Area rearrangement operator in discrete and continuous calculus

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Abstract

By interpreting sums as area we construct the area rearrangement operator which looks like $\Phi = -x\frac{d}{dx}$ in the continuous case and $\varphi = -x\Delta$ in the discrete case. We explore the properties of these operators, among them how they create a sequence of linearly independent functions all of which integrate/sum to the same value. Using the discrete operator, we discover a family of functions that satisfies those two properties, as well as one regarding their "finite diagonals". These three properties becomes the criteria for the Main Problem we will explore in this paper, where we search for a way to find other families of functions that satisfies this. This leads us to the "Main Solution", which itself can be seen as an operator, which exists both in discrete and continuous calculus, with its own interesting properties.

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Chapter 1

Preliminaries

1.1 Background and motivation

In this thesis we will look at infinite sums; in particular how to find different functions with identical infinite sums and how to combine and manipulate them in ways that preserves certain traits. As such sums are inherently discrete in the sense that we only plug in positive whole numbers, a majority of this thesis will deal with concepts from discrete calculus. A somewhat unusual notation that we sometimes see in this discrete calculus is letting $x \in \mathbb{N}$ (as opposed to being in the reals). This is a notation we will make use of whenever we are on the topic of discrete calculus. That said, we will occasionally make use of concepts from infinitesimal calculus. And later on, we will see what observations from the discrete case carries over to the continuous one (where we will let x be part of the reals, as usual). The other variables we make use of (n, m, k, etc. and their variations) will however always be part of the natural numbers.

1.2 Discrete calculus

In regular calculus, we are used to working with infinitesimal quantities and the infinite precision that real numbers offer. However, in discrete calculus, we no longer have access to such concepts or properties. Instead, we operate with a countable amount of elements and explore functions where we (usually) only plug in natural numbers. This has many consequences as to how we approach calculus. Take for instance the derivative $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, which is normally a limit procedure, where h gets arbitrarily close to 0, but never equal to 0. In discrete calculus, the smallest step we can take that is not 0 itself, is a step of 1 unit, hence the "derivative" of discrete calculus is of the following form:

Definition 1.2.1. *The forward difference (of step* 1*) of a function f is given by [6]*

$$\Delta f(x) = f(x+1) - f(x)$$

This is what we get if we set $h = 1$ in the continuous definition for the derivative. The forward difference shares many traits with the derivative; it is a linear map that reduces the degree of polynomials by 1 [6]. Although notably, it does this in a different way than the derivative. While the derivative reduces powers of x in a clean way, the forward difference instead does this with their discrete counterpart, which are called the "falling factorials", which are defined as such:

Definition 1.2.2. *We call the following expression the falling factorials of x*

$$(x)_n = x(x-1)(x-2)...(x-n+1) = \prod_{k=0}^{n-1}(x-k)$$

While the falling factorials serve the discrete version of x^n, we also have another similar version known as the "rising factorial", given by:

Definition 1.2.3. *We call the following expression the rising factorials of x*

$$x^{(n)} = x(x+1)(x+2)...(x+n-1) = \prod_{k=0}^{n-1}(x+k) \tag{1.1}$$

We will also deal with the binomial coefficient and the Stirling numbers of the second kind.

Definition 1.2.4. *The binomial coefficient is the the amount of ways to choose n (unordered) objects from k total objects, and is given by [6]*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.2}$$

The unsigned Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ are given by [6]

$$\left\{ {n \atop k} \right\} = [\frac{\Delta^n x^k}{n!}]_{x=0}$$

While their exacts values are not of particular interest to us, some of their properties most certainly are, the ones relevant for us being [6]

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \tag{1.3}$$

$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k} \right\} k + \left\{ {n \atop k-1} \right\} \tag{1.4}$$

$$\binom{n}{n+m} = \left\{ {n \atop n+m} \right\} = 0,\ m > 0 \tag{1.5}$$

$$\binom{n}{-m} = \left\{ {n \atop -m} \right\} = 0,\ m > 0 \tag{1.6}$$

$$\binom{n}{0} = \left\{ {n \atop 0} \right\} = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases} \tag{1.7}$$

$$\binom{n}{k} = \binom{n}{n-k} \tag{1.8}$$

We will also deal with the binomial transform, which can be related to the forward difference like so:

Definition 1.2.5. *let $f_n(x) = f(x+n)$ be some sequence of numbers generated by some arbitrary function, then its binomial transform is given by [3]*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+k) = (-1)^n \Delta^n f(x) \tag{1.9}$$

Finally, we will occasionally make use of the convention that the empty sum and empty product evaluates to 0 and 1 respectively, i.e.

$$\sum_{k=0}^{-1} f(k) = 0,\ \prod_{k=0}^{-1} f(k) = 1 \tag{1.10}$$

1.3 Important functions

We will encounter a number of both discrete and continuous functions in this thesis, so for ease of access we will define and discuss them here and reference them later when needed. The first of which being the harmonic numbers.

Definition 1.3.1. *The x-th harmonic number is defined as the sum of the first x reciprocals of the natural numbers, starting with* 1*, i.e.*

$$H_x = \sum_{k=1}^{x} \frac{1}{k} \tag{1.11}$$

Another useful function is the factorial function $x!$, which is the product of the x first positive integers. This discrete function can be extended to the reals via the gamma function.

Definition 1.3.2. *The gamma function* $\Gamma(x)$ *is given by [6]*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^t \, dt \tag{1.12}$$

In particular, if x is a non-negative integer we have $\Gamma(x) = (x-1)!$

From here, if we take the natural logarithm of it, and then its derivative we get what is known as the digamma function [6].

Definition 1.3.3. *The digamma function* $\psi(x)$ *is given by [6]*

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)} \tag{1.13}$$

The digamma function can also be expressed via the harmonic numbers for $x \in \mathbb{N}$ [6]

$$\psi(x) = H_{x-1} - \gamma \tag{1.14}$$

where γ is the Euler-Mascheroni constant (with approximate value 0.577). The digamma function is also part of the “polygamma functions”.

Definition 1.3.4. *The polygamma functions* $\psi^n(x)$ *of order n are defined as the n-th derivative of the digamma function, i.e. [6]*

$$\psi^n(x) = \frac{d^n}{dx^n} \psi(x) \tag{1.15}$$

Finally, the polygamma functions can be expressed by an infinite sum for $x \in \mathbb{N}$ [6]

$$\frac{\psi^n(x)}{(-1)^{n+1} n!} = \sum_{k=x}^{\infty} \frac{1}{k^{n+1}} \tag{1.16}$$

Chapter 2

Area rearrangement operator

2.1 Geometric intuition

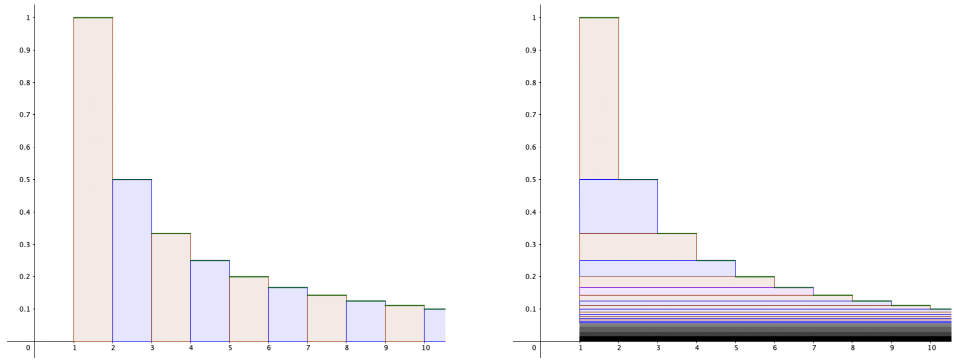
To build towards our new family of functions, we need an operator φ that takes in a function whose infinite sum converges and gives back another (different) function whose infinite sum converges to the same value. We will make use of the floor function to make a connection between sums and area under a curve.

Definition 2.1.1. *The floor function of x (denoted by $\lfloor x \rfloor$) is defined to be x rounded down to its nearest integer. We use also use the notation $f\lfloor x \rfloor := f(\lfloor x \rfloor)$ and refer to it as a floored function.*

Using the floor function we can view sums over integers as integrals of floored functions, as shown below, which we will then interpret geometrically to derive the operator of interest.

$$\sum_{x=a}^{b} f(x) = \int_{a}^{b+1} f\lfloor x \rfloor \, dx \tag{2.1}$$

This relation allows us to visualize sums as area underneath floored functions, where each term in the sum is now a 1 unit wide column. Assuming this area is finite, it should not matter how we measure it, as long as we ensure that we measure exactly the area we want. Obviously there are several ways to do this, but we want to focus on summing the areas in horizontal layers instead of the usual vertical ones.



(a) Counting the area in vertical strips

(b) Counting the area in horizontal strips

Figure 2.1: The area under $f(x) = 1/\lfloor x \rfloor$ counted in two different ways

When counting the area horizontally, notice how our 1st strip has the same height as $f(1)$, but is cut short by the height of $f(2)$. Likewise the 2nd slice has the height of $f(2)$, but cut short by the height of $f(3)$, but now with a width of 2. In fact, the x-th slice has the height of $f(x)$ (and assuming f is strictly decreasing) is always cut short by the height of $f(x+1)$, with a width of x. We thus get the following expression for our area rearrangement operator:

Definition 2.1.2. *Let $\mathbb{R}^{\mathbb{N}}$ denote the space of all real valued functions on $\mathbb{N}$. The area rearrangement operator $\varphi : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}^{\mathbb{N}}$ of some function $f(x)$ is given by*

$$\varphi f(x) = x(f(x) - f(x+1)) = -x\Delta f(x) \tag{2.2}$$

2.2 Properties

By construction, we have that if $f(x)$ has a converging infinite sum, then the infinite sum of $\varphi f(x)$ will converge to the same value, assuming (for now at least) that $f(x)$ is strictly decreasing. As we will see, this isn't actually a requirement.

Theorem 2.2.1. *If the infinite sum is well-defined and* $\lim_{x\to\infty} x f(x) = 0$, *then*

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} \varphi f(x)$$

Proof. Let $g(x) = \varphi f(x) = xf(x) - xf(x+1)$, then we have

$$\sum_{x=1}^{n} f(x) = f(1) + f(2) + ... + f(n)$$

$$\begin{aligned}\sum_{x=1}^{n} g(x) &= f(1) - f(2) + 2f(2) - 2f(3) + 3f(3) - 3f(4) + ... + nf(n) - nf(n+1) \\ &= f(1) + f(2) + f(3) + ... + f(n) + nf(n+1) \\ &= \sum_{x=1}^{n} f(x) + nf(n+1)\end{aligned}$$

Per assumption $nf(n+1) \to 0$ as $n \to \infty$ and we have our conclusion.
□

As the topic of infinite sums is vast, we will narrow our focus down to a subset of rational functions. This will allow us to focus on an "easy" set of functions that will simplify most proofs, several of which will generalize to other sufficiently nice functions. The functions of interest are defined as follows.

Definition 2.2.2. *Any rational function $f(x) = p(x)/q(x) \neq 0$ with integer coefficients that has no poles for any positive integers and $\deg(q) > 1 + \deg(p)$ is called "suitable" or a "suitable function". If we also allow $\deg(q) \geq 1 + \deg(p)$, then we call them "suitable*" (with a star).*

We can also use this definition to create vector spaces of interest.

Definition 2.2.3. *We define $\mathbb{S}$ (and $\mathbb{S}^*$) to be the vector space(s) defined by the span of all suitable (or suitable*) functions with scalars in $\mathbb{Q}$.*

It's easy to see that suitable functions must have a converging infinite sum, as the degree of the denominator is at least 2 more than the degree of the numerator, and with no poles for the positive integers there's nothing that prevents the sum from converging. Using the definition of suitable functions, we can get a corollary from the above theorem.

Corollary 2.2.4. *For a suitable $f(x)$ we have*

$$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} \varphi^n f(x)$$

Proof. We first note that φ of a rational function with rational coefficients is itself a rational function with integer coefficients. Couple this with the fact that if a suitable $f(x)$ has no poles for positive integers, then $\varphi f(x) = xf(x) - xf(x+1)$ does not have poles there either. We know that $\varphi f(x)$ has a converging infinite sum provided $f(x)$ has one, which means that the polynomial in the denominator and numerator of $\varphi f(x)$ must have a difference of degrees of at least 2. Hence, the function must stay suitable after applying φ to it, and we can use the above theorem as many times as we'd like. $\square$

From this, we have that any suitable function has infinitely many "cousins", that all converge to the exact same value. In this section, we will continue to explore the properties of φ.

Proposition 2.2.5. *φ takes suitable functions to suitable functions and suitable* functions to suitable* functions.*

Proof. The first part was shown in the proof for Corollary 2.2.4.

For suitable* functions we consider the partial fractional decomposition (pfd) of some suitable* function f. Since φ is linear (proven in Proposition 2.2.6), we can examine what φ does to each term of the pfd, and if all terms remains suitable*, then the sum of these terms must remain suitable*. Let's consider one term of the pfd, which will be of the following form:

$$\begin{aligned} f(x) &= \frac{p(x)}{q(x)^m} \\ \varphi f(x) &= \frac{xp(x)}{q(x)^m} - \frac{xp(x+1)}{q(x+1)^m} \\ &= \frac{x(p(x)q(x+1)^m - p(x+1)q(x)^m)}{q(x)^m q(x+1)^m} \end{aligned}$$

We let $Q(x) = p(x)q(x+1)^m - p(x+1)q(x)^m$ to get

$$\varphi f(x) = \frac{xQ(x)}{q(x)^m q(x+1)^m}$$

Let $\deg(p) = n$ and $\deg(q^m) = M$, for $n < M$, then $\deg(xQ) = n + M$ which is less than $\deg(q(x)^m q(x+1)^m) = 2M$, and hence we end up with another rational function where the degree of the polynomial in the denominator is greater than the one in the numerator. Finally we note that if $q(x)$ does not have any poles at the positive integers, then neither does $q(x+1)$, and thus the functions stays suitable* after applying φ to it.

□

Proposition 2.2.6. *The operator φ has the following properties.*

1. *φ is linear.*
2. *φ does not change the degree of non-constant polynomials, whereas constants are mapped to* 0.
3. *φ can be modified to become injective by only allowing functions that tend to* 0.

Proof. Consider two arbitrary functions $f(x)$ and $g(x)$ and some constant c.

1. To show linearity, we need to show that $\varphi(f+g) = \varphi(f) + \varphi(g)$ and $\varphi(c \cdot f) = c \cdot \varphi(f)$

$$\begin{aligned} &\varphi(f(x)+g(x)) \\ &= x[(f(x)+g(x)) - (f(x+1)+g(x+1))] \\ &= x[(f(x) - f(x+1) + (g(x) - g(x+1))] \\ &= x[f(x) - f(x+1)] + x[g(x) - g(x+1)] \\ &= \varphi(f) + \varphi(g) \end{aligned}$$

$\varphi(c \cdot f) = x[cf(x) - cf(x+1)] = c \cdot x[f(x) - f(x+1)] = c \cdot \varphi(f)$
Hence, φ is linear,

2. Since φ first takes the forward difference (before multiplying by $-x$), it will reduce the degree of polynomials by 1. Multiplying it with $-x$ will raise its power back to its original degree, unless it was a constant that was reduced to 0, which will remain at 0 even after multiplying by $-x$.

3. Since φ is linear and takes constants to 0, if φ takes $f(x) \to F(x)$, then it will also take $(f(x)+c) \to F(x)$, hence it is not injective. But if we restrict the inputs by requiring them to tend to 0 as x approaches

infinity, this no longer becomes a problem, as only a single choice of c will let $f(x) \to 0$. As we have already seen, if $f(x) \to 0$, then so does $\varphi(f)$. To check for injectivity, we now need to examine the kernel of φ. For φ to map something to 0, we need $f(x) = f(x+1)$. This is only satisfied by constant functions and periodic ones with a period of 1, the only one that fits our criteria that it has to tend to 0 is the zero function itself, hence φ is injective when we require that the input function tends to 0.

□

Theorem 2.2.7. *φ satisfies a discrete version of the Leibniz rule, i.e.*

$$\varphi(f(x)g(x)) = \varphi(f(x))g(x) + f(x+1)\varphi(g(x))$$

Proof.

$$\begin{aligned}
\varphi(f(x)g(x)) &= xf(x)g(x) - xf(x+1)g(x+1) \\
&= xf(x)g(x) - xf(x+1)g(x) + xf(x+1)g(x) - xf(x+1)g(x+1) \\
&= (xf(x) - xf(x+1))g(x) + f(x+1)(xg(x) - xg(x+1)) \\
&= \varphi(f(x))g(x) + f(x+1)\varphi(g(x))
\end{aligned}$$

□

As we can see, φ inherits a lot of its properties from the finite difference operator; it's linear, takes constants to 0 and satisfies this special case of the Leibniz rule. That said, it doesn't lower the degree of non-constant polynomials, but it does have an inverse.

Theorem 2.2.8. *The inverse of φ is given by the following:*

$$\varphi^{-1} f(x) = c - \sum_{k=1}^{x-1} \frac{f(k)}{k} = C + \sum_{k=x}^{\infty} \frac{f(k)}{k}$$

for some constants c, C.

Proof.

$$\begin{aligned}
\varphi f(x) &= xf(x) - xf(x+1) \\
\frac{\varphi f(x)}{x} &= f(x) - f(x+1) \\
\sum_{k=1}^{x-1} \frac{\varphi f(k)}{k} &= f(1) - f(2) + f(2) - f(3) + \ldots + f(x-1) - f(x) \\
-\sum_{k=1}^{x-1} \frac{\varphi f(k)}{k} &= -f(1) + f(x) \\
f(x) &= f(1) - \sum_{k=1}^{x-1} \frac{\varphi f(k)}{k}
\end{aligned}$$

where $f(1)$ is some constant. Since φ takes such constants to 0, any arbitrary constant c will work in place of (or in addition to) $f(1)$. Alternatively, we can do the infinite sum from $k = x$ to ∞, which also creates a telescoping sum, but this time $f(x) \to 0$ as $x \to \infty$, meaning we don't get an extra constant term when evaluating this sum. But since we know φ takes constants to 0, we should still add such a constant nonetheless. □

As mentioned earlier, we will mostly focus on rational functions, which then creates a problem for our inverse, as it takes some rational functions to non-rational functions. An example of this is functions of the form $f(x) = 1/x^m$ which gets mapped to some polygamma functions.

Proposition 2.2.9. *φ^{-1} maps functions of the form $1/x^m$ to $\frac{\psi^m(x)}{(-1)^{m+1}m!}$.*

Proof. Plugging $f(x) = 1/x^m$ into the inverse with the infinite sum we get the infinite sum formula for the polygamma functions (1.16).

$$\varphi^{-1} f(x) = \sum_{k=x}^{\infty} \frac{f(k)}{k} = \sum_{k=x}^{\infty} \frac{1}{k^{m+1}} = \frac{\psi^m(x)}{(-1)^{m+1}m!}$$

□

Theorem 2.2.10. *The set $\{f_n(x)\}_{n=0}^{\infty}$ defined by the sequence $f_{n+1}(x) = \varphi f_n(x)$, for a suitable* f_0 is linearly independent in $\mathbb{S}^*$.*

Proof. We will first show that whenever we apply φ to some suitable* $f(x) = p(x)/q(x)^m$ we end up with $\varphi f(x) = \frac{\hat{p}_1(x)}{q(x)^m} + \frac{\hat{p}_2(x)}{q(x+1)^m}$, where $q(x)$ is irreducible, $\deg(q^m) > \deg(p)$, $\operatorname{pgcd}(p,q) = 1$ and $\hat{p}_2(x) \neq 0$. From Proposition 2.2.5 we have:

$$\begin{aligned}\varphi f(x) &= \frac{x(p(x)q(x+1)^m - p(x+1)q(x)^m)}{q(x)^m q(x+1)^m} \\ &= xQ(x)/(q(x)^m q(x+1)^m)\end{aligned}$$

which we can apply partial fraction decomposition to and write in the form that we want, as long as $q(x+1)$ does not cancel out. Note that x cannot be a factor of $q(x+1)$, as this would imply that $q(x)$ has a pole at $x = 1$, which contradicts it coming from a suitable f. Now, since $q(x)$ is irreducible, this means that $q(x+1)$ is also irreducible [4], meaning that each of the m $q(x+1)$ in the denominator either fully cancels out, or doesn't cancel out at all.

Let $\bar{m}$ be such that $0 < \bar{m} \leq m$. We now assume that some of the $q(x+1)^m$ cancels out, leaving us with $\bar{m}$ of them. This means that $q(x+1)^{\bar{m}}$ is a factor of $Q(x)$, which implies $Q(x) = q(x+1)^{\bar{m}}\hat{Q}(x)$, where $\hat{Q}(x)$ is some polynomial. Then we have:

$$\begin{aligned}Q(x) &= q(x+1)^{\bar{m}}\hat{Q}(x) \\ p(x)q(x+1)^m - p(x+1)q(x)^m &= q(x+1)^{\bar{m}}\hat{Q}(x) \\ p(x)q(x+1)^{m-\bar{m}} - \frac{p(x+1)q(x)^m}{q(x+1)^{\bar{m}}} &= \hat{Q}(x)\end{aligned}$$

Since $p(x)q(x+1)^{m-\bar{m}}$ and $\hat{Q}(x)$ are polynomials, we must have that $\frac{p(x+1)q(x)^m}{q(x+1)^{\bar{m}}}$ is also a polynomial, which implies that $q(x+1)$ gets partially or fully cancelled out. Per assumption, $p(x+1)$ and $q(x+1)$ have no factors in common, so we must have that $q(x)$ divides $q(x+1)$ (or vice versa), but no non-constant polynomial satisfies this, and as $q(x)$ must have at least a degree of 1, this is impossible, and hence, our assumption that $q(x+1)$ could be cancelled out (either partially or fully) leads to a contradiction.

Now, we choose any suitable* f_0 and perform partial fractional decomposition (pfd) on it to get some finite amount of terms, all of which of the form $\frac{p(x)}{q(x)^m}$. We then choose such a term, such that $q(x+n)$ is not equal to any of the other $q(x)$ in the other terms for all n. This guarantees that whenever we go from $n \to n+1$, we will have a $q(x+n) \to q(x+n+1)$ which will always be unique (and doesn't "turn into" any of the other denominators found in the original pfd). Now we have for every n shown that f_n will contain a term in its pfd with a unique denominator, which none of the proceeding elements of the set contains, or can be turned into with a linear combination. This ensures that whenever we add the $(n+1)$'th term, the set remains linearly independent. Notably this argument also holds for suitable functions, meaning such a set will also be linearly independent in $\mathbb{S}$. $\square$

Chapter 3

A new family of functions

3.1 Construction & closed forms

With φ explored, it is time to move onto the main family of functions we want to explore in this thesis. Our goal is that given any function with a converging infinite sum to find different functions whose infinite sum converge to the same value. For the sake of simplicity, we will only look at suitable starting functions, the simplest of which being $f_0(x) = 1/x^2$. With the use of φ we already have infinitely many other functions that share this infinite sum. Furthermore, we can use linear combinations of these functions to create even more of them, so in a sense, our problem seems to already have been solved before it even started. But when we try to create an explicit formula for f_n, given the initial function $f_0(x) = 1/x^2$, something unexpected happens. Starting with with this choice of f_0 and letting $f_1 = \varphi(f_0)$ we have:

$$\begin{aligned} f_0(x) &\to f_1(x) \\ \frac{1}{x^2} &\to \frac{2x+1}{x(x+1)^2} \end{aligned}$$

We want to generalize the change from f_0 to f_1, such that the same change can be applied to f_1 to get f_2, such that it has the same infinite sum. If we ignore that some factors are squared, the change between the denominators has a simple pattern: $x \to x(x+1)$, which seems to follow a pattern similar to the rising factorial (1.1) like so:

$$x(x+1) \to x(x+1)(x+2) \to x(x+1)(x+2)(x+3) \to \ldots \to \prod_{k=0}^{n}(x+k)$$

but always with the "last" factor squared. As for the numerator, if we again ignore the denominator having a squared factor, then taking its derivative gives us what we need. In other words, for initial function $1/x^2$ we get the following family of functions:

Definition 3.1.1. *We call the following family of functions the "Basel functions" (named after the Basel problem [5], which is the base case when taking their infinite sums)*

$$b_n(x) = \frac{\frac{d}{dx}\prod_{k=0}^{n}(x+k)}{(x+n)\prod_{k=0}^{n}(x+k)}$$

As unrigorous as the construction of these functions was, it has in fact successfully created a family of functions, all of which converge to the same value (proven in Theorem 3.2.2). But even more interestingly, the Basel functions are a completely different family of functions than what we would get by repeatedly applying φ to $1/x^2$. Apart from the two first functions being the same, there is seemingly no overlap between them. And obviously, we can plug all of these new functions into φ to generate infinitely many more functions that sum to our target value. For now, we will continue to explore this new family of functions, to discover their most relevant properties. In their current closed form, they look rather ugly being expressed by products. Fortunately, it can be easily shown that they can be rewritten using sums (or the digamma function).

Theorem 3.1.2. *In addition to having a product formula, the Basel functions also has a summation and a digamma formula:*

$$b_n(x) = \frac{\frac{d}{dx}\prod_{k=0}^{n}(x+k)}{(x+n)\prod_{k=0}^{n}(x+k)} = \frac{1}{x+n}\sum_{k=0}^{n}\frac{1}{x+k} = \frac{\psi(x+n+1)-\psi(x)}{x+n} \tag{3.1}$$

Proof. We start with the summation expression. We do this by finding the derivative of $P_n(x) = \prod_{k=0}^{n}(x+k)$, via implicit differentiation:

$$\begin{aligned}
P_n(x) &= \prod_{k=0}^{n}(x+k) \\
\ln(P_n(x)) &= \ln(\prod_{k=0}^{n}(x+k)) = \sum_{k=0}^{n}\ln(x+k) \\
\frac{d}{dx}\ln(P_n(x)) &= \frac{d}{dx}\sum_{k=0}^{n}\ln(x+k) \\
\frac{1}{P_n(x)}\cdot\frac{d}{dx}P_n(x) &= \sum_{k=0}^{n}\frac{1}{x+k} \\
\frac{d}{dx}P_n(x) &= P_n(x)\sum_{k=0}^{n}\frac{1}{x+k}
\end{aligned}$$

Substituting this in on the LHS simplifies the expression to the summation form.

As for the digamma expression, we show that it is the same as the summation form. We cancel the common factor of $1/(x+n)$ and use the "harmonic number" definition of $\psi(x)$ (1.14) to arrive at the conclusion.

$$\begin{aligned}
&\psi(x+n+1) - \psi(x) \\
&= H_{x+n} - \gamma - (H_{x-1} - \gamma) \\
&= H_{x+n} - H_{x-1} \\
&= \sum_{k=1}^{x+n}\frac{1}{k} - \sum_{k=1}^{x-1}\frac{1}{k} \\
&= \sum_{k=x}^{x+n}\frac{1}{k} = \frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n} \\
&= \sum_{k=0}^{n}\frac{1}{x+k}
\end{aligned}$$

□

3.2 Properties

To prove that the Basel functions sum to the same value, we will first build towards a lemma, the idea of which is that if two infinite sums has a well defined difference, then their partial difference has to approach this value. Starting with $f_0(x) = 1/x^2$ and $f_1(x) = \varphi(f_0)$, we examine their first terms and define their partial difference.

$$\sum f_0(x) = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$
$$\sum f_1(x) = \frac{3}{4} + \frac{5}{18} + \frac{7}{48} + \dots$$

We define the partial difference between these two sums as the n first terms of $f_0(x)$ and $(n-1)$ first terms of $f_1(x)$. Let $d(x)$ be the function that describes this sequence of partial differences, where $d(1) = f_0(1) = 1$. To get the next value of $d(x)$, we take the previous value, add the next term from f_0, and subtract the next term from f_1, like so:

$$d(2) = d(1) + f_0(2) - f_1(1) = 1 + \frac{1}{4} - \frac{3}{4} = \frac{1}{2}$$
$$d(3) = d(2) + f_0(3) - f_1(2) = \frac{1}{2} + \frac{1}{9} - \frac{7}{48} = \frac{1}{3}$$

This process of generating the values of $d(x)$ is given by the following recursive formula:

$$d(x+1) = d(x) + f_0(x+1) - f_1(x),\ d(1) = f_0(1)$$

As we continue this process it looks like $d(x) = 1/x$. But equally important is to note that if $d(x) \to 0$, then the two sums must be the same.

Lemma 3.2.1. *Given $f(x)$ and $g(x)$ with converging infinite sums and $d(x)$ satisfying $d(x+1) = d(x) + f(x+1) - g(x)$, $d(1) = f(1)$, then*

$$\sum_{x=1}^{\infty} f(x) - \sum_{x=1}^{\infty} g(x) = \lim_{x\to\infty} d(x)$$

Proof. By construction of $d(x)$ the statement is true. Furthermore, if the two functions sum to the same value, then $d(x)$ must go to 0. Likewise if $d(x)$ goes to 0, then the sums must be equal. □

This lemma allows us to use knowledge of the infinite sum of one of them to infer the other's. Additionally it lets you choose which function to be "f" and which to be "g", as $d(x)$ will depend on this choice, potentially making it easier or harder to reach a desirable conclusion. A special case of this lemma is when the two sums are equal, which implies that the limit goes to 0, which is what we will use to show that our $f_n(x)$ all sum to the same value.

Theorem 3.2.2. *For the Basel functions $b_n(x)$ we have*

$$\sum_{x=1}^{\infty} b_n(x) = \sum_{x=1}^{\infty} b_0(x) = \frac{\pi^2}{6}$$

Proof. We will take for granted that $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$, as this specific value isn't the main interest of this proof, rather that all the sums are equal. We show this by comparing (the summation formula (3.1) for) $b_n(x)$ and $b_{n+1}(x)$, with a fitting $d_n(x)$ that records their partial difference. In this case we have $d_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} \frac{1}{x+k}$. Using our lemma, we need the following to hold for all n:

$$\begin{aligned} d_n(x+1) &= d_n(x) + b_n(x+1) - b_{n+1}(x) \\ d_n(x+1) - d_n(x) &= b_n(x+1) - b_{n+1}(x) \\ \frac{1}{n+1}\left(\sum_{k=0}^{n} \frac{1}{x+k+1} - \sum_{k=0}^{n} \frac{1}{x+k}\right) &= \frac{1}{x+n+1}\left(\sum_{k=0}^{n} \frac{1}{x+k+1} - \sum_{k=0}^{n+1} \frac{1}{x+k}\right) \\ \frac{1}{n+1}\left(\frac{1}{x+n+1} - \frac{1}{x}\right) &= \frac{1}{x+n+1}\left(-\frac{1}{x}\right) \\ \frac{1}{n+1} \cdot \frac{x-x-n-1}{x(x+n+1)} &= \frac{-1}{x(x+n+1)} \\ \frac{-n-1}{n+1} &= -1 \\ -1 &= -1 \end{aligned}$$

which clearly holds for all n. Since $d_n(x) \to 0$ (for all n), we have that $b_n(x)$ has the same infinite sum as $b_{n+1}(x)$, meaning they are all equal. □

Theorem 3.2.3. *The Basel functions are linearly independent in $\mathbb{S}$*

Proof. By using the summation formula $b_n(x) = \frac{1}{x+n}\sum_{k=0}^{n} \frac{1}{x+k}$, we see that for every n, there will be one and only one term that has a double pole, that being the term of the form $\frac{1}{(x+n)^2}$. This term will be unique for every n, and as such we cannot combine any of the other terms into these unique double pole terms, and therefore every $b_n(x)$ has a term that ensures it is not a linear combination of any of the others. □

Now, to give some further insight into the Basel functions, let's examine the first few values of n and x, although now re-indexed such that n starts at 1 as opposed to 0:

	$x=1$	$x=2$	$x=3$	$x=4$
$b_1(x)=\frac{1}{x^2}$	1	1/4	1/9	1/16
$b_2(x)=\frac{2x+1}{x(x+1)^2}$	3/4	5/18	7/48	9/100
$b_3(x)=\frac{3x^2+6x+2}{x(x+1)(x+2)^2}$	11/18	13/48	47/300	37/360
$b_4(x)=\frac{4x^3+18x^2+22x+6}{x(x+1)(x+2)(x+3)^2}$	25/48	77/300	19/120	319/2940

Viewing their values like this reveals a (very surprising) property, namely that their finite diagonals all sum to 1. This is the 3rd and final property that makes the Basel functions special (the other two being equal infinite sums and linear independence). We will later search for other families of functions that also satisfies these three criteria, but for now, we will continue to explore the Basel functions and build towards a proof of our "diagonal sum" property.

In the previous table we saw that each row was generated by each of our $b_n(x)$ functions, but this is not the only way to generate these values. If we go column by column instead, we can use the harmonic numbers to create the following family of functions: $h_n(x)=\frac{H_x-H_{n-1}}{x}$, which generates the same values (and some extra ones):

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$
$h_1(x)$	1	3/4	11/18	25/48	137/300	49/120
$h_2(x)$	0	1/4	5/18	13/48	77/300	29/120
$h_3(x)$	−1/2	0	1/9	7/48	47/300	19/120
$h_4(x)$	−5/6	−1/6	0	1/16	9/100	37/360
$h_5(x)$	−13/12	−7/24	−1/12	0	1/25	11/180

As we can see, we get $b_1(x)$ down the "first diagonal", and then as we move on to the diagonal to its right, we get the values of $b_2(x)$, and then $b_3(x)$ to the right of that, etc. What is new here is the diagonal of zeros, which separates the previous terms from some new negative terms. For those interested, the diagonals with negative terms are generated by the following function: $D_n(x)=\frac{1}{x}\sum_{k=1}^{n}\frac{1}{x+k}$, but apart from acknowledging its existence, it is not something we will explore further in this thesis.

Since we have a set of numbers that can be generated in two different ways, we can make a bijection between them.

Lemma 3.2.4. *If* $b_n(x) = b(n,x) = \frac{1}{x+n-1}\sum_{k=0}^{n-1}\frac{1}{x+k}$
and $h_n(x) = h(n,x) = \frac{1}{x}(H_x - H_{n-1}) = \frac{1}{x}(\sum_{k=1}^{x}\frac{1}{k} - \sum_{k=1}^{n-1}\frac{1}{k})$,
then $b(n,x) = h(x, x+n-1)$

Proof.

$$
\begin{aligned}
h(n,x) &= \frac{1}{x}(\sum_{k=1}^{x}\frac{1}{k} - \sum_{k=1}^{n-1}\frac{1}{k}) \\
h(x,x+n-1) &= \frac{1}{x+n-1}(\sum_{k=1}^{x+n-1}\frac{1}{k} - \sum_{k=1}^{x-1}\frac{1}{k}) \\
&= \frac{1}{x+n-1}(\sum_{k=1}^{x-1}\frac{1}{k} + \sum_{k=x}^{x+n-1}\frac{1}{k} - \sum_{k=1}^{x-1}\frac{1}{k}) \\
&= \frac{1}{x+n-1}\sum_{k=x}^{x+n-1}\frac{1}{k} \\
&= \frac{1}{x+n-1}(\frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{x+n-1}) \\
&= \frac{1}{x+n-1}\sum_{k=0}^{n-1}\frac{1}{x+k} = b(n,x)
\end{aligned}
$$

□

With this, we finally have everything we need to prove our claim that the sums of $b_n(x)$'s finite diagonals all equal 1.

Theorem 3.2.5. $b_n(x)$*'s finite diagonals sum to* 1, *i.e.*

$$\sum_{k=1}^{n} b_k(n-k+1) = 1$$

Proof.

$$\begin{aligned}\sum_{k=1}^{n} b_k(n-k+1) &= \sum_{k=1}^{n} b(k,n-k+1) = \sum_{k=1}^{n} h(n-k+1,n)\\ &= \sum_{k=1}^{n} \frac{1}{n}(\sum_{K=1}^{n} \frac{1}{K} - \sum_{K=1}^{n-k} \frac{1}{K}) = \sum_{k=1}^{n} \frac{1}{n}(H_n - H_{n-k})\\ &= \frac{1}{n}\sum_{k=1}^{n} H_n - \frac{1}{n}\sum_{k=1}^{n} H_{n-k} = \frac{n}{n}H_n - \frac{1}{n}(H_{n-1} + ... + H_1 + H_0)\\ &= H_n - \frac{1}{n}\sum_{k=0}^{n-1} H_k\end{aligned}$$

For our statement to hold, we need this to evaluate to 1. We then get:

$$H_n - \frac{1}{n}\sum_{k=0}^{n-1} H_k = 1 \iff \sum_{k=0}^{n-1} H_k = nH_n - n$$

which we will prove by induction. For the base case $n = 0$, both sides evaluates to 0. We let $n \to n+1$ to finish the proof.

$$\begin{aligned}\sum_{k=0}^{n} H_k &= (n+1)H_{n+1} - (n+1)\\ H_n + \sum_{k=0}^{n-1} H_k &= (n+1)H_{n+1} - (n+1)\\ H_n + nH_n - n &= (n+1)(H_n + \frac{1}{n+1}) - (n+1)\\ (n+1)H_n - n &= (n+1)H_n - n + 1 - 1\end{aligned}$$

This holds true, meaning the diagonals always sum to 1. □

Another interesting property of these functions is that the difference between the degree of the denominator and numerator looks to remain constant at 2, but some linear combinations of them gives a difference of 3 instead. If we let the coefficients sum to 1, the linear combination will have the same infinite sum, and as such, with a difference of degree 3 will make them converge faster than those with a degree difference of 2. Here's some examples of these:

$$
\begin{aligned}
b_0 + b_1 - b_2 &= \frac{6x^3 + 17x^2 + 14x + 4}{x^6 + 6x^5 + 13x^4 + 12x^3 + 4x^2} \\
b_0 + b_2 - b_3 &= \frac{9x^4 + 54x^3 + 113x^2 + 102x + 36}{x^7 + 11x^6 + 47x^5 + 97x^4 + 96x^3 + 36x^2} \\
b_1 + \varphi(b_0) - \varphi(b_1) &= \frac{9x^2 + 20x + 8}{x^5 + 6x^4 + 13x^3 + 12x^2 + 4x} \\
(b_1 + b_2 - b_4), &(b_0 + b_3 - b_4), (b_0 + b_4 - b_5)
\end{aligned}
$$

In general, it seems to hold for triplets of the form $(b_0 + b_{n-1} - b_n)$, but obviously it works for other triplets as well, and even more so if we allow applying φ to this family of functions.

Chapter 4

The Main Problem

The family of functions we explored in the previous section had three main properties; their infinite sums were the same, their (finite) diagonals summed to the same value and they were linearly independent. These three criteria are the basis of the "Main Problem" I want to explore in this thesis, in which we pick an arbitrary function $f_0(x)$ whose infinite sum converge, and try to find all $f_n(x)$ such that these three requirements are fulfilled.

Definition 4.0.1. *The Main Problem (very informally known as the Tveiten problem) is given by the following:*

Pick any suitable function $f_0(x)$. Find all $f_n(x)$ such that:

1. $\sum_{x=1}^{\infty} f_n(x) = \sum_{x=1}^{\infty} f_0(x)$ *(equal infinite sums)*
2. $\sum_{k=0}^{n} f_k(n-k+1) = f_0(1)$ *(equal diagonals)*
3. $f_n(x)$ *are linearly independent in* $\mathbb{S}$

Remark 4.0.2. The formula in the second criteria adds the diagonal terms from top/right to bottom/left, but this is equivalent to adding them in the opposite order, in which case the criteria would look like:
$\sum_{k=0}^{n} f_{n-k}(k+1) = f_0(1)$

4.1 Various approaches

As we have seen before, φ can be used to generate linearly independent functions that sum to the same value, and as such is a natural place to begin looking for a solution to this problem.

Proposition 4.1.1. *For any suitable f_0, φf_0 is a valid choice for f_1 (as it doesn't contradict any of the three criteria)*

Proof. We already know that this choice of f_1 satisfies the first and last criteria of the Main Problem. So we need to show the equal diagonals criteria holds true for the second diagonal:

$$f_0(2) + f_1(1) = f_0(2) + 1(f_0(1) - f_0(2)) = f_0(1)$$

□

Unfortunately, repeatedly applying φ to our function does not give a solution to our problem, as φ only respects the diagonal condition with respect to its input, and outside of this it has no reason to hold true with respect to the preceding functions. However, applying φ to an already known solution seems to generate a new solution. So before we attempt to solve for solutions, we explore the possibility of using previously known solutions to generate new ones.

Conjecture 4.1.2. *If $f_n(x)$ is a solution to the Main Problem for $f_0(x)$, then $g_n(x) = \varphi f_n(x)$ is a solution for $g_0(x) = \varphi f_0(x)$*

Proving this holds true is easy enough for the first and third criteria. We get the first one for free, as φ preserves the infinite sum of a suitable f_0. As for the third criteria, we have shown that φ is linear and injective with respect to suitable functions, meaning it preserves linear independence [7]. It is the second criteria that is troublesome, showing that our new set of functions satisfies the diagonal property. We will later see that this holds true for one type of solutions, but not for another (see Remark 5.2.7).

Although an illegal choice for the initial function, if we let $f_0 = 1/x$, we notice something special. If we apply φ to it we get $1/(x+1)$, which is $f_0(x)$ shifted by 1 unit. Applying φ again doesn't shift the function again, but as we already know, the solution to the Main Problem is not given by repeatedly applying φ, so if we were forced to guess the closed form in this case, it would have to be $f_n = 1/(x+n)$. These f_n satisfies the diagonal and linearly independent criteria, but their infinite sum doesn't converge, so it doesn't make sense to call this a solution, so instead we will refer to it as an "almost solution" for now. What's interesting with this sequence of functions is how it relates to the Basel functions. Recall how they were of the form $b_n = 1/(x+n)\sum_{k=0}^{n} 1/(x+k)$, which is notably entirely made up from our "almost solution". This brings us to our next conjecture.

Conjecture 4.1.3. *Let $f_n(x)$ be a solution to the Main Problem for $f_0(x)$, then $g_n(x) = \frac{1}{x+n}\sum_{k=0}^{n} f_k(x)$ is a solution for $g_0(x) = f_0(x)/x$*

This new family of function seems to respect the 2 last criteria, but proving that all of their infinite sums are the same is an elusive problem. Like in the previous conjecture, it is possible that this holds true for a specific kind of solution, but this remains unknown. Later on we construct a sequence where this doesn't work (see Remark 5.2.7).

If this were to be true (for the solutions of interest at the very least), it would give us the solutions to all initial functions of the form $f_0(x) = 1/x^m$, for $m \geq 2$, which could all be generated from the Basel functions, which itself can be generated from our "almost solution". If we create a table for these families of functions, we can show an equivalent procedure to generate all of these functions. Let $\zeta_{m,0}$ be the initial functions of the form $1/x^m$, then we have $\zeta_{1,n}$ be our almost solution, $\zeta_{2,n}$ be the Basel functions, and the rest can be constructed using the previous entries.

$\zeta_{1,0} = 1/x$	$\zeta_{1,1} = 1/(x+1)$	$\zeta_{1,2} = 1/(x+2)$	$\zeta_{1,3} = 1/(x+3)$	$\dots$
$\zeta_{2,0} = 1/x^2$	$\zeta_{2,1} = b_1(x)$	$\zeta_{2,2} = b_2(x)$	$\zeta_{2,3} = b_3(x)$	$\dots$
$\zeta_{3,0} = 1/x^3$	$\zeta_{3,1}$	$\zeta_{3,2}$	$\zeta_{3,3}$	$\dots$
$\zeta_{4,0} = 1/x^4$	$\zeta_{4,1}$	$\zeta_{4,2}$	$\zeta_{4,3}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

With the first row given, we can recursively find any $\zeta_{n,m}$ by adding together the m first functions in the row above and then multiply it by $1/(x+m) = \zeta_{1,m}$. If we fill in this table, then each row represents one solution to the Main Problem (except the first one being an "almost solution"), creating a family of solutions. Similar families of solutions could possibly be created for other solutions, given that the above conjecture holds (for certain solutions).

Another property of solutions to the Main Problem is that we would expect the sum of two solutions to itself be another solution. To show this, we first need the following definition.

Definition 4.1.4. *For $a,b \in \mathbb{R}$, $a \sim b$ if $a = \frac{p}{q}b$, for p,q integers, $q \neq 0$.*

Theorem 4.1.5. *if f_n and g_n are solutions to the Main Problem (for some f_0 and g_0 respectively) such that $\sum f_n = a \nsim b = \sum g_n$, then $h_n = c_1 f_n + c_2 g_n$ is a solution for $h_0 = c_1 f_0 + c_2 g_0$, for $c_1, c_2 \in \mathbb{Q}$*

Proof. Let $f_n = \{f_0, f_1, ...\}$ and $g_n = \{g_0, g_1, ...\}$ be two solutions to the Main Problem, meaning they are both (individually) linearly independent sets. If their union is not linearly independent, then we can linearly combine elements from f_n to get an element from g_n (or vice versa). But we have that $\sum f_n \nsim \sum g_n$, meaning no matter how we combine terms from f_n (with rational coefficients), their infinite sum will never be equal to the infinite sum of g_n, hence their union must also be linearly independent. □

Remark 4.1.6. We will later see that the constraint $a \nsim b$ is not actually necessary for the conclusion of this theorem, but the proof for this requires a general solution to the Main Problem.

This theorem (alongside the previous conjecture) allows us to potentially use “almost solutions” to find new solutions. In the same way that we can combine two known solutions to find a new one, we could in theory sometimes do this in reverse; splitting a solution into two different solutions. If one of these new solutions were of the form of a rational function with a sum diverging to infinity, we could show that these ”almost solutions” are in fact well-defined, and can be combined in specific ways to construct even more solutions. An example of this would be $f_0 = 2/(2x+x^2) = 1/x - 1/(x+2)$. If we granted that our “almost solution” for $f_0 = 1/x$ is well-defined, and found a solution for $f_0 = 2/(2x+x^2)$, we could use this to find a well-defined solution for $f_0 = 1/(x+2)$, which we could then use (with or without the previous conjecture) to find more solutions. As our known solutions grows, we would also get more “almost solutions”, which would give us more options for ways to combine and modify them to create more solutions.

When it comes to actually finding solutions for any given f_0, that looks to be difficult without some one-fits-all solution. The best hope we have is to apply φ to our first function, and hope there is an obvious change between

f_0 and f_1 that is easy to generalize. A potential example of such a case is $f_0 = 1/(x^2+x)$, which seems to have the following solution:

$$f_n(x) = \frac{n+1}{x^2+(2n+1)x+n(n+1)} = (n+1)(\frac{1}{x+n} - \frac{1}{x+n+1})$$

Another trick that simplifies the problem is that the denominators seems to have the following pattern: $q_{n+1}(x) = q_n(x+1)q_0(x)$ before any cancellation occurs, which if true, simplifies the problem from finding the right rational functions to finding the right polynomial for each of the denominators. But even with all of this, finding individual solutions is very difficult.

4.2 A note on linear independence

Before we move on to the Main Solution to the Main Problem, I want to stress the importance of the 3rd criteria; the linear independence of our functions, as this being a requirement isn't immediately obvious. If we allow linear dependence, then if we start with a suitable initial function f_0 and apply φ to it to get f_1, we can now combine them to create the other f_n. As f_0 and f_1 sum to the same value, we note that $c_0 f_0 + c_1 f_1$ will also sum to this same value, provided that $c_0 + c_1 = 1$. Using the fact that the sum of the diagonals equals $f_0(1)$, we can create another equation for c_0 and c_1, giving us 2 equations with 2 unknowns:

1. $c_0 + c_1 = 1$
2. $c_0 f_0(1) + c_1 f_1(1) = f_0(1) - (f_0(3) + f_1(2))$

Solving this gives you $f_2(x) = c_0 f_0(x) + c_1 f_1(x)$, which you can then use to create two new equations with two unknowns, that you solve to get $f_3(x)$ and so on and so forth. If we do this to the first two elements of the Basel functions the sequence continues like so:

$$\frac{23x^2+4x-5}{9x^2(x+1)^2}, \frac{487x^2-x-163}{162x^2(x+1)^2}, \frac{62027x^2-7352x-25577}{18225x^2(x+1)^2}$$

which obviously differ from the Basel functions and grows in complexity quicker (and notably looks a lot uglier).

The reason why such solutions are of little interest, is that the whole point of the Main Problem is to find **new** functions that sum to the same value while also satisfying the diagonal property. While a clever solution, allowing for linear combinations of previous functions to generate new ones doesn't really (in my opinion) create any new such functions, and are more like old functions in disguise. Requiring the family of functions to be linearly independent is a good enough requirement that the functions we get are "new" and different enough from the previous ones.

Chapter 5

Solution to the Main Problem

We will now examine a solution to the Main Problem, which as we will see, works for any suitable choice of f_0.

Definition 5.0.1. *For any suitable choice of f_0, we call the following family of functions the "Main Solution" (to the Main Problem)*

$$f_{n+1} = \frac{\varphi(f_0 + f_1 + ... + f_n)}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} \varphi(f_k)$$

5.1 Proving the Main Solution solves the Main Problem

Theorem 5.1.1. *The family of functions defined by the Main Solution satisfies the 1st criteria of the Main Problem (equal infinite sums)*

Proof. We show this by induction. For the base case we have

$$\sum_{x=1}^{\infty} f_0(x) = \sum_{x=1}^{\infty} \varphi f_0(x)$$

which holds true per the equal infinite sum property of φ. We assume our claim is true for some n, then check if it holds true for $n+1$.

$$\sum_{x=1}^{\infty} f_{n+1} = \sum_{x=1}^{\infty} \frac{\varphi(f_0+f_1+...+f_n)}{n+1} = \frac{1}{n+1}\sum_{x=1}^{\infty}(\varphi(f_0)+\varphi(f_1)...+\varphi(f_n))$$
$$= \frac{1}{n+1}(\sum_{x=1}^{\infty}\varphi(f_0)+\sum_{x=1}^{\infty}\varphi(f_1)+...+\sum_{x=1}^{\infty}\varphi(f_n))$$
$$= \frac{1}{n+1}(\sum_{x=1}^{\infty} f_0+\sum_{x=1}^{\infty} f_0+...+\sum_{x=1}^{\infty} f_0) = \frac{n+1}{n+1}\sum_{x=1}^{\infty} f_0 = \sum_{x=1}^{\infty} f_0$$

□

Theorem 5.1.2. *The Main Solution has the following recursive formula*

$$f_{n+1} = \frac{nf_n+\varphi(f_n)}{n+1} \tag{5.1}$$

Proof. We see that the formula for the definition can always be rearranged into our target formula for any n.

$$f_1 = \varphi(f_0) = \frac{0f_0+\varphi(f_0)}{1}$$
$$f_2 = \frac{\varphi(f_0+f_1)}{2} = \frac{\varphi(f_0)+\varphi(f_1)}{2} = \frac{f_1+\varphi(f_1)}{2}$$
$$f_3 = \frac{\varphi(f_0+f_1+f_2)}{3} = \frac{\varphi(f_0+f_1)+\varphi(f_2)}{3} = \frac{2f_2+\varphi(f_2)}{3}$$
$$...$$
$$f_{n+1} = \frac{\varphi(f_0+...+f_n)}{n+1} = \frac{\varphi(f_0+...+f_{n-1})+\varphi(f_n)}{n+1} = \frac{nf_n+\varphi(f_n)}{n+1}$$

□

Before we move onto the explicit formula for the Main Solution, we will establish a lemma which will be used in several proofs involving the explicit formula.

Lemma 5.1.3.

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}f(x+k) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k) - \sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k+1) \tag{5.2}$$

Proof. This follows from (1.3), (1.6) and (1.5):

$$\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}f(x+k)\\
&=\sum_{k=0}^{n+1}(-1)^k\binom{n}{k}f(x+k)+\sum_{k=0}^{n+1}(-1)^k\binom{n}{k-1}f(x+k)\\
&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k)+\sum_{k=1}^{n+1}(-1)^k\binom{n}{k-1}f(x+k)\\
&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k)-\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k+1)
\end{aligned}$$

□

Theorem 5.1.4. *The Main Solution has the following explicit formula, for any given f_0:*

$$f_n(x)=\frac{1}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k) \tag{5.3}$$

Proof. Using the definition of the Main Solution and φ we can find expressions for the first few functions in terms of f_0:

$$\begin{aligned}
f_0(x)&=f_0(x)\\
f_1(x)&=x(f_0(x)-f_0(x+1))\\
f_2(x)&=\frac{1}{2}x(x+1)(f_0(x)-2f_0(x+1)+f_0(x+2))\\
f_3(x)&=\frac{1}{6}x(x+1)(x+2)(f_0(x)-3f_0(x+1)+3f_0(x+2)-f_0(x+3))
\end{aligned}$$

which appears to be of the form:

$$f_n=\frac{1}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k)$$

We will show by induction that the two forms we have found are the same, i.e.

$$\frac{\varphi(f_0+f_1+\ldots+f_{n-1})}{n}=\frac{1}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k)$$

Let $n = 1$ be the base case:
$\frac{\varphi(f_0)}{1} = \frac{1}{1!}x(f_0(x) - f_0(x+1))$, which per the definition of φ is true.

We assume this holds for some n, and see if it still holds for $n+1$:

$$\frac{\varphi(f_0 + f_1 + ... + f_{n-1} + f_n)}{n+1} = \frac{1}{(n+1)!}\prod_{k=0}^{n}(x+k)\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}f_0(x+k)$$

We multiply both sides by $(n+1)$, then focus on the RHS, apply Lemma 5.1.3 and use the recursive formula for the Main Solution (5.1):

$$\begin{aligned}
&\frac{1}{n!}\prod_{k=0}^{n}(x+k)\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}f_0(x+k)\\
=&(x+n)\cdot\frac{1}{n!}\prod_{k=0}^{n-1}(x+k)[\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k) - \sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k+1)]\\
=&(x+n)\cdot\frac{\varphi(f_0+...+f_{n-1})}{n} - (x+n)\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k+1)\\
=&(x+n)f_n - x(x+n)\prod_{k=1}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k+1)
\end{aligned}$$

We use the recursive formula (5.1) on the LHS:

$$\begin{aligned}
\varphi(f_0 + f_1 + ... + f_{n-1} + f_n) &= nf_n + \varphi f_n\\
&= nf_n + xf_n - xf_n(x+1) = (x+n)f_n - xf_n(x+1)
\end{aligned}$$

We cancel the $(x+n)f_n$ term on both sides to get:

$$xf_n(x+1) = x(x+n)\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k+1)$$

Expanding the LHS we get:

$$\begin{aligned}
xf_n(x+1) &= x\cdot\frac{1}{n!}\prod_{k=0}^{n-1}(x+k+1)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k+1)\\
&= x(x+n)\frac{1}{n!}\prod_{k=1}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k+1)
\end{aligned}$$

which equals the RHS. □

Theorem 5.1.5. *The Main Solution satisfies the 2nd criteria of the Main Problem (equal diagonals)*

Proof. We need to show that f_n satisfies the diagonal property, i.e. $\sum_{K=0}^{N} f_{N-K}(K+1) = f_0(1)$. Let $f_0(x) = f(x)$, we then have

$$f_n(x) = \frac{1}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(x+k)$$
$$f_{N-K}(K+1) = \frac{1}{(N-K)!}\prod_{k=0}^{N-K-1}(K+k+1)\sum_{k=0}^{N-K}(-1)^k\binom{N-K}{k}f(K+k+1)$$

We can simplify everything in front the of the sum:

$$\frac{1}{(N-K)!}\prod_{k=0}^{N-K-1}(K+k+1)$$
$$= \frac{1}{(N-K)!}(K+1)(K+2)...(N)$$
$$= \frac{1}{(N-K)!}\cdot\frac{N!}{K!} = \frac{N!}{K!(N-K)!} = \binom{N}{K}$$

We then have:

$$\sum_{K=0}^{N} f_{N-K}(K+1) = \sum_{K=0}^{N}\binom{N}{K}\sum_{k=0}^{N-K}(-1)^k\binom{N-K}{k}f(K+k+1)$$
$$= \binom{N}{0}[\binom{N-0}{0}f(1) - \binom{N-0}{1}f(2) + ... + \binom{N-0}{N-1}f(N) - \binom{N}{N}f(N+1)]$$
$$+\binom{N}{1}[\binom{N-1}{0}f(2) - \binom{N-1}{1}f(3) + ... + \binom{N-1}{N-1}f(N+1)]$$
$$+\dots$$
$$+\binom{N}{N-1}[\binom{1}{0}f(N) - \binom{1}{1}f(N+1)]$$
$$+\binom{N}{N}f(N+1)$$
$$= \sum_{K=0}^{N} f(N-K+1)\sum_{k=0}^{N-K}(-1)^k\binom{K+k}{k}\binom{N}{K+k}$$

We expect all $f(N-K+1)$ to cancel out, with the sole exception of $f(1)$.

We get this if the inner sum evaluates to 1 when $K = N$ and 0 otherwise. The former is trivially true, and the latter can be done by induction, i.e. $\sum_{k=0}^{N-K}(-1)^k\binom{K+k}{k}\binom{N}{K+k} = 0$ for all K, N s.t. $0 \leq K < N$.

For now, we skip the base case and go directly to the induction hypothesis; we assume it holds for some N, and then see if it still holds for $N+1$:

$$\sum_{k=0}^{N-K+1}(-1)^k\binom{K+k}{k}\binom{N+1}{K+k}$$
$$= \sum_{k=0}^{N-K+1}(-1)^k\binom{K+k}{k}\binom{N}{K+k-1} + \sum_{k=0}^{N-K+1}(-1)^k\binom{K+k}{k}\binom{N}{K+k}$$
$$= \sum_{k=0}^{N-K+1}(-1)^k\binom{K+k}{k}\binom{N}{K+k-1} + (-1)^{N-K+1}\binom{N+1}{N-K+1}\binom{N}{N+1}$$
$$= \sum_{k=0}^{N-K+1}(-1)^k\binom{K+k}{k}\binom{N}{K+k-1}$$

Notice that after applying induction once, we end up with an almost identical expression, the only change being we have an extra term in our sum, and that the 2nd binomial coefficient's 2nd argument has been lowered by 1. In other words, showing that our original expression always sums to 0 is equivalent to showing that our new expression also sums to 0. If we apply induction again, we will once again add another term to our sum, and lower the argument by 1 once more. By induction, we can do this as many times as we want, until the 2nd binomial has a negative 2nd argument for all values of N and K. At this point it will always evaluate to 0, which means all of the terms in the sum evaluates to 0, meaning the sum itself also evaluates to 0. The only thing left to show is that the base cases hold every time we apply induction to our problem. Let $K = 0$ and $N = 1$ for each of the base cases, we then need to show:

$$\sum_{k=0}^{1+m}(-1)^k\binom{k}{k}\binom{1}{k-m} = \sum_{k=0}^{1+m}(-1)^k\binom{1}{k-m} = 0$$

For $m = 0$ we get:

$$\sum_{k=0}^{1}(-1)^k\binom{1}{k} = \binom{1}{0} - \binom{1}{1} = 0$$

Then we note for any $m > 0$, the last 2 terms will be the same as we get above (with or without reversed signs, which doesn't change anything), while all previous terms will have a binomial with a negative argument, meaning they all evaluate to 0, which completes our proof.

□

Theorem 5.1.6. *The Main Solution satisfies the third criteria of the Main Problem (linear independence)*

Proof. We know that the set defined by the sequence of repeatedly applying φ to some suitable f_0 is a linearly independent set. Recall the recursive formula for the Main Solution: $f_{n+1} = \frac{nf_n + \varphi(f_n)}{n+1}$, which is just a linear combination of f_n and $\varphi(f_n)$. By the same argument we used to show that the former set was linearly independent, we can show that the Main Solution is also linearly independent. We start with f_0, and add f_1, which is a linear combination of f_0 and f_1, where f_1 is guaranteed to have at least one unique denominator in its pfd, which makes our new set linearly independent. By the same reasoning, adding f_2 to our set also preserves linear independence. And by induction, we have that the entire Main Solution is a linearly independent set. □

With that, we have shown that the Main Solution solves the Main Problem for any suitable function $f_0(x)$.

5.2 Properties of the Main Solution

With the benefit of having a one-fits-all solution to our problem, we can improve some of our previous statements.

Theorem 5.2.1. *Let $\mathscr{M}$ be all sets of functions that are part of the Main Solution for suitable functions. Then for $f_n, g_n \in \mathscr{M}$ we have*

1. $\varphi(f_n) \in \mathscr{M}$
2. $f_n + g_n \in \mathscr{M}$ *provided* $f_0 + g_0$ *is suitable*

Proof. This is essentially an improved version of a previous conjecture and theorem. For the first claim, we need to show that the Main Solution generated by $\varphi(f_0) = xf_0(x) - xf_0(x+1)$ is the same as the sum of the Main Solutions for $xf_0(x)$ and $xf_0(x+1)$.

$$\begin{aligned}
xf_n(x) - xf_n(x+1) &= \frac{x}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k) \\
&\quad - \frac{x}{n!}\prod_{k=1}^{n}(x+k+1)\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_0(x+k+1) \\
&= \frac{x}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}(f_0(x+k) - f_0(x+k+1)) \\
\varphi(f_0(x)) &= \frac{1}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}x(f_0(x+k) - f_0(x+k+1)) \\
&= \frac{x}{n!}\prod_{k=0}^{n-1}(x+k)\sum_{k=0}^{n}(-1)^k\binom{n}{k}(f_0(x+k) - f_0(x+k+1))
\end{aligned}$$

As for the second claim, if $f_0 + g_0$ is a suitable function, then we can use this as our initial function and apply the Main Solution to it to get a family of functions that is in $\mathscr{M}$. □

Playing around with different choices of f_0 for the Main Solution is lots of fun, especially for "illegal" ones. The most obvious one is $f_0 = 0$, which (as expected) makes all $f_n = 0$, which violates the linear independence criteria. Another interesting initial choice is $f_0 = 1/x$, which doesn't have a converging infinite sum, but as previously guessed, the "almost solution" we had where $f_n = 1/(x+n)$ was correct.

Proposition 5.2.2. *Applying the Main Solution to $f_0(x) = 1/x$ gives us:*

$$f_n(x) = \frac{1}{x+n}$$

Proof. We know the Main Solution satisfies $f_{n+1} = \frac{nf_n + \varphi(f_n)}{n+1} +$, so if our closed form expression also satisfies this (for the same initial function, they must be the same).

$$
\begin{aligned}
f_n &= \frac{1}{x+n} \\
f_{n+1} &= \frac{n\frac{1}{x+n} + \frac{x}{x+n} - \frac{x}{x+n+1}}{n+1} = \frac{\frac{x+n}{x+n} - \frac{x}{x+n+1}}{n+1} \\
&= \frac{1 - \frac{x}{x+n+1}}{n+1} = \frac{n+1}{(x+n+1)(x+n)} = \frac{1}{x+n+1}
\end{aligned}
$$

□

Corollary 5.2.3.

$$\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{x+k} = n! \prod_{k=0}^{n} \frac{1}{x+k}$$

Proof. We know that plugging $f_0(x) = 1/x$ into the Main Solution is equal to $1/(x+n)$. Rearranging the equation gives us the conclusion. □

Another interesting question is whether or not the Main Solution is the only solution. One attempt at checking this is to see whether or not the Basel functions are the same as the Main Solution applied to $b_0(x) = 1/x^2$.

Theorem 5.2.4. *The Basel functions are the same as the Main Solution applied to $b_0(x) = 1/x^2$, i.e.*

$$b_n(x) = \frac{1}{x+n} \sum_{k=0}^{n} \frac{1}{x+k} = \frac{1}{n!} \prod_{k=0}^{n-1} (x+k) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(x+k)^2}$$

Proof. We use the same reasoning as before, checking if the Basel functions satisfies the same recursive formula.

$$\begin{aligned}
&\frac{1}{n+1}(nb_n+\varphi(b_n))\\
=&\frac{1}{n+1}\left(\frac{n}{x+n}\sum_{k=0}^{n}\frac{1}{x+k}+\frac{x}{x+n}\sum_{k=0}^{n}\frac{1}{x+k}-\frac{x}{x+n+1}\sum_{k=0}^{n}\frac{1}{x+k+1}\right)\\
=&\frac{1}{n+1}\left(\frac{x+n}{x+n}\sum_{k=0}^{n}\frac{1}{x+k}-\frac{x}{x+n+1}\sum_{k=0}^{n}\frac{1}{x+k+1}\right)\\
=&\frac{1}{n+1}\left(\frac{1}{x}+\sum_{k=1}^{n}\frac{1}{x+k}-\frac{x}{x+n+1}\left(\frac{1}{x+n+1}+\sum_{k=1}^{n}\frac{1}{x+k}\right)\right)\\
=&\frac{1}{n+1}\left(\frac{1}{x}-\frac{x}{(x+n+1)^2}+\left(1-\frac{x}{x+n+1}\right)\sum_{k=0}^{n}\frac{1}{x+k}\right)\\
=&\frac{1}{n+1}\left(\frac{1}{x}-\frac{x}{(x+n+1)^2}\right)+\frac{1}{n+1}\cdot\frac{n+1}{x+n+1}\sum_{k=1}^{n}\frac{1}{x+k}
\end{aligned}$$

We now set this equal to b_{n+1} and confirm that they are the same:

$$\begin{aligned}
\frac{1}{n+1}\left(\frac{1}{x}-\frac{x}{(x+n+1)^2}\right)+\frac{1}{x+n+1}\sum_{k=1}^{n}\frac{1}{x+k}&=\frac{1}{x+n+1}\sum_{k=0}^{n+1}\frac{1}{x+k}\\
\frac{x+n+1}{x(n+1)}-\frac{x}{(n+1)(x+n+1)}+\sum_{k=1}^{n}\frac{1}{x+k}&=\sum_{k=1}^{n}\frac{1}{x+k}+\frac{1}{x}+\frac{1}{x+n+1}\\
\frac{x+n+1}{x(n+1)}-\frac{x}{(n+1)(x+n+1)}&=\frac{1}{x}+\frac{1}{x+n+1}\\
\frac{(x+n+1)^2-x^2}{x(n+1)(x+n+1)}&=\frac{(2x+n+1)(n+1)}{x(n+1)(x+n+1)}\\
2nx+2x+n^2+2n+1&=2nx+2x+n^2+2n+1
\end{aligned}$$

□

Corollary 5.2.5.

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{(x+k)^2}=\frac{n!(x-1)!}{(x+n)!}(\psi(x+n+1)-\psi(x))$$

Proof. This comes as a result of the Basel functions being the same as the Main Solution for the initial function $f_0=1/x^2$. If we multiply both sides by $1/n!\prod_{k=0}^{n-1}(x+k)$ we get the Main Solution expression on the LHS, and if (and only if) our claim holds, the RHS will then evaluate to the digamma expression for the Basel functions.

$$\begin{aligned}
&\frac{1}{n!}(\prod_{k=0}^{n-1}(x+k))\frac{n!(x-1)!}{(x+n)!}(\psi(x+n+1)-\psi(x)) \\
&= x(x+1)...(x+n-1)\frac{(x-1)!}{(x+n)!}(\psi(x+n+1)-\psi(x)) \\
&= \frac{(x+n-1)!}{(x-1)!}\frac{(x-1)!}{(x+n)!}(\psi(x+n+1)-\psi(x)) \\
&= \frac{1}{x+n}(\psi(x+n+1)-\psi(x))
\end{aligned}$$

□

While the Basel functions equals a Main Solution, and thus doesn't help determine whether or not there exists other solutions, we can show that other solutions do exist.

Theorem 5.2.6. *The Main Solution is not the only solution to the Main Problem.*

Proof. Let $f_n(x)$ be a family of functions defined by the Main Solution. Then let $g(x)$ have a converging infinite sum such that $g(1) = 0 = g(2)$. Then $h(x) = g(x) - \varphi(g)$ will have an infinite sum equal to 0, and $h(0) = 0$. We now modify two consecutive elements in our family of functions like so: $f_m \to \hat{f}_m = f_m(x) + h(x)$ and $f_{m+1} \to \hat{f}_{m+1} = f_{m+1}(x) - h(x+1)$ for some fixed m, then both the infinite sum and the diagonal sum be preserved. If we show that there exists such an $h(x)$ that also preserves linear independence for some $f_n(x)$, then we are done.

Let $b_n(x)$ be the Basel functions, and $g(x) = (x-1)(x-2)/x^4$. This choice of $g(x)$ satisfies the criteria above, and hence $h(x) = g(x) - \varphi(g)$ has an infinite sum equal to 0 with $h(1) = 0$, and hence the first two criteria of the Main Problem are satisfied. Applying pfd to h we get:

$$h(x) = -\frac{1}{x} + \frac{1}{x+1} + \frac{4}{x^2} - \frac{4}{(x+1)^2} - \frac{5}{x^3} + \frac{5}{(x+1)^3} + \frac{2}{x^4} - \frac{2}{(x+1)^4}$$

We modify two consecutive elements from the Basel functions: $b_1 \to \hat{b}_1$ and $b_2 \to \hat{b}_2$ and use the same argument as in Theorem 3.2.3 to get our conclusion. That is, we note the n-th member of our modified family of functions contains a term in its pfd of the form $1/(x+n)^2$ that none of the proceeding functions had in its pfd, which remains true even in this modified version, and hence they are linearly independent, meaning this alternate family of function is another solution to the Main Problem for the initial function $f_0(x) = 1/x^2$. □

Remark 5.2.7. Such a modified version of the Main Solution is enough to disprove conjecture 4.1.2 and 4.1.3. That said, we have shown that the former holds true for the Main Solution, whereas the latter keeps its status as a conjecture even if we change "any solution" to "the Main Solution".

In this proof we modified the 2nd and 3rd entries of an already known solution to create our new solution, but this was arbitrarily chosen and can be replaced by any two consecutive functions in the sequence. Furthermore, we can do this multiple times and with different choices for $h(x)$, which strongly points to there being no bound to the amount of such solutions to the Main Problem. The most obvious consequence of such solutions are the earlier conjectures where we tried to use previous solutions to find new ones. The existence of these modified solutions provides us with counterexamples to these conjectures, and as such, these conjectures may strictly be qualities of the Main Solution itself, rather than of any particular solution.

Another interesting observation is what the Main Solution does to polynomials, as it seems to "kill" polynomials of degree m after $m+1$ iterations, meaning the first m f_n equals some polynomials, and then after that they all equal 0.

Theorem 5.2.8. *Let $f_0(x)$ be some polynomial of degree m, then the sequence $f_n(x)$ defined by the Main Solution is identically equal to* 0 *for all* $n > m$

Proof. We see that the Main Solution contains the binomial transformation (1.9), which is the same as the m-th forward difference with alternating signs. We know from [6] that the forward difference reduces polynomials by 1 degree, which means that it will be fully reduced to the 0-polynomial after $m+1$ iterations of the forward difference. After this, the forward difference will always return 0. □

A fun consequence of this theorem is that if we create a "Main Problem", but for an initial function f_0 some polynomial of degree m, and only care about the diagonal property, then this problem can be solved with (at most) m non-zero polynomials. By applying the Main Solution to the initial polynomial, it's obvious that the functions we get will always stay as a polynomial, and as the theorem says, we will only get the 0-polynomial after the first m additional functions in the sequence.

5.3 Main Solution on shifted functions

In this section we will explore how the Main Solution interacts with shifted functions. We define the Main Solution as an operator on suitable functions like so: $\mathscr{M} : f_0(x) \mapsto (f_n(x))_{n=0}^{\infty}$. We take some suitable function $f(x) = f_{0,0}(x)$ and shift it by m units to turn it into $f_{0,0}(x-m) = f_{0,m}(x)$, and then apply the Main Solution: $\mathscr{M} f_{0,m}(x) = f_{n,m}(x)$, where m is the amount we shifted the function by, and n is the n-th element after applying the Main Solution.

We now have a new family of functions that converge to some (usually) new value, even if we shift it back by the same amount. As we will see, its sum will now no longer be independent of n, but it will still be related to the original sum in some way. We start with the case where we shift the function by 1 unit.

Theorem 5.3.1. *Let $f(x)$ and $f(x-1)$ be suitable, then we have:*

$$\sum_{x=1}^{\infty} (f_{n,1}(x+1) - f_{n,0}(x)) = f(0) - f_{n,1}(1) = \sum_{k=0}^{n-1} f_{k,0}(1)$$

Proof. Let $\sum_{x=1}^{\infty} f(x) = A$, then $\sum_{x=1}^{\infty} f(x-1) = A + f(0)$. We then have $\sum_{x=1}^{\infty} f_{n,1}(x+1) = A + f(0) - f_{n,1}(1)$, which means $\sum_{x=1}^{\infty} f_{n,1}(x+1) - \sum_{x=1}^{\infty} f_n(x) = f(0) - f_{n,1}(1)$, which was the first equality we wanted to show. We use induction on the other equality:

$$f(0) - f_{n,1}(1) = \sum_{k=0}^{n-1} f_{k,0}(1)$$

For the base case $n = 0$ both sides evaluates to 0. We then assume our statement holds for some n, and sees if it holds for $n+1$:

$$
\begin{aligned}
f(0) - f_{n+1,1}(1) &= \sum_{k=0}^{n} f_{k,0}(1) \\
f(0) - f_{n+1,1}(1) &= f_{n,0}(1) + \sum_{k=0}^{n-1} f_{k,0}(1) \\
f(0) - f_{n+1,1}(1) &= f_{n,0}(1) + f(0) - f_{n,1}(1) \\
-f_{n+1,1}(1) &= f_{n,0}(1) - f_{n,1}(1)
\end{aligned}
$$

We then use the explicit formula for the Main Solution (5.3) for all the terms. Note that since all terms are evaluated at $x = 1$, the "$1/n!\prod(x+k)$" part of the Main Solution will cancel itself out. The LHS then becomes:

$$
\begin{aligned}
-f_{n+1,1}(1) &= -\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} f(k) \\
&= -\sum_{k=0}^{n+1}(-1)^k \binom{n}{k} f(k) - \sum_{k=0}^{n+1}(-1)^k \binom{n}{k-1} f(k) \\
&= -\sum_{k=0}^{n}(-1)^k \binom{n}{k} f(k) + \sum_{k=0}^{n}(-1)^k \binom{n}{k} f(k+1)
\end{aligned}
$$

which equals the RHS:

$$
f_{n,0}(1) - f_{n,1}(1) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f(k+1) - \sum_{k=0}^{n}(-1)^k \binom{n}{k} f(k)
$$

□

One of the main motivations for doing this is how it lets us apply the Main Solution to functions that diverge because it has a pole at $x = 1$. An example of this would be $f(x) = 1/(x-1)^2$. If we instead start with $1/x^2$, we shift it back by 1 to get to $1/(x-1)^2$, apply the Main Solution, and then shift it back, we will then have successfully applied the Main Solution to an otherwise illegal starting function, whose infinite sum we can now relate to another family of functions we are familiar with.

In the proof above we showed that $f(0) - f_{n,1}(1) = \sum_{k=0}^{n-1} f_{k,0}(1)$. While the terms on the LHS may not always be defined (due to poles), the sum on the RHS always is, which suggests that the equality (excluding the middle one) in Theorem 5.3.1 holds even if $f(x-1)$ isn't suitable, which would mean we can get the same, meaningful result regardless of $f(x-1)$ being suitable or not. Furthermore, the theorem seems to hold even for suitable* functions. If we define $g_n(x) = F_n(x+1) - f_n(x)$, then when we take its infinite sum, it appears we always end up with a telescoping sum, which again leaves us with some sum of rational numbers.

By extending theorem 5.3.1 to an arbitrary shift of length m we get the following theorem.

Theorem 5.3.2. *Let $f(x)$ and $f(x-m)$ be suitable, then we have*

$$\sum_{x=1}^{\infty}(f_{n,m}(x+m)-f_{n,0}(x)) = \sum_{k=0}^{m-1}(f(-k)-f_{n,m}(k+1)) = \sum_{k=0}^{n-1}\sum_{h=0}^{m-1} f_{k,h}(h+1)$$

Proof. Let $\sum_{x=1}^{\infty} f(x) = A \in \mathbb{R}$, then assuming $f(x-m)$ is suitable we have $\sum_{x=1}^{\infty} f(x-m) = A + f(0) + f(-1) + ... + f(-m+1) = A + \sum_{k=0}^{m-1} f(-k)$. We now apply the Main Solution: $f(x-m) = f_{0,m}(x) \xrightarrow{\mathscr{M}} f_{n,m}(x)$. We shift it back and take its infinite sum to get:

$$\begin{aligned}\sum_{x=1}^{\infty} f_{n,m}(x+m) &= A + \sum_{k=0}^{m-1} f(-k) - f_{n,m}(1) - f_{n,m}(2) - ... - f_{n,m}(m) \\ &= A + \sum_{k=0}^{m-1} f(-k) - \sum_{k=0}^{m-1} f(k+1)\end{aligned}$$

We subtract the infinite sum of $f(x)$ to get the first desired equality:

$$\sum_{x=1}^{\infty} f_{n,m}(x+m) - \sum_{x=1}^{\infty} f(x) = \sum_{k=0}^{m-1} f(-k) - \sum_{k=0}^{m-1} f(k+1)$$

Now we need to show $\sum_{k=0}^{m-1}(f(-k)-f_{n,m}(k+1)) = \sum_{k=0}^{n-1}\sum_{h=0}^{m-1} f_{k,h}(h+1)$. We apply induction and check the base case $n=0$:

$$\begin{aligned}\sum_{k=0}^{m-1} f(-k) - \sum_{k=0}^{m-1} f_{0,m}(k+1) &= 0 \\ \sum_{k=0}^{m-1} f(-k) &= \sum_{k=0}^{m-1} f_{0,0}(k+1-m) \\ f(0)+f(-1)+...+f(-m+1) &= f(-m+1)+...f(-1)+f(0)\end{aligned}$$

Which holds true for all m. Now we let $n \to n+1$:

$$\sum_{k=0}^{m-1}(f(-k)-f_{n+1,m}(k+1)) = \sum_{k=0}^{n}\sum_{h=0}^{m-1} f_{k,h}(h+1)$$

$$\sum_{k=0}^{m-1}(f(-k)-f_{n+1,m}(k+1)) = \sum_{k=0}^{n-1}\sum_{h=0}^{m-1} f_{k,h}(h+1) + \sum_{h=0}^{m-1} f_{n,h}(h+1)$$

$$\sum_{k=0}^{m-1}(f(-k)-f_{n+1,m}(k+1)) = \sum_{k=0}^{m-1}(f(-k)-f_{n,m}(k+1)) + \sum_{h=0}^{m-1} f_{n,h}(h+1)$$

$$\sum_{k=0}^{m-1} f_{n+1,m}(k+1) = \sum_{k=0}^{m-1} f_{n,m}(k+1) - \sum_{h=0}^{m-1} f_{n,h}(h+1) \tag{5.4}$$

We end up with three sums, which we will evaluate one by one. But before we do this, we need to establish some properties of the Main Solution. We want to express all of our terms in terms of $f_{n,0}(x) = f_n(x)$. To do this, we separate the explicit formula for the Main Solution into two parts; the factorial/product and the sum. If we start with the sum, the easy case is whenever we have a function of the form $f_{n,m}(x)$, as its summation part evaluates to $\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k+x-m)$, which shares its summation form with $f_{n,0}(x-m)$. The slightly trickier case is when we have $f_{n+1,m}$ whose summation expression evaluates to:

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}f(k+x-m)$$

which per Lemma 5.1.3 equals

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k+x-m) - \sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k+x-m+1)$$

which shares its summation expression with $f_{n,0}(x-m)+f_{n,0}(x-m+1)$.

Now, the factorial and product expression for $f_{n,m}(x)$ evaluates to

$$\begin{aligned}\frac{1}{n!}\prod_{k=0}^{n-1}(x+k) &= \frac{x(x+1)\dots(x+n-1)}{n!}\cdot\frac{(x-1)!}{(x-1)!}\\ &= \frac{(x+n-1)!}{n!(x-1)!} = \frac{(x+n-1)\dots(n+1)n!}{n!(x-1)!}\\ &= \frac{(n+1)(n+2)\dots(x+n-1)}{(x-1)!} = \frac{1}{(x-1)!}\prod_{k=1}^{x-1}(n+k)\end{aligned}$$

For the $f_{n+1,m}$ case, we simply let $n \to n+1$. We now use the properties of the product and summation part of the Main Solution to express the terms in (5.4) one by one, starting with the sum on the LHS.

$$\begin{aligned}\sum_{k=0}^{m-1} f_{n+1,m}(k+1) = f_{n+1,m}(1) &= f_n(1-m) - f_n(2-m)\\ + f_{n+1,m}(2) &= [f_n(2-m) - f_n(3-m)](n+2)\\ + f_{n+1,m}(3) &= [f_n(3-m) - f_n(4-m)]\frac{(n+2)(n+3)}{2}\\ &+ \ldots\\ + f_{n+1,m}(m) &= [f_n(0) - f_n(1)]\frac{(n+2)\ldots(n+m)}{(m-1)!}\end{aligned}$$

Now we do the same for the sums on the LHS:

$$\begin{aligned}\sum_{k=0}^{m-1} f_{n,m}(k+1) = f_{n,m}(1) &= f_n(1-m)\\ + f_{n,m}(2) &= f_n(2-m)(n+1)\\ + f_{n,m}(3) &= f_n(3-m)\frac{(n+1)(n+2)}{2}\\ &+ \ldots\\ + f_{n,m}(m) &= f_n(0)\frac{(n+1)\ldots(n+m)}{(m-1)!}\end{aligned}$$

$$\begin{aligned}-\sum_{h=0}^{m-1} f_{n,h}(h+1) = -f_{n,0}(1) &= -f_n(1)\\ - f_{n,1}(2) &= -f_n(1)(n+1)\\ - f_{n,2}(3) &= -f_n(1)\frac{(n+1)(n+2)}{2}\\ &- \ldots\\ - f_{n,m-1}(m) &= -f_n(1)\frac{(n+1)\ldots(n+m-1)}{(m-1)!}\end{aligned}$$

We need the coefficients for each $f_n(x)$ be equal on both sides of the equation. Starting with the coefficients for $f_n(1)$, we need to show the following:

$$1+(n+1)+\frac{(n+1)(n+2)}{2}+\ldots+\frac{(n+1)\ldots(n+m-1)}{(m-1)!}=\frac{(n+2)\ldots(n+m)}{(m-1)!}$$

For the base case $m=1$ we get $1=1$ (using the convention that the empty product on the RHS is 1). Now we let $m \to m+1$:

$$\frac{(n+2)...(n+m)}{(m-1)!} + \frac{(n+1)...(n+m)}{m!} = \frac{(n+2)...(n+m+1)}{m!}$$
$$(n+2)...(n+m)m + (n+1)...(n+m) = (n+2)...(n+m+1)$$
$$(n+2)...(n+m)(m+n+1) = (n+2)...(n+m)(n+m+1)$$

which is true. Finally, to show the coefficients of the other terms are equal, we need to show the following:

$$\frac{(n+2)...(n+m+1)}{m!} - \frac{(n+2)...(n+m)}{(m-1)!} = \frac{(n+1)...(n+m)}{m!}$$

which is equivalent to the previous expression. Therefore, all the coefficients are equal on both sides of the equation, which concludes the proof. □

Remark 5.3.3. Note that the first equality in the theorem can be simplified to:

$$\sum_{x=1}^{\infty} (f_{n,0}(x-m) - f_{n,m}(x+m)) = \sum_{k=1}^{m} f_{n,m}(k)$$

Once again, we note that while the middle term may not always be suitable, the LHS and RHS always are, regardless if $f(x-m)$ is suitable or not.

A reason why we care about these shifted functions, is how they relate to the sums over the negative integers for some suitable functions. As usual, we take the Basel functions as our example. For $n=0$ we have a pole at $x=0$, and for every n after this, we will also have poles at the negative integers up to n. So when we take our infinite sum, we want to avoid these. So for each n, we start our sum from $x=-n$ and from there go towards $-\infty$. If we let $x \to -x-n$, we can take the infinite sum from $x=1$ to ∞ like normal:

$$\sum_{x=1}^{\infty} b_n(-x-n)$$

Our goal is to evaluate this sum (or similar ones). If we use the previous theorem for $m=1$ on $b_0(x)$, we surprisingly seem to get an equality between their infinite sums, which if we are lucky, means that the two functions themselves are equal. We will show this is the case, not just for the Basel functions, but also for initial functions of the form $1/x^m$ (although with alternating signs for each m).

Theorem 5.3.4. *For $f_0(x) = 1/x^m$ we have $f_{n,1}(x+1) = (-1)^m f_n(-x-n)$*

Proof. We plug both expressions into the explicit formula for the Main Solution (5.3).

$$f_{n,1}(x+1) = (-1)^m f_n(-x-n)$$
$$\frac{1}{n!}\prod_{k=0}^{n-1}(x+k+1)\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{(x+k)^m} = \frac{(-1)^m}{n!}\prod_{k=0}^{n-1}(-x-n+k)\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{(-x-n+k)^m}$$

We cancel the $1/n!$ and focus on the RHS and use (1.8) to get:

$$(-1)^m\prod_{k=0}^{n-1}(-x-n+k)\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{(-x-n+k)^m}$$
$$(-1)^m\cdot(-1)^n\prod_{k=0}^{n-1}(x+n-k)\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{n-k}}{(-1)^m(x+n-k)^m}$$
$$(-1)^n\cdot(-1)^n\prod_{k=0}^{n-1}(x+k+1)\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{(x+k)^m}$$

which equals the LHS, and hence the functions are the same (up their sign). □

Now that we can express $f_n(-x-n)$ in terms of $f_{n,1}(x+1)$, we can apply the previous theorem(s) to establish the sum of our target functions. In fact, we can now establish the values of the sums of such functions over all the integers where the functions doesn't have poles.

Theorem 5.3.5. *Let $f_n(x)$ be the Main Solution for $f_0(x) = \frac{1}{x^m}$, and $\mathbb{Z}^*$ be the set of integers where $f_n(x)$ doesn't have any poles. If we assume that theorem 5.3.2 holds even when $f(x-1)$ isn't suitable, we then have:*

$$\sum_{x\in\mathbb{Z}^*} f_n(x) = 2\zeta(m) + \sum_{k=0}^{n-1} f_k(1)$$

Where $\zeta(m) = \sum_{x=1}^{\infty}\frac{1}{x^m}$ is the Riemann-zeta function.

Proof. We split the sum into two parts, and apply the previous theorem.

$$\begin{aligned}\sum_{x\in\mathbb{Z}^*} f_n(x) &= \sum_{x=1}^{\infty} f_n(x) + (-1)^m \sum_{x=1}^{\infty} f_n(-x-n) \\ &= \zeta(m) + (-1)^m \cdot (-1)^m \sum_{x=1}^{\infty} f_{n,1}(x+1)\end{aligned}$$

From theorem 5.3.2 we have that

$$\begin{aligned}\sum_{x=1}^{\infty} (f_{n,1}(x+1) - f_{n,0}(x)) &= \sum_{k=0}^{n-1} f_{k,0}(1) \\ \sum_{x=1}^{\infty} f_{n,1}(x+1) - \zeta(m) &= \sum_{k=0}^{n-1} f_k(1) \\ \sum_{x=1}^{\infty} f_{n,1}(x+1) &= \zeta(m) + \sum_{k=0}^{n-1} f_k(1)\end{aligned}$$

Substituting this into the above we get our conclusion. □

Unfortunately, theorem 5.3.4 only applies to initial functions of the form $f_0(x) = 1/x^m$, and we don't have a similar relation that works for an arbitrary suitable function. As such, (for now at least) we can't create such formulas for sums over all the integers (excluding the poles) for other suitable functions.

Chapter 6

Area rearrangement operator in continuous calculus

In this chapter we will look at the continuous version of the φ operator and explore its properties. Recall how $\varphi f(x) = -x\Delta f(x)$. The continuous version of the finite difference is the derivative, and as such, the continuous version of the area rearrangement operator is as follows:

Definition 6.0.1. *The continuous area rearrangement operator* $\Phi : C^\infty([1,\infty)) \mapsto C^\infty([1,\infty))$ *is given by*

$$\Phi f(x) = -xf'(x)$$

While we can plug in any smooth function into Φ, we will mostly consider functions that has a converging infinite integral. With this restriction, the operator should behave like it does in the discrete case, and for the sake of brevity we will skip proving claims that are trivial and/or equivalent to the discrete case. This includes (but is not limited to) Φ being linear, injective, taking constants to 0 and having an inverse. It should also still rearrange the area of interest in some way, by measuring the area in horizontal strips, rather than the usual vertical columns, but unlike the former claims, this is worth investigating further. Under certain conditions, we expect to end up with the same area of some function f after applying Φ to it. The geometric interpretation of measuring the area “sideways” is reminiscent of a Lebesgue integral, however these are not the same and should not be conflated. Whereas the Lebesgue integral always evaluates to the same value as the Riemann integral for continuous functions [9, p. 81-82], we will soon see that the Φ operator does not necessarily do this.

6.1 Identities

From the discrete case, we expect some function $f(x)$ to have the same area as $\Phi f(x)$, at least under some (as for now unknown) conditions. If we instead of focusing on a strict equality between them, consider the difference between them, we would then expect to get some meaningful result, which we do.

Theorem 6.1.1.

$$\int f(x) - \Phi f(x)\, dx = xf(x) + C$$

Proof. Using the definition of Φ we get:

$$\int f(x) - \Phi f(x)\, dx = \int f(x) - (-xf'(x))\, dx = \int f(x) + xf'(x)\, dx$$

Using partial fraction decomposition we get our result:

$$\begin{aligned} &\int f(x)\, dx + \int xf'(x)\, dx \\ &= \int f(x)\, dx + xf(x) - \int f(x)\, dx \\ &= xf(x) + C \end{aligned}$$

□

From this we see that the difference between the areas are not necessarily 0, and hence the integral of f does not always agree with the integral of $\Phi(f)$, meaning this is not equivalent to a Lebesgue integral. But from this identity, we can still find cases where the two must be the same, and as such, recover a continuous version of the discrete equal infinite sums identity.

Theorem 6.1.2. *Let $a = -\infty$ or 0 and $b = 0$ or $+\infty$, if $\int_a^b f dx$ is well-defined for a "suitable" f, we have*

$$\int_a^b f(x) - \Phi(f)\, dx = 0$$

Proof. If $a = 0 = b$ then the integral obviously evaluates to 0. Otherwise, we know the integral evaluates to $[xf(x)]_a^b = bf(b) - af(a)$. If either a or b is 0 and $f(a)$ or $f(b)$ is well-defined at this point, then their product evaluates to 0.

Lastly we need to show that $\lim_{x\to\pm\infty} xf(x) = 0$. We know an infinite sum converges if and only if the corresponding improper integral converges [10]. Per assumption, our improper integral is well-defined, which means so is the corresponding infinite sum, which we have shown earlier that if it is absolutely convergent satisfies the limit of interest, at least when $x \to +\infty$. For the negative case, we note that it is essentially equivalent to the positive case, and as such it also holds. Finally, we need to address the criteria that f needs to be absolutely convergent. If we let f be "suitable" like before (i.e. a rational function), we then know it can only change its sign a finite amount of times. After this, it will behave as either a non-negative (or non-positive) function, and hence the conclusion follows. □

Corollary 6.1.3. *For the same assumptions as before we have*

$$\int_a^b f(x) - \Phi^n f(x)\, dx = 0$$

Proof. We simply note that if the integrals are equal after applying Φ once, and that $\Phi f(x)$ necessarily has to remain "suitable", meaning the conditions of the above theorem holds no matter how many times we apply Φ, and hence all the integrals will be the same. □

And just like that we have our equal continuous improper integral version of the equal discrete infinite sums, by use of the area rearrangement operator. Like in the discrete case, there are some choices of f that are illegal and thus dictates what constitutes a "suitable" function. From the theorem, we noted that $f(x)$ has to be well-defined for $x = 0$, as we need $0f(0)$ to evaluate to 0 (or at the very least $\lim_{x\to 0^+} xf(x)$ needs to be 0), meaning any function with x^m in the denominator prevents this from happening. As such we arrive at the following definition for continuous "suitable" functions.

Definition 6.1.4. *A suitable function (in the continuous case) is any rational function* $f(x) = p(x)/q(x) \neq 0$ *with integer coefficients that has no poles for any non-negative integers. If we also allow* $\deg(q) \geq 1 + \deg(p)$*, then we call them "suitable*" (with a star).*

Definition 6.1.5. *We define $\mathbb{S}_C$ (and $\mathbb{S}_C^*$) to be the vector space(s) defined by the span of all continuously suitable (or suitable*) functions with scalars in $\mathbb{R}$.*

Now that we have the continuous equal area identity, a natural question to ask is whether or not we can have a continuous version of our Main Problem. While the Φ operator seems to generate linearly independent functions, there does not seem to be an obvious "equal finite diagonal sum" property to be found in the continuous case, which prevents us from translating the discrete problem into a continuous one.

That said, there is one short-coming φ had that Φ does not share, that being the ease of finding a closed form for repeatedly applying it to some function.

Theorem 6.1.6.

$$\Phi^n f(x) = (-1)^n \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k f^{(k)}(x)$$

where $\left\{ {n \atop k} \right\}$ is the unsigned Stirling numbers of the second kind

Proof. We can find the first few cases of $\Phi^n f(x)$ by hand:

$$\begin{aligned}
\Phi^0 f(x) &= +f(x) \\
\Phi^1 f(x) &= -xf'(x) \\
\Phi^2 f(x) &= +xf'(x) + x^2 f''(x) \\
\Phi^3 f(x) &= -xf'(x) - 3x^2 f''(x) - x^3 f'''(x) \\
\Phi^4 f(x) &= +xf'(x) + 7x^2 f''(x) + 6x^3 f'''(x) + x^4 f''''(x)
\end{aligned}$$

which seems to be of the above form. We know $\Phi^{n+1} f(x) = -x\frac{d}{dx}\Phi^n f(x)$, so if the expression we got satisfies this (for the same initial f), we have our conclusion. We begin with the RHS:

$$\begin{aligned}
-x\frac{d}{dx}\Phi^n f(x) &= -x(-1)^n \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \frac{d}{dx}[x^k f^{(k)}(x)] \\
&= (-1)^{n+1} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} kx^k f^{(k)}(x) + (-1)^{n+1} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^{k+1} f^{(k+1)}(x)
\end{aligned}$$

We now focus on the LHS and use the properties of the Stirling numbers (1.4), (1.5), (1.7), which yields a result reminiscent to Lemma 5.1.3.

$$\begin{aligned}\Phi^{n+1} f(x) &= (-1)^{n+1} \sum_{k=1}^{n+1} \left\{ {n+1 \atop k} \right\} x^k f^{(k)}(x) \\ &= (-1)^{n+1} \sum_{k=1}^{n+1} \left\{ {n \atop k} \right\} k x^k f^{(k)}(x) + (-1)^{n+1} \sum_{k=1}^{n+1} \left\{ {n \atop k-1} \right\} x^k f^{(k)}(x) \\ &= (-1)^{n+1} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} k x^k f^{(k)}(x) + (-1)^{n+1} \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^{k+1} f^{(k+1)}(x)\end{aligned}$$

which is the same as the RHS. □

Using the previous theorems we can find antiderivatives for $\Phi^m f(x) - \Phi^n f(x)$.

Theorem 6.1.7. *for $m < n$ we have*

$$\int \Phi^m f(x) - \Phi^n f(x)\, dx = x \sum_{k=m}^{n-1} \Phi^k f(x) + C$$

Proof. We first show the following: $\int f(x) - \Phi^n f(x) dx = x \sum_{k=0}^{n-1} \Phi^k f(x) + C$. We do this by subtracting and adding each of the $\Phi^m f(x)$ that lies "between" f and $\Phi^n f$, like so:

$$\begin{aligned}&\int f(x) - \Phi^n f(x)\, dx \\ &= \int f(x) - \Phi^1 f(x) + \Phi^1 f(x) - \Phi^2 f(x) + \dots + \Phi^{n-1} f(x) - \Phi^n f(x)\, dx \\ &= \int f(x) - \Phi^1 f(x)\, dx + \int \Phi^1 f(x) - \Phi^2 f(x)\, dx + \dots + \int \Phi^{n-1} f(x) - \Phi^n f(x)\, dx\end{aligned}$$

We now have n integrals, all of the form $\int g_n(x) + x g_n'(x)\, dx$, which we know evaluates to $x g_n(x) + C$. Substituting $\Phi^n f(x)$ in for $g(x)$ we get

$$x f(x) + x \Phi^1 f(x) + \dots + x \Phi^{n-1} f(x) + C = x \sum_{k=0}^{n-1} \Phi^n f(x) + C$$

From here we do this for some m and n and take the difference to get our conclusion

$$\int f(x) - \Phi^n f(x)\, dx - \int f(x) - \Phi^m f(x)\, dx = x \sum_{k=0}^{n-1} \Phi^k f(x) - x \sum_{k=0}^{m-1} \Phi^k f(x) + C$$

$$\int \Phi^m f(x) - \Phi^n f(x)\, dx = x \sum_{k=m}^{n-1} \Phi^k f(x) + C$$

□

6.2 Continuous Main Solution

Although it doesn't seem possible to create a continuous version of the Main Problem, we can still define a "Main Solution" for the continuous case, which has more or less the same properties as before.

Definition 6.2.1. *The continuous version of the Main Solution is given by*

$$f_{n+1}(x) = \frac{\Phi(f_0(x) + f_1(x) + ... + f_n(x))}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} \Phi f_k(x)$$

Like before, this definition can be rearranged into a recursive formula of the form $f_{n+1} = \frac{nf_n + \Phi(f_n)}{n+1}$, and obviously it still has the "equal integral" property that we explored previously. Although the proof is similar to the discrete case, I will show that the sequence generated by this Main Solution is a linearly independent set.

Theorem 6.2.2. *The sequence of functions defined by the Main Solution for a suitable f is a linearly independent set in $\mathbb{S}_C$.*

Proof. We take some suitable function and apply pfd on it to get a sum of suitable* functions. We choose one of them and see what happens when we apply Φ to it. Let $f(x) = \frac{p(x)}{q(x)^m}$, be a suitable* function s.t. $\text{pgcd}(p,q) = 1$ and $q(x)$ is irreducible.

$$\begin{aligned}\Phi(f) &= -x\frac{d}{dx}[p(x)q(x)^{-m}] \\ &= -x[p'(x)q(x)^{-m} - mp(x)q(x)^{-m-1}q'(x)] \\ &= \frac{-xp'(x)}{q(x)^m} + \frac{mxp(x)q'(x)}{q(x)^{m+1}}\end{aligned}$$

We ignore the first term with the $q(x)^m$ in the denominator, as that's the denominator we had to begin with. The other term has $q(x)^{m+1}$ in its denominator, which will ensure linear independence as long as this factor does not get cancelled out (by anything more than a constant). Since f was suitable* we have that x can not be a factor of $q(x)$. Likewise per assumption $p(x)$ does not divide $q(x)$. Hence we only have to show that $q'(x)$ cannot divide $q(x)$

(by anything more than a constant). Let's assume it does, which happens if and only if $q'(x)$ divides $q(x)$, meaning $\frac{q(x)}{q'(x)}$ must be a polynomial not equal to a constant multiple of $q(x)$. The only way for this to happen would be for this to happen is if $q(x)$ has any repeated roots, which contradicts it being irreducible, and hence we have $q'(x)$ does not divide $q(x)$ (by more than a constant).

From here the proof remains the same as in the discrete case; we pick a term in the pfd of f that ensures linear independence and note that the Main Solution can be recursively defined such that it inherits linear independence from the set defined by repeatedly applying Φ. $\square$

Like before, there also exists an explicit expression for the Main Solution.

Theorem 6.2.3. *The Main Solution has the following explicit formula*

$$f_n(x) = (-1)^n \frac{1}{n!} x^n f^{(n)}(x)$$

Proof. We know that our family of functions satisfies $f_{n+1} = \frac{nf_n + \Phi(f_n)}{n+1}$, so if our explicit formula also satisfies this (for the same initial function) they must be the same.

$$\begin{aligned}
f_{n+1} &= \frac{nf_n + \Phi(f_n)}{n+1} \\
\frac{x^{n+1} f^{(n+1)}}{(-1)^{n+1}(n+1)!} &= \frac{1}{n+1}\left(n \frac{x^n f^{(n)}}{(-1)^n n!} - x \frac{\frac{d}{dx}(x^n f^{(n)})}{(-1)^n n!}\right) \\
\frac{x^{n+1} f^{(n+1)}}{(-1)^{n+1}(n+1)!} &= \frac{nx^n f^{(n)}}{(-1)^n (n+1)!} - \frac{nx^n f^{(n)}}{(-1)^n (n+1)!} - \frac{x^{n+1} f^{(n+1)}}{(-1)^n (n+1)!} \\
\frac{x^{n+1} f^{(n+1)}}{(-1)^{n+1}(n+1)!} &= \frac{x^{n+1} f^{(n+1)}}{(-1)^{n+1}(n+1)!}
\end{aligned}$$

$\square$

Like before, we can find antiderivatives of differences of f_n.

Theorem 6.2.4. *For f_m and f_n in the Main Solution s.t. $m < n$ we have*

$$\int f_m(x) - f_n(x)\, dx = \sum_{k=m}^{n-1} \frac{x f_k(x)}{k+1} + C = \sum_{k=m}^{n-1} \frac{(-1)^k}{(k+1)!} x^{k+1} f^{(k)}(x) + C$$

Proof. First we find an expression for the integral of the difference of two consecutive functions in the Main Solution.

$$\int f_n - f_{n+1}\,dx = \int f_n - \frac{nf_n + \Phi(f_n)}{n+1}\,dx = \frac{1}{n+1}\int f_n - \Phi(f_n)\,dx = \frac{xf_n(x)}{n+1} + C$$

We now look at the integral of the difference between two arbitrary functions in the sequence, and subtract and add the missing functions "between" them.

$$\begin{aligned}
&\int f_m - f_n\,dx \\
=&\int f_m - f_{m+1} + f_{m+1} - f_{m+2} + \ldots + f_{n-1} - f_n\,dx \\
=&\int f_m - f_{m+1}\,dx + \int f_{m+1} - f_{m+2}\,dx + \ldots + \int f_{n-1} - f_n\,dx \\
=&\frac{xf_m(x)}{m+1} + \frac{xf_{m+1}(x)}{m+2} + \ldots + \frac{xf_{n-1}(x)}{n} + C \\
=&\sum_{k=m}^{n-1}\frac{xf_k(x)}{k+1} + C = \sum_{k=m}^{n-1}\frac{x(-1)^k x^k f^{(k)}(x)}{(k+1)k!} + C = \sum_{k=m}^{n-1}\frac{(-1)^k x^{k+1} f^{(k)}(x)}{(k+1)!} + C
\end{aligned}$$

□

Like before, the Main Solution "kills off" polynomials, which is even more obvious this time. The explicit formula includes the n-th derivative in its product, which reduces polynomials of degree m to 0 after $m+1$ iterations. What's more interesting is if we let $f_0 = x^m$, then the alternating binomial coefficients show up.

Theorem 6.2.5. *f_n's coefficients for $f_0 = x^m$ are the alternating binomial coefficients, i.e.*

$$\frac{(-1)^n}{n!}x^n\frac{d^n}{dx^n}x^m = (-1)^n\binom{m}{n}x^m$$

Proof. We cancel out the alternating signs on both sides, and use a closed form expression for the repeated derivative of x^m

$$\begin{aligned}
\frac{1}{n!}x^n\frac{m!}{(m-n)!}x^{m-n} &= \binom{m}{n}x^m \\
\frac{m!}{n!(m-n)!}x^m &= \binom{m}{n}x^m
\end{aligned}$$

□

The Main Solution can be used to find and evaluate sequences of integrals. Among them is the integral that extends the factorial function from the naturals to the reals.

Theorem 6.2.6. *The Main Solution can be used to derive the following integral formula for $n!$*

$$n! = \int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) \tag{6.1}$$

Proof. Let $f_0(x) = e^{-x}$ and plug it into the Main Solution to get $f_n(x) = \frac{x^n}{n!}e^{-x}$. Per the property of the Main Solution we have $\int_0^\infty \frac{x^n}{n!}e^{-x}\,dx = \int_0^\infty e^{-x}dx$. We evaluate the latter integral to get 1, and multiply both sides by $n!$ to get our conclusion. □

Another interesting function we can apply the continuous Main Solution to is $f_0(x) = \ln(x)/e^x$. While this function has a pole at $x = 0$, we still have $\lim_{x\to 0^+} xf(x) = 0$, and as its integral from 0 to ∞ still converges (to $-\gamma$ [2]), we have enough motivation to apply the Main Solution to it and see what happens. When we do this, we get a sequence of functions that begins like this:

$$\begin{aligned}
f_0(x) &= \frac{\ln(x)}{e^x} \\
f_1(x) &= \frac{x\ln(x)}{e^x} - \frac{1}{e^x} \\
f_2(x) &= \frac{x^2\ln(x)}{2e^x} - \frac{2x+1}{2e^x} \\
f_3(x) &= \frac{x^3\ln(x)}{6e^x} - \frac{3x^2+3x+2}{6e^x}
\end{aligned}$$

We notice that we seem to get one term with a natural log, and one with a polynomial in it (in their numerators). While the Main Solution tells us what the integral from 0 to ∞ their sum remains constant, we don't know what their integrals evaluates to individually, which is what we will explore here. The integral of the natural log term seems to be $H_n - \gamma$, whereas the one with the polynomial seems to evaluate to $-H_n$. Adding those values together we get our expected result. Now we just have to prove that these are the values the integrals evaluate to.

Theorem 6.2.7. *Assuming the integral from* 0 *to* ∞ *of the Main Solution applied to $f_0(x) = \ln(x)/e^x$ always evaluates to $-\gamma$, we have:*

1.

$$\int_0^\infty \frac{x^n \ln(x)}{n!e^x} dx = H_n - \gamma$$

2.

$$\int_0^\infty \frac{\sum_{k=1}^n (k-1)!\binom{n}{k} x^{n-k}}{n!e^x} dx = H_n$$

Proof. We start by adding the expression in the first integral to the negative of the second one, and show that this is what we get when we apply the Main Solution to our original function, which means we have to show the following:

$$(-1)^n \frac{x^n}{n!} \frac{d^n}{dx^n} \frac{\ln(x)}{e^x} = \frac{x^n \ln(x)}{n!e^x} - \frac{\sum_{k=1}^n (k-1)!\binom{n}{k} x^{n-k}}{n!e^x}$$

$$(-1)^n \frac{d^n}{dx^n} \frac{\ln(x)}{e^x} = \frac{\ln(x)}{e^x} - \frac{\sum_{k=1}^n (k-1)!\binom{n}{k} x^{-k}}{e^x}$$

We do this with proof by induction. For the base case $n = 0$ we get $\ln(x)/e^x$ on both sides. We let $n \to n+1$ and show that it still holds true.

$$(-1)^{n+1} \frac{d^{n+1}}{dx^{n+1}} \frac{\ln(x)}{e^x} = \frac{\ln(x)}{e^x} - \frac{\sum_{k=1}^{n+1} (k-1)!\binom{n+1}{k} x^{-k}}{e^x}$$

We focus on the LHS:

$$\begin{aligned}
&(-1)^{n+1} \frac{d^{n+1}}{dx^{n+1}} \frac{\ln(x)}{e^x} \\
=&(-1) \frac{d}{dx} (-1)^n \frac{d^n}{dx^n} \frac{\ln(x)}{e^x} \\
=&(-1) \frac{d}{dx} [\frac{\ln(x)}{e^x} - \frac{\sum_{k=1}^n (k-1)!\binom{n}{k} x^{-k}}{e^x}] \\
=&(-1) [\frac{1}{xe^x} - \frac{\ln(x)}{e^x} + \frac{\sum_{k=1}^n (k-1)!\binom{n}{k} x^{-k}}{e^x} + \frac{\sum_{k=1}^n k!\binom{n}{k} x^{-k-1}}{e^x}]
\end{aligned}$$

We cancel the $\ln(x)/e^x$ term on both sides and multiply by $-e^x$ to get:

$$\begin{aligned}
&\frac{1}{x} + \sum_{k=1}^n (k-1)!\binom{n}{k} x^{-k} + \sum_{k=1}^n k!\binom{n}{k} x^{-k-1} \\
&= \sum_{k=1}^n (k-1)!\binom{n}{k} x^{-k} + \sum_{k=0}^n k!\binom{n}{k} x^{-k-1}
\end{aligned}$$

Focusing on the RHS, we do something similar to Lemma 5.1.3 to get:

$$\begin{aligned}
&\sum_{k=1}^{n+1}(k-1)!\binom{n+1}{k}x^{-k}\\
&=\sum_{k=1}^{n+1}(k-1)!\binom{n}{k}x^{-k}+\sum_{k=1}^{n+1}(k-1)!\binom{n}{k-1}x^{-k}\\
&=\sum_{k=1}^{n}(k-1)!\binom{n}{k}x^{-k}+\sum_{k=0}^{n}k!\binom{n}{k}x^{-k-1}
\end{aligned}$$

which equals the LHS.

From here, we note that proving either of the statements in the theorem proves the other one, as per assumption the difference between them is $-\gamma$. We choose the second one and show that it is true. Per (1.2) we have $(k-1)!\binom{n}{k}/n! = 1/(k(n-k)!)$. We use this and expand the sum to get:

$$\begin{aligned}
&\int_0^\infty \frac{x^{n-1}}{(n-1)!e^x}+\frac{x^{n-2}}{2(n-2)!e^x}+\frac{x^{n-3}}{3(n-3)!e^x}+\dots+\frac{1}{ne^x}dx\\
=&\int_0^\infty \frac{x^{n-1}dx}{(n-1)!e^x}+\frac{1}{2}\int_0^\infty \frac{x^{n-2}dx}{(n-2)!e^x}+\frac{1}{3}\int_0^\infty \frac{x^{n-3}dx}{(n-3)!e^x}+\dots+\frac{1}{n}\int_0^\infty \frac{dx}{e^x}
\end{aligned}$$

From (6.1) we have that all of the above integrals evaluate to 1, meaning we end up with the sum of their coefficients, which is the sum of the n first reciprocals of the natural numbers, which is the definition of the Harmonic numbers. □

These two examples highlight how the Main Solution can do more than just find expressions whose integrals converge to some constant value. When we have a sequence of functions that converge to the same value, we can manipulate them to get interesting and non-trivial results. We only showed two examples of this, but there are almost surely many more to be found. As long as we can find a clean expression for the n-th derivative of the starting function, the possibility of finding interesting identities will remain.

Chapter 7

Final thoughts

7.1 Conjectures and further work

In this section we will examine ideas that had lots of potential, but ultimately were left under-explored or unproven. We already dealt with two conjectures in chapter 4, both concerning using known solutions of the Main Problem to find new ones. We later concluded that if we slightly changed the conditions of the conjecture, by replacing “any solution” to “the Main Solution”, we then get that the first one (4.1.2) holds true, while the other one (4.1.3) (for now at least) remains a conjecture. But as these have already been discussed, we will move on for now.

Another under-explored topic was how we managed to extend the table in section 3.2 to include a diagonals of zeros and negative terms. We relied on this observation for the proof of the diagonal property for the Basel functions, but we never explained how this was done to begin with. Unfortunately, the Basel functions just happened to have a pattern that was easy to extend, and as such, we don’t have an easy way to do the same for any arbitrary suitable function.

At last, we have a conjecture regarding the integrals of the discrete Main Solution functions. If we take any Main Solution $f_n(x)$ and plot their graphs, now letting x take on real values in the interval $[1,\infty)$, we notice that as n increases, the graphs seems to get “flatter”. This becomes a very important detail when we try to compare the infinite sum of these functions with their infinite integral. The idea is that as the curve gets flatter, the integral will become a better and better approximation of the original sum. Our conjecture being that in the limit, they become equal.

Conjecture 7.1.1. *If f_n is a solution to the Main Problem, then*

$$\sum_{x=1}^{\infty} f_n(x) = \lim_{n\to\infty} \int_1^{\infty} f_n(x)\, dx$$

For some extra motivation for this conjecture, we note that if it is true, then it becomes another solution to the problem of when we have some function whose sum equals its integral, like the "Sophomore's dream" [8] or certain expressions using the binomial coefficient [1].

Now, we can actually show that this holds for the Basel functions, much as a consequence of them having their summation formula.

Theorem 7.1.2. *For the Basel functions $b_n(x)$ we have*

$$\sum_{x=1}^{\infty} b_n(x) = \lim_{n\to\infty} \int_1^{\infty} b_n(x)\, dx = \frac{\pi^2}{6}$$

Proof. Let E_n be the difference between the sum and the integral. We then use (2.1) to turn our sum into an integral and make an expression for the difference between the sum and the integral:

$$\begin{aligned} E_n &= \sum_{x=1}^{\infty} b_n(x) - \int_1^{\infty} b_n(x)\, dx \\ &= \int_1^{\infty} b_n\lfloor x\rfloor\, dx - \int_1^{\infty} b_n(x)\, dx \\ &= \int_1^{\infty} b_n\lfloor x\rfloor - b_n(x)\, dx \end{aligned}$$

Since $b_n(x)$ is strictly decreasing (and always > 0) on our interval for all n, we have that $b_n\lfloor x\rfloor - b_n(x)$ will be bounded by $b_n(x) - b_n(x+1)$ on every interval of the type $[m, m+1)$, i.e.

$$E_n = \int_1^{\infty} b_n\lfloor x\rfloor - b_n(x)\, dx \le \sum_{x=1}^{\infty} (b_n(x) - b_n(x+1)) = b_n(1)$$

Hence, if we show that $b_n(1)$ can become arbitrarily small, we have our conclusion. We start by using the summation formula for the Basel functions (3.1):

$$b_n(1) = \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+1} = \frac{H_n}{n+1}$$

We take the limit as $n \to \infty$ and use L'Hôpital's rule:

$$\lim_{n\to\infty}\frac{H_n}{n+1} = \lim_{n\to\infty}\frac{\frac{d}{dn}H_n}{\frac{d}{dn}(n+1)}$$

The numerator becomes 1, and as such we only need to focus on the derivative of the harmonic numbers. We change the argument back to x, and Use the harmonic number definition of $\psi(x)$ (1.14) and the definition of the polygamma functions (1.15) to get:

$$\begin{aligned} H_{x-1} &= \psi(x) + \gamma \\ H_x &= \psi(x+1) + \gamma \\ \frac{d}{dx}H_x &= \frac{d}{dx}\psi(x+1) = \psi^1(x+1) \end{aligned}$$

From the summation formula for the polygamma functions (1.16) it is obvious that as $x \to \infty$, $\psi^1(x) \to 0$, and hence, $E_n \to 0$ as $n \to \infty$. □

Remark 7.1.3. The exact same argument will work for any functions that is always > 0 and strictly decreasing for $x \in [1, \infty)$, with $\lim_{n\to\infty} f_n(1) = 0$.

Using this theorem, we can get some interesting identities:

$$\begin{aligned} \lim_{n\to\infty}\int_1^{\infty}\frac{1}{x+n}\sum_{k=0}^{n}\frac{1}{x+k}\,dx = \sum_{x=1}^{\infty}\frac{1}{x+n}\sum_{k=0}^{n}\frac{1}{x+k} &= \frac{\pi^2}{6} \\ \lim_{n\to\infty}[\int_1^{\infty}\sum_{k=0}^{n-1}\frac{1}{(x+n)(x+k)}\,dx + \int_1^{\infty}\frac{1}{(x+n)^2}\,dx] &= \frac{\pi^2}{6} \\ \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\ln(n+1)-\ln(k+1)}{n-k} + \lim_{n\to\infty}\frac{1}{n} &= \frac{\pi^2}{6} \\ \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{\ln(n+1)-\ln(k+1)}{n-k} &= \frac{\pi^2}{6} \end{aligned}$$

If we split the sum into two parts and plug both sides into $\exp(x)$ we get another nice expression:

$$\lim_{n\to\infty}[\sum_{k=0}^{n-1}\frac{\ln(n+1)}{n-k}-\sum_{k=0}^{n-1}\frac{\ln(k+1)}{n-k}]=\frac{\pi^2}{6}$$

$$\lim_{n\to\infty}\frac{\exp(H_n\ln(n+1))}{\exp(\sum_{k=0}^{n-1}\ln(k+1)^{\frac{1}{n-k}})}=\exp(\frac{\pi^2}{6})$$

$$\lim_{n\to\infty}\prod_{k=0}^{n-1}(\frac{n+1}{k+1})^{\frac{1}{n-k}}=e^{\frac{\pi^2}{6}}$$

$$\lim_{n\to\infty}\frac{(n+1)^{H_n}}{\prod_{k=0}^{n-1}(k+1)^{\frac{1}{k-n}}}=e^{\frac{\pi^2}{6}}$$

Of course the Basel functions are just one family of functions out of infinitely many from the Main Solution, so it would be nice if the conjecture holds true for all of them, which would allow us to find similar expressions for any suitable function. Perhaps if we could find a general "summation" formula for the Main Solution, then it would be easier to prove. Or maybe strictly looking at rational functions has been a red herring this whole time, and that the true space of functions of interest are some combination of digamma (and polygamma) functions. The fact that the inverse of the φ operator takes certain rational functions into polygamma functions supports this idea. Perhaps if we could generalize these findings to a space of polygamma functions, it would allow us to find "easy to work with" summation formulas for more than just the Basel functions.

7.2 Conclusion

Although the $x\frac{d}{dx}$ and $x\Delta$ operators are known in the literature, they seem to remain unfamiliar to many, despite their simplicity. Perhaps as a consequence of this, there are many properties of these operators that have been mostly overlooked, among them how if we put a minus sign in front, we get an operator that preserves area by rearranging it, which is why we refer to it as the area rearrangement operator. While interesting on its own, it is also the basis for the Main Solution, which can also be seen as an operator, which shares the property of preserving the very same area. In addition to being great ways of finding more unique functions whose sum/integral remains the same, they are also very useful in finding various sum and integral expressions that naturally arise when working with these families of functions.

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