Research Agendas, Inquiries and Answers in Constructive Type Theory

Forskningsagendaer, forespørsler og svar i konstruktiv typeteori

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Mastergradsoppgave i filosofi

Vår 2017

Universitetet i Bergen
Institutt for filosofi og førstesemesterstudier
Masteroppgave i FILO350
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Résumé

La théorie constructive des types a été créée en 1984 par Per Martin-Löf [1]. L'objectif de cette théorie est de supprimer la distinction entre le signe et la signification. Elle est construite sur l'intuitionnisme de Brouwer développé au debut du vingtième siècle. Plus récemment, en 2006, Olsson [2] a publié un article où il explique comment interpréter les questions et les agendas de recherche dans une théorie la révision des croyances, appelée la théorie AGM. L'interpretation de la théorie de la révision des croyances à travers la théorie constructive des types que je vais utiliser est développée par Primiero [3]. Mon projet est d'expliquer comment on peut représenter cette conception des questions et des agendas de recherche dans la théorie constructive des types.

Un agenda de recherche est formé par des questions qui demandent une réponse. Ces questions sont définies comme des disjonctions exclusives. Je vais garder cette definition, mais établir une distinction entre la demande d'une réponse qu'on trouve dans les questions et la disjonction exclusive. Cette demande va être introduite par un opérateur d'enquête. Une enquête demande une réponse. Je vais donc, introduire un opérateur de réponse qui spécifie si l'enquête est résolue. Il y a trois variantes de l'opérateur d'enquête, une pour les declarations des types, une pour les supposition et une pour les definitions. Il y a trois opérateurs d'enquêtes, il y a trois opérateurs de réponses, une pour chaque opérateur d'enquête. Un agenda de recherche est déterminé par une stratégie qui explique comment dériver les enquêtes. Il est composé d'enquêtes résolues.

Avec les opérateurs d'enquêtes et de réponses, je vais expliquer comment ils peuvent être utilisé pour formaliser des questions du langage naturel. Les problèmes qui ont été introduit par Olsson [2] seront résolus dans la théorie constructive des types grâce aux questions et aux enquêtes.

Sammendrag

Konstruktiv typeteori ble introdusert av Per Martin-Löf i 1984 [1]. Bakgrunnen var å unngå skillet mellom form og mening. Konstruktiv typeteori ble dannet på bakgrunn av Brouwers intuisjonisme fra begynnelsen av 1900-tallet. Senere, i 2006, publiserer Olsson [2] en artikkel hvor han implementerer forskningsagendaer og spørsmål i AGM-teorien. Relasjonen mellom AGM-teorien og konstruktiv typeteori ble utviklet av Primiero [3]. Det er denne teorien jeg skal ta utgangspunkt i her. Mitt prosjekt er å forklare hvordan man kan forstå forskningsagendaer og spørsmål i konstruktiv typeteori.

En forskningsagenda består av spørsmål som må besvares. Disse spørsmålene representeres som eksklusive disjunksjoner. Jeg skal beholde denne definisjonen av spørsmål her, men skille dem fra det spørrende aspektet ved å stille et spørsmål. Dette aspektet introduseres ved hjelp av en forespørselsoperator. En forespørsel krever svar. Jeg skal introdusere en svaroperator som angir om en forespørsel er lukket eller ikke. Vi har tre varianter av denne forespørselsoperatoren, en for typedeklarasjoner, en for antagelser og en for definisjoner. Da vi har tre varianter av forespørselsoperatoren, vil vi også ha tre varianter av svaroperatoren, en for hver forespørselsoperator. En forskningsagenda kan beskrives av en strategi som bestemmer hvilke forespørsler som skal utledes. En forskningsagenda forstås som en samling ubesvarte forespørsler.

Ved hjelp av disse operatorene skal jeg forklare hvordan vi kan formalisere noen aspekter ved spørsmål og svar i naturlig språk. Disse operatorene tilsvarer spørsmål slik de er presentert av Olsson [2]. Problemene som førte til introduksjonen av forskningsagendaer vil løses i det konstruktive typeteoretiske rammeverket ved hjelp av spørsmål og forespørsler.

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31. January, 2017

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1 Introduction

1.1 Project description

The project of this paper is to represent research agendas and questions as introduced by Olsson [2], in a constructive type-theoretical approach, introduced by Martin-Löf [1]. Constructive type theory has traditionally not been closely associated with belief revision, but since it has an established difference between proved propositions and assumptions it seems to handle the difference between knowledge and belief well.

My project is to interpret questions in constructive type theory (CTT). I will do this by introducing three new operators. The first operator is an exclusive disjunction, that has the form of questions for Olsson. The second operator is an inquiry operator that contains the demanding aspect of a question. This operator is actually three distinct operators, one that demands a type declaration, one that demands a new assumption and one that demands a definition. I will also introduce an answer operator for each of the inquiry operators. The answer operator is not found in Olsson's work, but it is introduced in order to distinguish between inquiries that are answered and inquiries that are unanswered. I will represent a research agenda, simply as a collection of unanswered inquiries. The representation of questions, inquiries, answers and research agendas in CTT is my contribution to the existing literature on the topic.

1.2 Why constructive type theory?

Constructive type theory is a variant of the intuitionistic theory of logic and mathematics. Intuitionism was introduced by Brouwer [4]. The background was to consider mathematics and logic as an activity of the mind. [5] An intuitionistic proof is a construction that we make in our mind. When we write down a mathematical proof it only serves to communicate this mental construction to others so that they also can form this mental construction in their mind.

In intuitionism propositions are mental objects. A proof for a proposition is also a mental object that we construct in our minds. Belief and knowledge are concepts that we generally attribute to the mentality of an agent. We know a proposition if we have a proof of it. Knowledge and the intuitionistic notion of a proved proposition seems closely related.

Belief revision theory handles the changes of belief og knowledge for an agent. The dominant theory of belief revision is the AGM-theory. Constructive type theory offers an alternative to this theory. Belief revision in constructive type theory consists of changing a context. In a context judgements are ordered and it is explicit which judgements that are dependent on other judgements. We therefore already have an explicit ordering on the beliefs that we have in the context. A context can be changed by the introducing modal logic. A change from one context to another, keeps everything from the first context in the second context. Because of this persistence, nothing is ever lost. This also gives us a log over every previous change in the context itself. In a context, we do not only have the information that is contained, but we may also see how the context ended up being like it is in the context itself.

A proposition is true only if it has a proof. This means that only the propositions that are actually proved are committing. It does not require the agent to be committed to judgements that are derivable from his beliefs. Belief revision in constructive type theory does not require the agent to be committed to everything that follows from his beliefs, but only what have been actually proved.

An important aspect of constructive type theory is that it is typed. This means that every concept in the theory has to be declared a certain type. This declaration is done on the level of judgements and not on some higher-order definition like we often find in classical approaches. In some sense it makes us able to represent comprehensibility in the same way as we represent beliefs and knowledge.

Constructive type theory has two kinds of proofs for a proposition. We have the categorical judgement, $a : A$ and we have the hypothesis, $x : A$. The difference between these two is that the first one has a as its proof, and the second one does not have an explicit proof yet, but it has a potential proof. As mentioned earlier, this distinction seems to correspond well together with the distinction between belief and knowledge. This can be seen as an advantage of the theory, as it does not require any operator to distinguish between knowledge and belief.

In constructive type theory, every judgement states explicitly what it depends on. It gives us the possibility to represent belief that are dependent on other beliefs in an explicit way. This gives us a notion of relation between different judgements.

1.3 Structure of the paper

This paper will start by presenting the general framework of constructive type theory as presented by Martin-Löf. [1] Ranta [6] introduced a framework for modal logic in CTT. I will also present this, as it is fundamental for the conception of change that we have in belief revision theory. Primiero [3] have later developed a framework for representing belief revision based on Ranta's notion of modal logic. Afterwards I will present the research agendas and questions as they are introduced by Olsson. [2] This will be done in the AGM-theory, as this is the theory Olsson uses to explain his ideas. I will try to explain the important notions of a research agenda and not focus to much on how it actually gets interpreted in the AGM-theory.

After the presentations of the theories, I will introduce my contribution to the field, namely how to represent research agendas and questions in CTT. I will start by explaining how questions, inquiries, answers and research agendas should be understood in this paper. Afterwards I will explain what inquiries and answers mean in CTT. This includes explaining the difference of act in assertions, inquiries and answers. It also includes explaining how inquiries and answers can be seen as judgements in CTT. Afterwards there will be a presentation of the rules for the operators that I want to introduce, namely questions, inquiries and answers. This will be followed by explaining what non-formal aspects inquiries can be used to represent. I will explain how research agendas may be looked at as a strategy to make inquiries and what kind of formal problems inquiries may solve. Afterwards I will explain how this solution may solve the problems that are mentioned by Olsson [2, p. 167-168]

Part I
Presentation of Constructive Type Theory

2 Historical background of CTT

In 1980 Per Martin-Löf gave a series of lectures where he presented a constructive type-theoretical approach to mathematics. His idea behind it was to " ... avoid keeping form and meaning (content) apart. ... Thus we make explicit what is usually implicitly taken for granted." [1]. In particular, this means to not have a clear separation between syntax and semantics and to always explain what set of objects we speak about. This may be seen as a reaction to traditional logic, where the syntactical rules are clearly separated from the semantical ones, and where the set of objects is not explicit (meaning that it uses a universal domain). [7, p. 6]

3 The logic of CTT

3.1 Proposition as types

3.1.1 Syntax and semantics

In classical logic we separate clearly between syntax and semantics. First we make rules for what symbols we are allowed to put together and in which combinations, the syntactical rules. Then we give rules for how to understand the different combinations that we explained in the syntactical rules, the semantical rules. This is not the case in constructive type theory. Here both of these operations are made explicit in what we may call the inference rules. An inference rule in classical logic is a semantic rule in the this style:

$$\frac{A}{A \vee B}$$

It is implicit that A and B are propositions or well-formed formulas. If they are not they are not available formulas in the system. In CTT it has to be made explicit that A and B are propositions and that A is true [1].

$$\frac{A : prop \quad B : prop \quad A\; true}{A \vee B}$$

3.1.2 What is a proposition?

What does it mean to say that something A is a *prop*, $A : prop$? *prop* is one example amongst an infinite number number of types. $A : prop$ is therefore the judgement that A is of the type proposition. Similarly we can explain judgements of other sorts like $b : A$ meaning that b is an object of type A. $b : A$ is also the definition of A *true*, where it is read as b is a proof of A. These kinds of simple judgements are called categorical judgements and can be read in a number of ways in ordinary language [8, p. 2]. [1, p. 4]

A *set*	$a : A$	
A is a set	a is an element of the set A	A is non-empty
A is a proposition	a is a proof of the proposition A the proposition A	A is non-empty
A is an expectation	a is a method of realising the expectation A	A is realisable
A is a problem	a is a method of solving the problem A	A is solvable

3.1.3 Judgements

There are four different kinds of categorical judgements [8, p. 4].

1	A *set*	A is a set
2	$A = B$	A and B are equal sets
3	$a : A$	a is an element of the set A
4	$a = b : A$	a and b are equal elements of the set A

3.2 Hypothetical judgements

3.2.1 Hypothetical judgements

Hypothetical judgements are judgements where one part depends on the other part, the judgement depends on some assumptions. Hypothetical judgements has a form like this:

$$B : \mathit{type}\ (x : A)$$

where A is a type, $B : \mathit{type}$ depends on the assumption that $x : A$ and $x : A$ does not depend on any other assumptions. $x : A$ can be said to be the hypothesis for B [1]. We can also make hypothetical judgements of this form:

$$b : B\ (x : A)$$

This means that under the assumption that x is an element of A, b is an element of the set B. If x is substituted by an element A that yields an element c of B where $a = c : A$, in b, a and c are equal elements of B. [8, p. 4]

It is possible to introduce several assumptions, a list of hypotheses. Such a list of hypotheses is called a context Γ such that:

$$b : B\ (\Gamma)$$

Generally hypothetical judgements have this form [8, p. 4]:

$$x : A\ (x_1 : A_1,\ x_2 : A_2,\ \ldots,\ x_n : A_n)$$

Such that:

$$\frac{\begin{array}{c} A_1 \ \mathit{type} \\ A_2 \ \mathit{type} \ (x_1 : A_1) \\ A_n \ \mathit{type} \ (x_1 : A_1, \ x_2 : A_2, \ \ldots, \ x_{n-1} : A_{n-1}) \\ A : \mathit{type} \ (x_1 : A_1, \ x_2 : A_2, \ \ldots, \ x_n : A_n) \end{array}}{x : A \ (x_1 : A_1, \ x_2 : A_2, \ \ldots, \ x_n : A_n)}$$

3.2.2 Functions

Hypothetical judgements can be used to introduce functions. We can end up with a judgement with this form :

$$f(x) : B(x : A)$$

What happens here is that we introduce a function from A to B. $x : A$ is a hypothesis for $f(x) : B$ such that $f(x) : B(x : A)$ is a hypothetical judgement. This kind of hypothetical judgements that introduces functions can be read in several ways [7, p. 21] :

$f(x) : B$ for arbitrary $x : A$
$f(x) : B$ under the hypothesis $x : A$
$f(x) : B$ provided $x : A$
$f(x) : B$ given $x : A$
$f(x) : B$ if $x : A$
$f(x) : B$ in the context $x : A$

x can be substituted with an element a in A such that it yields $f(a)$ of B. If we substitute by equal elements in A it yields equal elements of B. We can see this by looking at the substitution rules. [7, 8]

$$\frac{\begin{array}{l} (x : A) \\ f(x) : B \quad a : A \end{array}}{f(a) : B}$$

$$\frac{\begin{array}{l} (x : A) \\ f(x) : B \quad a = b : A \end{array}}{f(a) = f(b) : B}$$

$(x : A)$ is the hypothesis for $f(x) : B$ and all the three premisses are hypotheses for the conclusion. [7, p. 21]

3.2.3 Introduction of propositions

If we apply these substitution rules for propositional functions, that are introduced by hypothetical judgements, on individuals we get the introduction for propositions. Hypothetical judgements introduce propositions by this form [7, p. 22] :

$$B(x) : prop\ (x : A)$$

Where these are the substitution rules :

$$\frac{\begin{matrix}(x : A) \\ B(x) : prop \quad a : A\end{matrix}}{B(a) : prop}$$

$$\frac{\begin{matrix}(x : A) \\ B(x) : prop \quad a = b : A\end{matrix}}{B(a) = B(b) : prop}$$

3.3 Rules in CTT

In CTT there are mainly three groups of rules. Formation rules F explain when a proposition can be made by the different operators. Introduction rules I explain the conditions needed to judge a proposition true. Elimination rules E explain what propositions we can judge true based on the condition that a formula containing an operator is true. In addition to this there are rules concerning identity judgements, reflexivity $refl$, symmetry *symm* and transitivity *trans*. There are also substitution rules *subst* concerning the individuals in a set and extensionality rules *ext* for sets. If there is a variable in a discharged hypothesis, we bind this variable in the conclusion, if the conclusion depends on some other hypothesis the variable cannot occur free there. We therefore do not assume anything other than that it is an element of the type. The introduction and elimination also depends on the premises for their corresponding formation rules. This is taken implicitly in order to improve readability. [7, p. 28-30]

3.3.1 Formation rules

$$\frac{}{\bot : prop}\ \bot\mathrm{F}$$

$$\frac{A : prop \quad B : prop}{A \wedge B : prop}\ \wedge\mathrm{F}$$

$$\frac{A : prop \quad B : prop}{A \vee B : prop}\ \vee\mathrm{F}$$

$$\frac{A : prop \quad B : prop}{A \to B : prop} \to \mathrm{F}$$

$$\frac{A : prop}{\neg A : prop}\ \neg\mathrm{F}$$

$$\frac{A : set \quad \begin{matrix}(x : A)\\ B(x) : prop\end{matrix}}{(\forall x : A)B(x) : prop}\ \forall\mathrm{F}$$

$$\frac{A : set \quad \begin{matrix}(x : A)\\ B(x) : prop\end{matrix}}{(\exists x : A)B(x) : prop}\ \exists\mathrm{F}$$

3.3.2 Introduction rules

$$\frac{A\ true \quad B\ true}{A \wedge B\ true}\ \wedge\mathrm{I}$$

$$\frac{A\ true}{A \vee B\ true}\ \vee\mathrm{I}$$

$$\frac{B\ true}{A \vee B\ true}\ \vee\mathrm{I}'$$

$$\frac{\begin{matrix}(A\ true)\\ B\ true\end{matrix}}{A \to B\ true}\ \to\mathrm{I}$$

$$\frac{\begin{matrix}(A\ true)\\ \bot\ true\end{matrix}}{\neg A\ true}\ \neg\mathrm{I}$$

$$\frac{\begin{matrix}(x : A)\\ B(x)\ true\end{matrix}}{(\forall x : A)B(x)\ true}\ \forall\mathrm{I}$$

$$\frac{a : A \quad B(a)\ true}{(\exists x : A)B(x)\ true}\ \exists\mathrm{I}$$

3.3.3 Elimination rules

$$\frac{\bot\ true}{A\ true}\ \bot\mathrm{E}$$

$$\frac{A \wedge B\ true}{A\ true}\ \wedge\mathrm{E}$$

$$\frac{A \wedge B\ true}{B\ true}\ \wedge\mathrm{E}'$$

$$\frac{A \vee B \quad \begin{matrix}(A\ true) \\ C\ true\end{matrix} \quad \begin{matrix}(B\ true) \\ C\ true\end{matrix}}{C\ true}\ \vee\mathrm{E}$$

$$\frac{A \rightarrow B \quad A\ true}{B\ true} \rightarrow \mathrm{E}$$

$$\frac{\neg A\ true \quad A\ true}{\bot\ true}\ \neg\mathrm{E}$$

$$\frac{(\forall x : A)B(x)\ true \quad a : A}{B(a)\ true}\ \forall\mathrm{E}$$

$$\frac{(\exists x : A)B(x)\ true \quad \begin{matrix}(x : A, B(x)\ true) \\ C\ true\end{matrix}}{C\ true}\ \exists\mathrm{E}$$

3.3.4 Rules for reflexivity, symmetry and transitivity

$$\frac{A : set}{A = A : set}\ \mathrm{refl1}$$

$$\frac{A = B : set}{B = A : set}\ \mathrm{symm1}$$

$$\frac{A = B : set \quad B = C : set}{A = C : set}\ \mathrm{trans1}$$

$$\frac{a : A}{a = a : A}\ \mathrm{refl2}$$

$$\frac{a = b : A}{b = a : A}\ \mathrm{symm2}$$

$$\frac{a = b : A \quad b = c : A}{a = c : A}\ \mathrm{trans2}$$

3.3.5 Extensionality rules and substitution rules

$$\frac{A = B : set \quad a : A}{a : B}\ \text{ext1}$$

$$\frac{A = B : set \quad a = b : A}{a = b : B}\ \text{ext2}$$

$$\frac{a : A \quad \begin{matrix}(x : A)\\ J(x)\end{matrix}}{J(a)}\ \text{subst1}$$

$$\frac{a = c : A \quad \begin{matrix}(x : A)\\ B(x) : set\end{matrix}}{B(a) = B(c) : set}\ \text{subst2}$$

$$\frac{a = c : A \quad \begin{matrix}(x : A)\\ b(x) : B(x)\end{matrix}}{b(a) = b(c) : B(a)}\ \text{subst3}$$

3.4 Proof objects and set-theoretical operators

In order to make a judgement that a proposition A is true, A *true*, we need to give a proof object a for A. This is written $a : A$. This judgement is under the condition that $A : prop$. A proof for a complex proposition such as $A \wedge B$ is a pair (a, b), namely the proof object a of A and the proof object b for B. [6, p. 80] To explain the relation between logical operators and set theory we need to introduce set-theoretical operators and define the logical operators by these set-theoretical operators.

In order to obtain the definitions of universal quantification and material implication we introduce a Π operator (cartesian product of a family of sets), to obtain the definitions of existential quantification and conjunction we introduce a Σ operator (disjoint union of a family of sets) and to obtain the definition of disjunction we introduce a + operator (disjoint union or coproduce of two sets). [8, p. 5-7].

3.4.1 Π operator

$$\frac{A : set \quad \begin{matrix}(x : A)\\ B(x) : set\end{matrix}}{(\Pi x : A)B(x) : set}\ \Pi\text{F}$$

$$\frac{\begin{array}{c}(x:A)\\ b(x):B(x)\end{array}}{(\lambda x)b(x):(\Pi x:A)B(x)}\ \Pi\mathrm{I}$$

$$\frac{c:(\Pi x:B(x))\quad a:A}{Ap(c,a):B(a)}\ \Pi\mathrm{E}$$

$$\frac{\begin{array}{cc} & (x:A)\\ a:A & b(x):B(x)\end{array}}{Ap((\lambda x)b(x),a)=b(a):B(a)}\ \Pi\mathrm{Eq1}$$

$$\frac{c:(\Pi x:A)B(x)}{c=(\lambda x)Ap(c,x):(\Pi x:A)B(x)}\ \Pi\mathrm{Eq2}$$

$(\lambda x)b(x)$ is the canonical element of $(\Pi x : A)B(x)$. $Ap(c, a)$ stands for application of c to a, and it gives a canonical element of $B(a)$. If we have $c : (\Pi x : A)B(x)$, where c is a method that yields a canonical element $(\lambda x)b(x)$ of $(\Pi x : A)B(x)$, and we know $b(a) : B(a)$ by substituting x with a by $a : A$. We then end up with a canonical element of $B(a)$ by applying the method c. [1, p. 15]

Universal quantification and material implication are then defined like this [8, p. 6]:

$$(\forall x:A)(B(x))\equiv(\Pi x:A)B(x):prop \text{ when } A:set \text{ and } B(x):prop(x:A)$$

$$(A\rightarrow B)\equiv(\Pi x:A)B:prop \text{ when } A:prop \text{ and } B:prop$$

3.4.2 Σ operator

$$\frac{\begin{array}{cc} & (x:A)\\ A:set & B(x):set\end{array}}{(\Sigma x:A)B(x):set}\ \Sigma\mathrm{F}$$

$$\frac{a:A\quad b:B(a)}{(a,b):(\Sigma x:A)B(x)}\ \Sigma\mathrm{I}$$

$$\frac{\begin{array}{cc} & (x:A,y:B(x))\\ A:set & B(x):set\end{array}}{E(c,(x,y)d(x,y)):C(c)}\ \Sigma\mathrm{E}$$

$$\frac{a : A \quad b : B(a) \quad \begin{matrix}(x : A, y : B(x)) \\ d(x,y) : C((x,y))\end{matrix}}{E(a, b, (x,y)d(x,y)) = d(a,b) : C((a,b))}\ \Sigma\text{Eq}$$

$E(c, (x, y)d(x, y)) : C(c)$ stands for execution of c so that it yields a canonical element, a pair, (a, b) and substitute it for x and y in the right part $d(x, y)$ so that $d(a, b) : C((a, b))$. We can execute $d(x, y)$ and it will yield a canonical element e of $C((a, b))$. We can obtain $c = (a, b) : (\Sigma x : A)B(x)$ because when c is executed it yielded (a, b) and when some object a yields another object b and $a : A$ it is also the case that $a = b : A$. By substitution we end up with $C(c) = C((a, b))$ and e is therefore also a canonical element of $C(c)$. [1, p. 21]

Existential quantification and conjunction are then defined like this [8, p. 6-7]:

$$(\exists x : A)B(x) \equiv (\Sigma x : A)B(x) : \mathit{prop} \text{ when } A : \mathit{set} \text{ and } B(x) : \mathit{prop}(x : A)$$

$$A \wedge B \equiv (\Sigma x : A)B : \mathit{prop} \text{ when } A : \mathit{prop} \text{ and } B : \mathit{prop}$$

3.4.3 + operator

$$\frac{A : \mathit{set} \quad B : \mathit{set}}{A + B : \mathit{set}}\ +\text{F}$$

$$\frac{a : A}{i(a) : A + B}\ +\text{I1}$$

$$\frac{b : B}{j(b) : A + B}\ +\text{I2}$$

$$\frac{c : A + B \quad \begin{matrix}(x : A) \\ d(x) : C(i(x))\end{matrix} \quad \begin{matrix}(y : B) \\ e(y) : C(j(y))\end{matrix}}{D(c, (x)d(x), (y)e(y)) : C(c)}\ +\text{E}$$

$$\frac{a : A \quad \begin{matrix}(x : A) \\ d(x) : C(i(x))\end{matrix} \quad \begin{matrix}(y : B) \\ e(y) : C(j(y))\end{matrix}}{D(i(a), (x)d(x), (y)e(y)) = d(a) : C(i(a))}\ +\text{Eq1}$$

$$\frac{b : B \quad \begin{matrix}(x : A) \\ d(x) : C(i(x))\end{matrix} \quad \begin{matrix}(y : B) \\ e(y) : C(j(y))\end{matrix}}{D(i(a), (x)d(x), (y)e(y)) = e(b) : C(j(b))}\ +\text{Eq2}$$

$D(c, (x)d(x), (y)e(y)) : C(c)$ stands for execution of c so that it either yields a canonical element $i(a)$ that we substitute x in $d(x)$ for a, or it yields a canonical element $j(b)$ that we substitute y in $e(y)$ for b. This gives us an understanding of whether an element is originating in A or B when we have it in $A + B$.

Disjunction is then defined like this [8, p. 7]:

$$A \vee B \equiv A + B : prop \text{ when } A : prop \text{ and } B : prop$$

4 Modal logic in CTT

In a traditional view modal logic is a way of systemising possible worlds. In CTT possible worlds are seen as specifications of hypotheses. The motivation for introducing modal logic in this project is therefore to be able to show different alternative specifications or changes of hypotheses. Later I will show how we can use research agendas to choose between these specifications.

We will see that modal logic in CTT, seen as specifications of hypotheses, ends up being reflexive and transitive and therefore very similar to the S4-system. This subchapter will be based on Ranta's article Constructing Possible Worlds [6]

4.1 Possible worlds as hypotheses

By introducing modal logic we can claim that judgements are done relative to a certain world. $a : A$ can for example said to be true in world w. A judgement "in world w" should be understood as a hypothetical judgement. Since we have four different kinds of judgements I will show how "in world w" should be understood in these [6, p. 83].

1	$A\ set$ in w	$A(x) : set(x : w)$
2	$A = B$ in w	$A(x) = B(x) : set(x : w)$
3	$a : A$ in w	$a(x) : A(x)(x : w)$
4	$a = b : A$ in w	$a(x) = b(x)(x : w)$

In CTT a world w_2 is accessible from another world w_1 when w_2 is a specification of w_1. This means that all the information contained in w_1 is also contained in w_2. If we are in w_1, w_2 would be a possible extension. Here we have introduced the notion of possibility and here this concept will best be understood as an epistemic possibility, namely different ways of getting more knowledge. Formally we have that when we see that $w\ true(x : w)$ is trivial because $x : w(x : w)$. The relation is reflexive. We can also see that when w_2 is a specification of w_1, it is the case that $w_1\ true(y : w_2)$ there is a function d [6, p. 85] :

$$d(y) : w_1(y : w_2)$$

There exists a function from w_2 to w_1. We can say that w_1 is realised when w_2 is realised and therefore have an implication from w_2 to w_1 [6, p. 85]. The relation is transitive, because if there is a w_3 that is a specification of w_2 it will contain the information contained in w_2 and therefore also the information contained in w_1. From this we can see that there is also a function from w_3 to w_1.

$$\frac{d(d(y)) : w_1(d(y) : w_2(y : w_3))}{d(y) : w_1(y : w_3)}$$

Ranta [6, p. 85] argues that specifications would be an infinitely long conjunctions where we add more and more knowledge. This seems to be closely related to scientific research, a never-ending task, but where we still get more and more specified knowledge. We also see that this concept of possibility and hypotheses can lead us very close to belief revision theory or epistemic logic.

4.2 Specifications of contexts

A possible world is therefore some kind of list of propositions or hypotheses. As mentioned earlier this kind of list can be called a context. What we do when we have a relation from w_1 to w_2 actually performs a specification, w_2, of the situation that is approximated in w_1. In order to explain the relation between two worlds w_1 and w_2 where there is a relation from w_1 to w_2, we explain the frames for how contexts are specified. [9, p. 9]

For contexts to be an acceptable view of a possible world it needs to be never-ending specifications. This means that one can never achieve complete or finished knowledge, but the situation still continues to be further and further specified. [6, p. 93] [9, p. 9] The notion of a situation or world where the specification is never-ending can be linked to Husserl [6, p. 98].

The formulation of judgements in contexts can be understood in this way [6, p. 87] :

A *set* in Γ		$A(x_1 \ \ldots, x_n) : set$ $(x_1 : A, \ \ldots, x_n : A_n(x, \ \ldots, x_{n-1}))$
$A = B$ in Γ	$A, B : set$ in Γ	$A(x_1 \ \ldots, x_n) = B(x_1 \ \ldots, x_n) : set$ $(x_1 : A, \ \ldots, x_n : A_n(x, \ \ldots, x_{n-1}))$
$a : A$ in Γ	$A : set$ in Γ	$a(x_1 \ \ldots, x_n) : A(x_1 \ \ldots, x_n)$ $(x_1 : A, \ \ldots, x_n : A_n(x, \ \ldots, x_{n-1}))$
$a = b : A$ in Γ	$A : set$ in Γ, $a, b : A$ in Γ	$a(x_1 \ \ldots, x_n) = b(x_1 \ \ldots, x_n) : A(x_1 \ \ldots, x_n)$ $(x_1 : A, \ \ldots, x_n : A_n(x, \ \ldots, x_{n-1}))$

where $\Gamma = x_1 : A_1, \ \ldots, x_n : A_n(x_1, \ \ldots, x_{n-1})$.

In order to explain the relation between possible worlds, we have to explain how a context can be specified.

Ranta uses vector notation to explain the relation between two contexts in less heavy notation, where one is a specification of the other one. It is used to describe quantity or sequence. It is written as boldface and it means that [6, p. 87-88]

$$J(x_1,\ \ldots, x_n)(x_1 : A_1,\ \ldots, x_n : A_n(x_1,\ \ldots, x_{n-1}))$$

can be written as

$$J(\mathbf{x})(\mathbf{x} : \Gamma)$$

Let Γ and Δ be contexts where Δ is a specification of Γ, $\Gamma \leq_{\mathbf{f}} \Delta$. When

$$\Gamma = x_1 : A_1,\ \ldots, x_n : A_n(x_1,\ \ldots, x_{n-1})$$

and

$$\Delta = y_1 : B_1,\ \ldots, y_m : B_m(y_1,\ \ldots, y_{m-1})$$

the function f from Δ to Γ is a sequence of functions, $\mathbf{f} : \Delta \to \Gamma$ where

$$\begin{array}{l} f_1(y_1,\ \ldots, y_m) : A_1(y_1 : B_1,\ \ldots, y_m : B_m(y_1,\ \ldots, y_{m-1})), \\ \ldots, \\ f_n(y_1,\ \ldots, y_m) : A_n(f_1(y_1,\ \ldots, y_m),\ \ldots, \\ f_{n-1}(y_1,\ \ldots, y_m))(y_1 : B_1,\ \ldots, y_m : B_m(y_1,\ \ldots, y_{m-1})) \end{array}$$

When $\Gamma \leq_{\mathbf{f}} \Delta$ any set A in Γ is also in Δ, $A(\mathbf{f}(\mathbf{y})) : set(\mathbf{y} : \Delta)$. The same goes for all judgements in Γ. [6, p. 88]

4.3 Specifications that introduce new questions

4.3.1 Propositions that are true in a part of a context

If we specify a situation further, new questions may arise from this specification. If we have a context where it is true that all objects are B. If we then specify this situation by adding a new object, a question whether or not also this new object is B. This happens from one situation Γ to a specification of the first situation Δ. If we combine this with what we saw from the previous subchapter, namely $A(\mathbf{f}(\mathbf{y})) : set(\mathbf{y} : \Delta)$, we see that it can pose a problem as one proposition can be true in Γ, but false in Δ. The explanation is that the operators quantify over the objects we find in Γ and not always all the objects that are in Δ. The propositions that quantify over objects in Γ are true in at least a part of Δ. We can see that such relations are relations between instances of a context and not between contexts themselves [6, p. 89].

4.3.2 New questions

Ranta [6, p. 92] mentions the fact that it seems possible to introduce questions in CTT. He does not, however, speak about how questions may occur in a systematic way or even how they may be represented in the framework, but it shows that already in the introduction of modal logic in CTT, that questions may play an important role. The example that Ranta uses is when introducing a new man to a context, the question whether he is black, white or yellow arises

[6, p. 92]. It does of course depend on the rest of the situation, or context if this is a relevant question, but the background is there. This introduces us to my project. As questions are not yet properly developed in CTT, it seems like a natural way to extend the framework.

5 CTT as a theory of belief revision

5.1 Theory of belief

A question that intuitively may be posed, is whether the constructive type theory really is a theory for beliefs. In itself it does not seem like a normal theory of belief representation. It does not seem to be an instance of the AGM-theory. It has on the other hand been argued that CTT may be a very good theory to handle beliefs. We can even find something close to this view in Ranta [6]. He argues that in CTT with modal logic is precisely a theory that handles knowledge. This view has been argued for by several texts in the last 15 years, for example by Primiero [3] and Borghuis, Kamareddine and Nederpelt [10].

In this part I will focus on the representation of belief revision theory as made by Primiero [3]. This can be seen as a linking aspects from the AGM-theory to CTT.

Primiero [3, p. 148-163] argues that in CTT we do not have an equivalent to belief set in the direct sense. What we have is an informational state, that contains all direct information, or data, that the agent has and can use in order to base his knowledge. A knowledge state is another state, that contains the judgements that can be derived from the informational state. The combination of the informational state and knowledge state is collected in a knowledge frame. This knowledge frame contains all the information and all the derivations done on this information. The contexts are collected in the states (and therefore also in the knowledge frame), and it is on these contexts that operations like update, expansion, contraction and rejection are done. They can however be done on an empty context, and in that way also introduce "new" information that are independent from information in other informational state.

Inside the context the different judgements may be dependent on each other, or more precisely, a judgement in a context may depend on all the preceding judgements in the same context, but not the other way around. This means that in the context, we do not only get a representation of the belief set, but we may also have some ordering on this belief set. This may be seen as some kind of entrenchment relation as some beliefs are dependent on other judgements. [3, p.126-127, 130] We should notice here that judgements in a context do not have to be dependent on all previous judgements. They may be independent from each other. For independent judgements it is not as obvious that we have an ordering. This ordering may therefore only be partial.

Primiero [3, p.129-130] mentions three different "translations", or presuppositions, that needs to be done in order to look at CTT as a theory for beliefs. The first is that types correspond to meaningfulness in the theory. Every judgement is dependent on that every object in the judgement is declared as a certain type. This corresponds to meaningfulness, that the information is understandable. The second is that justification of a judgement corresponds to a proof. The information that is needed to claim a certain judgement is always present to the agent. The third is that contexts are the basis of what judgements are made on. This means that it is the belief set that our judgement is relying on. As there are three kinds of judgements, and one of how judgements are related to each other. I will try to explain what the three different kind of judgements in CTT seems to correspond to when we look at it as a theory of beliefs.

5.1.1 Type declarations

A type declaration is a declaration of the form $A : type$. It, as mentioned earlier, corresponds to comprehension of an agent. It corresponds to understanding a concept for an agent. It is a presupposition to any judgement, hypothetical or categorical, that uses the expression A. There are no direct correspondence between a type declaration and any concept in the classical AGM-theory, since it is a particularity of CTT. The closest we get is a violation of the syntactical rules. An example of a type declaration as information to an agent may be that an agent learns a new concept, or understands a piece of information.

5.1.2 Hypotheses

Hypotheses are judgements of the form, $x : A$. It is a judgement where the proof object is not explicitly known, but it is assumed that A is the case. We presuppose in this text that this form of judgement is the one that corresponds the best to belief in the classical sense. In a more fundamental sense, we see this kind of judgement as something that is provable, but, as it has been argued by several articles, it is established a strong connection between hypotheses and beliefs or information. [3, p. 141] Any information or thing we believe that are not proven should in some sense be a hypothesis.

5.1.3 Categorical judgements

Categorical judgements are judgements of the form, $a : A$. They claim that a certain A is proven and a is its proof. The proof is therefore explicitly known. These claims are closely related to knowledge in the classical sense [3, p. 140]. Categorical judgements are judgements that are proven in the system, so they cannot be wrong without violating what we can call syntactical rules. However, they may depend on some other judgements, that can be wrong. Another aspect is that all categorical judgements (and hypotheses) are dependent on the type declaration of their expressions. A type declaration may also be shown to be wrong [3, p. 179]. In this sense even categorical judgements are fallible, but only when what they depend on is wrong. A categorical judgement has to remain the case as long as what it depends on is the case. This seems to correspond very well to what we normally call knowledge in the sense of belief revision theory.

5.1.4 Hypothetical judgements

Hypothetical judgements are not the same kind of judgements than the previous three. A hypothetical judgement is a judgement where the first part is dependent on the second part. It is of the form, $x(y) : A(y : B)$. Both the first part, $x(y) : A$ and the second part $y : B$ can be any of the three foregoing judgements. In belief revision theory this is closely related to information or knowledge that is based on some other information or knowledge. An example is that an agent gets some sort of information under the assumption that the information is meaningful to the agent. We may call this conditional belief or knowledge.

5.2 Internal structure of contexts

Because of the strong relation between contexts and belief representation I will explain some important aspects of contexts in order to better explain beliefs. The first element in a context is in itself another context. [1, p. 11] So in some sense the first element of a context is a context for the second. This means that a context with two elements $(x : A, y(x) : B)$ is equivalent to the hypothetical judgement $y(x) : B(x : A)$. A context that contains only one element is then in some sense a context for the empty context. This empty context is similar to the one we find for categorical judgements. A categorical judgement is a judgement that depends on an empty context.

The main point here is to explain that contexts are ordered in a way that every context can be reduced to a context consisting of all but the first element, that is dependent on the first element. Every context can be reduced to an assumption for the empty context, and this kind of assumption is what it means to be a context. A context is just writing in a clear way a judgement that depends on a judgement that depends on a judgement and so on. The earlier judgements in a context may be more fundamental than the latter ones, because all the latter ones may depend on the earlier ones, while the earlier cannot depend on the latter. It does not make any sense to do anything with an element of a context without taking in count what it is dependent on.

5.3 What is a belief set in CTT really?

It is not clear how a belief set should be represented in CTT. The alternatives are either to represent them as a context or to represent them as some kind of list or collection of judgements. Ranta [7, p. 153] argues that what corresponds to a belief state in CTT is a context. An agent has one context that contains everything he believes, and when he expands this context the later beliefs may depend on the earlier stated ones. In this system knowledge is what is derived from the context.

The other alternative is quite similar, but it does not require the agent to collect all his beliefs in one context. This alternative has been argued for by Primiero [11, p. 6-7] [3] and in this view (that I will explain more thoroughly) a belief set, knowledge state, is what can be derived from several belief states, informational states. An informational state is a context, and a knowledge state is derived from an informational state. The combination of informational states and knowledge states gives a knowledge frame. We see that it seems easier to explain independent beliefs as different informational states. A potential problem here would be exactly how the different informational states work together. I will focus on Primiero's [3] variant of belief representation in this text and I will use his terms to explain research agendas.

This project does not depend on any specific definition of belief set in CTT. It will likely be available for both variants, as long as the variant has a way of extracting a belief out from the belief set. What it depends on is a notion of expansion and a notion of contraction. Expansion has been very well explained in the book by Primiero [3]. Contraction is a slightly more controversial topic, but it is being developed at the time of writing. Even though the project depends on a notion of contraction, it is not dependent on any specific notion of contraction. It seems to be available to the several natural representations of contraction.

5.4 Organising information and knowledge

As presented briefly earlier, Primiero [3, p. 148-163] argues for collecting all our information in a knowledge frame that consists of informational states and knowledge states. They are not closed under logical consequence in the same way as a belief set in the AGM-theory may be.

5.4.1 Informational states

An informational state is a collection of type declarations, all meaningful concepts for the agent, and hypotheses, collected in contexts. An informational state contains judgements that can be used to derive categorical judgements, collected in knowledge states. An informational state contains an agents implicit knowledge, understood as the knowledge that the agent is not specifically interested in, but that he/she needs or uses in order to acquire knowledge. [3, p. 148] It may be seen as the unprocessed data that we get, on which we have not yet produced a result.

5.4.2 Knowledge states

A knowledge state is a collection of the derived judgements that are derived from the informational state. These judgements can be considered the knowledge that the agent has. It is categorical in the sense that it has a proof based on the assumptions in the informational state. [3, p. 149] A categorical judgement will always depend on some other judgement, as we will always need a type declaration, but if a categorical judgement depends only on type declarations, it can be considered a theorem of the system, meaning a logical truth. Categorical judgements depend on the assumptions that has to be made in order to construct a proof for the judgement. It may be understood as the results that we get by processing the data found in the informational state.

5.4.3 Knowledge frame

A knowledge frame is a collection of several knowledge states that are based on several informational states. [3, p. 149] It is maybe what corresponds best with a belief set in a classical sense. It contains every belief and every piece of knowledge and its constructive proof. Intuitively this can be seen as the result together with its explanation, or as a conclusion presented together with its evidence.

5.4.4 Relation between informational states, knowledge states and knowledge frames

A knowledge state ($k - state$) contains categorical judgements that are derived from an informational state ($i - state$). [3, p. 149]

$$\begin{array}{c} < i - state_1 > \\ \downarrow \\ k - state_1 \end{array}$$

Multiple knowledge states may be derived from multiple informational states.

$$\begin{array}{ccc} < i - state_1 > & \ldots & < i - state_n > \\ \downarrow & \downarrow & \downarrow \\ k - state_1 & \ldots & k - state_n \end{array}$$

Multiple knowledge states are collected in one single knowledge frame ($k - frame$).

$$\begin{array}{ccc} k - state_1 & \ldots & k - state_n \\ \searrow & \downarrow & \swarrow \\ & k - frame & \end{array}$$

If an informational state is internally inconsistent, the inconsistency is potentially derivable in the knowledge state, and an inconsistency between two or more knowledge states, seems to be derivable in separate knowledge state.

5.5 Theory of revision

The next important part for a theory of belief revision is the revision part. It means that it should be possible to handle changes of information. Traditionally this corresponds to two different operations, namely expansion and contraction. Expansion is the operation that adds a new belief to our belief set and contraction is the operation that removes a belief from our belief set. The process of dynamic CTT was proposed by Ranta [6] by introducing modal logic. Modal logic made it possible to introduce new judgements and it is introduced as a part of the context. Like explained by Ranta, for every new context Δ, that was introduced from another context Γ, it is the case that all true judgements in Γ are true in Δ. This means that it is not possible to remove a judgement from Γ to Δ.

The theory of belief revision is not yet completely finished. It has not yet been made into a general framework to interpret all concepts from the AGM-theory in CTT. This is however a very recent field of research, and it will be explored more and more thoroughly. In this text we do not need the theory to be finished in order to represent research agendas. We shortly present the proposal from Primiero [11], where he explains the most important aspects of a belief revision theory. The operations that will be introduced are introduction of a concept, declaring a new type, addition of a hypothesis, addition of a definition, rejection, and contraction. The operations are done on the informational state, but since the knowledge state and the knowledge frame are both based on the informational state, they also affect these. There are no operations for adding categorical judgements, as these judgements are made out of hypotheses and their proof object is a proof in the system. It does not seem to make any sense to introduce categorical judgements as a separate operation in the same way as the other two judgements. You can achieve a similar thing by an addition of a hypothesis followed by an addition of a definition of the object in the hypothesis.

5.5.1 Introduction of a concept

This is an operation that we do not normally find in belief revision theories. It consists of declaring a new type in a context. It means to interpret a context Γ into another context Δ by at least one new type declaration. We can see that it is somehow an operation that may allow new hypotheses to be added to the context. [11, p. 9] It is not found in most other theories simply because the type declaration itself is a particularity of CTT. It has very strong similarities with expansion.

A context Γ may be extended by introduction of a concept: [3, p. 155]

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1}))$$

to another context Δ

$$\Delta = (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1}), < A_{n+1} : \mathit{type} >)$$

5.5.2 Addition of a hypothesis

The most obvious operation for a theory of belief revision is the expansion operation. It simply consists of adding a new belief to our belief set. More precisely it means to interpret a context Γ into another context Δ by introducing at least one new hypothesis. Δ therefore contains every judgement that is contained in Γ plus at least one new one. The objects of the recently introduced hypothesis needs to already be declared types in the previous context Γ. [11, p. 9]

A context Γ may be extended by addition of a hypothesis: [3, p. 155]

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1}))$$

to another context Δ

$$\Delta = (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1}), x_{n+1} : A_{n+1}(x_1, \ldots, x_n))$$

5.5.3 Addition of a definition

The third case is also a particularity of CTT. It is the substitution of a proof object for a hypothesis with a categorical proof object. This operation corresponds to finding a proof for a certain belief. It means to interpret a context Γ into another context Δ by substituting a variable for a constant. [3, p. 156] It is the change, or rather the presupposition for the change, from belief to knowledge. It can be seen as the base in order to make a judgement, that is a part of the informational state, into a part of a knowledge state. The hypothetical proof object, variable, is replaced by a particular proof object. It stays a hypothetical judgement, as it will always depend on at least a type declaration.

A context Γ may be extended by addition of a hypothesis: [3, p. 156]

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1}))$$

to another context Δ (where $1 \leq k \leq n$)

$$\Delta = (\Gamma, x_k = a : A_k)$$

5.5.4 Modification of an assumption

We may distinguish a fourth case of informational update. This situation is explained by Ranta [6, p. 84-85]. It is a special case of the introduction of possible worlds as mentioned earlier. It means that a judgement or context is the case in some possible world. It is not a distinct operation, but a special case of the other operations.

Modification of an assumption should be understood as adding an assumption to a judgement that occurs in the context. This assumption may be a type declaration, hypothesis or a definition.

Assume that we have a context Γ_1. We call the context for Γ_1 for Γ_2.

$$\Gamma_1(\Gamma_2)$$

We perform an update on Γ_2 by an introduction of a concept, an addition of a hypothesis or an addition of a definition. Here we use addition of a hypothesis as an example $x : A$ that gives us Δ_2.

$$\Delta_2 = (\Gamma_2, x : A)$$

We then update Γ_1 with the new context Δ_2 that gives us Δ_1

$$\Delta_1 = (\Gamma_1(\Delta_2))$$

We end up with an updated context Δ_1 that can also be written in the normal way.

$$\Delta_1 = (\Delta_2, \Gamma_1)$$

What we have ended up with is a modification of a context that occur inside the original context. This can be done on any judgement that occurs inside the context. Γ_1 may be a context that occurs as a context for some other context and so on.

A judgement in a context may therefore be modified by such a procedure described here. A context Γ may be extended by addition of a hypothesis:

$$\Gamma = (x_1 : A_1, \ldots, x_n : A_n(x_1, \ldots, x_{n-1}))$$

to another context Δ

$$\Delta = (x : A, x_1 : A_1(x), \ldots x_n : A_n(x, x_1, \ldots, x_{n-1}))$$

This is not really an operation in the same sense as the other operations, as it only describes how we can use the other operations to add assumptions that will not occur at the end of the context. The reason that we mention this here is that it is a useful procedure for the inquiries that will be introduced.

5.5.5 Rejection

Rejection is a new operation that we do not find in classical belief revision theory. It is an operation of rejecting new information or derivation. There have been described three different motivations to reject a judgement.

According to Primiero [11, p. 10] this process is in some sense the inverted process of one of the other ones. It is actually to go backwards in the revision and reject previous changes. To do this he mentions three steps. If an inconsistency is found the first step is to check the derivations. This step also includes checking the addition of a definition. If all derivations are syntactically correct, the addition of hypotheses to the context should be checked. If the addition of hypotheses to the context is correct, the introduction of concepts should be checked. Primiero [11, p. 10]

Primiero [11, p. 16] makes a procedure for revising a context where he shows how we may go backwards to an earlier context in order to resolve an inconsistency. This procedure for contraction is meant to happen as a control for every change in the context. If there is an inconsistency, the revision is rejected and the context returns to its original state. This algorithm seems to prefer old judgements over new judgements and if an inconsistency occurs, it is automatically the new belief that is removed. In some sense this algorithm seems to provide a good view on belief revision, if the most important is to keep the belief set free for inconsistencies. Primiero mentions this in the last chapter

[11, p. 17-18]. An important aspect of this operation is that it considers old information more reliable than new information. If we try to update a state with new information and it ends up being inconsistent we cannot choose to remove the data we had from before, it only explains how to refuse the new data. In order to explain the refutation of some older piece of data we have to use a concept of contraction.

5.5.6 Contraction

The last operation is the most controversial and difficult one. It involves removing a belief from the belief set. In CTT it corresponds to removing a judgement from the context, categorical or hypothesis, in order to resolve a problem in the context. This is also the operation that lacks the most research in order to make a good theory of belief revision. Contraction is a result of some kind of error related to expansion or update. Primiero [11, p. 10] argues that contraction should be motivated by falling into inconsistency. The procedure of contraction is however slightly more intricate in CTT than it is in the AGM-theory.

Every new context should contain all elements of the older context. This means that we cannot simply remove an element in a context. In order to solve this we may look at a contraction operation as a judgement that we can add to our belief state. If we adopt this interpretation of contraction, it is nothing more than special instance of expansion. This involves several aspects to belief revision.

The first aspect is that "proper" contraction, in the sense as we remove an element, cannot be performed unless we are in a situation explained by the rejection rules. For every change from one state to another state the second will be equal or larger then the previous, even for operations of for example pure contraction, contraction that is not followed by expansion.

The second aspect is that since every state can only be enlarged we can still keep the persistence explained by Ranta [6, p. 86], even when we do a contraction. This is because when enlarging a state with a contraction it will still contain all judgements that were found in the previous state.

The third aspect is that it will keep a "log" of the operations performed on a state. If we perform a contraction on a judgement in a state we will keep the judgement that it was performed a contraction in all later states. This means that we can trace back operations that have previously been done without going out of the state. We can look at it as an agent that know not only what state she has now, but what moves she has done in order to get to this specific state. This seems to be a useful point as we may be very close to represent something that corresponds to learning from our previous mistakes or experiences.

This view of contraction also has some potentially negative aspects. We seem to end up with significantly larger states, as we cannot remove any judgements. Another question is whether it actually corresponds to what we call contraction. We do not remove a judgement properly, we simply add a new belief saying that it should no longer be considered. A third negative aspect is that we need a method to determine for each judgement that there are no corresponding contraction judgement before it is used. I am not sure that this is a logical problem, but when applied it would need more resources than a theory where contraction simply removes a judgement.

We can see that this view of contraction is actually quite close to how agents think. An agent knows her state, but she also knows how she got there. If we have an analogy to a chess game, a player knows the positions of all the pieces on the board in a certain state and can reason on potential moves to do. In addition she knows what moves that were done both by her and the opponent before that certain state, so she may choose a strategy that is not only based on the certain state of the board right now, but also on how the opponent played earlier in the game. This is an aspect that is very easily lost if we allow proper contraction on a state. In order to make a strategy based on previous moves we would often need to make operations over changes of states. But when including contraction as a judgement in the state, we may keep this strategy level in the belief state itself and can allow such reasoning for an agent in a rather nice way.

This way of looking at contraction as a judgement seems to correspond quite well to a constructive framework. When we perform a contraction of a judgement the judgement is not forgotten, it is simply not believed to be the case anymore. The information of how the belief state ended up being is contained in the belief state. It is therefore a working well together with constructive framework, as justification for the belief state is never lost.

Contraction operations in CTT seem to be very important to explain thoroughly, as it is a very important operation in a belief revision theory. This will hopefully be done in the near future.

5.6 Possible worlds

Possible worlds may be used to describe the relation between informational states and knowledge states. In constructive type theory, a possible world is a context. It is potential approximations from a certain context or judgement. [3, p. 150] To say that a certain judgement holds in a certain world, is in some sense to expand the judgement. In other words it is to get a more and more precise description of what is described. As we can see from this, enlarging a world may be seen as extending the belief set. There are two ways of enlarging a world. The first one is informational updating, operations that are manipulating contexts, and knowledge extension, derivations from some context. [3, p. 152]

5.6.1 Contexts as possible worlds

Possible worlds in a belief revision interpretation of CTT are not exactly the same as they are considered classically. In this interpretation they are rather considered as different stages of a knowledge process. [3, p. 152] That a possible world is accessible from another possible world should be understood as that the first possible world is a potential knowledge extension when an agent possess the knowledge from the second. Another important aspect with this is that knowledge in a possible world is kept in all the possible worlds that are accessible from it. This means that knowledge cannot be removed, at least unless there is shown to be made an error in the extension. Information, however, does not have the same status as knowledge, so it seems like removal may be performed, in a way explained by the contraction or rejection rules. The knowledge explicitly states what it depends on, and if we remove the judgements that the knowledge depends on, we are not committed to the derived piece of knowledge anymore. [3, p. 153]

5.6.2 Informational updating

The first kind of extension is what may be called informational updating. It means to do one of the operations described earlier on a context contained in an informational state. This context may strictly speaking be empty. This operation is what corresponds to the classical revision operators in the AGM-theory. It does add something new to the state, something that could not be deduced. [3, p. 154-157] The operations that are falling under this category are Addition of a concept, Addition of a hypothesis, Addition of a definition, Modification of an assumption and probably also the process of contraction and revision. Contraction and revision are operations that are not yet fully developed, and it is not sure exactly how they will work. What is likely, is that they will have to operate on the informational state, and not the knowledge state, and are therefore likely to be placed under this category.

5.6.3 Knowledge extension

The second kind of extension is not found in traditional belief revision theory. Often the belief set (or possibly a knowledge set) are closed under logical consequence. This would be quite controversial to do in a constructive framework. In CTT, a proposition is not known, in some sense neither true nor false, before it has been constructed a proof of it. We can therefore not look at the knowledge state simply as the logical consequence of the informational state. It would be a too wide definition, as we may have not yet managed to construct a proof of all logical consequences of a state. Constructively, it simply does not seem to be an acceptable position. We therefore limit the knowledge state to what we have managed to construct a proof for, our actual knowledge. The operation of knowledge extension, is therefore a derivation on a state that gives us a new state. It is the operation we do when we go from an informational state to a knowledge state. It gives us more knowledge than we already had in the earlier state, but everything we get is a logical consequence of what we already knew. It may be called analytic extension while the informational updating in some sense is a synthetic one. [3, p. 157]

5.7 Specifying worlds

The operation of extending a context corresponds to the one of specifying a situation. Each informational state or knowledge state needs to be finite. Each state needs to have an enumerable amount of elements. This is because constructivity does not allow infinity, but only finite approximations of it. [3, p. 158]

5.7.1 Function from one state to next

A change from one state to another state is a function. Every state is accessible from the previous, in the modal sense. For informational states, Primiero represents the change like this. [3, p. 160]

$$I = \{i - state_0 \xrightarrow{f_0} i - state_1 \xrightarrow{f_1} \dots \xrightarrow{f_k} i - state_{k+1}\}$$

I is then to be considered as all information and how it has been changed.
We can see a similar representation for knowledge states.

$$K = \{k - state_0 \xrightarrow{f_0^*} k - state_1 \xrightarrow{f_1^*} \dots \xrightarrow{f_k^*} k - state_{k+1}\}$$

K is to be considered as all knowledge and how it has been changed.

As mentioned, a state cannot be infinite, but the process of changing from one state to another may continue infinitely. The infinity mentioned here is not an actual infinity, but it should be considered a potential infinity. This means that we can continue as long as we may want, but at no point has we ever reached infinity, and the model will stay finite at every step. [3, p. 160-161]

5.7.2 Producing a knowledge frame

An important aspect of this theory is to explain what a knowledge frame consists of, how it is made and eventually changed. A knowledge frame may only be produced on a finite number of knowledge and informational states. A knowledge frame is an agent's knowledge and belief at a certain stage.

A knowledge frame is in some sense the totality of the knowledge of an agent. It is not infinite, but it is a finite approximation of a potentially infinite amount of knowledge or data. A knowledge frame contains all knowledge states, where the knowledge states are derived from informational states. The amount of knowledge states that are contained in a knowledge frame is always finite. Each knowledge frame has a relation to its successor, by a similar relation as we find in informational updating or knowledge extension. Every change from one knowledge frame to the next one consists of either an informational updating or a knowledge extension. There is therefore a function from the first to the second. [3, p. 160-161]

5.7.3 Infinite specification

The notion of infinity is very important. In this framework a state or a frame may not contain an infinite amount of judgements. As a constructive approach, it will always only be a finite approximation of a potentially infinite amount of judgements. A change from one state to another state is an ongoing process that may continue infinitely long, but each state has to be finite. This notion of infinite specification does not seem to be in conflict with the constructive approach, as long as we never try to define the "last state". Earlier states will be gradually more precise approximations of this "last state", but will always stay approximations. Such a notion of infinity may be represented as choice sequences, where there is a law that makes us able to choose what direction to go, and this law may continue infinitely. [3, p. 161] This captures the notion of infinity constructively, as we never operate on any infinite set or infinite amount, but only explains how one may continue to add new elements.

Part II

Presentation of research agendas

6 Why do we need research agendas?

A research agenda is a set of questions that are derived from a dataset and that should be answered in order to solve a problem in this dataset. This is a very simple explanation of what research agendas are. The view that this text would take on research agendas in general is the one presented in Olsson's article "On the role of the research agenda in epistemic change" [2]. Here is a theory of research agendas presented and partly implemented in the AGM theory for belief revision.

In this article he presents three different ways were we seem to need some information outside of what he calls the Quinean dogma. By the Quinean dogma he means that belief revision from one state to the other is a function from the belief set and the entrenchment order on this belief set of the first state to the second state. [2, p. 166-167] Olsson claims that these two kinds of information, belief set and entrenchment order, is not sufficient to explain all kinds of belief change. He lists three different situations were he claims the theory is not sufficient and he claims that implementing research agendas may, at least to some degree, solve these problems. The three kinds of change that he mentions are: [2, p. 167]

- Ending up in an inconsistent belief set
- Stop to believe something, without our belief set being inconsistent
- Accepting a belief, without any explicit justification for this belief, in order to keep another belief in our belief set

All of these three kinds of change need explanation in order to make any sense for the reader. This list is just an overview over the problems Olsson claims may occur.

6.1 Ending up with an inconsistent belief set

To expand our belief set with something that is inconsistent with our original belief set is not very hard to imagine. Assume that for some reason you think that you are in the police and you try to find out were a certain person is. You tracked his passport through international police and because of that you believe that he entered Rome in Italy at a certain time t. Afterwards you then get information about his Visa card, that you also tracked, that says that he used it to pay for gas in a gas station in New York ten minutes after t. You also know that it is not possible for someone to move from Rome to New York in ten minutes. In this case you have ended up with an inconsistent belief set, as you believe at the same time that this person is in Rome and in New York at time t.

In this situation something must be done, and it must be done now. In the "Quinean dogma", what should be done is to remove the less entrenched belief. In this situation it can be understood as choosing whether you trust Italian customs more that the Visa provider or opposite. What Olsson [2, p. 168] claims is that it seems irrational to fully believe this belief, based on what can be marginal differences in credibility between Italian customs and the Visa provider. Both of them are very credible, but maybe you believe that computer systems are infallible and therefore choose to believe that the person is in New York. In either case it does not seem optimal to fully believe something, that could be very important, based on such a small difference in credibility. You could very easy have believed the opposite. Maybe this person is accused of murder in New York at time t and proof of him being in Rome at t is giving him alibi.

Olsson argues that a more natural solution to this problem is to start an investigation of which of the beliefs is the correct one, or at least has the best evidence. This cannot, according to Olsson, be modelled in this Quinean system. As he proposes research agendas as a solution it will in this case mean to put the question whether this person entered Italy at time t or that he used his Visa card in New York ten minutes after time t on the research agenda. What is meant by research agenda will be explained later in this chapter.

6.2 Stop to believe something, without inconsistency

To stop believing something in the absence of inconsistency seems to be a very strange operation as we seem to loose information that we do not need to loose, and in general we want our belief set to be as extensive as possible. This can however be explained quite clearly with an example. The main idea is that our new belief is working as non-conclusive evidence against another belief in our belief set. [2, p. 168]

Say for example that you believe that Italian border controls are keeping track of who is entering and leaving the country to a precision of 99.5% for people that are registered to enter the country to that they actually did it. 0.3% are mistakenly registered because of computational (internal problems of the registration system) problems and 0.2% are mistakenly registered because of other reasons. Then you get reliable information saying that this person (in the last example) has previously been registered to enter Italy in the tracking system of Italian border controls, without him actually entering the country. Say in this example that there were several witnesses, that saw him in Bruxelles at that particular time t'. His colleagues spoke with him and he used his ID-card to enter his job. Your evidence is very good that he was in Bruxelles at time t'. It is however registered by Italian border control that he entered Italy at this time. It could of course be the case that this incident was a part of these 0.3% that are mistakenly registered, and from this point of view, you cannot say that you have an inconsistent belief set. You still however may think that this was a peculiar event, that it would happen to exactly this person that you are investigating. The "quinean" way of solving this problem would be to add the new belief that he was mistakenly registered in the systems and also keep the belief that the Italian border control system actually works. This last belief may be changed to be less entrenched that it was before (that depends on the theory of entrenchment), but you would still keep both beliefs in your belief set.

This Olsson claims to be not an optimal solution [2, p. 168]. When discovering something like that, that is discrediting another belief in your belief set, you should do further investigations to find out whether this person actually was one of these 0.3% that were mistakenly registered because of computational problems, or that there is another reason that he was mistakenly registered, for example that he somehow manipulated the system. It would not be reasonable to remove either of the beliefs completely out of the belief set, as you do not know which one is the right one, particularly if the precision level was lower. A research agenda seems however to catch this unsureness that you end up with in this situation.

6.3 Accepting an auxiliary hypothesis to keep another belief in your belief set

To accept a belief to keep another belief also seems like a slightly strange operation to do. It does only make sense if the belief we want to keep is somehow an important belief. This can often be understood as being well-entrenched. The idea is that you got a well-entrenched belief B and you get information B' that is not fitting with B in itself, but it does if a third belief B'' is added to your belief set. You do not have any explicit justification to believe B'', but this belief may save both B and B' in your belief set. B'' may be called an auxiliary hypothesis. The operation is similar to an operation sometimes called abduction, inference to the best explanation. This is not a very legitimate reasoning alone, but if it is followed up by a commitment to find explicit evidence for B'' it seems to be more reasonable. This process of commitment to a search is something that can be shown with research agendas. [2, p. 168-169]

If we return to our example with the suspect, a similar situation could be to introduce an auxiliary hypothesis that someone stole his Visa card earlier and used it at this gas station in New York. This belief is not something that you have any explicit evidence for, but it would be an explanation to the problem earlier. This should not qualify to put the hypothesis directly in your belief set, but if it is followed up by an investigation to find evidence for this specific hypothesis, it may in some cases be an acceptable approach. To find such a hypothesis like this, does not seem to be something that can be decided by our formal model, it seems to relate to the agents experience or independent knowledge of the area explored. This is because it seems to often occur a number of such plausible hypotheses. Another hypothesis in this example could be that his wife was in New York at time t and that he gave his Visa card to her. We can likely construct a very high number of different hypotheses like this, but both the construction of them and to choose witch ones to follow up on, assuming to some extent limited resources, seems to be highly determined of non-formal properties of the agent. This does not however prevent us to keep it outside the formal model of the belief set. If an agent has chosen such a hypothesis the process of searching for evidence for this hypothesis may be shown in the formal model.

This operation seems very similar to the first example, as both of them seems to be based on some kind of inconsistency, but a distinction is what kind of research agenda it is producing. In the first example it is simply asking which one of the inconsistent elements is the actual case, but this operation is different, as it produces a belief that is not one of the inconsistent elements and the process is then to find out whether this new belief is true or not. It is also an operation where the agent seem to have direct influence on the formal model, as we do not have a logical reason to add exactly this auxiliary belief.

7 A model theoretical explanation of belief change

7.1 What is a question?

A question can be seen as a set of potential answers according to Olsson [2, p. 169]. These potential answers are sentences in the language that is used where one and exactly one of them is true. This can be understood as there are as many ways of specifying a belief set (according to a certain question) as there are alternative answers to the question. Another aspect is that none of the potential answers should not be excluded logically based on the belief set. They have to be real possibilities for this belief set. [2, p. 170]

The kind of questions that is explained here is not any kind of questions. They are for example not questions asking "why?", "how?" or similar questions. It may be yes-or-no questions, where we have two mutually exclusive potential answers or they can be of a more expanded form with several potential answers. These questions may be reduced to questions with two potential answers, yes-or-no questions, like it can be done with traditional logical operators with two arguments.

7.2 Representation of an epistemic state

The idea to represent research agendas in model theory is to that a model of an epistemic state S is not a pair, $S = \langle K, E \rangle$, consisting of a belief set, K, and an entrenchment ordering, E, on the elements of K, but a triple, $S = \langle K, E, A \rangle$, where the third part is an agenda, A, where all the questions on the research agenda are represented. This A is based on the belief set, K, and may be called a K-agenda. [2, 169-170]

A question, $Q = \{\alpha_1, \ldots, \alpha_n\}$ is a K-question when the belief set K entails the exclusive disjunction of the elements of Q, $\alpha_1 \vee \ldots \vee \alpha_n$, where α is a formula in the system, and that there are no proper subset Q' of Q, $Q' \subset Q$, such that K entails Q'. [2, 170] This means to ensure that all potential answers are considered, and that all potential answers are also actual possibilities. Here $\vee$ means exclusive disjunction. When $\alpha \in K$, $\{\alpha\}$ is a K-question. This means that each element of K are K-questions with only one element. Q_K is the set of all K-questions. [2, p. 170]

From this we may represent research agendas, A, as relative to K is a subset of all Q_K, $A \subseteq Q_K$ and whenever a question is an element of A, it is what we may see as on the research agenda. [2, p. 170]

7.3 Expansion

In model theory, belief revision is very often represented by the AGM-theory. This theory claims that belief revision consists of two (or three) basic operations and that from one belief state to another belief state we do one of these basic operations on the first epistemic state and end up with the second epistemic state. The axioms for the AGM-theory will not be listed and explained here, as they do not seem to be necessary for this paper. However, I will explain how two of these operations, expansion and contraction may be understood in the terms of research agendas, as proposed by Olsson [2, 171]. Sometimes a third operation is distinguished as a basic operations in the AGM-theory, revision, but it is often assumed that this operation really is just a combination of the other two, and therefore may be reduced to these two operations. I will not focus on this third operations here, because of that point.

A change of epistemic state, $\langle K, E, A\rangle$, to another epistemic state, $\langle K', E', A'\rangle$, may be seen as a function consisting of one of the two operations described in the AGM-theory, expansion or contraction.

Expansion is the operation of adding a belief to a belief set. This operation is in many ways a very simple operation. We simply take the new belief and add it to our already existing set of beliefs. An axiom for the syntax is that an epistemic state S that is expanded with a belief α, $S + \alpha$, is an epistemic state. $+$ is a symbol used to denote expansion. The belief set, K, is update to another belief state, K', where α is an element. The entrenchment ordering, E, will also be updated to another entrenchment ordering, E', but exactly how that is supposed to happen depends on the theory of entrenchment that we want to use. The most relevant to explain here is how the research agenda should react to a change from one agenda, A, to another updated agenda, A'. The most important and intuitive aspect of this operation is that all questions in the research agenda where α is a potential answer or where the answers may be derived by the updated belief set, $K \cup \{\alpha\}$, should be removed. [2, 171-172]

Olsson proposes a definition of expansion of research agendas like this: [2, p.172]

$$S + \alpha = \langle Cn(K \cup \{\alpha\}), E', A'\rangle,$$
$$\text{Where } A' = \{M | M = Q/_K\alpha \text{ for some } Q \in A\}$$

Where $Q/_K\alpha$ is the operation called K-truncation of Q by α. It means for Q, we remove all potential answers, incompatible with our new belief set, $K \cup \{\alpha\}$, and it is defined like this: [2, p. 172]

$$Q/_K\alpha = \{\beta \in Q | K \cup \{\alpha\} \text{ does not entail } \neg\beta\}$$

With these definitions we end up with a state that removes all answers to questions that are inconsistent with our new belief set, and we are sure that all potential answers are relevant for the belief set, that none of them have a potential answer that may be derived from our belief set and containing more than one element. A question is called settled when it has only one potential answer, only one element, $\{\alpha\}$. [2, p172]

7.4 Contraction

The operation of contraction has traditionally been getting significantly more attention than the operation of expansion. It is a more complicate matter than the one for expansion. What contraction is doing, denoted by the symbol $\div$, is to remove a belief from an epistemic state. The reason that this is a lot more complicate, is because it is usually not something that is happening alone. Normally it is happening before, together with or after an expansion in order to make room for or fix an epistemic state. Exactly how this operation works is debated and many different theories of contraction has been proposed. [12] For the research agendas, it seems to be possible to develop properties for the research agendas after contraction without being dependent on a specific theory of contraction. A theory of contraction is of course necessary in order to do belief revision, but it is not based on any specific theory, and seem to be compatible with several different ones. [2, p. 173]

The first and possibly obvious axiom for contraction is that an epistemic state, S that is contracted with a belief, α, $S \div \alpha$, is an epistemic state. Intuitively, the main property of contracting an epistemic state, S, with a belief, α, in relation to the research agenda, is that all settled questions where the belief was an element, will get other potential answers as elements. Another axiom that Olsson [2, p. 173] is giving for contraction is when a simple belief is removed from an epistemic state:

$$S \div \alpha = \langle K \div \alpha, E', A' \rangle,$$
$$\text{Where } Q \in A',$$
$$\text{for some } Q \in Q_{K \div \alpha} \text{ such that } \alpha \in Q$$

This is a definition where a belief α is simply removed from the epistemic state. The effect on the research agenda that is described here is that when a belief is contracted from an epistemic state, there should be a new question where α is a potential answer. What is happening to the belief set and the entrenchment ordering is not defined by this definition. Olsson [2, p. 174] also introduces another postulate of contraction that is describing a contraction of a belief α with respect to some alternative hypothesis β. [2, p 173]. This definition can maybe be seen as a more specific case of the first one, but it captures something that seems to be more precise and it is possibly describing a more intuitive understanding of cases where contraction should actually be used. [2, p. 174]

$$S \div_\beta \alpha = \langle K \div \alpha, E', A' \rangle,$$
$$\text{Where } Q \in A',$$
$$\text{for some } Q \in Q_{K \div \alpha} \text{ such that } \alpha, \beta \in Q$$

What it is saying is that we contract our epistemic state with a belief α because of some other belief β and by doing that we should have a question on our new research agenda where α and β are potential answers. [2, p.174] It can intuitively be understood as considering the belief β, but by doing that we need to remove α from our belief set. Instead of simply switching them, putting β in our belief set and removing α, we put both of them in a research agenda, to find out which one of them is the right one.

The contraction, as opposed to expansion, opens new questions. It may also remove questions, but not by closing them. If a question is dependent on a belief, and we contract this belief, the question is no longer relevant to our belief set. For example, if we have a question on our research agenda asking whether a person drove a Toyota or a BMW when he was filling gas in the gas station in New York (from the earlier example), under the assumption that he was in New York at time t, we will have to remove this question from our research agenda if we find out that he happened to be in Italy at time t, and not in New York at all. The question is not settled or answered, it is simply removed from the research agenda. It is not relevant anymore. If we would keep the question on our research agenda it would not be appropriate for our belief set anymore. [2, 174]

Part III

Representing research agendas in CTT

8 Explanation of what research agendas mean in CTT

A research agenda consists according to Olsson [2, p. 169] of questions that a researcher wants answered. This means that in order to give a representation of research agendas, we have to represent the more fundamental part, namely questions. The definition of questions is similar to the one given by Olsson, namely mutually exclusive potential answers. I will argue here that questions can be seen as exclusive disjunctions. I will also argue that we should include two other operations, namely inquiry and answer. These operations correspond to the normative part of a question, meaning that by posing an inquiry, we should want an answer and that it should be answered if possible. In this part I will explain what inquiries and answers mean in constructive type theory, explain the formal rules and I will explain how questions, inquiries, answers and research agendas may be used to solve some problems in the framework. At the end of this part there will be a discussion on whether this representation actually does what we said it would, namely correspond to research agendas as they are understood by Olsson.

Ranta [7, p. 137-143] mentions the possibility to have questions in a constructive type-theoretical approach. He argues that a question should be seen as an alternative kind of judgement, and therefore have an operator on the judgement level. Here I distinguish between inquiries and questions. Inquiries are based on this idea. They operate on a judgement level in a similar way as proposed by Ranta [7, p. 138]. In Ranta's proposal, we should look at a question $?A$ as a demand for whether we can derive $\vdash A$ or $\vdash \neg A$, and they will be the potential answers for the question. This is similar to the approach that I will present, but in this paper I will distinguish between the form of the question and the act of asking. This act is what I will call inquiry. It is an operator on judgements, in the same way as for Ranta, but it also makes it possible to demand judgements without two potential answers. I will also introduce a corresponding operator for answers that makes it explicit if a certain inquiry is answered. Ranta pays most attention to how to represent linguistic aspects in constructive type theory. Here I will focus on how introducing questions, inquiries and answers can give us any new perspective on handling belief and knowledge, but the operations in the theory and their linguistic counterpart are related. [7, p. 138-143]

8.1 Questions

For Olsson [2, p. 170], questions are sets of potential answers where exactly one answer is the right one. Classically every disjunction of a proposition and its negation is valid. In intuitionism neither a judgement A or its negation $\neg A$ is true before we can construct a proof for one of them. When we construct a proof for either it is true that $A \vee \neg A$, but this is not the case before such a proof is constructed [13].

For a question with several potential answers we can only guarantee that there exists a proof for one of them after we have found this proof. For some potential answers, where we do not know which one is true, we cannot guarantee that there exists a proof for either. We can therefore not guarantee that exactly one of the potential answers to a question in Olsson's sense is the case intuitionistically. This simply means that the truth of a question is not implied by the state, but occurs as a potentially true judgement.

This means that the notion of question needs to be adapted a bit in order to represent them in constructive type theory. The potential answers will still be mutually exclusive, if a proof for one potential answer is found it will entail for the negation of all the other alternatives. For Olsson a question as an exclusive disjunction should be on the research agenda if it was implied by the belief set. If such a question is implied by the state in intuitionism it would mean that we already had a proof for one of the disjuncts. This is not what we want. In constructive type theory, a question that is object of an inquiry should be potentially true and not actually true.

Olsson [2, p. 170] explains that questions can be seen as exclusive disjunctions. Questions can in this sense be represented as judgements in the same way as other judgements are represented. An exclusive disjunction is a judgement where one, but not both of the disjuncts is true. In CTT a disjunction implies the truth of one of its conjuncts. The representation of questions in CTT would be judgements that look a little bit like disjunctions, and a little bit like conjunctions. I will call this operator a question operator. A judgement, $A \veebar B$ should here be understood as that any proof is either a proof for A and $\neg B$ or a proof for B and $\neg A$. This captures the fact that only one of the potential answers may be the case. Together the propositions would be inconsistent, as a negation is a transformation of a proof from the proposition to falsum.

The next problem is when questions and inquiries should be made. Olsson [2, p. 174] mentions that we may add more questions to the research agenda, and that this does not cause any problems. There are also some cases where we should add a question. That is in the case of contraction. [2, p. 174] If we remove one of our beliefs we should introduce a new question where the judgement we removed should be one of the potential answers. This is connected with the notion of contraction that Olsson operates with.

We can see from here that a question in itself is not really something demanding any answer, any more than other hypothetical judgements. The process of representing a question in CTT will first be a representation of the structure of the question, as a connector, then an operator for inquiry will represent the normative demand that we find in classical questions. In this text I will distinguish the term question from the term inquiry, where a question simply is an exclusive disjunction and an inquiry is the normative "demand" for an answer.

8.2 Inquiries

We see that the representation of questions and research agendas depends on a conception of inquiry. A research agenda can be said to be a collection of inquiries. In this text I will use the notation $?E$ for an inquiry where E is object. This corresponds to some extent to what Olsson [2] argues for, but it does allow more as it is not restricted to exclusive disjunctions. In classical belief revision theory we do not have different kinds of judgements in the same sense as we do in CTT. In CTT we have the ability to express categorical judgements, hypotheses, type declarations and definitions. It seems like it is not only questions (as defined earlier) that could yield an inquiry. Hypotheses do not have a categorial object, their object is only a hypothetical one. If we only have a hypothetical object for a set or a proposition it seems reasonable as a general rule (possibly with exceptions) to search for an explicit object for this set or proposition. In this sense we may argue that there should be an inquiry, not only for questions, but for all hypotheses. I will distinguish three different inquiry operators, depending on what kind of judgement the inquiry demands. This means that we have explicit rules for each of those three operations. Every inquiry operator is introduced by a hypothetical judgement. We can make inquiries for judgements that are used as assumptions for other judgements.

We may want inquiries to include type declarations. Primiero [11, p. 16] argues that a missing type declaration should yield a rejection of the judgement. This means that we should simply not accept the new information, because it is meaningless. Introducing an inquiry for type declarations may cause us to not automatically reject the information, even though we miss a type declaration. We may have an inquiry that asks whether we may add the corresponding type declaration to our state. In this paper I will define inquiries for type declarations, assumptions for hypotheses and definitions.

We may argue that inquiries demand either an informational update or a knowledge extension. An inquiry that demands a knowledge extension is an inquiry where the answer may be derived from the current informational state. An inquiry that demands an informational update is an inquiry that may be answered by introducing new judgements in the informational state by an informational update. We may not close the knowledge state under logical consequence, as we do not have a proof for all logical consequences. In CTT we do not have a proof before it has actually been constructed and an inquiry that demands a knowledge extension is actually not distinguishable from an inquiry that demands an informational update. They are the same inquiry, only that the judgement that answers the inquiry may be given by a different operation. A collection of open inquiries, the research agenda, may be given as a function from a context.

This view on inquiry seems to open up for adding an auxiliary hypothesis to save some other theory [2, p. 167]. It is acceptable to add a hypothesis and we would search for a proof for this hypothesis as an inquiry precisely because it is a hypothesis. By introducing a question, we seem to cover falling into inconsistency with this way of looking at inquiries, as every question should be an inquiry on the research agenda. It is not equally clear how it may solve the problem of stopping to believe something without an inconsistency. It does add something new compared to Olsson's approach, as we may have inquiries on derivations, type declarations and addition of a definition. It seems therefore that the inquiry that we find in CTT is more general than the question that we find in classical belief revision theory.

8.3 Answers

An inquiry demands an answer. We will therefore also introduce an operator for answers. An answer states that another judgement is the answer to some inquiry. An answer is not the judgement that answers the inquiry itself. In this sense we could say that it connects the judgement to the inquiry. An answer can therefore be made whenever we have an inquiry and the judgement that the inquiry demands. As we have three kinds of inquiry, we will also have three kinds of answers, one kind of answer to each kind of inquiry.

Olsson [2, p. 173] argues that we should distinguish between judgements that have been made as an answer to some question and those who have not. The distinction will be seen here as a judgement that has a corresponding inquiry and answer and those that does not. If a judgement can be derived to an answer to some inquiry, it is an answer to a question in Olsson's sense. The disadvantage, that seems to also be a disadvantage with Olsson's theory, is that we cannot distinguish whether the judgement actually was a result of the inquiry or if it has been updated or derived somehow independently of the inquiry. We would in this sense not be able to distinguish between those judgements that just happened to answer an inquiry from those who came into question because of the inquiry. This would seem like a very natural distinction to develop further, but it is not further developed in this paper.

Another aspect of answers is that it corresponds to a certain inquiry. Say that we have two similar inquiries in a context. We may have answered one without having answered the other. An answer is related to a certain inquiry. If there are two similar inquiries and one has been answered, the answer could be used to derive an answer to the second inquiry as well, but it simply has not been done yet.

Olsson [2] does not specifically speak about answers to questions or inquiries. This is because his framework allows logical closure. In the constructive type-theoretical framework this is more problematic. That something is provable does not mean that we have such a proof. That something is true means that we have a proof of it. We do at a certain stage only have a finite number of judgements in a state. There are potentially an infinite number of propositions that are provable, a potential infinite number of stages. At every stage there are propositions that are provable, but not yet proved. This means that we have propositions that are potentially true, but are not actually true at our stage. Logical closure is that all potentially true propositions are actually true and this is therefore not acceptable in our framework.

This is the reason why we seem to need answers in a constructive framework, while it is not sure that it is necessary in a classical framework. We could not simply say that all inquiries should be closed if the answers are implied by the state, as this would need logical closure. An inquiry should be closed when and only when an answer is derived in the state. This causes the constructive approach to be more complex than the classical approach. Logical closure may be easily defined in logical terms, but it may require big resources to actually apply. An answer is therefore simply the judgement that closes a certain inquiry.

8.4 Research agenda as a strategy

A research agenda is a collection of inquiries that are not yet answered. For Olsson [2, p. 170] a research agenda for a belief set is simply the set of questions that are related to this belief set. In this sense a research agenda can be represented as a function from a context to a collection of inquiries. This function should yield a collection of all open inquiries that can be derived based on a collection of judgements. The problem for a research agenda would in some sense be to decide what inquiries that should be derived in the first place. There may be several reasons for why some inquiries should not be derived, they can be for example unanswerable or irrelevant for the goal of the agent.

If an investigation department in the police is going to use this system of belief revision to represent an investigation, they will have a goal, to find out who did the crime. They will have some restrictions, for example limited resources (investigators, money, time...) or juridical limitations. I think that these aspects may play a systematic role in the derivation of inquiries. That something is on the research agenda means that it is an unanswered, open, inquiry. The police may have a different research agenda than a thief, even though they may share the same beliefs. We have the ability to modify the research agendas according to what the theory is meant to be used to. In some sense one might argue that the research agenda is not a direct part of the belief revision theory in itself, but simply a way to extract information, relevant inquiries, from the theory. This seems to be correct, but derivation of inquiries still plays an important role as a part of the system, because it is what the theory gives as output.

I want to argue for some restrictions on the derivation of inquiries, for example to only derive an inquiry once. Intuitively we may claim that an optimal research agenda contains all derivable inquiries. This is why we will explain something we called maximal inquiry, as the maximal amount of inquiries that can be made on a finite collection of judgements. In order to prevent an infinite number of extensions, we will make some restrictions.

If we incorporate research agendas like this, we see that the belief revision theory, CTT with research agendas, is a function. It will be possible to understand this theory as a program. This fits well together with Martin-Löf's [14] idea, to use CTT as a programming language. This belief revision interpretation of CTT together with questions, inquiries, answers and a function for research agendas, could be seen as a program in itself. The process explained here is focused on how to extract relevant research agendas based on some information and derivation.

9 The meaning of inquiries and answers

9.1 Acts for assertions and inquiries

When I say something, what I am saying has some mood, meaning that there is a certain relation between me and what I am saying. If I say "The bus is gone", there are several ways to understand this. It can be understood as an assertion, I simply state the fact that the bus is gone. It can be understood as a question, that I ask someone whether the bus is gone. It can be understood as a wish, that I hope that the bus is gone, and probably in many other ways. How it should be understood depends on other things like the situation that it was said and who said it and so on. The point is that the speaker may have different moods or attitudes toward the proposition. Here this attitude is understood as the act. Normally in type theory a judgement is an assertion. This corresponds to what may be called an indicative mood. In linguistics it has been argued that it is not only the indicative mood that is important, but also the interrogative and imperative moods. [15] It seems like all these three kinds of mood, are important for us when we comunicate. This should give us a reason to make an interpretation in constructive type theory, in order to connect it to semantical aspects or meaning of natural language. In this paper I will explain how the interrogative mood may be interpreted as inquiries. We will introduce an answer operator that makes us able to explicitly answer such an inquiry.

In constructive type theory the most important is the act and not the proposition. There is an important difference between a judgement and a proposition. A judgement does not have any force unless the act of the judgement that the proposition occurs in is taken in count. Martin-Löf [1, p. 4-6] distinguishes four forms of judgement, but there is no reason that type theory should be limited to only these four. They are all assertions, but to introduce other kind of judgements, like inquiries, seems like a natural way of extending the language. [7, p. 2]

An important notion when we speak about acts is force. Frege used an operator that was supposed to correspond to the force of a judgement, that we are committed to what follows. This operator was the $\vdash$. It means that we are committed to its content. The proposition is true. [15]

Ranta [7, p. 137-143] speaks about the difference between assertions and questions. In constructive type theory, the level of act is left implicit. It is presupposed that a judgement is an assertion. An assertion is the case where we are committed to the content. For an inquiry we are not committed to the content in the same way. By introducing inquiries we introduce an operator on judgements, not on propositions. An assertion of A in this sense corresponds to $\vdash A$. This operator is usually left implicit, as all judgements normally are assertions. In order to introduce inquiries a ? operator will be introduced. This is an operator at the same level as we find the $\vdash$ operator.

Martin-Löf makes a distinction between the act and the object. I will call a judgement where we are committed to the content, a judgement in the normal sense, an evident judgement. This notion comes from Martin-Löf [16, p. 26-27]. The object in an evident judgement is the object of knowledge and the act is how the subject relates to the object, "knows that ...". [16, p. 25]. The difference between an inquiry and an evident judgement is the act. The object is the same. An inquiry asks whether the object is the case, without any direct commitment to the object. One may argue that there is a commitment to investigate whether the object really is the case, but that is anyway another kind of commitment than the one we find in evident judgements.

9.2 Inquiries or judgements?

A question that may arise by explaining inquiries, is whether an inquiry really is a judgement at all. A judgement is traditionally understood as a denial or an affirmation. It has been argued that the definition of judgement should be slightly changed in type theory, to not only include affirmation or denial of a proposition but also to include that something is a proposition. The term judgement is then used in a new way here in constructive type theory. [16, p. 19]

Can an inquiry be said to fall under this definition of judgement? Does it requires a new extension of the definition or is it simply something different from a judgement. A judgement is an expression of knowledge. The act of judging is similar to the act of knowing and what is judged is similar to the object of knowledge. [16, p. 19] A judgement carries an epistemic force in a similar way as when we use the expression "I know that ..." in natural language, meaning that we are committed to what follows. [16, p. 25]

An inquiry is not an expression of "I know that ..." in the same way that judgements are, as it does not carry the same commitment to what follows. If I have an inquiry about a certain judgement, I am not committed to the content of the judgement. That would make inquiries lose all their meaning. Instead of the "I know that ..."-commitment of the evident judgement, an inquiry carries a demanding commitment, "I ask whether ...?". In this sense it seems like inquiries cannot be judgements, at least not in the normal sense.

When we introduce the inquiry we put it in the formation and introduction rules as the conclusion of the rules. According to Martin-Löf [16, p. 13], premises and conclusions are always judgements. From this point of view inquiries should also be considered as judgements. This is a reason for why inquiries should end up being judgements, but it does not really explain how or why this is so.

Inquiries expresses a different act than the one we find in evident judgements, but as they are defined here, they behave very similar. An inquiry bears a similar commitment as we find in the assumption for a hypothetical judgement. It also has the same kind of commitment as we find in the antecedent of a conditional. This is because inquiries have the same introduction and elimination rules as the universal quantifier and the conditional. From this point of view, we already have judgements that does not bear any direct commitment, only that they appear as assumptions for other judgements. What inquiries do is that they make these assumptions explicit.

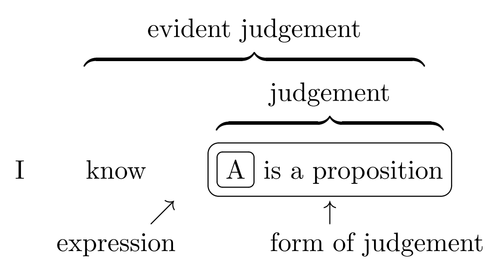
To return to the question of how to relate inquiries and judgements, an inquiry only expresses a commitment that we already find in the evident judgement. In a way the definition of judgement should be changed, as we introduce a new kind of act by introducing inquiries, asking or demanding. This change is not a very radical one, as it already contains the machinery to handle the behaviour of demands, at least in the sense that is introduced here. Because of the way we define inquiries, they behave similar as other judgements, by having rules for their formation, introduction and elimination, that are only based on other already introduced judgements.

In order to distinguish the difference of act in an evident judgement and the judgement we have when we have an inquiry, we can distinguish between an evident judgement and a demanding judgement. A demanding judgement is understood as a judgement where we do not have epistemic commitment to the content, the object. Both the evident judgement and the demanding judgement should be considered as judgements. They behave like judgements should behave, they may be assumptions and conclusions. Inquiries are captured by the definition of judgement if we change the definition slightly, but doing this should not be considered very controversial.

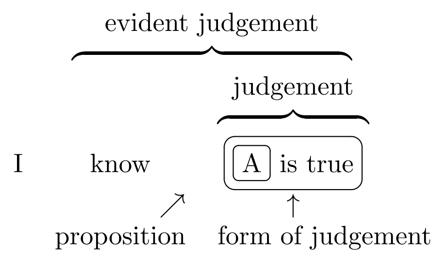
9.3 How can an inquiry be made evident to someone?

Inquiries are not propositions. One may not speak about an inquiry being true or false. We will introduce inquiries by formation, introduction and elimination rules. They are however not like the logical connectors, in the sense that they are not made to establish truth or falsity of complex expressions. How can the meaning of inquiries be grasped by someone? The distinction that is made here between the act of knowing, "I know ...", and the act of asking, "I ask whether ...", is the same distinction as mentioned by Ranta. [7, p. 26]

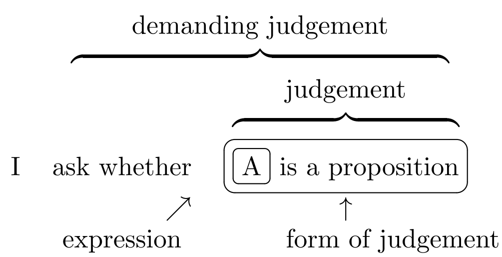
Martin-Löf [16, p. 26] explains the structure of a judgement of the form A is a proposition, $A : prop$, in this way:



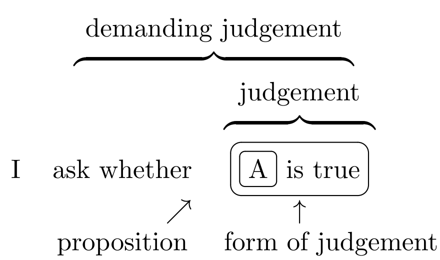
A is a complete expression. "... is a proposition" is the form of judgement. Together they make the judgement. The evident judgement is what we get by putting the judgement together with what we earlier called an act. This is what we normally refer to when we speak about judgements in constructive type theory. We include the act of the judgement. Similarly we have the structure of a judgement of the form A is true, A *true*, in this way [16, p. 27] :



Here the structure is the same, but A is no longer only an expression, but a proposition. This means that we must already have grasped what it means for A to be a proposition. We can make similar figures for inquiries that demands type declarations.



We can see that the structure is the same as for evident judgements. We still have the same structure for what is a judgement. The difference is the act. We find a similar structure for inquiries that demands the truth of a proposition.



This judgement presupposes that we have already grasped what it means that A is a proposition. As the inquiry originally is intended to be used in a belief-revisional interpretation of constructive type theory. The interpretations of the demanding judgements may be different than the ones presented here.

When we present the rules for the inquiries, we want them to capture not only whether A is a proposition, but also more generally what type A is. The demand whether A is a proposition can be seen as a special case of a more general inquiry. This also counts for the next example. We may not only ask whether a proposition is true, but whether a set has some element or if a set depends on some other set. These examples captures the relation between propositions and inquiries.

The exact formulation of the act in natural language may vary, just like we find many different ways to express evident judgements. The notion of inquiry is therefore not bound to the formulation "ask whether ...", but may have several interpretations.

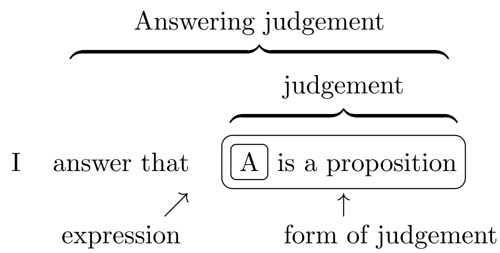
9.4 How can an answer be made evident to someone?

Answers are less different from assertions than inquiries. The important aspect of inquiries is that with an inquiry, we are not committed to the object of the act. With answers, we are committed to what follows. An answer is an evident judgement. The act of answers may be interpreted in a slightly different way than for other evident judgements, but the act carries the epistemic force that we find for the others.

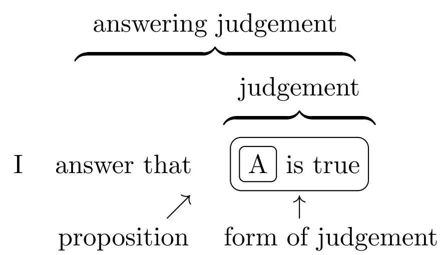
The rules for answers, state that they can be introduced when we have an inquiry and a judgement that is the object of the inquiry. It has similar rules as the rules for conjunction. If I state an answer, it means that there is an inquiry that the answer answers. If we do not wonder about anything, we cannot answer anything either. What it means to answer, is to answer an inquiry, or what we informally call a question. Answers are not answers independently of what was asked. Answers are related to what was asked. To ask is to state an inquiry. To answer is to answer an inquiry. An answer therefore implies the inquiry that it answers.

To answer an inquiry means to say that we possess a judgement that corresponds to the object of the inquiry. When we possess both the inquiry and this judgement we can state the answer. If we say that something is an answer to an inquiry, we have possess this something. When we have an answer, we also have the object of the answer. This means that the object of the answer can be derived from the answer itself. From this we can see that an answer has a similar relation to its object as we find in other evident judgements. We do, however, understand an answer in a slightly different way. This means that an answer is not an expression of knowledge, but an expression of answering. I will therefore call this an answering judgement, but keep in mind that this answering judgement is an evident judgement because of its epistemic commitment to its object.

We end up with the following structure for answering judgements where we have a type declaration as an object. The difference from the other evident judgements and demanding judgements is the act. The act is neither an act of knowledge, "I know that ...", nor an act of asking, "I ask whether ...", but an act of answering, "I answer that ..."



For answering judgements related to truth of a proposition we find a similar structure. Here the act is the same, but as with the inquiries, it presupposes that A is a proposition, and not only an expression.



Here it presupposes that we understand what it means for A to be a proposition. In the same way as mentioned with the inquiries, the judgements in these examples may be understood differently in a belief revision interpretation of constructive type theory. The formulation of the act in natural language, may differ and is not bound to the formulation "I answer that ...", but may be formulated in other ways.

To know something is to have a proof of it. To answer something is to answer an inquiry. It makes the connection between what we wonder about and what we know. It makes us stop demanding by stating that we have an answer.

9.5 Inquiries of three forms

We will end up with three kinds of inquiries and three kinds of answers. The three inquiries demand different things and are therefore made in different situations.

The first inquiry demands a type declaration. This seems to correspond to something we informally can call incomprehensibility. An example of such a situation can be seen as someone saying something we do not understand, and by understand it is meant using some term or name that are unfamiliar to us. If a scientist gets some unexpected data from a certain machine, he may not know if it is a faulty machine or if it is evidence for some undiscovered phenomena. This corresponds to the first kind of inquiry. In order for the researcher to do anything with the new data, he has to understand what kind of data he actually got. The type declaration inquiry corresponds to investigate what kind of data we have.

The second kind of inquiry demands whether a judgement may have some other assumptions. It demands whether we can reduce the judgement into smaller parts. This inquiry therefore demands whether it is possible to find a some other assumption. It presupposes that the expression in the judgement is a set. Suppose that I believe that a certain person robbed a bank under the assumption that he was in Bruxelles at the day of the robbery. I can try to reduce this judgement into more fundamental parts by demanding more assumptions. An inquiry of this kind simply demands whether our assumptions are based on some other assumptions. It seems natural if we consider the theory as an ongoing and never-ending process. This is the view we find in Ranta [7, p. 93]. This makes us able to explain an investigation of more and more specified information.

The third kind of inquiry demands the substitution of an object. It presupposes that the expression in the judgement is a set. This corresponds to a situation where we want to prove some proposition. It is the transformation from belief to knowledge. Say that the police believes that a certain person robbed the bank. This inquiry corresponds to investigate whether the hypothesis is correct. It presupposes that the hypothesis is meaningful for the police. In order for the person to be convicted the hypothesis should be ”proved beyond reasonable doubt”. It is not enough that the police believes that it was this person. They would need some proof of it. A constructive proof is of course not the same as a juridical proof, but there are similarities and seems adaptable to these situations.

10 Logic of research agendas

10.1 $\veebar$ operator for questions

The $\veebar$ operator can be said to be an exclusive disjunction. We will introduce this operator in order to represent the notion of question that we find in Olsson's paper. The exclusive disjunction that we find here is not the same as an exclusive disjunction in classical logic. A reason for this is the intuitionistic negation, as it is defined as a contradiction. This means that this rule requires the propositions to be actually inconsistent with each other, and not like with classical exclusive disjunction where the first proposition only requires the second proposition to be false and opposite. It should be considered like any other normal operator in this system. It does not have any demanding force like questions in the classical theory.

The formation and truth conditions can be given by following Tarski-style:

$$A \veebar B : prop \text{ if } A : prop \text{ and } B : prop$$

$$A \veebar B \text{ is true if}$$
$$A \text{ is true and } \bot \text{ is true provided that } B \text{ is true or}$$
$$B \text{ is true and } \bot \text{ is true provided that } A \text{ is true}$$

By these definitions we can introduce the operators formation, introduction and elimination rules. This operator is a specific case of the disjunction. This means that we can define in terms of the + operator, disjoint union or coproduce of two sets.

$$A \veebar B \equiv A + B : \mathit{prop} \text{ when } A : \mathit{prop} \text{ and } B : \mathit{prop}$$

We should be aware that each expression, A and B, acts as the conjunction of itself together with the negation of the other. A way to understand this operator would be as a disjunction of two conjunctions. It would seem like it is simply a more specific case of the disjunction.

$$(A \wedge \neg B) \vee (B \wedge \neg A) \ \mathit{true}$$

10.1.1 Formation rule

The formation of questions is the same as the formation for the other logical binary operators. This is because it is a propositional operator in the same way as other ones. Whenever A is a proposition and B is a proposition, $A \veebar B$ is a proposition.

$$\frac{A : \mathit{prop} \quad B : \mathit{prop}}{A \veebar B : \mathit{prop}} \ \veebar\mathrm{F}$$

10.1.2 Introduction rules

A question can be introduced in a very similar way as the disjunction. It is in fact an exclusive disjunction. It has two introduction rules. The difference between this operator and a normal disjunction is just the commitment to the negation of the right part if the left part is given. If we have the left part, we are also committed to the negation of the right part. The first introduction rule gives the left projection of the proof object and the second introduction rule gives the right projection of the proof object. The premises in the formation rule are also premises in the introduction rules.

$$\frac{B \ \mathit{true} \quad \neg A \ \mathit{true}}{A \veebar B \ \mathit{true}} \ \veebar\mathrm{I1}$$

$$\frac{A \ \mathit{true} \quad \neg B \ \mathit{true}}{A \veebar B \ \mathit{true}} \ \veebar\mathrm{I2}$$

10.1.3 Elimination rule

A question has a very similar elimination rule as the disjunction. The difference here is that we have two premises for each introduction rule and therefore also two assumptions for the hypothetical judgement in the second and third premise. If we can derive a certain judgement from the premises that occur in the first introduction rule and we can derive it from the premises that occur in the second introduction rule, we can eliminate the judgement and end up with the derived judgement.

The premises in the formation rule are also premises in the elimination rule.

$$\frac{A \veebar B \; true \qquad \begin{matrix}(A \; true, \neg B \; true)\\ C \; true\end{matrix} \qquad \begin{matrix}(B \; true, \neg A \; true)\\ C \; true\end{matrix}}{C \; true} \veebar\text{E}$$

10.2 ? operator for inquiries

We will see that inquiries have a very similar structure as the universal quantifier operator and the conditional. While a they claim that a proof for one judgement can be transformed into a proof for another judgement, an inquiry asks whether the first judgement actually is the case. The reason for the similarities is the relevance aspect of inquiries. It is not virtuous to make inquiries that does not have any relevance to our derivation. The consequent of the conditional corresponds to the reason for deriving the inquiry, the judgement that the inquiry is relevant for. It shows why we should pay attention to exactly that inquiry. An inquiry should be made only when it has some relevance to other judgements and this is limited by the rules given here. The inquiry operators have similar formation, introduction and elimination rules as the universal quantifier and in order to recover the thesis in the elimination rules, it is stated in the inquiry. It is written in square brackets, [and]. These brackets should not be considered like a context, but rather the judgement that the inquiry is in relation to. In the assumption inquiry and definition inquiry, the sets may be a propositions, because of the principle of propositions as sets.

By introducing the ? operator we introduce a difference on the judgement level and not only proposition level. The $\vdash$ operator is usually left implicit for judgements in constructive type theory, and we will continue to keep this implicit as it would give a heavier notation. The reader should be aware that for all judgements that do not use the ? operator, the $\vdash$ operator is implicit.

10.2.1 Formation rules

A judgement that occurs as an assumption for another judgement can be an inquiry. Type declarations, hypothetical judgements and definitions may be inquiries. J stands simply for judgement, as any judgement can be the thesis of a hypothetical judgement that we have in the formation rules. $J(x)$ should be understood as a judgement where x may occur. The notation under the bar states that the inquiry is a judgement.

The first formation rule is for type declaration inquiries. It states that whenever a judgement, $x : A$, is used as an assumption for another judgement, we can ask for its type. An inquiry for this type declaration is a judgement.

$$\frac{\begin{array}{c}(x : A)\\ J(x)\end{array}}{?_{type}A[J(x)]\ judgement}\ ?\mathrm{F1}$$

The second formation rule is for assumption inquiries. It states that whenever we have a set, $A : set$, and that we have a judgement, $J(x)$, that depends on this set, the inquiry for such an assumption is a judgement.

$$\frac{A : set \quad \begin{array}{c}(x : A)\\ J(x)\end{array}}{?_{ass}x : A[J(x)]\ judgement}\ ?\mathrm{F2}$$

The third formation rule is for definition inquiries. It states that whenever we have a set, $A : set$, and a judgement, $J(x)$, that has this set with a hypothetical object as an assumption, the inquiry for a definition is a judgement.

$$\frac{A : set \quad \begin{array}{c}(x : A)\\ J(x)\end{array}}{?_{def}x : A[J(x)]\ judgement}\ ?\mathrm{F3}$$

10.2.2 Introduction rules

We have three introduction rules for the inquiry operator. The first is an inquiry for a type declaration. The second is an inquiry for a hypothesis. The third is an inquiry for a definition. All introduction rules depend also on the premises for their respective formation rules.

The type declaration inquiry can be introduced when we have an assumption for a judgement where it occurs a hypothesis. To have a type declaration inquiry means to ask what type the expression is. A type declaration inquiry does not depend on that the expression is declared a type. An inquiry $?_{type}A$ does not depend on any other judgement $A : type$. That would be a circular argument. This rule is also implicitly dependent on that $J(x)$ is a judgement.

$$\frac{\begin{array}{c}(x : A)\\ J(x)\end{array}}{?_{type}A[J(x)]}\ ?\mathrm{I1}$$

The assumption inquiry can be introduced when a hypothesis occurs as an assumption for another judgement. What this means is not that we ask for a substitution of the hypothetical object with a categorical object, but that we should try to find another assumption that this assumption depends on. This rule is implicitly dependent on that A is resolved to a set, $A : set$, and that $J(x)$ is a judgement.

$$\frac{\begin{array}{c}(x:A)\\ J(x)\end{array}}{?_{ass}x:A[J(x)]}\ ?\mathrm{I2}$$

The definition inquiry can be introduced for all judgements that has a hypothetical object and occurs as an assumption for some other judgement. It means to demand a definition of an object, so we can perform a substitution from the hypothetical object to a categorical object. It is not an inquiry for actually substituting the hypothetical object. For the case $?_{def}x : A[J(x)]$, an inquiry for $x : A$, an answer would be a definition of the hypothetical object $x = a : A$, so that a substitution of a in the occurrences of x can be done. It is therefore an inquiry for a definition of an object and not for the substitution of an object in itself. The rule is dependent on that A is resolved to a set, $A : set$, and that $J(x)$ is a judgement.

$$\frac{\begin{array}{c}(x:A)\\ J(x)\end{array}}{?_{def}x:A[J(x)]}\ ?\mathrm{I3}$$

10.2.3 Elimination rules

All of the three elimination rules have the same structure. We can see from the elimination rules together with the introduction rules that inquiries have a very similar structure as universal quantification and conditionals.

The first elimination rule is for type declaration inquiries. If we have an inquiry for a type declaration that was motivated by a certain judgement, $J(x)$, and we have the type declaration that corresponds to what the inquiry demands, we can eliminate the inquiry and end up with the judgement that the inquiry was motivated by. This rule is also depending on that $J(x)$ is a judgement.

$$\frac{?_{type}A[J(x)] \quad a:A}{J(a)}\ ?\mathrm{E1}$$

The second elimination rule is for assumption inquiries. If we have an inquiry for an assumption that was motivated by a judgement, $J(x)$, and we have the judgement that the inquiry demands, we eliminate the inquiry with a new hypothetical proof object and end up with the judgement that motivated the inquiry. This rule is depending on that $J(x)$ is a judgement and that A is a set.

$$\frac{?_{ass}x:A[J(x)] \quad a:A}{J(a)}\ ?\mathrm{E2}$$

The third elimination rule is for definition inquiries. If we have an inquiry for a definition that was motivated by a judgement, $J(x)$, and we have a object for the same set, we can eliminate the inquiry and end up with the judgement that motivated the question. This is similar to the elimination rule for universal quantification. This rule is depending on that $J(x)$ is a judgement and that A is a set.

$$\frac{?_{def}x : A[J(x)] \quad a : A}{J(a)} \text{ ?E3}$$

We can see that all elimination rules also corresponds to the elimination rule for the universal quantifier. An inquiry may therefore be looked at as a conditional or a universal quantifier with a certain normative meaning. The universal quantifier and the conditional are both obtained by the Π operator. They have the same structure and are both expressed by the same set-theoretical operation.

10.3 ! operator for answers

An answer to an inquiry can also be seen as a judgement. This is not the same operation as removing an inquiry. When answering an inquiry we are given an answer. The inquiry should be considered solved and not removed from the agents state. The answer operator has similar formation, introduction and elimination rules as the conjunction.

Answering an inquiry means that the inquiry should not be considered open anymore. It takes an inquiry off the research agenda. They are, however, never really removed. That an inquiry is answered means that there has been derived an answer to this judgement.

10.3.1 Formation rules

All the formation rules for answers have the same structure. Whenever an inquiry is a judgement and the object of inquiry is a judgement, the answer to this inquiry is a judgement. Because we have three different kinds of inquiries we also have three different kinds of answer, one kind of answer to each kind of inquiry.

The first kind of answer is a type declaration answer, an answer to a type declaration inquiry. Whenever a type declaration inquiry is a judgement and the judgement that is the object of the inquiry is a judgement, the answer that corresponds to the type declaration inquiry is a judgement.

$$\frac{?_{type}A[J(x)] \ judgement \quad A : type \ judgement}{!A[J(x)]/A : type \ judgement} \text{ !F1}$$

The second kind of answer is an assumption answer, an answer to a assumption inquiry. Whenever an assumption inquiry is a judgement and the hypothetical judgement where the thesis is the object of the inquiry is a judgement, the answer that corresponds to the assumption inquiry is a judgement.

$$\frac{?_{ass}x : A[J(x)]\ judgement \quad \begin{matrix}(y : B)\\ x(y) : A\ judgement\end{matrix}}{!x : A[J(x)]/x(y) : A(y : B)\ judgement}\ !F2$$

The third kind of answer is a definition answer, an answer to a definition inquiry. Whenever a definition inquiry is a judgement and the judgement with definition of a hypothetical object for the set in the object of the inquiry is a judgement, the answer that corresponds to the definition inquiry is a judgement.

$$\frac{?_{def}x : A[J(x)]\ judgement \quad x = a : A\ judgement}{!x : A[J(x)]/x = a : A\ judgement}\ !F3$$

10.3.2 Introduction rules

The introduction rule for answers can be used whenever we have an inquiry for the judgement together with this judgement. We can close the inquiry by deriving the answer. Each introduction rule also depends on the premises for the corresponding formation rule.

The first kind of answer is a type declaration answer. Whenever we have a type declaration inquiry and we also have the judgement that is the object of the inquiry, we have an answer that corresponds to the type declaration inquiry.

$$\frac{?_{type}A[J(x)] \quad A : type}{!A[J(x)]/A : type}\ !I1$$

The second kind of answer is an assumption answer. Whenever we have an assumption inquiry and we also have a hypothetical judgement where the thesis is the object of the inquiry, we have an answer that corresponds to the assumption inquiry.

$$\frac{?_{ass}x : A[J(x)] \quad \begin{matrix}(y : B)\\ x(y) : A\end{matrix}}{!x : A[J(x)]/x(y) : A(y : B)}\ !I2$$

The third kind of answer is a definition answer. Whenever we have a definition inquiry and we also have the judgement with a definition of the hypothetical proof object for the set in the object of the inquiry, we have an answer that corresponds to the definition inquiry. The hypothetical object can be substituted with the categorical object that was found in the second premise.

$$\frac{?_{def}x : A[J(x)] \quad x = a : A}{!x : A[J(x)]/x = a : A}\ !I3$$

10.3.3 Elimination rules

As with conjunction, there are two elimination rules for each answer. The first elimination rule states that from an answer, we have the inquiry that it answers. The second elimination rule for answers states that we can derive the object of the answer. Each elimination rule also depends on the premises for the corresponding formation rule.

The first elimination rule for the first kind of answer, type declaration answer, states that whenever we have a type declaration answer, we also have a type declaration inquiry that the answer answered. This means that the inquiry has the same object as found in the first part of the answer. The inquiry is found as the first premise of the introduction rule for the first answer.

$$\frac{!A[J(x)]/A : type}{?_{type}A[J(x)]} \text{ !E1.1}$$

The second elimination rule for the first kind of answer, type declaration answer, states that whenever we have a type declaration answer, we also have the object of the answer, the type declaration that answers the inquiry. The type declaration is found as the second premise of the introduction rule for the first answer.

$$\frac{!A[J(x)]/A : type}{A : type} \text{ !E1.2}$$

The first elimination rule for the second kind of answer, assumption answer, states that whenever we have an assumption answer, we also have an assumption inquiry that the answer answered. This means that the thesis of the object of the answer is the same as the object of the inquiry. The inquiry is found as the first premise of the introduction rule for the second answer.

$$\frac{!x : A[J(x)]/x(y) : A(y : B)}{?_{ass}x = A[J(x)]} \text{ !E2.1}$$

The second elimination rule for the second kind of answer, assumption answer, states that whenever we have an assumption answer, we also have the object of the answer, the hypothetical judgement that answers the inquiry. The judgement is found as the second premise of the introduction rule for the second answer.

$$\frac{!x : A[J(x)]/x(y) : A(y : B)}{x(y) : A(y : B)} \text{ !E2.2}$$

The first elimination rule for the third kind of answer, definition answer, states that whenever we have a definition answer, we also have a definition inquiry that the answer answered. This means that the answer has categorical object that can be substituted with the hypothetical object found in the inquiry.

The inquiry is found as the first premise of the introduction rule for the third answer.

$$\frac{!x : A[J(x)]/x = a : A}{?_{def} x : A[J(x)]} \text{ !E3.1}$$

The second elimination rule for the third kind of answer, definition answer, states that whenever we have a definition answer, we also have the definition judgement with a categorical object to substitute the hypothetical object, the definition that answers the inquiry. The judgement is found as the second premise of the introduction rule for the third answer.

$$\frac{!x : A[J(x)]/x = a : A}{x = a : A} \text{ !E3.2}$$

Answers seem to be less controversial than inquiries. They operate in a similar way as the conjunction and could possibly also be replaced by it. It is easier to have an explicit operator that answers inquiries. It is then not to be confused by a conjunction judgements that may not be an explicit answer to some inquiry.

11 What inquiries and answers represent

11.1 What to do with the different inquiries?

11.1.1 Type declaration inquiry

I will argue that type declaration inquiries can be used for for two things. They can be used to demand a missing type declaration for an expression in a judgement and they can be used to represent a certain kind of question in a linguistic sense. The first alternative is a solution to what we will call a formal problem, a problem of derivation, and the second alternative is simply a way to formalise what we normally call questions.

If we update an informational state with judgement where the expression that occurs in the judgement does not have an appropriate type declaration, we can demand this type declaration by introducing a type declaration inquiry. If we have a judgement with a complex expression, an expression with an operator, we may not have an explicit type declaration for this complex expression in the informational state. If I derive a judgement $c : A \wedge B$ from an informational state, by the implicit assumptions from the formation rule in the introduction rule, I can only do that when $A : prop$ and $B : prop$. I do not need to have derived the explicit judgement $A \wedge B : prop$ in order to derive and update with $c : A \wedge B$. This means that if this judgement occurs as an assumption, it is not sure that we have the explicit type declaration for the expression $A \wedge B$. In order to get this explicit type declaration, we may introduce a type declaration inquiry for the expression. It is to make sure that the judgement has the proper formation. $\wedge$ may be any operator in the system and *prop* may be any type that corresponds of the formation rule to the operator.

Can this also happen for expressions that are not complex? Primiero [3, p. 155] argues that an update of the informational state presupposes that a proper type declaration already occurs in the informational state. This requirement prevents this from happening, but if we commit ourselves to a type declaration inquiry for the expression that occurs in the new judgement, it could seem acceptable to change or remove this requirement. A type declaration inquiry can commit us to investigate the type of a certain expression that occurs in the judgement. We seem to have a similar situation by the introduction of hypotheses in general. If we add a hypothesis, complex or not, to our informational state, we may not always have to be committed to a proper type declaration of every concept in the hypothesis if we commit ourselves to an investigation of this type declaration by a type declaration inquiry.

Type declaration inquiries may also be used to represent some situations in natural language. It can represent questions in a linguistic sense. In some sense we can use it to formalise questions. It can correspond to questions of the form "Can you explain ...?" or "What is ...?". Consider the following dialogue between two people, John and Mary:

John	:	"The butter is melting" is a proposition.
John	:	"We finished eating" is a proposition.
John	:	"The butter is put in the fridge" is a proposition.
John	:	The butter is put in the fridge if the butter is melting or we finished eating.
Mary	:	What is "The butter is melting or we finished eating"?
John	:	"The butter is melting or we finished eating" is a proposition.

The dialogue may not seem very natural. The reason for this is that we speak about propositions. Usually, when we speak we do not speak about propositions, but with propositions. One could imagine that it could occur for example in a course to learn english. John asserts that "The butter is melting", "We finished eating" and "The butter is put in the fridge" are propositions. John asserts that the butter is put in the fridge if the butter is melting or we finished eating. Mary asks what this last part, "the butter is melting or we finished eating", is. John answers that it is a proposition. Let M represent "The butter is melting, F represent "We finished eating" and B represent "The butter is put in the fridge". We can make a similar dialogue in this constructive type-theoretical framework.

[7, p. 169]

John	:	$M : prop$
John	:	$F : prop$
John	:	$B : prop$
John	:	$x(y) : B(y : M \vee F)$
Mary	:	$?_{type} M \vee F[x(y) : B]$
John	:	$!M \vee F[x(y) : B]/M \vee F : prop$

We may also represent this as a derivation by a tree. The assumptions in the tree represent John's informational state. This may be a representation of how John come to answer $!M \vee F[x(y) : B]/M \vee F : prop$. It can be derived from $M : prop$ and $F : prop$.

$$(M : prop^1, F : prop^2, B : prop, y : M \vee F)$$

$$\dfrac{\dfrac{x(y) : B}{?_{type} M \vee F[x(y) : B]}\,?I1 \qquad \dfrac{M : prop^1 \qquad F : prop^2}{M \vee F : prop}\,\vee F}{!M \vee F[x(y) : B]/M \vee F : prop}\,!I1$$

We can represent this process by updates of a context. Assume that we have a context where John's assertions occur. They could also have been introduces individually by several informational updates on an empty context.

$$\Gamma = (M : prop^1, F : prop^2, B : prop, y : M \vee F, x(y) : B)$$

We update Γ with a type declaration inquiry for the assumption by a knowledge extension.

$$\Gamma_1 = (\Gamma, ?_{type} M \vee F[x(y) : B])$$

We update Γ_1 with the a type declaration of the complex expression that is the object of the inquiry by a knowledge extension.

$$\Gamma_2 = (\Gamma_1, M \vee F : prop)$$

We update Γ_2 with a type declaration answer to the type declaration inquiry from the type declaration in the previous update by a knowledge extension.

$$\Gamma_3 = (\Gamma_2, !M \vee F[x(y) : B]/M \vee F : prop)$$

We end up with a context that contains all the information that is given by the dialogue. The proof tree represents the derivations that yields the knowledge extensions.

In this example we can see that both the formal problem and the formalisation are present. We have represented the dialogue in constructive type theory. We see that the question has been represented as a type declaration inquiry. The derivation also shows how a type declaration inquiry may demand a derivation of the type declaration of a complex expression.

Type declaration inquiries and type declaration answers does seem to correspond to some specific notion of what we call question and answer in a natural language. There are no upper limit for how many different types we may have. The connection between constructive type theory and linguistics has been developed by Ranta [7].

11.1.2 Assumption inquiry

As with the type declaration inquiries we can look at assumption inquiries in two different ways. They can be used to demand an assumption for a judgement and they can be used to represent another aspect of questions in a linguistic sense.

If we have a hypothesis $x : A$ that occurs in the informational state, we can make an assumption inquiry for this judgement. What happens is that we ask whether this hypothesis depends on some other hypothesis. It demands more and more specified information. By making an assumption inquiry of $x : A$, we ask if there are some other hypothesis $y : B$ that $x : A$ depends on $x(y) : A(y : B)$. In this situation we have ended up with more information about our original hypothesis $x : A$. It demands a specification of the original hypothesis. It does not demand sufficient conditions for the hypothesis to be true, but it demands at least one condition. We could demand more conditions by deriving a new assumption inquiry of the hypothesis when the first one has been answered.

Type declaration inquiries may represent a demand in the linguistic sense. It should be understood as a question of the form "Does ... depend on something else?". It seems to represent some kind of questions of the form "How ... ?" or "Why ... ?". Consider the following dialogue between two people, John and Mary:

John	:	The butter is put in the fridge if it is melting.
Mary	:	Why is butter melting?
John	:	Butter is melting if it is warm.

What we have here is a very similar situation as the one described by the assumption inquiry. John asserts that the butter is put in the fridge, under the assumption that it was melting. Mary asks why butter melts. John answers that butter melts if it is warm. Let W represent "it is warm". We can make a similar dialogue in this constructive type-theoretical framework:

John	:	$x(y) : B(y : M)$
Mary	:	$?_{ass} y : M[x(y) : B]$
John	:	$!y : M[x(y) : B]/y(u) : M(u : W)$

We can similarly make a tree for this situation. The assumptions in the tree represent John's informational state. $y(u) : M(u : W)$ may be seen as a hypothesis that is derivable in his informational state or that was added because of the inquiry.

$$\dfrac{\dfrac{\begin{matrix}(y : M)\\ x(y) : B\end{matrix}}{?_{ass} y : M[x(y) : B]}\,?I2 \qquad \begin{matrix}(u : W)\\ y(u) : M\end{matrix}}{!y : M[x(y) : B]/y(u) : M(u : W)}\,!I2$$

We can represent this process by the update of a context. Assume that we have a context where John's assertion occur. We assume that the appropriate type declarations for each expression occurs in Γ.

$$\Gamma_1 = (\Gamma, y : M, x(y) : B)$$

We update Γ_1 with an assumption inquiry for the assumption by a knowledge extension.

$$\Gamma_2 = (\Gamma_1, ?_{ass}y : M[x(y) : B])$$

We update Γ_2 with an assumption for $y : M$ by addition of hypothesis. This is actually several operations and is not the result of one single update. This complex operation is long and it is the operation described as modification of hypothesis. Here we look at this complex operation as one update in order to simplify the example.

$$\Gamma_3 = (\Gamma, u : W, y(u) : M, x(y(u)) : B, ?_{ass}y : M[x(y) : B])$$

We update Γ_3 with an assumption answer to the assumption inquiry from the addition of hypothesis in the previous update by a knowledge extension.

$$\Gamma_4 = (\Gamma_3, !y : M[x(y) : B]/y(u) : M(u : W))$$

We end up with a context that contains all the information that is given by the dialogue. The proof tree represents the derivations that yields the knowledge extensions. We see that assumption inquiries not only has a role in the belief revision framework, but that it also can be used to represent or formalise questions.

11.1.3 Definition inquiry

As with the two other inquiries the definition inquiry may be looked at as either solving a formal problem or representing a question in the linguistic sense.

A definition inquiry is a demand for a categorical object when we have a hypothetical object. This means that we want to turn our belief into knowledge. If we have a hypothesis $x : A$ that occurs as an assumption for some other judgement, $y(x) : B$, we can make a definition inquiry for a definition of this object. When we only have a hypothetical object for the assumption, we do not have an object to transform into an object for the thesis. In order to have such an object we will need a definition of the hypothetical object as a categorical object $x = a : A$. If we acquire this, we have an object a to transform into an object for the thesis $y(a) : B$. A definition inquiry represents the demand for such a definition.

A definition inquiry may represent a demand in the linguistic sense. It represents a question of the form "Is it true that ...?" or "Can you prove that ...? in natural language or simply an agent that investigates the truth of a proposition.

Consider the following dialogue between two people, John and Mary:

John	:	The butter was put in the fridge if it was melting.
Mary	:	Is it true that it was melting?
John	:	It is true that it was melting. There is a drop on the table.

John asserts that the butter was put in the fridge under the assumption that it was melting. Mary asks whether it is true that the butter was melting. John answers that it is true that the butter was melting and the drop is the proof of it. This proof has to be taken with caution. It is not a categorical proof in the constructive sense, but it could be considered an immediate proof in Martin-Löfs words [16, p. 29]. This means that it has to be grasped as a whole, not as an actual constructive demonstration. We can make a similar dialogue in this constructive type-theoretical framework:

$$\begin{array}{lcl} \text{John} & : & x(y) : B(y : M) \\ \text{Mary} & : & ?_{def} y : M[x(y) : B] \\ \text{John} & : & !y : M[x(y) : B]/y = m : M \end{array}$$

We can similarly make a tree for this situation. The assumptions in the tree represent John's informational state.

$$\dfrac{\dfrac{\begin{array}{c}(y : M) \\ x(y) : B\end{array}}{?_{def} y : M[x(y) : B]} \; ?I3 \qquad y = m : M}{!y : M[x(y) : B]/y = m : M} \; !I3$$

We can represent this process by the update of a context. Assume that we have a context where John's assertion occur. We assume that the appropriate type declarations for each expression occurs in Γ

$$\Gamma_1 = (\Gamma, y : M, x(y) : B)$$

We update Γ_1 with a definition inquiry for the assumption by a knowledge extension.

$$\Gamma_2 = (\Gamma_1, ?_{def} y : M[x(y) : B])$$

We update Γ_2 with a definition of y in $y : M$ by an addition of a definition, an informational update. This should yield a substitution of all occurrences of the object y by the object m as described by the addition of definition.

$$\Gamma_3 = (\Gamma_2, y = m : M)$$

We update Γ_3 with a definition answer to the definition inquiry from the addition of definition in the previous update by a knowledge extension.

$$\Gamma_4 = (\Gamma_3, !y : M[x(y) : B]/y = m : M)$$

We end up with a context that contains all the information that is given by the dialogue. The proof tree represents the derivations that yields the knowledge extensions. We see that definition inquiries not only has a role in the belief revision framework, but that it also can be used to represent or formalise questions.

11.1.4 Answers

In a belief revision interpretation of constructive type theory answers close inquiries. They are supposed to be thought of as judgements that make inquiries closed or answered. When there has been derived a suitable answer to an inquiry, we say that the inquiry is closed.

For a type declaration inquiry it means that the object of the answer is a type declaration where the expression is the same as the expression in the object of the type declaration inquiry. A suitable answer for an inquiry of the form $?_{type}A[J(x)]$ is an answer of the form $!A[J(x)]/A : type$.

For an assumption inquiry it means that the object of the answer is a hypothetical judgement where the thesis is the same judgement as the object of the assumption inquiry. A suitable answer for an inquiry of the form $?_{ass}x : A[J(x)]$ is an answer of the form $!x : A[J(x)]/x(y) : A(y : B)$.

For a definition inquiry it means that the object of the answer is a judgement where it occurs one categorical object and one hypothetical object that are equal elements of a set. This set occurs in the judgement that is the object for the definition inquiry and the hypothetical object in the answer is the same as the hypothetical object in the inquiry. A suitable answer for an inquiry of the form $?_{def}x : A[J(x)]$ is an answer of the form $!x : A[J(x)]/x = a : A$.

When no such answer has been derived, we say that the inquiry is open. This means that answers close one specific inquiry. Answers do not remove the inquiries, they simply close them. This gives us a log of how the state ended up being like it is in the state itself. An agent has direct access to all inquiries and answers that have been made up to that point in the state. We have not developed any particular way to use this log, but the idea of learning from our mistakes or represent experience seems to be closely related.

Answers may also be used to represent answers and not only be used as a way of closing inquiries. As we have seen in the earlier examples, we may use answers to represent an answer to a question in a linguistic sense. This shows us that representing answers in constructive type theory may not be only a formal term, but that it actually has some relation to what we normally call answers.

12 Research agendas and formal problems

12.1 Representing research agendas

Belief revision in constructive type theory seems to have a slightly more complex way of organising belief and knowledge than the classical AGM-theory. As a result of the persistence, a state can only be extended. We are committed to every judgement in the informational state that has not yet been contracted. We could say that we can stop being committed to a judgement, but never actually remove it. To distinguish between those judgements we are committed to and those we are not committed to, we would need an algorithm that finds all judgements that are not yet contracted.

A research agenda can be said to be a collection of inquiries that does not yet have an answer. In some sense one may argue that a research agenda consists of all open inquiries in a state. In the same way as with belief and knowledge, we would need an algorithm that finds all inquiries that have not been contracted or answered. This corresponds to the research agenda. As we will always have a finite amount of answers and inquiries, it should not be a problem to find all open inquiries. There are several ways to do this operation. In Primiero's terms an answer will normally occur as a result of a knowledge extension. It is a derived judgement.

12.1.1 A general limitation for inquiries and answers

When we should make inquiries is the most important aspect of representing research agendas. According to the introduction and formation rules we can make an inquiry whenever we have a hypothetical judgement where an expression occurs in the assumption or a set with a hypothetical object that occurs as an assumption for some other judgement. This means that we are allowed to make several inquiries for the same judgement, a potentially infinite amount of times. Even though we are allowed to do this by the rules, it does not seem to acquire anything. Intuitively it is not very fruitful to pose the same inquiry several times.

A general limitation is that we should not make an inquiry about a judgement when we already have an open (unanswered) inquiry about that very same judgement. We should only make an inquiry when there are not any other inquiries in the state with the same object. This prevents a potentially infinite amount of inquiries. As long as we have a finite number of judgements in our state, we can only have a finite number of inquiries. This does also seem to count for answers. We should not answer a closed (answered) inquiry. $?_{...}$ should be understood as either a type declaration inquiry, assumption inquiry or a definition inquiry. Suppose that E is an object of the inquiry. It can be of any of the forms that we have introduced. Suppose that we have a context where it occurs an inquiry.

$$\Gamma_1 = (\Gamma, ?_{...}E)$$

According to the general limitation we cannot update Γ_1 with an inquiry when it already occurs a similar open inquiry in Γ_1. We can therefore not update Γ_1 with $?_{...}E$ because it already occurs a similar open inquiry in Γ_1. We assume that the kind of judgement is the same. If they are different kinds of judgements, the limitation should not be applied.

Suppose J is a judgement that is the answer of an inquiry. Suppose that we have a context where it occurs an answer.

$$\Gamma_1 = (\Gamma, !E/J)$$

According to the general limitation we cannot update Γ_1 with an answer that already occurs in Γ_1 when there are no open inquiry with E as an object. We can therefore not update Γ_1 with $!E/J$ because it already occurs last in Γ_1. What kind of answer will depend on what judgement that occurs for J. We assume that the judgement has been correctly derived from Γ.

By the limitation mentioned, we have avoided a potentially infinite number of inquiries and answers for a finite number of judgements. A state contains in general only a finite number of judgements. This means that we have a finite number of potential inquiries for a state. It is possible to make an algorithm that gives all potential inquiries of a state. Under the restrictions mentioned earlier, this will give us all inquiries that we may derive from a state. The general limitation is understood as for any other judgement, if it is an inquiry with the same object as the one we try to derive, we should not derive the inquiry. This limitation together with the algorithm to derive can be considered like some strategy for an agent.

As there can only be a finite amount of judgements the algorithm will be ending. What we end up with is that all potential inquires have been made. It is the maximal inquiry of a state. Every research agenda should contain an equal or smaller amount than we get by this algorithm.

12.1.2 Specific limitations for inquiries

When using a theory for belief revision there may be several reasons to not have the maximal inquiry of a state. Exactly what limitations to have will depend on how the belief revision theory is used. I do not think it is virtuous to make some limitations that are intended to count for every potential use of the belief revision theory. I will present two limitations that seem very natural to impose on the derivation of inquiries.

The first limitation is that we should not make inquiries about judgements when we already have the answer. To have the answer in this sense means that the judgement is actually the case and not only potentially the case. This means that we should only make inquiries when a judgement that is needed in order to derive an answer to the inquiry is already found in the state. This limitation should only count when the object is the exact same judgement and not only implied by the other judgements. We may have an inquiry about a judgement that is implied by the state as long as it is not made explicit. This limitation prevents us from deriving an inquiry as soon as a similar inquiry has been answered. The general limitation only prevents us from deriving open inquiries, but by introducing this specific limitation, we impose a limitation on deriving inquiries that are already answered.

Suppose we have a context.

$$\Gamma = (A : type, x : A, y : B)$$

By this limitation we cannot update Γ with an inquiry of the form $?_{type}A[y : B]$. This is because $A : type$ is found in the state and it would, together with the inquiry, be sufficient to derive an answer to the inquiry $!A[y : B]/A : type$. Similar examples can be made for the other inquiries.

The second limitation is that if we have an assumption of a certain thesis where we may derive an inquiry, but the thesis is already proved categorically, we ought not to derive the inquiry. This means that we should not make an inquiry in order to prove a thesis that is already proved. This limitation presupposes that two proofs of a judgement are not any better than one, that one proof is sufficient. In some areas we may still want to investigate further, even though the result is already proved, for example that we will end up with a different proof. If it is considered an advantage with more than one proof, this limitation should be ignored. If it is enough with only one proof for a judgement, this limitation seems to hold. That something has a categorical object would mean that they would be resolved to a type, as the introduction rules also depend on the premises of the formation rule. This means that this limitation seems to count for definition inquiries, assumption inquiries and type declaration inquiries.

Suppose we have a context.

$$\Gamma = (a : A, x : A, y : B)$$

By this limitation we cannot update the Γ with inquiries of the form $?_{type}A[y : B]$ or $?_{...}x : A[y : B]$. We already have a categorical object for A.

This is just two limitations that we may put on the derivation of inquiries. It is not a complete list. We may find many other limitations that are useful in certain applications.

We may argue that we should make limitations when it is not possible, for example by practical reasons, for the agent to give an answer to the inquiry. Another limitation may be when answering the inquiry anyway would not get us any closer to reach our goal. These limitations are dependent on non-logical aspects that the agent takes in count. I do think that these limitations can have an effect on the formal representation, but not something we can give as general rules. These kinds of limitations are depending on the actual application of the belief revision theory. Limitations of this kind may seem useless from a logical point of view, but from a practical point of view, such limitations may be necessary in order to have a useful theory. If the limitations are done with caution they make the theory simpler and it can be more efficient to get the data that is actually needed. They seem to be relevant because of practical and not logical reasons. In this paper we will not take any standpoint on other limitations to impose, but simply mention that further limitations are possible.

A different way of doing this would be to ignore some inquiries, but as this involve making more judgements and derivations than necessary, it does not seem to be a better solution. In terms of the research agenda, the result seems to be the same.

12.1.3 Dependent inquiries

We may end up with several inquiries on the research agenda. Some judgements are dependent on other judgements. We may derive an inquiry for two judgements where one judgement depends on the other one. An inquiry for the first judgement would in some sense also depend on the inquiry for the second judgement. It seems like this could be reflected in the research agenda.

Suppose a judgement has two assumptions and the first assumption is dependent on the second assumption $(x : B, y(x) : C, a(y(x)) : A)$. In this situation we may have a definition inquiry for both the first assumption, $x : B$, and the second assumption, $y(x) : C$, but an answer to the inquiry of the first assumption would only give a proof for the second assumption and not proof for the thesis itself. This means that even if we answer an inquiry about the first assumption, we would have to give an answer to the second assumption in order to have a proof for the thesis. In this situation, where the first assumption is only an assumption for the thesis through the second assumption, it seems like we should give an inquiry of the second assumption higher priority than the inquiry of the first assumption.

This does not mean that an inquiry for the first assumption is useless, as it may be necessary in order to prove the second assumption, but it appears to be an indirect assumption for the thesis. This may give us a relation of importance between inquiries when they come from the same context. It is difficult to give a general rule for exactly how to make inquiries based on a certain context, as contexts may contain independent assumptions, but for the case with dependent assumptions, understood as a transitive relation, that every assumption is dependent on the the previous assumptions. We presuppose the general limitation. We also presuppose that the proper type declarations are found in Γ.

For the example we mentioned and with an assumption inquiry, we would end up with the following derivation.

The original context contains three judgements.

$$\Gamma_1 = (\Gamma, x : B, y(x) : C, a(y(x)) : A)$$

Here we make a definition inquiry about the first assumption, based on the original judgement. The update of the context is motivated by a knowledge extension, an introduction of the definition inquiry. It is introduced as a modification of an assumption.

$$\Gamma_2 = (x : B, y(x) : C, ?_{def} x : B[y(x) : C], a(y(x)) : A)$$

Here we make a definition inquiry about the second assumption. The update of the context is motivated by a knowledge extension, an introduction of the definition inquiry.

$$\Gamma_3 = (\Gamma_2, ?_{def} y(x) : C[a(y(x)) : A])$$

We can see that we ended up with that the inquiry for the second assumption not only depends on the original judgement, but it also depends on the inquiry for the first assumption. By making the inquiries in such a way, we may get a natural order on the inquiries. We can see that the the inquiry for $x : B$ occurs in the context of the inquiry for $y(x) : C$. If belief sets are represented simply as contexts, the ordering can be reflected between inquiries as well.

This is not the only ordering that could be given for inquiries. If a notion of relevance or some other ordering between judgements is made in the constructive type theory, this ordering can be reflected between inquiries as well. The point here is that inquiries behave as judgements an ordering of other judgements may be used for inquiries as well.

12.2 Inquiries and errors

Primiero argues [11, p. 7] [3, p. 179] that there are several kinds of errors that may occur in the constructive type-theoretical framework. We may distinguish between informational error and proper knowledge error. Proper knowledge error is an error caused by deriving in an incorrect way. Informational error is caused by extending the informational state in a way that causes problems. In order to solve these errors Primiero argues for a rejection rule that can prevent a state from being changed. [11] I will argue that some errors may be solved in a different way, by introducing inquiries. I do not think that inquiries can solve all of them, but that we may get a new approach to some informational errors.

The first kind of error occurs when a judgement is added to an informational state and that this judgement contains a variable that is not declared to an appropriate type. [3, p. 179] The addition of a hypothesis where an expression does not yet have an appropriate type declaration may be acceptable if it is followed up by an investigation for such a type declaration. The information is not meaningful for the agent yet, but by an introduction of such a type declaration the information may be shown to meaningful at a later stage. This kind of error may be solved by introducing a type declaration inquiry for the expression that lacks a type declaration.

$$\frac{\begin{array}{c}(x : A)\\ B : prop\end{array}}{?_{type}A[B : prop]}\ ?I1$$

We see that A lacks an appropriate type declaration. By introducing a type declaration inquiry we make this explicit. We can then later perform an introduction of a concept by the way described by modification of assumptions. Γ represents the non-updated contexts while Δ represents the updated contexts.

We call the context $(x : A)$ of $B : prop$ for Γ_1.

$$\Gamma_1 = (x : A)$$

We call the context for Γ_1 for Γ_2.

$$\Gamma_1(\Gamma_2)$$

This is the empty context for $x : A$.

$$\Gamma_2 = ()$$

We perform an update on Γ_2 with the type declaration $A : prop$ that gives us Δ_2.

$$\Delta_2 = (\Gamma_2, A : prop)$$

We then update Γ_1 with the new context Δ_2.

$$\Delta_1 = (\Gamma_1(\Delta_2))$$

We end up with a new context that we call Δ_1 that is the new context for $B : prop$.

$$\Delta_1 = (x : A(A : prop))$$

Δ_1 can also be written $(A : prop, x : A)$, which gives us the new judgement.

$$B : prop(A : prop, x : A)$$

This shows us how a type declaration may be added to the context at a later stage than the judgement that depends on the type declaration. By deriving an inquiry for the type declaration we make the investigation for this type declaration explicit.

A second kind of error occurs when a judgement that is added to the informational state and the proposition is inconsistent with other propositions. This means that the new judgement, together with another judgement in the informational state can give a proof object for $\perp$. [3, p. 179] A particular case of this may be where the proposition in the judgement is internally inconsistent. In that case it would seem reasonable to reject the judgement in the way Primiero suggests. If not it would seem like we can introduce a question and an inquiry for this question. The two judgements will be mutually exclusive and will therefore be compatible with the introduction of the question operator. This problem seems to be similar to the first problem that Olsson presents and can therefore be solved in a similar way.

We can see that introducing inquiries may give us a new way of solving errors in the constructive type-theoretical framework. To reject the whole judgement because some part of it lacks information or is not compatible with the rest of the state seems like a strong rule. It would seem that the rejection rule may be useful in some cases, as the case with proper knowledge error and internally inconsistent propositions, but in the other situations inquiries may open up to keep the judgements in the state by demanding that what is missing should be added at a later stage.

12.3 Contraction, inconsistency, questions and inquiries

Olsson [2, p. 173-174] mentions the effect contraction should have on the research agenda. He claims that when a judgement is the object of contraction, it should be made into a question together with the judgement that motivated the contraction, what it was inconsistent with. In constructive type theory the notion of contraction is not yet fully developed. We do not have a clear view of what should count as motivation for performing a contraction. It may be undermining the evidence for a proposition or that two or more propositions are inconsistent. Here we assume that at least inconsistency should give reason to perform a contraction. This is similar to the notion of contraction that Olsson uses in his paper.

There are four ways of falling into inconsistency. The first way is when a proposition is internally inconsistent.

$$(z : A \wedge (A \rightarrow \perp))$$

In this case we should simply reject the judgement. If it is done by the rejection operation or a contraction operation does not matter here. This should be a clear example of a judgement that is not acceptable to keep in the informational state, as we can prove that it leads to an inconsistency. The second way is that we can derive a categorical proof for $\perp$.

$$(a : A, b : A \rightarrow \perp)$$

If we can derive $\perp$ categorically, the system in itself is inconsistent. This is an example of where we can by conditional elimination get a categorical proof object for $\perp$. This is a situation that should never occur. If two propositions are inconsistent, they cannot both be categorically proved.

The third way is when we have two inconsistent propositions where one has a categorical proof and the other one has a hypothetical proof.

$$(a : A \rightarrow \perp, x : A)$$

In this situation we have a categorical proof that A leads to an inconsistency by the categorically proved proposition $A \rightarrow \perp$. This should, as the first example, be rejected or contracted because we can prove that A leads to an inconsistency.

The fourth way is when two propositions are inconsistent with each other and both have a hypothetical proof object.

$$(x : A, y : A \rightarrow \perp)$$

This is an inconsistency where we cannot keep both of the judgements, but we cannot prove either of them. A point worth noticing is that inconsistency in constructive type theory seems less critical than in the classical theory. If we have an inconsistency in the classical theory, we can use it to prove every expression in the language, by the principle of explosion. Principle of explosion is also valid in intuitionism, $\perp$-elimination, but the difference is that we only have a hypothetical proof for the propositions. This means that we can only prove $\perp$, and everything that follows, as a hypothetical. In this sense an inconsistency seems less urgent in constructive type theory than in the classical framework.

The fourth example is what we normally speak about when we speak about inconsistent propositions. As the propositions are inconsistent, they should motivate some contraction. Olsson argues that when a judgement is contracted with respect to some other judgement, each judgement should be a potential answer in a question on the research agenda. This can be done when they are mutually exclusive. [2, 173-174]

In constructive type theory there are more alternatives. We can represent a similar operation as the one Olsson mentions. When a judgement is contracted with respect to some other judgement and they are mutually exclusive. We will say that they motivate the contraction. The exact operation of contraction is not defined here. Every change from one state to another should be a function from one context to another one. This should also count for contraction. We therefore simply represent the operation of contraction as a change from one context to some other context without specifying what this change consists of. When a contraction is performed we can introduce a question ($\veebar$) with a hypothetical proof object and each proposition as a disjunct. From there we introduce a definition inquiry for the question.

Let us say we have a context with two inconsistent propositions with hypothetical proof objects. We presuppose that A and B are inconsistent propositions.

$$\Gamma = (x : A, y : B)$$

We perform some manipulation on the context, depending on how the definition of contraction has been developed. The contraction should be a manipulation from Γ to another context Γ_1.

$$\Gamma \to \Gamma_1$$

We assume that A and B are the inconsistent propositions that motivated the contraction. They are mutually exclusive so we can introduce a question with a hypothetical proof object and each proposition as a disjunct. That would update the context by the following question.

$$\Gamma_2 = (\Gamma_1, z : A \veebar B)$$

Presuppose that this occurs as a context for some other judgement $J(z)$.

$$\Gamma_3 = (\Gamma_2, J(z))$$

We then update the context with a definition inquiry for the question.

$$\Gamma_4 = (\Gamma_3, ?_{def} z : A \veebar B[J(z)])$$

This corresponds to how Olsson says questions and inquiries should be affected by a contraction. A difference here is that for Olsson, this operation is implicit in the definition of contraction, while here it is a strategy. This is the strategy we may use in order to solve Olsson's first problem. [2, p. 167-168] I have not formalised the operation of contraction here. I have not explained how the original context with the inconsistent propositions gets affected by the contraction, as the notion of contraction has not yet been completely developed. We may also derive an answer to the inquiry by the proper updates.

We have an inquiry in a context of this form.

$$\Gamma_4 = (\Gamma_3, ?_{def}z : A \veebar B[J(z)])$$

We update the context with a definition of the object for the question.

$$\Gamma_5 = (\Gamma_4, z = d : A \veebar B$$

We update the context by deriving an answer to the inquiry.

$$\Gamma_6 = (\Gamma_5, !z : A \veebar B[J(z)]/z = d : A \veebar B$$

The inquiry is now closed. d is either a left projection projection of the set $A \veebar B$, derived by the a proof of A and a proof of $\neg B$, or a right projection, derived by a proof of B and a proof of $\neg A$. We can derive a proof of A and a proof of $\neg B$ if d is a left projection. We can derive a proof of B and $\neg A$ if d is a right projection.

There seems to be three ways to proceed with a contraction and the introduction of a question when two propositions with hypothetical proof objects are inconsistent. We may contract both judgements and introduce a question with each proposition as a disjunct. We may contract one of the judgements and introduce a question in a similar way. We may keep both judgements and introduce a question in a similar way. In the last situation, it would seem necessary with a contraction of one judgement sooner or later (when the other judgement is proved), but we can delay it as long as possible. I will not argue for one of them here, but all of them seem to be reasonable alternatives and which one to choose will depend on the definition of contraction and on the application of the theory. No matter what alternative, we still have the introduction of a question in a similar way as done in the example. The difference is how the context should be affected and changed.

In constructive type theory, we are not only limited to introduce $\veebar$ operators in the case of contraction. We also have other operators that could be introduced in a similar way. If we have a notion of contraction that is not dependent on inconsistency, we could introduce an inclusive disjunction ($\vee$) in a similar way as shown in the example. If the two judgements are not inconsistent, but they are contracted because their justification has been undermined or something like that, this could be an alternative. With such a notion of contraction, we would be able to solve Olsson's [2, p. 168] second problem in this way. Another way of solving this second problem would be to simply make an inquiry for the judgement that has lost its justification.

What we see is that in order to use inquiries in a satisfactory way, we depend on the development of a theory of contraction in constructive type theory. This is under development at the time of writing. How contraction should be defined may also be affected by the fact that it is followed up by an inquiry. If we know that by performing a contraction, we also commit ourself to investigate the matter more thoroughly, a notion of contraction that is not only motivated by inconsistency may be more acceptable. If we make some assumptions for how such a theory of contraction may be, we can see that it has the potential to solve at least the first and possibly the second problem that has been presented by Olsson. An important difference between this approach and Olsson's approach is that we solve the problems by describing a strategy and not by the definition of contraction itself. This may change as this notion of contraction has not yet been properly developed.

12.4 Representing working hypotheses and explanations

Olsson's third problem involves the addition of a new working hypothesis. [2, p. 168-169] This means that we should be able to handle the addition of such hypotheses in the constructive type-theoretical framework. This operation is also important in order to perform abductive reasoning and to represent investigations in general. We will call a working hypothesis an explanation. An explanation can be looked at as a story where the conclusion is the thesis of a hypothetical judgement and the background for the story is the assumptions. Here we will explain how this framework may represent the creation of explanations and to some extent how we can choose between them.

12.4.1 Creating explanations

Constructive type theory offers a very nice way of representing the creation of new working hypotheses or explanations. It does not give a direct account of how these explanations should be made, but I will explain how they can be represented in constructive type theory as hypothetical judgements.

A hypothetical judgement can be looked at as a story. This means that the assumptions for a thesis explains the reasons for believing the thesis. Here it seems that we can get a notion of adequacy, at least to some extent. An adequate explanation is one where the thesis can be inferred from its assumptions. It is also presupposed that the explanation actually explains what it is supposed to explain. If the thesis cannot be inferred from the assumptions, it lacks some assumptions,. The story has holes. This aspect does not cover the whole notion of adequacy, but it tells us one time adequacy may fail. How to make explanations is not covered by logical rules, at least not the rules I have presented here. It would seem to be dependent on the situation and the properties of the agent, like experience or creativity.

Adding an explanation corresponds to adding a hypothetical judgement. In order to add a hypothetical judgement, we add the assumptions to the informational state by informational updates, and we perform a knowledge extension by deriving a conclusion from the assumptions that has been added. This can be done relatively safely, as we are not committed to our assumptions in the same way as in the classical approach. Intuitively judgements that we add by an informational update, should optimally be based on some evidence, that we have some reason to believe them. The addition of these assumptions can be compared to collecting evidence for some facts, where the facts are the judgements that are added. It is important to note that in the classical framework, if we add something that is similar to an assumption to our belief set, it bears a stronger commitment than a hypothesis that occurs in the informational state in the type-theoretical framework. This is because in the classical framework (Olsson's framework), it does not differ between an assumption in what we may call a conditional belief and a normal belief. They both have a similar commitment. Consider the following example.

Suppose that we have a context Γ.

$$\Gamma = (A : prop, B : prop)$$

We add a judgement $x : A$ by addition of a hypothesis.

$$\Gamma_1 = (\Gamma, x : A)$$

We add a judgement $y : A \to B$ by addition of a hypothesis.

$$\Gamma_2 = (\Gamma_1, y : A \to B)$$

We derive a judgement $b(x, y) : B$ from Γ_2 by a knowledge extension.

$$b(x, y) : B(\Gamma_2)$$

We end up with a hypothetical judgement that corresponds to an explanation. The informational updates may be looked at as data collection or assumptions that are made, while the knowledge extension may be looked at as what knowledge the data may give us. The combination of the data and the knowledge may be seen as an explanation.

Olsson argued that we may add an auxiliary hypothesis if it is followed up by an investigation of its truth. [2, p. 169] If we in this framework also commit ourselves to an inquiry of the assumptions that are made, the operation of adding a hypothesis to the informational state seems like an even more acceptable operation. On the other hand, if all the assumptions that occur in the context has a categorical object, we would also be committed to the thesis.

12.4.2 Deciding between the explanations

What the constructive type-theoretical approach seems to give us is the ability to explicitly give the assumptions for the explanation. If we have some hypotheses or explanations that we want to choose between, a natural way of doing this is to inquire about the assumptions for the explanations. It may be the case that we are committed to some of the assumptions because of other judgements. This is a strategy that can yield a proof for the explanation. If we end up with being committed to all assumptions in an explanation, we should also be committed to the explanation. In order to choose between two or more explanations we inquire about the assumptions for each explanation and see if it is possible to prove the thesis of the explanation. We continue the example from before.

We have the hypothetical judgement.

$$b(x, y) : B(\Gamma_2)$$

We update the context by a knowledge extension of a definition inquiry of $x : A$.

$$\Gamma_3 = (\Gamma_2, ?_{def}x : A[y : A \to B])$$

We update the context with a definition of A by an addition of a definition.

$$\Gamma_4 = (\Gamma_3, x = a : A)$$

We end up with a definition and a corresponding substitution of the proof object in the updated context.

$$b(a, y) : B(\Gamma_4)$$

We may derive an answer to the inquiry from the definition and update Γ_4 by a knowledge extension. The inquiry is then closed.

$$\Gamma_5 = (\Gamma_4, !x : A[y : A \to B]/x = a : A)$$

By doing the same with a proof object for $A \to B$ we end up by having a categorical proof object for the proposition in the thesis, $b : B$. This means that we have represented an investigation and the discovery of a proof of B. The inquiry corresponds to the demand of a proof object for the assumptions.

To use this strategy in order to prove a hypothesis can be more difficult in the actual application of the framework, as it may be difficult to find answers to the inquiries. This strategy does not explain how we should choose between competing explanations in the sense of abduction, but as the loose ends, unproven assumptions, are explicit, the background for choosing between explanations seems to be more explicit. We end up with explicit descriptions of what is proven in the explanation and what is unanswered or only assumed. To choose between different explanations when none of them are proven, is a project for development of abduction, but constructive type theory seems to give the situation of the assumptions of the explanation in an explicit way.

13 Relation to Olsson's research agendas

Olsson mentions three different problems to the standard theory of belief revision and he claims that by introducing research agendas, we can solve these problems. [2, p. 167] A question occurs whether this interpretation of research agendas really corresponds to Olsson's understanding of research agendas and if they solve his problems.

13.1 Are research agendas the same as they are for Olsson?

Research agendas are represented in a different theory and they will therefore not be the same as they are in the AGM-theory. However, this representation of research agendas seem to cover many of the aspects that Olsson's research agendas covers.

Olsson did not make a distinction between what we call question and inquiry. A question for Olsson is a disjunction of mutually exclusive expressions that also has some normative demanding force. A research agenda is a collection of all these questions that are implied by the belief set. [2, p. 169] The distinction between a question and an inquiry that is made here is the distinction between mutual exclusive expressions and their normative demanding force.

For Olsson the normative demanding force is implicit in the question. [2, p. 169] By making it explicit we open up for asking not only questions with mutually exclusive answers, but it makes it possible to ask for specific judgements without at the same time asking for its negation. We may also ask for type declarations. The difference is naturally found in the different frameworks that are used. In the intuitionistic framework we cannot expect to always end up answering either the judgement or its negation. In the classical framework this seems more natural. We therefore make explicit what judgements we want answered. Another difference between the frameworks is that in the type-theoretical framework the judgements mention explicitly what it depends on. This enables us to restrict the inquiries for judgements that are assumptions for other judgements. In the classical framework the notion of logical closure is not problematic, but constructively it is more controversial. We therefore avoid infinite amounts of inquiries. This seems also to be because of the different frameworks that the operations are defined within.

A question for Olsson demands an expansion of our belief set with one of its potential answers. [2, p. 170] This is understood as expanding with an expression and it will be a piece of knowledge or belief depending on the interpretation of the belief set. In this theory, we introduced different inquiries, whether it is belief, knowledge or a type declaration that is demanded. That Olsson's questions do not include type declarations in the questions is evident as this is a particularity for the type-theoretical approach.

An important difference between the theories is that inquiries are limited to judgements that occur as assumptions. This is not the case in the classical approach. Olsson's questions may demand expressions that have never occurred in any previous stage of the belief revision process. This is a necessary restriction in a constructive framework, as it could potentially include an infinite amount of inquiries.

Olsson did not include an explicit notion of answer in his paper. For him, an answered question was a question with only one potential answer. As mentioned earlier, it would seem like we could handle inquiries without an explicit answer operator, but introducing an explicit answer operator seems to give a more accessible and understandable theory. The answer operator does not seem like a very controversial operator, as it behaves in a very similar way as a conjunction.

13.2 Are Olsson's problems solved?

Olsson [2, p. 167] mentions three different problems that he thinks are not possible to solve in the traditional AGM-framework. He also claims that by introducing research agendas they can be solved. Research agendas in constructive type theory seem to solve some of the problems, but does not seem to solve all of them. The problems are more thoroughly explained earlier in the paper.

13.2.1 Ending up with an inconsistent belief set

The first problem is ending up with an inconsistent belief set. [2, p. 167] The constructive type-theoretical approach seems to solve this problem in a similar way as it is solved by Olsson. If we end up in an inconsistent state, there are two alternatives. The first alternative is the rejection rule as introduced by Primiero [11, p. 10] that should be applied if the judgement is internally inconsistent. This would simply reject the judgement and stop the attempted expansion. The second alternative is that it is inconsistent together with another judgement. This corresponds to Olsson's requirement for contraction [2, p. 173]. We assume that we are speaking about a potential inconsistency and not an actual inconsistency, namely that we have a categorical proof of $\perp$. An inconsistency of the second kind would not be solvable as it would be the system in itself that is inconsistent.

When we have two judgements that are inconsistent with each other we introduce the operator called question, with the two inconsistent propositions as disjuncts and derive a definition inquiry with this question judgement as an object. What we have ended up with is an inquiry that demands a definition of a proof object for the question. An answer to this inquiry would be a proof object for either the left side or the right side of the question. This seems like a very long procedure compared to Olsson's procedure, but it seems to solve the problem.

Olsson [2, p. 173-174] argues that inconsistency should motivate a contraction of at least one of the judgements that are inconsistent, and that this contraction will yield a question where both judgements occur as potential answers. In this approach we do not have a similar commitment to hypotheses as there are for beliefs in the classical framework. There are therefore three ways to proceed if two judgements are inconsistent. We may contract both judgements. We may contract one of the judgements. We may keep both judgements and delay the contraction. By introducing a question and an inquiry for this question, the problem seems to be solved, no matter how and when we decide to perform the contraction.

We will show how this is supposed to solve the problem by illustrating with an example. Suppose that we get information that a certain person is in Rome. We then get information that this person is in New York, meaning that he is not in Rome. We start an investigation of whether he is in Rome or not.

"The person is in Rome" is represented by A. "The person is not in Rome" is represented by $\neg A$. In the example we used a proposition and its negation in order to make the example simpler. If there had been a more complex proposition, we would need to also capture the fact that the propositions actually are inconsistent. This is what we end up with by updating a context with these two pieces of information. We have a context where we have two propositions that are inconsistent.

$$\Gamma = (A : prop, x : A, y : \neg A)$$

We perform some manipulation on the context, depending on how the definition of contraction has been developed. The contraction should be a manipulation from Γ to another context Γ_1.

$$\Gamma \to \Gamma_1$$

When we have two propositions that are inconsistent we introduce a question where each proposition occur as a disjunct.

$$\Gamma_2 = (\Gamma_1, z : A \veebar \neg A)$$

We assume that we use Γ_2 and update it with some judgement. This is done in order to introduce the inquiry operator. Another way to understand this is simply that Γ_2 occurs as a context for some judgement or other context.

$$\Gamma_3 = (\Gamma_2, B : prop)$$

We then derive an inquiry by a knowledge extension and update Γ_3.

$$\Gamma_4 = (\Gamma_3, ?_{def} z : A \veebar \neg A[B : prop])$$

We have a representation of an investigation for a definition of the proof object in the question based on a contraction. This corresponds to Olsson's solution of representing an investigation that was motivated by a contraction.

13.2.2 Stop to believe something, without inconsistency

The second problem is stop to believe something, without inconsistency. [2, p. 167] This is the problem that the constructive type-theoretical approach struggle a bit with. The situation is that we get justification for a belief that undermines the justification for another belief, but where they are not inconsistent with each other. What can be done here is to introduce an inquiry for the judgement that has been undermined. If we have a notion of contraction that is not depending on inconsistency, but can be applied if the justification of a judgement has been undermined, we may solve this problem in a very similar way as with the first problem. We cannot make a question as they are not inconsistent. We may introduce a normal disjunction, but it does not really capture the relation between the judgements. An alternative is simply to introduce an inquiry for both the judgement that has been undermined and the new judgement that undermines the other judgement, but that does not really explain the relation that the judgements have to each other in a good way. We can see that the problem is not directly connected to questions or inquiries, but the notion of contraction. If contraction shows to be motivated by for example undermining of justification, it could be argued that questions should have a similar structure and we could solve the problem in the exact same way as the first problem.

Suppose that B undermines the justification of A. They are not inconsistent, but there should be performed some contraction and followed up by an investigation. We have some problem in a context that motivates a contraction based on the propositions A and B.

$$\Gamma = (A : prop, B : prop, x : A, y : B)$$

This contraction operation corresponds to a change from Γ to another context Γ_1. This is a similar operation as in the previous example, only that the propositions are not inconsistent.

$$\Gamma \to \Gamma_1$$

We update the context with a disjunction where each proposition occurs as a disjunct.

$$\Gamma_2 = (\Gamma_1, z : A \vee B)$$

We assume that we use Γ_2 and update it with some judgement.

$$\Gamma_3 = (\Gamma_2, C : prop)$$

We update the context with a definition inquiry for the disjunction by a knowledge extension.

$$\Gamma_4 = (\Gamma_3, ?_{def} z : A \vee B[C : prop])$$

We have ended up with a definition inquiry for the proof object of a disjunction where each proposition is a disjunct. This depends on that we have a notion of contraction that can handle the situation. In our example we introduced a disjunction, as the propositions are not inconsistent. We could also have solved this situation by simply making a definition inquiry for each judgement in this way.

We have a context with the situation described earlier.

$$\Gamma = (A : prop, B : prop, x : A, y : B, C : prop)$$

We introduce a definition inquiry on the first judgement $x : A$.

$$\Gamma_1 = (A : prop, B : prop, x : A, y : B, ?_{def}x : A[y : B], C : prop)$$

We introduce a definition inquiry on the second judgement $y : B$.

$$\Gamma_2 = (\Gamma_1, ?_{def}y : B[C : prop])$$

We end up with a definition inquiry for both judgements, but this approach does not show the relation that the judgements have to each other. It seems like the first solution would be a better solution in most cases, but if we do not have a notion of contraction that handles this problem in a good way, the second solution is an alternative.

13.2.3 Accepting an auxiliary hypothesis to keep another belief in your belief set

The third problem is accepting an auxiliary hypothesis to keep another belief in your belief set. [2, p. 167] This simply corresponds to adding a hypothesis as explained earlier. This theory seems to capture this fairly well, as we always have explicit assumptions for a thesis. The process of solving this problem would be to simply add the thesis under the assumptions that we have made and make an inquiry for these assumptions. As we are not committed to our assumptions of a thesis in this framework in the same way as we may be committed to assumptions in Olsson's framework, the process of adding such an assumption is in itself not as controversial. The process of demanding an investigation of the theory afterwards, is simply to make inquiries about the assumptions.

We have some theory $b : B$ that depends on some assumption $y : C$. This theory runs into some kind of trouble, and we can save the theory by adding $x : A$ as an assumption for $y : C$. This can be solved in the following way by the type-theoretical framework.

Say that we have a context and some judgement derived from this context.

$$b(y) : B(\Gamma, y : C)$$

We update the context with modification of an assumption by stating a hypothesis that the thesis depends on.

$$b(y(x)) : B(\Gamma, x : A, y(x) : C)$$

We update the context with a definition inquiry for the hypothesis we just introduced.

$$b(y(x)) : B(\Gamma, x : A, y(x) : C, ?_{def}x : A[y(x) : C])$$

We end up with an inquiry for the proof object of the hypothesis we introduced and Olsson's problem is solved.

Research agendas in constructive type theory seems to solve the first, the third and partly the second problem. A good solution to the second problem depends on a notion of contraction that allows propositions that are not inconsistent to be a motivation for ocntraction. We do find a similar problem in the classical theory as well. The first problem has a slightly longer solution than the classical solution, but I think the constructive type-theoretical approach is slightly richer and that this is the reason for the length of the solution. The third problem seems to be less of a problem in this approach than it is in the classical framework, but also here we can offer a similar solution as the one Olsson proposes.

14 Conclusion

14.1 Summary

By representing research agendas in constructive type theory, we have introduced three kinds of operators. We have distinguished between the form of a question and the demanding aspect of a question. The question operator is simply an exclusive disjunction. The inquiry operators demand a certain judgement to be added to the context. The answer operators states whether an inquiry is closed or not. The inquiry are actually three distinct operators depending on what kind of judgement that is the object of the inquiry. We also have three answer operators, one for each inquiry operator.

By introducing these three operators we show how the problems that Olsson [2, p. 167] presented in his article can be solved in constructive type theory. We introduced a strategy for when inquiries can and should be made based on a context. By introducing inquiries we have a new way of representing investigations and interrogative speech acts in natural language. When we introduce a hypothetical judgement we have some commitment to investigate whether the assumptions actually hold. Inquiries seem to make this commitment explicit. Answering an inquiry makes the inquiry closed. The collection of open inquiries corresponds to a research agenda. This paper depends on the development of a proper notion of contraction and we have given some thoughts about how this notion may be developed and understood in this framework.

14.2 Further research

As mentioned, this work depends on a proper development of the notion of contraction. We have made some assumptions about the notion here, but exactly how this should be done in the constructive type-theoretical framework will need further research. Because of persistence between stages, it would seem that contraction could be looked at as a special kind of expansion.

One project to continue here is the representation of goals. An agent seems to have some goals to achieve and I argued shortly that such goals may be represented as inquiries. The notion of how to represent a goal for an agent should be further developed. It would seem natural to look at how agents actually form their goals in order to make a good representation of them.

By introducing questions, inquiries and answers we can see that we are closing in on a dialogical interpretation of the system. It would be a very natural extension and the rules that I have presented here does not seem to pose any particular problems in such a framework. By introducing operators as inquiry and answer to constructive type theory, we see that game-theoretical semantics may be a good framework to interpret these operators in. A foundation for this have already been made by Rahman and Clerbout. [8]

In the dialogical framework for constructive type theory [8] we see that many operations look similar to the ones that I have introduced here. They are not the same, as they are used on a game-theoretical interpretation of the proposition level, while the operators that are introduced here are used on a judgement level. In this framework we find some other operators that may be very relevant in order to extend the notion of the type declaration inquiry. We may not only ask for the type declaration, but also for the canonical elements, the generation method or the equality rules for sets. [8, p. 17] This would seem like a natural way to extend our notion of type declaration inquiry to include more precise inquiries related to sets.

In this paper we argued that inquiries and answers should not be seen only as logical operators, but that they have some counterpart in natural language, that they actually correspond to some kinds of inquiry and answer in the normal sense. This is a notion that should be developed further. It is partly done by Ranta [7], but his approach has a stricter notion of question than we have developed here by the inquiry. This means that it would have to be explained in the terms of the operators that I have introduced here and not only in the terms of his operators. The operators that I have introduce seem to correspond to some questions, inquiries or answers in the normal sense, but they do not cover all aspects of them. To introduce new variants that would cover other aspects of interrogative acts would also be a very relevant topic to continue developing.

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