

Exact solutions for shoaling waves in shallow water

Master thesis in Applied and Computational Mathematics

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Abstract

The dynamics of shallow-water waves at the surface of an inviscid and incompressible fluid over a background shear flow approaching a sloping beach are investigated. First, we derive the nonlinear shallow-water equations in the presence of both background shear flow and a sloping beach. In this case, the hyperbolic shallow-water equations are not reducible and it is not straightforward to find the Riemann invariants. However, using intuition gained from the case of a shear flow over a flat bed, Riemann invariants can nevertheless be found. The Riemann invariants provide a proper hodograph transformation which is combined with several additional changes of variables to put the equations in linear form. This linear equation can be solved using the method of separation of variables. In this way, we are able to find exact solutions which give us a prediction of the shoaling process and of the development of the waterline (run-up). Our work is inspired by the method of Carrier and Greenspan [4]. Therefore, a careful description of the Carrier-Greenspan method is presented first.

Table of Contents

| | |
|---|-----------|
| Outline and motivation | 1 |
| 1 Equations for surface water waves | 3 |
| 1.1 Introduction | 3 |
| 1.1.1 Conservation laws | 3 |
| 1.1.2 Wave theory | 6 |
| 1.2 Shallow-water flow over an inclined bottom | 11 |
| 1.3 Background shear flow for constant depth | 14 |
| 2 Irrotational long waves on a beach | 19 |
| 2.1 Derivation | 19 |
| 2.2 Method of Carrier and Greenspan | 22 |
| 2.2.1 Characteristic form | 23 |
| 2.2.2 Hodograph transformation | 25 |
| 2.2.3 Change of independent variables | 27 |
| 2.2.4 Potential function | 28 |
| 2.2.5 Exact solutions | 30 |
| 3 Shallow water dynamics on linear shear flows and plane beaches | 35 |
| Bibliography | 37 |

Outline and motivation

In this thesis we present exact solutions of the shallow-water equations over a shear flow for a flat and sloping bed, which to our knowledge has not been shown before. The method combines a hodograph transformation and several additional changes of variables to put the equations in linear form. We obtain exact solutions of the linear equations by the method of separation of variables.

The model for long waves over a shear flow approaching a sloping beach is a nontrivial modification of Carrier and Greenspan's model [4]. They treat the case of irrotational flow to find an exact solution of the run-up problem on a linear beach.

The motivation for adding a shear flow to the velocity field is due to the fact that background shear flow changes the behaviour of long surface waves in shallow water. One possible way in which such a shear flow can develop is through the Stokes drift of a wave train approaching a beach. The required return flow creates an undertow [13] which is flowing in the seaward direction from the shoreline. By the inclusion of shear flow we can obtain a better understanding of the shoaling process and the development of the waterline.

Chapter 1

We begin with some background and general theory. We introduce the equations of conservation of mass and conservation of momentum. We proceed with general wave theory, where the cases of linear approximation and shallow water are being described. The derivation of the shallow-water equations for a flow over an inclined bottom and for a background shear flow over a constant depth are presented.

Chapter 2

We explain the approach of Carrier and Greenspan, in order to set the stage for the exhibition to the shear flow.

Chapter 3

Chapter 3 is given in the form of our submitted paper. We show how the shallow-water equations can be solved exactly when a linear shear flow is present. We consider the case of shear flow over a flat bed as this case will give us important clues on how to proceed in the more difficult case of a shear flow over a sloping bed. The investigation of long waves propagating towards a sloping beach is the main focus.

Chapter 1

Equations for surface water waves

1.1 Introduction

In this section some basic theory based on [10] and [15] is presented. We begin by giving a short introduction of the conservation laws. In the second subsection we present an introduction of the theory of gravity waves, where the equations for the conservations laws will be adjusted and proper boundary conditions are defined. By linearizing, a solution can be achieved and the dispersion relation can be obtained. Finally, basic theory for shallow-water waves are presented.

1.1.1 Conservation laws

To describe the dynamics in fluids there are three conservation laws which are commonly used. These three laws state that the mass, momentum and energy are conserved. In this subsection we will derive the equations for the conservation of mass and momentum. The equation for the conservation of energy will not be focused on, however it will be shown at the end of next section how the equation can be obtained from the mass and momentum equation.

An Eulerian description will be applied, which implies that every quantity will depend on the position vector \mathbf{x} and time t . The spatial coordinates are

denoted by (x, y, z) and (u, v, w) are the corresponding components for the velocity vector $\mathbf{u}(\mathbf{x}, t)$.

The equations of motion can in terms of a control volume be written in integral form. A control volume could be a volume fixed in space or a volume which can change its shape and move within the fluid flow depending on time. The latter is called a material volume and such a volume is occupied by a specific collection of neighbouring fluid particles.

The law of conservation of mass states that the mass can neither be created nor destroyed. For a fixed volume V the integral equation for conservation of mass can be expressed directly from the fact that the change of mass inside a fixed volume equals the mass flux across the boundary surface A , hence

$$\frac{d}{dt} \int_V \rho dV = - \int_A \rho \mathbf{u} \cdot \mathbf{n} dA.$$

In the equation above, $\rho(\mathbf{x}, t)$ is the density at position \mathbf{x} at time t and the vector \mathbf{n} is normal to the surface of the volume. By the Leibniz's rule the time derivative can be moved inside the integral and by applying the Gauss divergence theorem, the equation becomes

$$\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) dV = 0.$$

The integrand must vanish at every point (\mathbf{x}, t) since the control volume V can be chosen arbitrary. Therefore the equation requires that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (1.1)$$

Further by using the relation $\nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho$ and the operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$, called material derivative, the equation can be written as

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0.$$

For incompressible fluid the material derivative of the density will be zero and the equation can be written as

$$\nabla \cdot \mathbf{u} = 0. \quad (1.2)$$

The law of conservation of momentum is developed from Newton's second law. The rate of change of the momentum in a control volume V equals the

sum of the external forces acting on the volume. The momentum equation can be written in integral form as

$$\int_V \frac{\partial}{\partial t}(\rho \mathbf{u}) dV + \int_A \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) dA = \int_V \rho \mathbf{g} dV + \int_A \mathbf{F} dA.$$

The first and the second terms are the rate of change of momentum inside V and the flux of momentum across the material surface A , where the vector \mathbf{n} is the outward normal vector. The two terms on the right hand side are the momentum created inside V by the body force \mathbf{g} and the surface force \mathbf{F} acting across the surface A . The surface force, force per unit area, can be written as $\mathbf{F} = n_i \cdot \tau_{ij}$, where the matrix τ_{ij} is called the stress tensor. The stress tensor can be split into the normal components and tangential components (shear stress) to the contact area. For a static fluid, the stress tensor reduces to $\tau_{ij} = -p\delta_{ij}$ where p is the pressure and δ_{ij} is the identity matrix. Because of viscosity there will be an additional shear tensor σ_{ij} when the fluid is in motion. The total stress tensor is then given as the sum of the normal stress and the shear stress. The momentum equation can now be written as

$$\int_V \frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) dV = \int_V \rho \mathbf{g} + \nabla \cdot (-p\delta_{ij} + \sigma_{ij}) dV,$$

where the Gauss divergence theorem has been applied. The first and the second terms can be expanded as

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = \rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \rho \frac{D\mathbf{u}}{Dt},$$

where the two terms inside the bracket is zero from the conservation of mass eq.(1.1). The integrand must vanish at every point (\mathbf{x}, t) since the control volume V can be chosen arbitrary. Therefore the equation requires that

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} + \nabla \cdot (-p\delta_{ij} + \sigma_{ij}). \quad (1.3)$$

For a Newtonian fluid the stress tensor is expressed as

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) + \left(\mu_\nu - \frac{2}{3}\mu \right) \frac{\partial u_m}{\partial x_m} \delta_{ij}. \quad (1.4)$$

For more details, see [10]. For an incompressible fluid the divergence of \mathbf{u} equals zero according to eq.(1.2). Thus, including eq.(1.4) in eq.(1.3), it becomes

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u}. \quad (1.5)$$

This is the Navier-Stokes equation for an incompressible fluid. The last term is referred to a net viscous force and the coefficient μ depends on the thermodynamic state.

1.1.2 Wave theory

We are considering surface water waves provided from the air-water interface, where gravity and surface tension are the two restoring forces. The fluid is assumed to be incompressible and inviscid, and we consider the gravitational field to be constant. The gravity force will be the only body force acting on the volume. This force is called conservative which means that it can be expressed by a potential function, that is $\rho \mathbf{g} = \rho \nabla \Phi$. Since the z-axis is pointing upwards, the potential function becomes $\Phi = -gz$, where g is the acceleration of gravity. The body force can thus be written as $\rho \mathbf{g} = -\rho g \mathbf{k}$, where \mathbf{k} is the unit vector in the z-direction. By assuming the fluid to be inviscid, the equations eq.(1.2) and eq.(1.5), are now

$$\nabla \cdot \mathbf{u} = 0, \quad (1.6)$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla p - g \mathbf{k}. \quad (1.7)$$

Equation (1.7) is the Euler equation. By considering the flow to be irrotational, i.e $\nabla \times \mathbf{u} = 0$, makes it possible to write the velocity vector as a potential function, $\mathbf{u} = \nabla \phi$. Eq.(1.6) for ϕ becomes the Laplace's equation $\Delta \phi = 0$. Then by the relation $(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{1}{2} \mathbf{u}^2 \right)$ and by integrating eq.(1.7) with respect to \mathbf{x} , we get the two equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0,$$

$$\phi_t + \frac{1}{2} (\nabla \phi)^2 = -\frac{p - p_0}{\rho} - gz + C(t),$$

where p_0 is a constant. The integration constant $C(t)$ can be absorbed into ϕ by choosing a new potential, so it can be ignored. To be able to solve these equations we need to obtain proper boundary conditions. Let the air-water interface be described by $f(x, y, z, t) = 0$. Further, it is convenient to describe the surface as $z = \eta(x, y, t)$ and therefore choose $f(x, y, z, t) = \eta(x, y, t) - z$. The fluid can not cross the interface, which means that the velocity of the fluid normal to the interface must be equal to the velocity of the interface normal to itself. Hence $(\mathbf{n} \cdot \mathbf{u})_{z=\eta} = \mathbf{n} \cdot \mathbf{U}_{\text{interface}}$ where \mathbf{n} is the normal vector denoted by $\mathbf{n} = \nabla f / |\nabla f|$ and $\mathbf{U}_{\text{interface}}$ can be considered purely vertical, $\mathbf{U}_{\text{interface}} = \eta_t \mathbf{k}$. If we neglect the motion of the air, the second boundary condition can be obtained by letting $p = p_0$ at the surface, where p is the water pressure and p_0 is the atmospheric pressure. The free surface boundary conditions can now be written as

$$\eta_t + \phi_x \eta_x + \phi_y \eta_y = \phi_z \quad \text{on } z = \eta(x, y, t), \quad (1.8)$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta = 0 \quad \text{on } z = \eta(x, y, t). \quad (1.9)$$

No fluid can cross the solid fixed boundary on the bottom $z = -h(x, y)$, so the velocity normal to the boundary must equal zero, thus $\mathbf{n} \cdot \mathbf{u} = 0$. The function $h(x, y)$ is the distance from the sea bed to the undisturbed water-surface located at $z = 0$. The third boundary condition becomes

$$\phi_z + \phi_x h_x + \phi_y h_y = 0 \quad \text{on } z = -h(x, y). \quad (1.10)$$

Linear theory

For small-slope, small-amplitude gravity waves on the free surface the equations can be linearized. This limitation implies that the wave amplitude is small compared to the wavelength. From now on the depth will be considered to be constant, denoted h_0 . For small perturbations on the water surface, η and ϕ are small and the free surface boundary conditions can be linearized to be

$$\eta_t = \phi_z, \quad \phi_t + g\eta = 0 \quad \text{on } z = \eta(x, y, t). \quad (1.11)$$

By differentiating the second equation with respect to t , the two equations reduce to

$$\phi_{tt} + g\phi_z = 0 \quad \text{on } z = 0, \quad (1.12)$$

where we have linearized further by applying these conditions on $z = 0$. Together with the Laplace's equation, the linear problem for constant depth are

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{on } z = -h_0 < z < 0, \quad (1.13)$$

$$\phi_{tt} + g\phi_z = 0 \quad \text{on } z = 0, \quad (1.14)$$

$$\phi_z = 0 \quad \text{on } z = -h_0. \quad (1.15)$$

Water waves are propagating horizontally, i.e they are oscillatory in $\mathbf{x} = (x, y)^T$ direction at time t . The function $\eta(\mathbf{x}, t)$ specifies the surface shape, so for a travelling wave we assume the wave to take the form as a $\cos(x)$ function

$$\eta = Ae^{i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t} + \text{c.c.}$$

Here A is the amplitude of the wave, $\boldsymbol{\kappa} = (k_1, k_2)^T$ is the wave number vector and ω is the angular frequency. The c.c-term indicate the complex conjugate, the surface function should not be complex. In the following, it is implicitly assumed that the complex conjugate is added to every expression. By eq.(1.11), we seek a solution of $\phi(x, y, z, t)$ to be on the form

$$\phi = Z(z) \exp^{i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t}. \quad (1.16)$$

Further, to calculate the coefficient $Z(z)$, substitute ϕ into the Laplace's equation, which gives us

$$Z'' - \kappa^2 Z = 0,$$

where $\kappa^2 = |\boldsymbol{\kappa}| = (k_1^2 + k_2^2)^{1/2}$. In addition to the uniform bottom boundary condition eq.(1.15) and the linearized free surface boundary conditions eq.(1.11) on $z = 0$, the coefficient becomes

$$Z(z) = -\frac{ig}{\omega} A \frac{\cosh \kappa(h_0 + z)}{\cosh \kappa h_0}$$

and ϕ is now

$$\phi = -\frac{ig}{\omega} A \frac{\cosh \kappa(h_0 + z)}{\cosh \kappa h_0} \exp^{i\boldsymbol{\kappa}\cdot\mathbf{x}-i\omega t}. \quad (1.17)$$

It is still an equation that has not been taken advantage of, that is eq.(1.12) and it gives us the expression

$$\omega^2 = g\kappa \tanh(\kappa h_0). \quad (1.18)$$

This is called the dispersion relation and represents the relation between the angular frequency ω and the wave number κ . The wave number is the spatial frequency $\kappa = 2\pi/\lambda$, where λ is the wavelength. The propagation speed (or phase speed) of these water waves is $c = \omega/\kappa$ and by the dispersion relation, we have that

$$c = \sqrt{\frac{g}{\kappa} \tanh(\kappa h_0)}.$$

This shows that the propagation speed depends on the wavelength, with longer wavelength the waves will propagate faster. These waves are called dispersive.

Shallow-water theory

Water waves are commonly split into the two categories named deep water and shallow water. Here, the shallow-water assumptions will be investigated. The waves are regarded as shallow-water waves (or long waves) when $\kappa h_0 \rightarrow 0$, which means that the wavelength is much larger than the undisturbed water depth h_0 . To consider the dispersion relation, we know that for small arguments the hyperbolic functions behaves as $\cosh(x) \approx 1$ and $\sinh(x) \approx \tanh(x) \approx x$. Then from eq.(1.18) we have the approximation $\omega^2 \approx g\kappa^2 h_0$, which gives the expression for the phase speed

$$c = \sqrt{gh_0}. \quad (1.19)$$

Shallow-water waves are non-dispersive since the propagation speed does not depend on the wavelength.

We will now consider a one-dimensional flow in shallow water. That is, the waves are only propagating along the x-coordinate axis. The velocity field can then be written as $\mathbf{u} = \mathbf{u}(u, w)$. To get an idea of how the velocity field behave by the $\kappa h_0 \rightarrow 0$ assumption, eq.(1.17) gives us the velocity components

$$u = \frac{g\kappa}{\omega} A \exp^{i\kappa x - i\omega t}, \quad w = -\frac{ig\kappa^2}{\omega} A(h_0 + z) \exp^{i\kappa x - i\omega t},$$

where it is interesting to see that the horizontal component is independent of the vertical z-coordinate. We have now seen from the equations in linear

theory that the shallow-water waves are nondispersive and that $u = u(x, t)$. It is now time to go back and include the nonlinear terms.

For long waves the vertical acceleration is small compare with the horizontal acceleration, so the first step is to approximate the material derivative of w . The vertical component of the Euler equation (1.7) is then

$$-\frac{1}{\rho} \frac{\partial p}{\partial z} - g = 0.$$

Integrating the equation above with respect to z and with the limits set from z to η , gives

$$p - p_0 = \rho g(\eta - z). \quad (1.20)$$

From the equation we can see that the pressure depends on z which means that the pressure field is completely hydrostatic in shallow-water waves. The horizontal component of the Euler equation then becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -g \frac{\partial \eta}{\partial x}.$$

As mentioned before, we can again see that since the right hand side is independent of z , the rate of change of u is independent of z , so the momentum equation is then

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x}.$$

One way to derive the shallow-water equation for the mass conservation law, can be to integrate eq.(1.6) as

$$\int_{-h_0}^{\eta} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} dz = 0.$$

Using the appropriate boundary conditions, eq.(1.8) and eq.(1.10), gives us the equation

$$\frac{\partial}{\partial x} \int_{-h_0}^{\eta} u dy + \frac{\partial \eta}{\partial t} = 0.$$

The shallow-water equations are then

$$\eta_t + [u(\eta + h_0)]_x = 0, \quad (1.21)$$

$$u_t + uu_x + g\eta_x = 0. \quad (1.22)$$

1.2 Shallow-water flow over an inclined bottom

The intention of this section is to show how the shallow-water equations can be derived more directly. We also show how the equation for the conservation of mechanical energy is connected to the mass and momentum equation. In this section we are following the theory of [1].

Consider a one-dimensional flow through a channel along the x -axis. The channel has a constant width b and an angle of inclination α , see figure (1.1). The undisturbed depth $h(x)$ is non-uniform and the total depth of the fluid, which is perpendicular to the bottom, is denoted by $H(x, t) = \eta(x, t) + h(x)$, where η is the surface elevation. As shown in Section 1.1.2, for fluids in

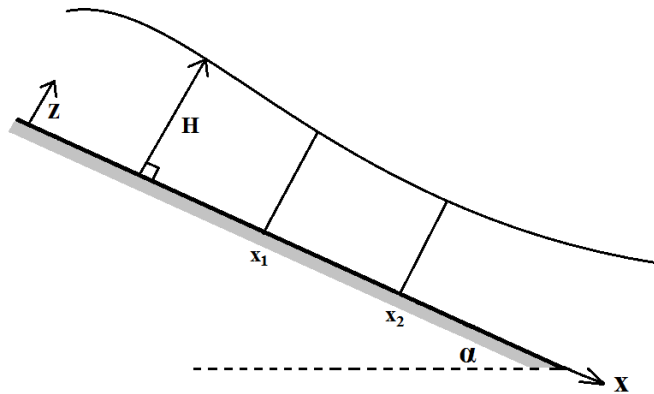


Figure 1.1: Shallow-water flow in a channel over an inclined bottom

shallow water it is commonly assumed that the vertical acceleration can be neglected and thus the pressure becomes hydrostatic. We assume the velocity component u to only depend on the spatial coordinate x and the time t . The fluid is considered to be homogeneous, incompressible, inviscid, and irrotational.

In Section 1.1.2 we derived the equation for conservation of mass, eq.(1.21), with the assumption of incompressibility, eq.(1.6), and defined proper boundary conditions. We will derive the conservation laws for a control volume

defined by the dimensions in fig.(1.1). The control volume is defined as:

$$\int_0^H \int_0^b \int_{x_1}^{x_2} dx dy dz.$$

where H , b , x_1 , x_2 is the total depth, width and the limits of the interval along the x-axis, respectively.

In the one-dimensional case we consider the conservation of mass in the fixed interval between x_1 and x_2 along the x-axis at time t as

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho H b dx + [\rho H b u]_{x_1}^{x_2} = 0,$$

where ρ is the constant density of the fluid. By the use of Leibniz's rule, the equation can be written as

$$\int_{x_1}^{x_2} (\rho H b)_t + (\rho H b u)_x dx = 0.$$

Since the equation above holds for any arbitrary points $x_1 < x_2$ for any time t , the integrand must vanish identically, so that the mass equation becomes

$$H_t + (uH)_x = 0, \quad (1.23)$$

where the constants ρ and b has been excluded.

For the same control volume, the equation for conservation of momentum is

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho H b u dx + [\rho H b u^2]_{x_1}^{x_2} + [p]_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho g H b \sin \alpha dx. \quad (1.24)$$

The first term is the change of momentum with respect to time, and the second term is the flux of momentum through the interval. The forces acting on the volume are the pressure force and the conservative body force which arises from the gravitational force field. The pressure times the area Hb is the pressure force. The pressure is assumed hydrostatic and therefore by the eq.(1.20), we have that $p - p_0 = \rho g(H - z)$. The free surface will be at H since $z = 0$ at the bottom, not η as in the formula from last section. The atmospheric pressure p_0 at the surface will be assumed to be zero. The expression for the pressure force is then

$$p = \int_0^H \rho g(H - z)b \cos \alpha dz = \frac{1}{2} \rho g H^2 b \cos \alpha, \quad (1.25)$$

where $\cos \alpha$ compensate to the fact that $H(x, t)$ is perpendicular to the bottom. The term on the right hand side of eq.(1.24) is the mass times the acceleration of gravity g . For our volume we divide the gravity force in two. The perpendicular gravity force to the bottom is equal to the normal force, so the only force left is the force parallel to the incline bottom. Hence, the momentum equation can now be written in the form

$$\int_{x_1}^{x_2} [(\rho H b u)_t + (\rho H b u^2)_x + (\frac{1}{2} \rho g H^2 b \cos \alpha)_x - \rho g H b \sin \alpha] dx = 0.$$

Again, since x_1 and x_2 are arbitrary, the integrand must vanish pointwise. The equation above can be written as

$$(Hu)_t + (Hu^2 + \frac{1}{2} \gamma H^2)_x = \gamma H \tan \alpha, \quad (1.26)$$

where $\gamma = g \cos \alpha$ and the constants ρ and b has been removed.

It is convenient to write the equation in another form. The left hand side can be written as

$$(Hu)_t + (Hu^2 + \frac{1}{2} \gamma H^2)_x = u[H_t + (Hu)_x] + H[u_t + (\frac{1}{2} u^2 + \gamma H)_x].$$

The terms in the first bracket on the right hand side equals zero by conservation of mass, eq.(1.23). The momentum equation can therefore be written in the form

$$u_t + (\frac{1}{2} u^2 + \gamma H)_x = \gamma \tan \alpha, \quad (1.27)$$

which is equivalent to eq.(1.26) for smooth solutions.

We have now derived equations for both the conservation of mass and momentum for shallow water. We will now proceed to consider the mechanical energy, that is the kinetic and potential energy. The equation of mechanical energy follows from the scalar product of $u(x, t)$ with the momentum equation,

$$\begin{aligned} \frac{d}{dt} \int_{x_1}^{x_2} \int_0^H (\frac{1}{2} \rho u^2 + \rho \gamma z) b dz dx + \left[\int_0^H (\frac{1}{2} \rho u^2 + \rho \gamma z) b u dz \right]_{x_1}^{x_2} \\ + \left[\int_0^H \rho \gamma d(H - z) u dz \right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho g H b u \sin \alpha dx. \end{aligned}$$

The first term is the rate of change of mechanical energy, while the second term is the flux of energy through the boundary. the two remaining terms

represent rate of work by the pressure force and rate of work by the gravity force, respectively. The equation can be written as:

$$\int_{x_1}^{x_2} [((\frac{1}{2}\rho Hu^2 + \frac{1}{2}\rho\gamma H^2)b)_t + ((\frac{1}{2}\rho Hu^3 + \frac{1}{2}\rho\gamma H^2u)b)_x + (\frac{1}{2}\rho\gamma H^2bu)_x - \rho\gamma Hbu \tan \alpha] dx = 0,$$

and since x_1 and x_2 are arbitrary and ρb are constant, we obtain the equation

$$(\frac{1}{2}Hu^2 + \frac{1}{2}\gamma H^2)_t + (\frac{1}{2}Hu^3 + \gamma H^2u)_x = \gamma Hu \tan \alpha.$$

This is the mechanical energy equation for a one-dimensional flow in shallow water. The presented equation is derived from the momentum equation and is not a separate principle. The mechanical energy equation is always satisfied when the conservation of mass and momentum are satisfied in the differential form. We can see this by writing the left hand side of the above equation as

$$\begin{aligned} (\frac{1}{2}Hu^2 + \frac{1}{2}\gamma H^2)_t + (\frac{1}{2}Hu^3 + \gamma H^2u)_x &= (\gamma Hu^2 + \frac{1}{2}u^2)[H_t + (Hu)_x] \\ &+ u[(Hu)_t + (Hu^2 + \frac{1}{2}\gamma H^2)_x]. \end{aligned}$$

1.3 Background shear flow for constant depth

Until now we have derived the shallow-water equations for surface waves with the assumption of irrotational flow. The presence of vorticity affects the behaviour of surface waves. For long waves in shallow water a good description of the behaviour can be obtained by considering the vorticity to be constant [11].

In coastal areas, a background current is often observed. The wave dynamics are affected by the presence of a background current [14] and it is therefore interesting to include a shear current in the velocity field. An approximation of the background current can be obtained by using a uniform shear flow [2][11]. In the case of a long waves, it is reasonable to assume the background current profile to only have a linear variation by depth.

One possible way in which such a shear flow can develop is through the Stokes drift of a wave train. Near shore, a required return flow creates an undertow [13] which is flowing in the direction opposing the wave propagation. This situation leads to a shear flow, such as shown in fig.(1.2).

The shallow-water equations over a shear flow for constant depth will now be derived. Even though the derivation seems standard, we have not found it in the literature. We consider a one-dimensional flow where the fluid is homogeneous, incompressible and inviscid. Due to the addition of shear flow, the velocity component in the x -direction will now depends on z as well. It is convenient to define the velocity component as

$$V(x, z, t) = U(z) + u(x, t). \quad (1.28)$$

For practical reasons the velocity component in the x -direction is denoted $V(x, z, t)$ instead of $u(x, z, t)$ as before. The $U(z)$ term denotes the linear shear flow and can be expressed as

$$U(z) = -\Gamma_0 + \Gamma_1 z, \quad (1.29)$$

where Γ_0 and Γ_1 are constants. The equation for vorticity , $\omega = \text{curl } \mathbf{u}$, gives us the vorticity as

$$\omega = -\frac{dU}{dz} + \frac{dw}{dx} = -\Gamma_1.$$

This is a clockwise constant vorticity which is perpendicular to the (x, z) -plane. The total depth is $H(x, t) = \eta(x, t) + h_0$, where h_0 is the undisturbed depth and $\eta(x, t)$ is the surface elevation, see fig.(1.2).

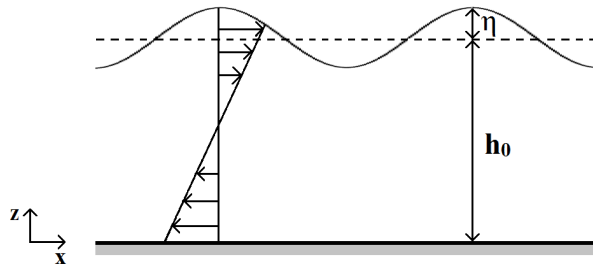


Figure 1.2: Background shear flow over a constant depth.

To derive the equation of conservation of mass, we consider a control volume between the interval x_1 and x_2 along the x -axis and the distance in the y -direction is the constant b . The integral equation for constant depth with uniform shear flow is then

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho H(x, t) b \, dx + \left[\int_0^{H(x, t)} \rho V(x, z, t) b \, dz \right]_{x_1}^{x_2} = 0.$$

Inserting for eq.(1.28) and eq.(1.29) gives

$$\int_{x_1}^{x_2} H_t dx + \left[\int_0^H -\Gamma_0 + \Gamma_1 z + u dz \right]_{x_1}^{x_2} = 0,$$

where the constants ρ and b have been excluded. By integrating, we obtain

$$\int_{x_1}^{x_2} H_t dx + \left[-\Gamma_0 H + \frac{\Gamma_1}{2} H^2 + uH \right]_{x_1}^{x_2} = 0,$$

which is

$$\int_{x_1}^{x_2} H_t + \left(-\Gamma_0 H + \frac{\Gamma_1}{2} H^2 + uH \right)_x dx = 0.$$

The integrand must vanish at every point (x, t) since the interval between x_1 and x_2 can be chosen arbitrary, thus

$$H_t + \left(-\Gamma_0 H + \frac{\Gamma_1}{2} H^2 + uH \right)_x = 0. \quad (1.30)$$

The hydrostatic pressure force is the only external force acting on the control volume. The expression for the pressure force is similar to eq.(1.25), except now H is parallel to the pressure p . Hence, the integral equation of conservation of momentum becomes

$$\begin{aligned} \frac{d}{dt} \int_{x_1}^{x_2} \int_0^{H(x,t)} \rho V(x, z, t) b dz dx + \left[\int_0^{H(x,t)} \rho V^2(x, z, t) b dz \right]_{x_1}^{x_2} \\ + \left[\int_0^{H(x,t)} \rho g(H(x, t) - z) b dz \right]_{x_1}^{x_2} = 0 \end{aligned}$$

and since ρ and b are constants, they can be neglected. By combining eq.(1.28) and eq.(1.29), yielding

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_0^H U + u dz dx + \left[\int_0^H U^2 + 2Uu + u^2 dz \right]_{x_1}^{x_2} + \left[\frac{g}{2} H^2 \right]_{x_1}^{x_2} = 0.$$

Since x_1 and x_2 are arbitrary, the equation can be written in a differential form as

$$\begin{aligned} \left(-\Gamma_0 H + \frac{\Gamma_1}{2} H^2 + uH \right)_t \\ + \left(\Gamma_0^2 H - \Gamma_0 \Gamma_1 H^2 + \frac{\Gamma_1^2}{3} H^3 - 2\Gamma_0 uH + \Gamma_1 uH^2 + u^2 H + \frac{g}{2} H^2 \right)_x = 0. \end{aligned}$$

By expanding the derivatives, we can use the conservation of mass eq.(1.30) and remove the terms $(-\Gamma_0 \cdot \text{eq.}(1.30))$ from the equation, obtaining:

$$\begin{aligned} \Gamma_1 H H_t + (uH)_t - \Gamma_0 \Gamma_1 H H_x + \Gamma_1^2 H^2 H_x - \Gamma_0 (uH)_x + \Gamma_1 u_x H^2 \\ + 2\Gamma_1 u H H_x + (u^2 H)_x + \left(\frac{g}{2} H^2\right)_x = 0. \end{aligned}$$

Then we can remove the terms $(\Gamma_1 H \cdot \text{eq.}(1.30))$, resulting in

$$u_t H + u H_t - \Gamma_0 u_x H - \Gamma_0 u H_x + \Gamma_1 u H H_x + 2u u_x H + u^2 H_x + g H H_x = 0$$

and finally remove the terms $(u \cdot \text{eq.}(1.30))$ and get

$$u_t H - \Gamma_0 u_x H + u u_x H + g H H_x = 0.$$

At last, the total depth $H(x, t)$ can be eliminated, which gives the equation in a different form as

$$u_t + \left(-\Gamma_0 u + \frac{1}{2} u^2 + g H \right)_x = 0. \quad (1.31)$$

By the inclusion of a uniform shear flow, the shallow-water equations, eq.(1.30) and eq.(1.31), for flat bed has been derived. These equations will be studied further and solved in Chapter 3.

Chapter 2

Irrotational long waves on a beach

The shallow-water equations for a one-dimensional irrotational flow can be used to obtain an idea of the behaviour of long waves propagating towards a sloping beach. Carrier and Greenspan [4] succeeded in obtaining explicit solutions for these shallow-water equations on a uniform sloping beach. They showed how the two nonlinear equations can be reduced to a linear equation by first applying the Riemann invariants to perform a proper hodograph transformation, and then change the independent variables to eliminate the nonlinear terms. The analysis of the explicit solutions shows the shoaling process and provide expressions for the maximum run-up and minimum run-down at the beach. In particular, Carrier and Greenspan showed that there exist long waves which do not break as they climb a linear beach profile.

In this chapter the methodology of [4] will be carefully investigated. We start off with the derivation of the shallow-water equations on a sloping beach. Even though the derivation seems standard, we have not found it in the literature.

2.1 Derivation

We consider a homogeneous, incompressible and inviscid fluid where the pressure is completely hydrostatic. The water waves are considered to propagate

in one dimension and the particle velocity component u is depended on the spatial coordinate x and the time t . The surface waves in shallow water are propagating towards a linear sloping beach, where the beach profile is denoted by $b(x) = \alpha x$. The total depth is given by

$$H(x, t) = \eta(x, t) + h(x), \quad (2.1)$$

where $\eta(x, t)$ is the surface elevation and $h(x)$ is the undisturbed water depth. The x -axis is positioned at the undisturbed level, see fig.(2.1).

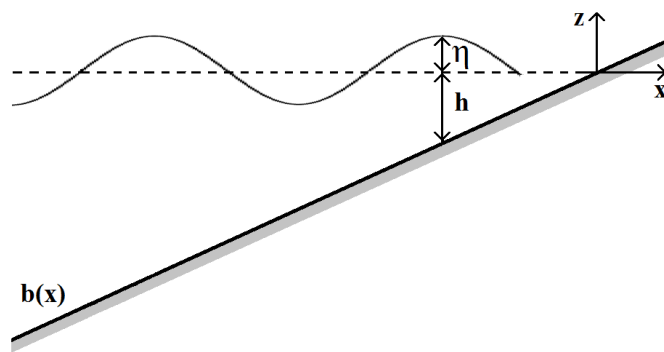


Figure 2.1: Shallow-water waves approaching a linear beach profile.

The derivation of the conservation of mass equation will be implemented as in the previous chapter. The height for the control volume is from $-h(x)$ to $\eta(x, t)$, the width is the constant $\int dy = b$ and the length is defined as the interval between x_1 and x_2 . The integral equation is then given as:

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{-h}^{\eta} \rho b \, dz dx + \left[\int_{-h}^{\eta} \rho u(x, t) b \, dz \right]_{x_1}^{x_2} = 0,$$

where ρ is the constant density and $u(x, t)$ is the fluid velocity. Equivalent to former integral equations, we obtain

$$\eta_t + [u(\eta + h)]_x = 0. \quad (2.2)$$

The second shallow-water equation which will be derived is the equation of conservation of momentum for an irrotational flow. The only forces acting on the control volume are the pressure force. The pressure is assumed to be hydrostatic and given by eq.(1.20):

$$\int_{-h}^{\eta} \rho g(\eta - z) b \, dz.$$

The one-dimensional flow in the x-direction will be affected by the non-uniform environment. There will be a pressure from the seabed into the flow, called bottom force, p_x . The bottom force p_x opposes the flow and is defined in the negative x-direction, see fig.(2.2). An expression for p_x can be

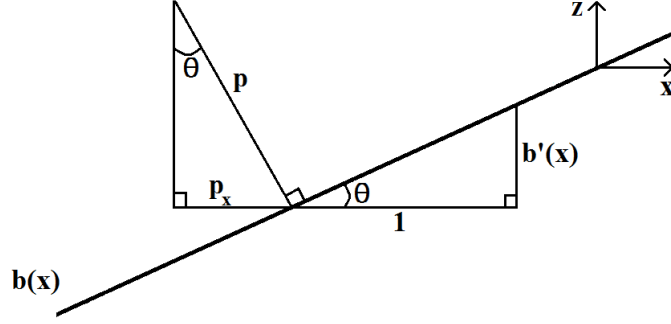


Figure 2.2: Sketch of the p_x geometry.

found by taking a look at the geometry. Fig.(2.2) shows the sloping beach $b(x) = \alpha x$ and two triangles. From the left triangle we have that $p_x = p \sin \theta$. An expression for $\sin \theta$ can be obtained from the definition of the derivative. Considering the right triangle, $\sin \theta$ can be expressed as

$$\sin \theta = \frac{b'(x)}{\sqrt{1 + (b'(x))^2}} \sim \alpha.$$

The last approximation need some more explanation. In shallow-water theory a common assumption is that the change of rate of the seabed is very small, in other words we assume that $b'(x) \ll 1$. We can now define the bottom force in the negative x-direction as the pressure times the area, where the area is between x_1 and x_2 times the constant width b , thus

$$- \int_{x_1}^{x_2} \alpha \rho g (\eta(x, t) - z) b \, dx \Big|_{z=-h}.$$

The momentum equation in integral form can be written as

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{-h}^{\eta} \rho u b \, dx \, dz + \left[\int_{-h}^{\eta} \rho u^2 b \, dz \right]_{x_1}^{x_2} + \left[\int_{-h}^{\eta} \rho g (\eta - z) b \, dz \right]_{x_1}^{x_2} = - \int_{x_1}^{x_2} \alpha \rho g (\eta + h) b \, dx, \quad (2.3)$$

where the density ρ and the width b are constants and can be removed. By using eq.(2.1), the integral in the third term results in

$$\int_{-h}^{\eta} g(\eta - z) dz = \frac{1}{2}gH^2$$

Since x_1 and x_2 are arbitrary, the integrand must vanish pointwise so eq.(2.3) requires that

$$(uH)_t + (u^2H)_x + (\frac{1}{2}gH^2)_x = -\alpha gH.$$

By the eq.(2.2), the equation above can be written as

$$u_t + uu_x + gH_x = -\alpha g.$$

The non-uniform depth is considered to be $h(x) = -\alpha x$. The minus sign makes sense in a way that the value of x will be negative on the left side of our coordinate system, meaning that the depth from the undisturbed level down to the seabed will be positive. Now we have that $H(x, t) = \eta(x, t) - \alpha x$ and the momentum equation becomes

$$u_t + uu_x + g\eta_x = 0. \quad (2.4)$$

Notice the geometric advantage of the model. Eq.(2.4) does not depend of the beach slope α , which simplify further calculations.

2.2 Method of Carrier and Greenspan

A presentation of the method of Carrier and Greenspan [4] will be given. In order to make eq.(2.2) and eq.(2.4) non-dimensional, we introduce the following non-dimensional variables as

$$u^* = \frac{u}{u_0}, \quad \eta^* = \frac{\eta}{\alpha l_0}, \quad h^* = \frac{h}{\alpha l_0}, \quad x^* = \frac{x}{l_0}, \quad t^* = \frac{t}{T},$$

where $T = \sqrt{l_0/\alpha g}$, $u_0 = \sqrt{gl_0\alpha}$ and the l_0 is a characteristic length. With these new variables, the non-dimensional shallow-water equations are

$$u_{t^*}^* + u^* u_{x^*}^* + \eta_{x^*}^* = 0, \quad (2.5)$$

$$\eta_{t^*}^* + [u^*(\eta^* - x^*)]_{x^*} = 0. \quad (2.6)$$

By this scaling, both α and g are eliminated from the equations. These equations are non-linear and we will now see how they can be transformed into a linear equation.

2.2.1 Characteristic form

In Section 1.1 we combined the shallow-water assumptions with the linear approach and obtained an expression for the propagation speed as $c = \sqrt{gh_0}$, eq.(1.19). Here, we are working with nonlinear equations for waves in shallow water where we define the characteristic speed [9] as $c = \sqrt{gH}$. The non-dimensional variable for the characteristic speed is $c^* = c/u_0$ and thereby the dimensionless expression for the wave speed is given as

$$c^*(x^*, t^*) = \sqrt{\eta^*(x^*, t^*) - x^*}. \quad (2.7)$$

For simplicity, the stars will from now on be disregarded. The shallow-water equations can be formulated by the wave speed $c(x, t)$ and the particle velocity $u(x, t)$ instead of $\eta(x, t)$ and $u(x, t)$. By the computation, $\eta = c^2 + x$, $\eta_t = 2cc_t$ and $\eta_x = 2cc_x + 1$, eq.(2.5) and eq.(2.6) becomes

$$u_t + uu_x + 2cc_x + 1 = 0, \quad (2.8)$$

$$2c_t + cu_x + 2uc_x = 0. \quad (2.9)$$

We can now see that it is convenient to include the wave speed since we could eliminate one c from eq.(2.9). By adding these two equations, and by subtracting eq.(2.9) from eq.(2.8), results in

$$(u + 2c)_t + u(u + 2c)_x + c(u + 2c)_x + 1 = 0,$$

$$(u - 2c)_t + u(u - 2c)_x - c(u - 2c)_x + 1 = 0.$$

In order to write these equations in a different form, we interpret 1 as $\partial t / \partial t$ and obtain

$$\left\{ \frac{\partial}{\partial t} + (u + c) \frac{\partial}{\partial x} \right\} (u + 2c + t) = 0, \quad (2.10)$$

$$\left\{ \frac{\partial}{\partial t} + (u - c) \frac{\partial}{\partial x} \right\} (u - 2c + t) = 0. \quad (2.11)$$

The structure in these equations is known as a characteristic form. By the method of characteristics [9] [12] the equations above give us the following expressions:

$$\begin{cases} u + 2c + t = \text{constant on curves } C^+ : \frac{dx}{dt} = u + c \\ u - 2c + t = \text{constant on curves } C^- : \frac{dx}{dt} = u - c. \end{cases} \quad (2.12)$$

These expressions need some more explanation. In general we have that the total derivative of a function $\phi(x(t), t)$ along a curve $x = x(t)$ is

$$\frac{d\phi}{dt} = \phi_t + \frac{dx}{dt} \phi_x,$$

where the curve in the (x, t) -plane has the slope $\frac{dx}{dt}$ at every point of it [15]. Now, if we consider a curve C^+ in the (x, t) -plane which satisfies $\frac{dx}{dt} = u + c$, then take the total derivative of the function $u + 2c + t$ along this curve, we get

$$\frac{d}{dt}(u + 2c + t) = \frac{\partial}{\partial t}(u + 2c + t) + (u + c) \frac{\partial}{\partial x}(u + 2c + t).$$

Combining this equation with eq.(2.10), it becomes

$$\frac{d}{dt}(u + 2c + t) = 0.$$

Hence, the function $u + 2c + t$ must remain constant for every point along the curve C^+ which satisfies $\frac{dx}{dt} = u + c$.

There will be a set of C^+ curves which are called characteristics and the functions which are constant on there respective curves are called Riemann invariants. Eq.(2.11) can be interpreted in the same way. The set of curves C^+ and C^- are distinct since $\frac{dx}{dt} = u + c$ and $\frac{dx}{dt} = u - c$ are different when $c \neq 0$. The curves describe the relation between the velocity u and the characteristic speed c . The expression in eq.(2.12) can be written as

$$\begin{cases} u + 2c + t = f(\alpha), \alpha \text{ constant on curves } C^+ : \frac{dx}{dt} = u + c \\ u - 2c + t = g(\beta), \beta \text{ constant on curves } C^- : \frac{dx}{dt} = u - c \end{cases},$$

where f and g are arbitrary functions and the variables α and β are called characteristic variables [9]. Since the functions f and g are arbitrary, we can choose them to be

$$\alpha = u + 2c + t, \tag{2.13}$$

$$-\beta = u - 2c + t. \tag{2.14}$$

In the following subsections we will see why this particular choice of the functions f and g are convenient.

2.2.2 Hodograph transformation

A hodograph transformation consists of interchanging the dependent and independent variables. It can be an efficient tool for converting nonlinear partial differential equations into linear partial differential equations [6] [9]. In the last subsection, we obtained the two nonlinear equations (2.10) and (2.11), which we clearly would like to convert to two linear equations. It is not obvious that a hodograph transformation will make this possible. However, Carrier and Greenspan [4] showed that this problem in fact can be linearised by an appropriate hodograph transformation.

There are two conditions to be aware of, the Jacobian determinate should in every equation be cancelled out [7] and according to the Inverse Function Theorem the transformation require the Jacobian determinant to be non-zero. In the equations (2.10) and (2.11), u and c are the dependent variables and x and t are the independent variables. The Jacobian determinate will not cancel out in the two equations if we try to perform the transformation $(x, t) \rightarrow (u, c)$. However, by applying the characteristic variables α and β to implement the hodograph transformation, we will see that the Jacobian determinate is cancelled out.

To invert the roles of the dependent variables α and β by the independent variables x and t , we transform $\alpha = \alpha(x, t)$ and $\beta = \beta(x, t)$ to

$$x = x(\alpha, \beta), \quad t = t(\alpha, \beta). \quad (2.15)$$

The Jacobian determinant of the transformation is

$$J = \frac{\partial(x, t)}{\partial(\alpha, \beta)} = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial t}{\partial \alpha} & \frac{\partial t}{\partial \beta} \end{vmatrix} = x_\alpha t_\beta - x_\beta t_\alpha.$$

To implement the transformation we differentiate each of eq.(2.15) with respect to x and obtain

$$1 = \frac{\partial x}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial x}{\partial \beta} \frac{\partial \beta}{\partial x} = x_\alpha (u_x + 2c_x) + x_\beta (-u_x + 2c_x),$$

$$0 = \frac{\partial t}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial t}{\partial \beta} \frac{\partial \beta}{\partial x} = t_\alpha (u_x + 2c_x) + t_\beta (-u_x + 2c_x).$$

Similarly, by differentiating with respect to t , gives us

$$0 = \frac{\partial x}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial x}{\partial \beta} \frac{\partial \beta}{\partial t} = x_\alpha (u_t + 2c_t + 1) + x_\beta (-u_t + 2c_t - 1),$$

$$1 = \frac{\partial t}{\partial \alpha} \frac{\partial \alpha}{\partial t} + \frac{\partial t}{\partial \beta} \frac{\partial \beta}{\partial t} = t_\alpha (u_t + 2c_t + 1) + t_\beta (-u_t + 2c_t - 1).$$

To calculate the latter parts of these equations, the definitions for the characteristic variables, eq.(2.13) and eq.(2.14), have been used. From these last four equations, we obtain the following expressions:

$$\begin{aligned} u_x &= \frac{t_\alpha + t_\beta}{2J}, & c_x &= \frac{t_\beta - t_\alpha}{4J}, \\ u_t &= -1 - \frac{x_\alpha + x_\beta}{2J}, & c_t &= \frac{x_\alpha - x_\beta}{4J}, \end{aligned}$$

which requires $J \neq 0$. Inserting these derivatives into eq.(2.10), gives us

$$-1 - \frac{x_\alpha + x_\beta}{2J} + 2\frac{x_\alpha - x_\beta}{4J} + 1 + (u + c) \left(\frac{t_\alpha + t_\beta}{2J} + 2\frac{t_\beta - t_\alpha}{4J} \right) = 0. \quad (2.16)$$

Here, we can see that we are quite lucky, actually, since the number 1 cancels out and we are therefore able to cancel out the Jacobian determinant as well. By inserting the same set of equations into eq.(2.11), gives

$$-1 - \frac{x_\alpha + x_\beta}{2J} - 2\frac{x_\alpha - x_\beta}{4J} + 1 + (u - c) \left(\frac{t_\alpha + t_\beta}{2J} - 2\frac{t_\beta - t_\alpha}{4J} \right) = 0. \quad (2.17)$$

Similarly, the number 1 luckily cancels out and the Jacobian determinant can be eliminated. Only now can we see why it is convenient to implement the hodograph transformation by this approach. Eq.(2.16) and eq.(2.17) can be written as

$$x_\beta - (u + c)t_\beta = 0, \quad (2.18)$$

$$x_\alpha - (u - c)t_\alpha = 0. \quad (2.19)$$

To see that these equations still are nonlinear in t , we can rewrite them by eq.(2.13) and eq.(2.14) as

$$x_\beta - \frac{1}{4}(3\alpha - \beta - 4t)t_\beta = 0,$$

$$x_\alpha - \frac{1}{4}(\alpha - 3\beta - 4t)t_\alpha = 0.$$

To reduce these nonlinear equations to a linear equation we have to do one last step, which is to change the independent variables.

2.2.3 Change of independent variables

By the hodograph transformation we obtained eq.(2.18) and eq.(2.19) where x , t , u and c are dependent variables, while α and β are independent variables. In order to obtain these equations in linear form, we introduce σ and λ as our new pair of independent variables. By a combination of the characteristic variables α and β , given in eq.(2.13) and eq.(2.14), σ and λ are defined as

$$u + t = \frac{\alpha - \beta}{2} = \frac{\lambda}{2}, \quad (2.20)$$

$$c = \frac{\alpha + \beta}{4} = \frac{\sigma}{4}. \quad (2.21)$$

To change the independent variables, we use the following differential expressions

$$\frac{\partial}{\partial \beta} = \frac{\partial}{\partial \lambda} \frac{\partial \lambda}{\partial \beta} + \frac{\partial}{\partial \sigma} \frac{\partial \sigma}{\partial \beta} = \frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \lambda},$$

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \lambda} \frac{\partial \lambda}{\partial \alpha} + \frac{\partial}{\partial \sigma} \frac{\partial \sigma}{\partial \alpha} = \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \sigma}.$$

By substitutions, eq.(2.18) and eq.(2.19) become

$$x_\sigma - ut_\sigma + ct_\lambda - x_\lambda + ut_\lambda - ct_\sigma = 0, \quad (2.22)$$

$$x_\lambda + ct_\sigma - ut_\lambda + x_\sigma - ut_\sigma + ct_\lambda = 0. \quad (2.23)$$

By linear algebra these equations can be reduced. One approach is to define $A = x_\sigma - ut_\sigma + ct_\lambda$ and $B = x_\lambda + ct_\sigma - ut_\lambda$ so that it can be written as a homogeneous system, that is

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the determinant for the 2×2 matrix is non-zero, the system must have a trivial solution. From this reasoning, eq.(2.22) and eq.(2.23) can be reduced to

$$x_\sigma - ut_\sigma + ct_\lambda = 0, \quad (2.24)$$

$$x_\lambda + ct_\sigma - ut_\lambda = 0. \quad (2.25)$$

The two terms ut_σ and ut_λ makes these equations nonlinear in t . A crucial observation given by Carrier and Greenspan [4], is that if we eliminate x from the equations, we also eliminate the nonlinear terms. In order to eliminate x , we differentiate the first equation with respect to λ and the second with respect to σ . We assume both x and t to be smooth functions, resulting in $x_{\sigma\lambda} = x_{\lambda\sigma}$ and $t_{\sigma\lambda} = t_{\lambda\sigma}$. By this assumption we obtain

$$u_\lambda t_\sigma - u_\sigma t_\lambda - c_\lambda t_\lambda + c_\sigma t_\sigma = c(t_{\lambda\lambda} - t_{\sigma\sigma}). \quad (2.26)$$

From eq.(2.20) and eq.(2.21), we find the following expressions: $u_\lambda = \frac{1}{2} - t_\lambda$, $u_\sigma = -t_\sigma$, $c_\lambda = 0$ and $c_\sigma = \frac{1}{4}$. Hence, equation (2.26) becomes

$$\sigma(t_{\lambda\lambda} - t_{\sigma\sigma}) - 3t_\sigma = 0, \quad (2.27)$$

which is a linear equation in t .

We have obtained a linear second-order partial differential equation from our two governing nonlinear shallow-water equations (2.2) and (2.4). The reduction from the two nonlinear equation to a linear equation is a remarkable simplification. The eq.(2.27) can be solved for $t(\sigma, \lambda)$ by separation of variables. However, it is problematic to find an expression of $x(\sigma, \lambda)$ from eq.(2.24) and eq.(2.25), with the solution obtained from eq.(2.27). Therefore, an additional step is required.

2.2.4 Potential function

In order to obtain an expression for $x(\sigma, \lambda)$ it is convenient to introduce a "potential" function $\phi(\sigma, \lambda)$ as

$$u(\sigma, \lambda) = \frac{1}{\sigma}\phi_\sigma(\sigma, \lambda). \quad (2.28)$$

Before we find an expression for $x(\sigma, \lambda)$, we begin by writing the eq.(2.27) due to the potential function. Using eq.(2.20), eq.(2.27) can be written in terms of the function $u(\sigma, \lambda)$ as

$$\sigma(u_{\sigma\sigma} - u_{\lambda\lambda}) + 3u_\sigma = 0. \quad (2.29)$$

Calculating u_σ , $u_{\sigma\sigma}$ and $u_{\lambda\lambda}$ by eq.(2.28), gives us eq.(2.29) in terms of the potential function, hence

$$-\frac{1}{\sigma^2}\phi_\sigma + \frac{1}{\sigma}\phi_{\sigma\sigma} + \phi_{\sigma\sigma\sigma} - \phi_{\sigma\lambda\lambda} = 0. \quad (2.30)$$

This equation can be simplified by applying the product rule to show that

$$\frac{1}{\sigma}\phi_{\sigma\sigma} = \frac{\partial}{\partial\sigma} \left(\frac{1}{\sigma}\phi_\sigma \right) + \frac{1}{\sigma^2}\phi_\sigma,$$

such that eq.(2.30) becomes

$$\frac{\partial}{\partial\sigma} \left(\frac{1}{\sigma}\phi_\sigma \right) + \frac{\partial}{\partial\sigma}\phi_{\sigma\sigma} - \frac{\partial}{\partial\sigma}\phi_{\lambda\lambda} = 0.$$

By integrating the above equation with respect to σ , we obtain

$$(\sigma\phi_\sigma)_\sigma - \sigma\phi_{\lambda\lambda} = 0. \quad (2.31)$$

Before we seek an exact solution to this equation, we will determine expressions for each dependent variable, $x(\sigma, \lambda)$, $\eta(\sigma, \lambda)$, $u(\sigma, \lambda)$ and $t(\sigma, \lambda)$.

An expression for $x(\sigma, \lambda)$ can be obtained by using eq.(2.24). Eq.(2.20) provides the expressions $t_\sigma = -u_\sigma$ and $t_\lambda = 1/2 - u_\lambda$, and the partial derivatives of $u(\sigma, \lambda)$ are obtained by eq.(2.28) as

$$u_\sigma = -\frac{1}{\sigma^2}\phi_\sigma + \frac{1}{\sigma}\phi_{\sigma\sigma}, \quad u_\lambda = \frac{1}{\sigma}\phi_{\sigma\lambda}.$$

By insertion and some calculation, eq.(2.24) becomes

$$x_\sigma - \frac{1}{\sigma^3}(\phi_\sigma)^2 + \frac{1}{\sigma^2}\phi_\sigma\phi_{\sigma\sigma} + \frac{\sigma}{8} - \frac{1}{4}\phi_{\sigma\lambda} = 0, \quad (2.32)$$

where we have used that $c = \sigma/4$. By the observation that

$$\frac{1}{\sigma^3}(\phi_\sigma)^2 + \frac{1}{\sigma^2}\phi_\sigma\phi_{\sigma\sigma} = -\frac{1}{2}\frac{\partial}{\partial\sigma} \left(\frac{1}{\sigma^2}(\phi_\sigma)^2 \right) = \frac{\partial}{\partial\sigma} \left(-\frac{u^2}{2} \right),$$

eq.(2.32) can be written as

$$\frac{\partial}{\partial\sigma}x = \frac{\partial}{\partial\sigma} \left(-\frac{u^2}{2} \right) - \frac{\partial}{\partial\sigma} \frac{\sigma^2}{16} + \frac{\partial}{\partial\sigma} \frac{\phi_\lambda}{4}.$$

By integrating with respect to σ , the expression for $x(\sigma, \lambda)$ becomes

$$x = \frac{\phi_\lambda}{4} - \frac{\sigma^2}{16} - \frac{u^2}{2}.$$

Expressions for $\eta(\sigma, \lambda)$, $t(\sigma, \lambda)$ and $u(\sigma, \lambda)$ are given by eq.(2.7), eq.(2.20) and eq.(2.28), respectively. To summarize, we have the following equations:

$$(\sigma\phi_\sigma)_\sigma - \sigma\phi_{\lambda\lambda} = 0 \quad (2.33)$$

and

$$\begin{aligned} u &= \frac{1}{\sigma}\phi_\sigma, \\ x &= \frac{\phi_\lambda}{4} - \frac{\sigma^2}{16} - \frac{u^2}{2}, \end{aligned} \quad (2.34)$$

$$\begin{aligned} \eta &= \frac{\phi_\lambda}{4} - \frac{u^2}{2}, \\ t &= \frac{\lambda}{2} - u. \end{aligned} \quad (2.35)$$

2.2.5 Exact solutions

Separation of variables is an efficient method for solving ordinary and partial differential equations. The linear equation given in eq.(2.33) is a standard cylindrical wave equation [7] which can be solved by this method. In order to separate the variables, we seek a solution with the form

$$\phi(\sigma, \lambda) = f(\sigma)g(\lambda). \quad (2.36)$$

By inserting the above equation into eq.(2.33), we obtain

$$\frac{\sigma f''(\sigma) + f'(\sigma)}{\sigma f(\sigma)} = \frac{g''(\lambda)}{g(\lambda)} = -\omega^2,$$

where ω is an arbitrary constant. The equation can be separated into two equations, yielding

$$\sigma f''(\sigma) + f'(\sigma) + \omega^2\sigma f(\sigma) = 0, \quad (2.37)$$

$$g''(\lambda) + \omega^2 g(\lambda) = 0. \quad (2.38)$$

If we multiply eq.(2.37) by σ and do a change of variable to $x = \omega\sigma$, we obtain

$$x^2 f''(x) + x f'(x) + x^2 f(x) = 0,$$

which is the Bessel's equation of order zero [8]. The general solution is a linear combination of the first kind of Bessel function, $J_0(x)$, and the second kind of Bessel function, $Y_0(x)$, that is

$$f(x) = c_1 J_0(x) + c_2 Y_0(x),$$

where c_1 and c_2 are arbitrary constants. By inserting for x , we get

$$f(\sigma) = c_1 J_0(\omega\sigma) + c_2 Y_0(\omega\sigma).$$

The Bessel function of second kind of order zero is not bounded when $\sigma \rightarrow 0$ and therefore we prefer the solution to be $f(\sigma) = c_1 J_0(\omega\sigma)$. The general solution of eq.(2.38) is

$$g(\lambda) = c_3 \cos(\omega\lambda) + c_4 \sin(\omega\lambda),$$

where c_3 and c_4 are arbitrary constants. Thus, a bounded solution of eq.(2.33) is

$$\phi(\sigma, \lambda) = A J_0(\omega\sigma) \cos(\omega\lambda), \quad (2.39)$$

where A is an arbitrary constant. Without loss of generality, we may consider $\omega = 1$.

The choice of ϕ provides expressions of η , u , x and t in terms of (σ, λ) -coordinates. It is not straightforward to understand these solutions in terms of (x, t) -coordinates. Therefore, in order to plot these exact solutions in terms of (x, t) -coordinates, a numerical approach is needed.

Let us give a short presentation of the numerical implementation. Expressions for ϕ , η , u , x and t , which are all functions of (σ, λ) , are used to fill arrays of numbers as c and λ run through certain prescribed sets of values. In order to plot the free surface elevation as $\eta = \eta(x, t)$, we use the two matrices for $t(\sigma, \lambda)$ and $x(\sigma, \lambda)$ as independent variables. First, we specify a value for t denoted t_1 , then we find all the indices for the matrix t where $|t - t_1|$ are smaller than a tolerance value. We can use the indices to take out the specific values from both the matrices x and η . All of these specific

values of x will, in the terms of the indices, be paired by the corresponding specific values of η . We can now sort these pairs of specific values by the terms of x . Thus, in a small time interval at time t_1 , one array for x and one array for η are obtained, i.e we are now in the position to plot $\eta = \eta(x, t)$. The visualization of the horizontal fluid velocity $u = u(x, t)$ can be done in a similar way.

The solutions $\eta(x, t)$ and $u(x, t)$ are single-valued as long as the Jacobian $\frac{\partial(x, t)}{\partial(\sigma, \lambda)}$ is nonzero. Therefore, the constant A and the independent variable σ are chosen to insure that a single-value solution is obtained. It can be shown by a simple numerical test that in $\sigma > 0$, the Jacobian determinant vanishes nowhere when $A \leq 1$. By this reasoning, Carrier and Greenspan showed that there exist long waves which do not break as they climb a sloping beach.

The position of the waterline is given when the total depth is zero, i.e $\sigma = 0$. By eq.(2.34), we have that

$$x(\sigma, \lambda) = -\frac{A}{4} J_0(\sigma) \sin \lambda - \frac{\sigma^2}{16} + \frac{1}{2} \left(\frac{A}{\sigma} J_1(\sigma) \cos \lambda \right)^2 \quad (2.40)$$

and since $J_0(0) = 1$ and $J_1(0) = 0$, we obtain

$$x(0, \lambda) = -\frac{A}{4} \sin \lambda. \quad (2.41)$$

This equation shows that the maximum run-up is $A/4$ and the minimum run-down is $-A/4$. From eq.(2.35) we have that the maximum run-up is at $t(0, \frac{3\pi}{2}) = \frac{3\pi}{4}$ and the minimum run-down is at $t(0, \frac{\pi}{2}) = \frac{\pi}{4}$. For two different A values, the free surface elevation $\eta(x, t)$ is shown in fig.(2.3) and fig.(2.4) as the wave is running up on a linear beach profile.

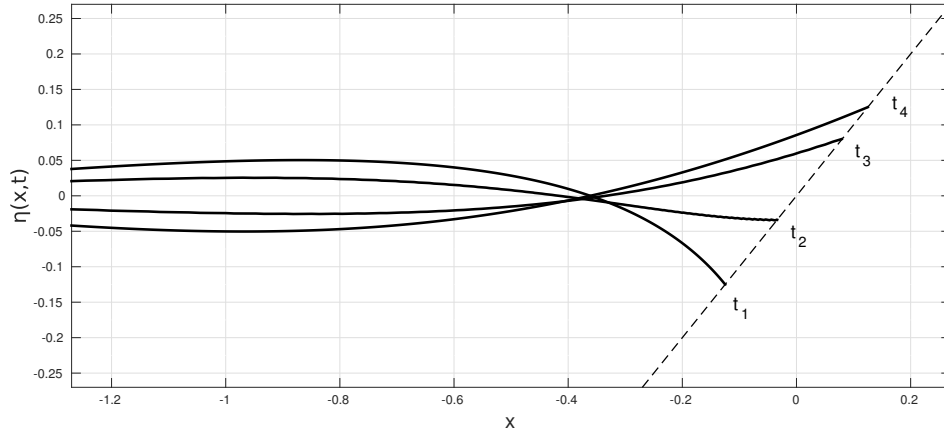


Figure 2.3: A irrotational long wave is running up on a sloping beach with $A = 0.5$. The solution is plotted at $t_1 = \frac{\pi}{4}$, $t_2 = \frac{5\pi}{12}$, $t_3 = \frac{7\pi}{12}$ and $t_4 = \frac{3\pi}{4}$.

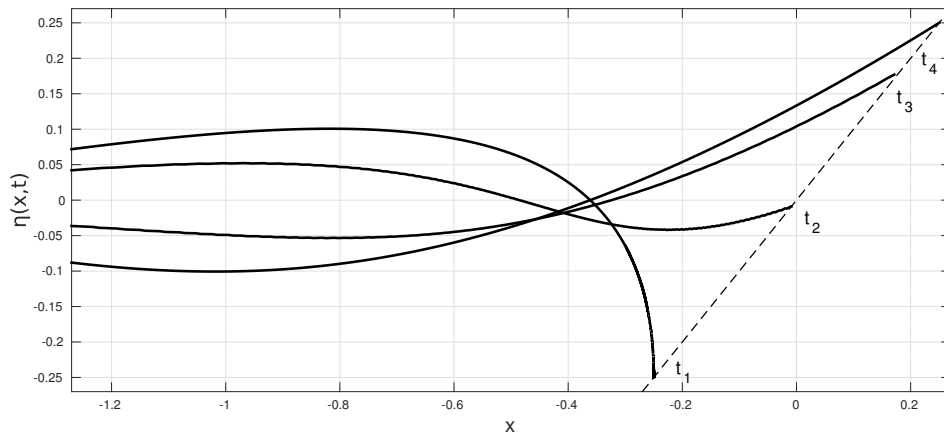


Figure 2.4: A irrotational long wave is running up on a sloping beach with $A = 1$. The solution is plotted at $t_1 = \frac{\pi}{4}$, $t_2 = \frac{5\pi}{12}$, $t_3 = \frac{7\pi}{12}$ and $t_4 = \frac{3\pi}{4}$.

One problem with the approach of Carrier and Greenspan is that it is difficult to treat the boundary-value problem. With the use of a hodograph transformation, the problem concerns the transfer of the boundary data from the (x, t) -coordinates to the (σ, λ) -coordinates. This problem was investigated in [3] and an approximate analytical solution of the boundary-value problem was obtained.

The exact solutions of the Carrier-Greenspan method provides a standing wave. A transient wave solution can be obtain from eq.(2.33) according to [3]. In addition, the initial-value problem can be solved exactly [5] where a propagating wave solution were obtained.

Chapter 3

Shallow water dynamics on linear shear flows and plane beaches

In this chapter, our submitted paper is presented in its entirety.

Shallow Water Dynamics on Linear Shear Flows and Plane Beaches

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Abstract

Long waves in shallow water propagating over a background shear flow towards a sloping beach are being investigated. The classical shallow-water equations are extended to incorporate both a background shear, and a plane beach, resulting in non-reducible hyperbolic system. Nevertheless, it is shown how several changes of variables based on the hodograph transform may be used to transform the system into a linear equation which may be solved exactly using the method of separation of variables. This method can be used to investigate the run-up of a long wave on a planar including the development of the shoreline.

1 Introduction

While many classical results in the theory of surface water waves have been obtained in the context of irrotational flow, this assumption is not always justified. Indeed, it is well known that vorticity may have a strong effect on the properties of surface waves, and there is now a growing literature on the effect of vorticity on the properties of surface waves. In mathematical studies focused on the influence of vorticity on the dynamics of a free surface, some simplifying assumptions are usually made. Examples of cases which have proved to be mathematically tractable include compactly supported vorticity, such as point vortices or vortex patches [11, 18], and the creation of vorticity through interaction with bathymetry [7] or through singular flow such as hydraulic jumps [16].

One important case which is particularly amenable to both analytic and numerical methods is the propagation of waves over a linear shear current. As noted in the classical paper [20], there is a certain scale separation between long surface waves and typical shear profiles which justifies the assumption that the shear is unaffected by the wave motion to the order of accuracy afforded by the model, and moreover, the precise profile of the shear flow may be approximated with a linear shear.

In the current work, we consider the case where a background shear current interacts with a sloping beach. In particular, suppose the seabed is given by $h(x) = -\alpha x$ (see Figure 1), and in addition a background shear flow $U(z) = \Gamma_0 + \Gamma_1 z$ is imposed. As shown in Appendix 1, for long surface waves, a set of shallow-water equations may be derived from first principles. The system has the form

$$\eta_t + \left(\Gamma_0(h + \eta) + \frac{\Gamma_1}{2}(\eta^2 - h^2) + u(h + \eta) \right)_x = 0, \quad (1)$$

$$u_t + \left(us + \frac{s^2}{2} + \frac{u^2}{2} + g\eta \right)_x = 0, \quad (2)$$

where $\eta(x, t)$ describes the deflection of the free surface at a point x and a time t , and $u(x, t)$ represents the horizontal fluid velocity. The function $s(x) = \Gamma_0 + \alpha\Gamma_1 x$, and in particular the coefficient $\alpha\Gamma_1$ represent the strength of the interaction between the sea-bed and the shear. Note that this system is hyperbolic, but the inclusion of non-trivial bathymetry makes the

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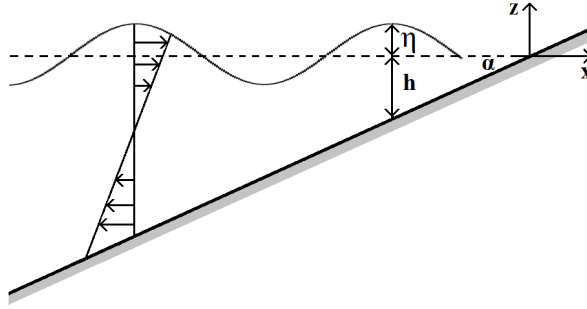


Figure 1: Sloping beach given by $h(x) = -\alpha x$.

system irreducible. Nevertheless it will be shown in the body of this paper that it is possible to employ a hodograph transform which aids in the construction of exact solutions of the system, and in particular allows us to make predictions of the development of the waterline.

The idea of exchanging the roles of dependent and independent variables originated in the theory of gas dynamics [10], and has been used in various special cases in hyperbolic equations, including the shallow-water equations. However, it was not until the work of Carrier and Greenspan [5] that it became possible to find exact solutions for the shallow-water equations in the case of non-constant bathymetry. Indeed, the real novelty of the work of Carrier and Greenspan lay in the fact that they succeeded in applying the hodograph transform in the case of a non-uniform environment. In particular, they obtained explicit solutions to the non-linear shallow-water equations on a linear beach profile, but without vorticity.

There are a few important variations on the method of Carrier and Greenspan. In particular, more general initial data were considered in [6], and physical properties such as mass and momentum fluxes related to the possible run-up of a tsunami were mapped out. Some generalizations of the CG method with regards to the shape of the beach profile were made in [12], where a convex bottom topography of the type $h(x) = x^{4/3}$ was considered. Also, three-dimensional effects were included in recent work [17], where a general approach was put forward to study the problem on a bay of arbitrary cross-section. The work laid down in [3] makes use of analysis techniques to estimate the Jacobian function associated to an arbitrary bottom profile, and thus proves that at least in theory, that the restriction to planar or convex beaches is not necessary.

One problematic issue with the approach of Carrier and Greenspan is that it is difficult to treat the boundary-value problem. For example, if wave and velocity data are known at a fixed location it is not straightforward to prescribe these as boundary data, and study the shoaling and run-up of the resulting shorewards propagating waves. This problem was investigated in-depth in [2], where it was shown how the boundary-value can be solved in the context of planar beaches.

As we stated above, the main purpose of the current work is to extend the Carrier-Greenspan approach to the case where background vorticity can be included in the flow. The need for such an extension arises from the fact that the propagation of water waves in coastal areas is often affected by the influence of currents. Previous works on this topic include the construction of periodic traveling waves over shear flows [9] in the Euler equations, numerical investigations [22] and the investigation of the pressure profile in asymptotic models [1, 23].

While a shear current may be a pre-existing condition, such as created by wind stress on the free surface it can also be induced directly by the wave motion itself. In particular, it is well known that a periodic wavetrain leads to mass transport through the classical Stokes drift. If the wave motion is directed towards a beach, the required return flow creates an undertow which is flowing in the seaward direction from the shoreline. In the extreme case where wave breaking occurs at the free surface, the mass transport is enhanced, and a stronger shear profile develops [19].

The plan of the paper is as follows. In section 2 we consider the case of a shear flow over a flat bed. While the inclusion of background vorticity into shallow-water models is known (see [13] for instance), it is not obvious how to find the Riemann invariants in this case. Even though they are known to exist in principle, it is not trivial to find closed-form expressions. In

section 3, we treat the case of a shear flow over a linear beach, and use intuition gained from the Riemann invariants from the flat-bed case to aid in the construction of the hodograph transform in the more difficult case of non-constant bathymetry. Finally, in Section 4, we explain how the equations may be solved exactly, and we include a few plots where we compare with the case without the background shear flow. Finally, the equations with both shear flow and an uneven bottom are derived in the Appendix.

2 Shear flow over a flat bed

We first look at the case of shear flow over a flat bed as this case will give us important clues on how to proceed in the more difficult case of a shear flow over a sloping bed. A sketch of the geometry is shown in Fig. 2. In particular, the total depth is $H(x, t) = \eta(x, t) + h_0$, where h_0 is the constant undisturbed depth. The vertical shear current is assumed to be of the form $U(z) = -\Gamma_0 + \Gamma_1 z$ which yields a background vorticity $-\Gamma_1$. Without loss of generality, we may assume that the density is constant, and the width of the channel is unity. The shallow-water equations for a flat bed are as follows:

$$H_t + \left(-\Gamma_0 H + \frac{\Gamma_1}{2} H^2 + uH \right)_x = 0, \quad (3)$$

$$u_t + \left(-\Gamma_0 u + \frac{1}{2} u^2 + gH \right)_x = 0. \quad (4)$$

In order to express the equations in non-dimensional variables, we introduce the following scaling: $u^* = \frac{u}{u_0}$, $\eta^* = \frac{\eta}{h_0}$, $x^* = \frac{x}{h_0}$, $t^* = \frac{t}{T}$, $\Gamma_0^* = \frac{\Gamma_0}{u_0}$, $\Gamma_1^* = \frac{\Gamma_1}{1/T}$ where $T = \sqrt{h_0/g}$, $u_0 = \sqrt{gh_0}$. The equations are then written in non-dimensional form as

$$H_{t^*}^* + \left(-\Gamma_0^* H^* + \frac{\Gamma_1^*}{2} H^{*2} + u^* H^* \right)_{x^*} = 0,$$

$$u_{t^*}^* + \left(-\Gamma_0^* u^* + \frac{1}{2} u^{*2} + H^* \right)_{x^*} = 0.$$

As is customary in shallow-water theory, the propagating speed of a wave is taken as $c = \sqrt{gH}$ (in non-dimensional variables $c^* = \sqrt{H^*}$ where $c^* = \frac{c}{u_0}$). Note that for easier reading, the stars on the non-dimensional variables will be omitted from now on. Adding and subtracting the two equations above, and using the speed c as an unknown, the equations can be written in so-called pre-characteristic form as

$$\begin{cases} \left\{ \frac{\partial}{\partial t} + (u - \Gamma_0 + c) \frac{\partial}{\partial x} \right\} (u + 2c) = -2\Gamma_1 c^2 c_x, \\ \left\{ \frac{\partial}{\partial t} + (u - \Gamma_0 - c) \frac{\partial}{\partial x} \right\} (u - 2c) = 2\Gamma_1 c^2 c_x. \end{cases}$$

This form may be useful in some situations connected to numerical integration of the equations, but is included here mainly as a stepping stone toward a similar set of equations in the case of the sloping bottom. In the current context, it is actually more advantageous to put the equations into proper characteristic form. However, since it is not easy to see how to eschew the $2\Gamma_1 c^2 c_x$ -terms on the right hand side, we will use a different approach to put the equations in characteristic form.

In vector notation, we can write eq.(3) and eq.(4) as

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0} \quad (5)$$

where $\mathbf{u} = [H, u]^T$. Further, $\mathbf{f}(\mathbf{u})_x = \mathbf{f}'(\mathbf{u})\mathbf{u}_x$, where $\mathbf{f}'(\mathbf{u})$ is the Jacobian matrix

$$\mathbf{f}'(\mathbf{u}) = \begin{bmatrix} -\Gamma_0 + \Gamma_1 H + u & H \\ 1 & -\Gamma_0 + u \end{bmatrix}.$$

The eigenvalues are

$$\xi_1 = u - \Gamma_0 + \frac{1}{2}\Gamma_1 H + \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H},$$

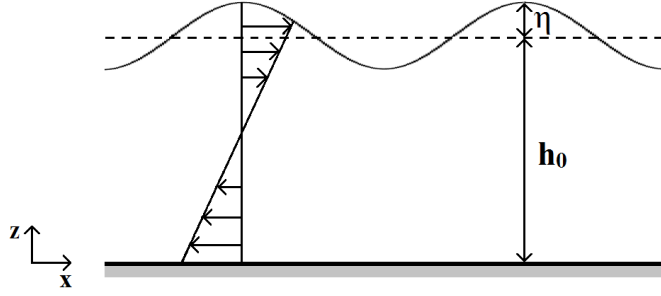


Figure 2: Background shear flow for constant depth.

$$\xi_2 = u - \Gamma_0 + \frac{1}{2}\Gamma_1 H - \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H}.$$

These eigenvalues are real and distinct which means that the system is strictly hyperbolic. Since the Jacobian matrix only depends on \mathbf{u} , and not x or t , the system is reducible, and Riemann invariants exist according to the standard theory [10]. However, finding exact expressions for the Riemann invariants is in general highly non-trivial.

In order to find the the Riemann invariants ω_1 and ω_2 , it will be convenient to define an eigenproblem $\mathbf{L}\mathbf{f}'(\mathbf{u}) = \mathbf{\Lambda}\mathbf{L}$ with the left eigenvectors

$$\mathbf{l}_1 = \begin{bmatrix} 2 \\ -\Gamma_1 H + \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} \end{bmatrix},$$

$$\mathbf{l}_2 = \begin{bmatrix} 2 \\ -\Gamma_1 H - \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} \end{bmatrix}.$$

Inserting the left eigenproblem in eq.(5), we can express eq.(5) as

$$\mathbf{l}_i^T \mathbf{u}_t + \xi_i \mathbf{l}_i^T \mathbf{u}_x = 0, \quad (6)$$

where $i = 1, 2$. If we now introduce the auxiliary function $\mu(\mathbf{u})$ satisfying

$$\nabla \omega_i(\mathbf{u}) = \left[\frac{\partial \omega_i}{\partial H}, \frac{\partial \omega_i}{\partial u} \right] = \mu_i(\mathbf{u}) \mathbf{l}_i^T, \quad (7)$$

the eq.(6) can be written as

$$\nabla \omega_i(\mathbf{u}) \mathbf{u}_t + \xi_i \nabla \omega_i(\mathbf{u}) \mathbf{u}_x = 0, \quad (8)$$

which is the same as

$$\left\{ \frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial x} \right\} \omega_i(\mathbf{u}) = 0.$$

The characteristic form in the latter equation shows that $\omega_i(\mathbf{u})$ is constant along the characteristics $\frac{dx}{dt} = \xi_i(\mathbf{u})$. The challenging part of this procedure is to find an expression for $\mu_i(\mathbf{u})$. To be able to proceed further, we start by assuming that $\mu_i(\mathbf{u})$ is chosen such that the relation $\frac{\partial^2 \omega_i}{\partial H \partial u} = \frac{\partial^2 \omega_i}{\partial u \partial H}$ is satisfied. First, to calculate $\mu_1(\mathbf{u})$, eq.(7) gives us

$$\frac{\partial \omega_1}{\partial H} = \mu_1(\mathbf{u}) 2$$

$$\frac{\partial \omega_1}{\partial u} = \mu_1(\mathbf{u}) \left(-\Gamma_1 H + \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} \right), \quad (9)$$

and if we let

$$\mu_1(\mathbf{u}) = \Gamma_1 + \frac{1}{H} \sqrt{(\Gamma_1 H)^2 + 4H}, \quad (10)$$

the assumption will be satisfied. Integration gives us

$$\omega_1 = 2\Gamma_1 H + 2\sqrt{(\Gamma_1 H)^2 + 4H} + \frac{8}{\Gamma_1} \sinh^{-1} \left(\frac{\Gamma_1 \sqrt{H}}{2} \right) + K_1(u),$$

$$\omega_1 = 4u + K_2(H),$$

where $K_1(H)$ and $K_2(u)$ are the constants of integration. By combining these, we obtain the first Riemann invariant

$$\omega_1 = u - \Gamma_0 + \frac{1}{2}\Gamma_1 H + \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} + \frac{2}{\Gamma_1} \sinh^{-1} \left(\frac{\Gamma_1 \sqrt{H}}{2} \right), \quad (11)$$

where we also have divided by 4 and subtracted by Γ_0 to simplify further work.

We can obtain the second Riemann invariant in a similar way. With the expression for the parameter μ_2 given by

$$\mu_2(\mathbf{u}) = \Gamma_1 - \frac{1}{H} \sqrt{(\Gamma_1 H)^2 + 4H},$$

we get

$$\omega_2 = u - \Gamma_0 + \frac{1}{2}\Gamma_1 H - \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} - \frac{2}{\Gamma_1} \sinh^{-1} \left(\frac{\Gamma_1 \sqrt{H}}{2} \right).$$

With these expressions in hand, the equations (3) and (4) can then be rewritten in characteristic form as

$$\begin{cases} \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} \end{cases} \omega_1 = 0,$$

$$\begin{cases} \frac{\partial}{\partial t} + \xi_2 \frac{\partial}{\partial x} \end{cases} \omega_2 = 0.$$

However, the equations are still nonlinear. Since one purpose of the present study is to obtain exact representations of solutions of (3) and (4), it will be convenient to perform yet another transformation to put the equations in linear form.

Switching dependent and independent variables via a hodograph transform from $\omega_1 = \omega_1(x, t)$ and $\omega_2 = \omega_2(x, t)$ to $x = x(\omega_1, \omega_2)$ and $t = t(\omega_1, \omega_2)$, results in

$$x_{\omega_2} - \xi_1 t_{\omega_2} = 0, \quad (12)$$

$$x_{\omega_1} - \xi_2 t_{\omega_1} = 0. \quad (13)$$

As long as the Jacobian matrix remains non-singular, linearity has been achieved and the equations can now be solved. We will come back to the solution in section 4.1.

3 Shear flow on a sloping bed

We will now consider the geometry in Fig. 1 with the total depth $H(x, t) = \eta(x, t) + h(x)$. The vertical shear current is assumed to be of the form $U(z) = \Gamma_0 + \Gamma_1 z$ with the vorticity $-\Gamma_1$. Note that the x -axis is now assumed to be aligned with the undisturbed free surface as this normalization is more convenient in the current setting.

To put equations (1) and (2) into non-dimensional form, we introduce new variables $u^* = \frac{u}{u_0}$, $\eta^* = \frac{\eta}{\alpha l_0}$, $x^* = \frac{x}{l_0}$, $t^* = \frac{t}{T}$, $\Gamma_0^* = \frac{\Gamma_0}{u_0}$, $\Gamma_1^* = \frac{\Gamma_1}{1/T}$ where $T = \sqrt{l_0/\alpha g}$, $u_0 = \sqrt{g l_0 \alpha}$ and l_0 is a characteristic length. We also define $s^* = \frac{s}{u_0}$. The equations then appear as

$$\eta_{t^*}^* + \left(\Gamma_0^* (\eta^* - x^*) + \frac{\alpha \Gamma_1^*}{2} (\eta^{*2} - x^{*2}) + u^* (\eta^* - x^*) \right)_{x^*} = 0, \quad (14)$$

$$u_{t^*}^* + \left(u^* s^* + \frac{s^{*2}}{2} + \frac{u^{*2}}{2} + \eta^* \right)_{x^*} = 0. \quad (15)$$

As in the previous section for the sake of readability, the stars will be disregarded in what follows. In an attempt to write the equations in characteristic form, one may insert the propagation speed in non-dimensional form $c = \sqrt{(\eta - x)}$, and then add and subtract them to obtain the pre-characteristic form

$$\begin{cases} \frac{\partial}{\partial t} + (u + s + c) \frac{\partial}{\partial x} \end{cases} (u + s + 2c + t) = -2s_x c^2 c_x,$$

$$\begin{cases} \frac{\partial}{\partial t} + (u + s - c) \frac{\partial}{\partial x} \end{cases} (u + s - 2c + t) = 2s_x c^2 c_x.$$

To be able to solve these equations, the difficulty lies in finding the Riemann invariants. We can write eq.(14) and eq.(15) as $\mathbf{u}_t + \mathbf{f}(\mathbf{u}, x)_x = \mathbf{0}$ where $\mathbf{u} = [\eta, u]^T$. The Jacobian matrix $\mathbf{f}'(\mathbf{u}, x)$ has the following eigenvalues

$$\begin{aligned}\xi_1 &= u + s + \frac{\alpha\Gamma_1}{2}c^2 + \frac{c}{2}\sqrt{(\alpha\Gamma_1c)^2 + 4}, \\ \xi_2 &= u + s + \frac{\alpha\Gamma_1}{2}c^2 - \frac{c}{2}\sqrt{(\alpha\Gamma_1c)^2 + 4}.\end{aligned}$$

Since the Jacobian matrix now depends on x , the system is not reducible, and it is not clear whether Riemann invariants can be found. In particular we cannot proceed in the same way as in section 2. However, when carefully combining the pre-characteristic form and the eigenvalues with the corresponding equations for the flat bed case, a bit of informed guessing points to defining the Riemann invariants as

$$\begin{aligned}\omega_1 &= u + s + \frac{1}{2}\alpha\Gamma_1c^2 + \frac{1}{2}c\sqrt{(\alpha\Gamma_1c)^2 + 4} + \frac{2}{\alpha\Gamma_1}\sinh^{-1}\left(\frac{\alpha\Gamma_1c}{2}\right) + t \\ \omega_2 &= u + s + \frac{1}{2}\alpha\Gamma_1c^2 - \frac{1}{2}c\sqrt{(\alpha\Gamma_1c)^2 + 4} - \frac{2}{\alpha\Gamma_1}\sinh^{-1}\left(\frac{\alpha\Gamma_1c}{2}\right) + t\end{aligned}$$

As it turns out, if these expressions are substituted into eq.(14) and eq.(15), the characteristic form

$$\begin{aligned}\left\{\frac{\partial}{\partial t} + \xi_1\frac{\partial}{\partial x}\right\}\omega_1 &= 0, \\ \left\{\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x}\right\}\omega_2 &= 0\end{aligned}$$

appears. These two equations are still nonlinear in t , so we continue by performing a hodograph transformation, changing $\omega_1 = \omega_1(x, t)$ and $\omega_2 = \omega_2(x, t)$ to $x = x(\omega_1, \omega_2)$ and $t = t(\omega_1, \omega_2)$, which results in the equations

$$\begin{aligned}x_{\omega_2} - \xi_1 t_{\omega_2} &= 0, \\ x_{\omega_1} - \xi_2 t_{\omega_1} &= 0.\end{aligned}$$

In contrast to the flat bed case, the equations are still nonlinear at this stage. Therefore, another step is required, and we introduce new variables $\omega_1 + \omega_2 = \lambda$ and $\omega_1 - \omega_2 = \sigma$. This change of variables give us

$$x_\lambda - At_\lambda + Bt_\sigma = 0, \quad (16)$$

$$x_\sigma - At_\sigma + Bt_\lambda = 0, \quad (17)$$

where to simplify, we have defined $A = u + s + \frac{\alpha\Gamma_1}{2}c^2$ and $B = \frac{c}{2}\sqrt{(\alpha\Gamma_1c)^2 + 4}$. Further, differentiating these equations, and using the identities $x_{\sigma\lambda} = x_{\lambda\sigma}$ and $t_{\sigma\lambda} = t_{\lambda\sigma}$ leads to

$$A_\lambda t_\sigma - A_\sigma t_\lambda - B_\lambda t_\lambda + B_\sigma t_\sigma = B(t_{\lambda\lambda} - t_{\sigma\sigma}). \quad (18)$$

In order to find expressions for the derivatives of A and B with respect to σ and λ , we start by writing the variables λ and σ as

$$\frac{\lambda}{2} = u + s + \frac{\alpha\Gamma_1}{2}c^2 + t, \quad (19)$$

$$\frac{\sigma}{2} = \frac{c}{2}\sqrt{(\alpha\Gamma_1c)^2 + 4} + \frac{2}{\alpha\Gamma_1}\sinh^{-1}\left(\frac{\alpha\Gamma_1c}{2}\right). \quad (20)$$

It is easy to see from eq.(19) that $A_\sigma = -t_\sigma$ and $A_\lambda = \frac{1}{2} - t_\lambda$. To calculate B_λ and B_σ , we start by differentiating B to find

$$B_\sigma = \frac{(\alpha\Gamma_1c)^2 + 2}{\sqrt{(\alpha\Gamma_1c)^2 + 4}}c_\sigma, \quad B_\lambda = \frac{(\alpha\Gamma_1c)^2 + 2}{\sqrt{(\alpha\Gamma_1c)^2 + 4}}c_\lambda,$$

where c_σ and c_λ are unknown. We can find an expression for these by differentiating eq.(20) implicitly with respect to σ and λ , yielding

$$\frac{1}{2} = c_\sigma \sqrt{(\alpha\Gamma_1 c)^2 + 4}, \quad 0 = c_\lambda \sqrt{(\alpha\Gamma_1 c)^2 + 4}. \quad (21)$$

Since the root cannot be zero, c_λ has to be zero. Thus, with these calculations eq.(18) becomes

$$\left(\frac{(\alpha\Gamma_1 c)^2 + 3}{(\alpha\Gamma_1 c)^2 + 4} \right) t_\sigma = \frac{c}{2} \sqrt{(\alpha\Gamma_1 c)^2 + 4} (t_{\lambda\lambda} - t_{\sigma\sigma}). \quad (22)$$

Unfortunately, the c is only given implicitly as a function of σ in eq.(20). However, notice that in eq.(20) both terms are increasing and monotone, so the relation can be inverted. Since we seek an expression for $c_{\sigma\sigma}$, we start by differentiating eq.(20) twice and get

$$0 = c_{\sigma\sigma} \sqrt{(\alpha\Gamma_1 c)^2 + 4} + \frac{(\alpha\Gamma_1 c_\sigma)^2 c}{\sqrt{(\alpha\Gamma_1 c)^2 + 4}}.$$

By inserting c_σ from eq.(21), we obtain the expression

$$c_{\sigma\sigma} = -\frac{(\alpha\Gamma_1)^2 c}{4((\alpha\Gamma_1 c)^2 + 4)^2}.$$

With some calculations eq.(22) then becomes

$$ct_{cc} + 3t_c = 4c((\alpha\Gamma_1 c)^2 + 4)t_{\lambda\lambda}, \quad (23)$$

which is a linear equation and can now be solved exactly.

4 Exact solutions of the equations

4.1 Flat bed

One way to solve eq.(12) and eq.(13) is to introduce new variables in the same way as shown above for the case of the sloping bed. Thus, introducing the variables $\lambda = \omega_1 + \omega_2$ and $\sigma = \omega_1 - \omega_2$, the equations can be written as

$$x_\lambda - (u - \Gamma_0 + \frac{1}{2}\Gamma_1 H)t_\lambda + \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} t_\sigma = 0, \quad (24)$$

$$x_\sigma - (u - \Gamma_0 + \frac{1}{2}\Gamma_1 H)t_\sigma + \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} t_\lambda = 0. \quad (25)$$

Moreover remembering the expressions for ω_1 and ω_2 from Section 2, λ and σ appear as

$$\frac{\lambda}{2} = u - \Gamma_0 + \frac{\Gamma_1}{2}H, \quad (26)$$

$$\frac{\sigma}{2} = \frac{1}{2}\sqrt{(\Gamma_1 H)^2 + 4H} + \frac{2}{\Gamma_1} \sinh^{-1} \left(\frac{\Gamma_1 \sqrt{H}}{2} \right). \quad (27)$$

Inverting the relation (27) results in the following linear equation for $t(H, \lambda)$:

$$Ht_{HH} + 2t_H = (\Gamma_1^2 H + 4)t_{\lambda\lambda}. \quad (28)$$

Before we solve this equation, notice that it is problematic to calculate $x(H, \lambda)$ without introducing a 'potential' function for $t(H, \lambda)$, i.e.

$$t = \frac{1}{\Gamma_1^2 H + 4} \phi_H. \quad (29)$$

However, if this potential is used, eq.(25) gives us an expression for $x(H, \lambda)$, viz.

$$x = \frac{\lambda}{2}t - \frac{1}{2}\phi_\lambda.$$

Eq.(28) can now be written due to eq.(29) as

$$H(\Gamma_1^2 H + 4)\phi_{HH} + 4\phi_H = (\Gamma_1^2 H + 4)\phi_{\lambda\lambda}$$

This equation can be solved using separation of variables, and the solution has the general form

$$\phi(H, \lambda) = A \cos(\omega\lambda) e^{-i\Gamma_1\omega H} [-\omega H(i\Gamma_1 - 2\omega)\mathcal{F}_1 + (i\Gamma_1\omega H - 1)\mathcal{F}_2]$$

where

$$\mathcal{F}_1 = {}_1F_1\left(\frac{2i\omega + 2\Gamma_1}{\Gamma_1}, 3, 2i\Gamma_1\omega H\right), \quad \mathcal{F}_2 = {}_1F_1\left(\frac{2i\omega + \Gamma_1}{\Gamma_1}, 2, 2i\Gamma_1\omega H\right)$$

are given in terms the generalized hypergeometric function ${}_1F_1$ [14]. Finally, the principal unknowns can be expressed in terms of λ and H as $u = \frac{\lambda}{2} + \Gamma_0 - \frac{\Gamma_1}{2}H$ and $\eta = H - h_0$.

4.2 Sloping bed

We now look at the more interesting case of exact solutions in the presence of the inclined bottom profile. To be able to solve for $x(c, \lambda)$, we will also here make use of a 'potential' function. Instead of introducing the potential function for $t(c, \lambda)$ directly, we rather start by defining

$$W(c, \lambda) = u(c, \lambda) + \alpha\Gamma_1 x(c, \lambda) + \frac{\alpha\Gamma_1}{2}c^2. \quad (30)$$

Combining the new function $W(c, \lambda)$ with eq.(19), we can rewrite eq.(23) and obtain

$$cW_{cc} + 3W_c = 4c((\alpha\Gamma_1 c)^2 + 4)W_{\lambda\lambda}. \quad (31)$$

If we now define the function $\phi(c, \lambda)$ by

$$W(c, \lambda) = \frac{1}{c((\alpha\Gamma_1 c)^2 + 4)} \phi_c(c, \lambda), \quad (32)$$

then eq.(31) becomes

$$c\phi_{cc} + \frac{4 - (\alpha\Gamma_1 c)^2}{4 + (\alpha\Gamma_1 c)^2}\phi_c = 4c((\alpha\Gamma_1 c)^2 + 4)\phi_{\lambda\lambda}. \quad (33)$$

We seek a solution in the form $\phi(c, \lambda) = f(c)g(\lambda)$, and thus separating the variables gives two equations of the form

$$c((\alpha\Gamma_1 c)^2 + 4)f''(c) + (4 - (\alpha\Gamma_1 c)^2)f'(c) + 4\omega^2 c((\alpha\Gamma_1 c)^2 + 4)^2 f(c) = 0, \\ g''(\lambda) + \omega^2 g(\lambda) = 0$$

where ω is a constant. The solution $\phi(c, \lambda)$ should be bounded as $c \rightarrow 0$, and the corresponding solution of (33) is

$$\phi(c, \lambda) = A \cos(\omega\lambda) e^{-i\alpha\Gamma_1\omega c^2} [-\omega c^2(i\alpha\Gamma_1 - 2\omega)\mathcal{F}_1 + (i\alpha\Gamma_1\omega c^2 - 1)\mathcal{F}_2],$$

where \mathcal{F}_1 and \mathcal{F}_2 are defined in terms of the generalized hypergeometric functions ${}_1F_1$, evaluated with the following arguments:

$$\mathcal{F}_1 = {}_1F_1\left(\frac{2i\omega + 2\alpha\Gamma_1}{\alpha\Gamma_1}, 3, 2i\alpha\Gamma_1\omega c^2\right), \quad \mathcal{F}_2 = {}_1F_1\left(\frac{2i\omega + \alpha\Gamma_1}{\alpha\Gamma_1}, 2, 2i\alpha\Gamma_1\omega c^2\right).$$

Using the function $W(c, \lambda)$, an expression for $t(c, \lambda)$ can be obtained from eq.(19):

$$t = \frac{\lambda}{2} - W - \Gamma_0.$$

Further, an expression for $x(c, \lambda)$ can be obtained from eq.(17). Inserting for $t(c, \lambda)$ from eq.(19) and eq.(30), results in

$$x_c = -WW_c - \Gamma_0 W_c - c((\alpha\Gamma_1 c)^2 + 4)\left(\frac{1}{2} - W_\lambda\right),$$

and in terms of the function W , it becomes

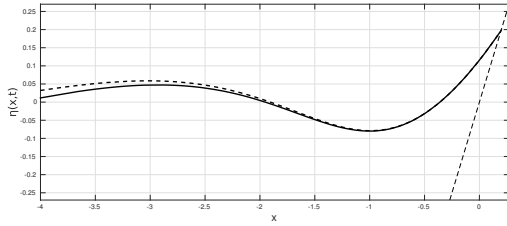
$$x = -\frac{W^2}{2} - \Gamma_0 W - \frac{c^2}{8} ((\alpha\Gamma_1 c)^2 + 8) + \phi_\lambda.$$

The equation for the propagation speed gives us the free surface elevation as $\eta(c, \lambda) = c^2 + x(c, \lambda)$, and an expression for the velocity component $u(c, \lambda)$ is given by

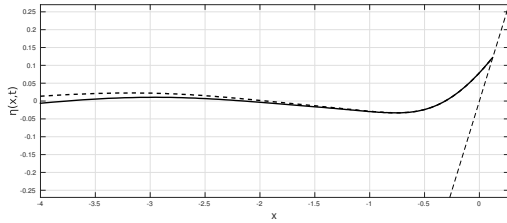
$$u = W - \alpha\Gamma_1 x - \frac{\alpha\Gamma_1}{2} c^2$$

from eq.(19) and eq.(30).

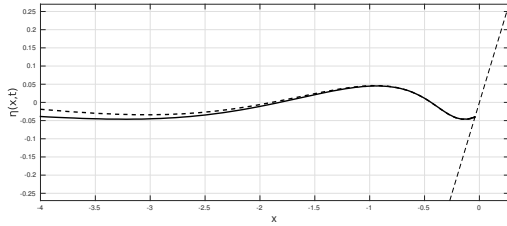
While these formulae give representations of solutions of (1) and (2), it is not completely straightforward to understand these solutions in (x, t) -coordinates. Indeed, in order to plot these exact solutions in terms of (x, t) -coordinates, a numerical approach is needed. A direct approach has been outlined for the problem without a shear flow [4], but it is unclear whether this method will work in the current situation with non-zero vorticity. Therefore, let us briefly outline the numerical implementation. First, expressions for ϕ , W , t , x , η , and u , which are all functions of (c, λ) are used to fill arrays of numbers as c and λ run through certain prescribed sets of values. In order to plot the free surface elevation as $\eta = \eta(x, t)$, we use the two matrices for $t(c, \lambda)$ and $x(c, \lambda)$ as independent variables, and tag the indices corresponding to certain values of x and t (to a prescribed tolerance). Then, we use these same indices in the matrix for η in order to find η as a function of x and t . With this



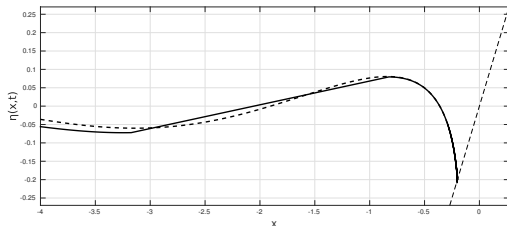
$$t_1 = 0.89$$



$$t_2 = 1.38$$



$$t_3 = 1.87$$



$$t_4 = 2.35$$

Figure 3: Free surface evolution with and without vorticity. The parameters are $A = 0.2$, $\Gamma_0 = 0.0025$, $\alpha\Gamma_1 = 0.1$, $\omega = 1$. The four plots are at time $t_1 = 0.89$, $t_2 = 1.38$, $t_3 = 1.87$ and $t_4 = 2.35$. The nonzero vorticity has the effect of introducing a slight setdown on the left-hand side.

simple scheme, plots of $\eta(x, t)$ are possible. The visualization of the horizontal fluid velocity $u = u(x, t)$ can be done in a similar way. The solution is single-valued so long as the Jacobian $\frac{\partial(x, t)}{\partial(c, \lambda)}$ is nonzero. Therefore, the constants $A, \alpha, \Gamma_0, \Gamma_1, \omega$ and the arrays of (c, λ) are all chosen so that a single-value solution is obtained.

Several plots are shown in Figures 3, 4 and 5. Figure 3 focuses on the comparison between the solutions found here with small $\alpha\Gamma_1$ (solid curves) and solutions found using the method of Carrier and Greenspan (dashed curves). It can be seen that the main effect of the background vorticity is to induce a small setdown on the left-hand side (a minor downward deflection of the mean water level). Note also that the construction laid down here depends on non-zero Γ_1 , so that the good agreement with the Carrier-Greenspan solutions validates our method. On the other hand, Figures 4 and 5 focus on the comparison of different strengths of vorticity. Here, it can be seen that while the run-up and run-down on the beach is identical, the amplitude of the wave on the left-hand side is smaller in the case of larger vorticity. Note that in these cases (as discussed in section 3), the parameter $\alpha\Gamma_1$ serves to measure the combined effect of the strength of the slope and the vorticity, since this parameter appears prominently in the non-dimensional version of the equations.

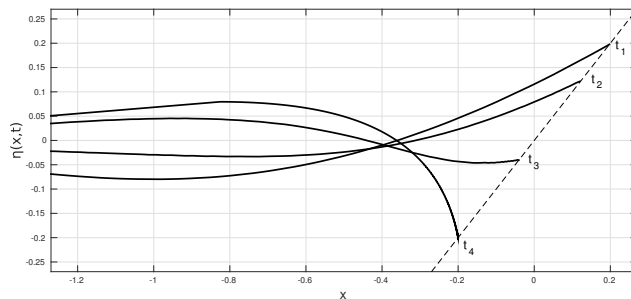


Figure 4: Solution of a wave running up on a sloping beach with $\alpha\Gamma_1 = 0.1$. The solution parameters are $A = 0.2, \Gamma_0 = 0.00025$ and $\omega = 1$. The solution is plotted at $t_1 = 0.89, t_2 = 1.38, t_3 = 1.87, t_4 = 2.35$.

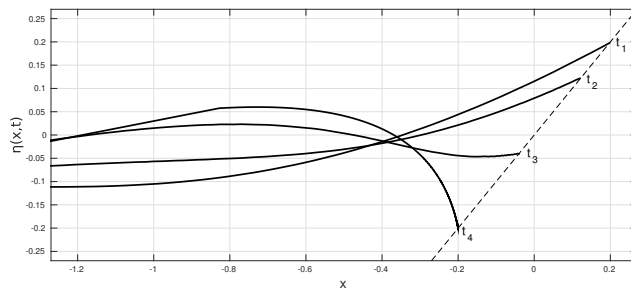


Figure 5: Solution of a wave running up on a sloping beach with $\alpha\Gamma_1 = 0.5$. The solution parameters are $A = 0.2, \Gamma_0 = 0.00025$ and $\omega = 1$. The solution is plotted at $t_1 = 0.89, t_2 = 1.38, t_3 = 1.87, t_4 = 2.35$.

5 Appendix

For the sake of completeness, the shallow-water equations with a background shear flow over a sloping beach will be derived. This derivation complements other already existing asymptotic models with background shear, such as presented in [1, 8, 21]. For a one dimensional flow we consider the velocity component to be $V(x, z, t) = U(z) + u(x, t)$, where the linear shear current is given by $U(z) = \Gamma_0 + \Gamma_1 z$ with Γ_0 and Γ_1 being constants. For an incompressible and inviscid fluid, the equation for conservation of mass for the control interval delimited by x_1 and x_2 on the x -axis is written as

$$\frac{d}{dt} \int_{x_1}^{x_2} H(x, t) dx + \left[\int_{-h(x)}^{\eta(x, t)} V(x, z, t) dz \right]_{x_1}^{x_2} = 0,$$

where $H(x, t) = \eta(x, t) + h(x)$ is the total depth. Integrating in z yields

$$\int_{x_1}^{x_2} \eta_t + \left(\Gamma_0 H + \frac{\Gamma_1}{2} (\eta^2 - h^2) + uH \right)_x dx = 0.$$

Since x_1 and x_2 are arbitrary, the integrand must vanish identically, so that we get the local mass balance equation

$$\eta_t + \left(\Gamma_0 H + \frac{\Gamma_1}{2} (\eta^2 - h^2) + uH \right)_x = 0. \quad (34)$$

For later use, we can rewrite this equation in a slightly different form as

$$\eta_t + \left(\Gamma_0 \eta + \frac{\Gamma_1}{2} \eta^2 + u\eta \right)_x + \left(\Gamma_0 h - \frac{\Gamma_1}{2} h^2 + uh \right)_x = 0. \quad (35)$$

Next, we will consider the momentum balance in the x -direction. Recall that the only forces acting on the control volume are the pressure force, and that the shallow-water approximation entails the assumption that the pressure is hydrostatic. The conservation of momentum is written as

$$\frac{d}{dt} \int_{x_1}^{x_2} \int_{-h}^{\eta} V dx dz + \left[\int_{-h}^{\eta} V^2 dz \right]_{x_1}^{x_2} + \left[\int_{-h}^{\eta} g(\eta - z) dz \right]_{x_1}^{x_2} = - \int_{x_1}^{x_2} \alpha g(\eta + h) dx, \quad (36)$$

where the second term on the left is the momentum flux through the lateral boundaries of the control volume at x_1 and x_2 , and the third term on the left is the pressure force on these lateral boundaries. The term on the right represents the pressure force in the negative x -direction due to the inclined bottom profile. The integral in the second term can be calculated to be

$$\int_{-h}^{\eta} V^2 dz = \Gamma_0^2 H + \Gamma_0 \Gamma_1 (\eta^2 - h^2) + \frac{\Gamma_1^2}{3} (\eta^3 + h^3) + 2\Gamma_0 uH + \Gamma_1 u(\eta^2 - h^2) + u^2 H.$$

Again, the integrand must vanish pointwise so eq.(36) requires that

$$\begin{aligned} & \left(\Gamma_0 H + \frac{\Gamma_1}{2} (\eta^2 - h^2) + uH \right)_t + \\ & \left(\Gamma_0^2 H + \Gamma_0 \Gamma_1 (\eta^2 - h^2) + \frac{\Gamma_1^2}{3} (\eta^3 + h^3) + 2\Gamma_0 uH + \Gamma_1 u(\eta^2 - h^2) + u^2 H + \frac{g}{2} H^2 \right)_x = -\alpha g H. \end{aligned}$$

This equation can be simplified significantly by combining it with eq.(34). Removing terms of the form $(\Gamma_0 \cdot \text{eq.}(34))$ and $(u \cdot \text{eq.}(34))$, the equation becomes

$$\begin{aligned} \Gamma_1 \left[\eta \eta_t + \frac{1}{2} \Gamma_0 \Gamma_1 (\eta^2 - h^2)_x + \frac{1}{3} \Gamma_1^2 (\eta^3 + h^3)_x + \Gamma_1 u_x (\eta^2 - h^2) + \frac{1}{2} \Gamma_1 u (\eta^2 - h^2)_x \right] \\ + H (u_t + \Gamma_0 u_x + uu_x + gH_x + \alpha g) = 0. \end{aligned}$$

After some algebra, the equation can be written as

$$\begin{aligned} \Gamma_1 \left[\eta \left(\eta_t + \left(\Gamma_0 \eta + \frac{\Gamma_1}{2} \eta^2 + u\eta \right)_x \right) - h \left(\Gamma_0 h - \frac{\Gamma_1}{2} h^2 + uh \right)_x \right] \\ + H \left(u_t + \Gamma_0 u_x + uu_x + gH_x + \alpha g \right) = 0. \end{aligned}$$

By eq (35), we obtain

$$\begin{aligned} \Gamma_1 \left[\eta \left(- \left(\Gamma_0 h - \frac{\Gamma_1}{2} h^2 + uh \right)_x \right) - h \left(\Gamma_0 h - \frac{\Gamma_1}{2} h^2 + uh \right)_x \right] \\ + H \left(u_t + \Gamma_0 u_x + uu_x + gH_x + \alpha g \right) = 0, \end{aligned}$$

which is the same as

$$\Gamma_1 H \left(- \Gamma_0 h + \frac{\Gamma_1}{2} h^2 - uh \right)_x + H \left(u_t + \Gamma_0 u_x + uu_x + gH_x + \alpha g \right) = 0.$$

Excluding $H(x, t)$ from the equation and by inserting the undisturbed water depth denoted by $h(x) = -\alpha x$, gives us

$$\alpha\Gamma_0\Gamma_1 + \alpha^2\Gamma_1^2x + \alpha\Gamma_1xu_x + \alpha\Gamma_1u + u_t + \Gamma_0u_x + uu_x + g\eta_x = 0.$$

Defining the function $s(x) = \Gamma_0 + \alpha\Gamma_1x$, leads to the equation

$$ss_x + u_xs + uu_x + us_x + g\eta_x + u_t = 0,$$

which can be written as

$$u_t + \left(us + \frac{s^2}{2} + \frac{u^2}{2} + g\eta \right)_x = 0.$$

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