UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Topology

Logarithmic Hochschild homology

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Qual è 'l geomètra che tutto s'affige
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond' elli indige,

tal era io a quella vista nova:
veder voleva come si convenne
l'imago al cerchio e come vi s'indova;

Dante Alighieri, *Divine Comedy*,
Paradiso, canto XXXIII

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Contents

Foreword

The purpose of this thesis is to analyse the logarithmic Hochschild homology for pre-log rings and to provide some tools to compute it in certain cases. The logarithmic Hochschild homology was recently introduced in [Rognes, 2009]; although a topological interpretation of this theory is also presented in Rognes's paper, we will only deal with its algebraic version.

In the framework of algebraic geometry, logarithmic structures on schemes were first defined by Fontaine and Illusie and outlined in [Kato, 1989]. Following Rognes's approach, we will define a pre-log ring (A, M) as a commutative ring A which we endow with a pre-log structure, i.e., with a commutative monoid M and a homomorphism from M to the underlying commutative monoid of A. Through an operation called "logification", we can extend M so that it contains an isomorphic copy of the units of A. From a certain point of view, a pre-log ring as such places itself in an intermediate position between A and the localization $A\big[M^{-1}\big]$ obtained by localizing the image of M through the pre-log structure homomorphism.

In this thesis, building upon the construction of the Hochschild homology for an algebra, we will reach the definition, as presented in [Rognes, 2009], of the log Hochschild homology of a pre-log ring, portraying it as a generalization of the Hochschild homology for algebras. The log Hochschild complex of (A, M), the homology of which will be considered, will be constructed by means of the Hochschild complex of A and a special simplicial commutative monoid built from M, called the replete bar construction of M. We will in particular consider pre-log rings where the commutative ring is a polynomial algebra in a finite number of variables.

One of the main strategies that we will employ to describe the log Hochschild homology will entail passing through the log Kähler differentials. The Kähler differentials Ω^1_A of a commutative ring A arise from the notion of derivations of A, which are, roughly speaking, additive maps defined on A satisfying the Leibniz derivation rule. The log Kähler differentials $\Omega^1_{(A,M)}$ of a pre-log ring (A, M) will have a broader set of generators, some of which – determined by the log structure of (A, M) – will feature distinct properties.

An additional technique that we can adopt to gather information about the log Hochschild homology of some specific pre-log rings is to interlock it in a long exact sequence, relating it to the ordinary Hochschild homology groups. An example in which this method applies nicely is the case where the remaining terms of the long exact sequence are the Hochschild homology of polynomial algebras in a finite number of variables, for which we try to provide an exhaustive description.

The thesis is structured as follows.

In Chapter 1 we will recollect some notions in commutative algebra, algebraic topology and category theory, fixing the notation for the objects later used in the rest of the thesis.

In Chapter 2 we will introduce the Hochschild homology $\mathrm{HH}_*(A)$ of a k-algebra A as the homology of the Hochschild complex of A. Special attention will be given to the A-module $\Omega^1_{A|k}$ of Kähler differentials and how to relate it, via an isomorphism, to the first Hochschild homology group. Using the language of derivations, we will moreover establish an isomorphism between the A-homomorphisms from the Kähler differentials to an A-module J and the derivations of A with values in J.

In Chapter 3 we will present some definitions about pre-log and log structures, explore the log Hochschild homology $\mathrm{HH}_*(A, M)$ and the log Kähler differentials $\Omega^1_{(A,M)}$ of a pre-log ring (A, M) and present results analogous to the ones shown in Chapter 2. The study of $\Omega^1_{(A,M)}$ will give a meaning to the title "logarithmic" for this theory. We will show how the inclusion of Ω^1_A in $\Omega^1_{A[M^{-1}]}$ factors through $\Omega^1_{(A,M)}$. We will also provide a description of the log Kähler differentials in terms of log derivations, ultimately to disclose that the log Kähler differentials of a pre-log ring is invariant under logification. An important section of this chapter will be devoted to the proof of the isomorphism between $\mathrm{HH}_1(A, M)$ and $\Omega^1_{(A,M)}$.

In Chapter 4 we will analyse the Hochschild homology and the log Hochschild homology in the particular situation where the considered ring is a polynomial algebra in a finite number of variables. After defining the graded algebra Ω^*_A of the differential forms of an algebra A, we will proceed to prove that there is a graded algebra isomorphism $\mathrm{HH}_*(A) \cong \Omega^*_A$ if A is a polynomial algebra in a finite number of variables. Other results in log Hochschild homology will be used to give a description of $\mathrm{HH}_*(\mathbb{Z}[x_1, \ldots, x_r], \langle x_1, \ldots, x_r \rangle)$.

Finally, in Chapter 5 we will show the existence of a long exact sequence in homology that will allow us to refine our knowledge of the log Hochschild homology in the case of a pre-log ring $(A, \langle x \rangle)$ where A is a flat $\mathbb{Z}[x]$-algebra.

Chapter 1

Basic notions

This chapter is a collection of the general notions in commutative algebra, algebraic topology and in category theory that are going to be used in the rest of the thesis.

1.1 Exact sequences and resolutions

Definitions and results from [Atiyah and Macdonald, 1969] and [Lang, 1993] are used as reference for this section.

Let k be a commutative ring. A sequence

$$\ldots \to M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \to \ldots$$

of k-modules M_i and k-module homomorphisms $f_i \colon M_i \to M_{i-1}$ for $i \in Z$ is **exact at** M_i if $\operatorname{im} f_{i+1} = \ker f_i$. The sequence is said to be **exact** if it is everywhere exact.

A k-module M is **free** if either it is the trivial module, or there exists a non-empty family of elements of M, called a **basis** for M, which is linearly indipendent and generates M.

Let M, N and P be k-modules. P is said to be a **projective module** if it has the (lifting) property that for any k-module homomorphism $f \colon P \to N$ and any surjective homomorphism $g \colon M \to N$ there exists a homomorphism $h \colon P \to M$ such that $f = gh$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "h"'] \arrow[d, "f"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

Many other properties are equivalent to this condition (see e.g. [Lang, 1993, Chapter III, Section 4]); for instance, a k-module is projective if and only if it is a direct summand of a free module. Hence, a free module is always projective.

Let N be a k-module. N is said to be a **flat module** if tensoring all the terms in any exact sequence of k-modules $\{M_i, f_i\}$ by $- \otimes_k N$ returns another exact sequence $\{M_i \otimes_k N, f_i \otimes \mathrm{id}_N\}$. One can show that any projective module is flat.

A **resolution** of a k-module M is an exact sequence

$$\ldots \to E_n \to E_{n-1} \to \ldots \to E_0 \to M \to 0$$

A resolution is said to have a property (e.g. to be projective, to be free) if every module in the resolution has it. Every module has a free resolution (see e.g. [Lang, 1993, Chapter XX, Section 1]).

Let M, N be k-modules; let

$$\ldots \to E_1 \to E_0 \to M \to 0$$

be a free or projective resolution of M. We define the **Tor** functor as follows: $\mathrm{Tor}_n^k(M, N)$ is the n-th homology group of the complex

$$\ldots \to E_1 \otimes_k N \to E_0 \otimes_k N \to 0$$

An important result states that different choices of the resolution of M yield the same $\mathrm{Tor}_n^k(M, N)$ up to isomorphism; moreover, it can be proved that $\mathrm{Tor}_n^k(M, N) \cong \mathrm{Tor}_n^k(N, M)$ (see e.g. [Dummit and Foote, 2004]).

1.2 Homology

The notions described in this section can be found in [Hatcher, 2002, Chapter 2].

Let k be a commutative ring. A **chain complex** $C_\bullet = (C_\bullet, b_\bullet)$ is a sequence of homomorphisms of k-modules C_n, $n \in \mathbb{Z}$, together with k-module homomorphisms $b_n \colon C_n \to C_{n-1}$ such that $b_n \circ b_{n+1} = 0$ for each $n \in \mathbb{Z}$. The homomorphisms b_n are called **boundary maps** of the complex. We will only consider chain complexes with $C_n = 0$ for $n < 0$ (so $b_n = 0$ for $n \leq 0$); a chain complex as such is then denoted explicitly as:

$$\mathrm{C}_\bullet : \ \ldots \longrightarrow C_n \xrightarrow{b_n} C_{n-1} \longrightarrow \ldots \xrightarrow{b_1} C_0 \xrightarrow{b_0} 0$$

The condition $b_n \circ b_{n+1} = 0$ implies that, for each n, there is an inclusion $\mathrm{im}\, b_{n+1} \subset \ker b_n$. We define the n-th **homology group** of the chain complex as the quotient group $\ker b_n / \operatorname{im} b_{n+1}$; it is generally denoted as $H_n(C_\bullet)$ in **degree** n. We shall denote with $H_*(C_\bullet)$ the graded abelian group defined by the sequence of the homology groups. Elements in C_n belonging to $\ker b_n$ are called **n-cycles**; elements in C_n belonging to $\operatorname{im} b_{n+1}$ are called **n-boundaries** (then, boundaries are cycles). Elements $[c] \in H_n(C_\bullet)$ are called **homology classes**.

Homology is a useful tool in algebraic geometry: it measures how "far" a chain complex is from the situation in which all cycles are boundaries, i.e., from being exact (see Section 1.1). Once agreed on how to associate a chain complex with an object (e.g. to a topological space), homology represents a helpful invariant to classify such objects; different choices of a complex and boundary maps for the initial object will then produce different kinds of homology. In this thesis we will deal with the homology of a specific chain complex associated to a pre-log ring, called the log Hochschild complex, the boundary maps of which will show some "cyclic" feature.

1.3 Basics in category theory

Although we will not be using ideas from category theory extensively in this thesis, we will sometimes deal with a terminology that can be useful to remind beforehand. The main reference for this section is [Mac Lane, 1998].

A **category** $\mathcal{C}$ consists of: a class of **objects**; a class of **arrows** (or morphisms) between objects (we denote the set of arrows between objects c_1 and c_2 with $\operatorname{Hom}_{\mathcal{C}}(c_1, c_2)$); an **identity** arrow $\mathrm{id}_c \colon c \to c$ for every object c; a law of **composition** $\operatorname{Hom}_{\mathcal{C}}(c_1, c_2) \times \operatorname{Hom}_{\mathcal{C}}(c_2, c_3) \to \operatorname{Hom}_{\mathcal{C}}(c_1, c_3)$ for any objects c_1, c_2 and c_3 (we denote with $g \circ f \colon c_1 \to c_3$ the composition of $f \colon c_1 \to c_2$ with $g \colon c_2 \to c_3$); which altogether satisfy the axioms of associativity and unit laws:

$$k \circ (g \circ f) = (k \circ g) \circ f$$
$$\mathrm{id}_b \circ f = f$$
$$g \circ id_b = g$$

for any objects a, b, c and d and for any arrows $f \colon a \to b$, $g \colon b \to c$ and $k \colon c \to d$.

Example 1.1. Categories relevant to this thesis are, for example, the category *CMon* of commutative monoids and monoid homomorphisms, and the category *CRing* of commutative rings and ring homomorphisms.

Example 1.2. For any $p \in \mathbb{N}$, let $[p] = \{0, 1, \dots, p\}$. We define the category Δ to have, as objects, sets $[p]$ for $p \in \mathbb{N}$ and, as arrows, weakly monotonic maps $\mu \colon [q] \to [p]$.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, a **(covariant) functor** $T\colon \mathcal{C} \to \mathcal{D}$ is a fumctor assigning to each object c of $\mathcal{C}$ an object Tc of $\mathcal{D}$, and to each arrow $f\colon c_1 \to c_2$ of $\mathcal{C}$ an arrow $Tf\colon Tc_1 \to Tc_2$ of $\mathcal{D}$, such that $T\,\mathrm{id}_c = \mathrm{id}_{Tc}$ and $T(g \circ f) = Tg \circ Tf$ for any object c and composable arrows f and g in $\mathcal{C}$.

Example 1.3. The functor $\mathbb{Z}[\,\cdot\,]\colon \mathit{CMon} \to \mathit{CRing}$ assigns to each commutative monoid M the commutative ring $\mathbb{Z}[M]$, i.e., the monoid ring on M, which consists of all the finite sums $\sum z_i m_i$ with $z_i \in \mathbb{Z}$, $m_i \in M$, under the product induced by the product in M. The identity on M is sent to the identity on $\mathbb{Z}[M]$; each diagram of commutative monoids (below, left diagram) is sent to the diagram of commutative rings (right diagram) with preserved direction of arrows.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "g\circ f"'] & N \arrow[d, "g"] & \mathbb{Z}[M] \arrow[r, "\overline{f}"] \arrow[dr, "\overline{g\circ f}"'] & \mathbb{Z}[N] \arrow[d, "\overline{g}"] \\
 & P & & \mathbb{Z}[P]
\end{tikzcd}$$

Given a category $\mathcal{C}$, its **opposite category** $\mathcal{C}^{\mathrm{op}}$ is the category with the objects of $\mathcal{C}$ as objects and arrows $f^{\mathrm{op}}\colon c_2 \to c_1$ for each arrow $f\colon c_1 \to c_2$ of $\mathcal{C}$.

A **contravariant functor** between two categories $\mathcal{C}$ and $\mathcal{D}$ is a morphisms $S\colon \mathcal{C} \to \mathcal{D}$ which assigns to each object c of $\mathcal{C}$ an object Sc of $\mathcal{D}$, and to each arrow $f\colon c_1 \to c_2$ of $\mathcal{C}$ an arrow $Sf\colon Sc_2 \to Sc_1$, such that $S\,\mathrm{id}_c = \mathrm{id}_{Sc}$ and $S(g \circ f) = Sf \circ Sg$ for any object c and composable arrows f and g in $\mathcal{C}$. A contravariant functor $S\colon \mathcal{C} \to \mathcal{D}$ is then a covariant functor $S\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

A functor that acts by forgetting some structure of an algebraic object is said to be **forgetful**.

Example 1.4. The functor $(-,\,\cdot\,)\colon \mathit{CRing} \to \mathit{CMon}$ assigning to each commutative ring A its underlying commutative monoid $(A,\,\cdot\,)$ is forgetful, since $(A,\,\cdot\,)$ ignores the abelian group structure of A.

Given a pair of arrows $f\colon a \to b$ and $g\colon a \to c$ in a category $\mathcal{C}$, a **pushout** of f and g is a commutative square

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "g'"] \\
c \arrow[r, "f'"'] & d
\end{tikzcd}$$

such that for each other commutative square as below (outer square)

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "g'"] \arrow[ddr, bend left, "g''"] & \\
c \arrow[r, "f'"'] \arrow[drr, bend right, "f''"'] & d \arrow[dr, dashed, "h"] & \\
& & e
\end{tikzcd}$$

there exists a unique $h\colon d \to e$ with $hf' = f''$ and $hg' = g''$. The pushout is, by construction, unique up to isomorphism.

Example 1.5. In the category *CRing* of commutative rings, the pushout is the **tensor product** of rings: for $f\colon R \to A$, $g\colon R \to B$ ring homomorphisms, the pushout of f and g is $A \otimes_R B$, where $f(r)a \otimes b = a \otimes g(r)b$ for $r \in R$, $a \in A$, $b \in B$. The maps completing the pushout diagram are $A \to A \otimes_R B$, $a \mapsto a \otimes 1_B$ and $B \to A \otimes_R B$, $b \mapsto 1_A \otimes b$.

Given a pair of arrows $f\colon b \to a$ and $g\colon c \to a$ in a category $\mathcal{C}$, a **pullback** of f and g is a commutative square

$$\begin{tikzcd}
d \arrow[r, "f'"] \arrow[d, "g'"'] & c \arrow[d, "g"] \\
b \arrow[r, "f"'] & a
\end{tikzcd}$$

such that for each other commutative square as below (outer square)

$$\begin{tikzcd}
e \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] \arrow[dr, dashed, "h"] & & \\
& d \arrow[r, "f'"] \arrow[d, "g'"'] & c \arrow[d, "g"] \\
& b \arrow[r, "f"'] & a
\end{tikzcd}$$

there exists a unique $h\colon e \to d$ with $f'h = f''$ and $g'h = g''$. By construction, the pullback is unique up to isomorphism.

Example 1.6. In the category *CMon* of commutative monoids, the pullback is the **fibered product** of monoids: for $f\colon N \to M$, $g\colon P \to M$ monoid homomorphisms, the pullback of f and g is $N \times_M P = \{(n,p) \in N \times P \mid f(n) = g(p)\}$. The maps completing the pullback diagram are the projections sending $(n,p) \in N \times_M P$ to $n \in N$ and $p \in P$ respectively.

A **natural transformation** between two functors $S, T\colon \mathcal{C} \to \mathcal{D}$ is a function assigning to each object c of $\mathcal{C}$ an arrow $Fc\colon Sc \to Tc$ such that for each arrow $h\colon c \to d$ of $\mathcal{C}$ the following square commutes:

$$\begin{tikzcd}
Sc \arrow[r, "Fc"] \arrow[d, "Sh"'] & Tc \arrow[d, "Th"] \\
Sd \arrow[r, "Fd"'] & Td
\end{tikzcd}$$

Given two categories $\mathcal{C}$ and $\mathcal{D}$, an **adjunction** between $\mathcal{C}$ and $\mathcal{D}$ is given by two functors $S\colon \mathcal{C} \to \mathcal{D}$ and $T\colon \mathcal{D} \to \mathcal{C}$ and a function ϕ which assigns, to each pair of objects $c \in \mathcal{C}$, $d \in \mathcal{D}$, a set bijection $\phi_{c,d} : \mathrm{Hom}_{\mathcal{D}}(Sc, d) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(c, Td)$ which is natural in c and d. The functor S is called **left adjoint**, while T is **right adjoint**. We also say that $h\colon Sc \to d$ is left adjoint to $\phi_{c,d}h\colon c \to Td$ (and $\phi_{c,d}h$ is right adjoint to h).

1.4 Simplicial objects

The content of [Mac Lane, 1963, Chapter VIII, Section 5] was used as reference for the main definitions. The reference for the section about the Eilenberg-Zilber theorem is [Mac Lane, 1963, Chapter VIII, Section 8]. For the Künneth formula, the reference is [Mac Lane, 1963, Chapter V, Section 10].

We defined in Example 1.2 the category Δ of sets $[p] = \{0, 1, \ldots, p\}$ and weakly monotonic maps $\mu\colon [q] \to [p]$. Let $\mathcal{C}$ be a category; a **simplicial object** in the category $\mathcal{C}$ is a contravariant functor $F\colon \Delta \to \mathcal{C}$. We will encounter, in this thesis, simplicial objects such as simplicial monoids and simplicial commutative rings. Equivalently, we can describe a simplicial object $S_\bullet = S$ in $\mathcal{C}$ as a family $\{S_q\}$, indexed by a **degree** $q \geq 0$, of objects in $\mathcal{C}$ together with two families of morphisms (arrows) of $\mathcal{C}$, namely **face maps** (or face operators) d_i, $i = 0, \ldots, q$, at each $q > 0$

$$d_i\colon S_q \to S_{q-1}$$

and **degeneracy maps** (or degeneracy operators) s_i, $i = 0, \ldots, q$, at each $q \geq 0$

$$s_i\colon S_q \to S_{q+1}$$

that satisfy, in every degree q where they are defined, the following identities:

$$d_i d_j = d_{j-1} d_i \qquad \text{if } i < j \tag{1.1a}$$

$$s_i s_j = s_{j+1} s_i \qquad \text{if } i \leq j \tag{1.1b}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ id_{S_q} & \text{if } i = j,\ i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases} \tag{1.1c}$$

A **simplicial map** $F\colon S \to T$ between two simplicial objects S and T in the same category $\mathcal{C}$ is a natural transformation between the contravariant functors S, $T\colon \Delta \to \mathcal{C}$. Equivalently, it is a family of arrows $F_q\colon S_q \to T_q$ of $\mathcal{C}$ such that the following two squares commute at each degree q and for every i, j where they are defined:

$$\begin{tikzcd}
S_{q+1} \arrow[d, "F_{q+1}"'] & S_q \arrow[l, "s_j"'] \arrow[r, "d_i"] \arrow[d, "F_q"] & S_{q-1} \arrow[d, "F_{q-1}"] \\
T_{q+1} & T_q \arrow[l, "s_j"] \arrow[r, "d_i"'] & T_{q-1}
\end{tikzcd}$$

The simplicial objects in a category $\mathcal{C}$ are themselves the objects of a category with the simplicial maps as arrows.

Let $M_\bullet$ be a simplicial module over a commutative ring k, with face operators d_i. Then $M_\bullet$ determines a chain complex, called the **Moore complex**:

$$M_\bullet : \ \ldots \longrightarrow M_n \xrightarrow{b_n} M_{n-1} \longrightarrow \ldots \xrightarrow{b_1} M_0 \xrightarrow{b_0} 0 \tag{1.2}$$

(also denoted with $M_\bullet$), setting

$$b_n = \sum_{i=0}^{n} (-1)^i d_i$$

In fact, for each n,

$$b_n \circ b_{n+1} = \left(\sum_{i=0}^{n} (-1)^i d_i\right)\left(\sum_{j=0}^{n+1} (-1)^j d_j\right)$$

that is, explicitly, the sum of the terms in the $n \times (n+1)$ table

$$\begin{array}{ccccc}
+d_0d_0 & -d_0d_1 & \ldots & \pm d_0d_n & \mp d_0d_{n+1} \\
-d_1d_0 & +d_1d_1 & \ldots & \mp d_1d_n & \pm d_1d_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\pm d_nd_0 & \mp d_nd_1 & \ldots & +d_nd_n & -d_nd_{n+1}
\end{array}$$

in which the rows of the upper-right triangle correspond term by term, by (1.1a), to the columns of the lower-left triangle with inverse sign. So $b_n \circ b_{n+1} = 0$ and $M_\bullet$ is a chain complex.

The Eilenberg-Zilber theorem

Let $U = U_\bullet$ and $V = V_\bullet$ be two simplicial modules over a commutative ring k. Each of them defines a chain complex as in (1.2). Tensoring $U_\bullet$ and $V_\bullet$ degreewise gives the **cartesian product simplicial module** $(U \boxtimes V)_\bullet$, with $(U \boxtimes V)_q =$

$U_q \otimes V_q$, and face and degeneracy maps given by the face and degeneracy maps for $U_\bullet$ and $V_\bullet$:

$$\begin{aligned} d_i(u \otimes v) &= d_i(u) \otimes d_i(v) \\ s_j(u \otimes v) &= s_j(u) \otimes s_j(v) \end{aligned}$$

This simplicial module, in turn, defines the chain complex (also) denoted as $(U \boxtimes V)_\bullet$, with boundary maps again given by $\partial_q = \sum_{i=0}^{q} (-1)^i d_i$. Moreover, the **tensor product of chain complexes** $U_\bullet \otimes V_\bullet = (U \otimes V)_\bullet$ is defined as

$$(U \otimes V)_\bullet : \; \dots \xrightarrow{\partial} \bigoplus_{p+q=2} U_p \otimes V_q \xrightarrow{\partial} \bigoplus_{p+q=1} U_p \otimes V_q \xrightarrow{\partial} U_0 \otimes V_0 \xrightarrow{\partial} 0$$

with boundary maps $\partial_{p+q}(u \otimes v) = \partial_p(u) \otimes v + (-1)^{\deg u}\, u \otimes \partial_q(v)$.

The **Eilenberg-Zilber theorem** states that there's a chain equivalence

$$(U \boxtimes V)_\bullet \underset{g}{\overset{f}{\rightleftarrows}} (U \otimes V)_\bullet$$

which will then give an isomorphism in homology. The chain map $f\colon (U \boxtimes V)_\bullet \to (U \otimes V)_\bullet$ is the **Alexander-Whitney map**, which is given by

$$\begin{aligned} f_n \colon U_n \otimes V_n &\to \bigoplus_{p+q=n} U_p \otimes V_q \\ u \otimes v &\mapsto \sum_{i=0}^{n} d_\star^{n-i}(u) \otimes d_0^i(v) \end{aligned} \tag{1.3}$$

where the d_j's are the face maps of the complexes and, at each degree q, $d_\star = d_q$. Its chain homotopy inverse $g\colon (U \otimes V)_\bullet \to (U \boxtimes V)_\bullet$ is called the **shuffle map**, defined in degree n for $u \in U_p$ and $v \in V_{n-p}$ by

$$\begin{aligned} g_n \colon \bigoplus_{p+q=n} U_p \otimes V_q &\to U_n \otimes V_n \\ u \otimes v &\mapsto \sum_{(\mu,\nu)} \operatorname{sgn}(\mu,\nu)(s_{\nu_q} \cdots s_{\nu_1}(u) \otimes s_{\mu_p} \cdots s_{\mu_1}(v)) \end{aligned} \tag{1.4}$$

where the s_j's are the degeneracy maps and the sum runs over all the (p,q)-shuffles (μ,ν), that is, over all the permutations of $p+q$ objects sending the set of indices $(0, \dots p+q-1)$ in a set $(\mu_1, \dots \mu_p, \nu_1, \dots \nu_q)$ such that $\mu_1 < \dots < \mu_p$ and $\nu_1 < \dots < \nu_q$. About the shuffle map, it is useful to specify that if $e\colon U_p \otimes V_q \to V_q \otimes U_p$ is the isomorphism $u \otimes v \mapsto v \otimes u$, the following diagram

$$\begin{tikzcd} U_p \otimes V_q \arrow[r, "g_{p+q}"] \arrow[d, "e"'] & U_{p+q} \otimes V_{p+q} \arrow[d, "(-1)^{pq}\, e"] \\ V_q \otimes U_p \arrow[r, "g_{p+q}"'] & V_{p+q} \otimes U_{p+q} \end{tikzcd}$$

commutes. In other words,

$$g \circ e(u \otimes v) = (-1)^{pq}\, e \circ g(u \otimes v) \tag{1.5}$$

In fact, the (p, q)-shuffles are in bijective correspondence with the (q, p)-shuffles:

$$\begin{aligned} \{(p, q)\text{-shuffles}\} &\to \{(q, p)\text{-shuffles}\} \\ \{\mu_1, \ldots \mu_p, \nu_1, \ldots \nu_q\} &\mapsto \{\nu_1, \ldots \nu_q, \mu_1, \ldots \mu_p\} \end{aligned} \tag{1.6}$$

The permutation that sends a (p, q)-shuffle to the correspondent (q, p)-shuffle is now evidently the product of $p \cdot q$ transpositions. In particular, for $p + q = 1$, $g \circ e = e \circ g$.
One can, moreover, verify that the shuffle map is associative.

The Künneth formula

Given $R_\bullet$ and $S_\bullet$ simplicial modules over a commutative ring k, the tensor product of chain complexes $(R \otimes S)_\bullet$ has boundary map

$$\partial(r \otimes s) = \partial(r) \otimes s + (-1)^{\deg r}\, r \otimes \partial(s)$$

This boundary map sends the tensor product of two cycles to a cycle, and the tensor product of a cycles and a boundary to a boundary. So, the homomorphism

$$\begin{aligned} \mathfrak{p}\colon\ \mathrm{H}_m(R_\bullet) \otimes \mathrm{H}_n(S_\bullet) &\to \mathrm{H}_{m+n}(R_\bullet \otimes S_\bullet) \\ r \otimes s &\mapsto r \otimes s \end{aligned} \tag{1.7}$$

is well-defined (see [Mac Lane, 1963, Chapter V, Section 10], "external homology product").

The **Künneth formula** states that if, at each degree n, the n-cycles and the n-boundaries of $R_\bullet$ are flat modules, then, for every n, there is a short exact sequence

$$\begin{aligned} 0 \longrightarrow \bigoplus_{p+q=n} \mathrm{H}_p(R_\bullet) \otimes_k \mathrm{H}_q(S_\bullet) \xrightarrow{\mathfrak{p}} \mathrm{H}_n((R \otimes S)_\bullet) \\ \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1^k(\mathrm{H}_p(R_\bullet), \mathrm{H}_q(S_\bullet)) \longrightarrow 0 \end{aligned}$$

where $\mathfrak{p}$ is the homology product in (1.7).

Another version of the Künneth formula applies under stronger conditions. If the n-cycles and the n-th homology of $R_\bullet$ are projective modules for each degree n, then, for every n, the homology product (1.7) induces an isomorphism

$$\bigoplus_{p+q=n} \mathrm{H}_p(R_\bullet) \otimes_k \mathrm{H}_q(S_\bullet) \cong \mathrm{H}_n((R \otimes S)_\bullet) \tag{1.8}$$

1.5 Spectral sequences

We will use an argument involving spectral sequences to prove, among other facts, the key theorem in Section 3.4. We will present some of the essential definitions; the reference for this section is [Mac Lane, 1963, Chapter XI, Sections 1, 3].

Let k be a commutative ring. A **spectral sequence** $E = \{E^r, d^r\}$, $r \in \mathbb{N}$ (we will consider $r \geq 2$), is a sequence of $\mathbb{Z}$-bigraded k-modules $E^r_{p,q}$, with a family of homomorphisms $d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ for each r, called **differentials**, such that $d \circ d = 0$, and with isomorphisms $E^{r+1} \cong \mathrm{H}(E^r)$ (where the homology refers to the boundary map given by the differential).

Since each term of the spectral sequence is the homology of the previous one, we can express any term as a quotient of cycles and boundaries. Using the isomorphism $E^{r+1} \cong H_*(E^r)$, we inductively define a tower of submodules

$$0 \subset B^2 \subset B^3 \subset \ldots \subset B^r \subset B^{r+1} \subset \ldots \subset C^{r+1} \subset C^r \subset \ldots \subset C^2 \subset E^2$$

such that $E^r \cong C_r/B_r$. This can be obtained defining C_2 and B_2 respectively as the bigraded modules of cycles and boundaries of E^2, and setting that $d^r \colon C_r/B_r \to C_r/B_r$ has kernel C_{r+1}/B_r and image B_{r+1}/B_r.

Let $C^\infty = \bigcap C^r$ and $B^\infty = \bigcup B^r$. Evidently $B^\infty \subset C^\infty$; we define $E^\infty = \{E^\infty_{p,q}\} = \{C^\infty_{p,q}/B^\infty_{p,q}\}$.

A **first quadrant** spectral sequence is a spectral sequence E such that $E^r_{p,q} = 0$ whenever $p < 0$ or $q < 0$. In a first quadrant spectral sequence, for fixed bidegree (p, q), the differentials $d^r_{p,q}$ and $d^r_{p+r,q-r+1}$ are ultimately 0 (for $r > \max(p, q+1)$); this implies that $E^r_{p,q} = E^{r+1}_{p,q} = E^\infty_{p,q}$ for large enough values of r.

A **filtration** of a k-module A is a family $F = \{F_pA \mid p \in \mathbb{Z}\}$ of submodules of A, with $F_{p-1} \subset F_p$ for each p. F determines an **associated graded module** $G^FA = \left\{\left(G^FA\right)_p\right\} = \{F_pA/F_{p-1}A\}$. A filtration of a graded k-module A_n is a family of sub-graded modules F_pA satisfying the same conditions; this determines at each n a filtration $\{F_pA_n\}$.

A spectral sequence $\{E^r, d^r\}$ is said to **converge** to a graded k-module A if there exists a filtration F of A and, at each p, isomorphisms of graded modules $E^\infty_{p,q} \cong F_pA_{p+q}/F_{p-1}A_{p+q}$ (graded by q); we denote with $E^2_p \Rightarrow A$ the convergence of E^r to A.

Chapter 2

The Hochschild homology

2.1 The Hochschild complex

We will give a definition of the Hochschild complex and we will build from it the Hochschild homology. The following definitions and results are based on the exposition given in [Loday, 1998]. In this chapter, k will denote a commutative ring.

Let A be a k-algebra and let M be a bimodule over A. Consider the modules $\mathrm{C}_n(A;M) := M \otimes A^{\otimes n}$ (all the tensor products are meant to be over k). We can define, for each $n \geq 0$, face and degeneracy operators as follows:

$$d_i(m, a_1, \ldots, a_n) = \begin{cases} (ma_1, a_2, \ldots, a_n) & \text{for } i = 0 \\ (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) & \text{for } 1 \leq i < n \\ (a_n m, a_1, \ldots, a_{n-1}) & \text{for } i = n \end{cases} \tag{2.1a}$$

$$s_j(m, a_1, \ldots, a_n) = \begin{cases} (m, 1, a_1, a_2, \ldots, a_n) & \text{for } j = 0 \\ (m, a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) & \text{for } 1 \leq j < n \\ (m, a_1, \ldots, a_n, 1) & \text{for } j = n \end{cases} \tag{2.1b}$$

Here, as we will often do, we used the notation $(x_1, \ldots, x_n)$ for the tensor product $x_1 \otimes \ldots \otimes x_n$.

One can easily compute that the face operators and the degeneracy operators as defined in (2.1) satisfy the conditions (1.1) for simplicial objects. This makes $\mathrm{C}_\bullet(A;M)$ a simplicial module; we can then define a k-linear **Hochschild boundary map** b: $M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ by setting

$$b_n = \sum_{i=0}^{n} (-1)^i d_i$$

Having a boundary map, we get a chain complex, called the **Hochschild complex**:

$$\mathrm{C}_\bullet(A;M):\ \ldots \xrightarrow{b} M\otimes A^{\otimes n} \xrightarrow{b} M\otimes A^{\otimes n-1} \xrightarrow{b} \ldots \xrightarrow{b} M\otimes A \xrightarrow{b} M \xrightarrow{b} 0$$

The n-th homology group $\mathrm{HH}_n(A;M)$ of the Hochschild complex is called the n-th **Hochschild homology group**. It is immediately seen that

$$\mathrm{HH}_0(A;M) = M/\{am - ma \mid a\in A,\ m\in M\}$$

We denote moreover with $\mathrm{HH}_*(A;M)$ the graded abelian group defined by the sequence $\mathrm{HH}_n(A;M)$, for $n\in\mathbb{N}$.

When treating Hochschild complexes and the Hochschild homology, we are often interested in the case when $M = A$. We will then denote $\mathrm{C}_\bullet(A) = \mathrm{C}_\bullet(A;A)$ and $\mathrm{HH}_*(A) = \mathrm{HH}_*(A;A)$.

Example 2.1. The Hochschild complex of $\mathbb{Z}$ is, under the isomorphism $\mathbb{Z}^{\otimes n}\cong\mathbb{Z}$, the following one:

$$\ldots \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

Then, easily,

$$\mathrm{HH}_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$$

Example 2.2. Let $A = \mathbb{Z}[x]/\big(x^2\big)$. To find the homology of $\mathrm{C}_\bullet(A)$, we can compute the homology of the "normalized Hochschild complex" instead. Precisely, we let $\overline{A} = A/\mathbb{Z} \cong \langle x\rangle/(x^2)$. The normalized Hochschild complex $\overline{\mathrm{C}}_\bullet(A)$ is defined degreewise as $\overline{\mathrm{C}}_n(A) = A\otimes\overline{A}^{\otimes n}$, with boundary maps induced by the boundary maps of the Hochschild complex. By [Loday, 1998, Proposition 1.1.15], $\mathrm{C}_\bullet(A)$ and $\overline{\mathrm{C}}_\bullet(A)$ give the same homology. We get:

$$\ldots \longrightarrow A\otimes\overline{A}^{\otimes 3} \xrightarrow{b_3} A\otimes\overline{A}^{\otimes 2} \xrightarrow{b_2} A\otimes\overline{A} \xrightarrow{b_1} A \xrightarrow{b_0} 0$$

where

$$b_n(a + bx\otimes x\otimes\ldots\otimes x) = \begin{cases} 0 & \text{for } n = 0 \\ 0 & \text{for odd } n \\ 2ax\otimes x\otimes\ldots\otimes x & \text{for even } n,\ n\geq 2 \end{cases}$$

This gives

$$\mathrm{HH}_n(\mathbb{Z}[x]/\big(x^2\big)) \cong \begin{cases} \mathbb{Z}[x]/\big(x^2\big) & \text{for } n = 0 \\ \mathbb{Z}[x]/\big(2x, x^2\big) & \text{for odd } n \\ \mathbb{Z}\{x\} & \text{for even } n,\ n\geq 2 \end{cases}$$

Remark 2.3. If A is a commutative k-algebra, one can check that $\mathrm{HH}_*(A;M)$ is a graded A-module, under the multiplication on the first coordinate A, which is compatible with the face and the boundary maps of $\mathrm{C}_\bullet(A;M)$.

2.2 Kähler differentials and derivations

For a commutative and unital k-algebra A, we define the **A-module of Kähler differentials** $\Omega^1_{A|k}$ (or just Ω^1_A) as the free A-module in the symbols $\{da \mid a \in A\}$ modulo the A-submodule generated by the relations $d(\lambda a + \mu b) = \lambda da + \mu db$ and $d(ab) = adb + bda$ for a, $b \in A$, λ, $\mu \in k$.

Example 2.4. The $\mathbb{Z}$-module of Kähler differentials of $\mathbb{Z}$ is the trivial module. In fact, by linearity, $dn = nd1$ for $dn \in \Omega^1_{\mathbb{Z}|\mathbb{Z}}$. But $d1 = d(1 \cdot 1) - d1 = 0$. The $\mathbb{Z}$-module $\Omega^1_{\mathbb{Q}|\mathbb{Z}}$ of Kähler differentials of $\mathbb{Q}$ is also the trivial module, being $d\frac{m}{n} = \frac{1}{n} \cdot nd\frac{m}{n} = \frac{1}{n}dm = \frac{m}{n}d1 = 0$ for m, $n \in \mathbb{Z}$.

Theorem 2.5. *For a commutative and unital k-algebra A, there is a canonical isomorphism of A-modules:*

$$\mathrm{HH}_1(A) \cong \Omega^1_{A|k}$$

Proof. Computing directly, we have that the boundary maps in degree 1 and 2 are as such:

$$\begin{aligned} b_1 \colon A \otimes A &\to A \\ a_1 \otimes a_2 &\mapsto a_1a_2 - a_2a_1 = 0 \end{aligned}$$

since A is commutative, making $\ker b_1 = A \otimes A$;

$$\begin{aligned} b_2 \colon A \otimes A \otimes A &\to A \otimes A \\ a_1 \otimes a_2 \otimes a_3 &\mapsto a_1a_2 \otimes a_3 - a_1 \otimes a_2a_3 + a_3a_1 \otimes a_2 \end{aligned}$$

Then, by definition,

$$\mathrm{HH}_1(A) = \frac{\ker b_1}{\operatorname{im} b_2} = \frac{A \otimes A}{\langle a_1a_2 \otimes a_3 - a_1 \otimes a_2a_3 + a_3a_1 \otimes a_2 \rangle} \tag{2.2}$$

Now, define

$$\begin{aligned} \tau \colon \ \mathrm{HH}_1(A) &\to \Omega^1_{A|k} \\ a_1 \otimes a_2 &\mapsto a_1da_2 \end{aligned} \tag{2.3}$$

We see that τ is a well-defined A-module homomorphism, since cycles in the same homology class have the same image. In fact, using the commutativity of

A, we have:

$$\begin{aligned}\tau(a_1a_2 \otimes a_3 - a_1 \otimes a_2a_3 + a_3a_1 \otimes a_2) &= a_1a_2d(a_3) - a_1d(a_2a_3) + a_3a_1d(a_2)\\ &= a_1a_2d(a_3) - a_1a_2d(a_3) - a_1a_3d(a_2) + a_3a_1d(a_2) = 0\end{aligned}$$

Moreover, once we define

$$\begin{aligned}\overline{\tau}\colon \Omega^1_{A|k} &\to \mathrm{HH}_1(A)\\ a_1da_2 &\mapsto [a_1 \otimes a_2] \qquad (2.4)\end{aligned}$$

we have that differentials in $\Omega^1_{A|k}$ are sent to cycles, since $A \otimes A = \ker b_1$. Also $\overline{\tau}$ is a well-defined A-module homomorphism, since

$$\overline{\tau}(d(a_1a_2)) = 1 \otimes a_1a_2 = a_1 \otimes a_2 + a_2 \otimes a_1 = \overline{\tau}(a_1da_2 + a_2da_1)$$

where the middle equality comes from the relation defined by $\operatorname{im} b_2$ in (2.2), choosing the first entry to be 1. Finally, we can easily see that $\tau\overline{\tau} = \mathrm{id}_{\Omega^1_{A|k}}$ and $\overline{\tau}\tau = \mathrm{id}_{\mathrm{HH}_1(A)}$. ■

We will now formulate another definition of the Kähler differentials in terms of an universal property on derivations.

For A again a commutative and unital k-algebra, and for J any A-module, a **derivation** of A with values in J is a k-linear map $D\colon A \to J$ such that $D(ab) = aD(b) + bD(a)$ for $a, b \in A$. We denote the A-**module of all derivations** of A with values in J with $\mathrm{Der}(A, J)$, or just $\mathrm{Der}(A)$ when $J = A$. The multiplication in the module is given by $A \times \mathrm{Der}(A, J) \to \mathrm{Der}(A, J)$, $c \times D \mapsto cD$ defined by $(cD)(a) = c \cdot D(a)$.
Alternatively, we can define the **square-zero extension** $A \oplus J$ as a commutative ring over A with multiplication map

$$\begin{aligned}\mu\colon (A \oplus J) \times (A \oplus J) &\to A \oplus J\\ (a_1 \oplus j_1, a_2 \oplus j_2) &\mapsto a_1a_2 \oplus (a_1j_2 + a_2j_1)\end{aligned}$$

In this way, $\mathrm{Der}(A, J)$ is isomorphic to the A-module of ring homomorphisms $\bar{D}\colon A \to A \oplus J$ over A. All of them have the form $\bar{D}(a) = a \oplus D(a)$, where D yet again satisfies the "Leibniz rule" $D(ab) = aD(b) + bD(a)$.

A derivation $d\colon A \to J$ is **universal** if, given any other derivation $\delta\colon A \to I$, δ factors over d, meaning that there is a unique A-linear map $\phi\colon J \to I$ that makes the following diagram commute:

$$\begin{tikzcd}A \arrow[r, "\delta"] \arrow[d, "d"'] & I \\ J \arrow[ur, "\phi"'] & \end{tikzcd}$$

In the following result we will see that this universal property is fulfilled by the Kähler differentials.

Proposition 2.6. *The derivation* $d\colon A \to \Omega^1_{A|k}$, $a \mapsto da$ *is universal, i.e., given a derivation* $\delta\colon A \to I$, *there is a unique* A*-linear map* $\phi\colon \Omega^1_{A|k} \to I$ *such that* $\delta = \phi \circ d$. *In detail,* $\phi(da) = \delta(a)$.

Proof. We just need to check that the declared map ϕ is well-defined. Immediately, we have that $\phi(d(\lambda a + \mu b) - \lambda da - \mu db) = \delta(\lambda a + \mu b) - \lambda\delta(a) - \mu\delta(b) = 0$ and $\phi(d(ab) - adb - bda) = \delta(ab) - a\delta(b) - b\delta(a) = 0$ since δ is a derivation. Since ϕ fits in the commutative diagram, it is also unique. ■

From this, we can get the following important result.

Corollary 2.7. *There is an isomorphism:*

$$\begin{aligned} \mathrm{Hom}_A\left(\Omega^1_{A|k}, J\right) &\xrightarrow{\sim} \mathrm{Der}(A, J) \\ f &\mapsto f \circ d \\ \phi &\mapsfrom \delta = \textit{(by universality)} = \phi \circ d \end{aligned}$$

In particular, taking $J = \mathrm{HH}_1(A)$, this implies that having an A-module homomorphism $f\colon \Omega^1_{A|k} \to \mathrm{HH}_1(A)$ is the same as having a derivation D of A with values in $\mathrm{HH}_1(A)$. We can use this to see that there effectively is such a homomorphism f. Consider, in fact,

$$\begin{aligned} D\colon A &\to \mathrm{HH}_1(A) = A \otimes A / \langle a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1 \rangle \\ a &\mapsto [1 \otimes a] \end{aligned} \tag{2.5}$$

for $a_i \in A$. The map D is a derivation, since

$$D(ab) = [1 \otimes ab] = [a \otimes b] + [b \otimes a] = aD(b) + D(a)b$$

Hence we get a homomorphism $f\colon \Omega^1_{A|k} \to \mathrm{HH}_1(A)$, $adx \mapsto aDx$; this is the same map as the map $\overline{\tau}$ described in Theorem 2.5.

From now on, we will refrain from denote homology classes with square brackets, unless necessary.

Chapter 3

The log Hochschild homology

We start by introducing the objects of our study, extensively following the theory described in [Rognes, 2009] for terminology, exposition and, often, notation. Throughout this thesis, a **commutative monoid** will be understood to be a set endowed with an associative and commutative multiplication and an identity element. Equivalently, using the notions in Section 1.3, a commutative monoid is a category with a single object, such that any two morphisms commute.

3.1 Log structures

Let A be a commutative ring. A **pre-log structure** on A is a pair (M, α) of a commutative monoid M and a monoid homomorphism $\alpha\colon M \to (A, \cdot\,)$ from M to the underlying commutative monoid of A. A **pre-log ring** (A, M, α), also denoted as (A, M) when the monoid homomorphism is clear, consists of a commutative ring A together with a pre-log structure (M, α) on A.

A **homomorphism of pre-log rings** $\left(f, f^\flat\right)\colon (A, M, \alpha) \to (B, N, \beta)$ is a ring homomorphism $f\colon A \to B$ together with a monoid homomorphism $f^\flat\colon M \to N$, such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "f^\flat"'] & (A, \cdot\,) \arrow[d, "(f, \cdot\,)"] \\
N \arrow[r, "\beta"'] & (B, \cdot\,)
\end{tikzcd}$$

Let $\iota : \mathrm{GL}_1(A) \hookrightarrow (A, \cdot\,)$ be the inclusion of the multiplicative group of units of A in A. Let $\alpha^{-1}\operatorname{GL}_1(A) \subseteq M$ be defined by the pullback square

$$\begin{tikzcd}
\alpha^{-1}\operatorname{GL}_1(A) \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\widetilde{\iota}"'] & \operatorname{GL}_1(A) \arrow[d, "\iota"] \\
M \arrow[r, "\alpha"'] & (A, \,\cdot\,)
\end{tikzcd}$$

If the restricted homomorphism $\widetilde{\alpha}$ in the diagram happens to be an isomorphism, then (M, α) is called a **log structure** on A, and (A, M, α), or just (A, M), is a **log ring**.

We can obtain a log structure from a pre-log structure in the following way. Let (A, M, α) be a pre-log ring. Its **associated log ring** $(A, M^{\mathrm{a}}, \alpha^{\mathrm{a}})$ is the log ring given by A with the log structure $(M, \alpha)^{\mathrm{a}} = (M^{\mathrm{a}}, \alpha^{\mathrm{a}})$, where M^{a} is defined by the pushout square

$$\begin{tikzcd}
\alpha^{-1}\operatorname{GL}_1(A) \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\widetilde{\iota}"'] & \operatorname{GL}_1(A) \arrow[d] \arrow[rdd, bend left, "\iota"] & \\
M \arrow[r] \arrow[rrd, bend right, "\alpha"'] & M^{\mathrm{a}} \arrow[rd, "\alpha^{\mathrm{a}}"] & \\
 & & (A, \,\cdot\,)
\end{tikzcd}$$

and $\alpha^{\mathrm{a}} \colon M^{\mathrm{a}} \to (A, \,\cdot\,)$ is the canonical homomorphism induced by α and ι. This is indeed a log ring: every unit $u \in \operatorname{GL}_1(A)$ has preimage $1 \oplus u$ through α^{a}, making $(\alpha^{\mathrm{a}})^{-1}\operatorname{GL}_1(A)$ isomorphic to $\operatorname{GL}_1(A)$. The transition from a pre-log structure to its associated log structure will be referred to as the "**logification**" of the pre-log ring.

Remark 3.1. Since we can always endow A with a trivial pre-log structure, taking $M = \{1\}$ and the unique $\alpha \colon \{1\} \to (A, \,\cdot\,)$, then we can also give A a log structure, taking the associated log structure to the trivial pre-log structure. In that case, $M^{\mathrm{a}} = \operatorname{GL}_1(A)$ and $\alpha^{\mathrm{a}} = \iota \colon \operatorname{GL}_1(A) \to (A, \,\cdot\,)$.

For a commutative monoid M, there is a **canonical pre-log structure** on its monoid ring $\mathbb{Z}[M]$, given by (M, ζ), where

$$\zeta \colon M \to \mathbb{Z}[M], \quad m \mapsto 1 \cdot m$$

This yields the **canonical log structure** on $\mathbb{Z}[M]$, given by $(M, \zeta)^{\mathrm{a}}$.

3.2 Bar constructions and the log Hochschild homology

In this thesis, when we are given a commutative monoid M, we will denote with M^{gp} its **group completion** and with $\gamma\colon M \to M^{\mathrm{gp}}$ the monoid homomorphism with the universal property that any other monoid homomorphism $\phi\colon M \to M'$, with M' abelian group, factors uniquely through γ. For the explicit construction of this (abelian) group, also called the **Grothendieck group** of M, see e.g. [Rosenberg, 1994, Theorem 1.1.3].

Once again, the terminology and the constructions that are going to follow are presented as described in [Rognes, 2009, Section 3].

Let $\epsilon\colon M \to P$ be a monoid homomorphism. ϵ is said to be **exact** if

$$\begin{tikzcd} M \arrow[r, "\gamma"] \arrow[d, "\epsilon"'] & M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\ P \arrow[r, "\gamma"'] & P^{\mathrm{gp}} \end{tikzcd}$$

is a pullback square.

If $\epsilon\colon M \to P$ is a homomorphism of commutative monoids, ϵ is said to be **virtually surjective** if $\epsilon^{\mathrm{gp}}\colon M^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is surjective.

Example 3.2. A first example of a non-surjective homomorphism of commutative monoids which is virtually surjective is the following. Consider $M = (\{1,\dots,m\},\cdot)$ and $P = (\{1,\dots,p\},\cdot)$ with $p > m$, where in both monoids the operation is defined such that $n_1 \cdot n_2 = \max\{n_1, n_2\}$. Let $\epsilon\colon M \to P$ be the inclusion; it is obviously a homomorphism and it is not surjective. Now, $M^{\mathrm{gp}} = \{1\}$. In fact, for any $n \in M$, $n \cdot n = n$, so $\gamma(n) = \gamma(n \cdot n) = \gamma(n)\gamma(n)$, being $\gamma\colon M \to M^{\mathrm{gp}}$ a monoid homomorphism. Since M^{gp} is a group, multiplying both the left- and the right-hand side by $\gamma(n)^{-1}$, we get $\gamma(n) = 1$. By the universal property of the group completion, $M^{\mathrm{gp}} \cong \{1\}$. By the same argument, also $P^{\mathrm{gp}} \cong \{1\}$, so $\epsilon^{\mathrm{gp}}\colon \{1\} \to \{1\}$ is surjective, making ϵ virtually surjective.

Example 3.3. We shall provide another example of a homomorphism of commutative monoids $\epsilon\colon M \to P$ which is virtually surjective, but not surjective, where, this time, the respective group completions are not trivial. Let $M = \langle 2, 3\rangle \subseteq (\mathbb{N}, +, 0) = P$, with $\epsilon\colon M \to P$ being the inclusion map (evidently not surjective). Clearly $P^{\mathrm{gp}} = \mathbb{Z}$. We claim that $\langle 2, 3\rangle^{\mathrm{gp}}$ is isomorphic to $\mathbb{Z}$. In fact, we can consider the inclusion $\iota\colon \langle 2,3\rangle \to \mathbb{Z}$. From the diagram

$$\begin{tikzcd}
\langle 2,3\rangle \arrow[r, "\gamma"] \arrow[d, "\iota"'] & \langle 2,3\rangle^{\mathrm{gp}} \arrow[dl, "\theta"] \\
\mathbb{Z} &
\end{tikzcd}$$

we know that ι factors uniquely as $\iota = \theta\gamma$ where θ is a homomorphism of abelian groups. Hence, $\theta\gamma(2) = \iota(2) = 2$. We also have that $\gamma(2)+2\gamma(2) = \gamma(2+2+2) = \gamma(3+3) = 2\gamma(3)$, so $\gamma(2) = -2\gamma(2)+2\gamma(3)$. We conclude that, in $\mathbb{Z}$, $2 = \theta\gamma(2) = \theta(-2\gamma(2) + 2\gamma(3)) = 2\theta(-\gamma(2) + \gamma(3))$, giving $\theta(-2\gamma(2) + \gamma(3)) = 1$. But then easily θ has an inverse homomorphism, given by $\mathbb{Z} \to \langle 2,3\rangle^{\mathrm{gp}}$, $1 \mapsto -2\gamma(2)+\gamma(3)$, so $\langle 2,3\rangle^{\mathrm{gp}} \cong \mathbb{Z}$. The group homomorphism $\epsilon^{\mathrm{gp}}\colon \mathbb{Z} \to \mathbb{Z}$ is the identity map, thus it is surjective. In conclusion, ϵ is virtually surjective, but not surjective.

A virtually surjective commutative monoid M over P, i.e. a commutative monoid M with a virtually surjective homomorphism $\epsilon\colon M \to P$, is called **replete** if the homomorphism ϵ is exact.

Given a virtually surjective $\epsilon\colon M \to P$, the **repletion** of M over P is the pullback $M^{\mathrm{rep}} = P \times_{P^{\mathrm{gp}}} M^{\mathrm{gp}}$ with the canonical map $\epsilon^{\mathrm{rep}}\colon M^{\mathrm{rep}} \to P$. We then get a commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\epsilon"'] & M^{\mathrm{rep}} \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}"] & M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\
P \arrow[r, "="] & P \arrow[r, "\gamma"] & P^{\mathrm{gp}}
\end{tikzcd}$$

where the map $\rho\colon M \to M^{\mathrm{rep}}$ in the diagram is called the **repletion map**. It is proven in [Rognes, 2009, Lemma 3.8] that M^{rep} is replete over P.

Given a commutative monoid M, the **bar construction** of M is the simplicial commutative monoid $\mathrm{B}M = \mathrm{B}_\bullet M$ given by q copies of M in degree q. Face operators d_i and degeneracy operators s_j in degree q are given as follows, for $0 \leq i, j \leq q$:

$$d_i(m_1,\ldots,m_q) = \begin{cases} (m_2,\ldots,m_q) & \text{for } i = 0 \\ (m_1,\ldots,m_i m_{i+1},\ldots,m_q) & \text{for } 1 \leq i \leq q-1 \\ (m_1,\ldots,m_{q-1}) & \text{for } i = q \end{cases}$$

$$s_j(m_1,\ldots,m_q) = \begin{cases} (1,m_1,\ldots,m_q) & \text{for } j = 0 \\ (m_1,\ldots,m_j,1,m_{j+1},\ldots,m_q) & \text{for } 1 \leq j \leq q-1 \\ (m_1,\ldots,m_q,1) & \text{for } j = q \end{cases}$$

For a commutative monoid M, the **cyclic bar construction** of M is the simplicial commutative monoid $\mathrm{B}^{\mathrm{cy}} M = \mathrm{B}^{\mathrm{cy}}_\bullet M$ which, in degree q, is given by

$q+1$ copies of M. With the usual notation, face and degeneracy operators for this simplicial commutative monoid are the following:

$$d_i(m_0,\dots,m_q)=\begin{cases}(m_0,\dots,m_im_{i+1},\dots,m_q) & \text{for } 0\leq i\leq q-1\\ (m_qm_0,m_1,\dots,m_{q-1}) & \text{for } i=q\end{cases}$$

$$s_j(m_0,\dots,m_q)=\begin{cases}(m_0,\dots,m_j,1,m_{j+1},\dots,m_q) & \text{for } 0\leq j\leq q-1\\ (m_0,\dots,m_q,1) & \text{for } j=q\end{cases}$$

A cyclic structure on $\mathrm{B}^{\mathrm{cy}}\,M$ is given by the operator:

$$\begin{gathered}t_q\colon \mathrm{B}^{\mathrm{cy}}_q\,M\to\mathrm{B}^{\mathrm{cy}}_q\,M\\ (m_0,\dots,m_{q-1},m_q)\mapsto(m_q,m_0,\dots,m_{q-1})\end{gathered}$$

The cyclic bar construction can be seen as the tensor product $S^1_\bullet\otimes M$, where $S^1_\bullet$ is the simplicial circle. So, the base point inclusion $*\to S^1_\bullet$ induces in each degree the inclusion map

$$\eta\colon M\to\mathrm{B}^{\mathrm{cy}}\,M,\ \ m\mapsto(m,1,\dots 1)$$

and the collapse map $S^1_\bullet\to *$ induces in each degree the map

$$\epsilon\colon \mathrm{B}^{\mathrm{cy}}\,M\to M,\ \ (m_0,m_1,\dots,m_q)\mapsto m_0m_1\cdots m_q$$

The **replete bar construction** $\mathrm{B}^{\mathrm{rep}}\,M=\mathrm{B}^{\mathrm{rep}}_\bullet\,M$ of a commutative monoid M is the repletion $(\mathrm{B}^{\mathrm{cy}}\,M)^{\mathrm{rep}}$ of the cyclic bar construction of M over M itself, which is the simplicial commutative monoid given by the pullback (bottom-right square) of simplicial commutative monoids:

$$\begin{tikzcd}
M \arrow[r, "="] \arrow[d, "\eta"'] & M \arrow[r, "\gamma"] \arrow[d, "\eta^{\mathrm{rep}}"] & M^{\mathrm{gp}} \arrow[d, "\eta^{\mathrm{gp}}"] \\
\mathrm{B}^{\mathrm{cy}}\,M \arrow[r, "\rho"] \arrow[d, "\epsilon"'] & \mathrm{B}^{\mathrm{rep}}\,M \arrow[r] \arrow[d, "\epsilon^{\mathrm{rep}}"] & \mathrm{B}^{\mathrm{cy}}\,M^{\mathrm{gp}} \arrow[d, "\epsilon^{\mathrm{gp}}"] \\
M \arrow[r, "="] & M \arrow[r, "\gamma"] & M^{\mathrm{gp}}
\end{tikzcd} \tag{3.1}$$

$\mathrm{B}^{\mathrm{rep}}_q\,M$ has elements $(m,g_0,\dots g_q)$, with $m\in M$ and $g_i\in M^{\mathrm{gp}}$, such that

$$\gamma(m)=\epsilon^{\mathrm{gp}}(g_0,\dots,g_q)=g_0\cdots g_q \tag{3.2}$$

Moreover, it has a natural cyclic structure, since both γ and ϵ^{gp} are cyclic maps (ϵ is a cyclic morphism giving M the constant cyclic structure). Such a structure is given by the operator

$$\begin{gathered}t_q\colon \mathrm{B}^{\mathrm{rep}}_q\,M\to\mathrm{B}^{\mathrm{rep}}_q\,M\\ (m,g_0,\dots,g_{q-1},g_q)\mapsto(m,g_q,g_0,\dots,g_{q-1})\end{gathered}$$

A simplicial structure for $\mathrm{B}^{\mathrm{rep}}\, M$ is given by face and degeneracy operators inherited from the face and degeneracy operators on the cyclic bar complex $\mathrm{B}^{\mathrm{cy}}\, M^{\mathrm{gp}}$, while being the identity on M:

$$d_i(m, g_0 \ldots, g_q) = \begin{cases} (m, g_0, \ldots, g_i g_{i+1}, \ldots, g_q) & \text{for } 0 \le i \le q-1 \\ (m, g_q g_0, g_1, \ldots, g_{q-1}) & \text{for } i = q \end{cases}$$

$$s_j(m, g_0, \ldots, g_q) = \begin{cases} (m, g_0, \ldots, g_j, 1, g_{j+1}, \ldots, g_q) & \text{for } 0 \le j \le q-1 \\ (m, g_1, \ldots, g_q, 1) & \text{for } j = q \end{cases}$$

The condition (3.2) gives an explicit formula for $g_0 = \gamma(m)(g_1 \cdots g_q)^{-1}$; by direct computation, one can show that the map

$$\begin{gathered} \mathrm{B}^{\mathrm{rep}}\, M \xrightarrow{\sim} M \times \mathrm{B}M^{\mathrm{gp}} \\ (m, \gamma(m)(g_1 \cdots g_q)^{-1}, g_1, \ldots, g_q) \longmapsto (m, g_1, \ldots, g_q) \end{gathered} \tag{3.3}$$

commutes with the face and degeneracy operators of the respective simplicial structures, providing thus an isomorphism of simplicial commutative monoids. With this identification, the repletion map ρ is as follows:

$$\begin{gathered} \rho\colon\ \mathrm{B}^{\mathrm{cy}}\, M \to \mathrm{B}^{\mathrm{rep}}\, M \cong M \times (M^{\mathrm{gp}})^q \\ (m_0, \ldots, m_q) \mapsto (m_0 \cdots m_q, \gamma(m_1), \ldots, \gamma(m_q)) \end{gathered} \tag{3.4}$$

The simplicial structure is now given by the face and degeneracy operators inherited from the face and degeneracy operators on the bar complex $\mathrm{B}M^{\mathrm{gp}}$, while still being the identity on M:

$$d_i(m, g_1 \ldots, g_q) = \begin{cases} (m, g_2, \ldots, g_q) & \text{for } i = 0 \\ (m, g_1, \ldots, g_i g_{i+1}, \ldots, g_q) & \text{for } 1 \le i \le q-1 \\ (m, g_1, \ldots, g_{q-1}) & \text{for } i = q \end{cases} \tag{3.5a}$$

$$s_j(m, g_1, \ldots, g_q) = \begin{cases} (m, 1, g_1, \ldots, g_q) & \text{for } j = 0 \\ (m, g_1, \ldots, g_j, 1, g_{j+1}, \ldots, g_q) & \text{for } 1 \le j \le q-1 \\ (m, g_1, \ldots, g_q, 1) & \text{for } j = q \end{cases} \tag{3.5b}$$

Let now (A, M, α) be a pre-log ring. With respect to the covariant functor

$$\mathbb{Z}[\,\cdot\,]\colon\ \mathit{CMon} \to \mathit{CRing},\ \ M \mapsto \mathbb{Z}[M]$$

from commutative monoids to commutative rings (as described in Example 1.3), the homomorphism $\alpha\colon M \to (A, \cdot\,)$ has left adjoint $\overline{\alpha}\colon \mathbb{Z}[M] \to A$. In degree q, $\mathbb{Z}\big[\mathrm{B}^{\mathrm{cy}}_q\, M\big] = \mathbb{Z}\big[M^{q+1}\big] \cong \mathbb{Z}[M]^{\otimes q+1}$ and $\mathrm{C}_q(A) = A^{\otimes q+1}$. Consider the simplicial map $S^1_\bullet \otimes \overline{\alpha}\colon \mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\, M] \to \mathrm{C}(A)$ (in degree q, $\overline{\alpha}^{\otimes q+1}\colon \mathbb{Z}[M]^{\otimes q+1} \to A^{\otimes q+1}$). Its right adjoint $\mathrm{B}^{\mathrm{cy}}\, M \to (\mathrm{C}(A), \cdot\,)$ defines degreewise a pre-log structure on the (simplicial) commutative ring $\mathrm{C}(A)$.

Definition 3.4 ([Rognes, 2009])**.** Let (A, M, α) be a pre-log ring; we shall at first work under the assumption that A is flat over $\mathbb{Z}[M]$. The **log Hochschild complex** of (A, M) is the replete simplicial pre-log ring $(\mathrm{C}_\bullet(A, M), \mathrm{B}^{\mathrm{rep}}_\bullet M, \xi)$ obtained by degreewise pushout of simplicial commutative rings:

$$\begin{tikzcd}
\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M] \arrow[r, "\mathbb{Z}[\rho]"] \arrow[d, "S^1_\bullet \otimes \overline{\alpha}"'] & \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M] \arrow[d, "\overline{\xi}"] \\
\mathrm{C}_\bullet(A) \arrow[r, "\overline{\psi}"'] & \mathrm{C}_\bullet(A, M)
\end{tikzcd}$$

where ρ is the repletion map figuring in (3.1). The pre-log structure map

$$\xi \colon \mathrm{B}^{\mathrm{rep}}_\bullet M \to (\mathrm{C}_\bullet(A, M), \,\cdot\,)$$

is then the right adjoint to the map $\overline{\xi}$ in the diagram.

In detail:

$$\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_n M] = \mathbb{Z}\big[M^{n+1}\big] \cong \mathbb{Z}[M]^{\otimes n+1}$$
$$\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_n M] \cong \mathbb{Z}[M \times (M^{\mathrm{gp}})^n] \cong \mathbb{Z}[M] \otimes \mathbb{Z}[M^{\mathrm{gp}}]^{\otimes n}$$

and $\mathrm{C}_n(A) = A^{\otimes n+1}$ as previously defined. Hence, in each degree n,

$$\mathrm{C}_n(A, M) \cong A^{\otimes n+1} \otimes_{\mathbb{Z}[M]^{\otimes n+1}} \Big(\mathbb{Z}[M] \otimes \mathbb{Z}[M^{\mathrm{gp}}]^{\otimes n}\Big)$$

The **log Hochschild homology groups** $\mathrm{HH}_*(A, M)$ are the homology groups of the Hochschild complex with the induced boundary maps. The log Hochschild boundary maps combines the boundary maps on the factors of the tensor product over $\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_n M]$. On the $\mathrm{C}_n(A)$ side, the face operators are the ones defined in (2.1a) for the Hochschild homology complex; on the $\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_n M]$ side, they are induced by the simplicial structure of the replete bar construction (shown in (3.5a)). Explicitly, we have:

$$\begin{aligned}
\mathrm{C}_\bullet(A, M) :\ \ldots &\xrightarrow{b_3} A^{\otimes 3} \otimes_{\mathbb{Z}[M]^{\otimes 3}} \Big(\mathbb{Z}[M] \otimes \mathbb{Z}[M^{\mathrm{gp}}]^{\otimes 2}\Big) \\
&\xrightarrow{b_2} A^{\otimes 2} \otimes_{\mathbb{Z}[M]^{\otimes 2}} (\mathbb{Z}[M] \otimes \mathbb{Z}[M^{\mathrm{gp}}]) \\
&\xrightarrow{b_1} A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M] \cong A \xrightarrow{b_0} 0 \qquad (3.6)
\end{aligned}$$

where, for $a_i \in A$, $m \in \mathbb{Z}[M]$ and $g_i \in \mathbb{Z}[M^{\mathrm{gp}}]$,

$$\begin{aligned}
&b_1((a_0 \otimes a_1) \otimes (m \otimes g_1)) = (a_0 a_1 \otimes m) - (a_1 a_0 \otimes m) = 0 \\
&b_2((a_0 \otimes a_1 \otimes a_2) \otimes (m \otimes g_1 \otimes g_2)) \\
&\quad = (a_0 a_1 \otimes a_2 \otimes m \otimes g_2) - (a_0 \otimes a_1 a_2 \otimes m \otimes g_1 g_2) + (a_2 a_0 \otimes a_1 \otimes m \otimes g_1)
\end{aligned}$$

and so on. We see that, for a pre-log ring (A, M) with A flat over $\mathbb{Z}[M]$, $\mathrm{HH}_0(A, M) = A$. Part of this thesis will be devoted to the investigation a more meaningful expression for $\mathrm{HH}_1(A, M)$.

Example 3.5. The Hochschild homology of a $\mathbb{Z}$-algebra A is trivially isomorphic to the log Hochschild homology of A endowed with the trivial pre-log structure. In this sense, we can consider the log Hochschild homology to be a generalization of the Hochschild homology.

Remark 3.6. We can provide a definition of the log Hochschild homology of a pre-log ring (A, M, α) also for the case in which A is not flat over $\mathbb{Z}[M]$. Let $X_\bullet$ be a simplicial resolution of A by flat $\mathbb{Z}[M]$-modules, i.e., a simplicial commutative algebra $X_\bullet = \{X_i\}$, $i \in \mathbb{N}$, such that X_i is flat over $\mathbb{Z}[M]$ for every i. For each i, let $\mathrm{C}_\bullet(X_i, M)$ be the log Hochschild complex of i. We define the n-th log Hochschild homology of (A, M, α) to be $\mathrm{HH}_n(A, M) = \mathrm{HH}_n(X_n, M)$. However, for simplicity we shall generally assume that A is flat over $\mathbb{Z}[M]$ when discussing log Hochschild homology.

3.3 Log Kähler differentials and log derivations

For simplicity, from this section onwards we will use the ring of integers $k = \mathbb{Z}$ as ground ring. For example, when A is a commutative ring, we will write Ω^1_A to denote $\Omega^1_{A|\mathbb{Z}}$.

We shall now define the "log" version for Kähler differentials. The module that we will obtain is going to be the pushout of two maps that we will now define.

For a pre-log ring (A, M, α), define the A-module homomorphism:

$$\begin{aligned} \psi\colon A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} &\to A \otimes M^{\mathrm{gp}} \\ a \otimes dm &\mapsto a\alpha(m) \otimes \gamma(m) \end{aligned} \tag{3.7}$$

To check that ψ is well-defined, we can consider the map $\delta\colon \mathbb{Z}[M] \to A \otimes M^{\mathrm{gp}}$ defined on M by $m \mapsto \alpha(m) \otimes \gamma(m)$ and extended linearly to $\mathbb{Z}[M]$. This is a derivation of $\mathbb{Z}[M]$ with values in $A \otimes M^{\mathrm{gp}}$, since

$$\begin{aligned} \delta(mn) &= \alpha(mn) \otimes \gamma(mn) = \alpha(m)\alpha(n) \otimes \gamma(m)\gamma(n) \\ &= \alpha(m)\alpha(n) \otimes \gamma(m) + \alpha(m)\alpha(n) \otimes \gamma(n) = \alpha(n)\delta(m) + \alpha(m)\delta(n) \end{aligned}$$

By Corollary 2.7, this derivation corresponds to the $\mathbb{Z}[M]$-module homomorphism $\Omega^1_{\mathbb{Z}[M]} \to A \otimes M^{\mathrm{gp}}$, which itself, by extensions of scalars, corresponds to the A-module homomorphism ψ.

Again, for a pre-log ring (A, M, α), we define another A-module homomorphism:

$$\begin{aligned} \phi\colon A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} &\to \Omega^1_A \\ a \otimes dm &\mapsto ad(\alpha(m)) \end{aligned} \tag{3.8}$$

Definition 3.7. For a pre-log ring (A, M, α), we define the A-module of **log Kähler differentials** $\Omega^1_{(A,M)}$ by the pushout of A-modules:

$$\begin{tikzcd} A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & A \otimes M^{\mathrm{gp}} \arrow[d, "\overline{\phi}"] \\ \Omega^1_A \arrow[r, "\overline{\psi}"'] & \Omega^1_{(A,M)} \end{tikzcd} \tag{3.9}$$

with A-module homomorphisms ψ and ϕ as defined respectively in (3.7) and (3.8).

In this way,

$$\Omega^1_{(A,M)} = \left(\Omega^1_A \oplus (A \otimes M^{\mathrm{gp}})\right)/\sim$$

where $\sim$ is A-linearly generated by the relation

$$d\alpha(m) \oplus 0 \sim 0 \oplus (\alpha(m) \otimes \gamma(m))$$

for $m \in M$. In $\Omega^1_{(A,M)}$, we will use the notation

$$da := \overline{\psi}(da), \qquad\qquad d\log m := \overline{\phi}(1 \otimes \gamma(m))$$

for $a \in A$ and $m \in M$. We then see that, for m, $n \in M$, $d\log(mn) = d\log m + d\log n$ (since $\overline{\phi}$ is a module homomorphism); moreover, $d\alpha(m) = \alpha(m) d\log m$.

Example 3.8. If $(\{1\}, \alpha)$ is the trivial pre-log structure on a commutative ring A, then the A-module homomorphisms ψ and $\overline{\psi}$ are isomorphisms. Hence, $\Omega^1_{(A,\{1\})} \cong \Omega^1_A$.

It is at this point convenient to delineate an isomorphism that will prove itself useful from now on.

Lemma 3.9. *For a commutative ring A and a commutative monoid M, there is an isomorphism of A-modules:*

$$A \otimes M^{\mathrm{gp}} \cong (A \otimes \mathbb{Z}[M^{\mathrm{gp}}])/\sim$$

where $\sim$ is A-linearly generated by the relation $a \otimes g_1 + a \otimes g_2 \sim a \otimes g_1 g_2$ for $a \in A$, g_1, $g_2 \in M^{\mathrm{gp}}$.

Proof. We will proceed to find two inverse A-module homomorphisms. In one direction, we define:

$$\begin{aligned} \vartheta\colon A \otimes M^{\mathrm{gp}} &\to (A \otimes \mathbb{Z}[M^{\mathrm{gp}}])/\sim \\ a \otimes g &\mapsto [a \otimes g] \end{aligned}$$

We then define, for $n_i \in \mathbb{Z}$ and $g_i \in M^{\mathrm{gp}}$:

$$\begin{aligned} \widetilde{\vartheta}\colon A \otimes \mathbb{Z}[M^{\mathrm{gp}}] &\to A \otimes M^{\mathrm{gp}} \\ a \otimes \sum_i n_i g_i &\mapsto a \otimes \prod_i g_i^{n_i} \end{aligned}$$

The submodule generated by $\sim$ lies in $\ker \widetilde{\vartheta}$, since

$$\widetilde{\vartheta}(a \otimes g_1 g_2 - a \otimes g_1 - a \otimes g_2) = a \otimes g_1 g_2 g_1^{-1} g_2^{-1} = a \otimes 1$$

Then there exists a unique A-module homomorphism

$$\overline{\vartheta}\colon (A \otimes \mathbb{Z}[M^{\mathrm{gp}}])/\sim \to A \otimes M^{\mathrm{gp}}$$

such that $\overline{\vartheta}([a \otimes \sum_i n_i g_i]) = a \otimes \prod_i g_i^{n_i}$. Now easily ϑ and $\overline{\vartheta}$ are inverse isomorphisms. ■

Example 3.10. Consider the pre-log ring (A, M, α) where $A = \mathbb{Z}[M]$ and α is the inclusion. The log Hochschild complex is defined with the pushout diagram in Definition 3.4, where now $\overline{\alpha}\colon \mathbb{Z}[M] \to A$ is the identity on A, so $\mathrm{C}_n(A, M) \cong A \otimes \mathbb{Z}[M^{\mathrm{gp}}]^{\otimes n}$. The boundary map in degree 1 is the zero-map, while $b_2(a \otimes g_1 \otimes g_2) = (a \otimes g_2) - (a \otimes g_1 g_2) + (a \otimes g_1)$ for $a \in A$, g_1, $g_2 \in \mathbb{Z}[M^{\mathrm{gp}}]$.
At this point, $\mathrm{HH}_1(A, M) \cong (A \otimes \mathbb{Z}[M^{\mathrm{gp}}])/\sim$ where $a \otimes g_1 + a \otimes g_2 \sim a \otimes g_1 g_2$ for $a \in A$, g_1, $g_2 \in M^{\mathrm{gp}}$. By Lemma 3.9, this is isomorphic to $A \otimes M^{\mathrm{gp}}$.
The A-module $A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]}$ is clearly isomorphic to Ω^1_A. This yields, computing the log Kähler differentials of (A, M) from the definition, that the bottom map in (3.9) is

$$\begin{aligned} \overline{\psi}\colon \Omega^1_A &\to \Omega^1_{(A,M)} \\ dm &\mapsto m d \log m \end{aligned}$$

and $\Omega^1_{(A,M)} \cong A \otimes M^{\mathrm{gp}}$. We then see that, for this pre-log ring,

$$\mathrm{HH}_1(A, M) \cong \Omega^1_{(A,M)}$$

We will prove in Theorem 3.22 that this isomorphism holds for any pre-log ring (A, M), provided that A is flat over $\mathbb{Z}[M]$, the condition required in the definition of the log Hochschild complex.

Example 3.11. Referring to Example 3.10, let (A, M, α) be a pre-log ring, where A is the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients, M is the free commutative monoid $\langle x\rangle = \{1, x, x^2, \dots\}$ and $\alpha\colon M \to (A, \cdot\,)$ is the inclusion. In this way, $A \cong \mathbb{Z}[M]$ and $\mathrm{HH}_1(\mathbb{Z}[x], \langle x\rangle) \cong \Omega^1_{(\mathbb{Z}[x],\langle x\rangle)}$. Explicitly, $\mathrm{HH}_1(\mathbb{Z}[x], \langle x\rangle) \cong \mathbb{Z}[x] \otimes \langle x\rangle^{\mathrm{gp}} \cong \mathbb{Z}[x] \otimes \mathbb{Z} \cong \mathbb{Z}[x]$. On the other hand, $\Omega^1_{\mathbb{Z}[x]} \cong \mathbb{Z}[x]\{dx\}$ and $\Omega^1_{(A,M)} \cong A \otimes M^{\mathrm{gp}} \cong \mathbb{Z}[x]\{d\log x\}$. The homomorphism $\overline{\psi}\colon \Omega^1_{\mathbb{Z}[x]} \to \Omega^1_{(\mathbb{Z}[x],\langle x\rangle)}$ maps $dx \mapsto x d\log x$ (thus it is not an isomorphism). More on this subject will be discussed in Chapter 4.

We are going to illustrate the functorial behaviour of the log Kähler differentials; in order to do so, we will need a lemma.

Lemma 3.12. *Let M be a commutative monoid and K an abelian group. There is a canonical bijective correspondence between the monoid homomorphisms $M \to K$ and the group homomorphisms $M^{\mathrm{gp}} \to K$.*

Proof. Let $f\colon M \to K$ be a monoid homomorphism. Then $f(1_M) = 1_K$. We can define a group homomorphism $f^{\mathrm{gp}}\colon M^{\mathrm{gp}} \to K$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\gamma"'] & K \arrow[d, equal] \\ M^{\mathrm{gp}} \arrow[r, "f^{\mathrm{gp}}"'] & K \end{tikzcd} \tag{3.10}$$

commutes, i.e., setting $f^{\mathrm{gp}}(\gamma(m)) = f(m)$ for m in M; the definition extends automatically to M^{gp} because f^{gp} is a group homomorphism, which implies, for any m in M,

$$1_K = f(1_M) = f^{\mathrm{gp}}(\gamma(1_M)) = f^{\mathrm{gp}}\big(\gamma(m)\gamma(m)^{-1}\big) = f^{\mathrm{gp}}(\gamma(m))f^{\mathrm{gp}}\big(\gamma(m)^{-1}\big)$$

returning $f^{\mathrm{gp}}\big(\gamma(m)^{-1}\big) = f^{\mathrm{gp}}(\gamma(m))^{-1}$ for any m in M. Conversely, given $g\colon M^{\mathrm{gp}} \to K$, one can define $\widetilde{g}\colon M \to K$, $m \mapsto g(\gamma(m))$. Clearly the correspondence

$$\begin{aligned} \mathrm{Hom}(M, K) &\longleftrightarrow \mathrm{Hom}(M^{\mathrm{gp}}, K) \\ f &\longmapsto f^{\mathrm{gp}} \\ \widetilde{g} &\longleftarrow\!\shortmid\; g \end{aligned}$$

is given by inverse isomorphisms. ■

In such a setting, when needed, we will use the short notation $f(m)$ implicitly meaning $f^{\mathrm{gp}}(\gamma(m))$.

Proposition 3.13. *The function* $\Omega^1_{(A,-)}$ *sending the pre-log ring* (A, M) *to* $\Omega^1_{(A,M)}$ *is a covariant functor on pre-log structures of* A *to* A*-modules.*

Proof. Given a homomorphism of pre-log rings $\big(\mathrm{id}, f^\flat\big) \colon (A, M, \alpha) \to (A, N, \beta)$, we need to find a A-module homomorphism $f_{*(M,N)} \colon \Omega^1_{(A,M)} \to \Omega^1_{(A,N)}$ that preserves identities and directions of arrows. By Lemma 3.12, we know that we can extend the monoid homomorphism $f^\flat \colon M \to N$ to the group homomorphism ${f^\flat}^{\mathrm{gp}} \colon M^{\mathrm{gp}} \to N^{\mathrm{gp}}$ between group completions, as in diagram (3.10):

$$\begin{tikzcd}
 & N \arrow[d, "\gamma_N"] \\
M \arrow[ur, "f^\flat"] \arrow[r, "\gamma_N \circ f^\flat"'] \arrow[d, "\gamma_M"'] & N^{\mathrm{gp}} \arrow[d, equal] \\
M^{\mathrm{gp}} \arrow[r, "{f^\flat}^{\mathrm{gp}}"'] & N^{\mathrm{gp}}
\end{tikzcd} \tag{3.11}$$

After the identification

$$\Omega^1_{(A,M)} = \big(\Omega^1_A \oplus (A \otimes M^{\mathrm{gp}})\big) / \sim_M$$

where $(d\alpha(m) \oplus 0) \sim_M (0 \oplus (\alpha(m) \otimes \gamma_M(m)))$ for $m \in M$, and similarly for $\Omega^1_{(A,N)}$, we can define

$$\begin{aligned} f_{*(M,N)} \colon \big(\Omega^1_A \oplus (A \otimes M^{\mathrm{gp}})\big) / \sim_M &\to \big(\Omega^1_A \oplus (A \otimes N^{\mathrm{gp}})\big) / \sim_N \\ da \oplus (1 \otimes m) &\mapsto da \oplus (1 \otimes {f^\flat}^{\mathrm{gp}}(m)) \end{aligned}$$

on a generator $da \oplus (1 \otimes m)$, then extended A-linearly. This is a well-defined A-module homomorphism. In fact, by the commutativity of (3.11) and by the relation $\alpha(m) = \beta f^\flat(m)$ for $m \in M$, we have

$$\begin{aligned} f_{*(M,N)}(d\alpha(m) \oplus 0) &= (d\alpha(m) \oplus 0) \\ &= (d\beta f^\flat(m) \oplus 0) \\ &\sim_N (0 \oplus (\beta f^\flat(m) \otimes \gamma_N f^\flat(m))) \\ &= (0 \oplus (\alpha(m) \otimes {f^\flat}^{\mathrm{gp}} \gamma_M(m))) \\ &= f_{*(M,N)}(0 \oplus (\alpha(m) \otimes \gamma_M(m)))) \end{aligned}$$

If $f^\flat$ is the identity on M, then ${f^\flat}^{\mathrm{gp}}$ is the identity on M^{gp} and $f_{*(M,M)}$ is the identity on $\Omega^1_{(A,M)}$. Moreover, if $\big(\mathrm{id}, g^\flat\big) \colon (A, N, \beta) \to (A, P, \upsilon)$ is another pre-log ring homomorphism, then easily

$$\Big(\mathrm{id}, (g \circ f)^\flat\Big) = \Big(\mathrm{id}, g^\flat \circ f^\flat\Big) \colon (A, M, \alpha) \to (A, P, \upsilon)$$

is a pre-log ring homomorphism and $g_{*(N,P)} \circ f_{*(M,N)} = (g \circ f)_{*(M,P)}$. Hence $\Omega^1_{(A,-)}$ is a covariant functor on pre-log structures of A to A-modules. ■

As for the case of Kähler differentials, we will give a description of the log Kähler differentials by means of a universal property, regarding, in this case, log derivations.

Let (A, M, α) be a pre-log ring and let J be an A-module. A **log derivation** $\big(D, D^\flat\big)$ of (A, M) with values in J consists of a derivation $D\colon A \to J$ of A with values in J and a monoid homomorphism $D^\flat\colon M \to (J, +)$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\alpha"] \arrow[d, "\mathrm{id} \times D^\flat"'] & (A, \cdot\,) \arrow[d, "(D, \cdot\,)"] \\
M \times (J, +) \arrow[r, "\alpha^\flat"'] & (J, \cdot\,)
\end{tikzcd}$$

where $(J, +)$ is the underlying abelian group of J and the lower arrow $\alpha^\flat$ maps $(m, x) \mapsto \alpha(m)x$; that is, $D^\flat$ is such that $\alpha(m)D^\flat(m) = D(\alpha(m))$. We note that, by Lemma 3.12, $D^\flat$ extends to $D^{\flat\,\mathrm{gp}}\colon M^{\mathrm{gp}} \to (J, +)$.

We denote the A-**module of log derivations** of (A, M) with values in J with $\mathrm{Der}((A, M), J)$. Our aim is now to show that, similarly to the case of Kähler differentials, there is a correspondence between the A-module homomorphisms from the log Kähler differentials and the log derivations.

Theorem 3.14. *There is an isomorphism of A-modules:*

$$\mathrm{Hom}_A\Big(\Omega^1_{(A,M)}, J\Big) \cong \mathrm{Der}((A, M), J)$$

Proof. We will make use of the universal property of the Kähler differentials described in Corollary 2.7.
Given $\big(D, D^\flat\big) \in \mathrm{Der}((A, M), J)$, consider the diagram

$$\begin{tikzcd}
 & A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & A \otimes M^{\mathrm{gp}} \arrow[d, "\overline{\phi}"] \arrow[dr, bend left, "h"] & \\
A \arrow[r, "d"] \arrow[rrr, bend right, "D"'] & \Omega^1_A \arrow[r, "\overline{\psi}"] \arrow[rr, bend right=15, "g"'] & \Omega^1_{(A,M)} \arrow[r, dashed, "f"] & J
\end{tikzcd}$$

where the A-module homomorphisms of the square are as in (3.9). The map $g\colon \Omega^1_A \to J$ is determined by the universal property of Kähler differentials, as the only homomorphism such that $D = g \circ d$, with $d\colon A \to \Omega^1_A$ the universal derivation. So $g(da) = D(a)$. The map $h\colon A \otimes M^{\mathrm{gp}} \to J$ is defined to be such that

$a \otimes x \mapsto aD^\flat(x)$, using the extension of $D^\flat$ to M^{gp} as described in Lemma 3.12. In this way,

$$\begin{aligned}
A \otimes_{\mathbb{Z}[M]} \Omega^1_{(A,M)} &\xrightarrow{\phi} \Omega^1_A \xrightarrow{g} J \\
a \otimes dm &\longmapsto ad(\alpha(m)) \longmapsto aD(\alpha(m))
\end{aligned}$$

while

$$\begin{aligned}
A \otimes_{\mathbb{Z}[M]} \Omega^1_{(A,M)} &\xrightarrow{\psi} A \otimes M^{\mathrm{gp}} \xrightarrow{h} J \\
a \otimes dm &\longmapsto a\alpha(m) \otimes \gamma(m) \longmapsto a\alpha(m)D^\flat(m) = aD(\alpha(m))
\end{aligned}$$

where the last equality comes from the definition of log derivation. We then determined a commutative square; being $\Omega^1_{(A,M)}$ defined as the pushout of the top and left maps, there exists a unique A-module homomorphism $f\colon \Omega^1_{(A,M)} \to J$ that makes the diagram commute, i.e., such that $f(da) = D(a)$ and $f(d\log m) = D^\flat(m)$.

On the other hand, given $f \in \mathrm{Hom}_A\left(\Omega^1_{(A,M)}, J\right)$, consider $g\colon \Omega^1_A \to J$, $g = f \circ \overline{\psi}$. Let $D\colon A \to J$ be defined as $D = g \circ d$, where $d\colon A \to \Omega^1_A$ is again the universal derivation, so $D(a) = g(da) = f(da)$. By the universal property of the Kähler differentials, D is a derivation of A with values in J. Setting $D^\flat\colon M \to (J, +)$, $D^\flat(m) = f(d\log m)$, we get

$$\begin{aligned}
D(\alpha(m)) &= g(d(\alpha(m))) = f\overline{\psi}(d(\alpha(m))) \\
&= f\overline{\psi}\phi(1 \otimes dm) = f\overline{\phi}\psi(1 \otimes dm) \\
&= f\overline{\phi}(\alpha(m) \otimes \gamma(m)) = \alpha(m) f(\overline{\phi}(1 \otimes \gamma(m))) \\
&= \alpha(m) f(d\log m) = \alpha(m) D^\flat(m)
\end{aligned}$$

Then $\left(D, D^\flat\right)$ is a log derivation of (A, M) with values in J. It is immediately seen that the described two maps

$$\begin{aligned}
\mathrm{Der}((A,M),J) &\to \mathrm{Hom}_A\left(\Omega^1_{(A,M)}, J\right) \\
\left(D, D^\flat\right) &\mapsto f \mid f(da) = D(a),\ f(d\log m) = D^\flat(m) \\
\mathrm{Hom}_A\left(\Omega^1_{(A,M)}, J\right) &\to \mathrm{Der}((A,M),J) \\
f &\mapsto \left(D, D^\flat\right) \mid D(a) = f(da),\ D^\flat(m) = f(d\log m) \qquad (3.12)
\end{aligned}$$

are inverse isomorphisms. ■

In this sense, the log derivation $\left(d, d^\flat\right)$ of (A, M) with values in $\Omega^1_{(A,M)}$ corresponding to the identity in $\Omega^1_{(A,M)}$ is a **universal log derivation**, detailed with $d(a) = da$, $d^\flat(m) = d\log m$ (thus $d(\alpha(m)) = \alpha^\flat(m, d\log m) = \alpha(m) d\log m$). In fact, the previous correspondence, along with the commutativity of the diagram

$$\begin{tikzcd}
\Omega^1_{(A,M)} \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & J \\
\Omega^1_{(A,M)} \arrow[ur, "f"'] &
\end{tikzcd}$$

shows that any other log derivation $\big(D, D^\flat\big)$ with values in J factors uniquely through $\big(d, d^\flat\big)$.

We saw that, by construction, the differentials of the form $d\log m$ formally behave as $a^{-1}da$ when $a = \alpha(m)$ (justifying the title "logarithmic" for these differentials). We can use the correspondence described in (3.12) as a help to prove the following theorem, the proof of which will perhaps allow us to get a more insightful view on these differentials.

Theorem 3.15. *Given a pre-log ring (A, M), its A-module of log Kähler differentials is invariant under logification of (A, M), i.e.*

$$\Omega^1_{(A,M)} \cong \Omega^1_{(A,M^{\mathrm{a}})}$$

Proof. We recall that

$$M^{\mathrm{a}} = \frac{M \oplus \mathrm{GL}_1(A)}{\langle n \oplus 1 - 1 \oplus \alpha(n) \rangle}$$

for $\alpha(n) \in \mathrm{GL}_1(A)$. $\mathrm{GL}_1(A)$ is a group, so

$$(M^{\mathrm{a}})^{\mathrm{gp}} = \frac{M^{\mathrm{gp}} \oplus \mathrm{GL}_1(A)}{\langle \gamma(n) \oplus 1 - 1 \oplus \alpha(n) \rangle}$$

for $\alpha(n) \in \mathrm{GL}_1(A)$, taking $\gamma(m \oplus u) \in (M^{\mathrm{a}})^{\mathrm{gp}}$ to be $\gamma(m) \oplus u$, (which has inverse $\gamma(m)^{-1} \oplus u^{-1}$); this allows us to consider the inclusion $M^{\mathrm{gp}} \to (M^{\mathrm{a}})^{\mathrm{gp}}$, $g \mapsto g \oplus 1$. We moreover recall that M^{a} is defined by pushout and $\alpha^{\mathrm{a}} \colon M^{\mathrm{a}} \to (A, \cdot\,)$ is such that $\alpha^{\mathrm{a}}(m \oplus u) = \alpha^{\mathrm{a}}(1 \oplus u)\alpha^{\mathrm{a}}(m \oplus 1) = \iota(u)\alpha(m) = u\alpha(m)$.
An A-module homomorphism between $\Omega^1_{(A,M)}$ and $\Omega^1_{(A,M^{\mathrm{a}})}$ is then immediately obtained. The pre-log ring homomorphism $(\mathrm{id}_A, \mathrm{id}_M \oplus 1) \colon (A, M) \to (A, M^{\mathrm{a}})$ gives, by Proposition 3.13, a homomorphism

$$\begin{gathered}
\theta \colon \Omega^1_{(A,M)} \to \Omega^1_{(A,M^{\mathrm{a}})} \\
da \mapsto da \\
d\log m \mapsto d\log(m \oplus 1)
\end{gathered}$$

Conversely, to get an A-module homomorphism in the opposite direction, we find a log derivation of (A, M^{a}) with values in $\Omega^1_{(A,M)}$. In this case we use

$$\begin{gathered}
D \colon A \longrightarrow \Omega^1_A \xrightarrow{\overline{\psi}} \Omega^1_{(A,M)} \\
a \longmapsto da \longmapsto da
\end{gathered}$$

which gives

$$\begin{aligned} D(\alpha^{\mathrm{a}}(m \oplus u)) &= D(u\alpha(m)) = d(u\alpha(m)) \\ &= \alpha(m)du + ud(\alpha(m)) \\ &= \alpha(m)du + u\alpha(m)d\log m \\ &= u\alpha(m)(u^{-1}du + d\log m) \end{aligned}$$

This suggests us a choice of an appropriate monoid homomorphism. Define:

$$\begin{aligned} D^\flat \colon M^{\mathrm{a}} \xrightarrow{\gamma} (M^{\mathrm{a}})^{\mathrm{gp}} &\xrightarrow{\sim} (M^{\mathrm{gp}} \oplus \mathrm{GL}_1(A))/\sim \longrightarrow \Omega^1_{(A,M)} \\ m \oplus u \longmapsto \gamma(m \oplus u) &\longmapsto \gamma(m) \oplus u \longmapsto u^{-1}du + d\log m \end{aligned}$$

To verify that $D^\flat$ is a well-defined homomorphism, we will use the universal property of M^{a}. In fact,

$$\begin{aligned} \zeta \colon \mathrm{GL}_1(A) &\to \Omega^1_{(A,M)} \\ u &\mapsto u^{-1}du \end{aligned}$$

is a homomorphism, since $\zeta(uv) = (uv)^{-1}d(uv) = u^{-1}du + v^{-1}dv = \zeta(u) + \zeta(v)$. Moreover, the diagram

$$\begin{tikzcd} \alpha^{-1}\,\mathrm{GL}_1(A) \arrow[r, "\widetilde{\alpha}"] \arrow[d, "\widetilde{\iota}"'] & \mathrm{GL}_1(A) \arrow[d, "\zeta"] \\ M \arrow[r, "d\log"'] & \Omega^1_{(A,M)} \end{tikzcd}$$

sending $m \mapsto \alpha(m) \mapsto \alpha(m)^{-1}d\alpha(m)$ (upper and right-hand side arrows) and $m \mapsto m \mapsto d\log m$ (left-hand side and lower arrows), commutes, by virtue of the relation $d\log m = \alpha(m)^{-1}d\alpha(m)$, for $\alpha(m)$ invertible. The homomorphisms ζ and $d\log$ then factor through $D^\flat \colon M^{\mathrm{a}} \to \Omega^1_{(A,M)}$ by the universal property of the pushout. In this way,

$$\begin{aligned} \alpha^{\mathrm{a}}(m \oplus u)D^\flat(m \oplus u) &= \alpha^{\mathrm{a}}(m \oplus u)(u^{-1}du + d\log m) \\ &= u\alpha(m)(u^{-1}du + d\log m) = D(\alpha^{\mathrm{a}}(m \oplus u)) \end{aligned}$$

so $\left(D, D^\flat\right)$ is a log derivation. We use the correspondence in (3.12) to find

$$\begin{aligned} \overline{\theta} \colon \Omega^1_{(A,M^{\mathrm{a}})} &\to \Omega^1_{(A,M)} \\ da &\mapsto da \\ d\log(m \oplus u) &\mapsto u^{-1}du + d\log m \end{aligned}$$

We shall now verify that θ and $\overline{\theta}$ are inverse isomorphisms. One direction is given by

$$\begin{aligned} \Omega^1_{(A,M)} \xrightarrow{\theta} \Omega^1_{(A,M^{\mathrm{a}})} &\xrightarrow{\overline{\theta}} \Omega^1_{(A,M)} \\ da \longmapsto da &\longmapsto da \\ d\log m \longmapsto d\log(m \oplus 1) &\longmapsto 1d1 + d\log m = d\log m \end{aligned}$$

(recalling $d1 = d(1 \cdot 1) - d1 = d1 - d1 = 0$). The other one is

$$\begin{tikzcd}
\Omega^1_{(A,M^{\mathrm{a}})} \arrow[r, "\overline{\theta}"] & \Omega^1_{(A,M)} \arrow[r, "\theta"] & \Omega^1_{(A,M^{\mathrm{a}})} \\
da \arrow[r, mapsto] & da \arrow[r, mapsto] & da \\
d\log(m \oplus u) \arrow[r, mapsto] & u^{-1}du + d\log m \arrow[r, mapsto] & u^{-1}du + d\log(m \oplus 1)
\end{tikzcd}$$

We use the fact that $(A, M^{\mathrm{a}}, \alpha^{\mathrm{a}})$ is a log ring to factor the inclusion $\mathrm{GL}_1(A) \to (A, \,\cdot\,)$ as

$$\begin{tikzcd}
\mathrm{GL}_1(A) \arrow[r] & M^{\mathrm{a}} \arrow[r, "\alpha^{\mathrm{a}}"] & (A, \,\cdot\,) \\
u \arrow[r, mapsto] & 1 \oplus u \arrow[r, mapsto] & \alpha^{\mathrm{a}}(1 \oplus u)
\end{tikzcd}$$

so that

$$\begin{aligned}
\theta\overline{\theta}(d\log(m \oplus u)) &= u^{-1}du + d\log(m \oplus 1) \\
&= \left(\alpha^{\mathrm{a}}(1 \oplus u)\right)^{-1} d\alpha^{\mathrm{a}}(1 \oplus u) + d\log(m \oplus 1) \\
&= d\log(1 \oplus u) + d\log(m \oplus 1) \\
&= d\log(m \oplus u)
\end{aligned}$$

making θ and $\overline{\theta}$ inverse isomorphisms. ■

With the result we just showed, one may choose to only consider log Kähler differentials of log rings, taking, for a pre-log ring, its logification. In the proof of Theorem 3.15 we used the fact that we could invert some elements in M^{a} (precisely, the invertible elements of A that came from M through α). The next example will show the features of the log Kähler differentials of a log ring in a case in which $\alpha(m)$ is always invertible.

Example 3.16. Given a pre-log ring (A, M), we define its **trivial locus** (as in [Rognes, 2009]) as the pre-log ring $\left(A\left[M^{-1}\right], M^{\mathrm{gp}}\right)$, where the ring is the localization $A\left[M^{-1}\right] = A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[M^{\mathrm{gp}}]$. In this case, $\alpha(m) \in \mathrm{GL}_1\left(A\left[M^{-1}\right]\right)$ for any $m \in M^{\mathrm{gp}}$, so $(M^{\mathrm{gp}})^{\mathrm{a}} \cong M^{\mathrm{gp}}$.
The log Kähler differentials of the trivial locus are generated by differentials da for $a \in A\left[M^{-1}\right]$ and $d\log m$ for $m \in M$, such that $d\alpha(m) = \alpha(m) d\log m$. Since $\alpha(m)$ is always invertible, one can express $d\log m = \alpha(m)^{-1} d\alpha(m)$, where both $\alpha(m)$ and $\alpha(m)^{-1}$ belong to $A\left[M^{-1}\right]$. This establishes an isomorphism $\Omega^1_{A[M^{-1}]} \cong \Omega^1_{(A[M^{-1}],M^{\mathrm{gp}})}$. We then see that the log Kähler differentials $\Omega^1_{(A,M)}$ place themselves in an intermediate position between Ω^1_A and $\Omega^1_{A[M^{-1}]}$: in $\Omega^1_{(A,M)}$ we only allow differentials of the form da or $d\log m$, the latter having the formal properties of $\alpha(m)^{-1} d\alpha(m)$, while in $\Omega^1_{A[M^{-1}]}$ there are also differentials of the form $\alpha(m)^{-1} d\alpha(n)$ for $m \neq n$.
In the following diagram we show the factorization $\Omega^1_A \to \Omega^1_{(A,M)} \to \Omega^1_{A[M^{-1}]}$;

the unlabeled arrows are the obvious inclusions, the upper-left square is a pushout and the outer square is commutative.

$$\begin{tikzcd}
A \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & A \otimes M^{\mathrm{gp}} \arrow[r] \arrow[d, "\overline{\phi}"] & A\big[M^{-1}\big] \otimes M^{\mathrm{gp}} \arrow[dd, "\overline{\phi}"] \\
\Omega^1_A \arrow[r, "\overline{\psi}"'] \arrow[d] & \Omega^1_{(A,M)} \arrow[rd, dashed] & \\
\Omega^1_{A[M^{-1}]} \arrow[rr, "\overline{\psi}"'] & & \Omega^1_{(A[M^{-1}],M^{\mathrm{gp}})} \cong \Omega^1_{A[M^{-1}]}
\end{tikzcd}$$

We will also present an easy example for the case in which $\alpha(m)$ is, on the contrary, not always invertible.

Example 3.17. In example 2.4 we saw that both the Kähler differentials of $\mathbb{Z}$ and $\mathbb{Q}$ are trivial. We will now compute the log Kähler differentials of the pre-log rings $(\mathbb{Z}, \langle p\rangle, \iota)$ and $(\mathbb{Q}, \langle p\rangle, \iota)$, with $\langle p\rangle = \big\{1, p, p^2, \dots\big\}$ for $p \in \mathbb{Z}$ and ι the inclusion.
In $\Omega^1_{(\mathbb{Q},\langle p\rangle)}$ there are differentials of the form dq for $q \in \mathbb{Q}$, with $dq = qd1 = 0$, and $d\log r$, for $r \in \iota\langle p\rangle$ invertible in $\mathbb{Q}$, with $d\log r = r^{-1}dr = 1d1 = 0$, so, immediately, $\Omega^1_{(\mathbb{Q},\langle p\rangle)}$ is trivial. As for $\Omega^1_{(\mathbb{Z},\langle p\rangle)}$, there are instead non-zero differentials of the form $d\log r$, for $r \in \iota\langle p\rangle$ not invertible in $\mathbb{Z}$. From Definition 3.7, we can look at the diagram:

$$\begin{tikzcd}
\mathbb{Z} \otimes_{\mathbb{Z}[\langle p\rangle]} \Omega^1_{\mathbb{Z}[\langle p\rangle]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathbb{Z} \otimes \langle p\rangle^{\mathrm{gp}} \arrow[d, "\overline{\phi}"] \\
\Omega^1_{\mathbb{Z}} \arrow[r, "\overline{\psi}"'] & \Omega^1_{(\mathbb{Z},\langle p\rangle)}
\end{tikzcd}$$

We know from Example 2.4 that $\Omega^1_{\mathbb{Z}} = \{1\}$; moreover, there are isomorphisms

$$\mathbb{Z} \otimes_{\mathbb{Z}[\langle p\rangle]} \Omega^1_{\mathbb{Z}[\langle p\rangle]} \to \mathbb{Z}, \;\; n \otimes dp \mapsto n$$

and

$$\mathbb{Z} \otimes \langle p\rangle^{\mathrm{gp}} \to \mathbb{Z}, \;\; n \otimes p^i \mapsto n \cdot i$$

such that the map $\tilde{\psi} \colon \mathbb{Z} \to \mathbb{Z}$ is actually the multiplication

$$1 \xmapsto{\sim} 1 \otimes dp \xmapsto{\psi} p \otimes p \xmapsto{\sim} p$$

This makes $\Omega^1_{(\mathbb{Z},\langle p\rangle)} \cong (\{1\} \oplus \mathbb{Z})/\sim$, where $1 \oplus 0 \sim 1 \oplus p$, so

$$\Omega^1_{(\mathbb{Z},\langle p\rangle)} \cong \mathbb{Z}/p\mathbb{Z}$$

in which the elements are $\{d\log p, 2d\log p, \dots, pd\log p = dp = 0\}$ (notice that $d\log p^m = md\log p$).

To go further with the analogy with the Kähler differentials, we now want to establish an isomorphism between the the log Kähler differentials and the first log Hochschild homology group. We will start by introducing an A-module homomorphism $\Omega^1_{(A,M)} \to \mathrm{HH}_1(A,M)$.

Proposition 3.18. *There exists an A-module homomorphism*

$$\overline{\omega}\colon \Omega^1_{(A,M)} \to \mathrm{HH}_1(A,M)$$

Proof. We will use the correspondence described in (3.12). An A-module homomorphism as such can be obtained once we find a log derivation $\left(D, D^\flat\right)$ of (A,M) with values in $\mathrm{HH}_1(A,M)$. We get a derivation $D\colon A \to \mathrm{HH}_1(A,M)$ passing through the derivation $A \to \mathrm{HH}_1(A)$, $a \mapsto 1 \otimes a$ described in (2.5), and composing with the homomorphism induced in homology from the map $\overline{\psi}$ in Definition 3.4. So, define:

$$\begin{aligned} D\colon A &\to \mathrm{HH}_1(A,M) \\ a &\mapsto (1 \otimes a) \otimes (1 \otimes 1) \end{aligned}$$

A monoid homomorphism $D^\flat\colon M \to \mathrm{HH}_1(A,M)$ can be obtained by composing the monoid homomorphism $M \to \mathrm{H}_1(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) \cong \mathbb{Z}[M] \otimes M^{\mathrm{gp}}$, $m \mapsto 1 \otimes \gamma(m)$ with the homomorphism induced in homology by the map $\overline{\xi}$ in Definition 3.4. We the get:

$$\begin{aligned} D^\flat\colon M &\to (\mathrm{HH}_1(A,M), +) \\ m &\mapsto (1 \otimes 1) \otimes (1 \otimes \gamma(m)) \end{aligned}$$

We see that

$$\begin{aligned} D(\alpha(m)) &= (1 \otimes \alpha(m)) \otimes (1 \otimes 1) \\ &= (1 \otimes 1) \otimes (m \otimes \gamma(m)) \\ &= (1 \otimes 1) \otimes ((m \otimes 1) \cdot (1 \otimes \gamma(m))) \\ &= (\alpha(m) \otimes 1) \otimes (1 \otimes \gamma(m)) \\ &= \alpha(m) \cdot (1 \otimes 1) \otimes (1 \otimes \gamma(m)) = \alpha(m) D^\flat(m) \end{aligned}$$

so $\left(D, D^\flat\right)$ is a log derivation. By Theorem 3.14, we get a homomorphism of A-modules:

$$\begin{aligned} \overline{\omega}\colon \Omega^1_{(A,M)} &\to \mathrm{HH}_1(A,M) \\ da &\mapsto D(a) = (1 \otimes a) \otimes (1 \otimes 1) \\ d\log m &\mapsto D^\flat(m) = (1 \otimes 1) \otimes (1 \otimes \gamma(m)) \end{aligned} \tag{3.13}$$

as we wanted to prove. ■

Though, as we will see, there actually is an isomorphism between $\Omega^1_{(A,M)}$ and $\mathrm{HH}_1(A,M)$, the map $\overline{\varpi}$ does not seem to be easily invertible at this point. In Section 3.4 we will prove that $\overline{\varpi}$ is indeed an isomorphism, under the assumption that A is flat over $\mathbb{Z}[M]$. The map we found will anyway be useful when dealing with the module of log differential n-forms and the graded commutative structure of HH_*; the latter will be explained in Lemma 3.19.

3.4 The isomorphism $\mathrm{HH}_1(A,M) \cong \Omega^1_{(A,M)}$

In Theorem 2.5 we showed that there is an isomorphism between the first Hochschild homology group of a k-algebra and its module of Kähler differentials, explicitly giving inverse module homomorphisms. We will use a different argument to show that, for a pre-log ring (A,M), there is an isomorphism $\mathrm{HH}_1(A,M) \cong \Omega^1_{(A,M)}$; this isomorphism will be conveyed, in one direction, by the homomorphism $\overline{\varpi}$ described in (3.13).

In this section we will encounter the notion of strictly commutative graded ring. A **graded ring** A_* is a sequence of abelian groups A_n, $n \geq 0$, with a bilinear, associative multiplication $\cdot : A \times A \to A$ and a unit $1 \in A_0$, such that $x \cdot y \in A_{m+n}$ if $x \in A_m$ and $y \in A_n$. A graded ring is **graded commutative** if $x \cdot y = (-1)^{mn} y \cdot x$ for $x \in A_m$ and $y \in A_n$. Such a ring is **strictly commutative** if moreover $x \cdot x = 0$ if $x \in A_n$, with n odd.

Lemma 3.19. *If $R_\bullet$ is a simplicial commutative ring, then $\mathrm{H}_*(R_\bullet)$ is a strictly commutative graded ring.*

Proof. We want to endow $\mathrm{H}_*(R_\bullet)$ with an associative and unital operation

$$\mathfrak{sh}(\,\cdot \otimes \cdot\,)\colon\ \mathrm{H}_m(R_\bullet) \otimes \mathrm{H}_n(R_\bullet) \to \mathrm{H}_{m+n}(R_\bullet)$$

such that, for $r \in \mathrm{H}_m(R_\bullet)$ and $s \in \mathrm{H}_n(R_\bullet)$,

$$\mathfrak{sh}(r \otimes s) = (-1)^{mn} \mathfrak{sh}(s \otimes r) \tag{3.14a}$$

$$\mathfrak{sh}(r \otimes r) = 0 \text{ for } r \text{ in odd degree} \tag{3.14b}$$

We saw in Section 1.4 that, given $R_\bullet$ and $S_\bullet$ simplicial commutative rings, the external homology product

$$\begin{aligned} \mathfrak{p}\colon\ \mathrm{H}_m(R_\bullet) \otimes \mathrm{H}_n(S_\bullet) &\to \mathrm{H}_{m+n}(R_\bullet \otimes S_\bullet) \\ r \otimes s &\mapsto r \otimes s \end{aligned} \tag{3.15}$$

is a well-defined homomorphism. The shuffle map described in (1.4) induces, by the Eilenberg-Zilber theorem, an isomorphism in homology

$$\mathfrak{g}\colon\ \mathrm{H}_{m+n}(R_\bullet \otimes S_\bullet) \xrightarrow{\sim} \mathrm{H}_{m+n}((R \boxtimes S)_\bullet)$$

Finally, for $S_\bullet = R_\bullet$, the multiplication map

$$\begin{aligned} m\colon R_q \times R_q &\to R_q \\ (r,s) &\mapsto rs \end{aligned}$$

induces a homomorphism $\mathfrak{m}\colon \mathrm{H}_{m+n}((R \boxtimes R)_\bullet) \to \mathrm{H}_{m+n}(R_\bullet)$ in homology. The composition $\mathfrak{sh} = \mathfrak{m} \circ \mathfrak{g} \circ \mathfrak{p}$ is the map we were looking for; the associativity of $\mathfrak{sh}$ comes from the associativity of the shuffle map. The sign in (3.14a) is determined by the shuffle map, as shown in (1.5).
We shall now explain why the condition (3.14b) is verified. For (μ,ν) a (p,p)-shuffle, consider the map

$$\begin{aligned} h_{(\mu,\nu)}\colon R_p \otimes R_p &\to R_{2p} \otimes R_{2p} \\ u \otimes v &\mapsto \operatorname{sgn}(\mu,\nu)(s_{\nu_q} \cdots s_{\nu_1}(u) \otimes s_{\mu_p} \cdots s_{\mu_1}(v)) \end{aligned}$$

Let (ν,μ) be the (p,p)-shuffle associated to (μ,ν) according to (1.6), where the permutation sending (μ,ν) to (ν,μ) is the product of $p \cdot p$ transpositions. One can easily see that, for $r \otimes r \in \mathrm{H}_{2p}((R \otimes R)_\bullet)$,

$$m \circ h_{(\nu,\mu)}(r \otimes r) = (-1)^{p^2} m \circ h_{(\mu,\nu)}(r \otimes r)$$

In particular, for p odd, we have $m \circ h_{(\nu,\mu)}(r \otimes r) = -m \circ h_{(\mu,\nu)}(r \otimes r)$. Since the shuffle map

$$\begin{aligned} g\colon R_p \otimes R_p &\to R_{2p} \otimes R_{2p} \\ r \otimes r &\mapsto \sum_{(\mu,\nu)} h_{(\mu,\nu)}(r \otimes r) \end{aligned}$$

is obtained as the sum of all the $h_{(\mu,\nu)}$ for (μ,ν) a (p,p)-shuffle, and such maps $h_{(\mu,\nu)}$ cancel out in pairs, we get $m \circ g(r \otimes r) = 0$, yielding, in homology, $\mathfrak{m} \circ \mathfrak{g}(r \otimes r) = 0$. Considering $r \otimes r \in \mathrm{H}_{2p}(R_\bullet)$ as the external homology product of $r \in \mathrm{H}_p(R_\bullet)$ and itself, we have that $\mathfrak{sh}(r \otimes r) = \mathfrak{m} \circ \mathfrak{g} \circ \mathfrak{p}(r \otimes r) = 0$ for r in odd degree. ■

From the definition of the log Hochschild complex as the degreewise pushout of a diagram of simplicial commutative rings (Definition 3.4), by Lemma 3.19 we obtain in homology a diagram of strictly commutative graded rings:

$$\begin{tikzcd} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M]) \arrow[r] \arrow[d] & \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) \arrow[d] \\ \mathrm{HH}_*(A) \arrow[r] & \mathrm{HH}_*(A, M) \end{tikzcd} \tag{3.16}$$

where, by definition, $\mathrm{HH}_*(A, M)$ is the homology of the log Hochschild complex.

For $R_\bullet$ a simplicial commutative ring and $X_\bullet$, $Y_\bullet$ respectively right and left simplicial R-modules, we will use the notation $(X \boxtimes_R Y)_\bullet$ to indicate the simplicial module (and its Moore complex) obtained by the degreewise pushout of given module homomorphisms $R_n \to X_n$, $R_n \to Y_n$. We can, then, write:

$$\mathrm{HH}_*(A, M) = \mathrm{H}_*\big(\big(\mathrm{C}(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}} M]} \mathbb{Z}[\mathrm{B}^{\mathrm{rep}} M]\big)_\bullet\big) \tag{3.17}$$

We recall that, for R_* a graded ring, and for X_* and Y_* respectively right and left graded R-modules, the graded module $X_* \otimes_{R_*} Y_*$ is defined in each degree n as the coequalizer of the two parallel multiplication maps

$$\begin{aligned} \bigoplus_{i+j+k=n} X_i \otimes R_j \otimes Y_k &\rightrightarrows \bigoplus_{i+j=n} X_i \otimes Y_j \\ x \otimes r \otimes y &\longmapsto xr \otimes y \\ &\longmapsto x \otimes ry \end{aligned} \tag{3.18}$$

From the diagram in (3.16) we obtain a map

$$\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) \longrightarrow \mathrm{HH}_*(A, M)$$

which, unfortunately, is not an isomorphism; to explicitly compute the homology in (3.17) is, moreover, not easy, even in degree 1. Nevertheless, it will prove itself useful to start by finding an expression for $\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])$ in degree 1.

Lemma 3.20. *Using the same notation,*

$$\big[\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])\big]_1 \cong \Omega^1_{(A,M)}$$

Proof. The module on the left-hand side is defined as the coequalizer of the two parallel multiplication maps, as in (3.18):

$$\begin{aligned} \bigoplus_{i+j+k=1} \mathrm{HH}_i(A) \otimes \mathrm{H}_j(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M]) \otimes \mathrm{H}_k(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) \rightrightarrows& \\ \bigoplus_{i+j=1} \mathrm{HH}_i(A) \otimes \mathrm{H}_j(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])& \end{aligned}$$

In order to compute it, we will first explicate in detail the objects involved with their degrees. $\mathrm{HH}_0(A) = A$ and $\mathrm{HH}_1(A) \cong \Omega^1_A$, as explained in Theorem 2.5. The homology of $\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M]$ is actually the Hochschild homology of $\mathbb{Z}[M]$, so $\mathrm{H}_0(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M]) = \mathbb{Z}[M]$ and $\mathrm{H}_1(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M]) \cong \Omega^1_{\mathbb{Z}[M]}$. As for $\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])$, we can easily compute from the complex

$$\ldots \longrightarrow \mathbb{Z}[M] \otimes \mathbb{Z}[M^{\mathrm{gp}}] \otimes \mathbb{Z}[M^{\mathrm{gp}}] \xrightarrow{b_2} \mathbb{Z}[M] \otimes \mathbb{Z}[M^{\mathrm{gp}}] \xrightarrow{b_1} \mathbb{Z}[M] \longrightarrow 0$$

with $b_1(m \otimes g_1) = 0$ and $b_2(m \otimes g_1 \otimes g_2) = (m \otimes g_2) - (m \otimes g_1 g_2) + (m \otimes g_1)$ for $m \in \mathbb{Z}[M]$ and $g_i \in \mathbb{Z}[M^{\mathrm{gp}}]$, that $\mathrm{H}_0(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) = \mathbb{Z}[M]$ and $\mathrm{H}_1(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) \cong \mathbb{Z}[M] \otimes M^{\mathrm{gp}}$ (using Lemma 3.9).
We start from the direct sum of these three tensor products:

$$\Omega^1_A \otimes \mathbb{Z}[M] \otimes \mathbb{Z}[M] \tag{3.19a}$$

$$A \otimes \Omega^1_{\mathbb{Z}[M]} \otimes \mathbb{Z}[M] \tag{3.19b}$$

$$A \otimes \mathbb{Z}[M] \otimes (\mathbb{Z}[M] \otimes M^{\mathrm{gp}}) \tag{3.19c}$$

Multiplying the central factor on the left or on the right, we land on the direct sum of these two tensor products:

$$\Omega^1_A \otimes \mathbb{Z}[M] \tag{3.20a}$$

$$A \otimes (\mathbb{Z}[M] \otimes M^{\mathrm{gp}}) \tag{3.20b}$$

Precisely, (3.19a) is mapped to (3.20a) and (3.19c) is mapped to (3.20b) through both the maps, while (3.19b) is mapped to (3.20a) or (3.20b) when multiplying the central factor on the left or on the right respectively. In detail:

$$\begin{aligned} \Omega^1_A \otimes \mathbb{Z}[M] \otimes \mathbb{Z}[M] &\rightrightarrows \Omega^1_A \otimes \mathbb{Z}[M] \\ da \otimes m \otimes n &\mapsto \alpha(m) da \otimes n \\ &\mapsto da \otimes mn \end{aligned}$$

has coequalizer $\big(\Omega^1_A \otimes \mathbb{Z}[M]\big)/\sim$, with $\alpha(m)da \otimes 1 \sim da \otimes m$, thus isomorphic to Ω^1_A, while

$$\begin{aligned} A \otimes \mathbb{Z}[M] \otimes (\mathbb{Z}[M] \otimes M^{\mathrm{gp}}) &\rightrightarrows A \otimes (\mathbb{Z}[M] \otimes M^{\mathrm{gp}}) \\ a \otimes m \otimes (n \otimes g) &\mapsto \alpha(m) a \otimes (n \otimes g) \\ &\mapsto a \otimes (mn \otimes g) \end{aligned}$$

has coequalizer $(A \otimes (\mathbb{Z}[M] \otimes M^{\mathrm{gp}}))/\sim$, with $\alpha(m)a \otimes 1 \otimes g \sim a \otimes m \otimes g$, thus isomorphic to $A \otimes M^{\mathrm{gp}}$. Finally, by what we just computed, the coequalizer of the two maps

$$A \otimes \Omega^1_{\mathbb{Z}[M]} \otimes \mathbb{Z}[M] \rightrightarrows \big(\Omega^1_A \otimes \mathbb{Z}[M]\big) \oplus (A \otimes (\mathbb{Z}[M] \otimes M^{\mathrm{gp}}))$$

can be identified with the coequalizer of

$$\begin{aligned} A \otimes \Omega^1_{\mathbb{Z}[M]} \otimes \mathbb{Z}[M] &\rightrightarrows \Omega^1_A \oplus (A \otimes M^{\mathrm{gp}}) \\ a \otimes dm \otimes n &\mapsto (a\alpha(n) d\alpha(m)) \oplus 0 \\ &\mapsto 0 \oplus (a\alpha(n)\alpha(m) \otimes \gamma(m)) \end{aligned}$$

which is then

$$\frac{\Omega^1_A \oplus (A \otimes M^{\mathrm{gp}})}{\langle a d\alpha(m) \oplus 0 = 0 \oplus (a\alpha(m) \otimes \gamma(m)) \rangle} \cong \Omega^1_{(A,M)}$$

Gathering together the summands in the direct sum, we obtain that

$$\left[\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])\right]_1 \cong \Omega^1_{(A,M)}$$

as we wanted to show. ■

In order to show that the first log Hochschild homology of a pre-log ring is isomorphic to the module of its log Kähler differentials, we will combine what we just proved with some of the results presented in [Quillen, 1967, Chapter II.6] (spectral sequence (b) in Theorem 6, p. 6.8; Corollary, p. 6.10), which we will now summarize.

Theorem 3.21 ([Quillen, 1967])**.** *Let $R_\bullet$ be a simplicial ring and let $X_\bullet$ and $Y_\bullet$ be respectively right and left simplicial R-modules. If $\mathrm{Tor}^{R_n}_q(X_n, Y_n) = 0$ for $q > 0$, then there is a canonical first quadrant spectral sequence*

$$E^2_{p,q} = \left[\mathrm{Tor}^{\mathrm{H}_*(R_\bullet)}_p(\mathrm{H}_*(X_\bullet), \mathrm{H}_*(Y_\bullet))\right]_q \Rightarrow \mathrm{H}_{p+q}((X \boxtimes_R Y)_\bullet)$$

We point out that in [Quillen, 1967] the notation used for the degreewise tensor product of simplicial modules is $X \otimes_R Y$ instead of $(X \boxtimes_R Y)_\bullet$.

Theorem 3.22. *For (A, M) pre-log ring, under the assumption that A is flat over $\mathbb{Z}[M]$, the map*

$$\overline{\omega}\colon \Omega^1_{(A,M)} \xrightarrow{\sim} \mathrm{HH}_1(A, M)$$

described in (3.13) is an isomorphism of A-modules.

Proof. Referring to Theorem 3.21, for our purposes, we consider $R_\bullet = \mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M]$, $X_\bullet = \mathrm{C}_\bullet(A)$ and $Y_\bullet = \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]$. We are interested in the case for $p + q = 1$. The condition on $\mathrm{Tor}^{R_n}_q(X_n, Y_n)$ in Theorem 3.21 is satisfied since we assume A flat over $\mathbb{Z}[M]$; so, $A^{\otimes n}$ is flat over $\mathbb{Z}[M]^{\otimes n}$ for every n (this result descends from [Eisenbud, 1995, Theorem A6.6]). We will consider the terms $E^2_{0,1}$ and $E^2_{1,0}$ of the spectral sequence.
Regarding $E^2_{0,1}$, we have immediately:

$$\begin{aligned} E^2_{0,1} &= \left[\mathrm{Tor}^{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])}_0(\mathrm{HH}_*(A), \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]))\right]_1 \\ &\cong \left[\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])\right]_1 \end{aligned} \tag{3.21}$$

About $E^2_{1,0}$, given a resolution by free $\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])$-modules

$$\ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]) \tag{3.22}$$

and tensoring it with $\otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A)$, we get a sequence

$$\begin{aligned} \ldots \longrightarrow F_2 \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A) \longrightarrow \\ \longrightarrow F_1 \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A) \longrightarrow F_0 \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A) \end{aligned} \quad (3.23)$$

the homology of which is the torsion we want to get. There is a resolution

$$\ldots \longrightarrow [F_2]_0 \longrightarrow [F_1]_0 \longrightarrow [F_0]_0 \longrightarrow \mathrm{H}_0(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])$$

of free $\mathrm{H}_0(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])$-modules given by the terms in degree 0 of each module in (3.22); so, taking (3.23) in degree 0, we get:

$$\begin{aligned} \ldots \longrightarrow \big[F_2 \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A)\big]_0 \longrightarrow \\ \longrightarrow \big[F_1 \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A)\big]_0 \longrightarrow \big[F_0 \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{HH}_*(A)\big]_0 \end{aligned}$$

Hence,

$$\begin{aligned} E^2_{1,0} = \Big[\mathrm{Tor}_1^{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])}(\mathrm{HH}_*(A), \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M]))\Big]_0 \\ \cong \mathrm{Tor}_1^{\mathrm{H}_0(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])}(\mathrm{HH}_0(A), \mathrm{H}_0(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])) \\ \cong \mathrm{Tor}_1^{\mathbb{Z}[M]}(A, \mathbb{Z}[M]) \cong 0 \end{aligned} \quad (3.24)$$

since $\mathbb{Z}[M]$ is itself a free $\mathbb{Z}[M]$-module. For the same reason, $E^2_{p,0} = 0$ for $p \geq 2$. Theorem 3.21 asserts that there is a short exact sequence:

$$0 \to E^\infty_{0,1} \to \mathrm{H}_{0+1}((X \boxtimes_R Y)_\bullet) \to E^\infty_{1,0} \to 0 \quad (3.25)$$

In our case, $E^\infty_{1,0} = E^2_{1,0}$ by definition, while

$$E^\infty_{0,1} = E^3_{0,1} \cong \ker d^2_{0,1} / \operatorname{im} d^2_{2,0} \cong E^2_{0,1}/0 \cong E^2_{0,1}$$

Hence, the short exact sequence (3.25) becomes

$$0 \to E^2_{0,1} \to \mathrm{H}_1((X \boxtimes_R Y)_\bullet) \to E^2_{1,0} \to 0$$

which is isomorphic, by (3.21) and (3.24), to

$$\begin{aligned} 0 \to \big[\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])\big]_1 \to \\ \to \mathrm{H}_1\big((\mathrm{C}(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}} M]} \mathbb{Z}[\mathrm{B}^{\mathrm{rep}} M])_\bullet\big) \to 0 \to 0 \end{aligned} \quad (3.26)$$

where the middle term is the first log Hochschild homology of (A, M), as described in (3.17). Moreover, Lemma 3.20 showed that the left term is isomorphic to $\Omega^1_{(A,M)}$. Explicitly,

$$\begin{aligned} \big[\mathrm{HH}_*(A) \otimes_{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet M])} \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet M])\big]_1 \to \Omega^1_{(A,M)} \\ (1 \otimes a) \otimes 1 \mapsto da \\ 1 \otimes (1 \otimes \gamma(m)) \mapsto d \log m \end{aligned} \quad (3.27)$$

The composition of the map in (3.27) with

$$\begin{aligned} \overline{\omega}\colon \Omega^1_{(A,M)} &\to \mathrm{HH}_1(A,M) \\ da &\mapsto (1\otimes a)\otimes(1\otimes 1) \\ d\log m &\mapsto (1\otimes 1)\otimes(1\otimes \gamma(m)) \end{aligned}$$

agrees with the natural inclusion $E^{\infty}_{0,1} \to \mathrm{HH}_1(A,M)$ in (3.26). Summarizing, we get the short exact sequence

$$0 \to \Omega^1_{(A,M)} \xrightarrow{\overline{\omega}} \mathrm{HH}_1(A,M) \to 0 \to 0$$

returning

$$\Omega^1_{(A,M)} \cong \mathrm{HH}_1(A,M)$$

as we wanted to prove. ■

Chapter 4

Polynomial algebras

4.1 Definitions and results on Hochschild homology

In this thesis, for a module V over k and a commutative and unital k-algebra A, we will denote by $\Lambda_A^n V$ the n-th **exterior power** of V, i.e. $V^{\otimes n}/\sim$, where the tensor product is over A and $v_1 \otimes \ldots \otimes v_n \sim 0$ if $v_i = v_j$ for some $i \neq j$ (we will also use the equivalent condition: $v_i = v_{i+1}$ for some i)[1]; we set moreover $\Lambda_A^0 V = A$. When $A = k$, we will omit A from the notation and write $\Lambda^n V$ instead. The class of $v_1 \otimes \ldots \otimes v_n$ in $\Lambda_A^n V$ is denoted as $v_1 \wedge \ldots \wedge v_n$. If $\sigma \in S_n$, $v_1 \wedge \ldots \wedge v_n = \operatorname{sgn}(\sigma) v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)}$.

The **exterior algebra** of V is $\Lambda_A^* V = \bigoplus_{n \in \mathbb{N}} \Lambda_A^n V$, where the multiplication $\wedge$ is induced by the product in the tensor algebra $V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes m+n}$ (so, by concatenation).

For a k-module V, the **symmetric algebra** over V is the algebra $S(V) = S^*(V)$, defined degreewise as $S^0(V) = k$ and $S^n(V) = V^{\otimes n}/\sim$ for $n > 0$, where $v_1 \otimes \ldots \otimes v_n \sim v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$ if $\sigma \in S_n$; multiplication is again given by concatenation. We will denote with $v_1 \ldots v_n$ the class of $v_1 \otimes \ldots \otimes v_n$. When V is free of dimension n and generated by $x_1, \ldots, x_n$, the symmetric algebra $S(V)$ is the **polynomial algebra** in the variables x_i with coefficients in k.

The following notation and part of the results about the differential n-forms are presented as in [Loday, 1998].

---

[1] By induction: assume $v_1 \wedge \ldots \wedge v_n = 0$ whenever $v_i = v_{i+k}$ for some $k < m$. Let $v_i = v_{i+m}$. Then $0 = v_1 \wedge \ldots \wedge (v_i + v_{i+1}) \wedge (v_i + v_{i+1}) \wedge \ldots \wedge v_{i+m} \wedge \ldots \wedge v_n$; expand and apply the inductive assumption.

Definition 4.1. Let A be a commutative and unital k-algebra. The A-module of **differential n-forms** is defined as the exterior product $\Omega_A^n = \Lambda_A^n \Omega_A^1$. We will write $a_0 da_1 \dots da_n$ to denote $a_0 da_1 \wedge \dots \wedge da_n \in \Omega_A^n$, for $a_i \in A$. We will use the notation Ω_A^* for the graded algebra of the differential forms.

In Theorem 2.5 we showed that there is an A-module isomorphism $\Omega_A^1 \cong \mathrm{HH}_1(A)$. In general, the same result does not hold in higher degree; we will show that, however, this holds in the case when A is a polynomial algebra in a finite number of variables.

Proposition 4.2. *There is a graded algebra homomorphism*

$$\overline{\tau}_* \colon \Omega_A^* \to \mathrm{HH}_*(A)$$

Proof. In the diagram

$$\begin{tikzcd}
\left(\Omega_A^1\right)^{\otimes n} \arrow[r, "\overline{\tau}^{\otimes n}"] \arrow[d, "\wedge"'] & (\mathrm{HH}_1(A))^{\otimes n} \arrow[d, "\mathfrak{sh}"] \\
\Omega_A^n \arrow[r, dashed] & \mathrm{HH}_n(A)
\end{tikzcd} \tag{4.1}$$

$\overline{\tau}$ is the isomorphism showed in (2.4) and $\mathfrak{sh}$ is the operation in $\mathrm{HH}_*(A)$ induced by the shuffle map as described in Lemma 3.19, making it a strictly commutative graded ring. Let I_n be the A-module generated by the elements $da_1 \otimes \dots \otimes da_n$ of $\left(\Omega_A^1\right)^{\otimes n}$ with $a_i = a_{i+1}$ for some i. Elements in I_n are sent to 0 by $\mathfrak{sh} \circ \overline{\tau}^{\otimes n}$ by virtue of (3.14b). The exterior product $\wedge$ quotients out those elements, so there exists a (unique) A-module homomorphism $\overline{\tau}_n \colon \Omega_A^n \to \mathrm{HH}_n(A)$ which makes the diagram commute. Moreover, $\left(\Omega_A^1\right)_*^{\otimes n}$ and $\mathrm{HH}_*(A)$ are graded A-algebras and $I^* = \coprod I_n$ is a graded ideal of $\left(\Omega_A^1\right)_*^{\otimes n}$ (since multiplication is given by concatenation, it is clear that the product of an element in I^* of degree n_1 by any element in Ω_A^* of degree n_2 lies in I^* and has degree $n_1 + n_2$). So we get a graded algebra homomorphism $\overline{\tau}_* \colon \Omega_A^* \to \mathrm{HH}_*(A)$. ■

We will later make use of the following description of the algebra of the differential forms of a polynomial algebra.

Proposition 4.3. *Let V be a free module over k. There is a canonical isomorphism of $S(V)$-modules:*

$$\begin{aligned} S(V) \otimes V &\to \Omega_{S(V)}^1 \\ a \otimes v &\mapsto adv \end{aligned} \tag{4.2}$$

Proof. The map

$$\begin{aligned} D\colon S(V) &\to S(V) \otimes V \\ v_1 \dots v_n &\mapsto \sum_i (v_1 \dots \hat{v_i} \dots v_n \otimes v_i) \end{aligned}$$

is a universal derivation of $S(V)$ with values in $S(V) \otimes V$. In fact, let $\delta\colon S(V) \to N$ be another derivation; this is determined completely on the value of δ on V. Now there exists a unique $S(V)$-linear map $\phi\colon S(V) \otimes V \to N$ such that $\delta = \phi \circ D$, given by $\phi(1 \otimes v) = \delta(v)$. So D is universal; by Proposition 2.6, $S(V) \to \Omega^1_{S(V)}$, $v \mapsto dv$ is also a universal derivation, so, by Corollary 2.7, the map in (4.2) is an isomorphism. ■

Corollary 4.4. *There is an isomorphism of $S(V)$-algebras:*

$$\begin{aligned} \Omega^*_{S(V)} &\xrightarrow{\sim} S(V) \otimes_k \Lambda^*_k V \\ adv_1 \dots dv_n &\mapsto a \otimes (v_1 \wedge \dots \wedge v_n) \end{aligned}$$

Proof. From Proposition 4.3, in each degree n we have:

$$\begin{aligned} \Omega^n_{S(V)} = \Lambda^n_{S(V)} \Omega^1_{S(V)} &\cong \Lambda^n_{S(V)}(S(V) \otimes_k V) \\ &\cong \big(S(V) \otimes_k V \otimes_{S(V)} \dots \otimes_{S(V)} S(V) \otimes_k V\big)/\sim \\ &\cong (S(V) \otimes_k V \otimes_k \dots \otimes_k V)/\sim \\ &\cong S(V) \otimes_k \Lambda^n_k V \end{aligned}$$

The multiplication in $S(V) \otimes_k \Lambda^*_k V$ is given by usual product on $S(V)$ and by concatenation on $\Lambda^*_k V$, so that

$$\begin{aligned} &(a \otimes (v_1 \wedge \dots \wedge v_n)) \cdot (b \otimes (w_1 \wedge \dots \wedge w_m)) \\ &\qquad = (a \cdot b \otimes (v_1 \wedge \dots \wedge v_n \wedge w_1 \wedge \dots \wedge w_m)) \end{aligned}$$

One can then easily see that the described degreewise isomorphism of $S(V)$-modules respects the respective $S(V)$-algebra structures. ■

Remark 4.5. We recall that, if $V = k\{v_1, \dots, v_r\}$ has finite dimension r, then $\Lambda^n V = 0$ for $n > r$, making $\Omega^n_{S(V)} \cong 0$ for $n > r$.

In order to show the isomorphism between the Hochschild homology and the algebra of differential forms in case of a polynomial algebra, we will need the following results. The first one, the proof of which we omit, appears in [Loday, 1998, Theorem 1.1.13].

Lemma 4.6. *If a unital algebra A is flat as a module over k, then there is an isomorphism*

$$\mathrm{HH}_n(A;M) \cong \mathrm{Tor}_n^{A\otimes A^{\mathrm{op}}}(M,A)$$

where A^{op} is the opposite algebra of A, in which the product is given by

$$A^{\mathrm{op}} \times A^{\mathrm{op}} \to A^{\mathrm{op}},\ (a,b) \mapsto ba$$

We will use Lemma 4.6 in the next lemma, which examines the case of a polynomial algebra in one variable only.

Lemma 4.7. *There is a graded algebra isomorphism*

$$\overline{\tau}_*\colon \Omega^*_{k[x]} \xrightarrow{\sim} \mathrm{HH}_*(k[x])$$

where $\overline{\tau}_*$ is the graded algebra homomorphism from Proposition 4.2.

Proof. We will start by computing the Hochschild homology of the polynomial algebra $k[x]$. By Lemma 4.6, we need a projective resolution of $k[x]$ in terms of $k[x] \otimes k[x] \cong k[x_1, x_2]$-modules. This is easy to find, after the identification

$$\begin{aligned} k[x] \otimes k\{x\} &\xrightarrow{\sim} k[x] \\ f(x) \otimes \lambda x &\mapsto \lambda \cdot f(x) \end{aligned}$$

for $\lambda \in k$. A free resolution of $k[x]$ is given by:

$$0 \longrightarrow k[x_1,x_2] \xrightarrow{\cdot\,(x_1 - x_2)} k[x_1,x_2] \xrightarrow{s} k[x]$$

where $s(x_1) = x = s(x_2)$. Tensoring the resolution by $\otimes_{k[x_1,x_2]} k[x]$, we get

$$0 \longrightarrow k[x_1,x_2] \otimes_{k[x_1,x_2]} k[x] \longrightarrow k[x_1,x_2] \otimes_{k[x_1,x_2]} k[x] \longrightarrow 0$$

Under isomorphism

$$\begin{aligned} k[x_1,x_2] \otimes_{k[x_1,x_2]} k[x] &\xrightarrow{\sim} k[x] \\ f(x_1,x_2) \otimes g(x) &\mapsto f(x,x)g(x) \\ 1 \otimes g(x) &\mapsfrom g(x) \end{aligned}$$

the chain complex becomes

$$0 \longrightarrow k[x] \longrightarrow k[x] \longrightarrow 0$$

where the middle map sends

$$g(x) \overset{\sim}{\longmapsto} 1 \otimes g(x) \mapsto (x_1 - x_2) \otimes g(x) \overset{\sim}{\longmapsto} (x - x) \otimes g(x) = 0$$

Hence, $\mathrm{HH}_0(k[x]) \cong k[x]$ as we knew; $\mathrm{HH}_1(k[x]) \cong k[x]$ as well; the homology is 0 in higher degree.
As for the differential n-forms, we have $\Omega^0_{k[x]} \cong k[x]$ and $\Omega^1_{k[x]} \cong k[x]\{dx\}$. In degree $n \geq 2$, $\Omega^n_{k[x]} \cong 0$, since, by Corollary 4.4, $\Omega^n_{k[x]} \cong k[x] \otimes_k \Lambda^n_k k\{x\}$ and $k\{x\}$ has dimension 1, making $\Lambda^n k\{x\} = 0$ in degree higher than 1.
As graded algebras, then,

$$\Omega^*_{k[x]} \cong k[x]\{1, dx\} \cong k[x]\{1, dx\}/\big((dx)^2\big)$$
$$\mathrm{HH}_*(k[x]) \cong k[x]\{1, dx\} \cong k[x]\{1, dx\}/\big((dx)^2\big)$$

where dx is the generator in degree 1.
We will now check that the graded algebra isomorphism is given by $\overline{\tau}_*$. This is trivial in degree 0 (because $\overline{\tau}^{\otimes 0}$ is the identity on $\Omega^0_{k[x]} = k[x]$) and in degree greater than 1 (because $\Omega^n_{k[x]} \cong \mathrm{HH}_n(k[x]) \cong 0$ for $n > 1$). In degree 1, we see that, via the described isomorphisms, the map $\overline{\tau}$

$$\begin{tikzcd}
\Omega^1_{k[x]} \arrow[r, "\overline{\tau}"] & \mathrm{HH}_1(k[x]) \arrow[r, "\sim"] & k[x]\{dx\} \otimes k\{x\} \arrow[r, "\sim"] & k[x]\{dx\} \\
dx \arrow[r, mapsto] & 1 \otimes x \arrow[r, mapsto] & dx \otimes x \arrow[r, mapsto] & dx
\end{tikzcd}$$

sends the generator dx of $\Omega^1_{k[x]}$ to the generator dx of $\mathrm{HH}_1(k[x])$. ■

Remark 4.8. We emphasize that, considering $k = \mathbb{Z}$, the isomorphism $\Omega^1_{\mathbb{Z}[x]} \xrightarrow{\sim} \mathbb{Z}[x]\{dx\}$, $dx \mapsto 1dx$ corresponds by Corollary 2.7 to the usual polynomial derivation $D \colon \mathbb{Z}[x] \to \mathbb{Z}[x]$, $D\big(\sum_i a_i x^i\big) = \sum_i a_i \cdot i \cdot x^{i-1} dx$.

We are now ready to prove the central theorem of this section.

Theorem 4.9. *Let $V = k\{x_1, \dots, x_r\}$ be a free, finitely generated k-module. There is a graded algebra isomorphism*

$$\overline{\tau}_* \colon \Omega^*_{S(V)} \xrightarrow{\sim} \mathrm{HH}_*(S(V))$$

Proof. Since the k-module V is finitely generated, we can decompose it as the product $V = k\{x_1\} \times \ldots \times k\{x_r\}$. We will use the general fact that there is an isomorphism of simplicial commutative monoids

$$\mathrm{B}^{\mathrm{cy}}_\bullet(M \times N) \cong \mathrm{B}^{\mathrm{cy}}_\bullet M \times \mathrm{B}^{\mathrm{cy}}_\bullet N$$

which is explicited in each degree by

$$\mathrm{B}^{\mathrm{cy}}_q(M \times N) \cong (M \times N)^{q+1} \cong M^{q+1} \times N^{q+1} \cong \mathrm{B}^{\mathrm{cy}}_q M \times \mathrm{B}^{\mathrm{cy}}_q N$$

Hence, $\mathrm{B}^{\mathrm{cy}}_{\bullet}\, V \cong \mathrm{B}^{\mathrm{cy}}_{\bullet}\, k\{x_1\} \times \ldots \times \mathrm{B}^{\mathrm{cy}}_{\bullet}\, k\{x_r\}$. The Hochschild complex of $k[V]$ is the Moore complex of $k[\mathrm{B}^{\mathrm{cy}}_{\bullet}\, V]$, so

$$\begin{aligned} \mathrm{C}_{\bullet}(k[x_1, \ldots, x_r]) &\cong k[\mathrm{B}^{\mathrm{cy}}_{\bullet}\, k\{x_1, \ldots, x_r\}] \\ &\cong k[\mathrm{B}^{\mathrm{cy}}_{\bullet}\, k\{x_1\}] \boxtimes \ldots \boxtimes k[\mathrm{B}^{\mathrm{cy}}_{\bullet}\, k\{x_r\}] \\ &\cong \mathrm{C}_{\bullet}(k[x_1]) \boxtimes \ldots \boxtimes \mathrm{C}_{\bullet}(k[x_r]) \end{aligned}$$

where the right-hand side is chain homotopic to $\mathrm{C}_{\bullet}(k[x_1]) \otimes \ldots \otimes \mathrm{C}_{\bullet}(k[x_r])$ by the Eilenberg-Zilber theorem. Taking homology, we have

$$\mathrm{HH}_*(k[x_1, \ldots, x_r]) \cong \mathrm{H}_*(\mathrm{C}_{\bullet}(k[x_1]) \otimes \ldots \otimes \mathrm{C}_{\bullet}(k[x_r]))$$

In each degree, the cycles and the homology of $\mathrm{C}_{\bullet}(k[x_i])$ are free, hence projective, k-modules for each i, so we can apply the Künneth formula as in (1.8), to get:

$$\mathrm{HH}_*(k[x_1, \ldots, x_r]) \cong \mathrm{HH}_*(k[x_1]) \otimes \ldots \otimes \mathrm{HH}_*(k[x_r])$$

By Lemma 4.7, $\overline{\tau}_*\colon \Omega^*_{k[x_i]} \to \mathrm{HH}_*(k[x_i])$ is an isomorphism of graded algebras for each i, giving:

$$\mathrm{HH}_*(k[x_1, \ldots, x_r]) \cong \Omega^*_{k[x_1]} \otimes \ldots \otimes \Omega^*_{k[x_r]} \tag{4.3}$$

The last step is to prove that

$$\Omega^*_{k[x_1]} \otimes \ldots \otimes \Omega^*_{k[x_r]} \cong \Omega^*_{k[x_1, \ldots, x_r]}$$

This can be done by induction. The base case is trivial; assume, as inductive hypothesis, that $\Omega^*_{k[x_1]} \otimes \ldots \otimes \Omega^*_{k[x_{s-1}]} \cong \Omega^*_{k[x_1, \ldots, x_{s-1}]}$ for a given s. Then, using the graded algebra isomorphism in Corollary 4.4, we get:

$$\begin{aligned} \Omega^*_{k[x_1]} \otimes \ldots \otimes \Omega^*_{k[x_{s-1}]} \otimes \Omega^*_{k[x_s]} &\cong \Omega^*_{k[x_1, \ldots, x_{s-1}]} \otimes \Omega^*_{k[x_s]} \\ &\cong k[x_1, \ldots, x_{s-1}] \otimes \Lambda^*_k\, k\{x_1, \ldots, x_{s-1}\} \otimes k[x_s] \otimes \Lambda^*_k\, k\{x_s\} \\ &\cong k[x_1, \ldots, x_s] \otimes \Lambda^*_k(k\{x_1, \ldots, x_{s-1}\} \oplus k\{x_s\}) \\ &\cong k[x_1, \ldots, x_s] \otimes \Lambda^*_k\, k\{x_1, \ldots, x_s\} \\ &\cong \Omega^*_{k[x_1, \ldots, x_s]} \end{aligned} \tag{4.4}$$

In here we used the graded isomorphism

$$\Lambda^n(k\{x_1, \ldots, x_s\} \oplus k\{x_s\}) \cong \bigoplus_{p+q=n} \Lambda^p\, k\{x_1, \ldots, x_s\} \otimes \Lambda^q\, k\{x_s\}$$

Replacing in (4.3) according to (4.4), we obtain:

$$\Omega^*_{k[x_1, \ldots, x_s]} \cong \mathrm{HH}_*(k[x_1, \ldots, x_r])$$

where the isomorphism is conveyed by $\overline{\tau}_*$, as we wanted to prove. ■

Using Theorem 4.9, we can proceed to explicitly compute the Hochschild homology of a polynomial k-algebra in a finite number of variables.

Example 4.10. Consider $k\{x, y\}$ as a k-module, generating the polynomial algebra $k[x, y]$. By Theorem 4.9, the Hochschild homology of $k[x, y]$ is isomorphic to the algebra of differential forms. We get:

$$\mathrm{HH}_n(k[x, y]) \cong \begin{cases} k[x, y] & \text{if } n = 0 \\ k[x, y]\{dx, dy\} & \text{if } n = 1 \\ k[x, y]\{dx \wedge dy\} & \text{if } n = 2 \\ 0 & \text{if } n > 2 \end{cases}$$

where in degree n the generators are the generators of $\Omega^n_{k[x]}$, i.e., the generators in degree n of $\Omega^*_{k[x]}$.

Example 4.11. In general, for the polynomial algebra $A = k[x_1, \dots, x_r]$ in r variables, Ω^*_A has $\binom{r}{n}$ generators in degree n, of the form $dx_{i_1} \wedge \dots \wedge dx_{i_n}$, for $1 \leq i_1 < \dots < i_n \leq r$. We than have that

$$\mathrm{HH}_n(A) \cong A^{\oplus \binom{r}{n}}$$

In the next section we will reach an analogous result in log Hochschild homology.

4.2 The logarithmic case

We will now try to study the behaviour of the log Hochschild homology and the log Kähler differentials, or the log differentials n-forms, for a polynomial $\mathbb{Z}$-algebra.

Definition 4.12. Let (A, M, α) be a pre-log ring. We define the A-module of **log differential n-forms** as the exterior product $\Omega^n_{(A,M)} = \Lambda^n_A \, \Omega^1_{(A,M)}$. We will use the notation $\Omega^*_{(A,M)}$ for the graded algebra of the differential forms.

Just as in the previous section, we have the following result.

Proposition 4.13. *There is a graded algebra homomorphism:*

$$\overline{\omega}_* \colon \Omega^*_{(A,M)} \to \mathrm{HH}_*(A, M)$$

Proof. The proof is identical to the one already seen in Proposition 4.2, since Lemma 3.19 holds for the log Hochschild homology too. In this case, the relevant diagram is

$$\begin{tikzcd}
\left(\Omega^1_{(A,M)}\right)^{\otimes n} \arrow[r, "\overline{\omega}^{\otimes n}"] \arrow[d, "\wedge"'] & (\mathrm{HH}_1(A,M))^{\otimes n} \arrow[d, "\mathfrak{sh}"] \\
\Omega^n_{(A,M)} \arrow[r, dashed] & \mathrm{HH}_n(A,M)
\end{tikzcd} \tag{4.5}$$

where $\overline{\omega}$ is the A-module homomorphism described in (3.13). Again, we set $\overline{\omega}_n$ to be the map induced on $\Omega^n_{(A,M)}$ by $\overline{\omega}^{\otimes n}$. ■

Except in degree 1, an isomorphism between the log differential n-forms and the n-th log Hochschild homology of a pre-log ring can generally not be found (as, generally, there is not an isomorphism between the Kähler differentials and the Hochschild homology of a k-algebra). We will, however, focus on some particular cases.

As our first example, we can consider the pre-log ring (A, M, α), with $A = \mathbb{Z}[x]$, $M = \langle x \rangle$ and α the inclusion; in this case, $A = \mathbb{Z}[M]$. We will need the following lemmas.

Lemma 4.14. *Given the canonical pre-log structure* $(\mathbb{Z}[M], M)$ *of a commutative monoid* M*, there is a graded algebra isomorphism*

$$\mathrm{HH}_*(\mathbb{Z}[M], M) \cong \mathbb{Z}[M] \otimes \mathrm{H}_*(\mathbb{Z}[\mathrm{B}_\bullet\, M^{\mathrm{gp}}])$$

Proof. Computing the log Hochschild complex of $(\mathbb{Z}[M], M)$, we get, from the definition,

$$\begin{tikzcd}
\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\, M] \arrow[r, "\overline{\rho}"] \arrow[d, "S^1_\bullet \otimes \overline{\alpha}"'] & \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\, M] \arrow[d, "\overline{\xi}"] \\
\mathrm{C}_\bullet(\mathbb{Z}[M]) \arrow[r, "\overline{\psi}"'] & \mathrm{C}_\bullet(\mathbb{Z}[M], M)
\end{tikzcd}$$

The map $S^1_\bullet \otimes \overline{\alpha}$ is now the identity; since the square is a pushout square, we obtain an isomorphism

$$\mathrm{C}_\bullet(\mathbb{Z}[M], M) \cong \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\, M] \tag{4.6}$$

We will use the isomorphism $\mathrm{B}^{\mathrm{rep}}\, M \cong M \times \mathrm{B}M^{\mathrm{gp}}$ described in (3.3) to get

$$\begin{aligned}
\mathrm{HH}_*(\mathbb{Z}[M], M) \cong \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\, M]) &\cong \mathrm{H}_*(\mathbb{Z}[M \times \mathrm{B}_\bullet\, M^{\mathrm{gp}}]) \\
&\cong \mathrm{H}_*(\mathbb{Z}[M] \otimes \mathbb{Z}[\mathrm{B}_\bullet\, M^{\mathrm{gp}}]) \\
&\cong \mathbb{Z}[M] \otimes \mathrm{H}_*(\mathbb{Z}[\mathrm{B}_\bullet\, M^{\mathrm{gp}}]) \qquad (4.7)
\end{aligned}$$

as we wanted to prove. ■

Lemma 4.15. *Let P be a commutative monoid. Then there is an isomorphism of graded algebras:*

$$\mathrm{H}_*(\mathbb{Z}[\mathrm{B}_\bullet P]) \cong \mathrm{Tor}_*^{\mathbb{Z}[P]}(\mathbb{Z},\mathbb{Z})$$

Proof. Consider the sequence

$$\ldots \to F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} \mathbb{Z}$$

where, for each n, $F_n = \mathbb{Z}[P]^{\otimes n+1}$ and ∂ is defined on the generators as

$$\begin{aligned} \partial\colon F_n &\to F_{n-1} \\ x_0 \otimes \ldots \otimes x_n &\mapsto \sum_{i=0}^{n-1} (-1)^i x_0 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x_n \\ &\quad + (-1)^n \epsilon(x_n) x_0 \otimes \ldots \otimes x_{n-1} \end{aligned}$$

with augmentation $\epsilon\colon F_0 \to Z$, $\sum_i n_i x_i \mapsto \sum_i n_i$, for $x_i \in P$, $n_i \in \mathbb{Z}$.
$F_\bullet$ is actually a free resolution of $\mathbb{Z}$, called **bar resolution**, in terms of $\mathbb{Z}[P]$-modules (the multiplication takes place on the first tensor factor); a proof for this can be found in [Mac Lane, 1963, Chapter IV, Theorem 5.1]. In order to compute $\mathrm{Tor}_*^{\mathbb{Z}[P]}(\mathbb{Z},\mathbb{Z})$, we apply $\mathbb{Z} \otimes_{\mathbb{Z}[P]} -$ to $F_\bullet$:

$$\begin{tikzcd}
\ldots \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P]^{\otimes 3} \arrow[r, "\partial"] \arrow[d, "\cong"'] & \mathbb{Z} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P]^{\otimes 2} \arrow[r, "\partial"] \arrow[d, "\cong"] & \mathbb{Z} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P] \arrow[d, "\cong"] \\
\ldots \arrow[r] & \mathbb{Z}[P]^{\otimes 2} \arrow[r, "\partial'"'] & \mathbb{Z}[P] \arrow[r, "\partial'"'] & \mathbb{Z}
\end{tikzcd}$$

We see that, via isomorphism

$$\begin{aligned} \mathbb{Z}[P]^{\otimes n} &\xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P]^{\otimes n+1} \\ x_1 \otimes \ldots \otimes x_n &\mapsto 1 \otimes 1 \otimes x_1 \otimes \ldots \otimes x_n \\ \epsilon(x_0) x_1 \otimes \ldots \otimes x_n &\mapsfrom 1 \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n \end{aligned}$$

the map ∂ induces the map ∂':

$$\begin{aligned} \partial'\colon \mathbb{Z}[P]^{\otimes n} &\to \mathbb{Z}[P]^{\otimes n-1} \\ x_1 \otimes \ldots \otimes x_n &\mapsto \epsilon(x_1) x_2 \otimes \ldots \otimes x_n \\ &\quad + \sum_{i=1}^{n-1} x_1 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x^n \\ &\quad + (-1)^n \epsilon(x_n) x_1 \otimes \ldots \otimes x_{n-1} \end{aligned}$$

making the lower line in the previous diagram indeed the Moore complex of $\mathbb{Z}[\mathrm{B}_\bullet P]$. Therefore, $\mathrm{Tor}_*^{\mathbb{Z}[P]}(\mathbb{Z},\mathbb{Z}) \cong \mathrm{H}_*(\mathbb{Z}[\mathrm{B}_\bullet P])$. ■

We then get the following expression of the log Hochschild homology of $(\mathbb{Z}[M], M)$.

Proposition 4.16. *Let M be a commutative monoid. There is a graded algebra isomorphism:*

$$\mathrm{HH}_*(\mathbb{Z}[M], M) \cong \mathbb{Z}[M] \otimes \mathrm{Tor}_*^{\mathbb{Z}[M^{\mathrm{gp}}]}(\mathbb{Z}, \mathbb{Z})$$

Proof. Immediate, from Lemma 4.14 and Lemma 4.15. ■

With the aid of Proposition 4.16, we are now ready to compare the log differential forms to the log Hochschild homology of the pre-log ring $(\mathbb{Z}[x], \langle x \rangle)$.

Proposition 4.17. *There is an isomorphism of graded algebras*

$$\overline{\omega}_* \colon \Omega^*_{(\mathbb{Z}[x], \langle x \rangle)} \xrightarrow{\sim} \mathrm{HH}_*(\mathbb{Z}[x], \langle x \rangle)$$

where $\overline{\omega}_*$ is the graded algebra homomorphism from Proposition 4.13.

Proof. We will start from the log differential forms. From Example 3.11, we have that $\Omega^1_{(\mathbb{Z}[x], \langle x \rangle)} \cong \mathbb{Z}[x]\{d \log x\}$. Hence

$$\Omega^n_{(\mathbb{Z}[x], \langle x \rangle)} \cong \Lambda^n_{\mathbb{Z}[x]} \Omega^1_{(\mathbb{Z}[x], \langle x \rangle)} \cong \begin{cases} \mathbb{Z}[x] & \text{if } n = 0 \\ \mathbb{Z}[x]\{d \log x\} \cong \mathbb{Z}[x] & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \tag{4.8}$$

where for the last case we used that, as a general fact, $\Lambda^n_A A \cong 0$ for $n \geq 2$. Our aim is to find an isomorphism between the log differential forms and the log Hochschild homology of $(\mathbb{Z}[x], \langle x \rangle)$. Applying Proposition 4.16 for $M = \langle x \rangle$, we get a graded algebra isomorphism:

$$\mathrm{HH}_*(\mathbb{Z}[x], \langle x \rangle) \cong \mathbb{Z}[x] \otimes \mathrm{Tor}_*^{\mathbb{Z}\left[x, x^{-1}\right]}(\mathbb{Z}, \mathbb{Z})$$

We are then interested in finding an explicit expression for $\mathrm{Tor}_*^{\mathbb{Z}\left[x, x^{-1}\right]}(\mathbb{Z}, \mathbb{Z})$. We will find a free resolution of $\mathbb{Z}$ in easier terms than the bar resolution described in Lemma 4.15. The sequence

$$0 \longrightarrow \mathbb{Z}\left[x, x^{-1}\right] \xrightarrow{f_1} \mathbb{Z}\left[x, x^{-1}\right] \xrightarrow{f_0} \mathbb{Z} \tag{4.9}$$

with homomorphisms defined by $f_1(p(x)) = (x-1)p(x)$ and $f_0(x) = 1$, is a free resolution of $\mathbb{Z}$ in terms of $\mathbb{Z}\left[x, x^{-1}\right]$-modules. In fact, f_0 is certainly surjective and $\mathbb{Z} \cong \mathbb{Z}\left[x, x^{-1}\right]/(x-1)$; as for f_1, to assume $0 = f_1(p(x)) = (x-1)p(x)$ for $p(x) = a_n x^n + \ldots + a_N x^N$ yields

$$-a_n x^n - \ldots - a_N x^N + a_n x^{n+1} + \ldots + a_N x^{N+1} = 0$$

so $a_N = 0$ and, by induction, $p(x) = 0$, making f_1 injective.
To get $\mathrm{Tor}_*^{\mathbb{Z}[x,x^{-1}]}(\mathbb{Z},\mathbb{Z})$, we apply $\mathbb{Z} \otimes_{\mathbb{Z}[x,x^{-1}]} -$ to (4.9), thus getting

$$0 \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[x,x^{-1}]} \mathbb{Z}\big[x, x^{-1}\big] \xrightarrow{\mathrm{id} \otimes f_1} \mathbb{Z} \otimes_{\mathbb{Z}[x,x^{-1}]} \mathbb{Z}\big[x, x^{-1}\big] \longrightarrow 0 \tag{4.10}$$

Under isomorphism

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}[x,x^{-1}]} \mathbb{Z}\big[x, x^{-1}\big] &\xrightarrow{\sim} \mathbb{Z} \\ s \otimes f(x) &\mapsto s \cdot f(1) \\ s \otimes 1 &\leftarrow\!\shortmid s \end{aligned}$$

the sequence in (4.10) becomes

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the middle map sends

$$1 \overset{\sim}{\longmapsto} 1 \otimes 1 \mapsto 1 \otimes (x-1) \overset{\sim}{\longmapsto} 1 - 1 = 0$$

Hence, taking homology, we have

$$\mathrm{Tor}_n^{\mathbb{Z}[x,x^{-1}]}(\mathbb{Z},\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \tag{4.11}$$

providing the sought expression for the log Hochschild homology of $(\mathbb{Z}[x], \langle x \rangle)$:

$$\mathrm{HH}_n(\mathbb{Z}[x], \langle x \rangle) \cong \mathbb{Z}[x] \otimes \mathrm{Tor}_n^{\mathbb{Z}[x,x^{-1}]}(\mathbb{Z},\mathbb{Z}) \cong \begin{cases} \mathbb{Z}[x] & \text{if } n = 0 \\ \mathbb{Z}[x] & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \tag{4.12}$$

Comparing the expressions in (4.12) with the differential forms in (4.8), we get a degreewise isomorphism

$$\Omega^*_{(\mathbb{Z}[x],\langle x \rangle)} \cong \mathrm{HH}_*(\mathbb{Z}[x], \langle x \rangle)$$

We will show that this isomorphism is induced by the homomorphism of graded algebras $\overline{\omega}_* \colon \Omega^*_{(\mathbb{Z}[x],\langle x \rangle)} \to \mathrm{HH}_*(\mathbb{Z}[x], \langle x \rangle)$ from Proposition 4.13. This is trivial in degree 0 and in degree greater than 1, while in degree 1 we get:

$$\begin{aligned} Z[x]\{d \log x\} \cong \Omega^1_{(Z[x],\langle x \rangle)} &\xrightarrow{\overline{\omega}} \mathrm{HH}_1(Z[x], \langle x \rangle) \\ d \log x &\mapsto (1 \otimes 1) \otimes (1 \otimes x) \end{aligned}$$

Via the isomorphism

$$\begin{tikzcd} \mathrm{HH}_1(\mathbb{Z}[x], \langle x \rangle) \arrow[r, "\sim"] & \mathrm{H}_1(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet \langle x \rangle]) \arrow[r, "\sim"] & \mathbb{Z}[x] \otimes \mathrm{H}_1(\mathbb{Z}[\mathrm{B}_\bullet \langle x \rangle^{\mathrm{gp}}]) \arrow[r, "\sim"] & \mathbb{Z}[x] \\ (1 \otimes 1) \otimes (1 \otimes x) \arrow[r, mapsto] & 1 \otimes x \arrow[r, mapsto] & 1 \otimes [x] \arrow[r, mapsto] & 1 \end{tikzcd}$$

in which we underline that the class of x is the class of 1 in $\mathrm{H}_1(\mathbb{Z}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]) \cong \mathbb{Z}$, we have that $\overline{\varpi}$ maps the generator $d\log x$ of $\Omega^1_{(Z[x],\langle x\rangle)}$ to the generator 1 of $\mathrm{HH}_1(Z[x],\langle x\rangle)$. Since, moreover, $\overline{\varpi}_*$ is a homomorphism of graded algebras, then it is actually an isomorphism of graded algebras. ■

We can extend the last result to polynomial algebras in more variables.

Theorem 4.18. *Let $M = \langle x_1,\dots,x_r\rangle$ be the commutative monoid generated by r elements. The log Hochschild homology of $(\mathbb{Z}[M], M, \alpha)$, where $\alpha\colon M \to (\mathbb{Z}[M], \cdot\,)$ is the inclusion, is computed as follows:*

$$\mathrm{HH}_n(\mathbb{Z}[M], M) \cong \mathbb{Z}[M]^{\oplus\binom{r}{n}}$$

Proof. We will proceed inductively. The base case $r = 1$ is verified in (4.12). Assume now that the statement is true for $M' = \langle x_1,\dots,x_{r-1}\rangle$; after isomorphisms

$$\mathbb{Z}[x_r]^{\otimes n} \otimes_{\mathbb{Z}[x_r]^{\otimes n}} \mathbb{Z}[x_r] \otimes \mathbb{Z}[\langle x_r\rangle^{\mathrm{gp}}]^{\otimes n-1} \cong \mathbb{Z}[x_r] \otimes \mathbb{Z}[\langle x_r\rangle^{\mathrm{gp}}]^{\otimes n-1}$$
$$\mathbb{Z}[M']^{\otimes n} \otimes_{\mathbb{Z}[M']^{\otimes n}} \mathbb{Z}[M'] \otimes \mathbb{Z}\big[M'^{\mathrm{gp}}\big]^{\otimes n-1} \cong \mathbb{Z}[M'] \otimes \mathbb{Z}\big[M'^{\mathrm{gp}}\big]^{\otimes n-1}$$

the log Hochschild complexes of $(\mathbb{Z}[x_r],\langle x_r\rangle)$ and $(\mathbb{Z}[M'], M')$ are, respectively:

$$\mathrm{C}_\bullet(\mathbb{Z}[x_r],\langle x_r\rangle):\ \ldots \to \mathbb{Z}[x_r]\otimes\mathbb{Z}[\langle x_r\rangle^{\mathrm{gp}}]^{\otimes 2} \to \mathbb{Z}[x_r]\otimes\mathbb{Z}[\langle x_r\rangle^{\mathrm{gp}}] \to \mathbb{Z}[x_r] \to 0$$
$$\mathrm{C}_\bullet(\mathbb{Z}[M'], M'):\ \ldots \to \mathbb{Z}[M']\otimes\mathbb{Z}\big[M'^{\mathrm{gp}}\big]^{\otimes 2} \to \mathbb{Z}[M']\otimes\mathbb{Z}\big[M'^{\mathrm{gp}}\big] \to \mathbb{Z}[M'] \to 0$$

The log Hochschild complex of $(\mathbb{Z}[M], M)$ is the cartesian (degreewise) product of chain complexes $\mathrm{C}_\bullet(\mathbb{Z}[M'], M') \boxtimes \mathrm{C}_\bullet(\mathbb{Z}[x_r],\langle x_r\rangle)$. The Eilenberg-Zilber theorem states that its homology is isomorphic to the homology of the tensor product of chain complexes $\mathrm{C}_\bullet(\mathbb{Z}[M'], M') \otimes \mathrm{C}_\bullet(\mathbb{Z}[x_r],\langle x_r\rangle)$. Since, in each degree, the cycles and the homology of $\mathrm{C}_\bullet(\mathbb{Z}[x_r],\langle x_r\rangle)$ are free (hence projective) $\mathbb{Z}$-modules, then we can apply the Künneth formula as in (1.8):

$$\bigoplus_{p+q=n} \mathrm{HH}_p(\mathbb{Z}[M'], M') \otimes_{\mathbb{Z}} \mathrm{HH}_q(\mathbb{Z}[x_r],\langle x_r\rangle) \cong \mathrm{HH}_n(\mathbb{Z}[M], M)$$

where the isomorphism is induced by the homology product $\mathfrak{p}$ as described in (3.15). Since $\mathrm{HH}_q(\mathbb{Z}[x_r],\langle x_r\rangle)$ is nontrivial only for $q = 0, 1$, we have:

$$\begin{aligned}\mathrm{HH}_n(\mathbb{Z}[M], M) &\cong \Big(\mathbb{Z}[M']^{\oplus\binom{r-1}{n-1}} \otimes \mathbb{Z}[x_r]^{\oplus\binom{1}{1}}\Big) \oplus \Big(\mathbb{Z}[M']^{\oplus\binom{r-1}{n}} \otimes \mathbb{Z}[x_r]^{\oplus\binom{1}{0}}\Big)\\ &\cong \mathbb{Z}[M]^{\oplus\binom{r-1}{n-1}+\binom{r-1}{n}}\\ &\cong \mathbb{Z}[M]^{\oplus\binom{r}{n}}\end{aligned}$$

■

Remark 4.19. In Theorem 4.18 we used a particular instance of the following general fact. If M and N are arbitrary commutative monoids, there is an isomorphism

$$\mathrm{B}^{\mathrm{rep}}_\bullet(M \times N) \cong \mathrm{B}^{\mathrm{rep}}_\bullet(M) \times \mathrm{B}^{\mathrm{rep}}_\bullet(N)$$

of simplicial commutative monoids, and hence an isomorphism

$$\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet(M \times N)] \cong \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet(M)] \boxtimes \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet(N)] \tag{4.13}$$

of simplicial commutative rings.

It is now natural to ask whether the homomorphism of graded algebras $\overline{\omega}_*$ described in Proposition 4.13 is an isomorphism if the pre-log ring considered is of the form $(\mathbb{Z}[M], M)$, with M as in Theorem 4.18. This will be proved in the following theorem, which also provides an alternative proof of Theorem 4.18 itself.

Theorem 4.20. *Let $M = \langle x_1, \dots, x_r\rangle$ be the commutative monoid generated by r elements. There is a graded algebra isomorphism*

$$\overline{\omega}_* \colon \Omega^*_{(\mathbb{Z}[M],M)} \xrightarrow{\sim} \mathrm{HH}_*(\mathbb{Z}[M], M)$$

Proof. The proof follows the one of Theorem 4.9. We use (4.6) and (4.13) to get

$$\begin{aligned}\mathrm{C}_\bullet(\mathbb{Z}[x_1,\dots,x_r], \langle x_1,\dots,x_r\rangle) &\cong \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x_1,\dots,x_r\rangle] \\ &\cong \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x_1\rangle] \boxtimes \dots \boxtimes \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x_r\rangle] \\ &\cong \mathrm{C}_\bullet(\mathbb{Z}[x_1], \langle x_1\rangle) \boxtimes \dots \boxtimes \mathrm{C}_\bullet(\mathbb{Z}[x_r], \langle x_r\rangle)\end{aligned}$$

the latter being chain homotopic to the usual tensor product of chain complexes, by the Eilenberg-Zilber theorem. Taking homology and applying the Künneth formula, we get:

$$\mathrm{HH}_*(\mathbb{Z}[x_1,\dots,x_r], \langle x_1,\dots,x_r\rangle) \cong \mathrm{HH}_*(\mathbb{Z}[x_1], \langle x_1\rangle) \otimes \dots \otimes \mathrm{HH}_*(\mathbb{Z}[x_r], \langle x_r\rangle)$$

By Proposition 4.17, $\overline{\omega}_* \colon \Omega^*_{(\mathbb{Z}[x_i],\langle x_i\rangle)} \to \mathrm{HH}_*(\mathbb{Z}[x_i], \langle x_i\rangle)$ is an isomorphism of graded algebras for each i; we then have:

$$\mathrm{HH}_*(\mathbb{Z}[x_1,\dots,x_r], \langle x_1,\dots,x_r\rangle) \cong \Omega^*_{(\mathbb{Z}[x_1],\langle x_1\rangle)} \otimes \dots \otimes \Omega^*_{(\mathbb{Z}[x_r],\langle x_r\rangle)}$$

We now want to show that there is a graded algebra isomorphism

$$\Omega^*_{(\mathbb{Z}[x_1],\langle x_1\rangle)} \otimes \dots \otimes \Omega^*_{(\mathbb{Z}[x_r],\langle x_r\rangle)} \cong \Omega^*_{(\mathbb{Z}[x_1,\dots,x_r],\langle x_1,\dots,x_r\rangle)} \tag{4.14}$$

In order to do so, we will first compute $\Omega^1_{(\mathbb{Z}[x_1,\dots,x_r],\langle x_1,\dots,x_r\rangle)}$. From the pushout diagram

$$\begin{tikzcd}
\mathbb{Z}[M] \otimes_{\mathbb{Z}[M]} \Omega^1_{\mathbb{Z}[M]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathbb{Z}[M] \otimes M^{\mathrm{gp}} \cong \mathbb{Z}[M] \arrow[d, "\overline{\phi}"] \\
\mathbb{Z}[M] \arrow[r, "\overline{\psi}"'] & \Omega^1_{(\mathbb{Z}[M],M)}
\end{tikzcd}$$

we obtain that $\Omega^1_{(\mathbb{Z}[x_1,\ldots,x_r],\langle x_1,\ldots,x_r\rangle)}$ is the $\mathbb{Z}[x_1,\ldots,x_r]$-module generated by elements dx_i and $d\log x_i$, for $1 \leq i \leq r$, subject to the relation $dx_i = x_i d\log x_i$. Hence

$$\begin{aligned}\Omega^1_{(\mathbb{Z}[x_1,\ldots,x_r],\langle x_1,\ldots,x_r\rangle)} &\cong \mathbb{Z}[x_1,\ldots,x_r]\{d\log x_1,\ldots,d\log x_r\}\\ &\cong \mathbb{Z}[x_1,\ldots,x_r]\otimes\mathbb{Z}\{d\log x_1,\ldots,d\log x_r\}\end{aligned}$$

Using the same argument as in Corollary 4.4, we get the graded algebra isomorphism

$$\Omega^*_{(\mathbb{Z}[x_1,\ldots,x_r],\langle x_1,\ldots,x_r\rangle)} \cong \mathbb{Z}[x_1,\ldots,x_r]\otimes_{\mathbb{Z}}\Lambda^*_{\mathbb{Z}}\,\mathbb{Z}\{d\log x_1,\ldots,d\log x_r\}$$

and applying the same inductive argument as in (4.4), we obtain the graded algebra isomorphism (4.14). Therefore,

$$\Omega^*_{(\mathbb{Z}[x_1,\ldots,x_r],\langle x_1,\ldots,x_r\rangle)} \cong \mathrm{HH}_*(\mathbb{Z}[x_1,\ldots,x_r],\langle x_1,\ldots,x_r\rangle)$$

where the isomorphism of graded algebras is conveyed by $\overline{\omega}_*$. ■

Chapter 5

A long exact sequence in log Hochschild homology

In Theorem 4.18 we found an explicit expression for the log Hochschild homology of a log ring $(A, M) = (\mathbb{Z}[x_1, \ldots, x_r], \langle x_1, \ldots, x_r\rangle)$, i.e., for the case in which A is the polynomial algebra in the variables given by the generators of M. Going further in our analysis, we are now interested in computing the log Hochschild homology when A is not the monoid ring of M. Specifically, let A be a commutative ring and let a be an element of A such that the map $\mathbb{Z}[x] \to A$, $x \mapsto a$, makes A a flat $\mathbb{Z}[x]$-algebra. We will show that the log Hochschild homology of $(A, \langle x\rangle)$ fits in the long exact sequence:

$$\begin{gathered} \ldots \longrightarrow \mathrm{HH}_i(A) \longrightarrow \mathrm{HH}_i(A, \langle x\rangle) \longrightarrow \mathrm{HH}_{i-1}(A/(a)) \longrightarrow \mathrm{HH}_{i-1}(A) \longrightarrow \ldots \\ \ldots \longrightarrow \mathrm{HH}_1(A, \langle x\rangle) \longrightarrow \mathrm{HH}_0(A/(a)) \longrightarrow \mathrm{HH}_0(A) \longrightarrow \mathrm{HH}_0(A, \langle x\rangle) \longrightarrow 0 \end{gathered}$$

5.1 A long exact sequence

Consider the commutative monoid $\langle x\rangle = \{1, x, x^2, \ldots\}$; its group completion $\gamma\colon \langle x\rangle \to \langle x\rangle^{\mathrm{gp}}$ is the inclusion. We recall from (3.3) the isomorphism of simplicial commutative monoids:

$$\begin{gathered} \mathrm{B}^{\mathrm{rep}}_{\bullet}\langle x\rangle \xrightarrow{\sim} \langle x\rangle \times \mathrm{B}_{\bullet}\langle x\rangle^{\mathrm{gp}} \\ (x^i, x^i(g_1 \cdots g_q)^{-1}, g_1, \ldots, g_q) \longmapsto (x^i, g_1, \ldots, g_q) \end{gathered} \tag{5.1}$$

Let now $\widehat{\mathrm{B}}^{\mathrm{rep}}_{\bullet}\langle x\rangle$ be the simplicial commutative monoid defined degreewise by

$$\widehat{\mathrm{B}}^{\mathrm{rep}}_{q}\langle x\rangle := \big\{(x^i, g_1, \ldots, g_q) \in \langle x\rangle \times (\langle x\rangle^{\mathrm{gp}})^q \mid i = 0 \Rightarrow (g_1, \ldots, g_q) = (1, \ldots, 1)\big\}$$

with face and degeneracy maps defined as those of the replete bar construction in (3.5). We see that $\widehat{\mathrm{B}}^{\mathrm{rep}}_{\bullet}\langle x\rangle$ is a then a simplicial commutative submonoid of $\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle$. Since $x^{i_1}\cdots x^{i_q} = 1$ for $x^{i_j} \in \langle x\rangle$ implies $i_j = 0$ for every j, and $\gamma(1) = 1$, the repletion map $\rho\colon \mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle \to \mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle$ described in (3.4) then factors as:

$$\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle \xrightarrow{\widehat{\rho}} \widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle \hookrightarrow \mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle \tag{5.2}$$

where $\widehat{\rho}$ is defined in the same way as ρ.

We will use the following result.

Lemma 5.1. *The map* $\widehat{\rho}\colon \mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle \to \widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle$ *as defined in (5.2) induces an isomorphism in homology:*

$$\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]) \xrightarrow{\sim} \mathrm{H}_*(\mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle\,]) \tag{5.3}$$

Furthermore, the induced map of commutative simplicial rings

$$\mathrm{C}_\bullet(A) \cong \mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle] \longrightarrow \mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle\,] \tag{5.4}$$

induces an isomorphism in homology.

Proof. The repletion maps ρ and $\widehat{\rho}$ are chain maps, thus they induce maps of homology groups. We have $\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]) = \mathrm{HH}_*(\mathbb{Z}[x])$; the isomorphism (5.1) gives $\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle]) = \mathrm{H}_*(\mathbb{Z}[\langle x\rangle \times \mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}])$. As for $\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle$, we see that in degrees higher than 0 its homology coincides with

$$\mathrm{H}_*(\mathbb{Z}[x\cdot\langle x\rangle \times \mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]) \cong x\cdot\mathbb{Z}[x] \otimes \mathrm{H}_*(\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}})$$

By Lemma 4.15, $\mathrm{H}_*(\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}) \cong \mathrm{Tor}_*^{\mathbb{Z}[x,x^{-1}]}(\mathbb{Z},\mathbb{Z})$, which we computed in (4.11) to be isomorphic to $\mathbb{Z}$ in degrees 0 and 1, while vanishing in higher degrees. In degree 0 the map induced in homology by $\widehat{\rho}$ is clearly an isomorphism; in degree 1 the generator x of $\mathrm{HH}_1([x]) \cong \mathbb{Z}[x]$ is sent to $x \cdot d\log x$, the generator of $\mathrm{H}_*(\,\mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle\,]\,) \cong x\cdot\mathbb{Z}[x]\{d\log x\}$, giving, again, an isomorphism.

About the second statement, we will again use Theorem 3.21. We get, for the left-hand side of (5.4), the spectral sequence

$$E^2_{p,q} = \Big[\mathrm{Tor}_p^{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle])}(\mathrm{HH}_*(A), \mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]))\Big]_q$$
$$\Rightarrow \mathrm{H}_{p+q}\big((\mathrm{C}(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\langle x\rangle]} \mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\langle x\rangle])_\bullet\big)$$

and, for the right-hand side of (5.4), the spectral sequence

$$E'^2_{p,q} = \Big[\mathrm{Tor}_p^{\mathrm{H}_*(\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle])}\Big(\mathrm{HH}_*(A), \mathrm{H}_*(\mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle\,])\Big)\Big]_q$$
$$\Rightarrow \mathrm{H}_{p+q}\Big((\mathrm{C}(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\langle x\rangle]} \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}\langle x\rangle\,])_\bullet\Big)$$

By (5.3), the map $\widehat{\rho}$ induces an isomorphism $\widehat{\rho}^{\,2}\colon E^2 \to {E'}^2$, so the two spectral sequences agree in every term, yielding (see e.g. [Mac Lane, 1963, Chapter XI, Theorem 3.4]) the isomorphism

$$\mathrm{H}_*\big((\mathrm{C}(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\langle x\rangle]} \mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\langle x\rangle])_\bullet\big) \cong \mathrm{H}_*\big((\mathrm{C}(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}\langle x\rangle]} \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}\langle x\rangle])_\bullet\big)$$

as we wanted to prove. ■

With the identification $\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle \cong \langle x\rangle \times \mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}$ as in (5.1), we now let $\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle$ act on $\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}$ by

$$(x^i, g_1, \ldots, g_q) \cdot (h_1, \ldots, h_q) = \begin{cases} (h_1, \ldots, h_q) & \text{if } i = 0 \\ (1, \ldots, 1) & \text{if } i \geq 1 \end{cases}$$

and on $\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle$ by the usual componentwise multiplication. We then consider the following map of simplicial sets defined degreewise as:

$$\begin{aligned} \sigma\colon\ \mathrm{B}^{\mathrm{rep}}_q\langle x\rangle &\to \mathrm{B}_q\langle x\rangle^{\mathrm{gp}} \\ (x^i, g_1, \ldots, g_q) &\mapsto \begin{cases} (g_1, \ldots, g_q) & \text{if } i = 0 \\ (1, \ldots, 1) & \text{if } i \geq 1 \end{cases} \end{aligned}$$

We see that σ respects the action of $\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle$. Since all the elements in $\widehat{\mathrm{B}}^{\mathrm{rep}}_q\langle x\rangle \subseteq \mathrm{B}^{\mathrm{rep}}_q\langle x\rangle$ are sent to $(1, \ldots, 1) \in \mathrm{B}_q\langle x\rangle^{\mathrm{gp}}$ by σ, this induces a well-defined map $\widehat{\sigma}$ from the quotient of simplicial subsets $\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle/\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle$ to $\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}$:

$$\begin{aligned} \widehat{\sigma}\colon\ \mathrm{B}^{\mathrm{rep}}_q\langle x\rangle/\widehat{\mathrm{B}}^{\mathrm{rep}}_q\langle x\rangle &\to \mathrm{B}_q\langle x\rangle^{\mathrm{gp}} \\ [x^i, g_1, \ldots, g_q] &\mapsto \sigma(x^i, g_1, \ldots, g_q) \end{aligned}$$

which is moreover an isomorphism of simplicial sets, with inverse

$$\begin{aligned} \widehat{\sigma}^{-1}\colon\ \mathrm{B}_q\langle x\rangle^{\mathrm{gp}} &\to \mathrm{B}^{\mathrm{rep}}_q\langle x\rangle/\widehat{\mathrm{B}}^{\mathrm{rep}}_q\langle x\rangle \\ (g_1, \ldots, g_q) &\mapsto [1, g_1, \ldots, g_q] \end{aligned}$$

As a general fact, given a simplicial set $X = X_\bullet$ and a simplicial subset $A = A_\bullet \subseteq X_\bullet$, there is a short exact sequence of simplicial abelian groups

$$0 \longrightarrow \mathbb{Z}[A] \longrightarrow \mathbb{Z}[X] \longrightarrow \widetilde{\mathbb{Z}}[X/A] \longrightarrow 0$$

where $\widetilde{\mathbb{Z}}[X/A] = \mathbb{Z}[X/A]/\mathbb{Z}\{A/A\}$ is the degreewise quotient of $\mathbb{Z}[X/A]$ by the subgroup $\mathbb{Z}\{A/A\} \cong \mathbb{Z}$ (see e.g. [Hatcher, 2002]). In our case, using the isomorphism $\widehat{\sigma}$, we get a short exact sequence of simplicial $\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]$-modules:

$$0 \longrightarrow \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle] \longrightarrow \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle] \longrightarrow \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}] \longrightarrow 0 \tag{5.5}$$

Since A is a flat $\mathbb{Z}[x]$-algebra, $A^{\otimes n}$ is a flat $\mathbb{Z}[x]^{\otimes n}$-algebra for every n (again, we use [Eisenbud, 1995, Theorem A6.6]). So, we get from (5.5) a short exact sequence of simplicial abelian groups

$$\begin{aligned} 0 \longrightarrow \mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle] \longrightarrow \mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle] \longrightarrow \\ \longrightarrow \mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}] \longrightarrow 0 \quad (5.6) \end{aligned}$$

We will compute the homology of the third term in (5.6) in the following lemma.

Lemma 5.2. *For every n, there is an isomorphism in homology:*

$$\mathrm{H}_n\Big(\mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]\Big) \cong \mathrm{HH}_{n-1}(A/(a))$$

Proof. We start by claiming that there is an isomorphism of simplicial abelian groups

$$\mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}] \cong \mathrm{C}_\bullet(A/(a)) \boxtimes \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}] \qquad (5.7)$$

In fact, we recall the isomorphism

$$\widetilde{\mathbb{Z}}[\mathrm{B}_q\langle x\rangle^{\mathrm{gp}}] \cong \frac{\mathbb{Z}[(\langle x\rangle \times (\langle x\rangle^{\mathrm{gp}})^p)/\sim]}{\mathbb{Z}\{(1,\dots,1)\}}$$

where $(x^i, g_1, \dots, g_q) \sim (1, \dots, 1)$ if $i > 0$. Let

$$\begin{gathered} (a_0 \otimes \dots \otimes a_n) \in C_n(A) \\ (x^{i_0} \otimes \dots \otimes x^{i_n}) \in \mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_n\langle x\rangle] \\ (1 \otimes g_1 \otimes \dots \otimes g_n) \in \widetilde{\mathbb{Z}}[\mathrm{B}_n\langle x\rangle^{\mathrm{gp}}] \end{gathered}$$

and assume $i_s > 0$ for some s. Then, in $\mathrm{C}_n(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_n\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_n\langle x\rangle^{\mathrm{gp}}]$, we have

$$\begin{aligned} &(a_0 a^{i_0} \otimes \dots \otimes a_n a^{i_n}) \otimes (1 \otimes g_1 \otimes \dots \otimes g_n) \\ &\qquad = (a_0 \otimes \dots \otimes a_n) \cdot (a^{i_0} \otimes \dots \otimes a^{i_n}) \otimes (1 \otimes g_1 \otimes \dots \otimes g_n) \\ &\qquad = (a_0 \otimes \dots \otimes a_n) \otimes (x^{i_0+\dots+i_n} \otimes x^{i_1} \otimes \dots \otimes x^{i_n}) \cdot (1 \otimes g_1 \otimes \dots \otimes g_n) \\ &\qquad = (a_0 \otimes \dots \otimes a_n) \otimes (x^{i_0+\dots+i_n} \otimes x^{i_1} g_1 \otimes \dots \otimes x^{i_n} g_n) \\ &\qquad = (a_0 \otimes \dots \otimes a_n) \otimes (1 \otimes 1 \otimes \dots \otimes 1) \\ &\qquad = (a_0 \otimes \dots \otimes a_n) \otimes 0 \end{aligned}$$

So we can see that we can quotient out the elements in $\mathrm{C}_\bullet((a))$ via $\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_n\langle x\rangle]$, obtaining the isomorphism in (5.7).
So, the homology of $\mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ is isomorphic to the homology of $\mathrm{C}_\bullet(A/(a)) \boxtimes \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$. By the Eilenberg-Zilber theorem we can compute the homology of $\mathrm{C}_\bullet(A/(a)) \otimes \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ instead. The homology of $\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ is

computed as the reduced homology of $\mathbb{Z}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$; we then get $\mathrm{H}_0(\mathbb{Z}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]) \cong \mathbb{Z} \oplus \mathrm{H}_0(\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}])$. The homology of $\mathbb{Z}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ has already been computed in (4.11) as $\mathrm{Tor}_*^{\mathbb{Z}[x,x^{-1}]}(\mathbb{Z},\mathbb{Z})$; we obtain:

$$\mathrm{H}_q\Big(\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]\Big) \cong \begin{cases} 0 & \text{if } n = 0 \\ \mathbb{Z} & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases} \tag{5.8}$$

We can apply the Künneth formula as in (1.8) to get:

$$\bigoplus_{p+q=n} \mathrm{HH}_p(A/(a)) \otimes \mathrm{H}_q(\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]) \cong \mathrm{H}_n\Big(\mathrm{C}_\bullet(A/(a)) \otimes \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]\Big)$$

where the left-hand side is nonzero only for $(p,q) = (n-1,1)$, thus giving

$$\begin{aligned} \mathrm{H}_n\Big(\mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]\Big) &\cong \mathrm{H}_n\Big(\mathrm{C}_\bullet(A/(a)) \otimes \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]\Big) \\ &\cong \mathrm{HH}_{n-1}(A/(a)) \otimes \mathbb{Z} \cong \mathrm{HH}_{n-1}(A/(a)) \end{aligned}$$

as we wanted to prove. ■

We are now ready to prove the central theorem of this chapter.

Theorem 5.3. *Let A be a commutative ring and let a be an element of A such that the map $\mathbb{Z}[x] \to A$, $x \mapsto a$, makes A a flat $\mathbb{Z}[x]$-algebra. Then there is a long exact sequence in homology:*

$$\ldots \longrightarrow \mathrm{HH}_i(A) \longrightarrow \mathrm{HH}_i(A,\langle x\rangle) \longrightarrow \mathrm{HH}_{i-1}(A/(a)) \longrightarrow \mathrm{HH}_{i-1}(A) \longrightarrow \ldots$$
$$\ldots \longrightarrow \mathrm{HH}_1(A,\langle x\rangle) \longrightarrow \mathrm{HH}_0(A/(a)) \longrightarrow \mathrm{HH}_0(A) \longrightarrow \mathrm{HH}_0(A,\langle x\rangle) \longrightarrow 0$$

Proof. From (5.6) we get a long exact sequence in homology. The homology of $\mathrm{C}_\bullet(A) \otimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle]$ is isomorphic to $\mathrm{HH}(A)$ by Lemma 5.1; the homology of $\mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle]$ is $\mathrm{HH}(A,\langle x\rangle)$ by definition. By Lemma 5.2, we have an isomorphism of homology groups

$$\mathrm{H}_n\Big(\mathrm{C}_\bullet(A) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]\Big) \cong \mathrm{HH}_{n-1}(A/(a))$$

So, the long exact sequence in homology is the sought one:

$$\begin{array}{cccccc} & \ldots & \longrightarrow & \mathrm{HH}_{i+1}(A,\langle x\rangle) & \longrightarrow & \mathrm{HH}_i(A/(a)) & \longrightarrow \\ \longrightarrow & \mathrm{HH}_i(A) & \longrightarrow & \mathrm{HH}_i(A,\langle x\rangle) & \longrightarrow & \mathrm{HH}_{i-1}(A/(a)) & \longrightarrow \\ & \vdots & & \vdots & & \vdots & \\ \longrightarrow & \mathrm{HH}_1(A) & \longrightarrow & \mathrm{HH}_1(A,\langle x\rangle) & \longrightarrow & \mathrm{HH}_0(A/(a)) & \longrightarrow \\ \longrightarrow & \mathrm{HH}_0(A) & \longrightarrow & \mathrm{HH}_0(A,\langle x\rangle) & \longrightarrow & 0 & \end{array} \tag{5.9}$$

■

5.2 Some examples

In this section we will apply Theorem 5.3 to the following pre-log rings:

$$(\mathbb{Z}[x], \langle x \rangle), \qquad (\mathbb{Z}[x], \langle x^2 \rangle), \qquad (\mathbb{Z}[x, y], \langle x \rangle)$$

For the first case, all the terms in the long exact sequence (5.9) are already known; for the other pre-log rings, the long exact sequence will help us to find an expression for the log Hochschild homology in degree greater than 1.

Example 5.4. As a first example, we consider the pre-log ring $(\mathbb{Z}[x], \langle x \rangle)$ with the pre-log structure map given by the inclusion. In this case, $\mathbb{Z}[x]/\langle x \rangle \cong \mathbb{Z}$. The long exact sequence in (5.9) gives:

$$\begin{aligned} \dots \qquad &\longrightarrow \mathrm{HH}_2(\mathbb{Z}[x], \langle x \rangle) \xrightarrow{\mathfrak{s}_2} \mathrm{HH}_1(\mathbb{Z}) \xrightarrow{\mathfrak{b}_1} \\ \longrightarrow \mathrm{HH}_1(\mathbb{Z}[x]) &\xrightarrow{\mathfrak{r}_1} \mathrm{HH}_1(\mathbb{Z}[x], \langle x \rangle) \xrightarrow{\mathfrak{s}_1} \mathrm{HH}_0(\mathbb{Z}) \xrightarrow{\mathfrak{b}_0} \\ \longrightarrow \mathrm{HH}_0(\mathbb{Z}[x]) &\xrightarrow{\mathfrak{r}_0} \mathrm{HH}_0(\mathbb{Z}[x], \langle x \rangle) \xrightarrow{\mathfrak{s}_0} 0 \end{aligned}$$

In Example 2.1 we found out that the Hochschild homology of $\mathbb{Z}$ is $\mathbb{Z}$ in degree 0 and vanishes in higher degree. From Lemma 4.7, we get that $\mathrm{HH}_0(\mathbb{Z}[x]) \cong \mathbb{Z}[x]$, $\mathrm{HH}_1(\mathbb{Z}[x]) \cong \mathbb{Z}[x]\{dx\}$ and $\mathrm{HH}_n(\mathbb{Z}[x]) \cong 0$ for $n \geq 2$. Moreover, there is an isomorphism $\mathfrak{r}_0\colon \mathrm{HH}_0(\mathbb{Z}[x]) \to \mathrm{HH}_0(\mathbb{Z}[x], \langle x \rangle)$, so $\mathfrak{b}_0$ is the zero map. Finally, we showed in Example 3.11 that $\mathrm{HH}_1(\mathbb{Z}[x], \langle x \rangle) \cong \mathbb{Z}[x]\{d\log x\}$ and that $\mathfrak{r}_1$ is the multiplication $dx \mapsto x d\log x$. The long exact sequence becomes:

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \\ \longrightarrow & \mathbb{Z}[x]\{dx\} & \xrightarrow{\cdot x} & \mathbb{Z}[x]\{d\log x\} & \xrightarrow{\mathfrak{s}_1} & \mathbb{Z} & \xrightarrow{0} \\ \longrightarrow & \mathbb{Z}[x] & \xrightarrow{\sim} & \mathbb{Z}[x] & \longrightarrow & 0 & \end{array}$$

where $\mathfrak{s}_1$ sends $d\log x$ to 1 (and $x d\log x$ to 0).

Example 5.5. Consider now the pre-log ring $(\mathbb{Z}[x], \langle x^2 \rangle)$ with the pre-log structure homomorphism given by the inclusion. The long exact sequence in (5.9) is:

$$\begin{aligned} \dots \qquad &\longrightarrow \mathrm{HH}_3(\mathbb{Z}[x], \langle x^2 \rangle) \xrightarrow{\mathfrak{s}_3} \mathrm{HH}_2(\mathbb{Z}[x]/(x^2)) \xrightarrow{\mathfrak{b}_2} \\ \longrightarrow \mathrm{HH}_2(\mathbb{Z}[x]) &\xrightarrow{\mathfrak{r}_2} \mathrm{HH}_2(\mathbb{Z}[x], \langle x^2 \rangle) \xrightarrow{\mathfrak{s}_2} \mathrm{HH}_1(\mathbb{Z}[x]/(x^2)) \xrightarrow{\mathfrak{b}_1} \\ \longrightarrow \mathrm{HH}_1(\mathbb{Z}[x]) &\xrightarrow{\mathfrak{r}_1} \mathrm{HH}_1(\mathbb{Z}[x], \langle x^2 \rangle) \xrightarrow{\mathfrak{s}_1} \mathrm{HH}_0(\mathbb{Z}[x]/(x^2)) \xrightarrow{\mathfrak{b}_0} \\ \longrightarrow \mathrm{HH}_0(\mathbb{Z}[x]) &\xrightarrow{\mathfrak{r}_0} \mathrm{HH}_0(\mathbb{Z}[x], \langle x^2 \rangle) \xrightarrow{\mathfrak{s}_0} 0 \end{aligned}$$

The isomorphism $\mathfrak{r}_0\colon \mathrm{HH}_0(\mathbb{Z}[x]) \to \mathrm{HH}_0(\mathbb{Z}[x], \langle x^2 \rangle)$ makes $\mathfrak{b}_0$ the zero map. Again, $\mathrm{HH}_0(\mathbb{Z}[x]) \cong \mathbb{Z}[x]$ and $\mathrm{HH}_1(\mathbb{Z}[x]) \cong \mathbb{Z}[x]\{dx\}$, while the Hochschild homology of $\mathbb{Z}[x]$ is 0 in higher degrees. However, we saw in Example 2.2 that the homology of $\mathbb{Z}[x]/(x^2)$ never vanishes, so $\mathrm{HH}_{n+1}(\mathbb{Z}[x], \langle x^2 \rangle) \cong \mathrm{HH}_n(\mathbb{Z}[x]/(x^2))$ for $n \geq 2$. More in detail,

$$\mathrm{HH}_n(\mathbb{Z}[x]/\big(x^2\big)) \cong \begin{cases} \mathbb{Z}[x]/\big(x^2\big) & \text{for } n = 0 \\ \mathbb{Z}[x]/\big(2x, x^2\big) & \text{for odd } n \\ \mathbb{Z}\{x\} & \text{for even } n,\ n \geq 2 \end{cases}$$

We see that the homomorphism of $\mathbb{Z}[x]$-modules

$$\mathfrak{b}_1\colon\ \mathrm{HH}_1(\mathbb{Z}[x]/(x^2)) \cong \mathbb{Z}[x]/\big(2x, x^2\big) \to \mathbb{Z}[x]\{dx\} \cong \mathrm{HH}_1(\mathbb{Z}[x])$$

must be the zero map. In fact, let $\mathfrak{b}_1(1) = f(x)$. Then

$$0 = \mathfrak{b}_1(0) = \mathfrak{b}_1(2x) = 2x \cdot f(x)$$

Since $\mathbb{Z}[x]\{dx\}$ is an integral domain, we get $f(x) = 0$. This implies that $\mathrm{HH}_2(\mathbb{Z}[x], \langle x^2 \rangle) \cong \mathrm{HH}_1(\mathbb{Z}[x]/(x^2))$. The only missing term in the long exact sequence is now $\mathrm{HH}_1(\mathbb{Z}[x], \langle x^2 \rangle)$, which we can compute, using Theorem 3.22, by means of $\Omega^1_{(\mathbb{Z}[x], \langle x^2 \rangle)}$. From the pushout diagram

$$\begin{tikzcd}
\mathbb{Z}[x] \cong \mathbb{Z}[x] \otimes_{\mathbb{Z}[x^2]} \Omega^1_{\mathbb{Z}[x^2]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathbb{Z}[x] \otimes \langle x^2 \rangle^{\mathrm{gp}} \cong \mathbb{Z}[x] \otimes \mathbb{Z} \arrow[d, "\overline{\phi}"] \\
\mathbb{Z}[x] \cong \Omega^1_{\mathbb{Z}[x]} \arrow[r, "\overline{\psi}"'] & \Omega^1_{(\mathbb{Z}[x], \langle x^2 \rangle)}
\end{tikzcd}$$

we get

$$\mathrm{HH}_1(\mathbb{Z}[x], \langle x^2 \rangle) \cong \Omega^1_{(\mathbb{Z}[x], \langle x^2 \rangle)} \cong (\mathbb{Z}[x]\{dx\} \oplus \mathbb{Z}[x]\big\{d \log x^2\big\})/\sim$$

where $\sim$ is $\mathbb{Z}[x]$-linearly generated by $2dx \oplus 0 \sim 0 \oplus x^2 d\log x^2$. In conclusion, the long exact sequence becomes

$$\begin{array}{ccccccc}
 & \dots & \longrightarrow & \mathbb{Z}\{x\} & \xrightarrow{\sim} & \mathbb{Z}\{x\} & \longrightarrow \\
\longrightarrow & 0 & \longrightarrow & \mathbb{Z}[x]/(2x, x^2) & \xrightarrow{\sim} & \mathbb{Z}[x]/(2x, x^2) & \xrightarrow{0} \\
\longrightarrow & \mathbb{Z}[x]\{dx\} & \xrightarrow{\mathfrak{r}_1} & \Omega^1_{(\mathbb{Z}[x], \langle x^2 \rangle)} & \xrightarrow{\mathfrak{s}_1} & \mathbb{Z}[x]/(x^2) & \xrightarrow{0} \\
\longrightarrow & \mathbb{Z}[x] & \xrightarrow{\sim} & \mathbb{Z}[x] & \longrightarrow & 0 &
\end{array}$$

with maps

$$\mathfrak{r}_1\colon \mathbb{Z}[x]\{dx\} \to \frac{\mathbb{Z}[x]\{dx\} \oplus \mathbb{Z}[x]\big\{d \log x^2\big\}}{\sim} \cong \mathrm{HH}_1(\mathbb{Z}[x], \langle x^2 \rangle)$$

sending $dx \mapsto dx \oplus 0$, and

$$\mathfrak{s}_1\colon\ \mathrm{HH}_1(\mathbb{Z}[x], \langle x^2 \rangle) \cong \frac{\mathbb{Z}[x]\{dx\} \oplus \mathbb{Z}[x]\big\{d \log x^2\big\}}{\sim} \to \mathbb{Z}[x]/(x^2)$$

sending $dx \mapsto 0$ and $d\log x^2 \mapsto 1$. We note that $x^2 d\log x^2$ is sent to 0 by $\mathfrak{s}_1$.

Example 5.6. Consider the pre-log ring $(\mathbb{Z}[x,y],\langle x\rangle)$, with the pre-log structure map given, again, by the inclusion. The long exact sequence in (5.9) is:

$$\begin{array}{cccccc}
\dots & \longrightarrow & \mathrm{HH}_3(\mathbb{Z}[x,y],\langle x\rangle) & \xrightarrow{\mathfrak{s}_3} & \mathrm{HH}_2(\mathbb{Z}[y]) & \xrightarrow{\mathfrak{b}_2} \\
\longrightarrow \mathrm{HH}_2(\mathbb{Z}[x,y]) & \xrightarrow{\mathfrak{r}_2} & \mathrm{HH}_2(\mathbb{Z}[x,y],\langle x\rangle) & \xrightarrow{\mathfrak{s}_2} & \mathrm{HH}_1(\mathbb{Z}[y]) & \xrightarrow{\mathfrak{b}_1} \\
\longrightarrow \mathrm{HH}_1(\mathbb{Z}[x,y]) & \xrightarrow{\mathfrak{r}_1} & \mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle) & \xrightarrow{\mathfrak{s}_1} & \mathrm{HH}_0(\mathbb{Z}[y]) & \xrightarrow{\mathfrak{b}_0} \\
\longrightarrow \mathrm{HH}_0(\mathbb{Z}[x,y]) & \xrightarrow{\mathfrak{r}_0} & \mathrm{HH}_0(\mathbb{Z}[x,y],\langle x\rangle) & \xrightarrow{\mathfrak{s}_0} & 0 &
\end{array}$$

As we know, $\mathrm{HH}_0(\mathbb{Z}[y])\cong\mathbb{Z}[y]$, $\mathrm{HH}_1(\mathbb{Z}[y])\cong\mathbb{Z}[y]\{dy\}$ and $\mathrm{HH}_n(\mathbb{Z}[y])\cong 0$ for $n\geq 2$. From Example 4.10 we have:

$$\mathrm{HH}_n(\mathbb{Z}[x,y])\cong\begin{cases}\mathbb{Z}[x,y] & \text{if } n=0\\ \mathbb{Z}[x,y]\{dx,dy\} & \text{if } n=1\\ \mathbb{Z}[x,y]\{dx\wedge dy\} & \text{if } n=2\\ 0 & \text{if } n>2\end{cases}$$

implying that also $\mathrm{HH}_n(\mathbb{Z}[x,y],\langle x\rangle)$ vanishes for $n>2$. The map

$$\mathfrak{r}_0\colon\ \mathrm{HH}_0(\mathbb{Z}[x,y])\to\mathrm{HH}_0(\mathbb{Z}[x,y],\langle x\rangle)$$

is an isomorphism of $\mathbb{Z}[x,y]$-modules, so $\mathfrak{b}_0$ is the zero map. The long exact sequence then becomes:

$$\begin{array}{cccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
\longrightarrow \mathbb{Z}[x,y]\{dx\wedge dy\} & \xrightarrow{\mathfrak{r}_2} & \mathrm{HH}_2(\mathbb{Z}[x,y],\langle x\rangle) & \xrightarrow{\mathfrak{s}_2} & \mathbb{Z}[y]\{dy\} & \xrightarrow{\mathfrak{b}_1} \\
\longrightarrow \mathbb{Z}[x,y]\{dx,dy\} & \xrightarrow{\mathfrak{r}_1} & \mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle) & \xrightarrow{\mathfrak{s}_1} & \mathbb{Z}[y] & \xrightarrow{0} \\
\longrightarrow \quad \mathbb{Z}[x,y] & \xrightarrow{\sim} & \mathbb{Z}[x,y] & \longrightarrow & 0 &
\end{array}$$

By Theorem 3.22, we can compute $\mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle)$ by means of $\Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)}$. From the pushout diagram

$$\begin{tikzcd}
\mathbb{Z}[x,y]\otimes_{\mathbb{Z}[x]}\Omega^1_{\mathbb{Z}[x]} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathbb{Z}[x,y]\otimes\langle x\rangle^{\mathrm{gp}}\cong\mathbb{Z}[x,y]\otimes\mathbb{Z} \arrow[d, "\overline{\phi}"] \\
\Omega^1_{\mathbb{Z}[x,y]} \arrow[r, "\overline{\psi}"'] & \Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)}
\end{tikzcd}$$

we obtain that $\Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)}$ is the $\mathbb{Z}[x,y]$-module generated by dx, dy and $d\log x$, subject to the relation $dx = xd\log x$. So $\mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle)\cong\Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)}\cong\mathbb{Z}[x,y]\{d\log x,dy\}$. We can now complete the long exact sequence with the map:

$$\begin{aligned}\mathfrak{r}_1\colon\ \mathrm{HH}_1(\mathbb{Z}[x,y])\cong\mathbb{Z}[x,y]\{dx,dy\}&\to\mathbb{Z}[x,y]\{d\log x,dy\}\cong\mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle)\\ dx&\mapsto xd\log x\\ dy&\mapsto dy\end{aligned}$$

Since we know that $\ker \mathfrak{s}_1 = \operatorname{im} \mathfrak{r}_1$ and $\operatorname{im} \mathfrak{s}_1 = \mathbb{Z}[y]$, we also have:

$$\begin{aligned} \mathfrak{s}_1 \colon \mathrm{HH}_1(\mathbb{Z}[x,y], \langle x \rangle) \cong \mathbb{Z}[x,y]\{d\log x, dy\} &\to \mathbb{Z}[y] \cong \mathrm{HH}_0(\mathbb{Z}[y]) \\ dy &\mapsto 0 \\ d\log x &\mapsto 1 \end{aligned}$$

We notice that $x d\log x$ is sent to 0 by $\mathfrak{s}_1$. Moreover, since $\mathfrak{r}_1$ is an injection, the map $\mathfrak{b}_1$ is the zero map. The map $\mathfrak{s}_2$ is then surjective, while $\mathfrak{r}_2$ is then injective. We get a short exact sequence:

$$0 \longrightarrow \mathbb{Z}[x,y]\{dx \wedge dy\} \xrightarrow{\mathfrak{r}_2} \mathrm{HH}_2(\mathbb{Z}[x,y], \langle x \rangle) \xrightarrow{\mathfrak{s}_2} \mathbb{Z}[y]\{dy\} \longrightarrow 0$$

Since $\mathbb{Z}[y]\{dy\}$ is a free as a group, the short exact sequence splits and, as a group, $\mathrm{HH}_2(\mathbb{Z}[x,y], \langle x \rangle) \cong \mathbb{Z}[x,y]\{dx \wedge dy\} \oplus \mathbb{Z}[y]\{dy\}$. Understanding what $\mathrm{HH}_2(\mathbb{Z}[x,y], \langle x \rangle)$ is isomorphic to as a $\mathbb{Z}[x,y]$-module will require some more effort.

Consider the following diagram of $\mathbb{Z}[x,y]$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2_{\mathbb{Z}[x,y]} \arrow[r, "w_1"] \arrow[d, "\overline{\tau}_2"', "\cong"] & \Omega^2_{(\mathbb{Z}[x,y],\langle x \rangle)} \arrow[r, "w_2"] \arrow[d, "\overline{\omega}_2"'] & \Omega^1_{\mathbb{Z}[y]} \arrow[r] \arrow[d, "\overline{\tau}"', "\cong"] & 0 \\
0 \arrow[r] & \mathrm{HH}_2(\mathbb{Z}[x,y]) \arrow[r, "\mathfrak{r}_2"'] & \mathrm{HH}_2(\mathbb{Z}[x,y], \langle x \rangle) \arrow[r, "\mathfrak{s}_2"'] & \mathrm{HH}_1(\mathbb{Z}[y]) \arrow[r] & 0
\end{tikzcd} \tag{5.10}$$

where

$$\begin{aligned} \Omega^2_{\mathbb{Z}[x,y]} &= \Lambda^2_{\mathbb{Z}[x,y]} \, \Omega^1_{\mathbb{Z}[x,y]} \cong \Lambda^2_{\mathbb{Z}[x,y]}(\mathbb{Z}[x,y]\{dx, dy\}) \cong \mathbb{Z}[x,y]\{dx \wedge dy\} \\ \Omega^2_{(\mathbb{Z}[x,y],\langle x \rangle)} &= \Lambda^2_{\mathbb{Z}[x,y]} \, \Omega^1_{(\mathbb{Z}[x,y],\langle x \rangle)} \cong \Lambda^2_{\mathbb{Z}[x,y]}(\mathbb{Z}[x,y]\{d\log x, dy\}) \\ &\qquad \cong \mathbb{Z}[x,y]\{d\log x \wedge dy\} \\ \Omega^1_{\mathbb{Z}[y]} &\cong \mathbb{Z}[y]\{dy\} \end{aligned}$$

and the $\mathbb{Z}[x,y]$-module homomorphisms in the upper row are defined on the generators

$$\begin{aligned} w_1(dx \wedge dy) &= x \cdot (d\log x \wedge dy) \\ w_2(d\log x \wedge dy) &= dy \end{aligned}$$

In particular, $w_2(x \cdot (d\log x \wedge dy)) = 0$. The two rows in (5.10) are then exact. The maps $\overline{\tau}_2 \colon \Omega^2_{\mathbb{Z}[x,y]} \to \mathrm{HH}_2(\mathbb{Z}[x,y])$ and $\overline{\tau} \colon \Omega^1_{\mathbb{Z}[y]} \to \mathrm{HH}_1(\mathbb{Z}[y])$ as in Proposition 4.2 are isomorphisms by Theorem 4.9. The map $\overline{\omega}_2 \colon \Omega^2_{(\mathbb{Z}[x,y],\langle x \rangle)} \to \mathrm{HH}_2(\mathbb{Z}[x,y], \langle x \rangle)$ is as described in Proposition 4.13. We will proceed to show that the diagram (5.10) is commutative.

To find an explicit expression for the isomorphism $\overline{\tau}_2$, we look at the commutative diagram in (4.1):

$$\begin{tikzcd}
\Omega^1_{\mathbb{Z}[x,y]} \otimes \Omega^1_{\mathbb{Z}[x,y]} \arrow[r, "\overline{\tau}^{\otimes 2}"] \arrow[d, "\wedge"'] & \mathrm{HH}_1(\mathbb{Z}[x,y]) \otimes \mathrm{HH}_1(\mathbb{Z}[x,y]) \arrow[d, "\mathfrak{sh}"] \\
\Omega^2_{\mathbb{Z}[x,y]} \arrow[r, "\overline{\tau}_2"'] & \mathrm{HH}_2(\mathbb{Z}[x,y])
\end{tikzcd}$$

The generator $dx \wedge dy$ of $\Omega^2_{\mathbb{Z}[x,y]}$ is the image of $dx \otimes dy \in \Omega^1_{\mathbb{Z}[x,y]} \otimes \Omega^1_{\mathbb{Z}[x,y]}$. The composition of the maps $\mathfrak{sh} \circ \overline{\tau}^{\otimes 2} = \mathfrak{m} \circ \mathfrak{g} \circ \mathfrak{p} \circ \overline{\tau}^{\otimes 2}$ gives:

$$\begin{aligned} \overline{\tau}^{\otimes 2} \colon \Omega^1_{\mathbb{Z}[x,y]} \otimes \Omega^1_{\mathbb{Z}[x,y]} &\to \mathrm{HH}_1(\mathbb{Z}[x,y]) \otimes \mathrm{HH}_1(\mathbb{Z}[x,y]) \\ dx \otimes dy &\mapsto (1 \otimes x) \otimes (1 \otimes y) \end{aligned}$$

$$\begin{aligned} \mathfrak{p} \colon \ \mathrm{HH}_1(\mathbb{Z}[x,y]) \otimes \mathrm{HH}_1(\mathbb{Z}[x,y]) &\to \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y]) \otimes \mathrm{C}_\bullet(\mathbb{Z}[x,y])) \\ \ldots &\mapsto (1 \otimes x) \otimes (1 \otimes y) \end{aligned}$$

$$\begin{aligned} \mathfrak{g} \colon \ \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y]) \otimes \mathrm{C}_\bullet(\mathbb{Z}[x,y])) &\to \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y]) \boxtimes \mathrm{C}_\bullet(\mathbb{Z}[x,y])) \\ \ldots &\mapsto (1 \otimes 1 \otimes x) \otimes (1 \otimes y \otimes 1) \\ &\qquad - (1 \otimes x \otimes 1) \otimes (1 \otimes 1 \otimes y) \end{aligned}$$

$$\begin{aligned} \mathfrak{m} \colon \ \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y]) \boxtimes \mathrm{C}_\bullet(\mathbb{Z}[x,y])) &\to \mathrm{HH}_2(\mathbb{Z}[x,y]) \\ \ldots &\mapsto (1 \otimes y \otimes x) - (1 \otimes x \otimes y) \end{aligned}$$

Considering $\mathrm{HH}_2(\mathbb{Z}[x,y])$ as the homology of $\mathrm{C}_\bullet(\mathbb{Z}[x,y]) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\widehat{\mathrm{B}}^{\mathrm{rep}}_\bullet\langle x\rangle]$, the class of $(1 \otimes y \otimes x) - (1 \otimes x \otimes y)$ corresponds to the class of

$$(1 \otimes y \otimes x) \otimes (1 \otimes 1 \otimes 1) - (1 \otimes x \otimes y) \otimes (1 \otimes 1 \otimes 1) \tag{5.11}$$

which is sent via $\mathfrak{r}_2$ to the same class in

$$\mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y]) \boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]} \mathbb{Z}[\mathrm{B}^{\mathrm{rep}}_\bullet\langle x\rangle]) \cong \mathrm{HH}_2(\mathbb{Z}[x,y], \langle x\rangle)$$

We will now find an explicit expression for the homomorphism $\overline{\omega}_2$. From the commutative diagram (4.5), we have:

$$\begin{tikzcd}
\Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)} \otimes \Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)} \arrow[r, "\overline{\omega}^{\otimes 2}"] \arrow[d, "\wedge"'] & \mathrm{HH}_1(\mathbb{Z}[x,y], \langle x\rangle) \otimes \mathrm{HH}_1(\mathbb{Z}[x,y], \langle x\rangle) \arrow[d, "\mathfrak{sh}"] \\
\Omega^2_{(\mathbb{Z}[x,y],\langle x\rangle)} \arrow[r, "\overline{\omega}_2"'] & \mathrm{HH}_2(\mathbb{Z}[x,y], \langle x\rangle)
\end{tikzcd}$$

We see that the generator $d\log x \wedge dy$ of $\Omega^2_{(\mathbb{Z}[x,y],\langle x\rangle)}$ comes from $d\log x \otimes dy$ in $\Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)} \otimes \Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)}$. Again, around the diagram, we get:

$$\begin{aligned} \overline{\omega}^{\otimes 2} \colon \Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)} \otimes \Omega^1_{(\mathbb{Z}[x,y],\langle x\rangle)} &\to \mathrm{HH}_1(\mathbb{Z}[x,y], \langle x\rangle) \otimes \mathrm{HH}_1(\mathbb{Z}[x,y], \langle x\rangle) \\ d\log x \otimes dy &\mapsto (1 \otimes 1 \otimes 1 \otimes x) \otimes (1 \otimes y \otimes 1 \otimes 1) \end{aligned}$$

$$\begin{aligned}\mathfrak{p}\colon \mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle)\otimes \mathrm{HH}_1(\mathbb{Z}[x,y],\langle x\rangle) &\to \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y],\langle x\rangle)^{\otimes 2})\\ \ldots &\mapsto (1\otimes 1\otimes 1\otimes x)\otimes(1\otimes y\otimes 1\otimes 1)\end{aligned}$$

$$\begin{aligned}\mathfrak{g}\colon \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y],\langle x\rangle)^{\otimes 2}) &\to \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y],\langle x\rangle)^{\boxtimes 2})\\ \ldots &\mapsto ((1\otimes 1\otimes 1)\otimes(1\otimes 1\otimes x))\otimes((1\otimes y\otimes 1)\otimes(1\otimes 1\otimes 1))\\ &\quad -((1\otimes 1\otimes 1)\otimes(1\otimes x\otimes 1))\otimes((1\otimes 1\otimes y)\otimes(1\otimes 1\otimes 1))\end{aligned}$$

$$\begin{aligned}\mathfrak{m}\colon \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y],\langle x\rangle)^{\boxtimes 2}) &\to \mathrm{HH}_2(\mathbb{Z}[x,y],\langle x\rangle)\\ \ldots &\mapsto (1\otimes y\otimes 1)\otimes(1\otimes 1\otimes x)\\ &\quad -(1\otimes 1\otimes y)\otimes(1\otimes x\otimes 1)\end{aligned}$$

We now see that the left square in (5.10) commutes, since

$$\begin{aligned}&\overline{\omega}_2\circ w_1(dx\wedge dy)=\overline{\omega}_2(x\cdot(d\log x\wedge dy))\\ &\quad=(x\otimes y\otimes 1)\otimes(1\otimes 1\otimes x)-(x\otimes 1\otimes y)\otimes(1\otimes x\otimes 1)\\ &\quad=(1\otimes y\otimes 1)\otimes(x\otimes 1\otimes x)-(1\otimes 1\otimes y)\otimes(x\otimes x\otimes 1)\\ &\quad=(1\otimes y\otimes x)\otimes(1\otimes 1\otimes 1)-(1\otimes x\otimes y)\otimes(1\otimes 1\otimes 1)\end{aligned}$$

which agrees with the expression in (5.11).
As for the right square in (5.10), we have

$$\overline{\tau}\circ w_2(d\log x\wedge dy)=\overline{\tau}(dy)=(1\otimes y)$$

from the expression of $\overline{\tau}$ in (2.4). On the other hand,

$$\begin{aligned}&\mathfrak{s}_2\circ\overline{\omega}_2(d\log x\wedge dy)=\\ &\quad=\mathfrak{s}_2((1\otimes y\otimes 1)\otimes(1\otimes 1\otimes x)-(1\otimes 1\otimes y)\otimes(1\otimes x\otimes 1))\\ &\quad=(1\otimes y\otimes 1)\otimes(1\otimes 1\otimes x)-(1\otimes 1\otimes y)\otimes(1\otimes x\otimes 1)=:e_1\end{aligned}$$

in $\mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[x,y])\boxtimes_{\mathbb{Z}[\mathrm{B}^{\mathrm{cy}}_\bullet\langle x\rangle]}\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}])$. This last homology has been computed in Lemma 5.2 by means of the homology of $\mathrm{C}_\bullet(\mathbb{Z}[y])\boxtimes\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ from the isomorphism (5.7); in the same lemma, the Eilenberg-Zilber theorem allowed us to compute the homology of $\mathrm{C}_\bullet(\mathbb{Z}[y])\otimes\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ instead. So, we apply the Alexander-Whitney map as in (1.3) to get, in homology:

$$\begin{aligned}\mathfrak{f}\colon \mathrm{H}_2(\mathrm{C}_\bullet(\mathbb{Z}[y])\boxtimes\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]) &\to \bigoplus_{p+q=2}\mathrm{H}_p(\mathrm{C}_\bullet(\mathbb{Z}[y]))\otimes\mathrm{H}_q(\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}])\\ e_1 &\mapsto y\otimes(1\otimes 1\otimes x)-y\otimes(1\otimes x\otimes 1)\\ &\quad+(1\otimes y)\otimes(1\otimes x)-(1\otimes y)\otimes(x\otimes 1)\\ &\quad+(1\otimes y\otimes 1)\otimes x-(1\otimes 1\otimes y)\otimes x\end{aligned}$$

From Lemma 5.2 we also get that the homology is zero everywhere but for $(p,q)=(1,1)$, so the only remaining terms are

$$(1\otimes y)\otimes(1\otimes x)-(1\otimes y)\otimes(x\otimes 1)$$

where $(1\otimes y)\otimes(x\otimes 1)=0$ since $(x\otimes 1)$ is quotiented out in $\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$. Moreover, since the homology of $\widetilde{\mathbb{Z}}[\mathrm{B}_\bullet\langle x\rangle^{\mathrm{gp}}]$ is $\mathbb{Z}$, the term $(1\otimes y)\otimes(1\otimes x)$ corresponds, in $\mathrm{HH}_1(\mathbb{Z}[y])$, to the class of $1\otimes y$.
Therefore, the right square in (5.10) commutes. By the five lemma (see e.g. [Mac Lane, 1963, Chapter I, Lemma 3.3]), the map $\overline{\omega}_2$ is an isomorphism of $\mathbb{Z}[x,y]$-modules, making

$$\mathrm{HH}_2(\mathbb{Z}[x,y],\langle x\rangle)\cong\mathbb{Z}[x,y]\{d\log x\wedge dy\}$$

Summarizing, the long exact sequence in homology is:

$$\begin{array}{ccccccc}
 & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
\longrightarrow & \mathbb{Z}[x,y]\{dx\wedge dy\} & \xrightarrow{\mathfrak{r}_2} & \mathbb{Z}[x,y]\{d\log x\wedge dy\} & \xrightarrow{\mathfrak{s}_2} & \mathbb{Z}[y]\{dy\} & \xrightarrow{0} \\
\longrightarrow & \mathbb{Z}[x,y]\{dx,dy\} & \xrightarrow{\mathfrak{r}_1} & \mathbb{Z}[x,y]\{d\log x,dy\} & \xrightarrow{\mathfrak{s}_1} & \mathbb{Z}[y] & \xrightarrow{0} \\
\longrightarrow & \mathbb{Z}[x,y] & \xrightarrow{\sim} & \mathbb{Z}[x,y] & \longrightarrow & 0 &
\end{array}$$

with maps $\mathfrak{r}_1$ and $\mathfrak{s}_1$ previously described and maps $\mathfrak{r}_2$ and $\mathfrak{s}_2$ explicited by

$$\begin{aligned}\mathfrak{r}_2\colon \mathbb{Z}[x,y]\{dx\wedge dy\}&\to\mathbb{Z}[x,y]\{d\log x\wedge dy\}\\ dx\wedge dy&\mapsto x\cdot(d\log x\wedge dy)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{s}_2\colon \mathbb{Z}[x,y]\{d\log x\wedge dy\}&\to\mathbb{Z}[y]\{dy\}\\ d\log x\wedge dy&\mapsto dy\end{aligned}$$

sending $x\cdot(d\log x\wedge dy)$ to 0.

Bibliography

[Atiyah and Macdonald, 1969] Atiyah, M. F. and Macdonald, I. G. (1969). *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont.

[Dummit and Foote, 2004] Dummit, D. S. and Foote, R. M. (2004). *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition.

[Eisenbud, 1995] Eisenbud, D. (1995). *Commutative algebra: With a view toward algebraic geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[Hatcher, 2002] Hatcher, A. (2002). *Algebraic topology*. Cambridge University Press, Cambridge.

[Kato, 1989] Kato, K. (1989). Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD.

[Lang, 1993] Lang, S. (1993). *Algebra*. Addison-Wesley, third edition.

[Loday, 1998] Loday, J.-L. (1998). *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Mac Lane, 1963] Mac Lane, S. (1963). *Homology*. Die Grundlehren der mathematischen Wissenschaften, Bd. 114. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg.

[Mac Lane, 1998] Mac Lane, S. (1998). *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition.

[Quillen, 1967] Quillen, D. G. (1967). *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York.

[Rognes, 2009] Rognes, J. (2009). Topological logarithmic structures. In *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 401–544. Geom. Topol. Publ., Coventry.

[Rosenberg, 1994] Rosenberg, J. (1994). *Algebraic K-Theory and Its Applications*. Algebraic K-theory and Its Applications. Springer.