Implications of Protected Areas to Optimal Management of Marine Resources

Master of Science Thesis in Applied Mathematics

Sturla Furunes Kvamsdal

Department of Mathematics
University of Bergen

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Abstract

The thesis explores how the introduction of a marine protected area influences the optimal management of a marine resource, i.e. a fish stock. Logistic growth functions are applied to a deterministic model with a migration term which is proportional to the difference in density in the two areas. A nonlinear profit function is adopted. The possibility and behavior of an optimal steady state equilibrium is investigated for different parameter values. An optimal feedback policy is computed using dynamic programming. The development of the optimally managed system through time is analyzed. The results indicate that protected areas decrease annual catches and revenues and increase the total standing stock. Several interesting effects of different parameters are revealed and discussed.

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Chapter 1

Introduction

This thesis discusses the management of renewable resources. The topic for the study is marine protected areas and application of the Hamilton-Jacobi-Bellman equation to a deterministic model. The next section gives a brief introduction to marine protected areas, what they are and why we investigate them. It also gives an overview of the rest of the thesis.

1.1 Marine Protected Areas

Fish is Norway's most important renewable resource, and the annual income from export is tens of billions of Norwegian kroner. The need to manage the resources in an effective and sustainable way is obvious, not only from an economic viewpoint, but also from an environmental perspective. Still, marine resources are hard to manage because of their uncertain nature. It is hard to predict catches and stock levels and certain policies can be difficult to enforce. The management strategies need to work despite of these problems. Hannesson argues for better methods as he comments on the collapse of the Canadian Northern Cod in his article *Marine Reserves: What would they accomplish?* [15]: *"The northern cod disaster is particularly disturbing since it took place despite a high degree of control over the harvest by the Canadian government, which was committed to a moderate rate of exploitation and whose marine science and scientists must be ranked as world class. Seen against this background, it would clearly be desirable to use fisheries management strategies that would work independently of incomplete information on stocks and catches and less-than-fully-effective enforcement policies."*

Marine protected areas (MPAs), also called marine reserves or sanctuaries, have received a lot of attention as an alternative fishery management tool in recent years. The topic has been subject to thorough research and numerous papers have been published on marine protected areas and related topics. Various areas have also been protected from fishing around the world. There are several different definitions and interpretations of a marine protected area, connected with different objectives and purposes. *"[...] the objectives of* *each MPA must be clearly stated, (i.e. whether to provide baseline research, protect habitat, protect particular species, or some combination of these purposes)."* Grader and Spain [12]. In our context a protected area is an area partly covering a fish stocks habitat which is closed to fishing. The main objective of many protected areas is management and conservation of fish stocks, and that is what we will concentrate on in this thesis. Amongst relevant issues we will *not* address is whether or not a single, large protected area is preferred over several smaller areas[1] and the socionomic and cultural implications of marine protected areas. So, how will we approach marine protected areas? The existing literature indicates that protected areas as the only management tool will not lead to higher revenues. *"[...] little would be gained by establishing marine reserves without applying some measures that constrain fishing capacity and effort."* Hannesson [15]. He continues further *"[...] marine reserves might provide a hedge against stock collapses [...]"*, which amongst others Conrad [8] also indicates. We will perform our analysis by suggesting a two-fold management strategy, namely by introducing an area which is closed to fishing and optimally manage the remaining area.

The main aim of this thesis is to find an optimal management policy for a fish stock where a part of its habitat has been protected from fishing. The nature of renewable resources such as fish stocks encourage to sustainability and we will pursue a sustainable development. We also investigate the consequences of applying the optimal policy, i.e. a possible steady state equilibrium.

In the next section we present the biological and economic submodels we apply in our model. The next chapter is dedicated to *Optimal Control Theory*, the theoretical framework for this thesis. Firstly in the third chapter we apply the theory to our model and secondly we examine some examples to which we implement our method. In the last chapter we discuss further the implications of the method and the results and we try to put our work in relation to previous research on marine protected areas and to the real-world of fisheries and decision making. Some derivations are done in the appendix together with program listing, before the bibliography is presented. Note that in Internet citations only the main URL-address is given, as URLs tend to be very long for specific pages.

Articles that have been important for this work, but have not been cited explicitly elsewhere, are Arnason *et al.* [2] and Sandal and Steinshamn [27, 29].

1.2 The model

The model we use is an aggregated deterministic model with logistic growth functions, formulated in a continuous time setting. We assume that the fish stock in question is dispersed uniformly over a known, limited geographical area. The main idea behind the

[1] Even though we can say that in some sense, our model cover this issue. We comment on it in the discussion of the model.

model is to divide the area into two zones, one which is closed to fishing and the other which is optimally managed through total allowable catch (TAC) quotas. We assume the migration between the two zones to be *density-dependent*, where the density is measured as total biomass relative to the carrying capacity in the area. Furthermore, we assume a rather simple economic framework with constant parameters.

We adopt the logistic growth function for the total stock z given by:

$$f(z, K) = rz\left(1 - \frac{z}{K}\right),$$

where r is the intrinsic growth rate and K is the carrying capacity. We let $s \in [0, 1]$ denote the fraction of the area which is protected and closed to fishing, and we assume that the natural growth in both areas follows the same growth rule as the entire stock. If we let x and y denote the biomass in the unprotected and the protected area respectively, the natural growth function for each area is given by:

$$f(x, K_x) = rx\left(1 - \frac{x}{(1-s)K}\right), \qquad f(y, K_y) = ry\left(1 - \frac{y}{sK}\right).$$

Note that by assuming $K_x = (1 - s)K$ we implicitly assume that whatever the carrying capacity consists of it is uniformly spread in the entire area.[2] For the rest of this thesis, if not otherwise is stated, we assume that the carrying capacity is given and that f has only one argument. The migration is a function of the relative densities in the two areas, and we assume that fish will migrate from the area with higher density to the area with lower density. If the density is equal, we assume the migration to be zero. Further, we assume the total migration to be proportional to the difference between the relative densities in the two areas. This results in a migration term of the following form:

$$\phi\left(\frac{y}{sK} - \frac{x}{(1-s)K}\right),$$

where ϕ is a biological constant. We have chosen the positive direction for the migration to point from the protected area and into the unprotected area. We are now ready to present the dynamics in the biological submodel. The change in stock in the protected area consists of natural growth subtracted migration, in the unprotected area it consists of natural growth and migration subtracted catch. Catch is denoted by h. This results in the following equations:[3]

$$\dot{x} = rx\left(1 - \frac{x}{(1-s)K}\right) + \phi\left(\frac{y}{sK} - \frac{x}{(1-s)K}\right) - h, \tag{1.1}$$

$$\dot{y} = ry\left(1 - \frac{y}{sK}\right) - \phi\left(\frac{y}{sK} - \frac{x}{(1-s)K}\right). \tag{1.2}$$

[2]The carrying capacity usually consists of food and habitat, spawning and nursery areas.

[3]Dot notation is used for the time derivative: $\dot{x} = \frac{dx}{dt}$.

We assume the utility, profit or objective function

$$\Pi(x,h) = p\left(1 - \frac{x_0}{x}\right)h - ch^2,$$

where p, c and x_0 are economic constants. This is the same function as Kugarajh, Sandal and Berge [22] uses and has to be comprehended as measuring *net* profit. The discount rate is δ and we assume the discount term to have the form $e^{-\delta t}$.

To ease the work and to underline the structure in the equations we change variables and parameters, the scaled variables are given by:

$$X = \frac{x}{(1-s)K}, \qquad Y = \frac{y}{sK}, \qquad U = \frac{h}{r(1-s)K}. \tag{1.3}$$

The proper scaling for the parameters is given by:

$$\tau = rt, \quad \gamma = \frac{\delta}{r}, \quad \omega = \frac{\phi}{rsK}, \quad \Omega = \frac{s\omega}{(1-s)}, \quad b = \frac{cr}{p}K(1-s), \quad X_0 = \frac{x_0}{(1-s)K}. \tag{1.4}$$

With this scaling the dynamic equations become:

$$\frac{dX}{d\tau} = X(1-X) + \Omega(Y-X) - U, \quad \frac{dY}{d\tau} = Y(1-Y) - \omega(Y-X).$$

We get the value from harvesting according to U by integrating the utility function over the time interval of harvesting. Since our focus is on a sustainable development, we choose infinity as the upper limit in the integral. The value is

$$\int_0^\infty e^{-\delta t}\, \Pi(x,h)\, \mathrm{d}t = p\int_0^\infty e^{-\gamma\tau}\, U \cdot \left(1 - \frac{X_0}{X} - bU\right) \mathrm{d}\tau.$$

We want to maximize this value. The maximizing problem is then stated as follows:

$$\max_{U\geq 0} \int_0^\infty e^{-\gamma\tau}\, U \cdot \left(1 - \frac{X_0}{X} - bU\right) \mathrm{d}\tau \tag{1.5}$$

under the conditions

$$\dot{X} = X(1-X) + \Omega(Y-X) - U, \tag{1.6}$$

$$\dot{Y} = Y(1-Y) - \omega(Y-X), \qquad X,\ Y,\ U \geq 0. \tag{1.7}$$

We need to add proper initial conditions for the problem to be well-defined. The experienced reader may recognize this as an optimal control problem in two dimensions. The next chapter provides necessary mathematical results required to deal with such problems.

It is important to understand the dynamics of the problem before moving on to the theory and the results. Therefore we provide a plot, Figure 1.1, of what the growth in the protected area typically will look like. In our model the natural growth in the protected area is not dependent on catch, so growth in y is equal for any level of catch. We see that it has a parabolic form in the y-direction and is linear in the x-direction. A cut along the line $x = y$ will provide the natural growth function $f(y, K_y)$.

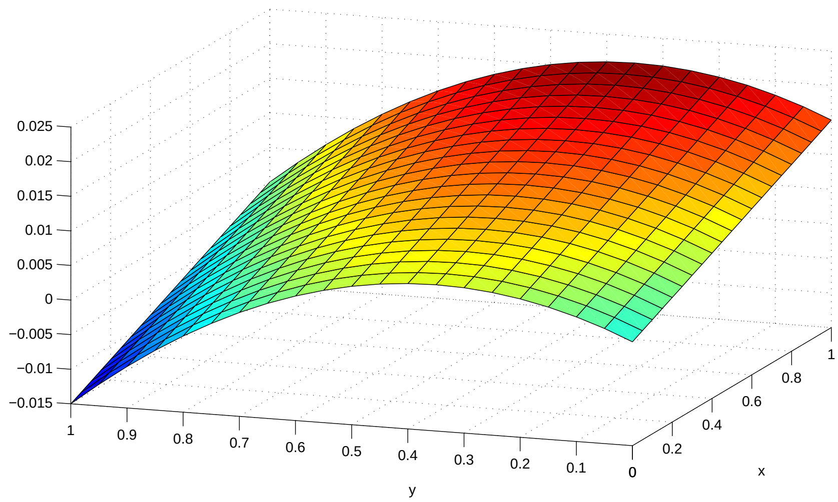

Figure 1.1: Demonstration of equation (1.7).

1.3 Discussion and critique

In this model we use logistic growth functions, please turn to Clark [7] for an enlightening discussion. In lack of more sophisticated biological models, we assume that migration is density dependent and proportional to the difference in density. The migration should be interpreted as *net* migration, fish will of course travel between the areas even though the density is the same in both areas, but assumably in the same amount in both directions. The idea is that the fish will seek a less congested habitat and to where it is easier to find and catch food. When the food is uniformly spread the desired area for the fish will be in the area where the density is lower, at least for extended periods of time. The time unit in the model is one year, which should be sufficient. Many authors swear to detailed cohort models, but the most important features of the dynamics of a fish stock is apparent in an aggregated model like ours and that is sufficient for our purpose.

The model we use is deterministic, and such models have provided unsatisfactory results in the literature. Conrad [8] claims that *"The value of a marine sanctuary [...] is not likely to be revealed in a deterministic model [...]"* and Holland and Brazee [18] joins in, *"If effort can be controlled, marine reserves provide little or negative benefits."* As long as the mere existence of fish in the sea is not given a value in any way, protected areas will not pay off. A stochastic model[4] is more likely to reveal the value of a marine protected area. Conrad [8] finds that in a stochastic model, the variance in biomass is reduced, which again should lead to less variance in catches. However, stochastic models are more complex

[4]Stochastic models covers uncertainties tied to several aspects of the modeling, i.e. shocks in the dynamics, stocks or catches or in prices.

and the solutions are harder to obtain. Feedback solutions deals with many of the issues stochastic models are designed to cover, but can be derived from deterministic models. One might need stochastics to reveal the value of marine protected areas, but you do not need stochastics to manage them properly! Feedback policies will do. Where time-dependent policies lose the thread and new calculations are needed when an unexpected event occurs, feedback policies pick it up immediately, without concern neither for how probable such an event is nor how large the disturbance is.

The x_0, in the profit function Π, can be interpreted as the open-access solution, and the term containing it is an economic protection of the stock. It is readily seen that for stock levels $x < x_0$, profit of any catch is negative. The open-access solution is characterized by zero profit. From the nonlinear term in our Π-function, we see that profit will be zero for even higher stock levels than x_0. This is seen in our results too, where a fishing moratorium is predicted before such low stock levels as x_0 is reached.

The limit $s \to 1$ reveals a weakness in our model. As s approaches 1, the stock in the protected area will move towards the carrying capacity, since everything is protected catch is zero, and the stock in the unprotected area will be driven to zero; No area – no fish. This is due to the nature-given continuity of the stock level and the fact that a stock which is not harvested is assumed to equal the carrying capacity. So, when s is close to 1, y will be close to the carrying capacity and x will be close to zero, that is in equilibrium. The model then predicts an enormous difference in density, which leads to a large migration into the unprotected area. Certain values of the migration constant ϕ will surely lead to very high stock levels in the unprotected area, at least when s is *close enough* to 1. The fish will then be very easy to catch and consequently supports a very profitable fishery. The problem in the model is that the equations still consider catch when x is very small. This is highly unrealistic and we should be careful to deduce anything from our results for s close to 1. The question that arises from this is how large s can be before the results become unrealistic. The answer is that when the areas that s measures reach a size where talking about densities does not make sense anymore, then the model will give bad results. The limit $s \to 0$ is also a potential pitfall. But in this end our model deals with the problem. Catch covers up for the potential migration into the smaller area.

We have formulated our model to consist of one area which is closed to fishing and one which is open; two areas. It is, of course, possible to introduce several protected areas and incorporate them into our model. One possible way of doing it is to collect all the protected areas in one large theoretical area, and adjust the migration coefficient according to the relative length of the border between the areas. Then the problem would still be two-dimensional and the method described applies. There is another issue regarding the two areas, and that is the natural growth. We have divided the stock into two stocks and assumed that the growth in each area follows the reproduction law for the entire stock with adjusted carrying capacity, without concern for the stock level in the other area. This seems perfectly correct since the natural growth depends on how much fish there is locally and not globally. It is, however, erroneous. We compare the natural growth for the entire stock against the stock divided into two equally large parts, that is, setting = 0.5, in Figure 1.2. Along $x = y$ the growth is equal for both cases, but for any other stock levels the growth in the entire stock is larger. The error is largest for the pathological cases of $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$.

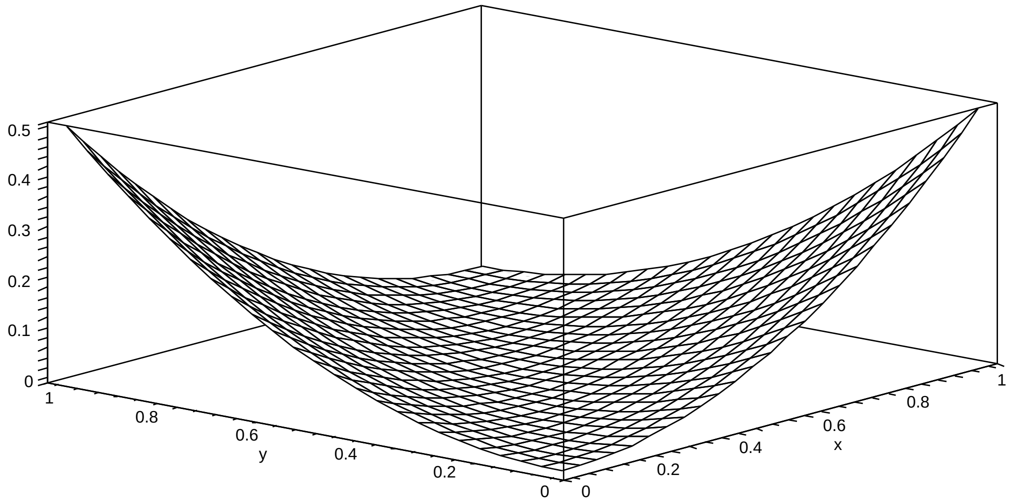

Figure 1.2: Comparison between local and global growth. The plot displays the function $F(x, y, K_x, K_y) = r\,(x + y)\left(1 - \frac{x+y}{K_x+K_y}\right) - \left(rx\left(1 - \frac{x}{K_x}\right) + ry\left(1 - \frac{y}{K_y}\right)\right)$, where $K_x = K_y = 1$ and $r = 1$.

Maybe a better way of modeling could be to use the growth function for the entire stock and letting the ratio between the stock level in each area and the total stock decide how much of the total growth should be added to the stock in the area. Then the growth term in the unprotected area would look like this:

$$\frac{x}{z} \cdot rz\left(1 - \frac{z}{K}\right) = rx\left(1 - \frac{z}{K}\right),$$

where $z = x + y$ and $K = K_x + K_y$ as before. Replacing x with y will yield the growth in the protected area.

The scaling in equations (1.3) and (1.4) is worth some discussion. The scaled variables and parameters are dimensionless, that is readily seen. Notable is the unit of the migration coefficient, biomass per time unit, which is perfect in accordance with the modeling. The scaled discount rate is interesting, $\gamma = \frac{\delta}{r}$. It reveals that neither the actual discount rate nor the intrinsic growth rate are crucial parameters. The decisive rate is the ratio between them. If it becomes large, the optimal harvest solution will approach the myopic solution, which is the solution that simply maximizes the profit function. What is meant by a large ratio will always depend on the actual problem. This implies that whether a fish stock can support a profitable fishery or not, depends on the general state of the surrounding economy, which is reflected through the discount rate.

When interpreting the results later, it is important to keep in mind the different scaling of the state variables and the catch variable. The scaled state variables are related to the density of fish in the actual area and is the ratio between the biomass and the carrying capacity. When $s = 0.5$ the carrying capacity in both areas is equal and only then we can compare the level of biomass in the two areas directly by comparing the scaled state variables. The catch variable is scaled towards the carrying capacity in the unprotected area and the intrinsic growth rate, hence it is not straight-forward to compare the scaled variables of catch and biomass either. Before we leave the issue of scaling it is significant to note that in the calculations we choose to scale X_0 to $X_0 = \frac{x_0}{K}$ and not as stated earlier. Thus a protected area do not imply a change in the open-access stock level.

Chapter 2

Optimal Control Theory

The main aim of this chapter is to give a brief introduction to the theory of deterministic optimal control of systems governed by ordinary differential equations. The chapter covers

- different problem formulations,
- the Pontryagin Maximum Principle,
- feedback controls,
- dynamic programming and
- economic interpretations of the most important equations.

It should be noted that this is merely a presentation of the necessary theoretical results, proofs and additional theory are not provided. Most of this chapter is inspired by Seierstad and Sydsæter [30] and Kamien and Schwartz [21] and the reader should consult those books for details.

2.1 Introduction to Optimal Control Theory

The vector $\mathbf{x} = (x_1, ..., x_n)$ defines the state of an economic system, the coordinates $x_1, ..., x_n$ are called the state variables and are usually functions of time. The state is typically describing some sort of capital. We assume that the development in the economy (and hence the $x_i(t)$ variables) can be controlled to some extent, that is; there are a number of control variables $u_i(t)$ that have influence on the economy. These form the control vector $\mathbf{u}(t)$. The control variables, or decision variables, policies or instruments, will typically describe different rents, effort levels or quotas. We also need to know the laws governing the development of the economy through time, that is we need to know the dynamics of the system. The dynamics are given by a set of differential equations on the form

$$\frac{dx_i}{dt} = \dot{x}_i = f_i(\mathbf{x}, \mathbf{u}, t), \qquad t \in (t_0, t_1),$$

where the f_i functions are known. Thus we assume that the rate of change of each state variable in general depends on all the state variables, all the control variables and on time explicitly. $[t_0, t_1]$ is the time interval for the project, in optimal control problems t_0 is fixed while t_1 can be fixed or free.

$\mathbf{x}(t_0)$ is assumed known as we usually know the present state of an economic system. For each control $\mathbf{u}(t)$ we can usually solve the system of first-order differential equations. Since $\mathbf{x}(t_0)$ is known the system describes an initial value problem which has a unique solution $\mathbf{x}(t)$ when we assume the considered functions to be continuously differentiable. Continuously is often too strict here, the Lipschitz condition is often sufficient for a unique solution. In resource management problems, however, the functions are usually continuously differentiable. In order to choose the most efficient control, or the optimal control, often denoted by $\mathbf{u}^*$, yielding the optimal path $\mathbf{x}^*$ in the state space, we need a way to measure the efficiency, or utility or profit. The efficiency is measured by the number

$$V = \int_{t_0}^{t_1} \Pi(\mathbf{x}, \mathbf{u}, t)\, \mathrm{d}t,$$

where Π is a given function. Π is known as the profit function for the problem, sometimes called the utility, revenue or objective function instead. In some problems the final state $\mathbf{x}(t_1)$ is subject to certain bounds or conditions. We want to maximize the efficiency, and thus the problem can be stated like this:

$$\max_{\mathbf{u}} \int_{t_0}^{t_1} \Pi(\mathbf{x}, \mathbf{u}, t)\, \mathrm{d}t, \qquad \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t), \quad \mathbf{x}(t_0) = \mathbf{x}^0, \tag{2.1}$$

For the problem to be well-defined we need to add the appropriate bounds and conditions on the different parameters and variables in the problem. Often neither the control nor the state can vary freely, they are also subject to certain bounds. Typically, only certain values for the state variable would make sense in the economic understanding, a control that lead to *illegal* values of the state variable would not be *admissible*. This leads to bounds on the variables. We say that the control $\mathbf{u}(t)$ has to belong to the *admissible set* U, the set of admissible controls are only those controls that lead to economic meaningful states. We also assume the control to be at least piecewise continuous.

2.2 The Pontryagin Maximum Principle

Here we state the Pontryagin Maximum Principle for fixed time intervals. It provides us with the necessary conditions for optimality of the following optimal control problem. The necessary conditions for optimum of the simplest problem in optimal control theory are derived in appendix A.

The problem is to find a piecewise continuous control vector function $\mathbf{u}(t) = (u_1(t), \ldots, u_r(t))$

and an associated continuous and piecewise differentiable state vector function $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$ defined on the fixed time interval $[t_0, t_1]$ that will

$$\text{maximize} \int_{t_0}^{t_1} \Pi(\mathbf{x}(t), \mathbf{u}(t), t)\, \mathrm{d}t \tag{2.2}$$

subject to the differential equations

$$\frac{dx_i(t)}{dt} = f_i(\mathbf{x}(t), \mathbf{u}(t), t), \quad i = 1, \ldots, n \tag{2.3}$$

initial conditions

$$x_i(t_0) = x_i^0, \quad i = 1, \ldots, n \tag{2.4}$$

terminal conditions

$$\left.\begin{array}{ll} x_i(t_1) = x_i^1 & \text{for } i = 1, \ldots, l \ (x_i^1 \text{ fixed}) \\ x_i(t_1) \geq x_i^1 & \text{for } i = l+1, \ldots, m \ (x_i^1 \text{ fixed}) \\ x_i(t_1) \text{ free} & \text{for } i = m+1, \ldots, n \end{array}\right\} \tag{2.5}$$

and control variable restriction

$$\mathbf{u}(t) = (u_1(t), \ldots, u_r(t)) \in U \subseteq \mathbb{R}^r. \tag{2.6}$$

A consequence of Pontryagins maximum principle is that the focus change from maximizing an integral with respect to one vector and several conditions to maximizing the Hamiltonian with respect to $\mathbf{u} \in U$. The Hamiltonian is given by

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = \Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t),$$

where the $\boldsymbol{\lambda}(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ are so-called Lagrange-multipliers. They are associated with the corresponding state variables as *shadowprices* or *costates*.

The Pontryagin Maximum Principle 2.1 *Let $\mathbf{u}^*(t)$ be a piecewise continuous control defined on $[t_0, t_1]$ which solves the problem described in (2.2-2.6). Let $\mathbf{x}^*(t)$ be the associated path. Then there exist a constant λ_0 and a continuous and piecewise continuously differentiable vector function $\boldsymbol{\lambda}(t) = (\lambda_1(t), \ldots, \lambda_n(t))$ such that for all $t \in [t_0, t_1]$,*

$$(\lambda_0, \lambda_1(t), \ldots, \lambda_n(t)) \neq (0, 0, \ldots, 0), \tag{2.7}$$

$\mathbf{u}^*(t)$ maximizes $H(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t)$ for $\mathbf{u}(t) \in U$, that is,

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t) \geq H(\mathbf{x}^*(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \qquad \forall\, \mathbf{u}(t) \in U. \tag{2.8}$$

Except at the points of discontinuities of $\mathbf{u}(t)$, for $i = 1, \ldots, n$:

$$\dot{\lambda}_i(t) = -\frac{\partial H^*}{\partial x_i} = -\frac{\partial H}{\partial x_i}(\mathbf{x}^*(t), \mathbf{u}^*(t), \boldsymbol{\lambda}(t), t). \tag{2.9}$$

Furthermore,

$$\lambda_0 = 1 \text{ or } \lambda_0 = 0, \tag{2.10}$$

and the following transversality conditions are satisfied,

$$\left.\begin{array}{rl} \lambda_i(t_1) \text{ no conditions} & \text{for } i = 1, \dots, l \\ \lambda_i(t_1) \geq 0 & \text{for } i = l+1, \dots, m \\ \lambda_i(t_1) = 0 & \text{for } i = m+1, \dots, n \end{array}\right\} \tag{2.11}$$

with $\lambda_i(t_1) = 0$ *if* $x_i^*(t_1) > x_i^1$ *for* $i = l+1, \dots, m$.

There are several ways to go about proving this result, see for example Pontryagin *et al.* [26] or Hestenes [17]. Clark [7] provides a so-called 'intuitive proof'. In terms of the Hamiltonian, we put what we call the first-order-conditions (FOC) for optimum of the problem like this, suppressing arguments:[1]

$$\left.\begin{array}{rcl} H_{\boldsymbol{\lambda}} & = & \dot{\mathbf{x}} \\ -H_{\mathbf{x}} & = & \dot{\boldsymbol{\lambda}} \\ \operatorname{argmax} H & = & \mathbf{u} \end{array}\right\} \tag{2.12}$$

Note that for *inner* solutions the last equation becomes $H_{\mathbf{u}} = 0$. Even though $\lambda_0 = 0$ is possible, we will always assume $\lambda_0 = 1$.

Most problems in economics include a discounting term, $e^{-\delta t}$. We say that we discount the profit back to time t_0 by multiplying the profit function by the discounting term. The Hamiltonian will then have the form

$$H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) = e^{-\delta t}\Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t).$$

If we introduce a *current value multiplier* $\mathbf{m}(t) = e^{\delta t}\boldsymbol{\lambda}(t)$, the *current value Hamiltonian* is given by

$$\begin{array}{rcl} \mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) & = & e^{\delta t} \cdot H(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t) \\ & = & \Pi(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{m}(t) \cdot f(\mathbf{x}(t), \mathbf{u}(t), t). \end{array}$$

The first-order conditions for optimum in current value terms becomes

$$\left.\begin{array}{rcl} \mathscr{H}_{\mathbf{m}} & = & \dot{\mathbf{x}} \\ \delta\mathbf{m} - \mathscr{H}_x & = & \dot{\mathbf{m}} \\ \operatorname{argmax} \mathscr{H} & = & \mathbf{u} \end{array}\right\} \tag{2.13}$$

For the rest of the thesis we will use the current value formulation.

[1] Subscripts denote partial derivatives; $H_{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{x}}$.

2.3 Feedback Controls

Next we will take a look at *feedback controls*. Usually the control in an optimal control problem is a function of states, costates and time explicitly. A feedback control is a function only of states and time. In autonomous problems a feedback control is only dependent on the state variables, and this is a strong feature. This means that once one has calculated the feedback solution, one immediately has the optimal policy for all possible states at hand. Thus management becomes easy, as the current state of a problem in optimal control problems usually is known. As Clark [7] puts it: *"Such control laws are simple to describe and to implement, and they are capable of responding to random fluctuations in the state variable and in parameters of the problem."* In other words, feedback controls represent adaptive management.[2] Hamiltonians that are linear in the control gives rise to so-called *bang-bang* policies, which dictate most rapid approach solutions (MRAP). A bang-bang policy is demonstrated in Figure 2.1, for the case of North-East Artic Cod (NEAC), where a marine protected area covering one half of the habitat is introduced. The calculations are done in relative numbers, x denotes the stock density in the unprotected area and y denotes the stock density in the protected area. We choose the migration parameter $\omega = 0.3$. The case of NEAC is further investigated in the examples (chapter 3.3).

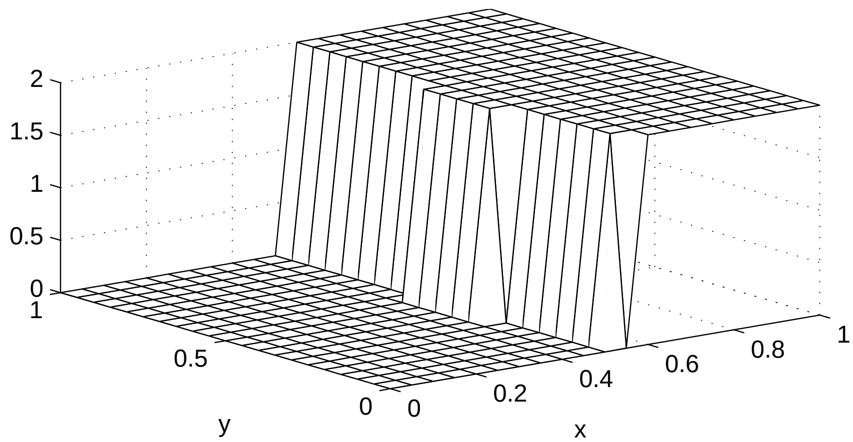

Figure 2.1: Bang-bang policy for NEAC, $s = 0.5$, $\omega = 0.3$. For NEAC-parameters and more, please turn to the examples in chapter 3.3. Note that this policy has been computed by simply disregarding the nonlinear term in the Hamiltonian, that is, setting $b = 0$. This is not the best linear approximation, however, the results clearly demonstrates the discontinuous bang-bang policy.

For one-dimensional problems one defines a *switch* equal to the partial derivative of the Hamiltonian with respect to the control, $S = H_u$. The switch is then a function independent of the control (assuming that the control enters linearly in both the dynamic equation

[2]Adaptive in this context refers to passively adaptive policies (PAP), as used by Walters [33].

and the utility function), and the optimal control shifts between the maximal and minimal possible control value when the switch changes sign. Typically bang-bang controls predict maximal control when the state is larger than the optimal state level, $x > x^*$, and minimal control when the state is lesser than the optimal state, $x < x^*$. This is the reason for the name bang-bang, the control shifts discontinuously between the extrema and those shifts are called 'bangs'. If the switch is zero on an interval, this method does not yield the optimal solution on the interval, and further investigation is needed. MRAP solutions move to the steady state equilibrium, or rather the optimal state position, as quickly as possible; this is because the control attains extreme values. These are the trivial feedback controls, and have been shown to give good results in some special cases, see Clark [7]. However, such controls are often unmanageable and unrealistic, and are usually a sign on oversimplification. Figure 2.2 shows how unrealistic results a bang-bang policy can yield, the policy applied is the policy shown in Figure 2.1. We see that the policy 'pulsates' between the maximum level, here set to $u = 2$ which in real-world values is close to the carrying capacity K, and the minimum level of zero. x pulsates heavily along with u while y only has small fluctuations, both approaches something that resembles a limit cycle steady state. The results are highly unrealistic, they predict maximal effort in very short time intervals and longer periods with no effort in between.[3]

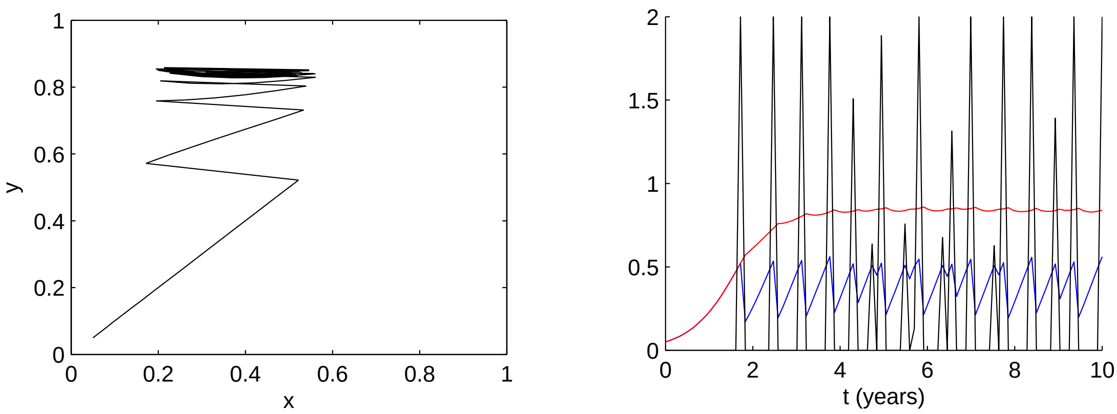

Figure 2.2: Development of state and catch for a bang-bang policy applied to the NEAC, $s = 0.5$ and $\omega = 0.3$. On the left we see the development through the state space. On the right we see the development through time, where the blue curve is the x variable (the density in the open area), the continuous red curve is the y variable (the density in the closed area) and the discontinuous black curve is the catch u. The initial position is $(x^0, y^0) = (0.05, 0.05)$, close to the open-access solution.

[3]We have to admit that we have manipulated the results a bit by enforcing that the policy only can change every second month (by adjusting the time step h when plotting the results, not in the calculations though). The calculations predict changes in the policy more often. We have done this to be able to clearly display the effects of a bang-bang policy.

A nontrivial feedback control is characterized by an asymptotic approach to the steady state equilibrium, this is optimal only when the Hamiltonian is nonlinear in the control. The utility function can be nonlinear in the control due to, for example, elastic demand or increasing marginal costs. The dynamic equations are usually linear in the control for resource management problems. Several nontrivial feedback controls are demonstrated in the examples.

2.4 Dynamic Programming

In the next sections we present the main result in dynamic programming and a discretization process which we apply to our problem. Dynamic programming was developed by *Richard Bellman*[4] and his name is tied to the *Hamilton-Jacobi-Bellman equation*. The equation is the fundamental differential equation for all problems in dynamic programming. In the following section we see how it can be derived.

2.4.1 The Hamilton-Jacobi-Bellman equation

Here we present the Hamilton-Jacobi-Bellman equation (HJB), which builds upon *the principle of optimality*. For more precise derivations, please turn to for example Kamien and Schwartz [21].

The principle of optimality can be informally stated like this: *An optimal path must at any time be optimal for the remaining problem, independent of the earlier decisions, and with the state resulting from the earlier decisions used as the initial condition.* We consider the following:

$$\left.\begin{aligned} V(x_0, t_0) \equiv \max_{\mathbf{u}} \int_{t_0}^{t_1} \Pi(\mathbf{x}(t), \mathbf{u}(t), t)\,\mathrm{d}t + \phi(\mathbf{x}(t_1), t_1), \\ \dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{x}(t_0) = x_0. \end{aligned}\right\} \tag{2.14}$$

The function $V(x,t)$ is defined for all $t \in (t_0, t_1)$ and for any possible x that might arise from decisions made before t. The definition leads directly to

$$V(\mathbf{x}(t_1), t_1) = \phi(\mathbf{x}(t_1), t_1).$$

We break up the integral in (2.14) as

$$V(x_0, t_0) = \max_{\mathbf{u}} \left(\int_{t_0}^{t_0+\Delta t} \Pi\,\mathrm{d}t + \int_{t_0+\Delta t}^{t_1} \Pi\,\mathrm{d}t + \phi \right)$$

where Δt is small and positive, arguments suppressed. Now the principle of optimality says that the control $\mathbf{u}$ should be optimal for the problem beginning at $t_0 + \Delta t$ in the

[4] Richard Bellman (1920-1984) received his PhD. at Princeton in 1946 and he invented dynamic programming in 1953. His many contributions to science and engineering won him many honors.

initial state $\mathbf{x}(t_0 + \Delta t) = x_0 + \Delta x$. This can be formulated as

$$\left.\begin{array}{c} V(x_0,t_0) = \max\limits_{\mathbf{u}(t), t\,\in\,[t_0,t_0+\Delta t]} \left(\displaystyle\int_{t_0}^{t_0+\Delta t} \Pi \,\mathrm{d}t + V(x_0+\Delta x, t_0+\Delta t) \right), \\ \dot{\mathbf{x}} = f, \quad \mathbf{x}(t) = x_0 + \Delta x. \end{array}\right\} \tag{2.15}$$

Now we need to do some approximations. First, we approximate the integral in (2.15) by $\Pi(x_0, \mathbf{u}, t_0) \cdot \Delta t$. We can do this when Δt is small enough, that is we consider the control to be constant on the interval of integration. We further assume that V is sufficiently smooth and perform a Taylor expansion of the second term on the right hand side. Moreover, we only consider the first-order terms. Next, subtracting $V(x_0, t_0)$ on each side, dividing through by Δt and letting $\Delta t \to 0$ yields

$$\max_{\mathbf{u}} \left(\Pi(\mathbf{x}, \mathbf{u}, t) + \frac{\partial V}{\partial t}(\mathbf{x}, t) + \frac{\partial V}{\partial x}(\mathbf{x}, t)\, \dot{\mathbf{x}} \right) = 0.$$

We have left out the zero subscripts since there is no ambiguity. Using the condition $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t)$ and since $\frac{\partial V}{\partial t}(\mathbf{x}, t)$ is independent of $\mathbf{u}$, we get the fundamental differential equation that govern the system.

$$\max_{\mathbf{u}} \left(\Pi(\mathbf{x}, \mathbf{u}, t) + \frac{\partial V}{\partial x}(\mathbf{x}, t) \cdot f(\mathbf{x}, \mathbf{u}, t) \right) + \frac{\partial V}{\partial t}(\mathbf{x}, t) = 0 \tag{2.16}$$

or, using subscripts for partial derivatives and suppressing arguments,

$$-V_t = \max_{\mathbf{u}} (\Pi + V_x \cdot f).$$

This equation is the Hamilton-Jacobi-Bellman equation for the continuous time setting. It can be shown that it is consistent with the first-order conditions for optimal solutions. The equation says that we have to equate the change in value over time against maximal possible profit today and change in value according to a change in the state. To be able to compute the solution to *our* problem, however, we will need to discretize the problem and apply equation (2.16) to the discrete problem.

The Hamilton-Jacobi-Bellman equation is a modified version of the Hamilton-Jacobi equation, well known from classical physics. The fundamental equation is in fact often referred to as the Hamilton-Jacobi equation when no stochastic processes are involved, as in our problem. The corresponding discrete-time equation is sometimes referred to as the Bellman equation. However, we will always refer to it as the Hamilton-Jacobi-Bellman equation.

2.4.2 Discretization

In this section we will present some results from the article *Using dynamic programming with adaptive grid scheme for optimal control problems in economics*, written by Grüne and Semmler, [14]. These are results that we will apply to our model to obtain numerical solutions. We will not, however, apply an adaptive grid, since we obtain good solutions without it. The problem is solved in a discrete-time setting. A discretization procedure is described in the article, we summarize it here since our model is formulated in continuous time. The discretization procedure goes back to Dolcetta [5] and Falcone [10], for further reference see also Bardi and Dolcetta [3] and Dolcetta and Falcone [6].

The problem considered is the following:

$$\left.\begin{aligned} V(\mathbf{x}) &= \max_{\mathbf{u}\in\mathscr{U}} \int_0^\infty e^{-\delta t}\Pi(\mathbf{x}(t),\mathbf{u}(t))\,\mathrm{d}t, \\ \frac{d}{dt}\mathbf{x}(t) &= f(\mathbf{x}(t),\mathbf{u}(t)), \quad \mathbf{x}(0)=\mathbf{x}_0\in\mathbb{R}^n. \end{aligned}\right\} \tag{2.17}$$

The basic discretizing technique is done using a two-step or semi-Lagrangian discretization, first in time and then in space. The first step is to replace the continuous optimal control problem (2.17) with a first order discrete time approximation. The approximation is given by

$$V_h(\mathbf{x}) = \max_{\mathfrak{u}\in\mathscr{U}_d} J_h(\mathbf{x},\mathfrak{u}), \quad J_h(\mathbf{x},\mathfrak{u}) = h\sum_{i=0}^{\infty}\beta^i\Pi(\mathbf{x}_{h,i},u_i), \tag{2.18}$$

where $\beta = 1-\delta h$ and $\mathbf{x}_h$ is defined by the discrete dynamics

$$\mathbf{x}_{h,0}=\mathbf{x}, \quad \mathbf{x}_{h,i+1}=\psi_h(\mathbf{x}_{h,i},u_i)\equiv\mathbf{x}_{h,i}+h\cdot f(\mathbf{x}_{h,i},u_i) \tag{2.19}$$

and $h>0$ is the discretization time step. $\mathscr{U}_d$ denotes the set of discrete control sequences $\mathfrak{u}=(u_1,u_2,\dots)$ for $u_i\in U\subseteq\mathscr{U}$. V_h is the solution of the Hamilton-Jacobi-Bellman equation for the discrete problem (2.18-2.19):

$$V_h(\mathbf{x}) = \max_{u\in U}\left[h\Pi(\mathbf{x},u)+\beta V_h(\psi(\mathbf{x},u))\right]. \tag{2.20}$$

We define the dynamic programming operator T_h by

$$T_h[V_h](\mathbf{x}) = \max_{u\in U}\left[h\Pi(\mathbf{x},u)+\beta V_h(\psi(\mathbf{x},u))\right].$$

Then V_h is the unique solution of the fixed point equation

$$V_h(\mathbf{x}) = T_h[V_h](\mathbf{x}) \qquad \forall\,\mathbf{x}\in\mathbb{R}^n.$$

Please turn to Grüne and Semmler [14] for further details. The second step approximates the solution on a grid Γ covering a *compact subset*[5] Θ of the state space. This is not a trivial requirement on the state space. However, for our problem all the interesting values of the state variables are contained in the closed quadratic $[0,1]\times[0,1]$ which is a compact

[5] A compact subset is a subset which is closed and bounded for any finite-dimensional space. For a more general and precise definition, check Munkres [23] book on topology or the great Internet resource *http://www.mathworld.com/*.

subset of $\mathbb{R}^2$. Let x_n be the nodes in the grid, $n = 1 \dots P$. We are searching for an approximation V_h^Γ which satisfies

$$V_h^\Gamma(x_n) = T_h[V_h^\Gamma(\psi)](x_n) \qquad \forall\, x_n \in \Gamma. \tag{2.21}$$

ψ is to be evaluated in x_n. When the value of $\psi(x_n, u)$ not *hits* a node in the grid (which it will not for most cases), we interpolate bilinearly in V_h^Γ. Note also that this approximation provides a feedback rule for the control, that is, the control is a function only of the state variable. In particular, this can be used to trace approximated optimal paths for $\mathbf{x}$.

It is shown that this procedure always converges to the correct solution for any initial conditions. For a rigorous analysis of the convergence of the discretization scheme see Bardi and Dolcetta [3] and Falcone and Giorgi [11].

2.5 Interpretations of Optimal Control Theory

In this last section of the chapter we submit some economic interpretations of the important equations presented earlier.

It can be shown (see for example Kamien and Schwartz [21]) that along the optimal path $\mathbf{x}^*(t)$ the costate variable $\boldsymbol{\lambda}(t)$ equals the rate of change in the value function, i.e. the gradient of $V(\mathbf{x}, \mathbf{u}, t)$. In economic terms, $\boldsymbol{\lambda}(t)$ is identified as the marginal value of the stock, or the marginal user cost, along the optimal trajectory. The terms in the Hamiltonian,

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, t) = \Pi(\mathbf{x}, \mathbf{u}, t) + \boldsymbol{\lambda}(t) \cdot f(\mathbf{x}, \mathbf{u}, t),$$

can be thought of as value flows. Π is the *cash flow* in the value function and f is the flow of investments, which pays off at price $\boldsymbol{\lambda}$. Furthermore, the Hamiltonian can be interpreted as the total rate of increase of total assets. The maximum principle asserts that the optimal control maximizes the rate of increase of total assets. For inner solutions, Dorfman [9] concludes that the optimality condition says that *"[...] along the optimal path of the decision variable at any time the marginal short-run effect of a change in decision must just counter-balance the effect of that decision on the total value of the capital stock an instant later."* Moreover, he says that the control should at every moment be chosen *"[...] so that the marginal immediate gain just equals the marginal long-run cost [...]"*.

$-\dot{\boldsymbol{\lambda}}$ can be understood as the rate of depreciation of capital (i.e. stock). The costate equation dictates that along the optimal path, the rate of depreciation must equal the sums of the marginal value flows, or the rate of change in the value flows. These interpretations make for example the condition $\boldsymbol{\lambda}(t_1) = 0$ for t_1 free easy to understand; when further investments have zero value, one terminates the project. As long as the costate is positive there is an incentive to exploit the associated resource. When the time horizon is infinite and $\mathbf{x}(t) \geq 0$ the condition is that $\boldsymbol{\lambda}(t) \to 0$. The interpretation is that it is optimal to extract all value from the resource during the time interval, but since the terminal time is never reached there will always be some value left in the resource.

The discounting term dates future income back to date zero to make them comparable to earlier income. Conrad [8] on discounting: *"It is tantamount to requiring the fish stock to earn a rate of return comparable to that earned by other assets elsewhere in a competitive economy."* Discounting forces a manager of a resource to make income earlier in time prior to those later. This is the same as valuing our generation's utility of the resource higher then a later generation's utility.

For more on economic interpretations of optimal control theory in particular and resource management in general, please turn to Hanley, Shoegren and White [16]. Dorfman [9] derives the maximum principle by means of economic analysis, and links capital theory and optimal control theory.

Chapter 3

Results

In this chapter we will look at the results from our efforts. First we apply the Pontryagin Maximum Principle to the model and obtain some purely analytical results. Next we explain how we apply the Hamilton-Jacobi-Bellman equation to the problem and finally we present some examples and comment upon the results. Note that we will only use lowercase letters to denote the different variables; whether or not it is the scaled variables that are referred to will be evident from the context. We will also relax the notation and write only m for both $m(t)$ and $m(x, y, u)$, which in fact are two different functions but they have the same value everywhere. The same goes for n and the other variables, there should be no ambiguity.

3.1 Analysis of the Equilibrium

Let us recapture the problem:

$$\max_{u \geq 0} \int_0^\infty e^{-\gamma t} \Pi(x, u) \, \mathrm{d}t, \quad \text{where } \Pi(x, u) = u \left(1 - \frac{x_0}{x} - bu\right),$$

under the conditions

$$\begin{aligned} \dot{x} &= f_1(x, y, u) = x(1 - x) + \Omega(y - x) - u, \\ \dot{y} &= f_2(x, y, u) = y(1 - y) - \omega(y - x), \\ & x, y, u \geq 0, \quad x(0) = x^0,\ y(0) = y^0. \end{aligned}$$

We have transformed our problem to an autonomous problem and therefore left the time-argument out of the equations. The current value Hamiltonian for this problem is given by:

$$\mathscr{H}(x, y, u) = \Pi(x, u) + m \cdot f_1(x, y, u) + n \cdot f_2(x, y, u),$$

where m and n are the current value multipliers for x and y respectively. The first-order conditions for optimum will then yield the following equations:

$$\dot{x} = x(1-x) + \Omega(y-x) - u \tag{3.1}$$
$$\dot{y} = y(1-y) - \omega(y-x) \tag{3.2}$$
$$\dot{m} = \gamma m - \Big(\frac{x_0 u}{x^2} + m(1-2x-\Omega) + n\omega\Big) \tag{3.3}$$
$$\dot{n} = \gamma n - (m\Omega + n(1-2y-\omega)) \tag{3.4}$$
$$u = \text{argmax } \mathscr{H} \tag{3.5}$$

We seek a steady-state equilibrium solution of these five equations, thus we put them all equal to zero. Note that equation (3.5) is replaced with $\mathscr{H}_u = 0$, which is the condition for an inner solution. From $\mathscr{H}$ it is readily seen that for $x \leq x_0$ both $\mathscr{H}_u = 0$ and $u = \text{argmax } \mathscr{H}$ returns $u = 0$. For $x > x_0$ we will have an inner solution whenever $m < 1 - \frac{x_0}{x}$. An upper bound on the shadowprice on x seems reasonable, if investments pays off at a certain level, no catch and thus maximal investment is optimal. This is a contradiction, because we assume equilibrium and u is assumed fixed. $u = 0$ in equilibrium indicates either $x = y = 0$ or $x = y = 1$. The reasoning holds in the first case, which is contained in $x < x_0$ anyway, but $u = 0$ in the latter case cannot be optimal. However, it is a trivial fact that zero catch never can be an optimal equilibrium as long as there is fish in the sea and densities are positive. Thus assuming an inner solution yields the optimal solution.

$$0 = x(1-x) + \Omega(y-x) - u \tag{3.6}$$
$$0 = y(1-y) - \omega(y-x) \tag{3.7}$$
$$0 = \gamma m - \Big(\frac{x_0 u}{x^2} + m(1-2x-\Omega) + n\omega\Big) \tag{3.8}$$
$$0 = \gamma n - (m\Omega + n(1-2y-\omega)) \tag{3.9}$$
$$0 = 1 - \frac{x_0}{x} - 2bu - m \tag{3.10}$$

We can now say a few things about the steady-state equilibrium.

- We have already argued for an upper limit on m. It follows from equation (3.10) and the requirement $u \geq 0$.
$$m \leq 1 - \frac{x_0}{x}.$$
- From equation (3.7) it is easily seen that in equilibrium we have $\omega = \frac{y(1-y)}{(y-x)}$. Since $\Omega = \frac{s\omega}{(1-s)}$ we put the expression for ω into equation (3.6) and we get the following:

$$\begin{aligned} u &= x(1-x) + \frac{s}{(1-s)} \cdot \frac{y(1-y)}{(y-x)} \cdot (y-x) \\ &= x(1-x) + \frac{s}{(1-s)} \cdot y(1-y). \end{aligned} \tag{3.11}$$

The term $\frac{s}{(1-s)}$ appears because of the scaling of the problem. If we perform the same reasoning on the unscaled problem it yields:

$$h = rx\left(1 - \frac{x}{(1-s)K}\right) + ry\left(1 - \frac{y}{sK}\right). \tag{3.12}$$

We see that the catch in equilibrium equals the total natural growth in both areas. One can be lead to think otherwise by the scaling term in equation (3.11).

- Since both x and y are densities, we assume them to take values in the interval $[0, 1]$. Then $y(1-y) \geq 0$ at all times. We also assume the constant ω to be positive. Starting with equation (3.7) this leads to

$$\dot{y} = y(1-y) - \omega(y-x) = 0 \Rightarrow \omega(y-x) > 0 \Rightarrow y > x$$

At equilibrium the density of fish will always be higher in the protected area and the migration term will be positive, that is fish will migrate from the protected area.

- From equation (3.8) we have that

$$\begin{aligned} \dot{m} &= \gamma m - \frac{x_0 u}{x^2} - m(1-2x-\Omega) - n\omega = 0 \\ \Rightarrow &\quad n\omega = \gamma m - \frac{x_0 u}{x^2} - m(1-2x-\Omega) \\ &\quad = m(\gamma - (1-2x) + \Omega) - \frac{x_0 u}{x^2} > 0 \\ \Rightarrow &\quad \gamma + \Omega - (1-2x) > 0 \\ \Rightarrow &\quad \gamma + \Omega > 1 - 2x. \end{aligned}$$

We have used the fact that $\frac{x_0 u}{x^2}$ is a positive term to arrive at this conclusion. In a similar fashion we obtain the following inequality from equation (3.9):

$$\gamma + \omega > 1 - 2y.$$

Note that the right hand side of the two last inequalities equals the derivative of the natural growth function, $\frac{d}{dz}f(z)$. As we assume that γ and ω are given, these inequalities yield lower bounds on x and y;

$$x > \frac{1}{2}(1-\gamma-\Omega), \quad y > \frac{1}{2}(1-\gamma-\omega).$$

The right hand side of these inequalities is smaller than $\frac{1}{2}$, which is the MSY level (Maximum Sustainable Yield). When analyzing resource management by optimal control theory it is often expected that the optimal stock level is larger than the MSY stock level. In our case the negative term $\frac{x_0}{x}$ in the profit function pushes the optimal stock level to the right and thus makes the term smaller. However, the examples provide proof that $x > x_{MSY}$ is not always optimal, in Figure 3.21 this is clearly not the case. But then again, the optimal *total* stock size is above the total MSY level.

- When we realize that an equilibrium only can exist when the natural growth in the protected area equals the migration, we get some interesting results. Equation (3.7) yields

$$y = \frac{1}{2}(1-\omega) \pm \frac{1}{2}\sqrt{(\omega-1)^2 + 4\omega x}. \tag{3.13}$$

  The interesting points are those decided by the plus sign in the equation. This defines an equilibrium curve in the state space $[0,1] \times [0,1]$. The equilibrium curve will always coincide with the point $(x, y) = (1, 1)$, when $\omega \geq 1$ it will also coincide with the point $(x, y) = (0, 0)$. Furthermore, we have that $x \leq y \leq 1$ when $x \in [0, 1]$. It is readily seen that $y = 1$ when $\omega = 0$ and that $y \to x$ when $\omega \to \infty$. In between these two we have $y = \sqrt{x}$ when $\omega = 1$. The curve is plotted to the left in Figure 3.1 for $\omega = 0.7$. From equation (3.11) we can calculate the harvest along the curve, this is done in the right plot in Figure 3.1 for the case $s = 0.5$. The harvest curve tells us where on the equilibrium curve the reproduction attains its maximum, which is the equilibrium-MSY level. This is, however, not the optimal equilibrium. The optimal equilibrium has to be calculated from the Hamiltonian and all the parameters have to be specified.

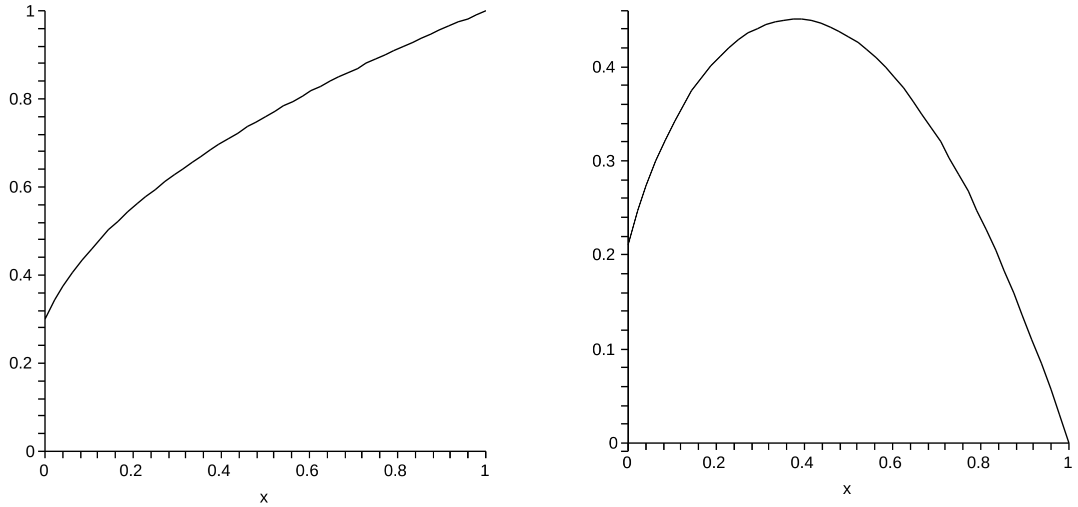

Figure 3.1: The left plot displays the curve given by equation (3.13), where $\omega = 0.7$. The corresponding catch along the curve is plotted to the right, $s = 0.5$.

The equations (3.6-3.10) consist of five equations and five unknowns and determine an equilibrium solution. By substitution we can reduce the system to one equation and one unknown. The roots of the yielded polynomial, called the *polynomial of equilibrium*, will be the mathematically possible equilibriums, from which we have to sort out those of interest. This can be used to study how the equilibrium depends on different parameters in the problem. However, the procedure is virtually impossible to do by hand without making mistakes. In this case we would end up with a polynomial of the tenth degree and it would be practically impossible find the roots by hand anyway.

Instead of analyzing the polynomial or the equations further, we will resort to numerics. We want to study the equilibrium closer and thus have to choose values for the parameters in the equations. Since we are not able to do this by hand, and since it is more convenient, we use the function `solve` which is built into the software *Maple* 9.50. The implementation of the problem into Maple is shown in appendix B.

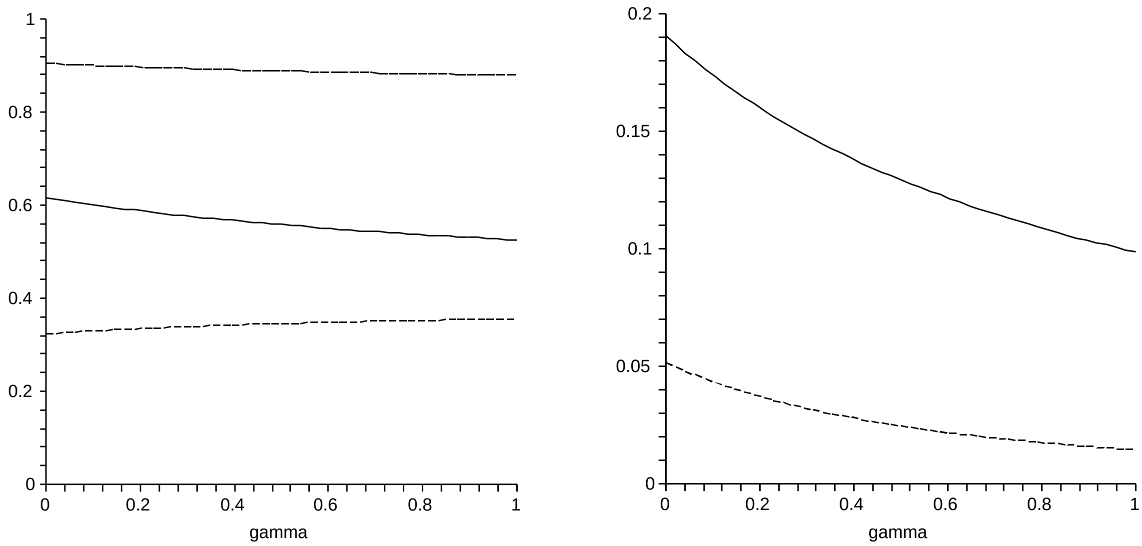

Figure 3.2: Equilibrium as a function of γ, the scaled discount rent. To the left we see the x, y and u equilibriums. y is the upper curve, x is the middle curve and u is the lower curve. To the right we see the costates m and n; m is the upper curve, n is the lower curve. The parameters are set at $\omega = 0.3$, $s = 0.5$, $x_0 = 0.1$ and $b = 1$.

In Figure 3.2 the equilibrium solution as a function of the scaled discount rate γ is displayed. We see that the standing stocks are declining and catch is increasing while the discount rate is increasing. This is what we expected; the larger the discount rate, the more important today is than tomorrow, and we want to increase catch today. However, it is shown by Sandal and Steinshamn [28] that an increasing discount rate can lead to higher standing stocks in nonlinear models, thus the effects should be investigated. The important feature about the effects is that the changes in the steady state stocks are relatively small with the increasing discount rate. We observe that the shadowprices fall with larger γ, that is, the marginal value of investments fall, an expected effect from discounting. The ratio between the scaled discount rate γ and the real discount rate δ is of order $\mathscr{O}(1)$, that is the plot shows results for an unrealistic span of discount rates. The parameters chosen for this example lie within what we consider to be realistic values. The exact values are listed in the figure.

3.2 Feedback solutions

In this section we explain how we apply the discretization-technique to our problem and how we obtain the feedback results from the Hamilton-Jacobi-Bellman equation.

First we have to discretize the problem. This is done by assuming that both the state variables and the control variables (singular in our case) can take values only in a defined set of finitely many discrete values. When using the Hamilton-Jacobi-Bellman equation one usually divides the procedure into two parts. We call the two parts *value iteration* and *policy iteration*. The equation for the discrete problem is given in equation (2.20), we recall it as:

$$V_h(\mathbf{x}) = \max_{u \in U} \left[h\,\Pi(\mathbf{x}, u) + \beta\, V_h(\psi(\mathbf{x}, u)) \right]. \tag{3.14}$$

where $\beta = 1 - h\gamma$, h is the *time-step*, γ is the discount rate and

$$\psi(\mathbf{x}, \mathbf{u}) = \mathbf{x} + h \cdot f(\mathbf{x}, \mathbf{u}).$$

Note that we also have discrete time now; time is divided into two periods, this instant (which could be a day, a week, a fishing season or a year) and the rest of eternity. The length of the first period is measured by h. In practice we choose small values for h since the problem is formulated in continuous time and we want the approximation to be as close as possible. The state in the second period is given by ψ; $\mathbf{x}_{i+1} = \psi(\mathbf{x}_i, \mathbf{u})$. From this point of view it is easy to understand the equation; the value is given by h times the current rate of profit (h is assumed to be so small that the rate does not change) plus the discounted future value of the resource, evaluated at the maximizing policy. In other words, the task is to maximize the sum of today's profit and future value. β is the linearization of $e^{-\gamma h}$ for small h. Larger h give smaller β, that is current profit counts for more of the total value for larger h. The value iteration is given by assuming that the policy is fixed and applying the Hamilton-Jacobi-Bellman equation, i.e.:

$$V_{i+1}(\mathbf{x}) = h\,\Pi(\mathbf{x}) + \beta\, V_i(\psi(\mathbf{x})) \tag{3.15}$$

We have left out the subscript h and the control argument. The maximizing term disappears since the policy is fixed. Note that we calculate the new value in every possible position in the state space for each iteration. In the policy iteration we first maximize the right hand side of equation (3.15) with respect to the control variable and assign the maximizing control as the new control and then assign the maximized value of the right hand side as the new value in each point. In other words, in the policy iteration we use the full Hamilton-Jacobi-Bellman equation, equation (3.14), while in the value iteration we only use a degenerate form of it, equation (3.15). It has been proved that this scheme converges for all initial conditions (see the section on dynamic programming) and we will arrive at the correct solution by only performing policy iterations. However, experience shows that it is much more efficient to perform a combination of value iterations and policy iterations. Often the most efficient way to perform the iterations is to perform value iterations until the value has 'settled down', and then perform policy iterations until the policy has 'settled down'. It usually takes a lot fewer iterations for the policy to settle than for the value to settle, but to compute the policy iterations takes a lot more time and resources because of the maximizing of the right hand side. Since the scheme converges for any initial value and policy, we can choose both V_0 and $\mathbf{u}_0$ equal to zero, displaying that we not have any knowledge about the solution in advance. However, we have to make sure that we do not only reproduce the zero solution. If that is the case, we can simply choose any other initial policy; the scheme will still converge. When the initial value is set to zero the first policy iteration simply maximizes the profit function and produces the so-called myopic solution. For large x, that is $x >> x_0$, $0 < u < \frac{x-x_0}{bx}$ is always better than $u = 0$, thus the myopic solution is never zero. The scheme will converge faster if the initial values are closer to the solution. Typically the computations will look like this, starting with zero initial conditions for both the value and the policy:

1. Perform one policy iteration to get away from the zero policy.
2. Perform several value iterations.
3. Perform a few policy iterations.
4. Repeat step 2 and 3 until both the policy and the value has settled.

We should note that when computing the last term in (3.14), we use bilinear interpolation in the table given by $V_i(\mathbf{x})$. For certain policy values the vector field given by $\psi(\mathbf{x}, \mathbf{u})$ will point out of the considered state space. If that is the case we approximate by choosing the intersecting point between the vector field and the boundary. We will use time-step $h = 0.05$ in all computations.

3.3 Examples

In this section we present results obtained by the methods explained earlier in this chapter. First we take a look at the most important Norwegian cod stock, namely the North-East Artic Cod. Thereafter we apply our method to the Canadian Northern Cod, a stock which is depleted. The reason we examine two cod stocks is not that our model is designed for cod only but a matter of convenience, since the relevant parameters for these stocks are readily available in the literature. Two examples would maybe seem unnecessary, but they provide a rich selection of different results and behaviors that needs explanation and completes the picture of MPAs in a better way than one single example can. We can also compare the results from the two cod stocks to a certain extent.

Before we dig into the results we find it appropriate to issue a few warnings. In the model we implicitly assume that the stock in question initially is uniformly dispersed over the habitat for the entire time period, i.e. without catch disturbing the balance. Neither of the stocks in the examples satisfy this assumption, and the results have to be understood as a demonstration of the method. Also, the values chosen for the parameters s and ω are not necessary realistic values. Our main focus will be on $s = 0.5$. First of all, this is convenient since the carrying capacity in both areas then is equal to $\frac{K}{2}$. Thus we can compare the two stock levels directly, when $y > x$ the biomass in the protected area is larger than the biomass in the unprotected area and vice versa. This is not necessarily true for other s values. Secondly, s has to be of a certain size for the effects to become apparent, at the same time s cannot be too close to 1. Regarding ω no data is available to us for calibration. ω too has to be of a certain size for the dynamics between the areas to become apparent, small values lead to a policy close to the one dimensional case. We somewhat randomly choose to focus on $\omega = 0.3$.

3.3.1 North-East Artic Cod

The North-East Artic Cod (*gadus morhua* in Latin, NEAC in short) is the most important cod stock in the Norwegian fisheries. Its main habitat is the Barents Sea, between Spitsbergen and Novaja Semlja, and it spawns along the Norwegian coast. These areas are covered by the ICES[1] fishing areas I, IIa and IIb (see Figure 3.3). The environmental factors forming the basis for the rich production in the Barents Sea are amongst others the large, shallow ocean area (average depth is 230m), the blending of water masses, cold and warm currents meeting and the ice melting and retreating.[2]

Parameter	Value	Explanation
r	0.4649	Intrinsic growth rate
K	5735.213	Carrying capacity
p	10.527	Price parameter
c	0.005973	Cost parameter
x_0	842.03	Free-entry limit

Table 3.1: NEAC parameter values found in Kugarajh, Sandal and Berge [22].

Parameter values for North-East Artic Cod are taken from Kugarajh, Sandal and Berge [22], who also study the same fishery. The values used are found in Table 3.1. The discount rate is set at 5 per cent. When we scale our problem, we get new parameters. We have

[1]The International Council for Exploration of the Sea (ICES) is the organization that coordinates and promotes marine research in the North Atlantic, [20].

[2]For more on both NEAC and the Barents Sea, please turn to ICES [20] or IMR [19].

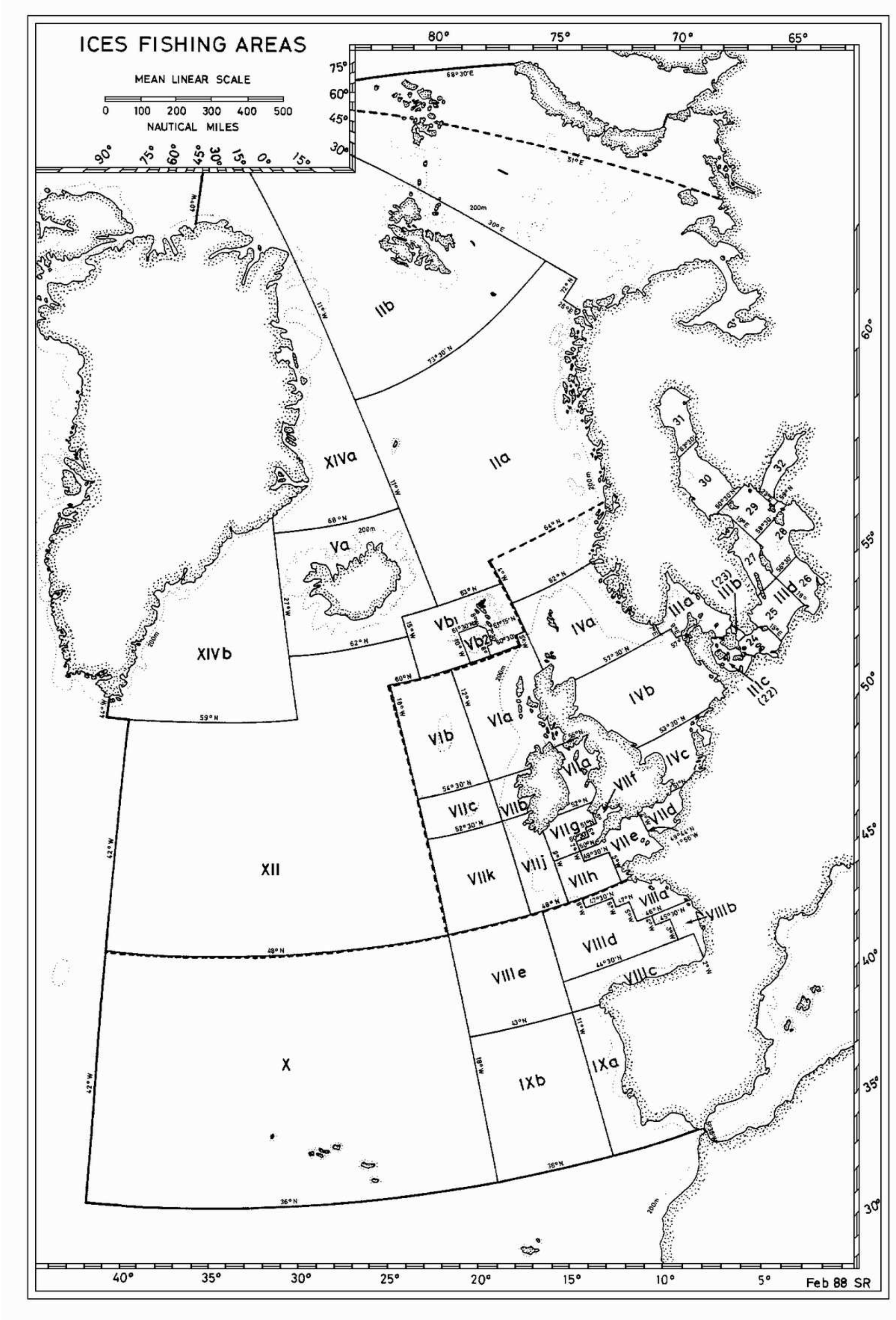

Figure 3.3: ICES fishing areas, [20].

calculated their values in Table 3.2. x_0 has to be scaled in the same way that x is. The scaling of x is $X = \frac{x}{(1-s)K}$, but we choose to scale x_0 such that $X_0 = \frac{x_0}{K}$. This will introduce a flaw in our calculations, $\frac{x_0}{K} < \frac{x_0}{(1-s)K}$, thus we calculate with a free-entry limit which is below the actual limit. Both the parameters s and ω enter the calculations, but we do not set these values. s is a decision variable and we want to study the effect it has upon the system. As good measures or data to calibrate ω are not available to us, we study the case of $\omega = 0.3$.

Parameter	Formulae	Value
b	$\frac{cr}{p}K(1-s)$	1.513 $(1-s)$
γ	$\frac{\delta}{r}$	0.1076
X_0	$\frac{x_0}{K}$	0.1468

Table 3.2: NEAC scaled parameters.

Now we turn to the results from our calculations. When studying these results, it is important to keep in mind that what are displayed are scaled variables if nothing else is stated. The first result we take a look at is how the equilibrium solution develops for different sizes of the protected area. These results are presented in Figure 3.4. Note that the figure shows relevant development according to s, and neither the true densities nor stock sizes. As mentioned; $\omega = 0.3$. We can think of this value in the following way: From one period to the next, the difference in density will be reduced by 30 per cent just because of migration between the two areas.

The left plot in Figure 3.4 shows the development of the stocks and catch; the plotted curves are $(1-s)x$, sy and $(1-s)u$. Plotting x, y and u would make no sense because the scaling is dependent of s. The bad behavior of the model for the limit $s \rightarrow 1$ is readily visible in Figure 3.4. We expect that as s moves to 1, y would also move to 1. The plot shows something else. It is of course true that if $s = 1$, then we should have $y = 1$; if everything is protected from catch and left alone, the stock moves to the carrying capacity, which is exactly $y = 1$ since we are speaking of relative numbers. However, if some area is left unprotected and s is close to 1, it would be possible to catch fish in the unprotected area, not because of the stock or natural growth in that area, note that x is driven to zero, but because a lot of fish would migrate into the unprotected area.[3] In equilibrium, the difference in density is large and the catch which is indicated in Figure 3.4 equals the migration, which again is the natural growth in the protected area for s close to 1. As $\omega = 0.3$, 30 per cent of the difference will migrate, which consequently is equal to the natural growth since we are in equilibrium. The numbers in the figure do not add up the same way, however, because u is scaled differently from the state variables.

[3]This explains the results according to the modeling, so in that sense we can say that the results are correct. But, we should keep in mind that the model does not give realistic results close to the limit $s = 1$, as pointed out earlier.

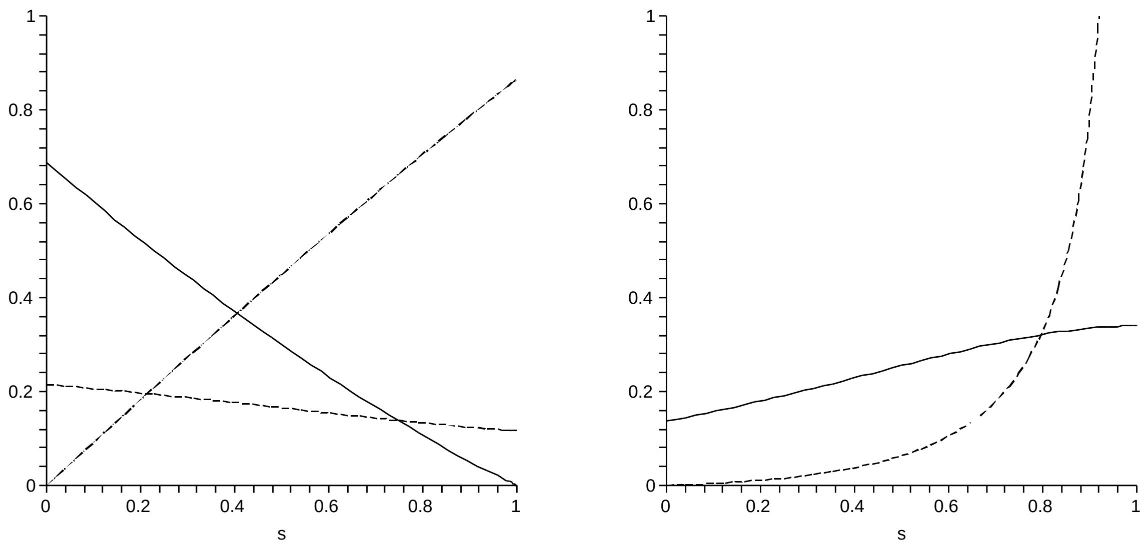

Figure 3.4: NEAC equilibrium as a function of s, $\omega = 0.3$. The quantities plotted on the left are $(1-s)x$, sy and $(1-s)u$. Thus at $s = 0$, x is the upper curve, u is the middle curve and y is the lower curve. To the right the costates m and n are plotted; m is the upper curve and n is the lower curve at $s = 0$.

The right plot in Figure 3.4 shows the equilibrium shadowprices as functions of s, m and n is associated with x and y respectively. Both shadowprices are increasing. The increase in n is obvious. As for $s = 0$, there is no protected area, no fish and thus no value. When the stock in the protected area represents a larger share of the total stock and a larger share of the catch, the marginal value increases. Remember that catch is equal to total growth at equilibrium. n approaching infinity at $s = 1$ is not realistic and is a further demonstration of the bad behavior of the model at that end; there should be an upper limit on the shadowprice. m increases because the more of the stock is protected, the lesser of the total stock is available for harvesting and the larger value has the fish left in the unprotected area to the margin. The effect could be compared to that of scarcity.

In Figure 3.5 we see the equilibrium solution as a function of ω, in the case where $s = 0.5$. The left plot shows the solutions for x, y and u. These are interesting results because x is not monotonic for ω close to zero. This effect is, however, not readily visible in the plot; the corresponding plot in the next example shows this more clearly. This behavior is also a reason for choosing $\omega = 0.3$ in the calculations; we do not want to be too close to an extremum. Apart from the behavior of x for small ω values, the results are as expected. When $\omega = 0$ there is no migration between the areas and we can perceive the system as two independent stocks, one which is not being harvested and one which is managed optimally.

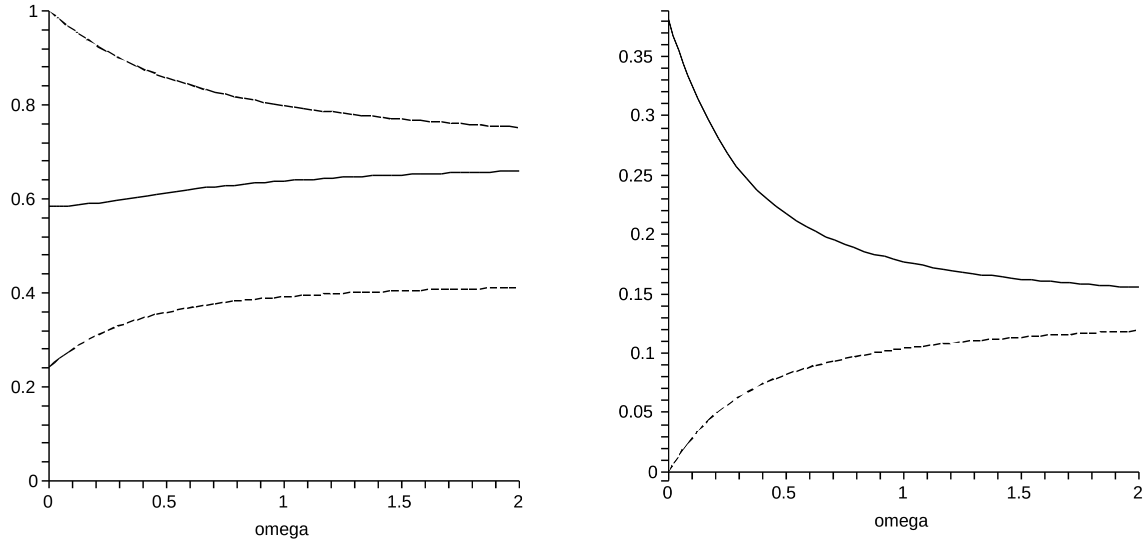

Figure 3.5: NEAC equilibrium as a function of ω, $s = 0.5$. In the left plot y is the upper curve, x is the middle curve and u is the lower curve. In the right plot m is the upper curve and n is the lower curve.

When ω increases from zero we expect that y will decline because of migration and that x will increase according to that. Catch will also increase, not because of the increase in x (note that $x \geq x_{MSY}$, which implies that the natural growth in x is declining), but because of the migration. The increase in catch can also explain the non-monotonic behavior of x close to zero. Remember that catch at equilibrium equals the natural growth in both areas,[4] see equation (3.12). As ω increases towards infinity, the catch stabilizes and the states converge to a common level. This is also reasonable behavior. For large ω values, any difference in the density will be corrected immediately and in practice the density in the two areas will be equal. From equation 3.13 we remember that $y \to x$ for $\omega \to \infty$. This is equivalent to the entire stock being accumulated in the unprotected area and thus managed as if there were no protected area. The same effort would be necessary to yield the same catch as in the case of no protected area, but the effort would be concentrated in the unprotected area instead of being spread out on the entire area; that is, the density would not change even though we imagine that the entire stock is situated in one area. If the dynamics in the model was divided into periods, where fishing occurred in one period and natural growth in another, as Anderson [1] has approached it, large ω values would probably overcompensate for differences in the densities, and in the worst consequence empty one of the areas in a very short time. That is not very realistic, but then again, very large ω values are not very realistic either. We do not encounter such problems in our aggregated model.

[4] Not the case for for example $\omega = 0$ or $s = 1$.

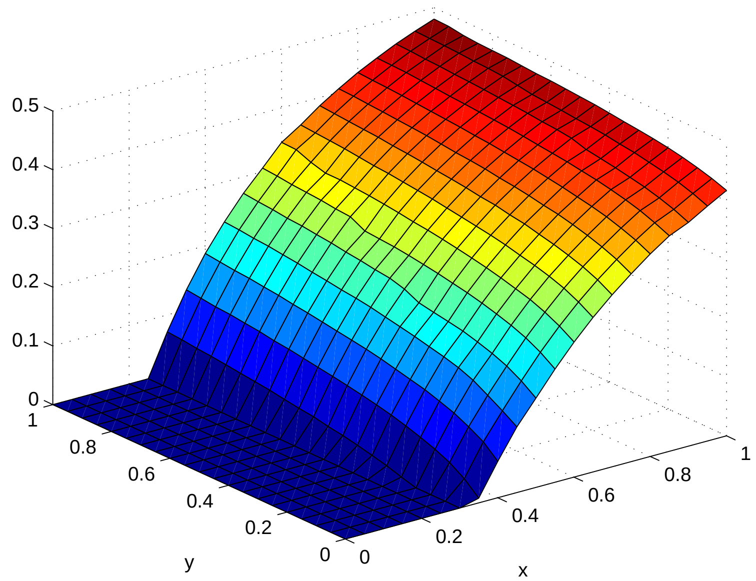

Figure 3.6: NEAC optimal policy; $s = 0.5$ and $\omega = 0.3$.

In the right plot in Figure 3.5 the shadowprices are displayed. When $\omega = 0$ there is no migration, and the stock in the protected area cannot be harvested from, and thus has no marginal value, $n = 0$. As ω increases, a larger part of the total stock is catchable, and the stock in the unprotected area increases. Both lead to a smaller marginal value of the unprotected stock, because it gets 'competition' from the stock in the protected area and from itself through the increase. Meanwhile, the protected stock decreases because of migration and the corresponding shadowprice increases, both because of the reduction in stock and because it is available for harvesting. For large ω values the shadowprices converge towards a common level, and there is no economic differences between the two areas, the stock is managed as if there were no protected area. In some aspects, s and ω work in 'opposite directions', that is, a larger ω has the effect that more of the total stock can be harvested from. A larger s has the opposite effect. The effects of different ω values are further discussed in the next example.

In the next figure, Figure 3.6, we see the optimal harvesting policy for NEAC as a feedback policy; $s = 0.5$ and $\omega = 0.3$. The policy generates a vector field in the state space; it is observed in Figure 3.7. The state will move around according to the vector field when the optimal policy is applied. It is evident from the vector field that an equilibrium occurs at $(x, y) = (0.5988, 0.8996)$, more data on the steady state in Table 3.3. The resulting policy is as expected, increasing with both x and y. This gives good reason to claim that a deterministic model is not sufficient to reveal the value of a marine protected area, since a nonzero protected area reduces catch. For the record, the shape of the policy in the x direction is in accordance with the results of Kugarajh, Sandal and Berge [22].

If we consider the results from the value-calculations (see Figure 3.8) which also are done within the Hamilton-Jacobi-Bellman iterations, we see that the value of the fishery will increase with both x and y. The value results look very flat in the figure, but the colors on the surface tells us that there is a slight increase in the y-direction. We can also see that the state has to be quite close to zero (each 'cell' is 0.05×0.05) before the value dramatically drops to zero. The reason for the steepness around the origin is that our model is logistic without *depensation*, which means that as long as the stock level is positive, there is positive natural growth in the stock, and the stock can recover from any stock level.[5] Value of the fishery for states close to the origin is small because of the discounting term and because the catch is zero close to the origin. Zero catch yields zero profit. The 'speed' of the natural growth is small for small stocks, and the time needed to reach a level which justifies nonzero catch grows longer as the stock gets closer to the origin, thus the 'corrosion' from the discounting term on future income is larger. Note that any nonzero stock level still has the potential to reach the optimal steady state equilibrium. The value from the fishery descends from the path to the equilibrium, and not from the equilibrium itself. In theory, the state never reaches the equilibrium, but it converges towards it. When we use the formulation *"the stock reaches the steady state"* we mean that the stock is close to the theoretic steady state, and in practice is equal to it. The value in the origin is of course zero; an empty sea has no value for a fisherman. With a zero discount rate, the calculations will not converge because the value will move to infinity (moving to equilibrium and staying there forever has infinite value when the discount rate is zero). But in Figure 3.9 we have plotted the value for a very small discount rate of 0.1 per cent. The results are as expected. The fishery has a much larger value for all possible states unless in the origin, where, of course, the value is still zero. These results confirm well established results in economic literature. We also see more clearly that the value increases with y; naturally, more fish in the sea has a larger potential value.

From Figure 3.7 we can deduce how the state will move around under the impetus of the optimal policy. The state will move towards the equilibrium in a monotone way for most possible state positions. We see, however, that for small x values and large y values (upper left corner of the plot) the stock in the protected area will first decay away from the equilibrium and increase later on for larger x values. Data on the steady state is found in Table 3.3. The density y for $s = 0$ is the value in the limit $s \rightarrow 0$. It predicts a total catch of 441 000 tonnes per annum. The total allowable catch (TAC) for 2005 is set at 485 000 tonnes by the Joint Russian-Norwegian Fisheries Commission (JRNC). What we

[5]This is not a good assumption. Very low stock levels will for most species, particularly those which are spread over vast areas, lead to negative natural growth; simply because they are not able to find each other and spawn. However, the assumption is good enough for our purpose. See Clark [7] for more on depensation.

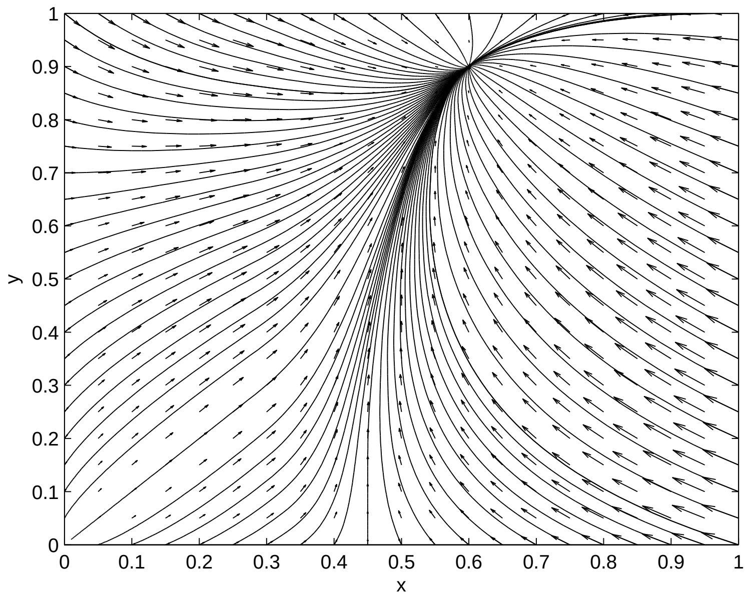

Figure 3.7: NEAC vector field, field lines are indicated; $s = 0.5$ and $\omega = 0.3$.

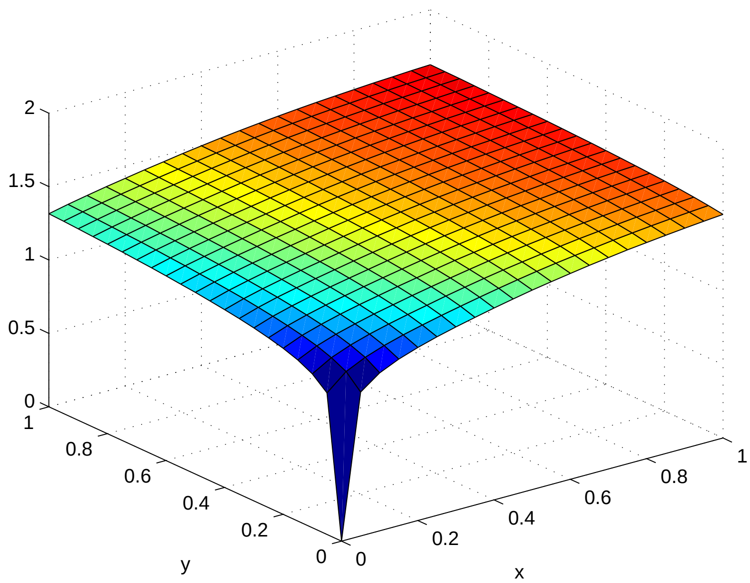

Figure 3.8: NEAC value of fishery; $s = 0.5$ and $\omega = 0.3$.

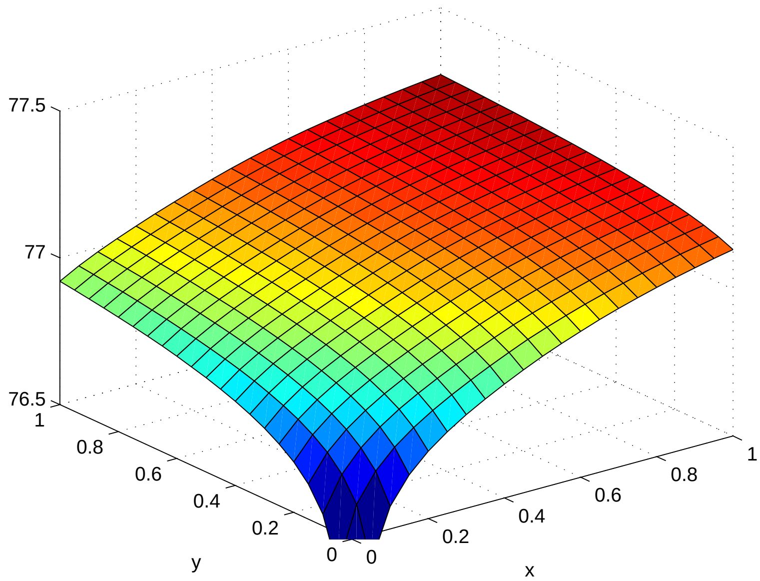

Figure 3.9: NEAC value of fishery; $s = 0.5$, $\omega = 0.3$ and $\delta = 0.1\%$.

cannot deduce from Figure 3.7 is *how fast* the system approaches the steady state. This is visualized in Figure 3.10. Note that the plot show real-world values. The unit along the axes in the first plot is 1000 tonnes, in the second plot the unit along the horizontal axis is years and along the vertical axis the unit is 1000 tonnes for the stocks and 1000 tonnes per annum for the catch. The figure shows the development through state-space and time for the initial condition $(x^0, y^0) = (100, 100)$. Maybe a better initial position would be $(x^0, y^0) = (400, 400)$, closer to the free-entry limit, which is $x_0 = 842$. The idea is then that the fishery has reached the open-access solution before any protected area is introduced. Thus the density in both areas would be the same and adding up to the free-entry limit. One might argue that the resource is well managed already, but according

		$s = 0.5$	$s = 0$	Unit
Relative unprotected stock	:	0.5988	0.6882	
Relative protected stock	:	0.8996	0.9236	
Total standing stock	:	4 296 000	3 947 000	tonnes
Unprotected standing stock	:	1 717 000	3 947 000	tonnes
Protected standing stock	:	2 579 000	0	tonnes
Annual catch (TAC)	:	441 000	572 000	tonnes/year

Table 3.3: NEAC steady state equilibrium data, comparing the cases $s = 0.5$ and $s = 0$ when $\omega = 0.3$.

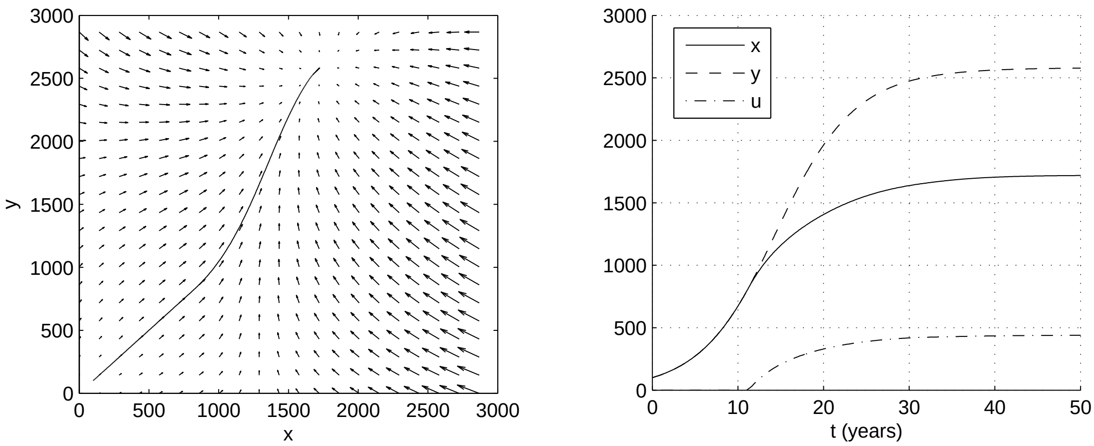

Figure 3.10: NEAC vector field and state development; $s = 0.5$ and $\omega = 0.3$. These plots show real-world values. On the left, the development of the state through the state space is displayed. On the right, the same development through time is shown; x is the solid curve, y is the upper dashed curve and u is the lower dashed curve. The unit along the axes in the left plot is 1000 tonnes. The units along the vertical axis are 1000 tonnes for the stocks and 1000 tonnes per annum for the catch in the right plot. The initial position is $(x^0, y^0) = (100, 100)$.

to Kugarajh, Sandal and Berge [22] *"[...] the bliss harvest path [...] tracks the historical harvests quite closely."*[6] Thus we can expect that today's stock level is close to the free-entry limit. We are expecting that the initial position is close to the line $x = y$ (equal densities) before the protected area is deployed, and a position close to zero reveals more of the development. We observe that x and y have equal growth in the beginning, when catch is zero. As soon as catch is positive, the development in x is slowed down. The situation reaches equilibrium after about 40 years. A total stock level of 200 000 tonnes would probably not be characterized as *within safe biological limits,* and we see that the stock needs several years to reach a stock level which can support a profitable fishery.

Next, we take a look at the results from choosing $s = 0.35$. The results are presented in Figures 3.11 - 3.13. Unfortunately, we cannot directly compare Figures 3.6 and 3.11, which represent NEAC catch for the cases where $s = 0.5$ and $s = 0.35$, respectively, because the scaling of the catch is dependent on s. Appropriate plots would show that catch is lower on a general basis for a larger protected area.[7] We can, however, observe that catch increases more rapidly as y rises for the first case. The vector field in Figure 3.12 follows

[6]The bliss solution maximizes the integrand at each point in time, and is equivalent to assuming an infinite discount rate, which again gives orders to fish like there is no tomorrow. This leads the system to the open-access solution, where catching one more unit does not increase profit.

[7]This is also the case for the value; it decreases for all possible states with a larger s value. A value plot is not included.

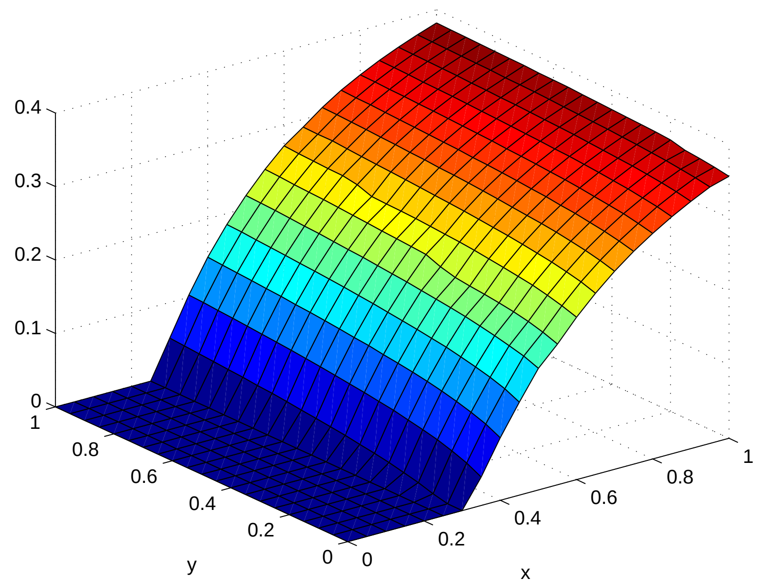

Figure 3.11: NEAC optimal policy; $s = 0.35$ and $\omega = 0.3$.

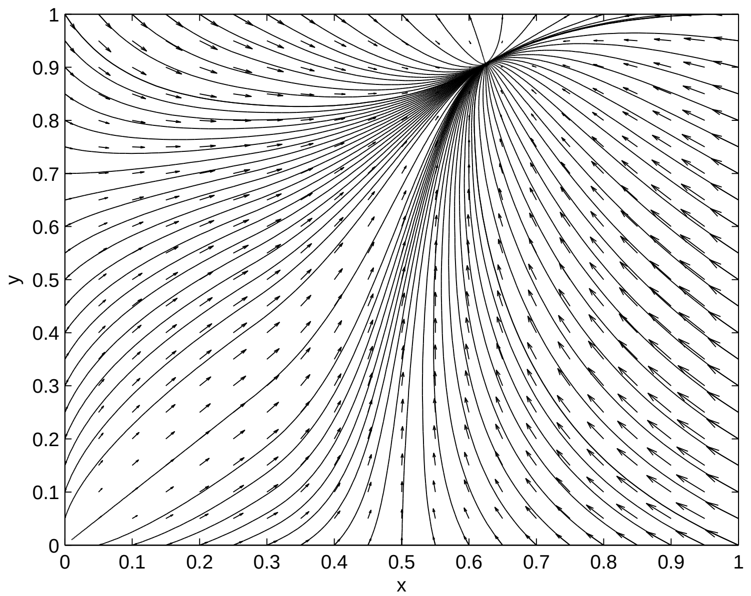

Figure 3.12: NEAC vector field, field lines are indicated; $s = 0.35$ and $\omega = 0.3$.

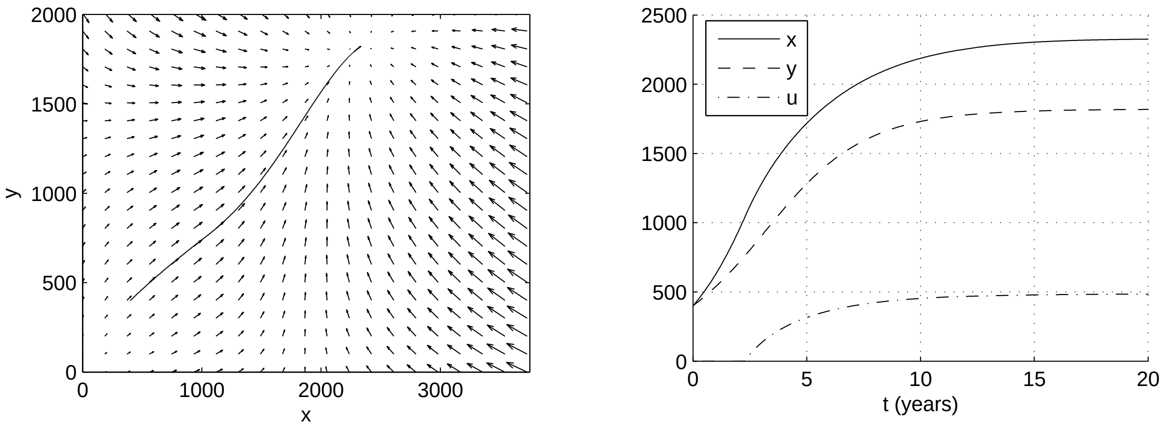

Figure 3.13: NEAC vector field and state-development in real-world values, starting from $(x^0, y^0) = (400, 400)$; $s = 0.35$ and $\omega = 0.3$. To the left the development of the state variable through space is displayed. To the right the development through time is displayed; x is the solid curve, y is the upper, dashed curve and u is the lower, dashed curve. In the left plot the unit along the axis is 1000 tonnes. The unit along the right vertical axis is 1000 tonnes for the stocks and 1000 tonnes per annum for the catch.

the same pattern as the vector field for $s = 0.5$ (Figure 3.7). A steady state equilibrium is indicated at $(x, y) = (0.6266, 0.9072)$, in fact very close to the steady state for the $s = 0.5$ case, which was situated at $(x, y) = (0.5988, 0.8996)$. More data on the steady state is found in Table 3.4. Figure 3.13 shows a somewhat different behavior of the development of the system through time than the equivalent development shown in Figure 3.10. Starting from $(x^0, y^0) = (400, 400)$ we observe that x increases at a higher rate than y when catch is zero. The larger standing stock is found in the unprotected area, but the density is higher in the protected area, as always in equilibrium. Comparing Figures 3.10 and 3.13 suggests that a system with a larger protected area needs more time to reach the equilibrium.

		$s = 0.35$	$s = 0$	Unit
Relative unprotected stock	:	0.6266	0.6882	
Relative protected stock	:	0.9072	0.9236	
Total standing stock	:	4 157 000	3 947 000	tonnes
Unprotected standing stock	:	2 336 000	3 947 000	tonnes
Protected standing stock	:	1 821 000	0	tonnes
Annual catch (TAC)	:	484 000	572 000	tonnes/year

Table 3.4: NEAC steady state equilibrium data, comparing the cases $s = 0.35$ and $s = 0$, where $\omega = 0.3$.

Finally, before moving on to the next example, we will provide plots that show the behavior of the steady state equilibrium as the b-parameter varies. Recall that b is the coefficient of the nonlinear term in Π, and $b = 0$ leads to MRAP solutions and trivial bang-bang controls, as demonstrated in chapter 2.3. The parameters are set at $s = 0.5$ and $\omega = 0.3$, as earlier; the plots are displayed in Figure 3.14. Note that b usually varies with s, but here we keep s fixed and let b vary independently. If we let s vary too, we would get the same results as those in Figure 3.4, where the effects of s was investigated. The results show that when b increase, the standing stocks increases while the optimal annual catch declines together with the shadowprices. b is the coefficient of a negative term in Π, which can be perceived as a cost term. Larger costs indicate smaller production levels and reduced profit. Our results agree with this. When b approaches infinity harvest yields very high costs, thus the harvest will approach zero and the standing stocks approach the carrying capacity. Increasing standing stocks yield falling marginal values.

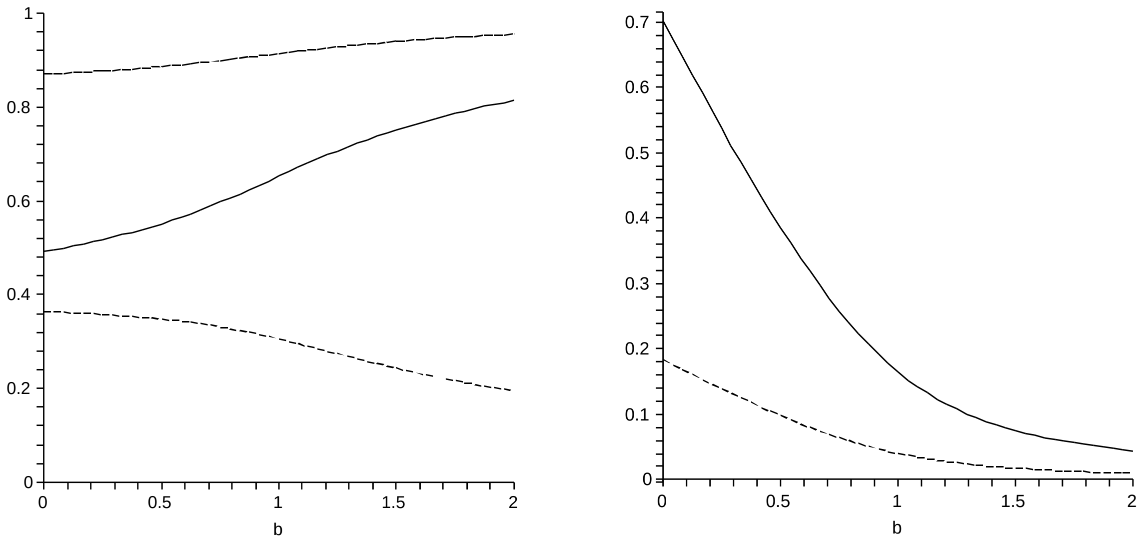

Figure 3.14: NEAC equilibrium as a function of b, $s = 0.5$ and $\omega = 0.3$. In the left plot, y is the upper curve, x is the middle curve and u is the lower curve. In the right plot, m is the upper curve and n is the lower curve.

3.3.2 Canadian Northern Cod

Northern Cod (NC in short) is a cod stock which is found east and north-east of Newfoundland in Canada, in the NAFO[8] regions 2J, 3K and 3L. The stock has been commercially

[8]North-west Atlantic Fisheries Organization, [24].

exploited for centuries and was once amongst the most productive and valuable fisheries in the world. Grafton, Sandal and Steinshamn study this fishery in [13]. After a linearization of their demand-function, based on the data on p. 577 in [13], the profit function matches our own profit function, which is

$$\Pi(h, x) = p\left(1 - \frac{x_0}{x}\right) h - ch^2.$$

We end up with the parameter values given in Table 3.5. Note that the linear price function is a very simplified approach to the original function, and the results should be interpreted with caution. In [13] a generalized form of the logistic growth function is applied; we approximate it by setting $\alpha = 1$. α is the generalizing parameter and $\alpha = 1$ yields the logistic growth functions. Different values of α skews the MSY stock level in the natural growth function to either left or right. Please turn to Clark [7] for more on generalized growth functions.

Parameter	Value	Explanation
r	0.30355	Intrinsic growth rate
K	3200.000	Carrying capacity
p	1.375	Price-parameter
c	0.001110	Cost-parameter
x_0	146.007	Free-entry limit

Table 3.5: NC parameter values derived from Grafton, Sandal and Steinshamn [13].

We set the discount rate at 5 per cent as for the NEAC case. The scaled parameter values are computed in Table 3.6. As with the NEAC, we choose to scale x_0 against the carrying capacity, K, instead of the carrying capacity in the unprotected area, $K(1-s)$.

Parameter	Formulae	Value
b	$\frac{cr}{p}K(1-s)$	$0.7842\ (1-s)$
γ	$\frac{\delta}{r}$	0.16472
X_0	$\frac{x_0}{K}$	0.04565

Table 3.6: NC scaled parameters.

Again we choose to consider the results when $s = 0.5$ and $\omega = 0.3$. The steady-state equilibrium occurs at $(x, y) = (0.3976, 0.8417)$, and the predicted catch in equilibrium is $u = 0.3727$. Data on the predicted steady state is found in Table 3.7. An interesting feature about the steady state is that the standing stock level is below the local MSY stock level, 'local' referring to the actual area and not the entire habitat. The corresponding figures for the case $s = 0$ are also present in the same table, and we observe that the standing stock level for no protected area is also below the MSY stock level. This is not in agreement with the results of the analysis in [13], where the equilibrium is indicated above the MSY level.

In Figure 3.15 the equilibrium solution as a function of ω is displayed. The behavior is much the same as the behavior of the NEAC case. Catch takes values in the same area in both cases, and standing stocks and marginal values seek to a common level for large ω values. Apart from different densities and marginal values in the two cases, the stronger non-monotonic behavior of x is worth noticing. When $\omega > 0$, a smaller x will induce a higher migration. The density y decline in a monotonic manner and for a certain ω value (in the neighborhood of $\omega = \frac{1}{2}$) the migration cannot compensate for a decreasing x and x increases beyond that point.

In Figure 3.16 we see the optimal policy for the NC fishery with a protected area size of $s = 0.5$. The corresponding value function is displayed in Figure 3.17. Both the policy and the value is increasing with both x and y, as for the NEAC case. The state will move around according to the vector field displayed in Figure 3.18. We observe that the vector field behaves differently than the typical pattern seen for the NEAC case. Again we observe that the steady state equilibrium occurs for $x < x_{MSY}$. In Figure 3.19, we see the path of the state through time, with the starting position in $(x^0, y^0) = (70, 70)$, which is close to the open-access solution. The model predicts a moratorium on fishing for more than 5 years and then approaches a level quite close to the equilibrium solution during the next 10 years. We observe that the stocks need longer time to settle down, without affecting the catch too much. There is a dramatic change in the growth of x as the fishing moratorium is removed.

		$s = 0.5$	$s = 0$	Unit
Relative unprotected stock	:	0.3976	0.4683	
Relative protected stock	:	0.8417	0.8628	
Total standing stock	:	1 983 000	1 498 000	tonnes
Unprotected standing stock	:	636 000	1 498 000	tonnes
Protected standing stock	:	1 347 000	0	tonnes
Annual catch (TAC)	:	181 000	242 000	tonnes/year

Table 3.7: NC steady state equilibrium data, comparing the cases $s = 0.5$ and $s = 0$ when ω is kept fixed at $\omega = 0.3$.

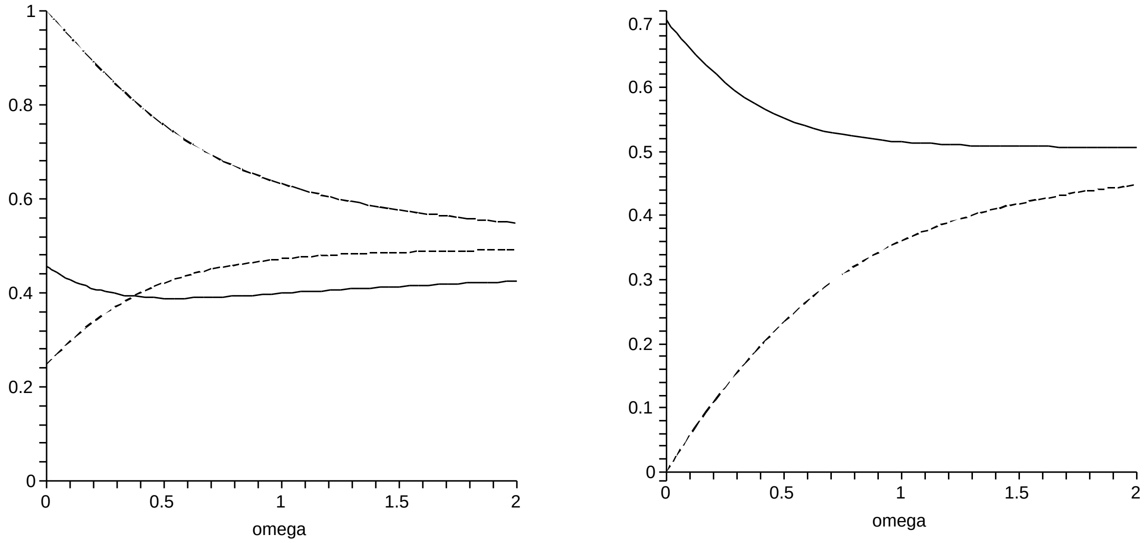

Figure 3.15: NC equilibrium as a function of ω, $s = 0.5$. In the left plot, y is the upper curve, u is the middle curve and x is the lower curve at the right hand side of the plot. In the right plot, m is the upper curve and n is the lower curve.

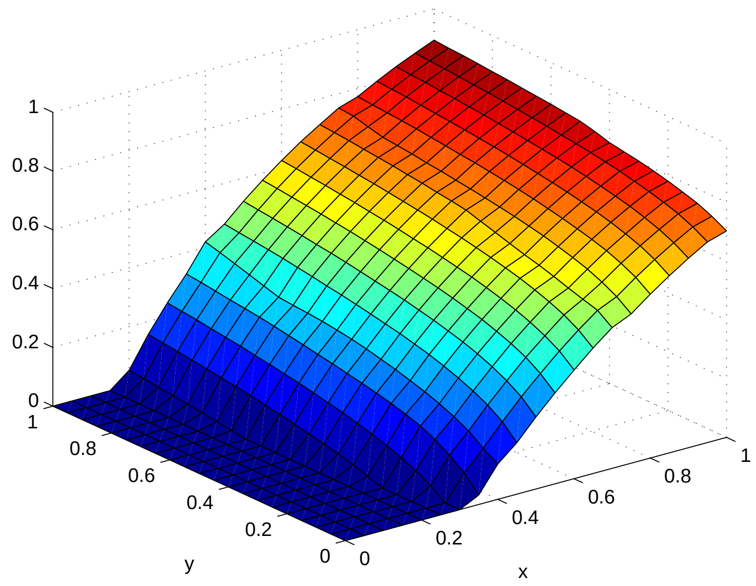

Figure 3.16: NC optimal policy; $s = 0.5$ and $\omega = 0.3$.

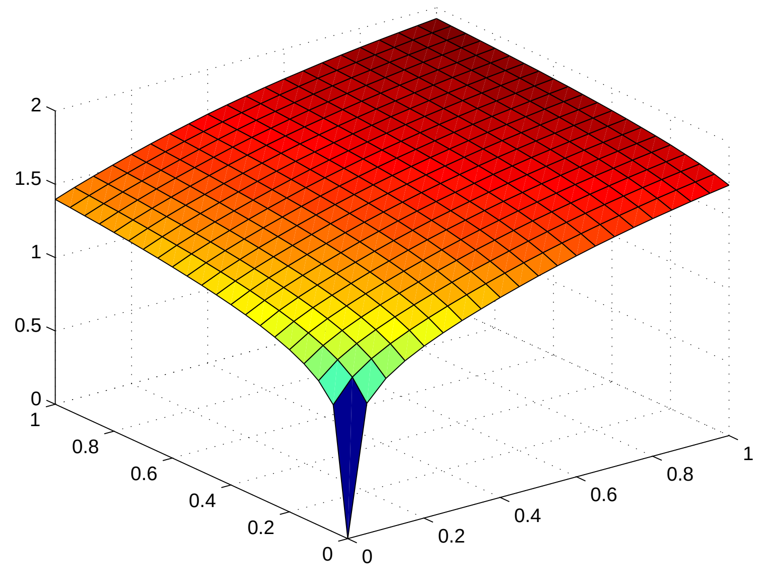

Figure 3.17: NC value of fishery; $s = 0.5$ and $\omega = 0.3$.

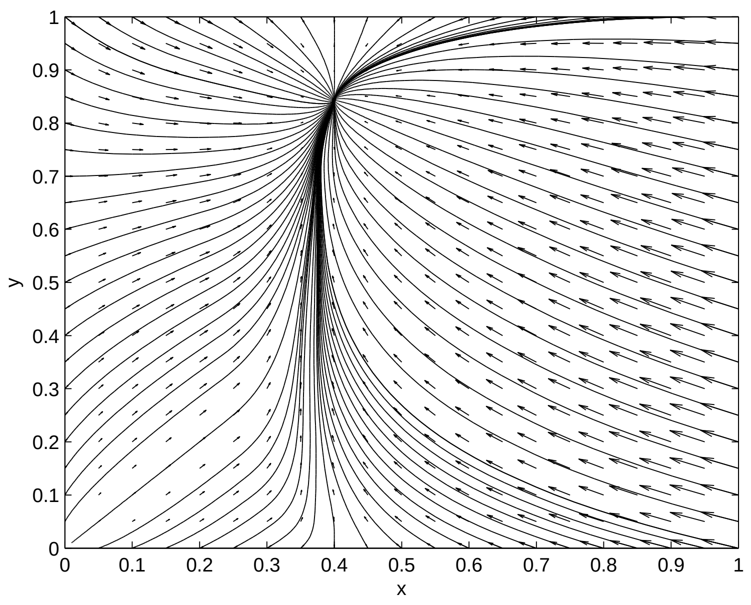

Figure 3.18: NC state vector field, field lines are indicated; $s = 0.5$ and $\omega = 0.3$.

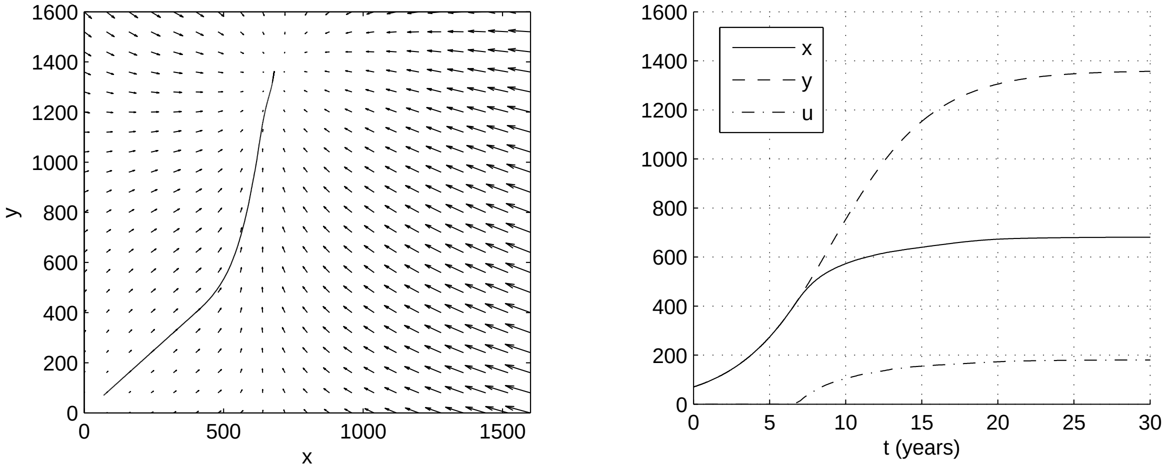

Figure 3.19: NC vector field and state-development in real world values, starting from $(x^0, y^0) = (70, 70)$; $s = 0.5$ and $\omega = 0.3$. In the left plot, the development of the state variable through the state space is displayed. In the right plot, the same development through time is shown; x is the solid curve in the middle, y is the upper dashed curve and u is the lower dashed curve. The unit along the vertical axis is 1000 tonnes for the stocks and 1000 tonnes per annum for the catch.

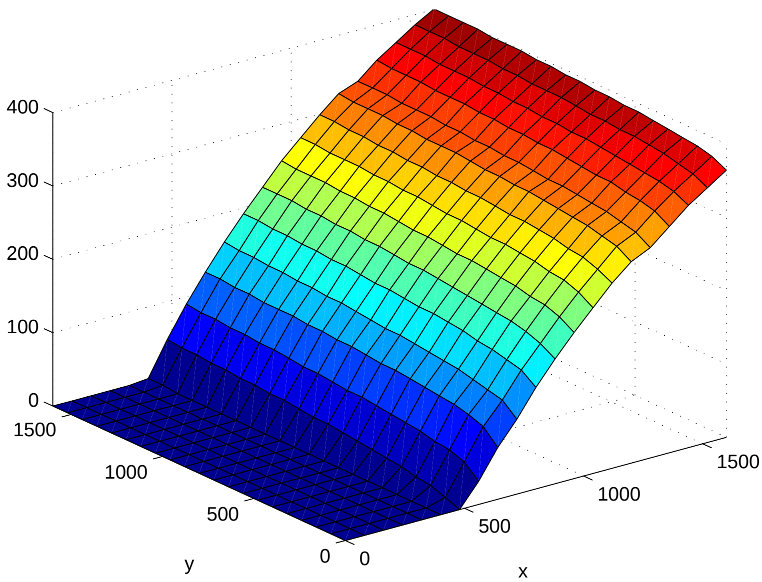

Figure 3.20: NC optimal policy; $s = 0.5$ and $\omega = 0.1$.

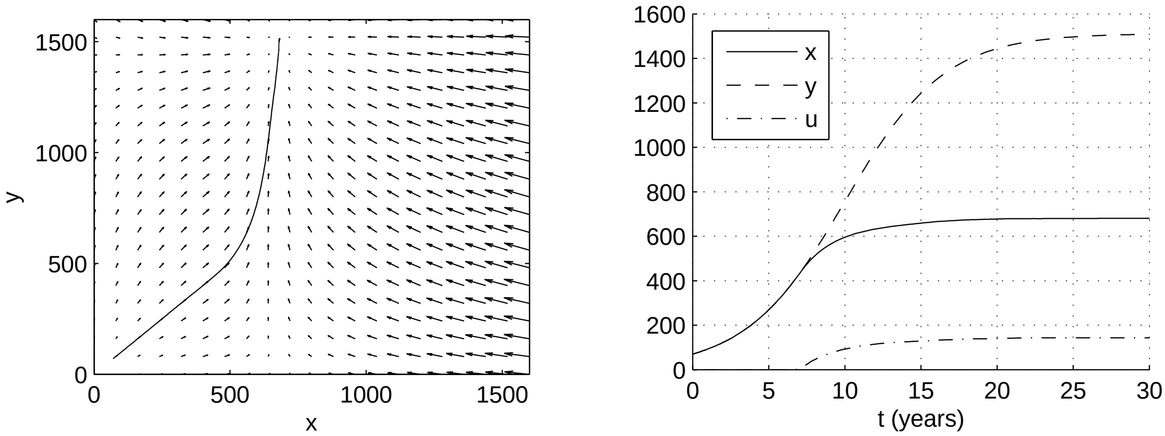

Figure 3.21: NC vector field and state-development in real world values, starting from $(x^0, y^0) = (70, 70)$; $s = 0.5$ and $\omega = 0.1$. In the left plot, the development of the state variable through the state space is displayed. In the right plot, the same development through time is shown; x is the solid curve in the middle, y is the upper dashed curve and u is the lower dashed curve. The unit along the axes in the left plot is 1000 tonnes. In the right plot, the unit along the vertical axis is 1000 tonnes, per annum for the catch.

The last result we include in this chapter is obtained when setting $\omega = 0.1$, while $s = 0.5$. The results are shown in Figures 3.20 and 3.21, where we see the optimal policy and the development of the state variable through space and time respectively. Note that both the plots are shown in real-world values. As earlier established, a smaller ω value yields smaller changes in the policy in the y direction. Also, the catch is smaller for all possible states for smaller ω values. From the development plots we see that the stock in the habitat almost reaches the carrying capacity, that is $\frac{K}{2} = 1\ 600\ 000$ tonnes, as the migration has lesser impact. From an initial total stock of 140 000 tonnes it takes more than 5 years before the moratorium is lifted, and the annual steady state catch is 144 000 tonnes. Comparing Figures 3.19 and 3.21, it is readily seen that a smaller migration coefficient yields larger standing stocks and smaller annual catches. Thus the results agree with Figure 3.15.

Chapter 4

Conclusions

In the first section of this chapter, we summarize and discuss our results and try to view them in light of the literature that exists on marine protected areas and related topics. In the last section, we point out possible interesting extensions of the model.

4.1 Discussion

In Chapter 3, several equilibrium solutions and feedback policies are demonstrated. One of the unexpected behaviors we have observed is the non-monotonicity of x for small values of ω. We expected that x would increase because $y > x$ in equilibrium and $\omega > 0$ yields migration into the open area. We also expected that u would increase because a larger ω leaves a larger part of the total stock available for harvesting. The declining x for small ω yields an increasing negative term in Π; thus as long as the harvest from the migration compensates for the increasing costs, it will be optimal to reduce the stock in the open area. Also, a smaller x leads to a larger migration, that is, a reduction in x yields increasing costs and increasing benefits. Regarding the migration, numerical investigations indicate that as ω increases, the difference in density is decreasing in a monotonic fashion, but the migration increases because of the increasing coefficient. The difference in density will never become zero; remember that we are discussing the situation in equilibrium and thus $y > x$.

Further reflections on the migration coefficient lead to a natural upper limit of 1 for the parameter. When $\omega > 1$, it yields a migration which is larger than the actual difference in density. Thus the investigated scenarios of $\omega > 1$ are presumably pathological. One could probably argue for larger ω values in special cases, for example that large differences in densities could initiate a 'race' after the preferred areas with smaller density and thus lead to a migration larger than $\omega = 1$ predicts.

In the tables where the data on the different equilibriums is found, we can observe that the density in the protected area is positive when $s = 0$, i.e. there is no protected area. When there is no area, density makes no sense. The computed density must be apprehended as density in the limit $s \to 1$. The model implicitly assumes that there exist two areas, and the cases where there is only one area ($s = 0$, $s = 1$) has to be treated carefully. The results when $s = 0$ do, however, match the results from treating the problem as one dimensional with no protected area.

We have observed that the optimal steady state stock level for the Canadian Northern Cod is $(x, y) = (0.3976, 0.8417)$ for $s = 0.5$. Even though the aggregated density is above the aggregated MSY stock level, i.e. $x + y > 1$, one might want to question this solution. Since harvest equals natural growth in equilibrium, a position exists such that $x > x_{MSY} = \frac{1}{2}$ and it yields the exact same catch as at the computed equilibrium. We can even calculate this position; it is $(x, y) = (0.6024, 0.8417)$. This position leads to a larger current profit, since the negative term $\frac{x_0}{x}$ is smaller, which could be comprehended as the fish being easier to catch and the same harvest is obtained with a smaller effort level. The explanation to this seemingly strange behavior rests with the migration term. Note that the difference in density is much larger in the inferior position, and thus the migration. An equilibrium simply cannot exist in the superior position, because the migration is not equal to the natural growth in the protected area. The harvest in equilibrium must equal the natural growth in the open area plus the migration, and the migration must equal the natural growth in the closed area, thus the harvest equals the total natural growth. In other words, the superior position does not coincide with the equilibrium curve, as demonstrated in Figure 3.1.

In the feedback policies we observe that the moratorium level changes with y. This seems perfectly reasonable, as a larger total stock induces a larger catch. We discussed the equation $\mathscr{H}_u = 0$ for the event of equilibrium, and found that when $x > x_0$, optimal catch is zero if $m > 1 - \frac{x_0}{x}$. The argument holds when not at equilibrium too and indicates that the stock level in the closed area has a decisive impact on the shadowprice m. This is no surprise, we have seen that the equations governing the system are all related. But it gives us insight in how the system works, and demonstrates that a larger protected stock yields a smaller shadowprice m.

We have explored marine protected areas by introducing them together with an optimal feedback policy in the unprotected area. The literature suggests that protected areas decrease revenues, see for example Anderson [1]; *"With marine reserves the sustainable yield for any total stock size will always be less than or equal to the status quo sustainable yield for that total stock size. The lower migration rate, the lower will be the sustainable yield. Seen from the other side, the cost of producing any sustainable level of catch will be higher with marine reserves."* Holland and Brazee [18] and Hannesson [15] agree. While revenues and catches decline, the literature also suggests that protected areas increase the total standing stocks, see Sumaila and Charles [32], Botsford *et al.* [4] and all of those above. Our analysis agrees with all of this.

We still believe that protected areas can be an effective way of management. *"[...] why use* *reserves? One answer to this question is that they should be used because effort is so very difficult to control because of a constant resistance to catch limitations. Reserves provide a buffer against increasing effort [...]"*, Botsford *et al.* [4]. Another argument in favor of protected areas is that *"[...] checking that no one is fishing on a reserve is about the simplest possible scheme to enforce [...]"*, Pezzey, Roberts and Urdal [25]. Moreover, *"[...] 'marine protected areas' (MPAs) [...] may provide a potentially positive tool. If used properly, and if drawn up in ways that involve and make biological sense for fishermen, MPAs may also become an effective tool in sustaining our fisheries by protecting key habitat and nursery areas."* Spain [31]. Even though the value of marine protected areas is somewhat unclear, the literature signals a firm belief that protected areas has the potential to be a strong management tool if used in the right way.

Grafton, Sandal and Steinshamn analyze feedback solutions in [13], and conclude that *"The results do not imply that the optimal management [...] involves only the application of a feedback rule."* This thesis suggests that protected areas and feedback rules couple nicely together as management tools. MPAs provides stability in a nondeterministic world, and feedback rules take care of the potential shocks that may occur on an irregular basis. As mentioned above, an MPA is a management tool that is easy to enforce; TAC policies are usually not. MPAs can work as a buffer against economic fluctuations when enforcing the TAC policy fails. We have demonstrated that small protected areas have little influence on steady state catches. The question is: Has the stability provided by a protected area a larger value than the reduction in profit? More research is needed to answer this question.

To summarize our work, we have modeled the dynamics of a fish stock whose habitat is partly covered by a protected area. We have analyzed the characteristics of a steady state equilibrium, computed the equilibrium curve in the state space and demonstrated a method for determining the optimal equilibrium. More importantly, we have demonstrated a method that yields the optimal policy which leads to the optimal steady state for all possible stock levels. Furthermore, we have investigated how the system develops through space and time. The next section discusses several possible extensions of the work.

4.2 Further work

There are several extensions of this model that could be interesting research topics. One necessary issue that would need attention if this work should be continued is optimization of the computer programs developed.

Maybe the most obvious extension is to introduce stochastic growth functions, even though we have spent some time arguing why we did not consider stochastics in this approach. There are signs in the literature indicating that stochastic models leave the marine protected areas a larger value than what we have seen in this thesis. Stochastics would, however, make the results harder to obtain. An adaptive grid, as described in Grüne and

Semmler [14], would not only then be an interesting tool to speed up the computations, but could also in our model speed up computations and even give more accurate solutions.

Another natural extension would be to introduce a more general profit function Π. This would make the model more interesting in a real-world management setting. The model could then readily be applied to a vast range of examples of renewable resources, not only marine ones.

There are a few extensions of the model that would make it a better approximation of reality, and which better could simulate real-world decisions. One is to assume that the parameter s, which gives the relative size of the protected area, is subject to changes. In our approach we have assumed s given, but in reality it is a decision parameter. In the model, s could then be a control parameter. Another extension would be to make the migration coefficient ψ dependent on s. This is an environmental parameter which has to be measured. In our model, a migration coefficient dependent on s would give more realistic results in the limit $s \to 1$, and the model would provide a better foundation for decision making. Alternatively, one could assume that the stock was not uniformly dispersed throughout its habitat. The migration would then depend on where the boundaries between the protected and the unprotected areas were drawn. Other amendments of the model are discussed in Chapter 1.3.

The last extension we mention is to introduce catch in both zones, where for example one of the areas is optimally managed through TAC quotas and the other is subject to an open access situation. Such an extension would make the model a lot more complex, but then it could maybe be applied to other scenarios than the sole-owner assumption, on which this entire thesis in fact rests upon.

Appendix A

Derivations

A.1 Necessary Conditions for Optimum

In this section, we derive the necessary conditions for optimum in the simplest optimal control problem. The derivations follow Kamien and Schwartz [21].

The system we want to solve is given in (2.1). We will derive the conditions for one state variable and one control variable. The extensions to several states and controls is straight forward. Let $\lambda(t)$ be a continuously differentiable function, called a Lagrange-multiplier. We have

$$\int_{t_0}^{t_1} \Pi(x(t), u(t), t)\, \mathrm{d}t = \int_{t_0}^{t_1} \left[\Pi(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) - \lambda(t)\dot{x}(t)\right] \mathrm{d}t.$$

When we integrate the last term by parts, we get

$$-\int_{t_0}^{t_1} \lambda(t)\dot{x}(t)\, \mathrm{d}t = -\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0) + \int_{t_0}^{t_1} x(t)\dot{\lambda}(t)\, \mathrm{d}t.$$

which gives

$$\begin{aligned}\int_{t_0}^{t_1} \Pi(x(t), u(t), t)\, \mathrm{d}t \;\; = \;\; & \int_{t_0}^{t_1} \left[\Pi(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t) + x(t)\dot{\lambda}(t)\right] \mathrm{d}t \\ & -\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0).\end{aligned}$$

Assume $u^*(t)$ to be the optimal path for the control, and that $x^*(t)$ is the associated optimal path for the state variable. We construct a family of comparison controls $v(t, a) = u^*(t) + ah(t)$, where $h(t)$ is a fixed function. Let $y(t, a)$ be the associated comparison path. We assume that $y(t, a)$ is a smooth function of both its arguments. All the comparison paths satisfy the following:

$$y(t, 0) = x^*(t), \quad y(t_0, a) = x_0.$$

Now we write

$$\begin{aligned} V(a) &= \int_{t_0}^{t_1} \Pi(y(t,a), v(t,a), t)\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \Big[\Pi(y(t,a), v(t,a), t) + \lambda(t) f(y(t,a), v(t,a), t) + y(t,a)\dot{\lambda}(t)\Big]\,\mathrm{d}t \\ &\quad -\lambda(t_1) y(t_1, a) + \lambda(t_0) y(t_0, a). \end{aligned}$$

$u^*(t)$ maximizes $V(a)$ so $V_a(0) = 0$.[1] Thus,

$$V_a(0) = \int_{t_0}^{t_1} \Big[(\Pi_x + \lambda f_x + \dot{\lambda}) y_a + (\Pi_u + \lambda f_u) h\Big]\,dt - \lambda(t_1) y_a(t_1, 0) = 0. \tag{A.1}$$

We suppress arguments in the last equation so the structure is more visible. Remember that $y(t_0, a) = x_0$ so $y_a(t_0, a) = 0$. If we let

$$\begin{aligned} &\dot{\lambda}(t) = -\Pi_x(x^*(t), u^*(t), t) - \lambda(t) f_x(x^*(t), u^*(t), t), \quad \lambda(t_1) = 0 \\ &\Pi_u(x^*(t), u^*(t), t) + \lambda(t) f_u(x^*(t), u^*(t), t) = 0, \end{aligned}$$

it is readily seen that (A.1) holds for any function $h(t)$. Together with the equation for $\dot{x}$, these conditions are necessary for optimum. Kamien and Schwartz [21] show that these conditions also are sufficient.

If we form the Hamiltonian,

$$H(x(t), u(t), t) = \Pi(x(t), u(t), t) + \lambda(t) f(x(t), u(t), t),$$

the necessary conditions are summed up in what we call the first-order-conditions (FOC) for optimality. We suppress all arguments:

$$\left.\begin{aligned} H_\lambda &= \dot{x} \\ H_x &= -\dot{\lambda} \\ H_u &= 0 \end{aligned}\right\} \tag{A.2}$$

To extend these results to problems in several variables we need to assign a multiplier $\lambda_i(t)$ to each state variable.

[1] Subscript is denoting the derivative, $V_a(a) = \frac{dV(a)}{da}$, or the partial derivative, $\Pi_x = \frac{\partial \Pi}{\partial x}$.

Appendix B

Program Listing

Here, all the program codes developed to solve the problems are listed. In the first section, one finds a Maple code used to examine the steady state equilibrium of the system. In the second section, one finds the Matlab m-files which compute the feedback solutions.

B.1 Computing the Equilibrium Solution

The following Maple commands define a procedure `lsn`, which computes the steady state equilibrium for a set of parameter values. `lsn` can be used to find specific numerical solutions or to plot the equilibrium as a function of one of the parameters. After the declaration beneath, the call `lsn(gamma,b,x0,s,omega)`, with numeric values assigned to the parameters, returns the solution `[x,y,u,m,n]`.

```
Initiation.
>restart;
 unprotect(gamma);

Define the equilibrium equations.
>lkv:=(x,y,u,m,n,gamma,b,x0,s,omega)->
      {x*(1-x)+s/(1-s)*omega*(y-x)-u=0,
      y*(1-y)-omega*(y-x)=0,
      gamma*m-(u*x0/x+m*(1-2*x-s/(1-s)*omega)+n*omega)=0,
      gamma*n-m*omega*s/(1-s)-n*(1-2*y-omega)=0,
      1-x0/x-2*b*u-m=0,
      x>=0,y>0,u>=0,m>=0,n>=0,x<=1,y<=1,u<=1,m<=10,n<=10};

Create a procedure that finds the solutions and picks the right one.
>lsn:=proc(gamma,b,x0,s,omega)
      local test,x,y,u,m,n;
      if not type([gamma,b,x0,s,omega],list(numeric)) then
       return('lsn(gamma,b,x0,s,omega)')
```

```
else
 test:=solve(lkv(x,y,u,m,n,gamma,b,x0,s,omega),{x,y,u,m,n});
end if;
assign(test);
[x,y,u,m,n];
end proc;
```

B.2 Computing Feedback Solutions

In this section the programs used to compute the feedback solutions are listed. These are all Matlab m-files. `HJBit` is the main program, and is the one called from the command line in Matlab. It calls the programs `Pmatrix2`, `policyit` and `valueiteration`. `HJBit` runs through the Hamilton-Jacobi-Bellman iteration scheme until needed precision is reached, where the assumed error is the largest change in the value and policy matrices before and after an iteration. Note that `HJBit` does not check that the initial policy-iteration takes the policy away from the zero policy. This should be implemented if the procedure is used for other problems. The flaw has not caused problems for our tasks, as the first iteration has always produced a nonzero policy. All the Matlab functions described in this chapter have explanatory comments after the declaration, so that for example the call `help HJBit` in Matlab will return the comments. These comments are necessary to understand how the programs works and what they do. Thorough knowledge of the model is also required. Note that we actually work on a 3-dimensional grid, where the stocks x and y and the policy u are the dimensions. We want to find the optimal level of u in each possible (x, y) position. `HJBit` also prints the last estimated error to screen as it runs through the iteration-scheme. The routines should be easy to modify to fit other problems.

HJBit.m:

```
function [V,u,X,Y,U,M,vfeil,ufeil]=
      HJBit(x1,y1,u1,h,beta,f,g,P,vtol,utol,d,x_eq,y_eq)
% [V,u,X,Y,U,M,vfeil,ufeil]=HJBit(x1,y1,u1,h,beta,f,g,P,vtol,utol,d,x_eq,y_eq)
% is a implementation of the HJB equation on a marine protected area
% problem. The routine has given valid results for other problems too, but
% needs more testing. Beneath follows a short explanation of the in and out
% parameters.
%
% Output: V          is a matrix holding the VALUE in every grid position.
%         u          is a matrix holding the optimal POLICY in every grid
%                    position.
%         X, Y and U are 3-dimensional matrices holding information of the
%                    grid, X is holding the positions in the 1st direction
%                    and so on. These are all made by the Matlab function
```

```
%                    MESHGRID, see help MESHGRID for more.
%          M         This 4-dimensional matrix holds the vector field of the
%                    state and the value of the profit function in each
%                    grid position.
%          vfeil     is a matrix holding the development of the change in
%                    the V matrix during the iterations. For every 100th
%                    valueiteration and every 5th policyiteration, the
%                    largest absolute change in V is added at the end of
%                    vfeil and printed to the screen. Used as
%                    estimated error for V.
%          ufeil     is a matrix holding the development of the change in
%                    the u matrix during the iterations. For every 5th
%                    policyiteration, the largest absolute change in u is
%                    added at the end of ufeil and printed to
%                    screen. Used as estimated error for u.
%
% Input: x1, y1 and u1 are row vectors which holds the grid positions, x1
%                      in the 1st direction and so on. These are put into
%                      MESHGRID and produces X, Y and U, see help MESHGRID
%                      for more.
%        h, beta       are parameters in the iteration, h is the
%                      'time-step' and beta is the discounting term, given
%                      by beta=1-h*gamma.
%        f,g and P     are the dynamic growth functions and the profit
%                      function. They are put into PMATRIX2 which produces
%                      M, see help PMATRIX2 for more.
%        vtol and utol are error measures that has to be satisfied before
%                      the iterations stop. As long as the last element in
%                      vfeil is larger than vtol the valueiterations will
%                      go on, the same goes for the last element in ufeil,
%                      utol and the policyiterations.
%        d             is only used in the procedure VEKTORFELT if it is
%                      activated. VEKTORFELT plots the resulting
%                      vector field from a given policy u with the
%                      Matlab function QUIVER, and d tells if every grid
%                      position should be displayed (d=1) or every 2nd position
%                      (d=2) and so on. d is optional and d=1 is the
%                      default value.
%        x_eq and y_eq gives the exact steady state equilibrium of the
%                      system and enters VEKTORFELT if it is activated.
%                      x_eq and y_eq are optional and the default value is
%                      (x_eq,y_eq)=(0,0).
```

```
% Measures the time for the procedure (tic-toc).
tic;

% Computes all the psi- and Pi-values by Pmatrix2.
[X,Y,U,M]=Pmatrix2(x1,y1,u1,h,f,g,P);

% Initialization of V and u.
[m,n,o]=size(X);
V=zeros(m,n); u=zeros(m,n);

% Runs one policyiteration first to get away from the zero policy.
[V,u]=policyit(V,u,X,Y,U,M,h,beta);

% Defines error- and help-vectors. Note that the estimated error in the
% first policy is not stored in vfeil and ufeil. The first iteration
% produces the myopic solution when V=0 and is not assumed optimal.
vfeil=10*vtol; ufeil=10*utol;
W=V; w1=u;
it=0;

% Iterates until the needed precision is gained. Makes sure that several
% rounds of value- and policyiterations are run (by the variable 'it').
% Take the largest change in the variables as an error measure.
while (vfeil(end)>vtol) || (ufeil(end)>utol) || (it<3)
    it=it+1;
    vfeil(end+1)=10*vtol;
    % Valueiteration
    while (vfeil(end)>vtol)
        for i=1:100
            W=valueiteration(W,u,X,Y,U,M,h,beta);
        end
        vfeil(end+1)=max(max(abs(W-V)));
        disp(vfeil(end))
        V=W;
    end
    ufeil(end+1)=10*utol;
    % Policyiteration.
    while (ufeil(end)>utol)
        for i=1:5
            [W,w1]=policyit(W,w1,X,Y,U,M,h,beta);
        end
        ufeil(end+1)=max(max(abs(w1-u)));
        vfeil(end+1)=max(max(abs(W-V)));
```

```
        disp([vfeil(end),ufeil(end)])
        u=w1;  V=W;
    end
end

% Checks if exact steady state equilibrium is given in (x_eq,y_eq), is
% replaced by (0,0) if not. If 'd' is not given, d is set to d=1.
if (nargin==10)
    d=1;
elseif (nargin==11)
    x_eq=0; y_eq=0;
end

% Plots results (TURNED OFF).
%figure, surf(X(1,:,1),Y(:,1,1),u), xlabel('x'), ylabel('y'), zlabel('u')
%figure, surf(X(1,:,1),Y(:,1,1),V), xlabel('x'), ylabel('y'), zlabel('V')
%vektorfelt(X,Y,U,M,u,d,x_eq,y_eq)

toc;
```

Next follows the program `Pmatrix2`. It is called in line 6 of `HJBit` and computes all the ψ and Π values in all the nodes of the grid and for all possible policies. All the values are stored in the 4-dimensional matrix M; the 3-dimensional grid is stored in the matrices X, Y and U. These are computed by the Matlab function `MESHGRID`. Values from M are needed in every iteration in `HJBit` and we compute all the values once, before entering the iterations, instead of computing only those we need for each computation. The procedure assumes that the growth function in the y direction is independent of u and computes the ψ values for y only for the first policy value, and copies them for all the other policy values. The procedure also checks if the ψ values, which describe the vector field in the state space, point out of the defined area of the state space (the gridded area). If that is the case, the intersecting point between the vector field and the boundary is replaced with the point outside the gridded area.

Pmatrix2.m:

```
function [X,Y,U,M]=Pmatrix2(x1,y1,u1,h,f,g,P)
% [X,Y,U,M]=Pmatrix2(x1,y1,u1,h,f,g,P) defines a 3-dimensional grid from
% the vectors x1, y1 and u1 and computes the values of the functions f, g
% and P in every node in the grid.
%
% Output: X,Y and U are the grid computed by the Matlab-function MESHGRID,
%         X gives the x-coordinates, Y the y-coordinates and U the
%         z-coordinates. M is a matrix which holds the values of the
%         functions psi and Pi, which, on component form, is given by:
%           psi(x,y,u)=[x+h*f(x,u),y+h*g(y,u)]
%           Pi(x,y,u)=P(x,y,u)
%         M is a four-dimensional matrix and is organized in the following
%         way:
%           - the three first coordinates refer to a position in the grid
%           given by X, Y and U.
%           - the fourth coordinate equal to 1 gives psi(x,y,u)[1]
%                                             2 gives psi(x,y,u)[2]
%                                             3 gives Pi(x,y,u)
%
% Input:  x1 gives the nodes in the x-direction.
%         y1 gives the nodes in the y-direction.
%         u1 gives the nodes in the z-direction.
%         h is the 'time step' in the discretization of phi.
%         f is the function giving dx/dt (inline or function handle).
%         g is the function giving dy/dt (inline or function handle).
%         P is the utility function Pi (inline or function handle).

% Computes the grid.
```

```
[X,Y,U]=meshgrid(x1,y1,u1);
[m,n,o]=size(X);

% Initiation.
M=zeros(m,n,o,3);

% Computes the values in M. Since psi(x,y,u)[2] is independent of u,
% we calculate it once for the first u value and only copy it for all the other
% u values.
for i=1:m
    for j=1:n
        M(i,j,1,2)=h*g(X(i,j,1),Y(i,j,1),U(i,j,1));
        for k=1:o
            M(i,j,k,1)=h*f(X(i,j,k),Y(i,j,k),U(i,j,k));
            M(i,j,k,2)=M(i,j,1,2);
            M(i,j,k,3)=P(X(i,j,k),Y(i,j,k),U(i,j,k));
            % These if-statements checks if any of the vectors in
            % M(:,:,:,1) and M(:,:,:,2) points out of the considered
            % state space (defined by [x1(1),x1(end)]x[y1(1),y1(end)]. If
            % that is the case, the intersecting point between the vector
            % and the boundary is chosen instead. This part of the
            % program could surely be speeded up.
            if (X(i,j,k)+M(i,j,k,1)<x1(1))
                M(i,j,k,2)=(M(i,j,k,2)*(x1(1)-X(i,j,k)))/M(i,j,k,1);
                M(i,j,k,1)=x1(1)-X(i,j,k);
            elseif (X(i,j,k)+M(i,j,k,1)>x1(end))
                M(i,j,k,2)=(M(i,j,k,2)*(x1(end)-X(i,j,k)))/M(i,j,k,1);
                M(i,j,k,1)=x1(end)-X(i,j,k);
            end
            if (Y(i,j,k)+M(i,j,k,2)<y1(1))
                M(i,j,k,1)=(M(i,j,k,1)*(y1(1)-Y(i,j,k)))/M(i,j,k,2);
                M(i,j,k,2)=y1(1)-Y(i,j,k);
            elseif (Y(i,j,k)+M(i,j,k,2)>y1(end))
                M(i,j,k,1)=(M(i,j,k,1)*(y1(end)-Y(i,j,k)))/M(i,j,k,2);
                M(i,j,k,2)=y1(end)-Y(i,j,k);
            end
        end
    end
end
```

Next follows the program `Policyit`. It is called several times in `HJBit`, first time on line 11. It computes a policy-iteration in the Hamilton-Jacobi-Bellman scheme and updates both the policy matrix and the value matrix. The local variable VI holds the values in the grid in the next state position, decided by ψ and computed by `Pmatrix2`. The values in VI are interpolated from the values in the value matrix V.

Policyit.m:

```
function [V,u]=policyit(v,u,X,Y,U,M,h,beta)
% [V,u]=policyit(v,u,X,Y,U,M,h,beta) computes a policy iteration in the
% Hamilton-Jacobi-Bellman-scheme for a discretized problem.
%
% Output: V is the new value matrix, can be plotted against X and Y.
%         u is the new policy matrix, can be plotted against. X and Y.
%
% Input:  v     is the value matrix before the iteration.
%         u     gives the current policy in every node.
%         X,Y,U gives the grid for the problem, see PMATRIX2.
%         M     gives psi- and Pi-values for the problem, see PMATRIX2.
%         h     gives the 'time step' in the discretization.
%         beta  is the discount term.

% Initialization.
[m,n,o]=size(X);
w=zeros(o,1);
V=v;

% Interpolate in v to find the values in the points given by psi (in
% M(:,:,:,1:2)), store them in VI.
VI=zeros(m,n,o);
VI(:,:,1:o)=interp2(X(:,:,1),Y(:,:,1),v,X(:,:,1:o) +
            M(:,:,1:o,1),Y(:,:,1:o) + M(:,:,1:o,2),'linear');

% Iteration. Computes first the value for every possible policy and picks
% then the policy that gives the largest value in each node. Stores the
% optimal policy and the largest value.
for i=1:m
   for j=1:n
        for k=1:o
            w(k)=h*M(i,j,k,3)+beta*VI(i,j,k); % The HJB-equation!
        end
        [wmax,k]=max(w);
        u(i,j)=U(1,1,k);
```

```
        V(i,j)=wmax;
    end
end
```

Finally follows the program `valueiteration`. It is called on line 25 of `HJBit` and computes a value-iteration in the Hamilton-Jacobi-Bellman scheme. It only updates the value matrix and assumes the policy matrix to be fixed. The local variable VI holds the values in the grid in the next state position, decided by ψ and computed by Pmatrix2. The values in VI are interpolated from the values in the value matrix V.

valueiteration.m:

```
function V=valueiteration(v,u,X,Y,U,M,h,beta)
% V=valueiteration(v,u,X,Y,U,M,h,beta) computes a value iteration in the
% Hamilton-Jacobi-Bellman-scheme for a discretized problem.
%
% Output: V is the new value matrix, can be plotted against X and Y.
%         u is the new policy matrix, can be plotted against. X and Y.
%
% Input:  v     is the value matrix before the iteration.
%         u     gives the current policy in every node.
%         X,Y,U gives the grid for the problem, see PMATRIX2.
%         M     gives psi- and Pi-values for the problem, see PMATRIX2.
%         h     gives the 'time step' in the discretization.
%         beta  is the discount term.

% Initialization.
[m,n,o]=size(X);
V=v;

% Interpolate in v to find the values in the points given by psi (in
% M(:,:,:,1:2)), store them in VI.
VI=zeros(m,n,o);
VI(:,:,1:o)=interp2(X(1,:,1),Y(:,1,1),v,X(:,:,1:o) +
            M(:,:,1:o,1),Y(:,:,1:o) + M(:,:,1:o,2),'linear');

% Iterates. Finds first the current policy (while-loop), computes then the
% new value.
for i=1:m
    for j=1:n
        k=1;
        while (U(i,j,k)<u(i,j))
            k=k+1;
        end
        V(i,j)=h*M(i,j,k,3)+beta*VI(i,j,k); % The HJB-equation!
    end
end
```

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