. The functions f and g in Theorem 1.3 are, in general, "immense". In fact, the function g is asymptotically equal to the function f_{ul} of the Unique Linkage Theorem [39, 55] and $f(k)$ is asymptotically equal to $(g(k))^2$. This function f_{ul} was improved to a single-exponential function on k in the case of planar graphs [2]. In [47], Mazoit improved f_{ul} to a single exponential function on $k + g$ for graphs embedded in a surface of Euler genus at most g.

We say that a graph class $\mathcal{G}$ is *linkage reducible* if it is hereditary (i.e., if $G \in \mathcal{G}$ then for every $S \subseteq V(G)$, the subgraph of G induced by the vertices in S belongs to $\mathcal{G}$) and if there is a function $f_{\mathcal{G}} : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every $G \in \mathcal{G}$, if the treewidth of G is at least $f_{\mathcal{G}}(k)$ and G contains a linkage L of size at most k, then there is a vertex $v \in V(G)$ such that $G \setminus v$ contains a linkage L' that is equivalent to L. Proposition 1.1 implies that the class of all graphs is linkage reducible.

Our results are modulable for every linkage reducible graph class (see the statement of Theorem 3.1 in Subsection 3.2). In this sense, the result of Mazoit [47] implies single-exponential upper bounds on $k + g$ for the functions f and g of Theorem 1.3, if we restrict ourselves to graphs embedded in surfaces of Euler genus at most g.

Scattered linkages. A interesting variant of DPP is the one where we demand the vertex-disjoint paths $P_1, \ldots, P_k$ between the terminals to be *induced*, i.e., for every $i \in [k]$ and every $v \in V(P_i)$, v has no neighbors in $\bigcup_{j \neq i} V(P_j)$. This problem has been extensively studied [40, 31, 17, 29] and is NP-complete even for $k = 2$ [29].

In general, given an integer $r \geq 0$, we say that a linkage is *r-scattered* if for every $i \in [k]$ and every $v \in V(P_i)$, there is no vertex in $\bigcup_{j \neq i} V(P_j)$ that is in distance at most r from v. Therefore, in a 0-scattered linkage we ask its paths to be vertex-disjont, while in a 1-scattered linkage we ask its paths to be induced.

We prove our results in terms of r-scattered linkages (Theorem 1.3 is the special case where $r = 0$) for classes that are r-linkage reducible, i.e., defining linkage reducibility by demanding that the equivalent linkages in this definition are also r-scattered.

THEOREM 1.4. *There are two functions $g, f : \mathbb{N}^2 \to \mathbb{N}$, such that for every $r, k \in \mathbb{N}$, if $\mathcal{G}$ is an r-linkage reducible graph class, Δ is a closed annulus, G is a partially Δ-embedded graph that belongs to $\mathcal{G}$, $\mathcal{A} = (\mathcal{C}, P)$ is a Δ-embedded railed annulus where $|\mathcal{C}| \geq f(k, r)$ and P has size at least $g(k, r)$, and L is an r-scattered linkage of size at most k whose terminals are outside Δ, then there is an r-scattered linkage L' equivalent to L that is combed in $\mathcal{A}$.*

For general positive values of r, the only known r-linkage reducible graph class is the class of planar graphs for $r = 1$ and this is implied by the results of Kawarabayashi and Kobayashi [29]. Using the ideas of [29] and the version of the Unique Linkage Theorem for surfaces of Mazoit [47], we complement our results by proving the following:

THEOREM 1.5. *For every $r \in \mathbb{N}$ and every $g \in \mathbb{N}$, the class of graphs embeddable on a surface of Euler genus g is r-linkage reducible.*

Organization of the paper. In Section 2, we present an overview of the proofs of Theorem 1.4 and Theorem 1.5. Then, in Section 3, we provide formal definitions and statements of our main results. The proof of Theorem 1.4 is presented in Section 4 and the proof of Theorem 1.5 is presented in Section 5.

2. Overview of the two main proofs. In this section, we present a high-level description of the proofs of Theorem 1.4 and Theorem 1.5.

2.1. Linkage Combing Lemma. In Theorem 1.4 we are given an r-linkage reducible graph class $\mathcal{G}$, a graph $G \in \mathcal{G}$ that is partially Δ-embedded, for some closed annulus Δ, and a Δ-embedded railed annulus $\mathcal{A} = (\mathcal{C}, P)$, where $\mathcal{C} = [C_1, \ldots, C_{2p-1}]$, and an r-scattered linkage L of G of size at most k, whose terminals avoid Δ (from now on, we call such a linkage Δ-*avoiding*). Our objective is to reroute the different paths of L in a way that, if they intersect the "central" cycle C_p of $\mathcal{A}$, then this intersection should be part of the rails P.

Minimal linkages. A crucial notion in our arguments is the one of *minimal linkages*. Intuitively, a *minimal linkage* with respect to L and $\mathcal{C}$ is a linkage L' that is equivalent to L and the number of edges of L' that are not edges of cycles in $\mathcal{C}$ is minimal among all linkages equivalent to L (see Subsection 4.1 for a formal definition). Such a minimal linkage "diverges" the least possible from $\mathcal{C}$. The minimality of L' implies that the treewidth of the graph $H_{L',\mathcal{C}}$ obtained by the union of L' and the cycles in $\mathcal{C}$ is bounded by $f_{\mathcal{G},r}(k)$, where $f_{\mathcal{G},r}$ is the function bounding the treewidth of the graphs in the definition of r-linkage reducibility of $\mathcal{G}$(see Lemma 4.2).

In the rest of this subsection we distinguish the subpaths of the paths of a linkage that are "inside" the annulus into three main types. First, we have *rivers* that are parts of the linkage that intersect both the inner and the outer cycle of $\mathcal{C}$ and are maximal subpaths of a path of L that is inside Δ. Also, the parts of the paths of the linkage that "enter" and "exit" the annulus from the same side and never intersect the other side are called *mountains* and *valleys*, depending whether the terminals of the corresponding path of L are from the "side" of the outer cycle or of the inner cycle, respectively. In what follows, we sketch how to prove that a minimal linkage L' has at most $f_{\mathcal{G},r}(k)$ rivers and its mountains (resp. valleys) have "height" (resp. "depth") at most $f_{\mathcal{G},r}(k)$, using the fact that the treewidth of $H_{L',\mathcal{C}}$ is at most $f_{\mathcal{G},r}(k)$ and then how to reroute the (few) rivers of L' in order to comb L' through the central cycle of $\mathcal{A}$.

Minimal linkages have few rivers. Assuming that the size of $\mathcal{C}$ is larger than $f_{\mathcal{G},r}(k)$, one can observe that if L' has more than $f_{\mathcal{G},r}(k)$ rivers, then $H_{L',\mathcal{C}}$ contains a bramble of size larger than $f_{\mathcal{G},r}(k)$, a structure that forces the treewidth to be larger than $f_{\mathcal{G},r}(k)$, implying a contradiction to the fact that the treewidth of $H_{L',\mathcal{C}}$ is at most $f_{\mathcal{G},r}(k)$ (see Subsection 4.2 and in particular Lemma 4.4 and Lemma 4.5).

Minimal linkages do not have high mountains or deep valleys. The *height* (resp. *depth*) of a mountain (resp. valley) of L' measures the "intrusion" of the mountain (resp. valley) in $\mathcal{C}$, i.e., how many cycles it intersects (see Subsection 4.3 for formal definitions of mountains and valleys). To prove that mountains (resp. valleys) have height (resp. depth) at most $f_{\mathcal{G},r}(k)$, we first show that all mountains and valleys of L' are *tight*, i.e., they cannot be "compressed" so as to intersect less cycles of $\mathcal{C}$ (see Lemma 4.6). Then, since the existence of a *tight* mountain (resp. valley) of "big enough" height (resp. depth) also implies the existence of a "large enough" bramble, we obtain an upper bound on the height (resp. depth) of mountains (resp. valleys) of L' (see Lemma 4.7 and Lemma 4.8). This also implies that L' cannot intersect the inner part of "sufficiently insulated" sequence of nested closed disks (Lemma 4.9).

Combing the linkage. Having all above tools, the proof of Theorem 1.4 works as follows: We consider a minimal linkage L' with respect to L and $\mathcal{C}$. We also consider the sequence of nested cycles obtained from $\mathcal{A}$ as in Figure 11, which bound a sequence of disks that are subsets of the closed annulus Δ. The size of $\mathcal{C}$ and P allows to take a sufficiently large such sequence so as the inner disk defined in this sequence, denoted by D, is not intersected by any mountain or valley of L'. The latter follows from the fact that the mountains and the valleys of L' have "small" height and depth, respectively. The disk D is situated in the "center" of $\mathcal{A}$, in the sense that it intersects only some "central" cycles of $\mathcal{C}$. These cycles are also intersected by the (few) rivers of L'. Using the railed annulus infrastructure, we can reroute the rivers of L' inside D and then prove that they can be "combed" through the rails of $\mathcal{A}$ (see Lemma 4.11).

2.2. Irrelevant vertices for scattered linkages. After proving Theorem 1.4, our task is to provide a proof for Theorem 1.5. The proof of Theorem 1.5 is based on ideas used in [29] to deal with the Induced Disjoint Paths Problem in planar graphs. We generalize these ideas for r-scattered linkages and for graphs embedded in more general surfaces. For this, we make use of the version of the Unique Linkage Theorem for graphs embedded in surfaces proved in [47].

The proof of Theorem 1.5 boils down to the proof of the following result: there is a function $f : \mathbb{N}^3 \to \mathbb{N}$ such that for every $r, g, k \in \mathbb{N}$, if G is a graph embedded on some surface Σ of Euler genus g, L is a Δ-avoiding r-scattered linkage of G of size at most k, where Δ is an open disk of Σ, and $\mathcal{C}$ is a Δ-nested sequence of $f(r, k, g)$ cycles of G, and v is a vertex of G that is inside the disk "cropped" by the inner cycle of $\mathcal{C}$, then there is an r-scattered linkage L' of $G \setminus v$ that is equivalent to L. The case where L does not contain v directly implies the result so we focus on the case where L contains v.

In order to prove the above result, we proceed to define a notion of minimal r-scattered linkages, minimizing its number of bridges and crossings with respect to $\mathcal{C}$ and we show that such a linkage can be rerouted in order to avoid v.

Bridges and crossings. Given an open disk Δ of a surface Σ, a graph G embedded in Σ and a Δ-avoiding linkage L of G, we define the *bridges* of L to be the maximal subpaths of L that do not contain any vertex inside Δ. A *crossing* of L with a Δ-nested sequence of cycles $\mathcal{C}$ of G is a subpath of L that "transverses" a cycle of $\mathcal{C}$. We consider a Δ-nested sequence of cycles $\mathcal{C}$ of G such that every two consecutive cycles in the sequence have distance at least r such that v is inside the disk "cropped" by the inner cycle of $\mathcal{C}$ that is inside Δ. Also, we consider *BC-minimal linkages around* v, that are linkages that are equivalent to L that contain v and have minimum number of bridges and crossings with respect to the Δ-nested sequence of cycles $\mathcal{C}$. We observe that, if the number of bridges of a BC-minimal linkage L' is at most $|\mathcal{C}| - k - 1$, then L' has at most $|\mathcal{C}| - 1$ components in Δ and therefore L' can be rerouted in the graph obtained from the union of L' and the cycles in $\mathcal{C}$ in order to obtained an r-scattered linkage equivalent to L' that also avoids v (Lemma 5.4).

Rainbows. We aim to show that any given BC-minimal linkage around v has at most $|\mathcal{C}| - k - 1$ bridges. In order to do this, we classify bridges in *rainbows* around Δ. A Δ-*rainbow* of L is a sequence of Δ-bridges of L that are *homotopic*, in the sense that they pairwise bound closed disk of Σ (there is not handle of the surface interfering between them). Note that the number of homotopy classes of cycles in a surface is bounded by a (linear) function on the Euler genus (see Proposition 5.5 and Lemma 5.6). Suppose, towards a contradiction, that there are "many enough" Δ-bridges of L. Then "sufficiently large" collection $\mathcal{B}$ of them should have the following property: they belong to the same homotopy class and the two "marginal" bridges in this homotopy class bound an open disk that contains the rest and does not contain any terminal of L'. The Δ-bridges in $\mathcal{B}$ together with parts of the cycles of $\mathcal{C}$ can define a sequence of nested cycles around the "central" bridge in $\mathcal{B}$. Then, using the result of [47], we can prove that there is a linkage equivalent to L' that avoids this central bridge, implying a contradiction to the BC-minimality of L'. In fact, to impose r-scatteredness, we contract some edges of L and we apply the result of [47], using a trick appeared in [29] for the case where $r = 1$, which we generalize to arbitrary $r \in \mathbb{N}$.

3. Preliminaries. In this section, we provide formal definitions and statements of our results. In Subsection 3.1, we provide some basic definitions on graphs, (partial) embeddability of graphs in subsets of the plane, and railed annuli. Then, in Subsection 3.2, we introduce some definitions concerning linkages and we state our main result (Theorem 3.1). In Theorem 3.1, the size perquisites on the underlying combinatorial structure of the given graph G are governed by a function $f_{\mathcal{G},r}$, where $\mathcal{G}$ is a class where $\mathcal{G}$ belongs that enjoys an irrelevant-vertex-type reduction of its r-scattered linkages if the treewidth of G is at least $f_{\mathcal{G},r}(|L|)$. By the Unique Linkage Theorem [39, 55] (see also Proposition 3.2), when considering linkages that are 0-scattered, there is such a function f for the class of all graphs. This function is huge but it can become single exponential for the class of graphs embedded in some fixed surface; and this holds for linkages of any given scatteredness. We present this in more detail in Subsection 3.3, where we state the analogous versions of Theorem 3.1 for the two aforementioned compromises between graph classes, scatteredness, and size dependence.

3.1. Definitions. We denote by $\mathbb{N}$ the set of non-negative integers. Given two integers p and q, the set $[p, q]$ refers to the set of every integer r such that $p \leq r \leq q$. For an integer $p \geq 1$, we set $[p] = [1, p]$ and $\mathbb{N}_{\geq p} = \mathbb{N} \setminus [0, p-1]$.

Basic concepts on graphs. All graphs in this paper are undirected, finite, and they do not have loops or multiple edges. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs, then we denote $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. Also, given a graph G and a set $S \subseteq V(G)$, we denote by $G \setminus S$ the graph obtained if we remove from G the vertices in S, along with their incident edges. A *path* (resp. *cycle*) in a graph G is a connected subgraph with all vertices of degree at most (resp. exactly) 2. A path is *trivial* if it has only one vertex and it is *empty* if it is the empty graph (i.e., the graph with empty vertex set). Given a graph G, an $r \in \mathbb{N}$, and a $v \in V(G)$, we define the *r-neighborhood* $N_G^{(\leq r)}[v]$ of v in G to be the set of all vertices $u \in V(G)$ such that there is a path of length at most r between u and v in G. Also, given a set $S \subseteq V(G)$, we use $N_G^{(\leq r)}[S]$ to denote the union over all $v \in S$ of $N_G^{(\leq r)}[v]$.

Disk and annuli on the plane. A *cycle* is a set homeomorphic to the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. We define a *closed disk* (resp. *open disk*) to be a set homeomorphic to the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ (resp. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$) and a *closed annulus* (resp. *open annulus*) to be a set homeomorphic to the set $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2\}$ (resp. $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$). Given a closed disk or a closed annulus X, we use $\mathbf{bor}(X)$ to denote the boundary of X (i.e., the set of points of X for which every neighborhood around them contains some point not in X). Notice that if X is a closed disk then $\mathbf{bor}(X)$ is a cycle, while if X is a closed annulus then $\mathbf{bor}(X) = C_1 \cup C_2$ where C_1, C_2 are the two unique connected components of $\mathbf{bor}(X)$ and C_1, C_2 are two disjoint cycles. We call C_1 and C_2 *boundaries* of X. We call C_1 the *left boundary* of X and C_2 the *right boundary* of X. Also given a closed disk (resp. closed annulus) X, we use $\mathsf{int}(X)$ to denote the open disk (resp. open annulus) $X \setminus \mathbf{bor}(X)$. When we embed a graph G in the plane, in a closed disk, or in a closed annulus, we treat G as a set of points. This permits us to make set operations between graphs and sets of points.

Partially embedded graphs. Given a graph G, we say that a pair $(L, R) \in 2^{V(G)} \times 2^{V(G)}$ is a *separation* of G if $L \cup R = V(G)$ and there is no edge in G between a vertex in $L \setminus R$ and a vertex in $R \setminus L$. We say that two separations (X_1, Y_1) and (X_2, Y_2) of a graph G are *laminar* if $Y_1 \subseteq Y_2$ and $X_2 \subseteq X_1$.

Given a closed disk Δ, we say that a graph G is *partially Δ-embedded*, if there is some subgraph K of G that is embedded in Δ such that $\mathbf{bor}(\Delta)$ is a cycle of K and $(V(G) \cap \Delta, V(G) \setminus \mathsf{int}(\Delta))$ is a separation of G. Similarly, given a closed annulus Δ, we say that a graph G is *partially Δ-embedded*, if there is some subgraph K of G that is embedded in Δ such that $\mathbf{bor}(\Delta)$ is the disjoint union of two cycles of K and there are two laminar separations (X_1, Y_1) and (X_2, Y_2) of G such that $X_1 \cap Y_2 = V(G) \cap \Delta$. In both above cases, given a graph G that is partially Δ-embedded and a subgraph K of G that is embedded in Δ and has the aforementioned properties, we call K the *compass* of (this particular partial embedding of) G. Moreover, we always assume that a partially Δ-embedded graph G is accompanied with an embedding of its compass in Δ.

Parallel cycles. Let Δ be a closed annulus with boundaries B_1, B_2, and let G be a partially Δ-embedded graph. Also, let $\mathcal{C} = [C_1, \ldots, C_p]$, $p \geq 2$ be a collection of vertex disjoint cycles of G that are embedded in Δ. We say that $\mathcal{C}$ is a *Δ-parallel sequence of cycles* of G if $C_1 = B_1$, $C_p = B_2$ and, for every $i \in [p-1]$, C_i and C_p are the boundaries of a closed annulus, denoted by D_i, that is a subset of Δ such that

$\Delta = D_1 \supseteq \cdots \supseteq D_{p-1}$. From now on, each Δ-parallel sequence $\mathcal{C} = [C_1, \ldots, C_p]$, $p \geq 2$ of cycles will be accompanied with the sequence $[D_1, \ldots, D_{p-1}]$ of the corresponding closed annuli.

If Δ is a closed disk, then a sequence $\mathcal{C}$ of vertex disjoint cycles of a partially Δ-embedded graph G is called Δ-*nested*, if every C_i is the boundary of a closed disk D_i of Δ such that $\Delta \supseteq D_1 \supseteq \cdots \supseteq D_p$. Each Δ-nested sequence $\mathcal{C}$ will be accompanied with the sequence $[D_1, \ldots, D_p]$ of the corresponding closed disks.

We stress that D_i is a closed annulus (resp. disk) if Δ is a closed annulus (resp. disk). In both above cases (i.e., either Δ is a closed annulus or a closed disk), we call $|\mathcal{C}|$ the *size* of $\mathcal{C}$ and, given $i, j \in [p-1]$, where $i \leq j$, we call the set $D_i \setminus \mathsf{int}(D_j)$ (i,j)-*annulus* of $\mathcal{C}$ and we denote it by $\mathsf{ann}(\mathcal{C}, i, j)$. Also, for every $i \in [p-1]$, we set D_i to be the (i,p)-*annulus* of $\mathcal{C}$ and we also denote it by $\mathsf{ann}(\mathcal{C}, i, p)$.

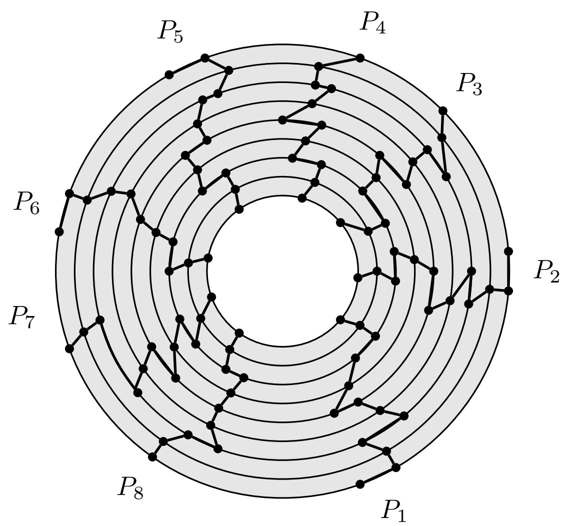

FIGURE 1. *An example of an annulus Δ depicted in grey and a Δ-embedded $(9,8)$-railed annulus $\mathcal{A} = (\mathcal{C}, \mathcal{P})$.*

Railed annuli. Let Δ be a closed annulus and let G be a partially Δ-embedded graph. Also, let $p \in \mathbb{N}_{\geq 3}$ and $q \in \mathbb{N}_{\geq 3}$ and assume that p is an odd number. A Δ-*embedded* (p,q)-*railed annulus* of G is a pair $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ where $\mathcal{C} = [C_1, \ldots, C_p]$ is a Δ-parallel sequence of cycles of G and $\mathcal{P} = [P_1, \ldots, P_q]$ is a collection of pairwise vertex-disjoint paths in G such that

- For every $j \in [q]$, $P_j \subseteq \Delta$.
- For every $(i,j) \in [p] \times [q]$, $C_i \cap P_j$ is a non-empty path, that we denote $P_{i,j}$.

We refer to the paths of $\mathcal{P}$ as the *rails* of $\mathcal{A}$ and to the cycles of $\mathcal{C}$ as the *cycles* of $\mathcal{A}$. We use $\mathsf{ann}(\mathcal{A})$ to denote $\mathsf{ann}(\mathcal{C}, 1, p)$. See Figure 1 for an example.

Treewidth. A *tree decomposition* of a graph G is a pair (T, χ) where T is a tree and $\chi : V(T) \to 2^{V(G)}$ such that

- $\bigcup_{t \in V(T)} \chi(t) = V(G)$,
- for every edge e of G there is a $t \in V(T)$ such that $\chi(t)$ contains both endpoints of e, and
- for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

The *width* of (T, χ) is equal to $\max\{\, |\chi(t)| - 1 \mid t \in V(T)\}$ and the *treewidth* of G is the minimum width over all tree decompositions of G.

3.2. Main result. Our goal is to prove Theorem 3.1. Intuitively, we show that, given a graph G that is partially embedded on an annulus, and a "large enough" railed annulus $\mathcal{A}$, any linkage of G can be "combed" through some rails of $\mathcal{A}$. We start with some definitions on linkages.

Linkages. A *linkage* in a graph G is a subgraph L of G whose connected components are non-trivial paths. The *paths* of a linkage are its connected components and we denote them by $\mathcal{P}(L)$. We call $|\mathcal{P}|$ the *size* of $\mathcal{P}(L)$. The *terminals* of a linkage L, denoted by $T(L)$, are the endpoints of the paths of L, and the *pattern* of L is the set $\{\{s,t\} \mid \mathcal{P}(L)$ contains some (s,t)-path$\}$ (in case L contains a trivial path we may see its unique endpoint as a singleton). Two linkages L_1, L_2 of G are *equivalent* if they have the same pattern and we denote this fact by $L_1 \equiv L_2$. Let Δ be a closed annulus or a closed disk, let G be a partially Δ-embedded graph, L be a linkage of G, and D be a subset of Δ. We say that L is *D-avoiding* if $T(L) \cap D = \emptyset$ and we say that L is *D-free* if $D \cap L = \emptyset$ (see Figure 2).

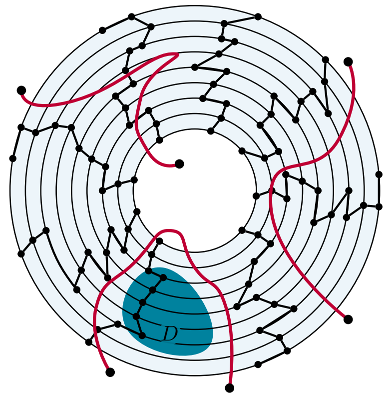

FIGURE 2. *An example of a railed annulus $\mathcal{A}$, a closed disk D (depicted in blue) and a linkage L (depicted in red) that is D-free and $\mathsf{ann}(\mathcal{A})$-avoiding.*

Given an $r \in \mathbb{N}$, we say that a linkage L of G is *r-scattered* if for every $v \in V(L)$ it holds that $N_G^{(\leq r)}[v] \cap (V(L) \setminus V(C_v)) = \emptyset$, where C_v is the connected component of L that contains v. Observe that, since $N_G^{(\leq 0)}[v] = \{v\}$, every linkage is 0-scattered.

Linkages combed in annuli. Let $t \in \mathbb{N}_{\geq 1}$, let $p = 2t+1$, and let $s \in [p]$ where $s = 2t'+1$. Also, let Δ be a closed annulus and $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a Δ-embedded (p,q)-railed annulus of a partially Δ-embedded graph G. Given some $I \subseteq [q]$, we say that a linkage L of G is *(s, I)-combed in $\mathcal{A}$* if

$$L \cap \mathsf{ann}(\mathcal{C}, t+1-t', t+1+t') \subseteq \bigcup_{i \in I} P_i.$$

Linkage reducible graph classes. Let $\mathcal{G}$ be a graph class and $r \in \mathbb{N}$. We say that $\mathcal{G}$ is *r-linkage reducible* if it is hereditary (i.e., if $G \in \mathcal{G}$ then for every $S \subseteq V(G)$, $G[S] \in \mathcal{G}$) and if there is a function $f_{\mathcal{G},r} : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every $G \in \mathcal{G}$, if $\mathbf{tw}(G) \geq f_{\mathcal{G},r}(k)$ and G contains an r-scattered linkage L of size at most k, then there is a vertex $v \in V(G)$ such that $G \setminus v$ contains an r-scattered linkage L' that is equivalent to L. We say that $f_{\mathcal{G},r}$ *certifies* that $\mathcal{G}$ is r-linkage reducible.

We are now ready to state the main result of our paper.

THEOREM 3.1. *There exists a function* $f_1 : \mathbb{N}^2 \to \mathbb{N}$*, where the images of* f_1 *are even, such that for every odd* $s \in \mathbb{N}_{\geq 1}$ *and every* $r, k \in \mathbb{N}$*, if*

- $\mathcal{G}$ *is an* r*-linkage reducible graph class certified by a function* $f_{\mathcal{G},r}$*,*
- Δ *is a closed annulus,*
- G *is a partially* Δ*-embedded graph that belongs to* $\mathcal{G}$*,*
- $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ *is a* Δ*-embedded* (p, q)*-railed annulus of* G*, where* $p = f_1(m, r) + s$ *and* $q \geq \frac{2r+5}{2} \cdot m$*, where* $m = f_{\mathcal{G},r}(k)$*,*
- L *is a* Δ*-avoiding* r*-scattered linkage of size at most* k*, and*
- $I \subseteq [q]$*, where* $|I| > m \cdot (r + 1)$*,*

then G *contains an* r*-scattered linkage* $\tilde{L}$ *where* $\tilde{L} \equiv L$*,* $\tilde{L} \setminus \Delta \subseteq L \setminus \Delta$*, and* $\tilde{L}$ *is* (s, I)*-combed in* $\mathcal{A}$*. Moreover,* $f_1(m, r) = \mathcal{O}(m^2 + mr + r)$*.*

The proof of Theorem 3.1 is presented in Section 4.

3.3. Implications of Theorem 3.1. As Theorem 3.1 is stated for graphs that belong to r-linkage reducible graph classes, our principal goal is to inspect which graph classes are r-linkage reducible and what is the function $f_{\mathcal{G},r}$ certifying this.

We say that a function is *even* if its images are even numbers. We state the following result.

PROPOSITION 3.2 ([39, 55]). *There exists an even function* $f_2 : \mathbb{N} \to \mathbb{N}$ *such that for every* $k \in \mathbb{N}$ *if* G *is a graph and* L *is a linkage of* G *of size at most* k *and* $\mathbf{tw}(G) \geq f_2(k)$*, then there is a vertex* $v \in V(G)$ *such that* $G \setminus v$ *contains a linkage* L' *that is equivalent to* L*.*

In terms of linkage reducible graph classes, the above proposition can be interpreted as:

Observation 3.3. The class $\mathcal{G}_{\mathsf{all}}$ of all graphs is 0-linkage reducible, certified by the function f_2.

Therefore, from Theorem 3.1 and Proposition 3.2, we derive the following:

COROLLARY 3.4. *There exist two functions* $f_2, f_3 : \mathbb{N} \to \mathbb{N}$ *such that for every odd* $s \in \mathbb{N}_{\geq 1}$ *and every* $k \in \mathbb{N}$*, if*

- Δ *is a closed annulus,*
- G *is a graph that is partially* Δ*-embedded,*
- $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ *is a* Δ*-embedded* (p, q)*-railed annulus of* G*, where* $p \geq f_3(k) + s$ *and* $q \geq 5/2 \cdot f_2(k)$*,*
- L *is a* Δ*-avoiding linkage of size at most* k*, and*
- $I \subseteq [q]$*, where* $|I| > f_2(k)$*,*

then G *contains a linkage* $\tilde{L}$ *where* $\tilde{L} \equiv L$*,* $\tilde{L} \setminus \Delta \subseteq L \setminus \Delta$*, and* $\tilde{L}$ *is* (s, I)*-combed in* $\mathcal{A}$*. Moreover,* $f_3(k) = \mathcal{O}((f_2(k))^2)$*.*

In [31], Kawarabayashi and Kobayashi proved that planar graphs are 1-linkage reducible. The function that certifies that planar graphs are 1-linkage reducible of planar graphs can be made single-exponential using the results of [2]. Based on the techniques of [31] and the single-exponential bound of [47], we also prove that the class of all graphs embedded on a surface of genus g is r-linkage reducible, certified by a single-exponential function.

THEOREM 3.5. *There is a function* $f_4 : \mathbb{N}^3 \to \mathbb{N}$ *such that for every* $r, k, g \in \mathbb{N}$ *if* G *is a graph embedded on a surface* Σ *of genus* g*,* L *is an* r*-scattered linkage of* G *of size at most* k*, and* $\mathbf{tw}(G) \geq f_4(r, k, g)$*, then there is a vertex* $v \in V(G)$ *such that* $G \setminus v$ *contains a linkage* L' *that is equivalent to* L*. Moreover, it holds that* $f_4(r, k, g) = r \cdot 2^{\mathcal{O}(k+g)}$*.*

The proof of Theorem 3.5 is presented in Section 5. Using Theorem 3.1 and Theorem 3.5, we can derive the following.

Corollary 3.6. *There exist two functions $f_5, f_4 : \mathbb{N}^3 \to \mathbb{N}$ such that for every odd $s \in \mathbb{N}_{\geq 1}$ and every $r, k, g \in \mathbb{N}$, if*

- *Σ is a surface of Euler genus g,*
- *Δ is a closed annulus of Σ,*
- *G is a graph embedded in Σ*
- *$\mathcal{A} = (\mathcal{C}, \mathcal{P})$ is a Δ-embedded (p, q)-railed annulus of G, where $p \geq f_5(r, g, k)+s$ and $q \geq \frac{2r+5}{2} \cdot f_4(r, k, g)$*
- *L is a Δ-avoiding r-scattered linkage of size at most k, and*
- *$I \subseteq [q]$, where $|I| > f_4(r, k, g) \cdot (r + 1)$,*

then G contains an r-scattered linkage $\tilde{L}$ where $\tilde{L} \equiv L$, $\tilde{L} \setminus \Delta \subseteq L \setminus \Delta$, and $\tilde{L}$ is (s, I)-combed in $\mathcal{A}$. Moreover, $f_5(r, k, g) = \mathcal{O}((f_4(r, k, g))^2)$.

4. How to comb a linkage. This section is dedicated to the proof of Theorem 3.1. We develop a series of definitions that allow us to "comb" a given linkage through the rails of a railed annulus. First, in Subsection 4.1, we define a notion of minimal linkages. Then, in Subsection 4.2, we prove that minimal linkages have few rivers, that means that they traverse the annulus few times and in Subsection 4.3 we prove that minimal linkages do not have high mountains or deep valleys, which means that there are not "deep enough" same-side laminar intrusions of the linkage in the annulus. Also, in Subsection 4.4 we present how to route an r-scattered linkage through a railed annulus in a "combed" way, given that the terminals of the linkage are vertices in the intersection of the rails and the cycles of the railed annulus and are scattered enough. This constitutes the core of the rerouting arguments that we develop, which are all tied together in Subsection 4.5, in order to prove Theorem 3.1.

4.1. Minimal linkages. In this subsection, we aim to define the notion of minimal linkages with respect to a given linkage L and a collection of cycles $\mathcal{C}$, that intuitively corresponds to linkages that are equivalent to L and are "diverging" from $\mathcal{C}$ in the minimum (in number of edges) possible way. Then, we prove that, intuitively, given a graph G, a linkage L of G, and a collection of cycles $\mathcal{C}$ of G, if L is a minimal linkage with respect to L and $\mathcal{C}$, then the union of L' and $\mathcal{C}$ is a graph of bounded treewidth. The bound on the treewidth is given by the function $f_{\mathcal{G},r}$ of the r-linkage reducible class $\mathcal{G}$ in which G belongs.

LB-pairs. Given a graph G, a *LB-pair* of G is a pair (L, B) where B is a subgraph of G with maximum degree 2 and L is a linkage of G. We define $\mathsf{diff}(L, B) = |E(L) \setminus E(B)|$ (i.e., the number of linkage edges that are not edges of B).

Lemma 4.1. *Let $r \in \mathbb{N}$, let $\mathcal{G}$ be an r-linkage reducible graph class, certified by a function $f_{\mathcal{G},r}$, let $G \in \mathcal{G}$, and let (L, B) be an LB-pair of G, where L is r-scattered in G. If $\mathbf{tw}(L \cup B) > f_{\mathcal{G},r}(|L|)$, then G contains a linkage L' where*

1. *$\mathsf{diff}(L', B) < \mathsf{diff}(L, B)$,*
2. *$L' \equiv L$,*
3. *$L' \subseteq L \cup B$.*

Proof. Let $H = L \cup B$. Since $G \in \mathcal{G}$ and $\mathcal{G}$ is r-linkage reducible, there is a vertex $v \in V(H)$ such that $H \setminus \{v\}$ contains a linkage L' that is equivalent to L. Notice that $E(L') \setminus E(B) \subseteq E(L) \setminus E(B)$. It remains to prove that this inclusion is proper. Let $\{x, y\}$ be a member of the common pattern of L and L' such that the (x, y)-path P of L is different than the (x, y)-path P' of L'. Clearly, P and P', when oriented from x to y, have a common part P^*. Formally, this is the connected component of $P \cap P'$

that contains x. Let e be the $(m+1)$-th edge of P, starting from x, where m is the length of P^*. Notice that $e \in E(L) \setminus E(B)$, while $e \notin E(L') \setminus E(B)$. We conclude that $E(L') \setminus E(B) \subsetneq E(L) \setminus E(B)$, therefore $|E(L') \setminus E(B)| < |E(L) \setminus E(B)|$, as required. ∎

Minimal linkages. Let $r \in \mathbb{N}$, let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, $D \subseteq \Delta$, L be a Δ-avoiding and D-free r-scattered linkage of G. We say that an r-scattered linkage L' of G is $(\mathcal{C}, D, L)$-*minimal* if, among all the Δ-avoiding r-scattered linkages of G that are equivalent to L and are subgraphs of $L \cup (\bigcup \mathcal{C} \setminus D)$, L' is one where the quantity $\mathsf{diff}(L', \bigcup \mathcal{C} \setminus D)$ is minimized.

Lemma 4.2. *Let $r \in \mathbb{N}$ and let $\mathcal{G}$ be an r-linkage reducible graph class, certified by a function $f_{\mathcal{G},r}$. Let Δ be a closed annulus, let G be a partially Δ-embedded graphthat also belongs to $\mathcal{G}$, let $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, let $D \subseteq \Delta$, and let L be a Δ-avoiding and D-free r-scattered linkage of G. If L' is a $(\mathcal{C}, D, L)$-minimal linkage of G, then $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) \leq f_{\mathcal{G},r}(|L'|)$.*

Proof. Let $B = \bigcup \mathcal{C} \setminus D$ and note that (L', B) is an LB-pair of G. If $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) > f_{\mathcal{G},r}(|L'|)$, then by Lemma 4.1, G contains a linkage L'' that is equivalent to L' where $\mathsf{diff}(L'', B) < \mathsf{diff}(L', B)$ and $L'' \subseteq L' \cup B$. This contradicts the assumption that L' is a $(\mathcal{C}, D, L)$-minimal linkage of G. ∎

4.2. Minimal linkages have few rivers. In this subsection we deal with *streams* and *rivers* of annuli-avoiding linkages. Intuitively, a stream is a minimal part of a given linkage L, that traverses a given annulus that L-avoids. We show that, given a linkage L and a collection of cycles $\mathcal{C}$, the minimum between the size of $\mathcal{C}$ and the number of different streams of L is a lower bound to the treewidth of the graph obtained by the union of L and $\mathcal{C}$ (Lemma 4.4). This implies an upper bound on the number of rivers of a minimal linkage (Lemma 4.5).

Streams and rivers. Let Δ be a closed annulus. Let G be a partially Δ-embedded graph, $\mathcal{C} = [C_1, \ldots, C_p]$ be Δ-parallel sequence of cycles of G, and L be a Δ-avoiding linkage of G. A $(\mathcal{C}, L)$-*stream* of G is a subpath of L that is a subset P of Δ and such that $V(P \cap C_1)$ consists of the one endpoint of P and $V(P \cap C_p)$ consists of the other. A *disjoint collection of* $(\mathcal{C}, L)$-*streams* of G is a collection $\mathcal{R}$ of $(\mathcal{C}, L)$-streams such that $\bigcup \mathcal{R}$ is a linkage of G. A $(\mathcal{C}, L)$-*river* of G is a $(\mathcal{C}, L)$-stream that is a subpath of a connected component of $L \cap \Delta$ that has one of its endpoints in C_1 and the other in C_p. Notice that not each $(\mathcal{C}, L)$-stream of G is a $(\mathcal{C}, L)$-river and any collection of $(\mathcal{C}, L)$-rivers is a disjoint collection of $(\mathcal{C}, L)$-streams (see Figure 3).

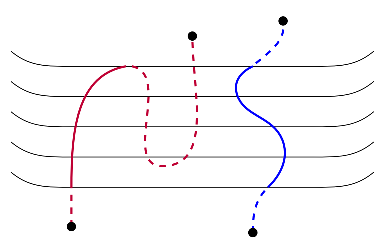

Figure 3. *An example of a $(\mathcal{C}, L)$-stream (depicted in solid red) and a $(\mathcal{C}, L)$-river (depicted in solid blue).*

We now introduce a notion of *ordering* of streams that will be used to consider certain collections of consecutive streams of a linkage.

Orderings of streams. If $\mathcal{Z}$ is a disjoint collection of $(\mathcal{C}, L)$-streams of G we define its D-*ordering* as follows: Consider the sequence $[Z_1, \ldots, Z_d]$, where $d = |\mathcal{Z}|$ and

$Z_i \in \mathcal{Z}$, $i \in [d]$, such that for each $i \in [d]$, one, say D_i, of the two connected components of $\Delta \setminus (Z_i \cup Z_{i+1})$ does not intersect $\bigcup \mathcal{Z}$ (here Z_{d+1} denotes Z_1). Among all $(d-1!)$ such sequences we insist that $[Z_1, \ldots, Z_d]$ is the unique one where $D \subseteq D_q$ and that the order of $\mathcal{Z}$ is the counter-clockwise order in which its elements appear around Δ (see Figure 4). We call $[Z_1, \ldots, Z_d]$ the *D-ordering* of $\mathcal{Z}$.

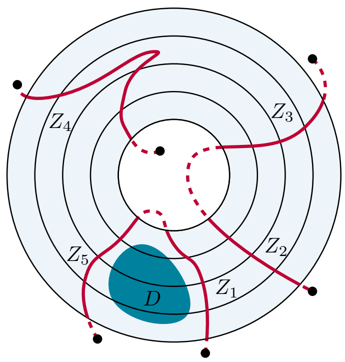

FIGURE 4. *An example of a Δ-parallel sequence $\mathcal{C}$ of cycles, an open disk $D \subseteq \Delta$ (depicted in blue), a linkage L (depicted in red) that is D-free and* $\mathsf{ann}(\mathcal{C})$*-avoiding, a disjoint collection $\mathcal{Z}$ of $(\mathcal{C}, L)$-streams, and the D-ordering $[Z_1, \ldots, Z_5]$ of $\mathcal{Z}$.*

In order to prove our next result (Lemma 4.4), we will use the equivalent definition of treewidth in terms of brambles.

Brambles. Given a graph G, we say that a subset S of $V(G)$ is *connected* if $G[S]$ is connected. Given $S_1, S_2 \subseteq V(G)$, we say that S_1 and S_2 *touch* if either $S_1 \cap S_2 \neq \emptyset$ or there is an edge $e \in E(G)$ where $e \cap S_1 \neq \emptyset$ and $e \cap S_2 \neq \emptyset$. A *bramble* in G is a collection $\mathcal{B}$ of pairwise touching connected subsets of $V(G)$. The *order* of a bramble $\mathcal{B}$ is the minimum number of vertices that intersect all of its elements.

PROPOSITION 4.3 ([62]). *Let $k \in \mathbb{N}$. A graph G has a bramble of order $k+1$ if and only if $\mathbf{tw}(G) \geq k$.*

We are now ready to prove the following result.

LEMMA 4.4. *Let Δ be a closed annulus. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, D be an open disk where $D \subseteq \Delta$, L be a Δ-avoiding and D-free linkage of G, and $\mathcal{Z}$ be a disjoint collection of $(\mathcal{C}, L)$-streams of G. Then $\mathbf{tw}(L \cup (\bigcup \mathcal{C} \setminus D)) \geq \min\{|\mathcal{C}|, |\mathcal{Z}|\}$.*

Proof. Let $[Z_1, \ldots, Z_d]$ be the D-ordering of $\mathcal{Z}$ and let D' be the connected component of $\mathsf{ann}(\mathcal{C}) \setminus (Z_d \cup Z_1)$ that contains D. Let $r = \min\{|\mathcal{C}|, |\mathcal{Z}|\}$, and let $[Z_1, \ldots, Z_r]$ be the sequence consisting of the first r elements of the D-ordering of $\mathcal{Z}$. Let also $\mathcal{C}'$ be the sequence consisting of the first r elements of $\mathcal{C}$. Notice that there is a disjoint collection $\mathcal{Z}' = [Z'_1, \ldots, Z'_r]$ of $(\mathcal{C}', L)$-streams of G such that for each $i \in [r]$, $Z'_i \subseteq Z_i$.

We now set $\mathcal{B} = \mathcal{C}' \setminus D'$, denote $\mathcal{B} = [B_1, \ldots, B_r]$, and notice that both $\mathcal{B}$ and $\mathcal{Z}'$ are sequences of paths in G, such that both $\bigcup \mathcal{B}$ and $\bigcup \mathcal{Z}'$ are linkages of G. Consider now the graph $Q = \bigcup \mathcal{B} \cup \bigcup \mathcal{Z}'$ and notice that $C = B_1 \cup Z'_1 \cup B_r \cup Z'_r$ is a cycle of G.

As $Q \subseteq L \cup (\bigcup \mathcal{C} \setminus D)$, it remains to prove that $\mathbf{tw}(Q) \geq r$. For this, because of Proposition 4.3, it suffices to give a bramble of Q of order $r+1$. For each $(i,j) \in [2, r-1]^2$ we define $X^{(i,j)} = V(B_i \cup Z'_j) \setminus V(C)$. It is easy to check that the set collection $\mathcal{X} = \{X^{(i,j)} \mid (i,j) \in [2, r-1]^2\}$ is a bramble of Q of order $\geq r-2$. Let also

$X^{(1)} = V(Z'_1) \setminus V(B_1)$, $X^{(2)} = V(B_1) \setminus V(Z'_r)$, and $X^{(3)} = V(Z'_r) \cup V(B_r \setminus V(Z'_1))$. See Figure 5 for an illustration of the above sets.

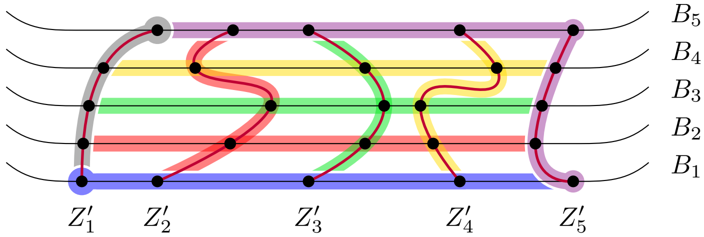

FIGURE 5. *An example of the construction of a bramble of Q, where $|\mathcal{B}| = 5$ and $|\mathcal{Z}'| = 5$. Here, $X^{(2,2)}, X^{(3,3)}, X^{(4,4)}$ are depicted in red, green, and yellow, respectively, while $X^{(1)}, X^{(2)}, X^{(3)}$ are depicted in grey, blue, and violet, respectively.*

Notice that $\mathcal{X} \cup \{X^{(1)}, X^{(2)}, X^{(3)}\}$ is also a bramble of Q since for every distinct $i, j \in [3]$, it holds that there is some edge of Q with endpoints in both $V(X^{(i)})$ and $V(X^{(j)})$ and some edge with endpoints in $V(X^{(i)})$ and X, for all $X \in \mathcal{X}$. Also, observe that since $X^{(1)}, X^{(2)}, X^{(3)}$ are disjoint, the order of the bramble $\mathcal{X} \cup \{X^{(1)}, X^{(2)}, X^{(3)}\}$ is the order of $\mathcal{X}$ incremented by three. Therefore Q contains a bramble of order at least $r + 1$, as required. ◻

Using Lemma 4.4, we can prove that minimal linkages have few rivers.

LEMMA 4.5. *Let Δ be a closed annulus and let $r \in \mathbb{N}_{\geq 0}$. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, $D \subseteq \Delta$, and L be a Δ-avoiding and D-free r-scattered linkage of G. If L' is a $(\mathcal{C}, D, L)$-minimal linkage and $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) < |\mathcal{C}|$, then L' has at most $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D))$ $\mathcal{A}$-rivers.*

Proof. Let $m = \mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D))$. Assume that G contains a collection $\mathcal{Z}$ of $(\mathcal{C}, L')$-rivers where $|\mathcal{Z}| > m$. Recall that $\mathcal{Z}$ is a disjoint collection of $(\mathcal{C}, L')$-streams of G. From Lemma 4.4, $\mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D)) \geq \min\{|\mathcal{C}|, |\mathcal{Z}|\} > m$, a contradiction. ◻

4.3. Minimal linkages do not have high mountains or deep valleys. In this subsection, we introduce another type of structure in linkages, that are *mountains* and *valleys*. Intuitively, given a linkage L and a collection $\mathcal{C}$ of cycles, a mountain (resp. valley) of L is a subpath of a path of L that crosses twice a cycle C_i in $\mathcal{C}$ while possibly crossing only cycles of larger (resp. smaller) indices. We then define *tight* mountains and valleys, that are mountains (resp. valleys) that cannot be "pushed away" towards their bases, due to the existence of a sequence of laminar mountains (resp. valleys) below (resp. above) them. We prove that all mountains and valleys of minimal linkages are tight (Lemma 4.6) and that tight mountains (resp. valleys) have small height (resp. depth) (Lemma 4.7). This implies that minimal linkages have mountains and valleys of small height and depth, respectively (Lemma 4.8). The latter implies that if all terminals of a linkage are outside of a disk that contains many nested cycles, then a minimal linkage would be disjoint of an "inner area" of this disk (Lemma 4.9).

Mountains and valleys. Let Δ be a closed annulus. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G of size $p \in \mathbb{N}$, D be an open disk where $D \subseteq \Delta$, L be a Δ-avoiding and D-free linkage of G. Recall that, by definition, if $\mathcal{C} = [C_1, \ldots, C_p]$, then for every $i \in [p-1]$, C_i and C_p are the boundaries of a closed annulus D_i that is a subset of Δ such that $\Delta = D_1 \supseteq \cdots \supseteq D_{p-1}$. We

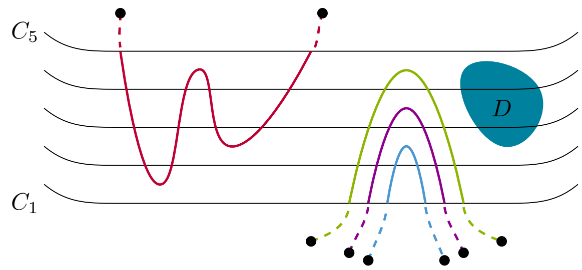

FIGURE 6. *An example of a $(\mathcal{C}, D, L)$-valley (depicted in solid red), and some $(\mathcal{C}, D, L)$-mountains based on C_1 (depicted in solid colors). Notice that all three $(\mathcal{C}, D, L)$-mountain based on C_1 that depicted here are tight.*

additionally set $D_p = C_p$ and $\mathsf{int}(D_p) = \emptyset$. Let $i \in [p]$. An *$(\mathcal{C}, D, L)$-mountain (resp.* $(\mathcal{C}, D, L)$-valley*) of G based on C_i* is a non-trivial subpath P of some path of L where

1. $P \subseteq D_i$ (resp. $P \subseteq \Delta \setminus \mathsf{int}(D_i)$),
2. $P \cap C_p = \emptyset$ (resp. $P \cap C_1) = \emptyset$),
3. $P \cap C_i$ has two connected components, each containing exactly one of the endpoints of P,
4. if D' is the closure of the connected component of $D_i \setminus P$ (resp. $(\Delta \setminus \mathsf{int}(D_i)) \setminus P$) that does not contain C_p (resp. C_1), then $D' \cap T(L) = \emptyset$ and $D' \cap D = \emptyset$.

We refer to Figure 6 for an illustration of the above definition. Clearly, in (4), D' is a closed disk. We call it, the *disk* of the $(\mathcal{C}, D, L)$-mountain (resp. valley) P and we denote it by $\mathsf{disk}(P)$. Notice that there is no $(\mathcal{C}, D, L)$-mountain based on C_p and there is no $(\mathcal{C}, D, L)$-valley based on C_1.

A $(\mathcal{C}, D, L)$*-mountain* (resp. $(\mathcal{C}, D, L)$*-valley*) of G is any $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) of G based on some of the cycles of $\mathcal{C}$.

The *height* (resp. *depth*) of a $(\mathcal{C}, D, L)$*-mountain* (resp. $(\mathcal{C}, D, L)$*-valley*) P that is based on C_i is the maximum j such that C_{i+j-1} (resp. C_{i-j+1}) intersects P and, in both cases, we denote it by $\mathsf{dehe}(P)$. Moreover, the height (resp. depth) of P is at least 1 and at most $p - 1$.

Tight mountains and valleys. Let $r \in \mathbb{N}_{\geq 0}$. Let Δ be a closed annulus. Let G be a partially Δ-embedded graph, $\mathcal{C} = [C_1, \ldots, C_p]$ be Δ-parallel sequence of cycles of G, and L be a Δ-avoiding r-scattered linkage of G. Also, let $D \subseteq \Delta$. Let P be a $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) based on C_i and let d be the maximum among all $d' \in \mathbb{N}_{\geq 0}$ such that $\mathsf{dehe}(P) \geq d' \cdot (r+1) + 2$. We say that the $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) P is *tight* if there is a sequence $[P_0, \ldots, P_d]$ of $(\mathcal{C}, D, L)$-mountains (resp. $(\mathcal{C}, D, L)$-valleys) based on C_i such that

- $P = P_d$,
- for every $j \in [0, d]$, $\mathsf{dehe}(P_j) \geq j \cdot (r + 1) + 2$, and
- for every $j \in [0, d - 1]$, $P_j \subseteq \mathsf{disk}(P_{j+1})$.

See Figure 6 for an illustration of some tight $(\mathcal{C}, D, L)$-mountains.

LEMMA 4.6. *Let Δ be a closed annulus and let $r \in \mathbb{N}_{\geq 0}$. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, $D \subseteq \Delta$, L be a Δ-avoiding and D-free r-scattered linkage of G. Let also L' be a $(\mathcal{C}, D, L)$-minimal r-scattered linkage of G. Then all $(\mathcal{C}, D, L')$-mountains (resp. $(\mathcal{C}, D, L')$-valleys) of G are tight.*

Proof. Let $B = \bigcup \mathcal{C} \setminus D$. We present the proof for the case of $(\mathcal{C}, D, L')$-mountains as the case of $(\mathcal{C}, D, L')$-valleys is symmetric.

Claim: Let $i \in \mathbb{N}_{\geq 1}, j \in \mathbb{N}_{\geq 1}$. If P_j is a $(\mathcal{C}, D, L')$-mountain of G based on C_i such that $\mathsf{dehe}(P_j) \geq j \cdot (r+1)+2$, then there exists a $(\mathcal{C}, D, L')$-mountain P' based on C_i such that $\mathsf{dehe}(P') \geq (j-1)\cdot(r+1)+2$ and $P' \subseteq \mathsf{disk}(P_j)$.

Proof of Claim: Suppose to the contrary that there does not exist a $(\mathcal{C}, D, L')$-mountain P' based on C_i such that $\mathsf{dehe}(P') \geq (j-1)\cdot(r+1)+2$ and $P' \subseteq \mathsf{disk}(P_j)$. Let $P_j^\star = (P_j \setminus D_{i+j\cdot(r+1)}) \cup (C_{i+j\cdot(r+1)} \cap \mathsf{disk}(P_j))$ and notice that $\mathsf{dehe}(P_j^\star) = j\cdot(r+1)+1$ (see Figure 7).

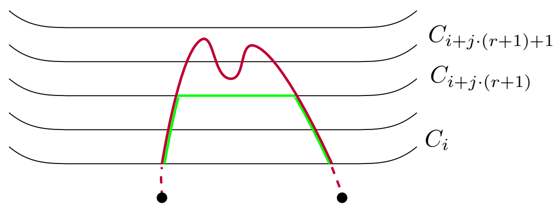

FIGURE 7. *An example of a $(\mathcal{C}, D, L')$-mountain P_j of G based on C_i such that $\mathsf{dehe}(P_j) \geq j\cdot(r+1)+2$ (depicted in red) and the $(\mathcal{C}, D, L')$-mountain $P_j^\star$ (depicted in green).*

Observe that the linkage $L'' = (L' \setminus P_j) \cup P_j^\star$ is equivalent to L. To see why L'' is also r-scattered, first keep in mind that $N_G^{(\leq r)}[V(C_{i+j\cdot(r+1)})] \cap V(C_{i+(j-1)\cdot(r+1)+1}) = \emptyset$. Also, recall that, by assumption, there is no $(\mathcal{C}, D, L')$-mountain P' based on C_i such that $\mathsf{dehe}(P') \geq (j-1)\cdot(r+1)+2$ and $P' \subseteq \mathsf{disk}(P_j)$. Thus, since $\mathsf{dehe}(P') \geq (j-1)\cdot(r+1)+2$ implies that P' intersects the cycle $C_{i+(j-1)\cdot(r+1)+1}$, we have that there is no $(\mathcal{C}, D, L')$-mountain P' based on C_i that intersects the cycle $C_{i+(j-1)\cdot(r+1)+1}$ and $P' \subseteq \mathsf{disk}(P_j)$. Therefore, taking into account that $P_j^\star \subseteq \mathsf{disk}(P_j)$, we have that, for every path $\tilde{P}$ in $L' \setminus P_j$, $N_G^{(\leq r)}[V(P_j^\star)] \cap V(\tilde{P}) = \emptyset$. This implies that L'' is an r-scattered linkage. Moreover, notice that $\mathsf{diff}(L'', B) < \mathsf{diff}(L', B)$ and $L'' \subseteq L' \cup B$. This contradicts the choice of L' as a $(\mathcal{C}, D, L)$-minimal r-scattered linkage of G. The claim follows. $\diamond$

Let P be a $(\mathcal{C}, D, L')$-mountain of G based on C_i and let d be the maximum among all $d' \in \mathbb{N}_{\geq 0}$ such that $\mathsf{dehe}(P) \geq d' \cdot (r+1)+2$. The fact that P is tight follows by recursively applying the Claim above. $\square$

We now prove that the height (resp. depth) of a tight mountain (resp. valley) is "small".

LEMMA 4.7. *Let Δ be a closed annulus and let $r \in \mathbb{N}_{\geq 0}$. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, D be a connected subset of Δ, and L be a Δ-avoiding and D-free r-scattered linkage of G. If P is a tight $(\mathcal{C}, D, L)$-mountain (resp. $(\mathcal{C}, D, L)$-valley) of G, then $\mathsf{dehe}(P) \leq \frac{3}{2}\cdot\mathbf{tw}(L \cup (\bigcup \mathcal{C} \setminus D))$.*

Proof. Let d be the maximum among all $d' \in \mathbb{N}_{\geq 0}$ such that $\mathsf{dehe}(P) \geq d' \cdot (r+1)+2$. We examine the non-trivial case where $d \geq 1$. We present the proof for the case where P is a $(\mathcal{C}, D, L)$-mountain as the case where P is a $(\mathcal{C}, D, L)$-valley is symmetric.

Let $\mathcal{C} = [C_1, \ldots, C_p]$. We assume that P is based on C_i, for some $i \in [p]$. By the definition of tightness, there is a sequence $\mathcal{P} = [P_0, \ldots, P_d = P]$ of $(\mathcal{C}, D, L)$-mountains based on C_i such that

- $P = P_d$,
- for every $j \in [0, d]$, $\mathsf{dehe}(P_j) \geq j \cdot (r+1)+2$, and

- for every $j \in [0, d-1]$, $P_j \subseteq \mathsf{disk}(P_{j+1})$.

For every $j \in [0, d]$, we denote $\mathcal{C}^{(j)} = [C_i, \ldots, C_{i+j\cdot(r+1)+1}]$ and by $\Delta^{(j)}$, the closure of the connected component of $\Delta \setminus (C_i \cup C_{i+j\cdot(r+1)+1})$ that is a subset of Δ. Notice that $\Delta^{(j)}$ is a closed annulus and $\mathcal{C}^{(j)}$ is a $\Delta^{(j)}$-parallel sequence of cycles of G. Notice that for every $j \in [0, d]$, L is an $\Delta^{(j)}$-avoiding and D-free (r-scattered) linkage of G.

Claim: For every $j \in [0, d-1]$ there exists a disjoint collection $\mathcal{Z}_j$ of $(\mathcal{C}^{(j)}, L)$-streams of G where $|\mathcal{Z}_j| \geq 2(d-j)+1$.

Proof of Claim: Let $j \in [0, d-1]$. Observe that for each $h \in [j+1, d]$ exactly two of the connected components of $\Delta^{(j)} \cap P_h$ are $(\mathcal{C}^{(j)}, L)$-rivers in G. This implies that there is a collection $\mathcal{R}_j$ of at least $2(d-j)$ many $(\mathcal{C}^{(j)}, L)$-rivers in G. Recall that $\mathcal{R}_j$ is a disjoint collection of $(\mathcal{C}^{(j)}, L)$-streams of G. Observe also that we can pick some subpath of $\Delta^{(j)} \cap P_j$ that has one endpoint in C_i and the other in $C_{i+j\cdot(r+1)+1}$. As this path does not share vertices with any of the paths in $\mathcal{R}_j$ we can add it in $\mathcal{R}_j$ and obtain a disjoint collection $\mathcal{Z}_j$ of $(\mathcal{C}^{(j)}, L)$-streams of G where $|\mathcal{Z}_j| \geq 2(d-j)+1$. Claim follows (see Figure 8).

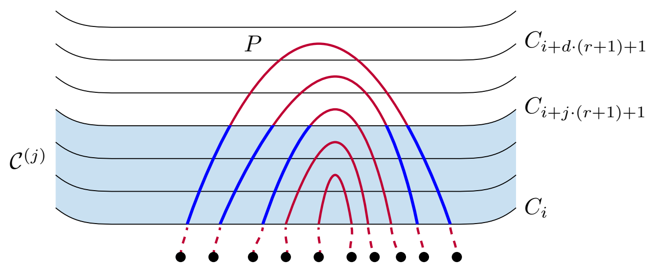

FIGURE 8. *An example of a tight $(\mathcal{C}, D, L)$-mountain P based on C_i of height d and the respective sequence of $(\mathcal{C}, D, L)$-mountains based on C_i (depicted in red), an annulus $\mathcal{C}^{(j)}$ (depicted in cyan), for some $j \in [2, d]$, and a disjoint collection $\mathcal{Z}_j$ (depicted in blue) of $(\mathcal{C}^{(j)}, L)$-streams of G.*

We now set $j' = \lfloor (2d+1)/3 \rfloor$ and observe that $0 \leq j' \leq d-1$. The above claim implies that there exists a disjoint collection $\mathcal{Z}_{j'}$ of $(\mathcal{C}^{(j')}, L)$-streams of G such that $|\mathcal{Z}_{j'}| \geq 2(d-j')+1 \geq j' = |\mathcal{C}^{(j')}|$. Therefore, we can apply Lemma 4.4 on $\mathcal{C}^{(j')}$ and deduce that $\mathbf{tw}(L \cup (\bigcup \mathcal{C}^{(j')} \setminus D)) \geq j'$. The Lemma follows as $L \cup (\bigcup \mathcal{C}^{(j')} \setminus D) \subseteq L \cup (\bigcup \mathcal{C} \setminus D)$ and $\lfloor (2d+1)/3 \rfloor \geq 2d/3$. ◻

Using Lemma 4.6 and Lemma 4.7, we prove that the mountains (resp. valleys) of minimal linkages have "small" height (resp. depth).

LEMMA 4.8. *Let Δ be a closed annulus and $r \in \mathbb{N}_{\geq 0}$. Let G be a partially Δ-embedded graph, $\mathcal{C}$ be a Δ-parallel sequence of cycles of G, D be a connected subset of Δ, L be a Δ-avoiding and D-free r-scattered linkage of G, and L' be a $(\mathcal{C}, D, L)$-minimal r-scattered linkage of G. Then all $(\mathcal{C}, D, L')$-mountains (resp. $(\mathcal{C}, D, L')$-valleys) of G have height (resp. depth) at most $\frac{3}{2} \cdot \mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D))$.*

Proof. We set $B = \bigcup \mathcal{C} \setminus D$. Let $\mathcal{C} = [C_1, \ldots, C_p]$. Let P be a $(\mathcal{C}, D, L')$-mountain (resp. $(\mathcal{C}, D, L')$-valley) of G based on C_i, for some $i \in [p-1]$ (resp. $i \in [2, p]$). From Lemma 4.6, P should be tight and, from Lemma 4.7, $\mathbf{tw}(L' \cup B) \geq \frac{2}{3} \cdot \mathsf{dehe}(P)$. Therefore, $\mathsf{dehe}(P) \leq \frac{3}{2} \cdot \mathbf{tw}(L' \cup B)$. ◻

Before concluding this section, we show that, given a closed *disk* Δ, a graph G partially Δ-embedded, a *nested* sequence of cycles $\mathcal{C}$ (i.e., sequence of cycles that crop nested disks of Δ), and a linkage L whose terminals are outside Δ, every minimal linkage with respect to L and $\mathcal{C}$ does not intersect any "deep enough" insulation layer of $\mathcal{C}$.

LEMMA 4.9. *Let Δ be a closed disk and let $r \in \mathbb{N}_{\geq 0}$. Let G be a partially Δ-embedded graph, $\mathcal{C} = [C_1, \dots, C_p]$, where $p \geq 3m/2 + 1$, be a Δ-nested collection of cycles of G, and L be a Δ-avoiding r-scattered linkage. Every r-scattered $(\mathcal{C}, \emptyset, L)$-minimal linkage L' of G is $\overline{D}_{3m/2+1}$-free, where $m = \mathbf{tw}(L' \cup (\bigcup \mathcal{C} \setminus D))$.*

Proof. Let L' be a $(\mathcal{C}, \emptyset, L)$-minimal linkage of G. Assume to the contrary that L' is a linkage of G that is intersecting $\overline{D}_{3m/2+1}$. As L' is a Δ-avoiding linkage of G we obtain that G contains some $(\mathcal{C}, \emptyset, L')$-mountain P, based on C_1 where $\mathsf{dehe}(P) > 3m/2$. We set Δ' to be the closed annulus $\Delta \setminus \mathsf{int}(D_p)$. As L is a Δ-avoiding linkage of G, it is also a Δ'-avoiding linkage of G. Therefore, we can apply Lemma 4.8, on G, $\mathcal{C}$, $\emptyset$, L, and L' and obtain that $\mathsf{dehe}(P) \leq 3m/2$, a contradiction. ∎

4.4. Routing combed paths. In this subsection, we aim to prove Lemma 4.11, that intuitively shows how to use the infrastructure of a railed annulus $\mathcal{A}$ of a given graph G, in order to obtain an r-scattered linkage of G whose endpoints lie in particular parts of $\mathcal{A}$ and it is combed in $\mathcal{A}$. The obtained routing will appear in the core of the proof of Theorem 3.1.

Let $k, r \in \mathbb{N}$. The $(k \times r)$-*grid* is the graph whose vertex set is $[k] \times [r]$ and two vertices (i, j) and (i', j') are adjacent if and only if $|i - i'| + |j - j'| = 1$. From now on, we assume that each vertex $(i, j) \in [k] \times [r]$ of the $(k \times r)$-grid is embedded at the point (i, j) in a coordinate system whose horizontal axis refers to the first coordinate, whose vertical axis refers to the second coordinate, and each edge of the grid is represented by a straight line segment. The *higher horizontal line* of the grid is the path of the grid whose vertices, in order of appearance, are $(1, r), (2, r), \dots, (k, r)$. Also, the *lower horizontal line* of the grid is the path of the grid whose vertices, in order of appearance, are $(1, 1), (2, 1), \dots, (k, 1)$.

The following proposition can be derived from the proof of [2, Lemma 7].

PROPOSITION 4.10. *Let $r \in \mathbb{N}$ and let $k, k', d \in \mathbb{N}$ such that $1 \leq d \cdot (r+1) \leq k' \leq k$. Let Γ be a $(k \times k')$-grid and let $\{p_1^{\text{up}}, \dots, p_d^{\text{up}}\}$ (resp. $\{p_1^{\text{down}}, \dots, p_d^{\text{down}}\}$) be vertices of the higher (resp. lower) horizontal line arranged as they appear in it from left to right. If for every $i \in [d]$ $N_\Gamma^{(\leq r)}[p_i^{\text{down}}] \cap \{p_{i-1}^{\text{down}}, p_{i+1}^{\text{down}}\} = \emptyset$ and $N_\Gamma^{(\leq r)}[p_i^{\text{up}}] \cap \{p_{i-1}^{\text{up}}, p_{i+1}^{\text{up}}\} = \emptyset$, then the grid Γ contains d paths $P_1, \dots, P_d$ such that, for every $i \in [d]$, the endpoints of P_i are p_i^{up} and p_i^{down} and for every $i, j \in [d]$, $i \neq j$, $N_\Gamma^{(\leq r)}[V(P_i)] \cap V(P_j) = \emptyset$.*

Given two vertex disjoint paths P_1 and P_2 of G, we say that an (P_1, P_2)-path of G is a path whose one endpoint is a vertex of P_1 the other endpoint is a vertex of P_2 and contains all edges of $P_1 \cup P_2$. We now prove the following:

LEMMA 4.11. *Let $r \in \mathbb{N}$, $p, q, s \in \mathbb{N}_{\geq 3}$, $b, d \in \mathbb{N}_{\geq 1}$, such that $p \geq s + 2b$ and $q \geq b + d \cdot (r + 1)$, where p and s are odd numbers. Also, let Δ be a closed annulus. If G is a partially Δ-embedded graph, $\mathcal{A}$ is a Δ-embedded (p, q)-railed annulus of G, $I \subseteq [q]$ where $|I| \geq d \cdot (r + 1)$, then there is an r-scattered linkage K of G such that,*

(a) there is an ordering $[K_1, \dots, K_d]$ of $\mathcal{P}(K)$, where for every $i \in [d]$,

$$K_i \text{ is a } (P_{1,b+(i-1)\cdot(r+1)+1}, P_{p,b+(i-1)\cdot(r+1)+1})\text{-path of } G \text{ and}$$

(b) K is (s, I)-combed in $\mathcal{A}$.

Proof. Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$, let $t = \lfloor p/2 \rfloor$ and $t' = \lfloor s/2 \rfloor$. Also, let $\{i_1, \dots, i_{d \cdot (r+1)}\} \subseteq I$ such that $\forall j \in [d \cdot (r + 1) - 1], i_j < i_{j+1}$.

Claim: There is a collection of paths $\mathcal{P}^{\mathsf{down}} = \{P_1^{\mathsf{down}}, \ldots, P_d^{\mathsf{down}}\}$ such that, for every $h \in [d]$, P_h^{down} is a $(P_{1,b+(h-1)\cdot(r+1)+1}, P_{b,i_{h\cdot(r+1)}})$-path and for every $h, j \in [d]$ where $h \neq j$, it holds that $N_G^{(\leq r)}[V(P_i^{\mathsf{down}})] \cap V(P_j^{\mathsf{down}}) = \emptyset$.

Proof of Claim: For $i \in [b], j \in [q]$ let $p_{i,j}$ be the vertex obtained after contracting all edges in $P_{i,j}$. We also define $E_{\mathsf{horizontal}} = \bigcup_{(i,j)\in[b]\times[q-1]} \{p_{i,j}, p_{i,j+1}\}$ and $E_{\mathsf{vertical}} = \bigcup_{(i,j)\in[b-1]\times[q]} \{p_{i,j}, p_{i+1,j}\}$.

Let H be the graph where $V(H) = \{p_{i,j} \mid (i,j) \in [b]\times[q]\}$ and $E(H) = E_{\mathsf{horizontal}} \cup E_{\mathsf{vertical}}$. Observe that H is a minor of G that is isomorphic to a $(q \times b)$-grid (see Figure 9). For $h \in [d]$, let p_h^{low} (resp. p_h^{high}) be the vertex $p_{1,b+(h-1)\cdot(r+1)+1}$ (resp. $p_{b,i_{h\cdot(r+1)}}$).

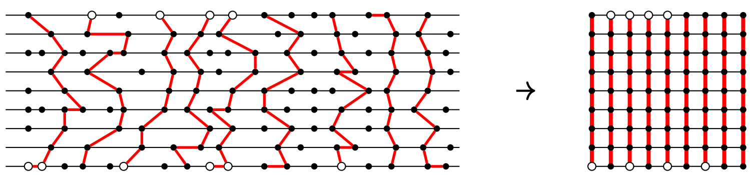

FIGURE 9. *An example showing the construction of the graph H. For every $h \in [d]$, the resulting vertices p_h^{low} and p_h^{high} (corresponding to the vertices of the paths $P_{1,b+(h-1)\cdot(r+1)+1}$ and P_{b,i_h}, respectively) are depicted in white.*

Due to Proposition 4.10, H contains d paths $P_1, \ldots, P_d$ such that, for every $h \in [d]$, the endpoints of P_h are p_h^{low} and p_h^{high} and for every $i, j \in [d]$, $i \neq j$, $N_H^{(\leq r)}[V(P_i)] \cap V(P_j) = \emptyset$. Therefore, if we substitute every vertex of each P_i with the edges that where contracted in G in order to obtain it in H, we obtain the claimed result. $\diamond$

Applying the previous claim symmetrically, we can find a collection of paths $\mathcal{P}^{\mathsf{up}} = \{P_1^{\mathsf{up}}, \ldots, P_d^{\mathsf{up}}\}$ such that, for every $j \in [d]$, P_j^{up} is a $(P_{p-b+1,i_{j\cdot(r+1)}}, P_{p,b+(j-1)\cdot(r+1)+1})$-path and for every $j, h \in [d]$ where $j \neq h$, it holds that $N_G^{(\leq r)}[V(P_i^{\mathsf{up}})] \cap V(P_h^{\mathsf{up}}) = \emptyset$.

For every $h \in [d]$, let $P_h^{\mathsf{mid}} = \mathsf{ann}(\mathcal{C}, b, p-b+1) \cap P_{i_{h\cdot(r+1)}}$ and let $K_h = P_h^{\mathsf{down}} \cup P_h^{\mathsf{mid}} \cup P_h^{\mathsf{up}}$. Observe that $K = \bigcup_{i\in[d]} K_i$ is an r-scattered linkage of G and for every $i \in [d]$, K_i is a $(P_{1,b+(i-1)\cdot(r+1)+1}, P_{p,b+(i-1)\cdot(r+1)+1})$-path of G. Since $p \geq s + 2b$, $t = \lfloor p/2 \rfloor$, and $t' = \lfloor s/2 \rfloor$, $\mathsf{ann}(\mathcal{C}, t+1-t', t+1+t') \subseteq \mathsf{ann}(\mathcal{C}, b, p-b+1)$ and therefore K is (s, I)-combed in $\mathcal{A}$. $\square$

4.5. Combing the linkage - Proof of Theorem 3.1. Before we proceed with the proof of Theorem 3.1 we need some more definitions. Let Δ be a closed annulus and let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a Δ-embedded (p, q)-railed annulus of a partially Δ-embedded graph G.

We refer the reader to Figure 10 for an illustration of the following definitions. For every $i \in [p]$, we define $F_{\mathcal{A}}^{(i)}$ as the edge set of the unique $(P_{i,q}, P_{i,1})$-path that does not contain any vertex from P_2. We also set $F_{\mathcal{A}} = \bigcup_{i\in[p]} F^{(i)}$. Let $(i, j, j') \in [p] \times [q]^2$ where $j \neq j'$. We denote by $L_{i,j\to j'}$ the shortest path in C_i starting from a vertex of $P_{i,j}$ and finishing to a vertex of $P_{i,j'}$ and that does not contain any edge from $F_{\mathcal{A}}$. Let $(i, i', j) \in [p]^2 \times [q]$ where $i \neq i'$. We denote by $R_{i\to i',j}$ the shortest path in P_j starting from a vertex of $P_{i,j}$ and finishing to a vertex of $P_{i',j}$. Let $(i, i', j, j') \in [p]^2 \times [q]^2$ such that $i < i'$ and $j < j'$. We define $\Delta_{i,i',j,j'}$ as the closed disk bounded by the unique cycle in the graph

$$P_{i,j} \cup L_{i,j\to j'} \cup P_{i,j'} \cup R_{i\to i',j'} \cup P_{i',j'} \cup L_{i',j'\to j} \cup P_{i',j} \cup R_{i'\to i,j}.$$

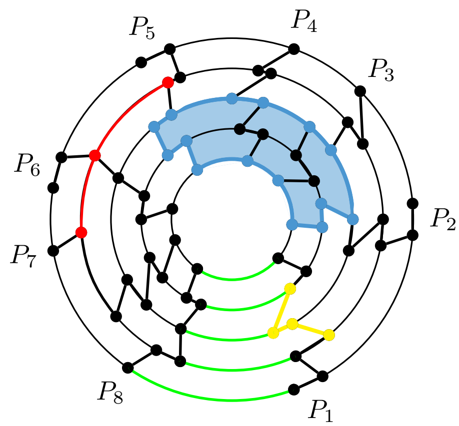

FIGURE 10. *An example of a* $(5, 8)$*-railed annulus* $\mathcal{A}$*, the set* $F_{\mathcal{A}}$ *(depicted in green), and the graphs* $L_{2,5\to 7}$ *(depicted in red),* $R_{2\to 4,1}$ *(depicted in yellow), and* $\Delta_{3,5,2,5}$ *(depicted in blue).*

Let Δ be a closed annulus. Let $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ be a (p, q)-railed annulus of a partially Δ-embedded graph G. We set $z = \lfloor \min\{p, q\}/2 \rfloor$. For each $i \in [z]$, we define

$$C_i^{\mathcal{A}} = \mathbf{bor}(\Delta_{i,p-i+1,i,q-i+1}).$$

We set $\mathcal{C}_{\mathcal{A}} = [C_1^{\mathcal{A}}, \ldots, C_z^{\mathcal{A}}]$. If $p, q \geq 5$, we use $\Delta_{\mathcal{C}_{\mathcal{A}}}$ to denote the closed disk $\mathsf{ann}(\mathcal{C}^{\mathcal{A}}, 1, z) = \Delta_{1,p,1,q}$. Notice that if $p, q \geq 5$, then $\mathcal{C}_{\mathcal{A}}$ is a $\Delta_{\mathcal{C}_{\mathcal{A}}}$-nested collection of cycles of G (see Figure 11).

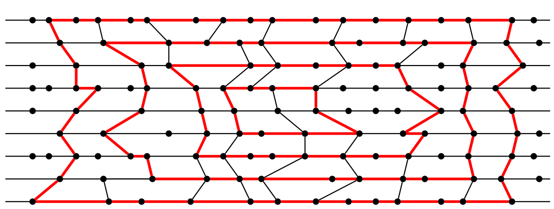

FIGURE 11. *A* $(9, 9)$*-railed annulus* $\mathcal{A} = (\mathcal{C}, \mathcal{P})$ *and the sequence* $C_{\mathcal{A}}$ *(depicted in red).*

Proof of Theorem 3.1. We set $m = f_{\mathcal{G},r}(k)$. We define $f_3(m, r) := 3m^2 + 6m + 2rm + 2r + 2$. Let also $b = 3m/2$, and keep in mind that

$$p \geq f_3(m, r) + s = 3m^2 + 6m + 2rm + 2r + 2 + s = 2 \cdot (m + 1) \cdot (b + r) + 2 + s + 2b$$

and that $|I| \geq m \cdot (r + 1)$.

Recall that $\mathcal{C}_{\mathcal{A}} = [C'_1, \ldots, C'_z]$, where $z = \lfloor \min\{p, q\}/2 \rfloor$, is a $\Delta_{\mathcal{C}_{\mathcal{A}}}$-nested collection of cycles of G. For each $i \in [z]$, we denote by D'_i the closed disk corresponding to C'_i. Let also $D := D'_{b+1}$. Keep in mind that $D'_1 = \Delta_{1,p,1,q} = \Delta_{\mathcal{C}_{\mathcal{A}}}$ and $D = \Delta_{b+1,p-b,b+1,q-b}$.

Observe now that L is a $\Delta_{\mathcal{C}_{\mathcal{A}}}$-avoiding linkage. Let L' be a $(\mathcal{C}_{\mathcal{A}}, \Delta_{\mathcal{C}_{\mathcal{A}}}, L)$-minimal r-scattered linkage. Given that, by definition, $\mathcal{G}$ is hereditary, we have that since $G \in \mathcal{G}$ and $L' \cup (\bigcup \mathcal{C}_{\mathcal{A}} \setminus \Delta_{\mathcal{C}_{\mathcal{A}}})$ is a subgraph of G, $L' \cup (\bigcup \mathcal{C}_{\mathcal{A}} \setminus \Delta_{\mathcal{C}_{\mathcal{A}}}) \in \mathcal{G}$. Therefore, by definition of $\mathcal{G}$ and Lemma 4.2, $\mathbf{tw}(L' \cup (\bigcup \mathcal{C}_{\mathcal{A}} \setminus \Delta_{\mathcal{C}_{\mathcal{A}}})) \leq m$. By applying Lemma 4.9 on G, $\mathcal{C}_{\mathcal{A}}$, L, and $\Delta_{\mathcal{C}_{\mathcal{A}}}$, we obtain that L' is a D-free linkage of G.

It is easy to verify that L' is Δ-avoiding, $p > m$, $D \subseteq \Delta$, and $|L'| = |L| \leq k$. Consider a $(\mathcal{C}, D, L')$-minimal r-scattered linkage L'' of G. Again, heredity of $\mathcal{G}$ implies that $L'' \cup (\bigcup \mathcal{C} \setminus D) \in \mathcal{G}$. Therefore, by definition of $\mathcal{G}$ and Lemma 4.2, $\mathbf{tw}(L'' \cup (\bigcup \mathcal{C} \setminus D)) \leq m$. We may now apply Lemma 4.8 and Lemma 4.5 on k, G, $\mathcal{A}$, D, and L' and deduce that

(i.) All $(\mathcal{C}, D, L'')$-mountains/valleys of G have height/depth at most b.

(ii.) L'' has at most m $\mathcal{A}$-rivers of G,

Let $\mathcal{Z} = [Z_1, \ldots, Z_d]$ be the D-ordering of the $\mathcal{A}$-rivers of L'' in G and keep in mind that, from (ii.), $d \leq m$. Also, since for every $i \in [d]$, Z_i is a subgraph of L'' and L'' is an r-scattered linkage, it holds that for every $i, j \in [d]$ where $i \neq j$, $N_G^{(\leq r)}[V(Z_i)] \cap V(Z_j) = \emptyset$.

Recall that $m \geq r$. For every $i \in [d]$, we define x_i^{down} as the vertex in the path $C_{(i+1)\cdot b+1} \setminus \mathsf{int}(D)$ that belongs in Z_i and has the minimum possible distance (in $C_{(i+1)\cdot b+1} \setminus \mathsf{int}(D)$) to the vertices of the path $P_{(i+1)\cdot b+1, q-b}$ and we denote by Q_i^{down} the path certifying this minimum distance. Similarly, we define x_i^{up} as the vertex in the path $C_{p-(i+1)\cdot b} \setminus \mathsf{int}(D)$ that belongs in Z_i and has the minimum possible distance (in $C_{p-(i+1)\cdot b} \setminus \mathsf{int}(D)$) to the vertices of the path $P_{p-(i+1)\cdot b, q-b}$ and we denote by Q_i^{up} the path certifying this minimum distance. Since $m \geq r$ and $b = 3m/2$, we have that the union of $Q_1^{\text{down}}, \ldots, Q_d^{\text{down}}, Q_1^{\text{up}}, \ldots, Q_d^{\text{up}}$ is an r-scattered linkage.

For $i \in [d]$, let Z_i^{down} and Z_i^{up} be the two connected components of the graph obtained from Z_i if we remove the edges of its $(x_i^{\text{down}}, x_i^{\text{up}})$-subpath (see Figure 12). We choose Z_i^{down} (resp. Z_i^{up}) so that it intersects C_1 (resp. C_p).

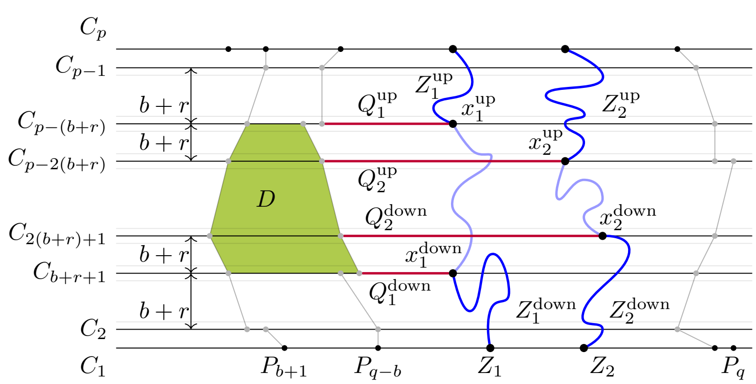

FIGURE 12. *Visualization of an (p, q)-railed annulus and the notations introduced above.*

Claim: For every $i \in [d]$, it holds that $V(Z_{i-1}^{\text{down}}) \cap N_G^{(\leq r)}[V(Q_i^{\text{down}})] = \emptyset$ and $V(Z_{i-1}^{\text{up}}) \cap N_G^{(\leq r)}[V(Q_i^{\text{up}})] = \emptyset$, where Z_0^{down} (resp. Z_0^{up}) denotes Z_q.

Proof of claim: If $V(Z_{i-1}^{\text{down}}) \cap N_G^{(\leq r)}[V(Q_i^{\text{down}})] \neq \emptyset$ for some $i \in [d]$, then there is a connected component F of $Z_{i-1}^{\text{down}} \cap \overline{D}_{(i-1)\cdot(b+r)+1}$ that is a path with endpoints in $V(C_{(i-1)\cdot(b+r)+1})$ and intersects $N_G^{(\leq r)}[V(Q_i^{\text{down}})]$. Observe that F is a $(\mathcal{C}, D, L'')$-mountain of G based on $C_{(i-1)\cdot(b+r)+1}$. Since $V(F) \cap N_G^{(\leq r)}[V(Q_i^{\text{down}})] \neq \emptyset$ and $V(Q_i^{\text{down}}) \subseteq V(C_{i\cdot(b+r)+1})$, we have that $V(F) \cap V(C_{(i-1)\cdot(b+r)+b+1})) \neq \emptyset$ and therefore F is of height $> b$, a contradiction to (i.). Thus, for every $i \in [d]$,

$V(Z_{i-1}^{\mathrm{down}}) \cap N_G^{(\leq r)}[V(Q_i^{\mathrm{down}})] = \emptyset$. Similarly, suppose, towards a contradiction, that $V(Z_{i-1}^{\mathrm{up}}) \cap N_G^{(\leq r)}[V(Q_i^{\mathrm{up}})] \neq \emptyset$ for some $i \in [d]$. Then there is a connected component F' of $Z_{i-1}^{\mathrm{up}} \cap (\Delta \setminus D_{p-(i-1)\cdot(b+r)})$ that is a path with endpoints in $V(C_{p-(i-1)\cdot(b+r)})$ and intersects $N_G^{(\leq r)}[V(Q_i^{\mathrm{up}})]$. Observe that F' is a $(\mathcal{C}, D, L'')$-valley of G based on $C_{p-(i-1)\cdot(b+r)}$. Since $V(F') \cap N_G^{(\leq r)}[V(Q_i^{\mathrm{up}})] \neq \emptyset$ and $V(Q_i^{\mathrm{up}}) \subseteq V(C_{p-i\cdot(b+r)})$, we have that $V(F') \cap V(C_{p-((i-1)\cdot(b+r)-b)})) \neq \emptyset$ and therefore F' is of depth $> b$, a contradiction to (i.). Claim follows. $\diamond$

Because of the above claim, it follows that the paths $Q_i^{\mathrm{down}} \cup Z_i^{\mathrm{down}}$ (resp. $Q_i^{\mathrm{up}} \cup Z_i^{\mathrm{up}}$), $i \in [d]$ are $(Z_i \cap C_1, P_{i\cdot(b+r)+1,q-b})$-paths (resp. $(Z_i \cap C_p, P_{p-i\cdot(b+r),q-b})$-paths) in G that do not intersect the open disk $\mathsf{int}(D)$ and the graph $\bigcup_{i\in[d]} Q_i^{\mathrm{down}} \cup Z_i^{\mathrm{down}} \cup Q_i^{\mathrm{up}} \cup Z_i^{\mathrm{up}}$ is an r-scattered linkage of G.

Let $w = (m+1)\cdot(b+r)+2$ and $w' = p-(m+1)\cdot(b+r)-1$. For $i \in [d]$, we now define $c_i = b+(i-1)\cdot(r+1)+1$ and (see Figure 13)

$$Y_i^{\mathrm{down}} = P_{i\cdot(b+r)+1,q-b} \cup L_{i\cdot(b+r)+1,q-b\to c_i} \cup P_{i\cdot(b+r)+1,c_i} \cup R_{i\cdot(b+r)+1\to w,c_i} \cup P_{w,c_i},$$
$$Y_i^{\mathrm{up}} = P_{p-i\cdot(b+r),q-b} \cup L_{p-i\cdot(b+r),q-b\to c_i} \cup P_{p-i\cdot(b+r),c_i} \cup R_{p-i\cdot(b+r)\to w',c_i} \cup P_{w',c_i}.$$

Note that Y_i^{down} is a $(P_{i\cdot(b+r)+1,q-b}, P_{w,c_i})$-path and Y_i^{up} is a $(P_{p-i\cdot(b+r),q-b}, P_{w',c_i})$-path.

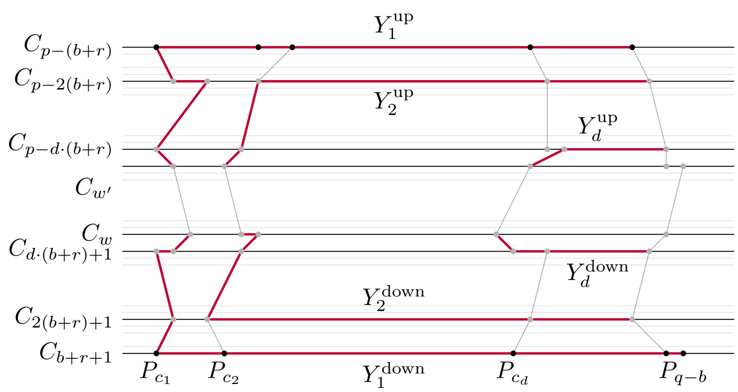

FIGURE 13. *Visualization of the definition of* Y_i^{up} *and* Y_i^{down}, $i \in [d]$.

By the definition of Y_i^{down} and Y_i^{up}, the graphs $X_i^{\mathrm{down}} = Z_i^{\mathrm{down}} \cup Q_i^{\mathrm{down}} \cup Y_i^{\mathrm{down}}$ and $X_i^{\mathrm{up}} = Z_i^{\mathrm{up}} \cup Q_i^{\mathrm{up}} \cup Y_i^{\mathrm{up}}$, $i \in [d]$, are paths and $\bigcup_{i\in[d]} X_i^{\mathrm{down}} \cup X_i^{\mathrm{up}}$ is an r-scattered linkage. In particular, we have that

$$X_i^{\mathrm{down}} \text{ is a } (Z_i \cap C_1, P_{w,c_i})\text{-path and} \tag{4.1}$$
$$X_i^{\mathrm{up}} \text{ is a } (Z_i \cap C_p, P_{w',c_i})\text{-path} \tag{4.2}$$

Let $\Omega = \mathsf{ann}(\mathcal{C}, w, w')$ and $K' = \bigcup_{i\in[d]} X_i^{\mathrm{down}} \cup X_i^{\mathrm{up}}$. Observe that, since for every $i \in [d]$ P_{w,c_i} is a subpath of Y_i^{down} and P_{w',c_i} is a subpath of Y_i^{up}, we have

$$K' \cap \Omega = \bigcup_{i\in[d]} (V(P_{w,c_i}) \cup V(P_{w',c_i})). \tag{4.3}$$

Let $\bar{\mathcal{A}} = (\bar{\mathcal{C}}, \bar{\mathcal{P}})$, where $\bar{\mathcal{C}} = [C_w, \ldots, C_{w'}]$ and $\bar{\mathcal{P}} = \mathcal{P} \cap \Omega$. Notice that $|\bar{\mathcal{C}}| = w' - w + 1 = p - 2(m+1)\cdot(b+r) - 2 \geq s + 2b$. Notice also that $d \leq |I|$ and $I \subseteq [q]$.

Finally, $b = 3/2m$ and $d \leq m$ imply that $b + d \cdot (r+1) \leq \frac{2r+5}{2} \cdot m \leq q$. We can now apply Lemma 4.11 for p, q, s, b, d, $\bar{\mathcal{A}}$, and I and obtain a linkage K of $\bar{\mathcal{A}}$ satisfying properties (a) and (b) of Lemma 4.11.

From Property (a) we can write $\mathcal{P}(K) = [K_1, \ldots, K_d]$ and, using (4.3), we deduce that, for $i \in [d]$, K_i is a (P_{w,c_i}, P_{w',c_i})-path of G. This, together with (4.1), (4.2), and (4.3), implies that $K \cup K'$ is a linkage of G where $K \cup K' \subseteq \mathsf{ann}(\mathcal{C})$. From Property (b), K is (s, I)-combed in $\bar{\mathcal{A}}$, therefore, from (4.3), we get that $K \cup K'$ is (s, I)-combed in $\mathcal{A}$. Observe also that each of the d paths of $\mathcal{P}(K \cup K')$ is a $(Z_i \cap C_1, Z_i \cap C_p)$-path of G for some $i \in [d]$. We define

$$\tilde{L} = (L \setminus A') \cup K \cup K'$$

where $A' = \mathsf{int}(\mathsf{ann}(\mathcal{C}))$. By definition $\tilde{L}$ is an r-scattered linkage of G where $\tilde{L} \equiv L$ and $\tilde{L} \setminus \mathsf{ann}(\mathcal{C}) \subseteq L \setminus \mathsf{ann}(\mathcal{C})$. Finally, as $K \cup K'$ is (s, I)-combed in $\mathcal{A}$, then $\tilde{L}$ (s, I)-combed in $\mathcal{A}$ as well. □

5. Linkage reducibility of surface embeddable graphs. In this section, our goal is to prove Theorem 3.5. Our approach is based on the technique developed by Kawarabayashi and Kobayashi for solving the induced paths problem on planar graphs [31]. We generalize this technique to two directions. The first (and rather straightforward) task is to adapt their results for general r-scattered linkages (keep in mind that induced linkages are 1-scattered linkages). The second, and more intricate, task is to lift their technique from graphs embedded on the plane to graphs embedded to a surface of fixed genus. The main difference is that the number of homotopic loops in a graph embedded in a surface is a linear function of the genus of the surface (see Proposition 5.5). Making use of this, we can enhance the arguments of [31] to deal with the presence of crosscaps and handles outside a fixed disk of the surface. To obtain a single-exponential dependency on k, we use the result of [47] (see Proposition 5.8) to route linkages in graphs embedded in surfaces. To ease readability, in Subsection 5.1 we start with some definitions, we state an intermediate result (Lemma 5.1), and we show how this result implies Theorem 3.5. Then, in Subsection 5.2, we present the proof of Lemma 5.1.

5.1. An intermediate step. To prove Theorem 3.5, we will show Lemma 5.2, which intuitively states that given a graph G embedded on a surface and some "large enough" collection $\mathcal{C}$ of nested cycles of G embedded in a disk, every (scattered) linkage of G can be rerouted "away" from vertices inside the disk bounded by the innermost cycle of $\mathcal{C}$. We start with some additional definitions.

Surfaces. A *surface* is a compact connected 2-manifold without boundary. It is known (see e.g., [49]) that any surface Σ can be obtained, up to homeomorphism, by adding $\mathbf{eg}(\Sigma)$ crosscaps to the sphere, where $\mathbf{eg}(\Sigma)$ is called the Euler genus of Σ.

Isolated vertices. Let Δ be a closed disk, let G be a graph, and let $\mathcal{C}$ be a Δ-nested sequence of cycles of G. Given a vertex set $S \subseteq V(G)$, we say that $\mathcal{C}$ *isolates* S if $S \subseteq \mathsf{int}(D_{|\mathcal{C}|})$. Also, given an $\ell \in \mathbb{N}$, an open disk Δ of Σ, and a set $S \subseteq V(G)$, we say that S is (ℓ, Δ)-*isolated* in G if there is a Δ-nested sequence of cycles of G of size ℓ that isolates S.

Let $t \in \mathbb{N}_{\geq 2}$. Given a vertex $v \in V(G)$ and an $r \in \mathbb{N}$, we say that a Δ-nested sequence of cycles $\mathcal{C} = [C_1, \ldots, C_t]$ of G is r-*tight around* v if $\mathcal{C}$ isolates $\{v\}$ and for every $i \in [t]$, there is no cycle $C'_i \neq C_i$ of G contained in $D_i \setminus D_{i+1}$ such that $N_G^{(\leq r)}[V(C_{i+1})] \cap V(C'_i) = \emptyset$, where $C_{t+1} = \{v\}$.

Bridges. Let Σ be a surface and Δ be an open disk of Σ. Let G be a graph embedded in Σ and L be a Δ-avoiding linkage of G. We call a connected component

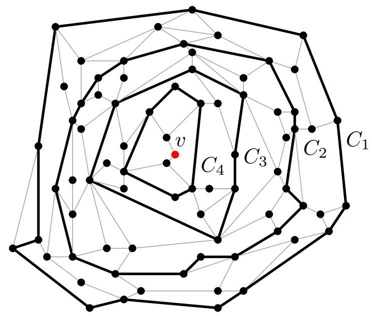

FIGURE 14. *A graph G, a Δ-nested sequence of cycles $\mathcal{C} = [C_1, \ldots, C_4]$ that is 1-tight around the vertex v (depicted in red).*

B of $L \setminus \Delta$ a Δ-*bridge* of L if $B \cap T(L) = \emptyset$ and $B \subsetneq \mathbf{bor}(\Delta)$ (see Figure 15 for an illustration). Observe that every Δ-bridge of L is a subpath of a path of L. The *endpoints* of a Δ-bridge B of L are its (two) vertices that are incident to exactly one edge in B. We denote by $\mathcal{B}_\Delta(L)$ the set of all Δ-bridges of L and use $\mathsf{bridges}_\Delta(L)$ to denote $|\mathcal{B}_\Delta(L)|$.

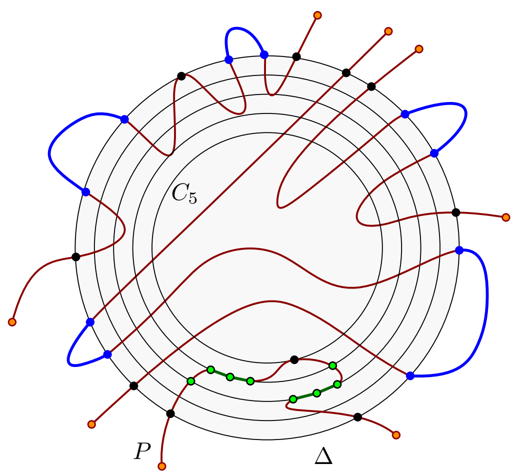

FIGURE 15. *A closed disk Δ of a surface Σ, a Δ-nested sequence of cycles $\mathcal{C} = [C_1, \ldots, C_5]$, and a $\mathsf{int}(\Delta)$-avoiding linkage L. The $\mathsf{int}(\Delta)$-bridges of L are depicted as blue segments of the linkage L while the crossings of the path $P \in \mathcal{P}(L)$ with C_3 and C_4 are depicted in green. Note that P does not cross C_5.*

Crossings. Let $\mathcal{C} = [C_1, \ldots, C_t], t \geq 2$ be a Δ-nested sequence of cycles of G. For $i \in [t]$, we say that a path $P = [v_0 e_1 v_1 \cdots v_\ell]$ of L *crosses* C_i if there exist integers q, r with $0 < q \leq r < \ell$ such that the subpath $P' = [v_q \ldots v_r]$ of P is contained in C_i, e_q and e_{r+1} are not in C_i, and exactly one of e_q and e_{r+1} is in D_i. In this case, we say that P crosses C_i at P'. See Figure 15 for an illustration. We also define $\mathsf{cross}_\mathcal{C}(L)$ to be the total number of crossings of L with $\mathcal{C}$. More formally,

$$\mathsf{cross}_\mathcal{C}(L) = |\{P' \mid \exists P \in \mathcal{P}(L) \text{ and } \exists C \in \mathcal{C} \text{ such that } P \text{ crosses } C \text{ at } P'\}|.$$

BC-minimal linkages. Let G be a graph embedded on Σ, let Δ be an open disk of Σ, let v be a vertex in $\Delta \cap V(G)$, and let $\mathcal{C}$ be a Δ-nested sequence of cycles of

G that is r-tight around v. We say that a Δ-avoiding r-scattered linkage L of G is *BC-minimal around* v, if $v \in V(L)$ and for every Δ-avoiding r-scattered linkage L' of G such that $v \in V(L')$ and $L \equiv L'$, it holds that $\mathsf{bridges}_\Delta(L) \leq \mathsf{bridges}_\Delta(L')$ and, under this condition, $\mathsf{cross}_\mathcal{C}(L) \leq \mathsf{cross}_\mathcal{C}(L')$.

We now state the following results that intuitively says that given a graph G that is embedded on a fixed surface, a "big enough" nested sequence $\mathcal{C}$ of cycles of G, and an r-scattered linkage L that contains a vertex v that is isolated from $\mathcal{C}$ and has minimal number of bridges and crossings, there is an r-scattered linkage L' of $G \setminus v$ that is equivalent to L.

LEMMA 5.1. *There exists a function $f_6 : \mathbb{N}^3 \to \mathbb{N}$ such that for every $r, k, g \in \mathbb{N}$, if Σ is a surface of genus g, G is a graph embedded on Σ, Δ is an open disk of Σ, v is a vertex in $\Delta \cap V(G)$, $\mathcal{C}$ is a Δ-nested sequence of cycles of G, where $|\mathcal{C}| \geq f_6(r, k, g)$, that is r-tight around v, and L is a Δ-avoiding r-scattered linkage of G of size k that is BC-minimal around v, then there is a Δ-avoiding r-scattered linkage L' of $G \setminus v$ such that $L \equiv L'$, $L \setminus \Delta = L' \setminus \Delta$, and $L' \subseteq L \cup \bigcup \mathcal{C}$.*

The proof of Lemma 5.1 is postponed to Subsection 5.2. We now show how to use Lemma 5.1 to prove the following result.

LEMMA 5.2. *There is a function $f_7 : \mathbb{N}^3 \to \mathbb{N}$ such that for every $r, k, g \in \mathbb{N}$ if Σ is a surface of genus g, G is a graph embedded on Σ, Δ is an open disk of Σ, L is a Δ-avoiding r-scattered linkage of G of size at most k, $\mathcal{C}$ is a Δ-nested sequence of cycles of size $f_7(r, k, g)$, v is a vertex of G that is isolated in G by $\mathcal{C}$, then there is a Δ-avoiding r-scattered linkage L' of $G \setminus v$ that is equivalent to L such that $L' \subseteq L \cup \bigcup \mathcal{C}$. Moreover, it holds that $f_7(r, k, g) = r \cdot 2^{\mathcal{O}(k+g)}$.*

Proof. Let $r, k, g \in \mathbb{N}$. We set $f_7(r, k, g) = r \cdot f_6(r, k, g)$. Let Σ be a surface of genus g, G be a graph embedded in Σ, Δ be an open disk of Σ, L be a Δ-avoiding r-scattered linkage of G of size at most k, and v be a vertex of G that is $(f_7(r, k, g), \Delta)$-isolated in G. We assume that $v \in V(L)$, since otherwise the theorem holds trivially.

Since v is $(f_7(r, k, g), \Delta)$-isolated in G and $f_7(r, k, g) = r \cdot f_6(r, k, g)$, there is a nested sequence $\mathcal{C}$ of cycles of G of size at least $f_6(r, k, g)$ that is r-tight around v and whose outer disk Δ' is a subset of Δ. Among all Δ-avoiding r-scattered linkages of G that are equivalent to L and contain v, let $\tilde{L}$ be the one that minimizes the quantities $\mathsf{cross}_\mathcal{C}(\tilde{L})$ and $\mathsf{bridges}_\Delta(\tilde{L})$. Observe that, since $\Delta' \subseteq \Delta$, $\tilde{L}$ is also Δ'-avoiding. By Lemma 5.1, there is a Δ'-avoiding r-scattered linkage $\tilde{L}'$ of $G \setminus v$ that is equivalent to $\tilde{L}$ and moreover $\tilde{L} \setminus \Delta' = \tilde{L}' \setminus \Delta'$. Notice that since $\tilde{L}$ is Δ-avoiding, $\Delta' \subseteq \Delta$, and $\tilde{L} \setminus \Delta' = \tilde{L}' \setminus \Delta'$, it follows that $\tilde{L}'$ is also Δ-avoiding. Therefore, $L' := \tilde{L}'$ is the claimed linkage. ◻

We next show how to prove Theorem 3.5 using Lemma 5.2. Before this, we introduce some additional definitions concerning *walls*.

Walls. An *elementary r-wall*, for some odd integer $r \geq 3$, is the graph obtained from a $(2r \times r)$-grid with vertices $(x, y) \in [2r] \times [r]$, after the removal of the "vertical" edges $\{(x, y), (x, y + 1)\}$ for odd $x + y$, and then the removal of all vertices of degree one. Notice that, as $r \geq 3$, an elementary r-wall is a planar graph that has a unique (up to topological isomorphism) embedding in the plane such that all its finite faces are incident to exactly six edges. The *perimeter* of an elementary r-wall is the cycle bounding its infinite face.

An *r-wall* is any graph W obtained from an elementary r-wall $\bar{W}$ after a series of edge subdivisions (given an edge $e = \{u, v\} \in E(G)$, we define the *subdivision* of e to be the operation of deleting e, adding a new vertex w and making it adjacent to u and v). The *perimeter* of W, denoted by $\mathsf{perim}(W)$, is the cycle of W whose non-subdivision vertices are the vertices of the perimeter of $\bar{W}$.

Given an elementary r-wall $\bar{W}$, some odd $i \in \{1, 3, \ldots, 2r-1\}$, and $i' = (i+1)/2$, the *i'-th vertical path* of $\bar{W}$ is the one whose vertices, in order of appearance, are $(i, 1), (i, 2), (i+1, 2), (i+1, 3), (i, 3), (i, 4), (i+1, 4), (i+1, 5), (i, 5), \ldots, (i, r-2), (i, r-1), (i+1, r-1), (i+1, r)$. Also, given some $j \in [2, r-1]$ the *j-th horizontal path* of $\bar{W}$ is the one whose vertices, in order of appearance, are $(1, j), (2, j), \ldots, (2r, j)$. A *vertical* (resp. *horizontal*) path of W is one that is a subdivision of a vertical (resp. horizontal) path of $\bar{W}$. Notice that the perimeter of an r-wall W is uniquely defined regardless of the choice of the elementary r-wall $\bar{W}$. A *subwall* of W is any subgraph W' of W that is an r'-wall, with $r' \leq r$, and such the vertical (resp. horizontal) paths of W' are subpaths of the vertical (resp. horizontal) paths of W.

Let an odd integer $r \geq 3$. Let W be an r-wall of a graph G and K' be the connected component of $G \setminus \mathsf{perim}(W)$ that contains $W \setminus \mathsf{perim}(W)$. The *compass* of W, denoted by $\mathsf{Compass}(W)$, is the graph $G[V(K') \cup V(\mathsf{perim}(W))]$. Observe that W is a subgraph of $\mathsf{Compass}(W)$ and $\mathsf{Compass}(W)$ is connected.

The *layers* of an r-wall W are recursively defined as follows. The first layer of W is its perimeter. For $i = 2, \ldots, (r-1)/2$, the i-th layer of W is the $(i-1)$-th layer of the subwall W' obtained from W after removing from W its perimeter and all occurring vertices of degree one. Notice that each $(2r+1)$-wall has r layers. The *central vertices* of W are the two non-subdivision vertices of W that do not belong to any of its layers and that are connected by a path of W that does not intersect any layer.

We will use the following relation between the treewidth of a graph G embedded on a surface of fixed genus and a wall of G, derived from [7, Theorem 4.12].

PROPOSITION 5.3. *There is a function $f_8 : \mathbb{N}^2 \to \mathbb{N}$ such that for every $r, g \in \mathbb{N}$, if G is a graph embedded on a surface Σ of genus g and $\mathbf{tw}(G) > f_8(r, g)$, then G contains an r-wall as a subgraph. Moreover, $f_8(r, g) = \mathcal{O}(r \cdot g)$.*

Proof of Theorem 3.5. We set $m = \sqrt{2k+1} \cdot (2 \cdot f_7(r, k, g) + 1)$ and $f_4 = f_8(g, m)$. Let G be a graph embedded on a surface Σ of genus g and let L be an r-scattered linkage of G of size at most k. Also, suppose that $\mathbf{tw}(G) \geq f_4(r, k, g)$. Since $\mathbf{tw}(G) \geq f_4(r, k, g)$ and $f_4(r, k, g) = f_8(g, m)$, by Proposition 5.3 we have that G contains an m-wall W as a subgraph. Also, since L is an r-scattered linkage of G of size at most k and $m = \sqrt{2k+1} \cdot (2 \cdot f_7(r, k, g) + 1)$, there is a subwall W' of W of height $2 \cdot f_7(r, k, g) + 1$ such that W is embedded in a closed disk Δ of Σ and $\mathsf{Compass}(W')$ does not contain any terminal of L. Therefore, L is Δ-avoiding. Let $\mathcal{C}$ be the collection of the layers of W', let v be a central vertex of W', and observe that $|\mathcal{C}| = f_7(r, k, g)$ and v is isolated in G by $\mathcal{C}$. By Lemma 5.2, there is a Δ-avoiding r-scattered linkage L' of $G \setminus v$ that is equivalent to L. Moreover, since $f_7(r, k, g) = r \cdot 2^{\mathcal{O}(k+g)}$, we also have that $f_4(r, k, g) = r \cdot 2^{\mathcal{O}(k+g)}$. □

5.2. Proof of Lemma 5.1. In order to show Lemma 5.1, we first observe that if the given linkage has few bridges, then we can find an equivalent linkage that avoids an isolated vertex v. This is formulated in the following lemma that can be derived from the proof of [31, Theorem 7]. We prove it here for completeness.

LEMMA 5.4. *Let $\ell, r, k \in \mathbb{N}$. Let Δ be a closed annulus. If G is a partially Δ-embedded graph, v is a vertex of G, $\mathcal{C}$ is a Δ-nested sequence of cycles of G of size at least ℓ that is r-tight around v, and L is a Δ-avoiding r-scattered linkage of G of size at most k that is BC-minimal around v and $\mathsf{bridges}_\Delta(L) \leq \ell - k - 1$, then there is a*

Δ-avoiding r-scattered linkage L' of $G \setminus v$ that is equivalent to L, $L \setminus \Delta = L' \setminus \Delta$, and $L' \subseteq L \cup \bigcup \mathcal{C}$.

Proof. Since L is Δ-avoiding, for every $P \in \mathcal{P}(L)$, the number of components in $P \cap \mathsf{int}(\Delta)$ is equal to $\mathsf{bridges}_\Delta(P) + 1$. Therefore, the number of components in $L \cap \mathsf{int}(\Delta)$ is equal to $\mathsf{bridges}_\Delta(L) + |\mathcal{P}(L)|$. Since $\mathsf{bridges}_\Delta(L) \leq \ell - k - 1$ and $|\mathcal{P}(L)| \leq k$, we have that there exist at most $\ell - 1$ components in $L \cap \mathsf{int}(\Delta)$. Now, it is easy to see that the fact that $\mathcal{C}$ is a Δ-nested sequence of cycles of G of size at least ℓ implies that there is a Δ-avoiding r-scattered linkage L' of $G \setminus v$ that is equivalent to L and $L \setminus \Delta = L' \setminus \Delta$ (intuitively, L' can be obtained by shortcuting every component of $L \cap \mathsf{int}(\Delta)$ to pass through the cycles of $\mathcal{C}$). ◻

In the rest of this section, our goal is to argue that one can always reduce the number of bridges. We start with some additional definitions.

Rainbows. Let Σ be a surface and let Δ be an open disk of Σ. We set $\overline{\Delta} = \Sigma \setminus \Delta$. Let also G be a $\overline{\Delta}$-embedded 1-regular graph such that $V(G) \subseteq \mathbf{bd}(\Delta)$ and $\bigcup E(G) \cap \mathbf{bd}(\Delta) = \emptyset$. We call G a Δ-*outer matching*. Let C be the embedded cycle whose vertices are the vertices of G and whose edges are the connected components of $\mathbf{bd}(\Delta) \setminus V(G)$. We denote $G^+ = G \cup C$ and observe that G^+ is a Σ-embedded 3-regular multigraph.

The *facets* of G are the connected components of $\Sigma \setminus G^+$ that are different than Δ. We call a facet of G a *bar* if it is homeomorphic to an open disk D whose boundary is the union of two edges of G, two edges of C, and the 4 endpoints of those edges. We denote by $\mathsf{Bars}(G)$ the set of all bars of G. Given a $b \in \mathsf{Bars}(G)$, we denote by $\mathsf{bd\text{-}edges}(b)$ the set of edges of G that are contained in the boundary of b and observe that for every $b \in \mathsf{Bars}(G)$, $|\mathsf{bd\text{-}edges}(b)| = 2$.

Given two edges $e_1, e_2 \in E(G)$, we say that they are *neighboring*, if $e_1 = e_2$ or there is a $b \in \mathsf{Bars}(G)$ such that $\mathsf{bd\text{-}edges}(b) = \{e_1, e_2\}$. We consider the transitive closure of the neighboring relation and let $\mathcal{E}(G)$ be the partition of $E(G)$ defined be the resulting equivalence relation. We call $\mathcal{E}(G)$ the *homotopy-partition* of $E(G)$ and each $E \in \mathcal{E}(G)$ a Δ-*rainbow* of G. We say that a Δ-rainbow E of G is *trivial* if $|E| = 1$. See Figure 16 for an illustration of the above notions.

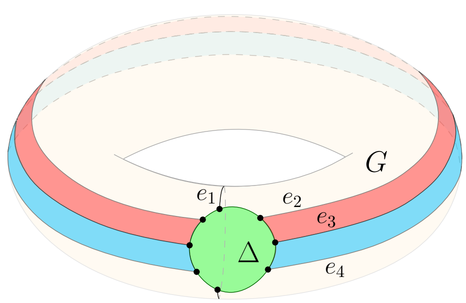

FIGURE 16. *A surface Σ of Euler genus one, an open disk Δ of Σ (depicted in green), and a Δ-outer matching G. The bars of G are depicted in red and blue. The edges e_2 and e_3 of G and the edges e_3 and e_4 are neighboring. The set $\{e_2, e_3, e_4\}$ is a Δ-rainbow of G. The singleton $\{e_1\}$ is a trivial Δ-rainbow of G.*

Given a non-trivial Δ-rainbow E of G, we define the *span* of E in Σ, denoted by $\mathsf{Span}_\Sigma(E)$, to be the set

$$\bigcup \{b \cup \mathbf{bor}(b) \mid b \in \mathsf{Bars}(G) \text{ and } \mathsf{bd\text{-}edges}(b) \subseteq E\}.$$

Notice that the span of every non-trivial Δ-rainbow of G is homeomorphic to a closed disk of $\overline{\Delta}$. We call an edge of E *peripheral* if it is a subset of the boundary of $\mathsf{Span}_\Sigma(E)$. Observe that every non-trivial Δ-rainbow of G has exactly two peripheral edges.

The following result is [49, Proposition 4.2.7].

PROPOSITION 5.5. *Let Σ be a surface, G be a graph embedded in Σ, and u, v be vertices of G (possibly $u = v$). If $P_0, P_1, \ldots, P_k$ are pairwise internally disjoint paths (or cycles) from u to v such that no two of them are homotopic, then*

$$k \leq \begin{cases} \mathbf{eg}(\Sigma), & \text{if } \mathbf{eg}(\Sigma) \leq 1 \\ 3\mathbf{eg}(\Sigma) - 3, & \text{if } \mathbf{eg}(\Sigma) \geq 2. \end{cases}$$

We now use Proposition 5.5 in order to prove the following result, which provides a bound on the size of the homotopy-partition of a given Δ-outer matching. This gives an upper bound on the number of different Δ-rainbows of an open disk Δ of a surface Σ as a function of $\mathbf{eg}(\Sigma)$.

LEMMA 5.6. *Let $g \in \mathbb{N}$, Σ be a surface of genus g, Δ be an open disk of Σ, M be a Δ-outer matching, and $\mathcal{E}(M)$ be the homotopy-partition of M. It holds that $|\mathcal{E}(M)| \leq 3g - 2$.*

Proof. Let $\mathcal{E}(M) = \{E_1, \ldots, E_r\}$ be the homotopy-partition of $E(M)$. We will prove that $r \leq 3g - 3$. For every $i \in [r]$, let e_i be an edge in E_i and $E = \bigcup_{i \in [r]} e_i$. Let x be a point of Δ and f be a homomorphism from Σ to Σ that maps every point of $\Delta \cup \mathbf{bor}(\Delta)$ to x and leaves everything else untouched. Observe that since Δ is an open disk, then $\mathbf{eg}(f(\Sigma)) = \mathbf{eg}(\Sigma) = g$ and if M' is the graph obtained from M after identifying all vertices of M, then M' is a $f(\Sigma)$-embedded graph (whose edges are all loops). Since M is a Δ-outer matching, $f(E)$ is a set of pairwise non-crossing loops that are incident to x. Observe that for every $e, e' \in E$, there is an $i \in [r]$ such that $e, e' \in E_i$ if and only if the loops $f(e)$ and $f(e')$ are non-homotopic. This means that r is equal to the number of non-homotopic elements of $f(E)$, that, by Proposition 5.5, are at most $3g - 2$. Therefore, $r \leq 3g - 2$. ◻

Next, we define rainbows in linkages. Here, in this context, the Δ-outer matching will correspond to the set of Δ-bridges of a given linkage L (in fact, to the contraction of bridges to single edges).

Rainbows in linkages. Let L be a Δ-avoiding linkage of a graph G embedded in a surface Σ, where Δ is an open disk of Σ. We define the *bridge representative* H of L to be the graph obtained from $\bigcup \mathcal{B}_\Delta(L)$ after dissolving every internal vertex of every Δ-bridge of L (which means deleting every such vertex and making its two neighbors adjacent). Observe that H is a Δ-outer matching. This observation will allow us to refer to *homotopy-partitions* and *Δ-rainbows* of the bridge representative of a linkage.

Let $\mathcal{B}$ be a subset of $\mathcal{B}_\Delta(L)$ and H be the bridge representative of L. We set $E_\mathcal{B}$ to be the set of edges of H that correspond to the Δ-bridges in $\mathcal{B}$. We say that $\mathcal{B}$ is a *Δ-rainbow of L* if $E_\mathcal{B}$ is a non-trivial Δ-rainbow of H. Moreover, if $E_\mathcal{B}$ is a non-trivial Δ-rainbow of H and e_1, e_2 are the peripheral edges of $E_\mathcal{B}$, then we call the corresponding bridges B_1, B_2 *peripheral bridges* of the Δ-rainbow $\mathcal{B}$ of L. If $\mathcal{B}$ is a Δ-rainbow of L such that $|\mathcal{B}| \geq 3$ and B_1, B_2 are its peripheral bridges, we denote by $\Delta_\mathcal{B}$ the (unique) connected component of $\Sigma \setminus (\Delta \cup B_1 \cup B_2)$ that intersects $\mathcal{B}$. For example, in Figure 16, if the "red-blue" Δ-rainbow E is equal to $E_\mathcal{B}$, for some $\mathcal{B} \subseteq \mathcal{B}_\Delta(L)$ of some linkage L, then $\Delta_\mathcal{B}$ corresponds to the open disk "cropped" by the peripheral edges of E, i.e., the union of the red and the blue open disk together with the edge e_3 that is incident to both of them. We say that $\mathcal{B}$ is *clear* if $\Delta_{\mathcal{B}} \cap T(L) = \emptyset$.

Let G be a graph, H be a subgraph of G, and $F \subseteq E(H)$. Given a graph $J \subseteq G/F$, we say that H is an *F-expansion* of J if J is obtained from H by contracting all edges in F.

The following result can be derived from the proof of [31, Theorem 7]. It intuitively states that in the presence of a large enough Δ-rainbow of L, one can either find an equivalent linkage with less Δ-bridges or a minor of G that has the following properties: 1) it contains a sequence of nested cycles that isolate a bridge of L and 2) every linkage of this minor can be "expanded" to an r-scattered linkage of G. The two latter properties will allow us to "shift" from r-scattered linkages to 0-scattered linkages and apply the result of Mazoit [47] (see Proposition 5.8) to reroute any given linkage away from the isolated bridge of L. This rerouting allows us to obtain again a linkage with less bridges.

Proposition 5.7. *There exists a function $f_9 : \mathbb{N}^2 \to \mathbb{N}$ such that for every $r, \ell \in \mathbb{N}$, if G is a partially Δ-embedded graph, v is a vertex of G, $\mathcal{C}$ is a Δ-nested sequence of cycles of G of size at least $f_9(r, \ell)$ that is r-tight around v, and L is a Δ-avoiding r-scattered linkage of G such that*

- $v \in V(L)$,
- *for every Δ-avoiding r-scattered linkage L' of G such that $v \in V(L')$ and $L \equiv L'$, it holds that $\mathsf{cross}_{\mathcal{C}}(L) \leq \mathsf{cross}_{\mathcal{C}}(L')$, and*
- *there is a clear Δ-rainbow of L of size at least $f_9(r, \ell)$,*

then

1. *either there is a Δ-avoiding r-scattered linkage L' of G such that $v \in V(L')$, $L \equiv L'$, and $\mathsf{bridges}_{\Delta}(L') < \mathsf{bridges}_{\Delta}(L)$, or*
2. *there exist*
   - *a graph $H \subseteq \bigcup \mathcal{C}$ and an edge-set $F \subseteq E(L) \cap \mathsf{int}(\Delta)$ such that if $\tilde{L}$ is a linkage of $(L \cup H)/F$, then the F-expansion L' of $\tilde{L}$ is an r-scattered linkage of G, and*
   - *a nested sequence of cycles $\mathcal{C}'$ of $(L \cup H)/F$ of size ℓ that isolates a Δ-bridge of L.*

Moreover, it holds that $f_9(r, \ell) = \mathcal{O}(r \cdot \ell)$.

Before presenting the proof of Lemma 5.1, we state the main result of [47].

Proposition 5.8. *There is a function $f_{10} : \mathbb{N}^2 \to \mathbb{N}$ such that for every $k, g \in \mathbb{N}$ if Σ is a surface of genus g, G is a graph embedded on Σ, L is a linkage of G of size at most k, v is a vertex of G, and $\mathcal{C}$ is a nested sequence of cycles of G of size $f_{10}(k, g)$ that isolates $\{v\}$, then there is a linkage L' of $G \setminus v$ that is equivalent to L. Moreover, it holds that $f_{10}(k, g) = 2^{\mathcal{O}(k+g)}$.*

We conclude this section with the proof of Lemma 5.1.

Proof of Lemma 5.1. Let $r, k, g \in \mathbb{N}$. We set

$$m = (2k+1) \cdot f_9(r, f_{10}(k, g)),$$
$$f_6(r, k, g) = 3g \cdot m + k + 1.$$

Let Σ be a surface of genus g, Δ be an open disk of Σ, G be a Σ-embedded graph, and v be a vertex in $\Delta \cap V(G)$. Also, let $\mathcal{C}$ be a Δ-nested sequence of cycles of G of size at least $f_6(r, k, g)$ that is r-tight around v, and L be a Δ-avoiding r-scattered linkage of G of size k such that

(a) for every Δ-avoiding r-scattered linkage L' of G such that $v \in V(L')$ and $L \equiv L'$, it holds that $\mathsf{cross}_{\mathcal{C}}(L) \leq \mathsf{cross}_{\mathcal{C}}(L')$ and
(b) for every Δ-avoiding r-scattered linkage L' of G such that $v \in V(L')$ and $L \equiv L'$, it holds that $\mathsf{bridges}_{\Delta}(L) \leq \mathsf{bridges}_{\Delta}(L')$.

We assume that $v \in V(L)$, since otherwise the lemma holds for $L' = L$.

We aim to prove that $\mathsf{bridges}_{\Delta}(L) \leq 3gm = f_6(r,k,g) - k - 1$, since in this case, by Lemma 5.4, we deduce that there is a Δ-avoiding r-scattered linkage L' of $G \setminus v$ such that $L' \equiv L$ and $L \setminus \Delta = L' \setminus \Delta$.

Suppose, towards a contradiction, that $\mathsf{bridges}_{\Delta}(L) > 3gm$. We will use property (a) from above in order to apply Proposition 5.7 and obtain a contradiction to property (b). Let J be the bridge representative of L and recall that J is a Δ-outer matching. Therefore, by Lemma 5.6, it follows that $|\mathcal{E}(J)| \leq 3g$ and therefore there exists an $E \in \mathcal{E}(J)$ such that $|E| \geq m$. This implies the existence of a Δ-rainbow of L of size at least m. Notice that since $m = (2k+1) \cdot f_9(r, f_{10}(k,g))$ and $|T(L)| = 2k$, there exists a clear Δ-rainbow of L of size at least $f_9(r, f_{10}(k,g))$.

Now, by Proposition 5.7, one of the following holds:

(i) either there is a Δ-avoiding r-scattered linkage L' of G such that $v \in V(L')$, $L \equiv L'$, and $\mathsf{bridges}_{\Delta}(L') < \mathsf{bridges}_{\Delta}(L)$, or
(ii) there exist
  - a graph $H \subseteq \bigcup \mathcal{C}$ and an edge-set $F \subseteq E(L) \cap \mathsf{int}(\Delta)$ such that if $\tilde{L}$ is a linkge of $(L \cup H)/F$ then the F-expansion L' of $\tilde{L}$ is an r-scattered linkage of G, and
  - a nested sequence of cycles $\mathcal{C}'$ of $(L \cup H)/F$ of size $f_{10}(k,g)$ that isolates a Δ-bridge of L.

Observe that if (i) holds, then we arrive to a contradiction to property (b) of L. In the case that (ii) holds, let B be a Δ-bridge of L that $\mathcal{C}'$ isolates. Notice that B is also a Δ-bridge of the linkage L/F. Let v_B be the vertex obtained by contracting every edge of B and let L^* be the graph obtained from L/F after applying the same contractions. Observe that L^* is equivalent to L. Now, by Proposition 5.8 for $L^* \cup H$, L^*, v_B, and $\mathcal{C}'$, we deduce the existence of a linkage $\tilde{L}$ of $(L^* \cup H) \setminus v_B$ that is equivalent to L^*. Notice that $\tilde{L}$ is also a linkage of $(L \cup H)/F$ that does not intersect B and is equivalent to L. Consider now the F-expansion L' of $\tilde{L}$ and observe that, since $\tilde{L}$ is equivalent to L, the same holds for L'. Moreover, L' is an r-scattered linkage in G. The fact that L' is an r-scattered linkage in G that is equivalent to L, contains v and does not intersect B, contradicts property (b) of L. Therefore, we have that $\mathsf{bridges}_{\Delta}(L) \leq 3gm$ and this concludes the proof of the lemma. ☐

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