UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Analysis

On extremal function in one class of mapping with finite distortion

Bhagyashri Ingale

Spring 2017

Acknowledgments

Firstly, I would like to thank my supervisor Professor Irina Markina for her continuous helping hands to solve all difficulties as well as step-by-step guidance, support and patience to complete my thesis. Also I would like to thank Professor Alexander Vasil'ev for his guidance.

I also thankful to friendly and supportive environment of analysis group, specially Francesca Azzolini, Anja Eidsheim, Stine Marie Berge for their support and suggestions throughout the master studies.

I wish to express my sincere gratitude and respect to my family and friends for their unconditional love, moral support and eager throughout my study. Especially I would like to thank my husband Nilesh Ingale for his encouragement, guidance and endless help in my work.

Contents

Introduction

The notion of the module of a family of curves goes back to the works of L. Alhfors and A. Beurling [2], where they introduced the notion of extremal length to study the analytic functions of one complex variable. Later on it was discovered that the module of a family of curves is invariant under conformal transformation. Recall that the linear map given by the differential map of a conformal maps transforms circles to circles.

Two dimensional quasiconformal mappings were introduced by Grötzsch [4] in 1928. A rather comprehensive treatment can be found in [1, 8]. A homeomorphic quasiconformal mapping differs from a conformal mapping by the fact that the distortion coefficient is not equal to one anymore, but instead of being uniformly bounded. Geometrically it means that, at a point the derivative of a quasiconformal map transforms an infinitesimal circle to an infinitesimal ellipse for which the ratio between the major axis and the minor axis is bounded by the coefficient of quasiconformality. The quasiconformal maps due to it's more flexible structure are widely used in mathematics, for example in studies of certain elliptic partial differential equations [19], extremal problems on the Riemannian surfaces and Kleinian groups [18]. The modulus of a family of curves is invariant under the conformal maps but is not anymore invariant under the quasiconformal maps, but nevertheless it changes according to the coefficient of the quasiconformality.

Extremal problems of mappings of finite distortion was initiated by Astala, Iwaniec, Martin and Onninen in [6]. Whereas mappings of finite distortion are generalisations of quasiconformal mappings, these do not have uniform bound on their distortion. In the extremal problems for quasiconformal mappings, we minimize the maximal distortion, whereas for mappings of finite distortion we minimize the mean distortion functional. Zoltan M. Balogh, Katrin Fässler, and Ioannis D. Platis in [11] developed a separate method to study minimisation problems for the mean distortion functional in the class of finite distortion homeomorphisms by modulus of a family of curves. Furthermore, they went on to prove extremality of the spiral stretch mapping defined on annulus in the complex plane.

In 2005, Rodin's gives a method in [17] for finding the extremal function of the module of a extremal family of curves explicitly. In our thesis we consider minimization problems for the mean distortion functional in the class of finite distortion homeomorphisms by modulus of family of curve, in particular, we prove that extremal spiral stretch map in [11], can be obtained also by an explicit formula which is based on Rodin's theorem in

[17].

The extremal quasiconformal map f is a solution of the Beltrami equation with the Beltrami coefficient μ_f given by the function $k\frac{\overline{\varphi}}{|\varphi|}$, where k is arbitrary constant and $\varphi(\zeta)d\zeta^2$ is a holomorphic quadratic differential in a corresponding domain. The quasiconformal solutions of the Beltrami equations with this special Beltrami coefficients (associated to a quadratic differential) are called the Teichmüller maps. In [11], it is also shown that the quasiconformal spiral stretch map f_N in the class of finite distortion maps is the minimizer of the mean distortion functional. In our thesis we prove that the above extremal function f_N is a Teichmüller map, which has Beltrami coefficient of form

$$\mu_f(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|},$$

where $\varphi(z)dz^2$ is holomorphic quadratic differential. This relation to the extremal spiral stretch map f_N induces the length element given by $|\sqrt{\varphi}||dz| = \frac{k|dz|}{|z|}$. Moreover, the length element is up to a constant defined by the extremal metric $\rho = \frac{k|dz|}{|z|}$ associated to f_N.

The aim of the thesis is to study one example of an extremal problem in a class of quasiconformal homeomorphic mappings in [16] and also for class of finite distortion in [11], acting from an annulus domain to another annulus domain with some part of the boundary fixed. We explain how the extremal function in the class of homeomorphic mappings with finite distortion is related to the Teichmüller map by finding the corresponding quadratic differential.

In Chapter 1, we start with the basic idea and definitions on the theory of quasiconformal maps which can be found in [1, 8, 14] and with the definition of the modulus of a family of curves. Some basic definition required for modulus of a family of curves and some related theorem can be found [9]. Also we present examples of the module of some families of curves. In Chapter 2 we present an example of an extremal problem for quasiconformal maps and for mappings with finite distortion, for which we define spiral stretch map. In Chapter 3 will give some definitions that can be found in [3, 5, 12, 13, 15] and we prove the spiral stretch map is a Teichmüller map. In Chapter 4, lastly, in addition, we state Rodin's theorem for finding the some modules of a family of curves and extremal functions. Also we will give two explicit formulas for finding module and extremal function and we will prove that how modules of a family of curves and extremal functions obtained by using of Rodin's explicit formulas and method developed in [11] coincide with each other. We conclude the thesis by providing some relation between the shear map and the spiral map.

Chapter 1

Preliminary Introduction

In this chapter we will give an overview of the quasiconformal theory and present different notions of a quasiconformal map. We also show that notion of quasiconformality is originated from a notion of conformality. We define the modulus of a curve family, and give some example on modulus of family of curves.

1.1 Quasiconformal map

Quasiconformal mappings are generalisations of conformal mappings. They are less rigid than conformal mappings and are therefore more applicable. The concept of quasiconformal mappings was introduced by Grötzsch [4] in 1928. He introduced the first extremal problem which led to the notion of quasiconformality, where he considered the following problem: Let Q be a square and R be a rectangle (that is not a square). Then there is no a conformal mapping that maps vertices to vertices. Instead Grötzsch asked for "the mapping closest to being conformal homeomorphic mapping" $f\colon Q \to R$ that maps vertices to vertices.

Before we explain the Grötzsch problem by using simple example, see Example 1.14, we will give some necessary definitions.

Definition 1.1. Let Ω be an open set in $\mathbb{C}$. A map $f\colon \Omega \to \mathbb{C}$ is called **conformal** if f is bijective, holomorphic and there exists a homeomorphic extension $\tilde{f}$ to the boundary of the domain Ω.

Let $z \in \mathbb{C}$ be written as $z = x + iy$ and $w = f(z) = u + iv$. Assuming that f is differentiable in the variables (x, y) we obtain

$$df = du + idv = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) dy. \tag{1.1}$$

Using the relations $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, $dx = \frac{dz+d\bar{z}}{2}$, $dy = \frac{dz-d\bar{z}}{2i}$, equation (1.1) can be rewritten as

$$\begin{aligned} df &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\bar{z} \\ &= \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z} = f_z dz + f_{\bar{z}} d\bar{z}. \end{aligned} \tag{1.2}$$

The partial derivatives are

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.3}$$

Recall that if f is $\mathbb{R}$-differentiable at z_0, then there exists a matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ such that

$$f(z) \quad = \quad f(z_0) + \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(z - z_0).$$

If $f \in C^1$ at $z_0 = x_0 + iy_0$, then using the complex notation, we get

$$f(z) = f(z_0) + u_x(x - x_0) + u_y(y - y_0) + iv_x(x - x_0) + iv_y(y - y_0) + o(z - z_0). \tag{1.4}$$

We modify equation (1.4) as follows,

$$\begin{aligned} f(z) \quad &= \quad f(z_0) + \frac{u_x}{2}(z + \bar{z} - z_0 - \bar{z_0}) + \frac{u_y}{2i}(z - \bar{z} - z_0 + \bar{z_0}) + \\ &\qquad i\frac{v_x}{2}(z + \bar{z} - z_0 - \bar{z_0}) + i\frac{v_x}{2i}(z - \bar{z} - z_0 - \bar{z_0}) + o(z - z_0) \\ &= \quad f(z_0) + \left(\frac{u_x}{2} + \frac{u_y}{2i} + i\frac{v_x}{2} + \frac{v_y}{2}\right)(z - z_0) + \\ &\qquad \left(\frac{u_x}{2} - \frac{u_y}{2i} + i\frac{v_x}{2} - \frac{v_y}{2}\right)(\bar{z} - \bar{z_0}) + o(z - z_0). \end{aligned}$$

Thus

$$f(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\bar{z}}(z_0)(\bar{z} - \bar{z_0}) + o(z - z_0). \tag{1.5}$$

Here a function $f \in C^1$ can be well approximated near z_0 by

$$L(z) = f(z_0) + f_z(z_0)(z - z_0) + f_{\bar{z}}(z_0)(\bar{z} - \bar{z_0}).$$

Note that L can be written as the composition $T_{f(z_0)} \circ L_1 \circ T_{-z_0}$ where $T_{-z_0} = z - z_0$, $T_{f(z_0)} = z + f(z_0)$ are translations, and $L_1(z) = f_z(z_0)z + f_{\bar{z}}(z_0)\bar{z}$ is a linear map. If we consider f as a function of two variables, it's Jacobian at $z_0 = x_0 + iy_0$ is given by

$$J_f(z_0) = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f_z|^2 - |f_{\bar{z}}|^2.$$

Proposition 1.2. *Let f be an **orientation preserving function** that is $|f_z| > |f_{\bar{z}}|$. Then the linear map*

$$L_1(z) = f_z(z_0)z + f_{\bar{z}}(z_0)\bar{z} \tag{1.6}$$

maps circles to the ellipses.

Proof. For simplicity we first consider the function

$$L_{1\mu}(z) = z + \mu\bar{z}, \quad \text{where} \quad \mu(z_0) = \frac{f_{\bar{z}}(z_0)}{f_z(z_0)},$$

since f is orientation preserving so $f_z(z_0) \neq 0$. Then $L_1(z) = f_z(z_0)L_{1\mu}$. Consider a circle $x^2 + y^2 = r^2$ of radius r centred at the origin, or in polar coordinates $re^{i\theta}$.

Then

$$\begin{aligned} L_{1\mu}(z) &= z + \mu\bar{z} = x + iy + \mu x + i\mu y \\ &= (1+\mu)x + i(1+\mu)y \end{aligned} \tag{1.7}$$

maps (x, y) to $(u, v) = ((1+\mu)x, (1-\mu)y)$, which implies that $x = \frac{u}{1+\mu}$ and $y = \frac{v}{1-\mu}$. Hence, the image of the circle $x^2 + y^2 = r^2$ under $L_{1\mu}(z)$ is the ellipse $\left(\frac{u}{(1+\mu)r}\right)^2 + \left(\frac{v}{(1-\mu)r}\right)^2 = 1$ where $r(1-\mu)$ is the minor axis and $r(1+\mu)$ is the major axis. Our aim is to find the image under the function $L_1(z)$. We use polar coordinates to write $z = |z|e^{i\theta} = re^{i\theta}$, $f_z = |f_z|e^{i\theta_1}$, and $\mu = |\mu|e^{i\theta_2}$. Thus

$$\begin{aligned} L_1(z) &= f_z(z + \mu\bar{z}) = |f_z|e^{i\theta_1}\left(re^{i\theta} + |\mu|e^{i\theta_2}re^{-i\theta}\right) \\ &= |f_z|re^{i\left(\theta_1 + \frac{\theta_2}{2}\right)}\left(e^{i\left(\theta - \frac{\theta_2}{2}\right)} + |\mu|e^{-i\left(\theta - \frac{\theta_2}{2}\right)}\right). \end{aligned}$$

If we denote by $g(z) = |f_z|re^{i\left(\theta_1 + \frac{\theta_2}{2}\right)}$ and $h(z) = re^{-i\frac{\theta_2}{2}}$, then we observe that $g(z)$ is a rotation and a dilation and $h(z)$ is simply a rotation. Therefore, $L_1(z)$ becomes $L_1(z) = g \circ L_{1|\mu|} \circ h$ and it also maps a circle to an ellipse with the minor axis $|f_z|r(1-|\mu|)$ and the major axis $|f_z|r(1+|\mu|)|$, and the ratio of major axis to minor axis is $\frac{1+|\mu|}{1-|\mu|}$. □

Hence we conclude that f maps infinitesimal circles centred at z_0 to infinitesimal ellipses where the ratio of the major and minor axis is

$$D_f(z_0) := \frac{1 + |\mu|(z_0)}{1 - |\mu|(z_0)} = \frac{1 + \frac{|f_{\bar{z}}(z_0)|}{|f_z(z_0)|}}{1 - \frac{|f_{\bar{z}}(z_0)|}{|f_z(z_0)|}} = \frac{|f_z(z_0)| + |f_{\bar{z}}(z_0)|}{|f_z(z_0)| - |f_{\bar{z}}(z_0)|} \geq 1. \tag{1.8}$$

The number $D_f(z_0)$ is called the **dilatation (or distortion)** of f at z_0, and $\mu(z_0) = \frac{f_{\bar{z}}}{f_z}$ is called the **complex dilatation** of f at z_0.

Remark 1.3. We have $|\mu| = \frac{|f_z|}{|f_{\bar{z}}|} < 1$ since f is an orientation preserving maps, defined in Proposition 1.2. Moreover, from definition D_f we obtain $|\mu| = \frac{D_f - 1}{D_f + 1}$, therefore $D_f \leq K$ for some K if and only if $|\mu| \leq \frac{K-1}{K+1}$.

We now formulate a preliminary definition of a quasiconformal map.

Definition 1.4. A smooth map $f \colon \Omega \to \mathbb{C}$ is **quasiconformal** if $\tilde{k}_f := \sup_{z \in \Omega} D_f(z) < \infty$. We say that f is K **quasiconformal** if $\tilde{k}_f \leq K$.

Remark 1.5. The constant $\tilde{k}_f \geq 1$, since $D_f \geq 1$. Hence $K \geq \tilde{k}_f \geq 1$.
If f is conformal then we know $f_{\bar{z}} = 0$. Therefore, $D_f = 1$ which implies $\tilde{k}_f = 1$.

Here we shown one of the implication of the following theorem,

Theorem 1.6. *[1, page 16] A map f is conformal if and only if f is 1-quasiconformal.*

Remark 1.7. If f is quasiconformal, then the number $\tilde{k}_f$ measures how close f is to being conformal.

We can weaken the requirement of C^1 differentiability of f, and simply ask that f has distributional derivatives or it is from ACL class, the space of absolutely continuous functions on almost all lines parallel to the coordinate axes. Thus, if a map $f\colon \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ is smooth and

$$\frac{f_{\bar{z}}}{f_z} \leq k < 1,$$

then the map f is quasiconformal. Let $\mu\colon \Omega \to \mathbb{C}$ be an arbitrary measurable function such that $\|\mu\|_\infty = k < 1$, or in other words $\mu \in B(0,1) \subset L^\infty(\Omega)$.

Definition 1.8 (**Analytic definition of a quasiconformal map**)**.** Let Ω be an open set in $\mathbb{C}$. A map $f\colon \Omega \to \mathbb{C}$ is called quasiconformal if f is belongs in ACL in Ω, and satisfies the Beltrami equation

$$f_{\bar{z}} = \mu(z) f_z \tag{1.9}$$

almost everywhere in Ω. The homeomorphism f is said to be $\tilde{K}$-quasiconformal if

$$\| \mu(z) \|_\infty = \operatorname*{ess\,sup}_{z\in\Omega} |\mu(z)| \leq k < 1, \qquad \text{with} \quad \tilde{K} = \frac{1+k}{1-k}.$$

The coefficient $\tilde{K}$ is an analogous to the dilatation D_f. If f is conformal, then $\mu(z) \equiv 0$. The function μ is called the Beltrami coefficient and measures how far f is from being a conformal map at every point.

Theorem 1.9. *[3] Let $f\colon \Omega \to \mathbb{C}$ be a solution to the Beltrami equation (1.9). Assume that $f_{\bar{z}}, f_z \in L^2(\Omega)$. Denote by f^μ the solution of (1.9) in $\Omega = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that $f(0) = 0$, $f(1) = 1$, and $f(\infty) = \infty$. We say that f is* ***normalised at*** *$0, 1, \infty$. Then the set of normalised solutions f^μ is in one-to-one correspondence with the Beltrami coefficients from the open unit ball $B(0,1) \subset L^\infty(\bar{\mathbb{C}})$. Moreover, the solutions f^μ depend holomorphically on μ and for any $R > 0$ there exists $\delta > 0$ and a constant $C_R > 0$ such that*

$$|f^{t\mu}(z) - z - tF(z)| \leq C_R t^2 \quad \text{for} \quad |z| < R \quad \text{and} \quad |t| < \delta,$$

where

$$F(z) = -\frac{z(z-1)}{\pi} \int_{\mathbb{C}} \frac{\mu(\zeta)\, d\xi\, d\eta}{\zeta(\zeta-1)(\zeta-z)}, \qquad \zeta = \xi + i\eta.$$

We also have the following convergences, see [3].

1. *If $\{\mu_n(z)\}$ is a sequence in $B(0,1) \subset L^\infty(\bar{\mathbb{C}})$ converging to $\mu(z)$ pointwise almost everywhere and the f^{μ_n} and f^μ are the corresponding (unique) solutions to the Beltrami equations in $\bar{\mathbb{C}}$, then f^{μ_n} converges to f^μ uniformly on compact subsets of $\mathbb{C}$.*
2. *If $\{f_n\}$ is any sequence of normalised (at $0, 1, \infty$) quasiconformal homeomorphisms of $\bar{\mathbb{C}}$ such that their dilatations are uniformly bounded: $K(f_n) \leq K_0$ for every n, then there is a subsequence f_{n_k} converging uniformly on compact subsets of $\mathbb{C}$ to a normalised quasiconformal map f with $K(f) \leq K_0$.*

Let us introduce a geometric definition of quasiconformality, which does not require differentiability of f.

Definition 1.10. A **quadrilateral** $Q(z_1, z_2, z_3, z_4)$ is a Jordan domain with four distinct vertices z_1, z_2, z_3, z_4 with cyclic order on boundary.

Denote by $R(a, b)$ the rectangle with vertices $0, a, a + ib, ib$.

Theorem 1.11. *[14] For any quadrilateral $Q(z_1, z_2, z_3, z_4)$ there exist unique $a > 0$, $b > 0$, and a conformal map $h\colon Q(z_1, z_2, z_3, z_4) \to R(a, b)$ such that $h(z_1) = 0$, $h(z_2) = a$, $h(z_3) = a + ib$, and $h(z_4) = ib$.*

Remark 1.12. The rectangle $R(a, b)$ from Theorem 1.11 is called the **canonical rectangle** for $Q(z_1, z_2, z_3, z_4)$.
The number $\frac{a}{b}$ is called **conformal modulus** of the quadrilateral $Q(z_1, z_2, z_3, z_4)$ and is denoted by $M(Q(z_1, z_2, z_3, z_4)) := \frac{a}{b}$. We provide a more general definition of the modulus of a family of curves in Section 1.2. This construction immediately implies that if there is a conformal map from Theorem 1.11 $\varphi\colon Q(z_1, z_2, z_3, z_4) \to Q'(z'_1, z'_2, z'_3, z'_4)$, then both quadrilaterals have conformal modulus equal to $\frac{a}{b}$. This shows that the conformal modulus is invariant under the conformal transformations.

Definition 1.13 (Geometric definition of a quasiconformal map). Let Ω, Ω_1 be domains in the complex plane, $f\colon \Omega \to \Omega_1$ a homeomorphism, and $K \geq 1$ a given constant. The map f is called K **quasiconformal on** Ω if

$$M(f(Q)) \leq KM(Q), \tag{1.10}$$

for any quadrilateral Q, such that $\bar{Q} \subset \Omega$.

We now explain the Grötzsch problem by an example:

Example 1.14. Let Q be a square and R be a rectangle (not square). Consider a map $f\colon Q \to R$, such that

$$f(x, y) = (ax, y) = ax + iy,$$

where $a > 1$ or $0 < a < 1$, see Figure 1.1

- *Case $a > 1$*
  We compute $\frac{\partial f}{\partial x} = a$ and $\frac{\partial f}{\partial y} = i$, therefore, by equation (1.3) we obtain

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(a - ii) = \frac{a+1}{2}$$

$$|f_z| = \sqrt{f_z\overline{f_z}} = \sqrt{\left(\frac{a+1}{2}\right)\left(\frac{a+1}{2}\right)} = \frac{|a+1|}{2} = \frac{a+1}{2},$$

  since $a > 1 > 0$. Similarly, we get

$$|f_{\bar{z}}| = \frac{|a-1|}{2} = \frac{a-1}{2},$$

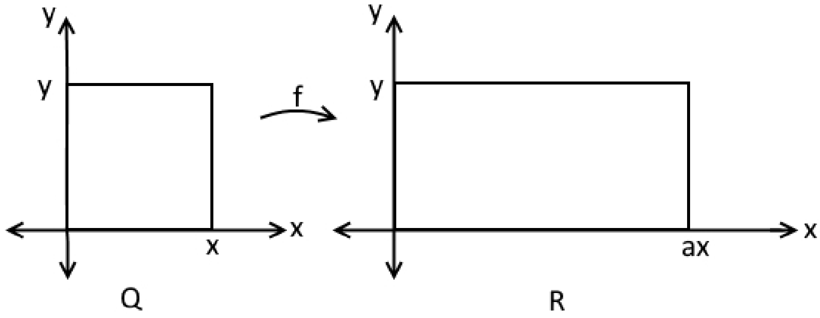


Figure 1.1

since $a > 1$. By Definition 1.8 of distortion we get

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{a+1+a-1}{a+1-a+1} = a > 1.$$

- *Case* $0 < a < 1$

  We compute

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(a - ii) = \frac{a+1}{2}$$

$$|f_z| = \sqrt{f_z\overline{f_z}} = \sqrt{\left(\frac{a+1}{2}\right)\left(\frac{a+1}{2}\right)} = \frac{|a+1|}{2} = \frac{a+1}{2},$$

  since $0 < a < 1$. Similarly, we get

$$|f_{\bar{z}}| = \frac{|a-1|}{2} = \frac{1-a}{2},$$

  because $0 < a < 1$. By Definition 1.8 of distortion we get

$$D_f = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} = \frac{a+1+1-a}{a+1-1+a} = \frac{1}{a} > 1.$$

Hence f is quasiconformal map, but not conformal by Theorem 1.6.

1.2 The module of a family of curves and extremal length

The dilatation of a quasiconformal map can be measured along curves, producing so called *linear dilatation*. This leads to the notion of extremal length and the length-area method, which is widely used in the theory of conformal and quasiconformal mappings on Riemann surfaces. The extremal length has its far going generalisation: the outer measure, called the module of a family of curves, that is one of the main tools in the theory of spacial quasiconformal, quasiregular mapping, mappings with finite distortion and other functional spaces requiring that the dilatation function $k(z)$ is an element of some integrability class.

1.2.1 Curves

We start by fixing the notation and listing some basic definitions

Definition 1.15. Let $I \subset \mathbb{R}^1$ be an interval, which is either open, closed or semi open. A continuous mapping $\gamma\colon I \to \mathbb{R}^n$, is called a **curve**. A curve is referred to as open or closed if I is open or closed, respectively.

Definition 1.16. Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a curve and let $a = t_0 \le t_1 \le \cdots \le t_k = b$ be a subdivision of $[a,b]$. The **length of curve** γ is defined by

$$l(\gamma) = \sup_{\gamma \in [a,b]} \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})|.$$

Remark 1.17. The length of the curve γ satisfies $0 \le l(\gamma) \le \infty$ and $l(\gamma) = 0$ if and only if γ is constant.

Definition 1.18. If $l(\gamma) < \infty$, then $\gamma\colon [a,b] \to \mathbb{R}^n$ is called a **rectifiable curve**. If $l(\gamma) = \infty$, then γ is called a **non-rectifiable curve**.

Definition 1.19. A curve $\gamma\colon I \to \mathbb{R}^n$ is called **locally rectifiable** if γ restricted to each closed subinterval of I is rectifiable.

Theorem 1.20. *[9, page 8] If $\gamma\colon (a,b) \to \mathbb{R}^n$ is absolutely continuous on every closed subinterval of (a,b), then*

$$l(\gamma) = \int_a^b \left|\frac{d\gamma(t)}{dt}\right| dt.$$

Definition 1.21. The curve $\gamma^0\colon [0, l(\gamma)] \to \mathbb{R}^n$ is called the **normal representation** of γ, or the **parametrization of** γ by means of **arc length**.

Definition 1.22. For each rectifiable curve $\gamma\colon [a,b] \to A$, where $A \subset \mathbb{R}^n$ is a Borel set, we define the **line integral over** γ of a non-negative Borel function $\rho\colon A \to \mathbb{R}$, by

$$\int_\gamma \rho\, ds = \int_0^{l(\gamma)} \rho(\gamma^0(t))\, dt,$$

where γ^0 is the normal representation of γ.

Theorem 1.23. *[9, page 9] If $\gamma\colon [a,b] \to A$ is absolutely continuous, then*

$$\int_\gamma \rho\, ds = \int_a^b \rho(\gamma(t)) \left|\frac{d\gamma(t)}{dt}\right| dt.$$

Remark 1.24. Let $\gamma\colon I \to \Omega$, $\Omega \subset \mathbb{C}$, Then we use complex notation and write

$$\int_\gamma \rho ds = \int_\gamma \rho |dz|.$$

1.2.2 Extremal length and module of a family of curves

Definition 1.25. Let Γ be a family of locally rectifiable curves in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A function ρ is said to be an **admissible metric** if $\rho \geq 0$ is measurable, and

$$A(\rho) = \iint \rho^2 \, d\mathcal{L}^2(z) \neq 0, \infty.$$

The length of $\gamma \in \Gamma$ with respect to ρ is

$$L_\gamma(\rho) = \int_\gamma \rho \, |dz|.$$

We denote

$$L(\rho) := \inf_{\gamma \in \Gamma} L_\gamma(\rho).$$

The **extremal length** is defined by

$$\lambda(\Gamma) := \sup_\rho \frac{L(\rho)^2}{A(\rho)},$$

where the supremum is taken over all admissible metrics ρ.

Note that $L_\gamma(1)$ coincides with the length $l(\gamma)$ defined Definition in 1.16, because of Theorem 1.20.

Definition 1.26. Let Γ be a family of locally rectifiable curves in $\Omega \subset \mathbb{R}^n$. We define

$$F(\Gamma) \colon = \left\{ \rho \colon \mathbb{R}^n \to \mathbb{R} \colon \ \rho \text{ is non-negative Borel function, and } \int_\gamma \rho \, ds \geq 1 \text{ for all } \ \gamma \in \Gamma \right\}.$$

We call $\rho \in F(\Gamma)$ an **admissible function**. For each $p \geq 1$ we set

$$M_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \iint \rho^p \, d\mathcal{L}^2(z),$$

where $d\mathcal{L}^2(z)$ is the two dimensional Lebesgue measure. The number $M_p(\Gamma)$ is called the **p-module of the family** Γ. If $p = n$, then $M_n(\Gamma)$ is called **conformal module of** Γ. In the our thesis we only consider the case $p = 2$.

Remark 1.27. The extremal length of Γ is equal to $\frac{1}{M_2(\Gamma)}$ for a family of curves Γ in $\Omega \subset \overline{\mathbb{C}}$.

We can exemplify $\lambda(\Gamma) = \frac{1}{M_2(\Gamma)}$ in the case of mappings from quadrilateral to rectangle by making use the geometric definition of quasiconformality.

We consider a quasiconformal map f mapping the quadrilateral $Q(z_1, z_2, z_3, z_4)$ onto the rectangle $R = \{u + iv, 0 < u < a, 0 < v < b\}$. Then

$$\iint_Q |f'(z)|^2 \, dxdy = ab.$$

Let Γ be the family of all locally rectifiable Jordan curves in Q which joins the sides (z_1, z_2) and (z_3, z_4). Then

$$\int_\gamma |f'(z)| \, |dz| \geq b$$

for every $\gamma \in \Gamma$, with equality if γ is the inverse image of a vertical line segment of R joining its horizontal sides. Hence

$$\frac{1}{\lambda(\Gamma)} = \frac{\iint_Q |f'(z)|^2 \, dxdy}{\inf_{\gamma \in \Gamma} \left(\int_\gamma |f'(z)| \, |dz| \right)^2} = \frac{ab}{b^2} = \frac{a}{b}. \tag{1.11}$$

We know $M_2(Q) = \frac{a}{b}$. Here we can get rid of the chosen quasiconformal map f and introduce the set $F(\Gamma)$, whose elements ρ are non negative, Borel measurable functions on $Q(z_1, z_2, z_3, z_4)$ and $\int_\gamma \rho |dz| \geq 1$ for every $\gamma \in F(\Gamma)$. Thus we will obtain

$$\inf_{\rho \in F(\Gamma)} \iint_Q \rho^2 dxdy = M_2(\Gamma)$$

and by equation (1.11). Hence we conclude that $\lambda(\Gamma) = \frac{1}{M_2(\Gamma)}$.

Theorem 1.28. *[9, page 16] The p-module of a family of curves for $p \geq 1$ has the following properties,*

1) $M_p(\emptyset) = 0$.

2) If $\Gamma_1 \subset \Gamma_2$, then $M_p(\Gamma_1)) \leq M_p(\Gamma_2)$.

3) $M_p\left(\bigcup_{k=1}^{\infty} \Gamma_k\right) \leq \sum_{k=1}^{\infty} M_p(\Gamma_k)$.

This shows that the modulus is an outer measure on a set of continuous maps on an interval.

Theorem 1.29. *For a map $f\colon \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ the following is true:*

1) [1, page 11 with $K = 1$] If f is conformal, then $M_2(\Gamma) = M_2(f(\Gamma))$;

2) [1, page 11] If f is K quasiconformal, then $\frac{1}{K} M_2(\Gamma) \leq M_2(f(\Gamma)) \leq K M_2(\Gamma)$.

Remark 1.30. [8] Module of a family of non-rectifiable curves $\tilde{\Gamma}$ is zero that is $M_2(\tilde{\Gamma}) = 0$.

1.2.3 Examples of the module of some families of curves.

We provide a couple of examples on how one can numerically compute module of families of curves. We, moreover, show $M_2(\Gamma) = M_2(f(\Gamma))$ for a conformal map f.

Example 1.31. Let $R = \{(x, y) \in \mathbb{R}^2, 0 \leq x \leq \alpha, 0 \leq y \leq \beta\}$. We want to calculate the module $M_2(\Gamma)$ of a family of locally rectifiable curves Γ connecting horizontal sides of the rectangle R, see Figure 1.2

Thus, we need to calculate

$$M_2(\Gamma) = \inf \left\{ \iint_R \rho^2 \, d\mathcal{L}^2(z) \, ; \text{ over all } \rho \in F(\Gamma) \right\}.$$

For the function $\rho = \frac{1}{\beta}$ and an arbitrary $\gamma \in \Gamma$, we obtain from Definition 1.21 and Theorems 1.20 and 1.23,

$$\int_\gamma \rho \, ds = \int_0^{l(\gamma)} \frac{dt}{\beta} = \frac{1}{\beta} \int_0^{l(\gamma)} dt = \frac{l(\gamma)}{\beta} \geq 1. \tag{1.12}$$

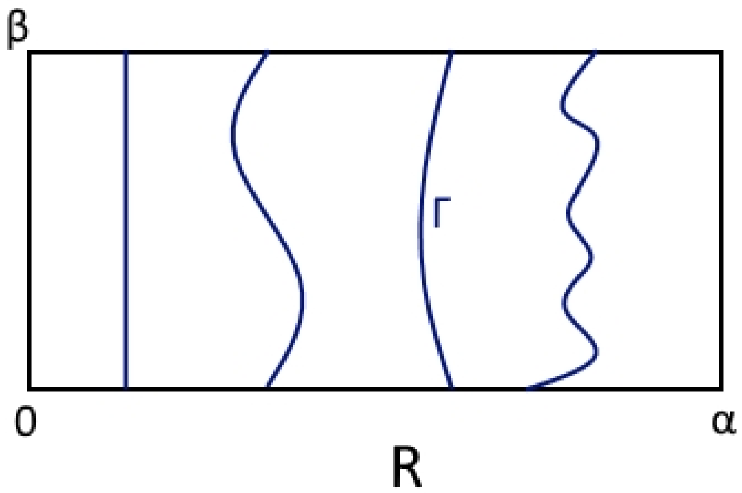


Figure 1.2

since $\beta = l(\gamma_0)$, where γ_0 is a vertical segment connecting horizontal sides of the rectangle R. Relation (1.12) implies that the function $\rho = \frac{1}{\beta}$ is admissible for Γ. Let $\Gamma_0 \subset \Gamma$ be a family of vertical segments connecting horizontal sides of R. Then for $\rho = \frac{1}{\beta}$, and for any $\gamma_0 \in \Gamma_0$ we obtain $\int_{\gamma_0} \rho\, ds = 1$ equation from (1.12). Moreover, for any $\rho \in F(\Gamma)$, and for any $\gamma \in \Gamma$ such that $\gamma \colon [a, b] \to \mathbb{R}$ we get

$$\begin{aligned} 1 \quad &\leq \quad \int_\gamma \rho\, ds = \int_a^b \rho(\gamma(t))\Big|\frac{d\gamma(t)}{dt}\Big|\, dt = \int_a^b \rho(\gamma(t))\Big|\frac{d\gamma(t)}{dt}\Big|^{\frac{1}{2}}\,\Big|\frac{d\gamma(t)}{dt}\Big|^{\frac{1}{2}}\, dt \\ &\leq \quad \left(\int_a^b \rho^2(\gamma(t))\Big|\frac{d\gamma(t)}{dt}\Big| dt\right)^{\frac{1}{2}} \left(\int_a^b \Big|\frac{d\gamma(t)}{dt}\Big| dt\right)^{\frac{1}{2}} \leq \left(\int_\gamma \rho^2\, ds\right)^{\frac{1}{2}} l(\gamma)^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality and Theorem 1.20. Thus we obtain

$$\frac{1}{l(\gamma)} \leq \int_\gamma \rho^2\, ds,$$

by squaring and dividing by $l(\gamma)$. Particularly, for $\gamma_0 \in \Gamma_0$ we deduce

$$\frac{1}{\beta} \leq \int_0^\beta \rho^2(t)\, dt. \tag{1.13}$$

Integrating both parts of inequality (1.13) over $[0, \alpha]$, we obtain

$$\int_0^\alpha \frac{1}{\beta} ds = \frac{\alpha}{\beta} \leq \int_0^\alpha \int_0^\beta \rho^2(t)\, dt\, ds = \iint_R \rho^2\, d\mathcal{L}^2(z) \tag{1.14}$$

for any $\rho \in F(\Gamma_0)$. Thus we conclude that $\frac{\alpha}{\beta} \leq M_2(\Gamma_0)$ by taking infimum over all functions $\rho \in F(\Gamma_0)$ in equation (1.14). Hence we obtain a lower bound for $M_2(\Gamma)$:

$$\frac{\alpha}{\beta} \leq M_2(\Gamma_0) \leq M_2(\Gamma) \tag{1.15}$$

by Theorem 1.28. Now we find an upper bound for $M_2(\Gamma)$. Since $\rho_0 = \frac{1}{\beta}$ is admissible for Γ, we have

$$M_2(\Gamma) = \inf_{\rho \in F(\Gamma)} \iint_R \rho^2 \, d\mathcal{L}^2(z) \leq \iint_R \rho_0^2 \, d\mathcal{L}^2(z) = \frac{1}{\beta^2}\, \alpha\, \beta = \frac{\alpha}{\beta}. \tag{1.16}$$

Hence we obtain

$$\frac{\alpha}{\beta} \leq M_2(\Gamma) \leq \frac{\alpha}{\beta}$$

from equation (1.15) and equation (1.16), which implies $M_2(\Gamma) = \frac{\alpha}{\beta}$.

Remark 1.32. If we consider a family of locally rectifiable curves Σ connecting vertical sides of the rectangle R, see Figure 1.3.

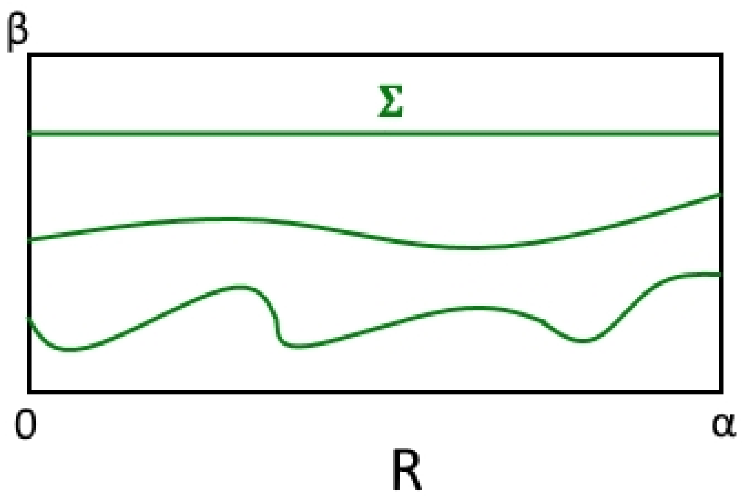


Figure 1.3

Then we obtain from an analogous way as in Example 1.31 $M_2\,(\Sigma) = \frac{\beta}{\alpha}$. This implies $M_2(\Gamma) M_2\,(\Sigma) = 1$.

Example 1.33. Let $A = \{z = re^{i\theta} \colon r_1 < |z| = r < r_2, 0 < \theta \leq 2\pi\}$ and let Γ be the family of all locally rectifiable curves $\gamma \colon [r_1, r_2] \to A$, joining the boundaries in annulus A, see Figure 1.4.

For the function $\rho = \frac{1}{r\,\log(\frac{r_2}{r_1})}$ and an arbitrary $\gamma \in \Gamma$, we obtain

$$\begin{aligned} \int_\gamma \rho \, ds &= \int_\gamma \frac{dt}{r\,\log\left(\frac{r_2}{r_1}\right)} = \frac{1}{\log\left(\frac{r_2}{r_1}\right)} \int_\gamma \frac{dt}{r} = \frac{1}{\log\left(\frac{r_2}{r_1}\right)} \int_{r_1}^{r_2} \frac{|\dot{\gamma}(t)|}{|\gamma(t)|}\, dt \\ &= \frac{1}{\log\left(\frac{r_2}{r_1}\right)} \int_{r_1}^{r_2} \left|\frac{\dot{\gamma}(t)}{\gamma(t)}\right| dt \geq \frac{1}{\log\left(\frac{r_2}{r_1}\right)} \left|\int_{r_1}^{r_2} \frac{\dot{\gamma}(t)}{\gamma(t)}\, dt\right| \\ &= \frac{1}{\log\left(\frac{r_2}{r_1}\right)} \left|\log\left(\frac{r_2}{r_1}\right)\right| = 1. \end{aligned} \tag{1.17}$$

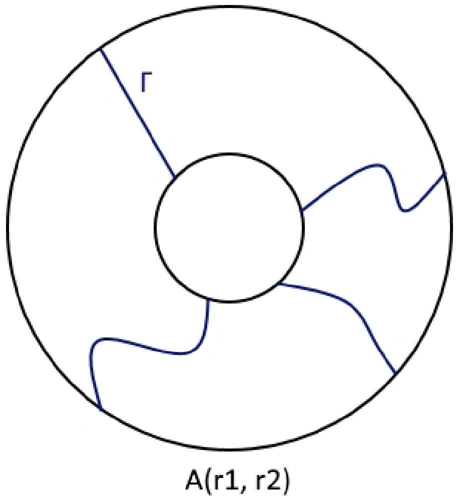


Figure 1.4

We deduce that the function $\rho = \frac{1}{r\,\log(\frac{r_2}{r_1})}$ is admissible for Γ. Let $\Gamma_0 \subset \Gamma$, where Γ_0 is the family of radial curves connecting the boundaries $|z| = r_1$ and $|z| = r_2$ in the annulus A. Let $\gamma_0 \in \Gamma_0$ and consider the parametrization of $\gamma_0 = (x_1, x_2)$ given by $(x_1 = t\cos\theta,\ x_2 = t\sin\theta)$, $t \in [r_1, r_2]$. Then $\frac{d\gamma_0(t)}{dt} = \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}\right) = (\cos\theta, \sin\theta)$ and

$$\left|\frac{d\gamma_0(t)}{dt}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1. \tag{1.18}$$

If we set function $\rho = \frac{1}{r(\log\left(\frac{r_2}{r_1}\right))}$ in Theorem 1.23 and use equation (1.18), then we get

$$\int_{\gamma_0} \rho\, ds = \int_{r_1}^{r_2} \frac{1}{r\,\log\left(\frac{r_2}{r_1}\right)}\, dt = \frac{1}{\log\left(\frac{r_2}{r_1}\right)} \int_{r_1}^{r_2} \frac{1}{r}\, dt = 1,$$

for any $\gamma_0 \in \Gamma_0$. Moreover

$$1 \le \int_\gamma \rho\, dr = \int_{r_1}^{r_2} \rho(\gamma(r))\left|\frac{d\gamma(r)}{dr}\right| dr = \int_{r_1}^{r_2} \rho(re^{i\theta})\, dr = \int_{r_1}^{r_2} \rho(re^{i\theta}) r^{\frac{1}{2}} r^{\frac{-1}{2}}\, dr$$

for any $\rho \in F(\Gamma)$ and for any $\gamma \in \Gamma$ such that $\gamma\colon [r_1, r_2] \to A$. When we integrate over $[0, 2\pi]$ and obtain

$$\int_0^{2\pi} 1 d\theta \le \int_0^{2\pi} \int_{r_1}^{r_2} \rho(re^{i\theta}) r^{\frac{1}{2}} r^{\frac{-1}{2}}\, dr d\theta.$$

Applying the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} 2\pi &\le \left(\int_0^{2\pi}\int_{r_1}^{r_2} \rho^2(re^{i\theta}) r\, dr d\theta\right)^{1/2} \left(\int_0^{2\pi}\int_{r_1}^{r_2} r^{-1}\, dr d\theta\right)^{1/2} \\ &= \left(\int_0^{2\pi}\int_{r_1}^{r_2} \rho^2\, r dr d\theta\right)^{1/2} \left(2\pi \log\left(\frac{r_2}{r_1}\right)\right)^{1/2}. \end{aligned}$$

Squaring both sides we obtain

$$4\pi^2 \leq \left(\int_0^{2\pi} \int_{r_1}^{r_2} \rho^2 \, r dr d\theta \right) \left(2\pi \log \left(\frac{r_2}{r_1} \right) \right).$$

We then deduce

$$\frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)} \leq \iint_A \rho^2 \, d\mathcal{L}^2(z) \quad \text{and} \quad \frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)} \leq M_2(\Gamma) \tag{1.19}$$

by taking infimum over all function $\rho \in F(\Gamma)$. Since the function $\rho_0 = \frac{1}{r \log\left(\frac{r_2}{r_1}\right)}$ is admissible for Γ, we also have

$$M_2(\Gamma) = \inf_{\rho \in F(\Gamma)} \iint \rho^2 \, d\mathcal{L}^2(z) \leq \iint_A \rho_0^2 \, d\mathcal{L}^2(z) = \frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)}, \tag{1.20}$$

from equation (1.17). Hence from equation (1.19) and equation (1.20) we conclude that $M_2(\Gamma) = \frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)}$.

Remark 1.34. If we consider a family of locally rectifiable curves Σ separating the boundaries in annulus A, see Figure 1.5,

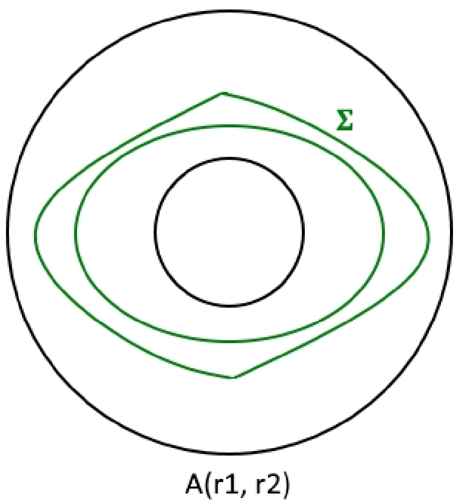


Figure 1.5

then we get $M_2(\Sigma) = \frac{\log\left(\frac{r_2}{r_1}\right)}{2\pi}$. This implies that we have $M_2(\Gamma) M_2(\Sigma) = 1$.

Example 1.35. In this example we show that module is invariant under a conformal map. Let $A = \{z = re^{i\theta} \colon 1 < |z| = r < e^{\alpha}, 0 < \theta \leq 2\pi\}$. We consider a family Γ of locally rectifiable curves connecting the boundaries in an annulus A, see Figure 1.6. By using Example 1.33 we obtain

$$M_2(\Gamma) = \frac{2\pi}{\log\left(\frac{e^{\alpha}}{1}\right)} = \frac{2\pi}{\alpha}. \tag{1.21}$$

Now, consider a conformal mapping $f\colon A \to f(A)$ given by $f(z) = \log z$. If $z = re^{i\theta}$ then $f(re^{i\theta}) = \log(re^{i\theta}) = \log r + i\theta$. When $r = 1$ then $f(re^{i\theta}) = i\theta$ and when $r = e^{\alpha}$ then $f(re^{i\theta}) = \alpha + i\theta$. This means that the image $f(A)$ is the rectangle $R = \{(x, y) \in \mathbb{R}^2, 0 \leq x \leq \alpha, 0 \leq y \leq 2\pi\}$. We also obtain that the image $f(\Gamma)$ is the family of locally rectifiable curves connecting the vertical sides of rectangle R see Figure 1.6. So from Example 1.31

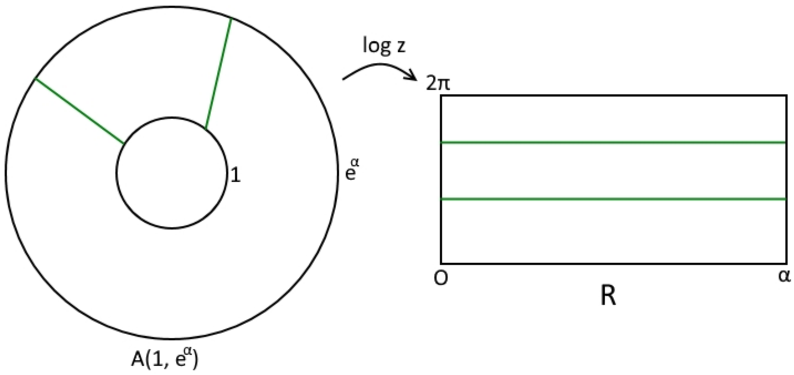


Figure 1.6

we know

$$M_2(f(\Gamma)) = \frac{2\pi}{\alpha} = \frac{2\pi}{\log\left(\frac{e^{\alpha}}{1}\right)}. \tag{1.22}$$

From equation (1.21) and equation (1.22) we conclude

$$M_2(\Gamma) = M_2(f(\Gamma)).$$

Chapter 2

Extremal problems for functional spaces

In this Chapter we describe Grötzsch's problem for quasiconformal maps formulated for annula domains. Moreover, we study the extremal problem for quasiconformal maps and for mappings with finite distortion.

2.1 Classes of mappings

We summarise the following properties of the homeomorphic maps $f\colon \Omega \to \mathbb{C}$ defined on some open set $\Omega \subset \mathbb{C}$. Let $f_z,\ f_{\bar{z}} \in \mathrm{L}^2_{loc}(\Omega)$. Assume that there exists a measurable function $k\colon \Omega \to \mathbb{R}^+$ such that

$$\|Df(z)\|^2 \leq k(z)J(f,z)a.e., \tag{2.1}$$

$$\text{where} \quad J(f,z) = \det(Df(z)), \quad ||Df(z)|| = \max\{|Df(z)\nu| : |\nu| = 1\}.$$

Then

1) $f\colon \Omega \to \mathbb{C}$ is **conformal** if $k(z) = 1$.
2) $f\colon \Omega \to \mathbb{C}$ is **K quasiconformal** if $k(z)$ is uniformly bounded, that is there is a constant $K \geq 1$ such that $|k(z)| \leq K$ for all $z \in \Omega$.
3) $f\colon \Omega \to \mathbb{C}$ is called of **finite distortion** if $k(z)$ is finite, that is $k(z) < \infty$ for all $z \in \Omega$.

Definition 2.1. The **linear distortion** function for f is defined by

$$k(z,f)\colon = \begin{cases} \frac{\|Df(z)\|^2}{J(f,z)} & \text{if} \quad J(f,z)\ > 0 \\ 1 & \text{otherwise} \end{cases} = \begin{cases} \frac{|f_z(z)|+|f_{\bar{z}}(z)|}{|f_z(z)|-|f_{\bar{z}}(z)|} & \text{if} \quad J(f,z)\ > 0, \\ 1 & \text{otherwise.} \end{cases}$$

We need to show that $\frac{\|Df(z)\|^2}{J(f,z)} = \frac{|f_z(z)|+|f_{\bar{z}}(z)|}{|f_z(z)|-|f_{\bar{z}}(z)|}$.
We know that

$$Df(z)\nu = f_z\nu + f_{\bar{z}}\bar{\nu} = Df(z)e^{i\theta} = f_z(z)e^{i\theta} + f_{\bar{z}}(z)e^{-i\theta}$$

since $\nu = |\nu|e^{i\theta}$, and the Jacobian $J(f,z) = |f_z|^2 - |f_{\bar{z}}|^2$. Also $f_z, f_{\bar{z}}$ can be written as $f_z = |f_z|e^{i\alpha}$ and $f_{\bar{z}} = |f_{\bar{z}}|e^{i\beta}$. Therefore,

$$\begin{aligned} |Df(z)e^{i\theta}| &= |\,|f_z(z)|e^{i\alpha}e^{i\theta} + |f_{\bar{z}}(z)|e^{\beta}e^{-i\theta}| \\ &= |\,|f_z(z)|e^{i(\theta+\alpha)} + |f_{\bar{z}}(z)|e^{i(\beta-\theta)}|. \end{aligned} \tag{2.2}$$

Now, $|Df(z)|$ has maximum length when $e^{i(\theta+\alpha)} = e^{i(\beta-\theta)}$,

$$\begin{aligned} \|Df(z)\| &= \max_{|\nu|=1} |Df(z)| \\ &= |\,|f_z(z)|e^{i(\theta+\alpha)} + |f_{\bar{z}}(z)|e^{i(\theta+\alpha)}| \\ &= |f_z(z)| + |f_{\bar{z}}(z)|. \end{aligned}$$

Hence we get,

$$\begin{aligned} \frac{\|Df(z)\|^2}{J(f,z)} &= \frac{(|f_z(z)| + |f_{\bar{z}}(z)|)^2}{|f_z(z)|^2 - |f_{\bar{z}}(z)|^2} \\ &= \frac{(|f_z(z)| + |f_{\bar{z}}(z)|)^2}{(|f_z(z)| + |f_{\bar{z}}(z)|)(|f_z(z)| - |f_{\bar{z}}(z)|)} \\ &= \frac{(|f_z(z)| + |f_{\bar{z}}(z)|)}{(|f_z(z)| - |f_{\bar{z}}(z)|)}. \end{aligned}$$

Definition 2.2. The **maximal distortion** of a conformal or quasiconformal map is given by $k_f := \operatorname{ess\,sup}_{z\in\Omega} k(z,f)$.

2.2 One extremal problem for quasiconformal maps

We state the Grötzsch problem for quasiconformal maps formulated for annula domains. Let $A(q,1) = \{z \mid q < |z| < 1\}$ and $A(q^{k_1},1) = \{z \mid q_1^k < |z| < 1\}$ be two ring domains. There are no conformal map from $A(q,1)$ to $A(q^{k_1},1)$ unless $k_1 = 1$. It can be proved by using the conformal invariance of the modulus of the family of curves Γ connecting the circles: $M_2(\Gamma) = -\frac{2\pi}{\log q}$ for $A(q,1)$ and $M_2(\Gamma) = -\frac{2\pi}{k_1 \log g}$ for $A(q^{k_1},1)$.

Grötzsch's problem is to find the most nearly conformal mapping from $A(q,1)$ to $A(q^{k_1},1)$. There is a question about what is the best way to define "most nearly conformal". One way is to ask for a mapping with the smallest possible maximal dilation, which is the more classical one and it is difficult to generalise. A second way is to ask for a mapping with the minimum average dilatation. This can be applied not only to quasiconformal mappings but also to the mappings with finite distortion or mappings having other restrictions for the dilatation. We illustrate the second approach for two different classes of the mappings. In both cases the solution in the class of quasiconformal maps is unique (up to postcomposition by conformal self-mapping of $A(q^{k_1},1)$) if the Beltrami coefficient has a special form: namely $\mu_f = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}$. It is an analogue of the Teichmüller uniqueness theorem.

We consider a mapping from the annulus $A(q,1)$ to the annulus $A(q^{k_1},1)$, $k_1 > 0$, in the complex plane which preserves the outer boundary $|z| = 1$ of annula $A(q,1)$ and $A(q^{k_1},1)$ and will rotates by an angle the inner boundary.

Definition 2.3. We denote by $\mathcal{C}$ the class of quasiconformal homeomorphisms mappings $f\colon A(q,1) \to A(q^{k_1},1)$ with $f_z, f_{\bar{z}} \in L^2_{Loc}(A(q,1))$ with $k(z,f) \subset L^1(A(q,1))$ and extension to the boundary

$$f(z) = \begin{cases} z, & |z| = 1 \\ q^{k_1-1} z e^{i\theta}, & |z| = q, \end{cases} \tag{2.3}$$

where $\theta \in [-\pi, \pi]$.

The maximal distortion k_f in Definition 2.2, is finite for quasiconformal mappings. To find a function that minimize this maximal distortion is the classical problem [16]. Let $\mathcal{C}$ be as in Definition 2.3. Then the minimization problem can be written as:

$$\textit{Find } f_0 \in \mathcal{C} \textit{ such that } k_{f_0} \le k_f \textit{ for all } f \in \mathcal{C} \textit{ and } z \in \Omega,$$

or

$$\textit{Find } f_0 \in \mathcal{C} \textit{ such that } \int_{A(q,1)} \Psi(k_{f_0})\rho^2(z)d\mathcal{L}^2(z) \le \int_{A(q,1)} \Psi(k_f)\rho^2(z)d\mathcal{L}^2(z) \textit{ for all } f \in \mathcal{C},$$

where Ψ is a continuous function and ρ is some measurable non negative function. We can solve this problem by following five steps. Consider a map $f\colon A(q,1) \to A(q^{k_1},1)$ in the class $\mathcal{C}$. We denote by "z" the plane containing $A(q,1)$ and by "w" the plane containing $A(q^{k_1},1)$.

Step 1 Let $\tilde{\gamma}$ be a radial segment in the w-plane. We find a curve $\tilde{\Gamma}$ in the z-plane which is the pre-image of the curve $\tilde{\gamma}$, that is $f(\tilde{\Gamma}) = \tilde{\gamma}$, see Figure 2.1.

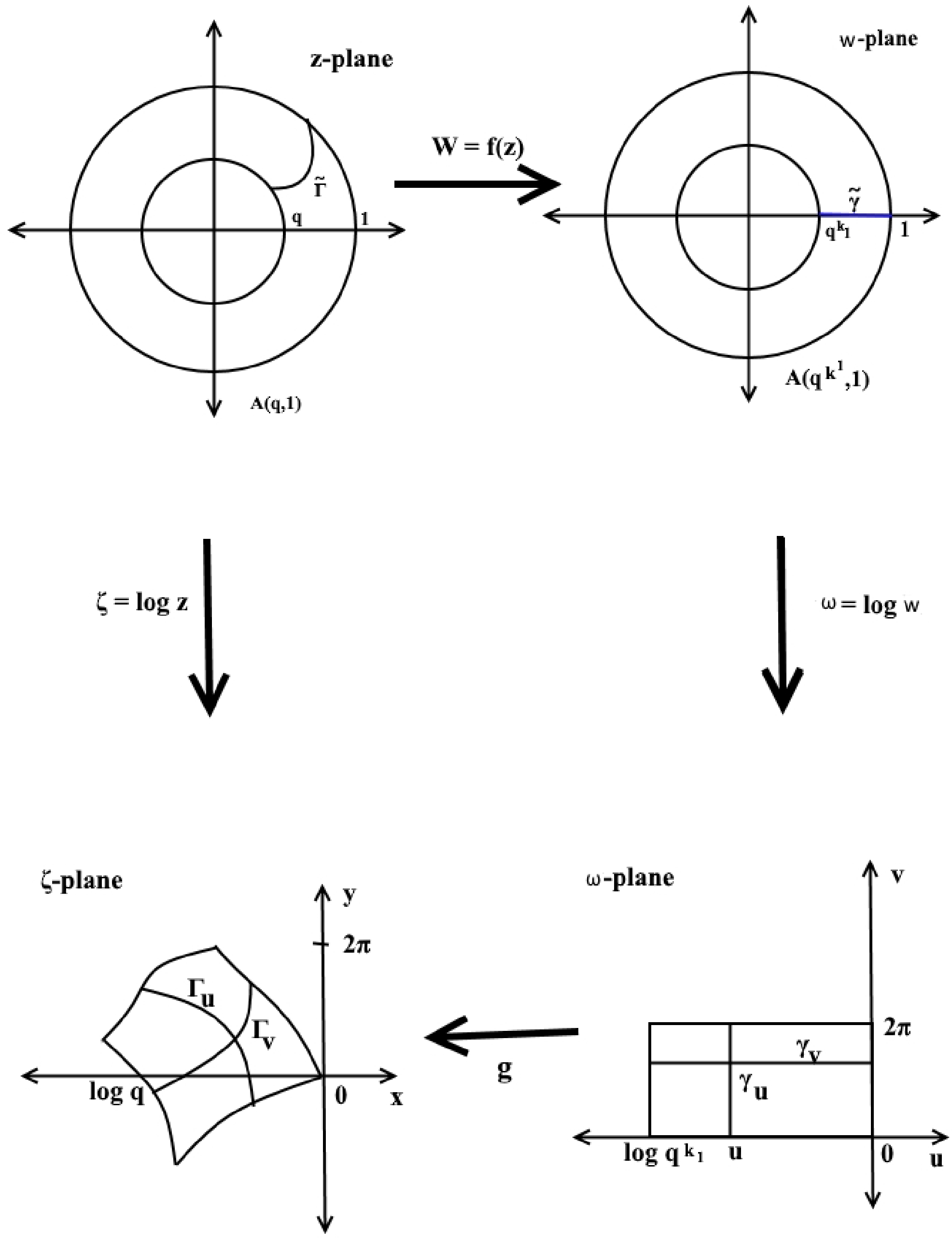

z-plane
W = f(z)
w-plane
q
1
A(q,1)
1
A(q k1,1)
ζ = log z
ω = log w
ζ-plane
y
2π
Γu
Γv
log q
0
x
g
ω-plane
v
2π
γv
γu
log q k1
u
0
u

Figure 2.1

We move from the z-plane to a new ζ-plane by using the map $\zeta = \log z$, where $\zeta = x + iy$, by the concept of branch cut. Similarly by branch cut we move from the w-plane to the ω-plane by using map $\omega = \log w$, where $\omega = u + iv$, see Figure 2.1. Now we consider the images $\zeta(\tilde{\Gamma}) = \Gamma$ and $\omega(\tilde{\gamma}) = \gamma$ as it is shown in Figure 2.1.

Step 2 Let $\gamma_u(v)$ be a vertical curve in the ω-plane for some fixed value $u \in [\log q^{k_1}, 0]$ with $0 \le v \le 2\pi$ as a parameter. The curve $\gamma_u(v)$ has length 2π. Consider the image $\Gamma_u(v)$ in ζ-plane, and observe that the length of $\Gamma_u(v)$ is always greater than 2π. Therefore, we obtain the following relation

$$2\pi \le l(\Gamma_u(v)) = \int_0^{2\pi} \left| \frac{\partial \Gamma_u(v)}{\partial v} \right| dv, \tag{2.4}$$

by using Theorem 1.20.

Step 3 Applying the Holder inequality and squaring, we get

$$4\pi^2 \le 2\pi \left(\int_0^{2\pi} \left| \frac{\partial \Gamma_u(v)}{\partial v} \right|^2 dv \right).$$

By using relation (2.1), we obtain

$$2\pi \le \int_0^{2\pi} k(\omega) J(g, \omega) dv,$$

where $k(\omega)$ is w.r.t. to g in equation (2.1). Now we integrate on the interval $[\log q^{k_1}, 0]$ and use a change of variable. We obtain

$$2\pi \log \left(\frac{1}{q^{k_1}} \right) \le \iint_{A(q,1)} k(z, f) \frac{1}{|z|^2} \, d\mathcal{L}^2(z).$$

Subtracting $2\pi \log(\frac{1}{q}) = \iint_{A(q,1)} \frac{1}{|z|^2} d\mathcal{L}^2(z)$ from both sides, we deduce the following inequality

$$-\frac{1}{2\pi} \iint_{A(q,1)} \frac{k(z, f) - 1}{|z|^2} \, d\mathcal{L}^2(z) \le \log \left(\frac{q^{k_1}}{q} \right). \tag{2.5}$$

Step 4 Now we take a horizontal curve γ_v in ω-plane and obtain Γ_v in ζ-plane. We observe that

$$\log \left(\frac{1}{q} \right) \le \log \left(\frac{1}{q^{k_1}} \right) \le l(\Gamma_v(u)) = \int_{\log q^{k_1}}^{0} \left| \frac{\partial d(\Gamma_v(u))}{\partial u} \right| du.$$

We repeat procedure of *Step 3* and deduce

$$2\pi \left(\log \left(\frac{1}{q} \right) \right)^2 \le \log \left(\frac{1}{q^{k_1}} \right) \iint_{A(q,1)} \frac{k(z, f)}{|z|^2} \, d\mathcal{L}^2(z).$$

We subtract

$$2\pi \log \left(\frac{1}{q} \right) \log \left(\frac{1}{q^{k_1}} \right) = \log \left(\frac{1}{q^{k_1}} \right) \iint_{A(q,1)} \frac{1}{|z|^2} \, d\mathcal{L}^2(z)$$

from both sides and obtain the inequality

$$\log \left(\frac{q^{k_1}}{q} \right) \le \frac{\log q^{k_1}}{2\pi \log q} \iint_{A(q,1)} \frac{k(z, f) - 1}{|z|^2} \, d\mathcal{L}^2(z). \tag{2.6}$$

Step 5 After combining (2.5) and (2.6) we derive the inequality

$$-\frac{1}{2\pi}\iint_{A(q,1)}\frac{k(z,f)-1}{|z|^2}\,d\mathcal{L}^2(z)\le\log\left(\frac{q^{k_1}}{q}\right)\le\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{k(z,f)-1}{|z|^2}\,d\mathcal{L}^2(z). \tag{2.7}$$

In the relation (2.7), the first inequality becomes an equality for the extremal function $w=e^{i\beta}z\varphi(|z|)$, where φ is arbitrary positive real function. Indeed, we first find the value $k(z,f)$ then we put that value in the left hand side of (2.7). Therefore, and setting $w=e^{i\beta}z\varphi(|z|)$ we obtain

$$\begin{aligned}
\frac{\partial w}{\partial z} &= e^{i\beta}\frac{\partial z\varphi(|z|)}{\partial z}=e^{i\beta}\left(\varphi(|z|)+z\varphi'(|z|)\frac{\bar z}{2|z|}\right)\\
&= e^{i\beta}\left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right),\\
\overline{\frac{\partial w}{\partial z}} &= e^{-i\beta}\left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right)\\
\left|\frac{\partial w}{\partial z}\right| &= \sqrt{\frac{\partial w}{\partial z}\overline{\frac{\partial w}{\partial z}}}\\
&= \sqrt{\left(e^{i\beta}\left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right)\right)\left(e^{-i\beta}\left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right)\right)}\\
&= \left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right).
\end{aligned}$$

Similarly, we can find

$$\left|\frac{\partial w}{\partial \bar z}\right|=\frac{|z|}{2}\varphi'(|z|).$$

Therefore,

$$\begin{aligned}
k(z,f) &= \frac{\left|\frac{\partial w}{\partial z}\right|+\left|\frac{\partial w}{\partial \bar z}\right|}{\left|\frac{\partial w}{\partial z}\right|-\left|\frac{\partial w}{\partial \bar z}\right|}=\frac{\left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right)+\frac{|z|}{2}\varphi'(|z|)}{\left(\varphi(|z|)+\frac{|z|}{2}\varphi'(|z|)\right)-\frac{|z|}{2}\varphi'(|z|)}\\
&= 1+\frac{|z|\varphi'(|z|)}{\varphi(|z|)}.
\end{aligned}$$

Hence

$$\begin{aligned}
-\frac{1}{2\pi}\iint_{A(q,1)}\frac{k(z,f)-1}{|z|^2}\,d\mathcal{L}^2(z) &= -\frac{1}{2\pi}\iint_{A(q,1)}\frac{1+\frac{|z|\varphi'(|z|)}{\varphi(|z|)}-1}{|z|^2}\,d\mathcal{L}^2(z)\\
&= -\frac{1}{2\pi}\int_0^{2\pi}\int_q^1\frac{\frac{s\varphi'(s)}{\varphi(s)}}{s^2}sdsd\theta=-\frac{1}{2\pi}\int_0^{2\pi}\int_q^1\frac{\varphi'(s)}{\varphi(s)}dsd\theta\\
&= -\frac{1}{2\pi}2\pi\log\left(\frac{\varphi(1)}{\varphi(q)}\right)=-\log(\varphi(1))+\log(\varphi(q)).
\end{aligned}$$

We know that $|w|=|e^{i\beta}||z||\varphi(|z|)|=|z||\varphi(|z|)|$. When $|z|=1$ then $|w|=|\varphi(1)|=1$ and when $|z|=q$ then $|w|=q|\varphi(q)|=q^{k_1}$. Therefore,

$$-\frac{1}{2\pi}\iint_{A(q,1)}\frac{k(z,f)-1}{|z|^2}\,d\mathcal{L}^2(z)=\log(q^{k_1-1}),$$

and

$$\log\left(\frac{q^{k_1}}{q}\right) = \log\left(\frac{q\varphi(q)}{q}\right) = \log(\varphi(q)) = \log(q^{k_1-1}).$$

Now, the second inequality in (2.7) becomes an equality if we use the extremal function $w = |z|^\alpha e^{i\psi(argz)}$, where ψ is an arbitrary positive real function. This also can be shown as above, but for simplicity we do it for a particular function. So, let consider the extremal function

$$w = |z|^\alpha e^{i(argz+\theta)} = z^{\frac{\alpha}{2}}\,\bar{z}^{\frac{\alpha}{2}}\,e^{i(argz+\theta)} \quad \text{or} \quad w = z^{\frac{\alpha}{2}}\,\bar{z}^{\frac{\alpha}{2}}\,e^{i(-arg\bar{z}+\theta)}.$$

Then

$$\begin{aligned}\frac{\partial w}{\partial z} &= \frac{\alpha}{2z}|z|^\alpha e^{i(argz+\theta)} + |z|^\alpha e^{i(argz+\theta)} i\frac{-i}{2z}\\ &= \frac{\alpha}{2z}|z|^\alpha e^{i(argz+\theta)} + |z|^\alpha e^{i(argz+\theta)}\frac{1}{2z},\end{aligned}$$

and

$$\overline{\frac{\partial w}{\partial z}} = \frac{\alpha}{2\bar{z}}|z|^\alpha e^{-i(argz+\theta)} + |z|^\alpha e^{-i(argz+\theta)}\frac{1}{2\bar{z}}.$$

Therefore,

$$\begin{aligned}\left|\frac{\partial w}{\partial z}\right| &= \sqrt{\frac{\alpha^2|z|^{2\alpha}}{4|z|^2} + \frac{\alpha|z|^\alpha}{4|z|^2} + \frac{\alpha|z|^\alpha}{4|z|^2} + \frac{|z|^\alpha}{4|z|^2}}\\ &= \frac{|z|^\alpha}{|z|}\sqrt{\frac{\alpha^2}{4} + \frac{\alpha}{2} + \frac{1}{4}} = \frac{|z|^\alpha}{|z|}\left(\frac{\alpha+1}{2}\right).\end{aligned}$$

Similarly, we find

$$\begin{aligned}\frac{\partial w}{\partial \bar{z}} &= \frac{\alpha}{2\bar{z}}|z|^\alpha e^{i(-arg\bar{z})+\theta} + |z|^\alpha e^{i(-arg\bar{z}+\theta)} i\frac{i}{2\bar{z}}\\ &= \frac{\alpha}{2\bar{z}}|z|^\alpha e^{i(-arg\bar{z}+\theta)} - |z|^\alpha e^{i(-arg\bar{z}+\theta)}\frac{1}{2\bar{z}},\end{aligned}$$

$$\overline{\frac{\partial w}{\partial \bar{z}}} = \frac{\alpha}{2z}|z|^\alpha e^{-i(-arg\bar{z}+\theta)} - |z|^\alpha e^{-i(-arg\bar{z}+\theta)}\frac{1}{2z}.$$

Therefore,

$$\begin{aligned}\left|\frac{\partial w}{\partial \bar{z}}\right| &= \sqrt{\frac{\alpha^2|z|^{2\alpha}}{4|z|^2} - \frac{\alpha|z|^\alpha}{4|z|^2} - \frac{\alpha|z|^\alpha}{4|z|^2} + \frac{|z|^\alpha}{4|z|^2}}\\ &= \frac{|z|^\alpha}{|z|}\sqrt{\frac{\alpha^2}{4} - \frac{\alpha}{2} + \frac{1}{4}} = \frac{|z|^\alpha}{|z|}\left(\frac{1-\alpha}{2}\right).\end{aligned}$$

Hence,

$$k(z,f) = \frac{|\frac{\partial w}{\partial z}| + |\frac{\partial w}{\partial \bar{z}}|}{|\frac{\partial w}{\partial z}| - |\frac{\partial w}{\partial \bar{z}}|} = \frac{\frac{|z|^\alpha}{|z|}\left(\frac{1+\alpha}{2}\right) + \frac{|z|^\alpha}{|z|}\left(\frac{1-\alpha}{2}\right)}{\frac{|z|^\alpha}{|z|}\left(\frac{1+\alpha}{2}\right) - \frac{|z|^\alpha}{|z|}\left(\frac{1-\alpha}{2}\right)} = \frac{1}{\alpha}.$$

we know that $|w| = ||z|^\alpha||e^{i(argz+\theta)}| = |z|^\alpha$. When $|z| = 1$ then $|w| = 1$ and when $|z| = q$ then $|w| = q^\alpha = q^{k_1}$. Hence,

$$\frac{\log q^{k_1}}{2\pi\log q} = \frac{\log q^\alpha}{2\pi\log q} = \frac{\alpha\log q}{2\pi\log q} = \frac{\alpha}{2\pi}.$$

Therefore,

$$\begin{aligned}\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{k(z,f)-1}{|z|^2}\,d\mathcal{L}^2(z) &= \frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)-\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{1}{|z|^2}\,d\mathcal{L}^2(z)\\ &= \frac{\alpha}{2\pi}\iint_{A(q,1)}\frac{1}{\alpha|z|^2}\,d\mathcal{L}^2(z)-\frac{\alpha}{2\pi}\iint_{A(q,1)}\frac{1}{|z|^2}\,d\mathcal{L}^2(z)\\ &= \frac{\alpha}{2\pi\alpha}2\pi\log\left(\frac{1}{q}\right)-\frac{\alpha}{2\pi}2\pi\log\left(\frac{1}{q}\right)=(\alpha-1)\log q,\end{aligned}$$

and since $k_1=\alpha$ we obtain

$$\log\left(\frac{q^{k_1}}{q}\right)=(\alpha-1)\log q.$$

Hence we have equality for this particular choice of extremal function.

Re-writing the right hand side of equation (2.7) we get

$$\log\left(\frac{q^{k_1}}{q}\right)\le\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)-\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{1}{|z|^2}\,d\mathcal{L}^2(z)$$

$$\log\left(\frac{q^{k_1}}{q}\right)\le\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)-\frac{\log q^{k_1}}{2\pi\log q}\left(2\pi\log\left(\frac{1}{q}\right)\right)$$

$$\log(q^{k_1})-\log(q)\le\frac{\log q^{k_1}}{2\pi\log q}\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)+\log q^{k_1}$$

$$-\frac{2\pi(\log q)^2}{\log q^{k_1}}\le\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)$$

$$\frac{2\pi}{-\log q^{k_1}}\le\frac{1}{(\log(q))^2}\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)$$

$$M_2(f(\Gamma))\le\frac{1}{(\log(q))^2}\iint_{A(q,1)}\frac{k(z,f)}{|z|^2}\,d\mathcal{L}^2(z)$$

which implies,

$$M_2(f(\Gamma))\le\iint_{A(q,1)}k(z,f)\rho^2(z)\,d\mathcal{L}^2(z) \tag{2.8}$$

where $\rho=\frac{1}{-\log q|z|}$. The inequality (2.8) is analogous to the inequality (2.14) proved in [10] for mappings with finite distortion.

2.3 The spiral stretch map

Definition 2.4. The **N-th spiral stretch map** f_N is defined by

$$f_N(z):=z|z|^{k_1-1}e^{ik_2\log|z|},\quad z\in A(q,1),$$

where $k_1>0$ and $k_2:=\frac{\theta+2\pi N}{\log q}$, $\theta\in[-\pi,\pi],N\in\mathbb{Z}$.

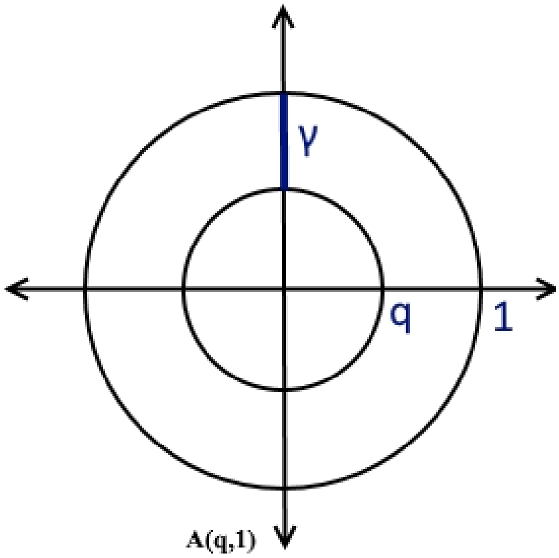


Figure 2.2

Example 2.5. We want to find the image of a radial curve $z = \gamma = re^{it}$, where $t = \frac{\pi}{2}$ is fixed and $0.4 \leq r \leq 1$ in annulus $A(q, 1)$ see Figure 2.2, under the map given by Definition 2.4.

For any general $z = re^{it}$ we obtain

$$f_N(z) = re^{it} r^{k_1 - 1} e^{i\left(\frac{\theta + 2\pi N}{\log q} \log r\right)} = r^{k_1} e^{i\left(t + \left(\frac{\theta + 2\pi N}{\log q} \log r\right)\right)}.$$

We set $k_1 = 2$, $q = 0.4$, $\theta = \pi/2$ and present the images of the ray $\gamma = re^{i\frac{\pi}{2}}$, $0.4 \leq r \leq 1$ under the map f_N for $N = 0, 1, 2$ on Figures 2.3, 2.4, 2.5 respectively.

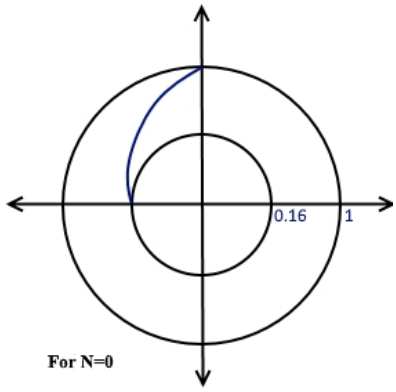


Figure 2.3

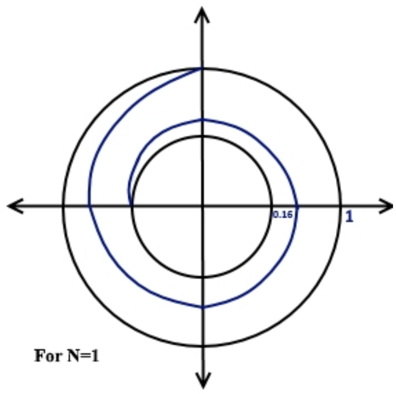


Figure 2.4

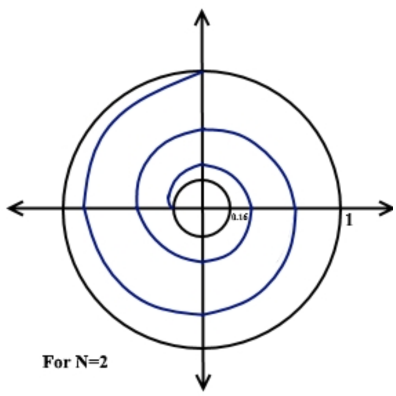


Figure 2.5

Spiral stretch map f_N has constant distortion $k(z, f_N)$ which can be calculated by making use of Definition 2.1. We write $f_N(z)$ as

$$f_N(z) = z(z\bar{z})^{\frac{k_1-1}{2}} e^{ik_2 \log(z\bar{z})^{1/2}}.$$

First we find $|\frac{\partial f_N}{\partial z}| = \sqrt{\frac{\partial f_N}{\partial z}\overline{\frac{\partial f_N}{\partial z}}}$. We calculate

$$\frac{\partial f_N}{\partial z} = |z|^{k_1-1}\, e^{i\frac{k_2}{2}\log(z\bar{z})}\, [\frac{1}{2}(k_1+1+ik_2)],$$

and

$$\overline{\frac{\partial f_N}{\partial z}} = |z|^{k_1-1}\, e^{-i\frac{k_2}{2}\log(z\bar{z})}\, [\frac{1}{2}(k_1+1-ik_2)].$$

Hence,

$$\frac{\partial f_N}{\partial z}\overline{\frac{\partial f_N}{\partial z}} = \left(\frac{k_1+1}{2}\right)^2 \left(|z|^{k_1-1}\right)^2 + \frac{k_2^2}{4}\left(|z|^{k_1-1}\right)^2,$$

and

$$\left|\frac{\partial f_N}{\partial z}\right| = |z|^{k_1-1}\sqrt{\left(\frac{k_1+1}{2}\right)^2 + \left(\frac{k_2}{2}\right)^2}. \tag{2.9}$$

In a similar way we find

$$\left|\frac{\partial f_N}{\partial \bar{z}}\right| = |z|^{k_1-1}\sqrt{\left(\frac{k_1-1}{2}\right)^2 + \left(\frac{k_2}{2}\right)^2}. \tag{2.10}$$

Therefore

$$\begin{aligned} k(z,f_N) &= \frac{|\frac{\partial f_N}{\partial z}| + |\frac{\partial f_N}{\partial \bar{z}}|}{|\frac{\partial f_N}{\partial z}| - |\frac{\partial f_N}{\partial \bar{z}}|} \\ &= \frac{|z|^{k_1-1}\sqrt{(\frac{k_1+1}{2})^2+(\frac{k_2}{2})^2} + |z|^{k_1-1}\sqrt{(\frac{k_1-1}{2})^2+(\frac{k_2}{2})^2}}{|z|^{k_1-1}\sqrt{(\frac{k_1+1}{2})^2+(\frac{k_2}{2})^2} - |z|^{k_1-1}\sqrt{(\frac{k_1-1}{2})^2+(\frac{k_2}{2})^2}} \\ &= \frac{\sqrt{(k_1+1)^2+(k_2)^2} + \sqrt{(k_1-1)^2+(k_2)^2}}{\sqrt{(k_1+1)^2+(k_2)^2} - \sqrt{(k_1-1)^2+(k_2)^2}}. \end{aligned}$$

2.4 One extremal Problem for maps of finite distortion

If we change the class $\mathcal{C}$ of quasiconformal mappings in Definition 2.3 into the class $\tilde{\mathcal{C}}$ of mapping $f\colon A(q,1) \to A(q^{k_1},1)$ of finite distortion satisfying boundary condition (2.3), then $k(z,f)$ may be unbounded. We hence introduce the definition of mean distortion functional

Definition 2.6. Let $f\colon \Omega \to \mathbb{C}$ be a homeomorphism with finite distortion. The **mean distortion functional** is the map

$$f \mapsto \int_\Omega k(z,f)\rho^2(z)d\mathcal{L}^2(z),$$

where ρ is some non negative Borel function and $d\mathcal{L}^2(z)$ is the Lebesgue measure on the open set $\Omega \subset \mathbb{C}$.

If $\tilde{\mathcal{C}}$ is a class of mapping as defined above, then instead of minimising the maximal distortion coefficient, one has to minimise the mean distortion functional. Hence we obtain the following problem [11]:

Find $f_0 \in \tilde{\mathcal{C}}$ *such that*

$$\int_\Omega k(z, f_0)\rho^2(z)d\mathcal{L}^2(z) \leq \int_\Omega k(z, f)\rho^2(z)d\mathcal{L}^2(z) \quad \text{for all} \quad f \in \tilde{\mathcal{C}}. \tag{2.11}$$

We will explain the method which can be found in [11]. To solve this problem, we first divide the class $\tilde{\mathcal{C}}$ in homotopic subclasses $\tilde{\mathcal{C}_N}$ such that $\bigcup_{N\in\mathbb{Z}} \tilde{\mathcal{C}_N} = \tilde{\mathcal{C}}$. For every mapping $f \in \tilde{\mathcal{C}}$ we can find $\tilde{f}_N \in \tilde{\mathcal{C}}$ which is homotopic to f. Then there exists an extremal map f_N in $\tilde{\mathcal{C}}_N$ for each N. Moreover the mapping f_N are quasiconformal and therefore have bounded distortions k_{f_N} has been proven in [11]. We obtain the following relation

$$\int_\Omega k(z, f_N)\rho^2(z)d\mathcal{L}^2(z) \leq \int_\Omega k(z, f)\rho^2(z)d\mathcal{L}^2(z) \quad \text{for all} \quad f \in \tilde{\mathcal{C}}_N. \tag{2.12}$$

It turns out that the map f_0 is an extremal among the sequence $\{f_N\}$, $N \in \mathbb{Z}$, in the sense that it minimises the distortion functions: that is $k_{f_0} \leq k_{f_N}$ for all $N \in \mathbb{Z}$. So we conclude that

$$\int_\Omega k(z, f_0)\rho^2(z)d\mathcal{L}^2(z) \leq \int_\Omega k(z, f_N)\rho^2(z)d\mathcal{L}^2(z) \quad \text{for all} \quad N \in \mathbb{Z}. \tag{2.13}$$

If we combine inequalities (2.12) and (2.13), then we obtain inequality (2.11). We present some steps that show inequality (2.12).

Step 1 We use the inequality proved in [10]:

$$M_2(f(\Gamma)) \leq \int_\Omega k(z, f)\rho^2(z)d\mathcal{L}^2(z) \text{ for all } \rho \in F(\Gamma). \tag{2.14}$$

Step 2 The following step is proved in [11] A quasiconformal map $g\colon \Omega \to \Omega$ that gives the maximal stretching will possess the following properties. If Γ_0 is a family of absolutely continuous curves in Ω satisfying the condition:

a) $\frac{g_{\bar{z}}(\gamma(s))\dot{\gamma}(s)}{g_z(\gamma(s))\dot{\gamma}(s)} < 0$ for almost every s and for every $\gamma \in \Gamma_0$ and
b) if there exists $\rho_0 \in F(\Gamma_0)$ such that

$$M_2(\Gamma_0) = \int_\Omega \rho_0^2(z)\, d\mathcal{L}^2(z), \tag{2.15}$$

then

$$M_2(g(\Gamma_0)) = \int_\Omega k(z, g)\rho_0^2(z)d\mathcal{L}^2(z) \quad \text{for all} \quad \rho_0 \in F(\Gamma_0). \tag{2.16}$$

Step 3 In this step it is shown that the quasiconformal spiral strech map f_N with the coefficient $k(z, f_N)$ and the family of absolutely continuous curves $\Gamma_{0,N}$ satisfy the requirements of *Step 2*. Moreover

$$M_2\left(f_N(\Gamma_{0,N})\right) = \frac{1}{(-\log q)^2(1 + C_N^2)} \int_{A(q,1)} k(z, f_N)\frac{1}{|z|^2} d\mathcal{L}^2(z), \tag{2.17}$$

where C_N is some constant associated with the curves from the family $\Gamma_{0,N}$.

Step 4 We enlarge the family $\Gamma_{0,N}$ by considering all the curves homotopic to the curves from $\Gamma_{0,N}$. Namely, we denote by Γ_N the family of absolutely continuous curves $\gamma\colon [q,1] \to \overline{A(q,1)}$ that are homotopic to $\gamma_\Phi^{C_N} \in \Gamma_{0,N}$, see equation (2.18).

Step 5 Comparing $M_2(f_N(\Gamma_{0,N}))$ and $M_2(f(\Gamma_N))$ for all $f \in \tilde{\mathcal{C}}_N$, one can deduce that

$$M_2(f_N(\Gamma_{0,N})) \le M_2(f(\Gamma_N)).$$

This inequality yields the result

$$\int_\Omega k(z,f_N)\rho^2(z)d\mathcal{L}^2(z) = M_2(f_N(\Gamma_{0,N})) \le M_2(f(\Gamma_N)) \le \int_\Omega k(z,f)\rho^2(z)d\mathcal{L}^2(z)$$

for all $f \in \tilde{\mathcal{C}}_N$.

Example 2.7. In this example we present the details of the proof of *Step 3*. Define

$$\gamma_\Phi^t(s) = se^{i(t\log s+\Phi)}, \quad s \in [q,1], \quad t \in \mathbb{R}, \quad \Phi \in [0,2\pi).$$

Consider the family

$$\Gamma_{0,N} = \{\gamma_\Phi^{C_N} : \Phi \in [0,2\pi)\}, \tag{2.18}$$

where

$$C_N = C(k_1,k_2) = \frac{-k_1^2 - k_2^2 + 1 - \sqrt{(k_1^2+k_2^2-1)^2 + 4k_2^2}}{2k_2}, \quad k_2 = k_2(N) \neq 0. \tag{2.19}$$

Definition 2.8. Define a non-negative Borel function

$$\rho_{0,N}(z) = \frac{1}{-\log q\sqrt{1+C_N^2}}\frac{1}{|z|}, \qquad z \in A(q,1). \tag{2.20}$$

We will prove now that $\rho_{0,N}$ is admissible for $\Gamma_{0,N}$ and

$$M_2(\Gamma_{0,N}) = \int_{A(q,1)} \rho_{0,N}^2(z)\, d\mathcal{L}^2(z) = \frac{-2\pi}{(1+C_N^2)\log q}$$

by making use of the procedure in the examples in Section 1.2.3. For any function $\rho_{0,N}$ as in (2.20) and for any absolutely continuous curve $\gamma\colon [q,1] \to \overline{A(q,1)}$ written as $\gamma(s) = |\gamma(s)|e^{i\varphi(s)}$, $\varphi\colon [q,1] \to \mathbb{R}$, we obtain

$$\begin{aligned}
\int_\gamma \rho_{0,N}(z)\, ds &= \frac{1}{-\log q\sqrt{1+C_N^2}}\int_\gamma \frac{ds}{|z|} = \frac{1}{-\log q\sqrt{1+C_N^2}}\int_q^1 \frac{|\dot\gamma(s)|}{|\gamma(s)|}\, ds \\
&\ge \frac{1}{-\log q\sqrt{1+C_N^2}}\left|\int_q^1 \frac{\dot\gamma(s)}{\gamma(s)}\, ds\right| \\
&= \frac{1}{-\log q\sqrt{1+C_N^2}}\left|\int_q^1 \frac{\frac{d}{ds}|\gamma(s)|}{|\gamma(s)|} + i\dot\varphi(s)\, ds\right| \\
&= \frac{1}{-\log q\sqrt{1+C_N^2}}\left|\log\left(\frac{|\gamma(1)|}{|\gamma(q)|}\right) + i(\varphi(1)-\varphi(q))\right|.
\end{aligned} \tag{2.21}$$

In particular, for $\gamma = \gamma_{\Phi}^{C_N}$ we have

$$|\gamma(1)| = 1, \quad |\gamma(q)| = q, \quad \text{and} \quad \varphi(1) = \Phi, \quad \varphi(q) = C_N \log q + \Phi.$$

Hence,

$$\begin{aligned}\int_{\gamma_{\Phi}^{C_N}} \rho_{0,N}(z)\, ds &\geq \frac{1}{-\log q\sqrt{1+C_N^2}}\left|\log\left(\frac{1}{q}\right) + i(\Phi - (C_N \log q + \Phi))\right| \\ &= \frac{|-\log q\,|\,|1+iC_N|}{-\log q\sqrt{1+C_N^2}} = 1.\end{aligned}$$

Therefore,

$$\int_{\gamma_{\Phi}^{C_N}} \rho_{0,N}(z)\, ds \geq 1, \quad \text{for any} \quad \Phi \in [0, 2\pi).$$

Hence $\rho_{0,N}$ is admissible for $\Gamma_{0,N}$. Furthermore

$$1 \leq \int_{\gamma_{\Phi}^{C_N}} \rho(z)\, ds = \int_q^1 \rho(\gamma_{\Phi}^{C_N})|\dot{\gamma}_{\Phi}^{C_N}(s)|ds = \int_q^1 \rho(\gamma_{\Phi}^{C_N})\sqrt{1+C_N^2}\, ds$$

for an arbitrary $\rho \in F(\Gamma_{0,N})$. We integrate both sides from 0 to 2π with respect to Φ and obtain

$$\begin{aligned}2\pi &\leq \sqrt{1+C_N^2}\int_0^{2\pi}\int_q^1 \rho(se^{i(C_N\log s+\Phi)})dsd\Phi \\ &\leq \sqrt{1+C_N^2}\int_0^{2\pi}\int_q^1 \rho(se^{i(C_N\log s+\Phi)})\frac{1}{\sqrt{s}}\sqrt{s}\, dsd\Phi.\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}2\pi &\leq \sqrt{1+C_N^2}\left(\int_0^{2\pi}\int_q^1 (\rho(se^{i(C_N\log s+\Phi)}))^2 sdsd\Phi\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\int_q^1 \frac{1}{s}dsd\Phi\right)^{\frac{1}{2}} \\ &= \sqrt{1+C_N^2}\left(\int_{A(q,1)} (\rho(g(z))^2 d\mathcal{L}^2(z)\right)^{\frac{1}{2}}(-2\pi\log q)^{\frac{1}{2}} \\ &= \sqrt{1+C_N^2}\left(\int_{A(q,1)} (\rho(g(z))^2 |J(g,z)| d\mathcal{L}^2(z)\right)^{\frac{1}{2}}(-2\pi\log q)^{\frac{1}{2}}.\end{aligned}$$

because $|J(g,z)| = 1$, see Remark 2.11 and where $g(z) = ze^{iC_N\log|z|}$. Using the change of variables under the map $g(z)$, we get

$$2\pi \leq \sqrt{1+C_N^2}\left(\int_{A(q,1)} \rho^2(\zeta)d\mathcal{L}^2(\zeta)\right)^{\frac{1}{2}}(-2\pi\log q)^{\frac{1}{2}}.$$

Finally, we obtain

$$\frac{-2\pi}{(1+C_N^2)\log q} \leq \int_{A(q,1)} \rho^2(\zeta)d\mathcal{L}^2(\zeta). \tag{2.22}$$

Taking infimum over all $\rho \in F(\Gamma_{0,N})$, we deduce

$$\frac{-2\pi}{(1+C_N^2)\log q} \le M_2(\Gamma_{0,N}). \tag{2.23}$$

Since $\rho_{0,N} \in F(\Gamma_{0,N})$ from equation (2.21), we have

$$\begin{aligned} M_2(\Gamma_{0,N}) &= \inf_{\rho\in F(\Gamma_{0,N})} \iint_{A(q,1)} \rho^2\, d\mathcal{L}^2(z) \le \int_{A(q,1)} \rho_{0,N}^2(z)\, d\mathcal{L}^2(z) \\ &= \frac{1}{(-\log q)^2(1+C_N^2)} \int_0^{2\pi}\int_q^1 \frac{1}{r^2} r dr d\theta = \frac{-2\pi}{(1+C_N^2)\log q}. \end{aligned} \tag{2.24}$$

Hence, from inequality (2.22) and (2.24) we obtain

$$M_2(\Gamma_{0,N}) = \int_{A(q,1)} \rho_{0,N}^2(z)\, d\mathcal{L}^2(z) = \frac{-2\pi}{(1+C_N^2)\log q}.$$

Let $\gamma \in \Gamma_N$ be homotopic to $\gamma_\Phi^{C_N} \in \Gamma_{0,N}$. From equation (2.21) we obtain

$$\int_\gamma \rho_{0,N}(z)\, d\mathcal{L}^2(z) \ge \int_{\gamma_\Phi^{C_N}} \rho_{0,N}(z)\, d\mathcal{L}^2(z) \ge 1,$$

which implies that $\rho_{0,N}$ is admissible for Γ_N. We know that $\Gamma_{0,N} \subset \Gamma_N$. By Theorem 1.28 and by equation (2.23) we have

$$\frac{-2\pi}{(1+C_N^2)\log q} = M_2(\Gamma_{0,N}) \le M_2(\Gamma_N)\ . \tag{2.25}$$

Since $\rho_{0,N} \in F(\Gamma_N)$, equation (2.24) implies

$$\begin{aligned} M_2(\Gamma_{0,N}) \le M_2(\Gamma_N) &= \inf_{\rho\in F(\Gamma_{0,N})} \iint_{A(q,1)} \rho^2\, d\mathcal{L}^2(z) \le \int_{A(q,1)} \rho_{0,N}^2(z)\, d\mathcal{L}^2(z) \\ &= \frac{-2\pi}{(1+C_N^2)\log q}. \end{aligned} \tag{2.26}$$

Hence from equation (2.25) and equation (2.26) we conclude that

$$M_2(\Gamma_{0,N}) = M_2(\Gamma_N) = \frac{-2\pi}{(1+C_N^2)\log q},$$

where Γ_N is the family of curve which are homotopic to the family $\Gamma_{0,N}$.

Remark 2.9. Notice that if $C_N = 0$, then the curves $\gamma_\Phi^0(s)$ are radial curves in the annulus $A(q,1)$ and we obtain the result of Example (1.33). We also observe that the family $\Gamma_{0,N}$ is the family of curves, that are tangent to the directions of the largest shrinking of the map f_N. In other words $|df_N| = (|f_{Nz}| - |f_{N\bar{z}}|)|dz|$ along the curves from $\Gamma_{0,N}$, this can be obtain by applying analogous calculation in Lemma 4.5 . This property imply that $M_2(f_N(\Gamma_{0,N})) = \int_{A(q,1)} k(f,z)\rho_{0,N}^2\, d\mathcal{L}^2(z)$, where $\rho_{0,N} = \frac{1}{(-\log q)\sqrt{1+C_N^2}}\frac{1}{|z|}$ is such that $\int_\gamma \rho_{0,N}\, ds \ge 1$ for any $\gamma \in \Gamma_{0,N}$.

Remark 2.10. We also observe that in the family of curves Γ_N homotopic to the curves from $\Gamma_{0,N}$ the function $\rho_{0,N} = \frac{1}{(-\log q)\sqrt{1+C_N^2}}\frac{1}{|z|}$ is the extremal function in the sense that it realises infimum in finding the module $M_2(\Gamma_N)$. Moreover, the family $\Gamma_{0,N}$ is extremal for the same module in the sense that $M_2(\Gamma_N) = M_2(\Gamma_{0,N})$. The same condition was satisfied in Examples 1.31 and 1.33 for the family of the vertical lines connecting the horizontal sides of the rectangular and for the family of the radial curves connecting boundaries of the annulus domain, respectively.

Remark 2.11. It is rest to show that $|J(g,z)| = 1$. Then

$$\begin{aligned} g(z) &= ze^{iC_N \log|z|} = (x+iy)e^{iC_N \log(\sqrt{x^2+y^2})} \\ &= x\cos(C_N \log(\sqrt{x^2+y^2})) - y\sin(C_N \log(\sqrt{x^2+y^2})) + \\ &\quad i\left(x\sin(C_N \log(\sqrt{x^2+y^2})) + y\cos(C_N \log(\sqrt{x^2+y^2}))\right). \end{aligned}$$

We denote by p and q the functions

$$p = Reg(z) = x\cos(C_N \log(\sqrt{x^2+y^2})) - y\sin(C_N \log(\sqrt{x^2+y^2})),$$

and

$$q = Img(z) = x\sin(C_N \log(\sqrt{x^2+y^2})) + y\cos(C_N \log(\sqrt{x^2+y^2})).$$

For simplicity we denote $A = C_N \log(\sqrt{x^2+y^2})$, then

$$\begin{aligned} \frac{dp}{dx} &= \cos(A) - x\sin(A)\frac{xC_N}{x^2+y^2} - y\cos(A)\frac{xC_N}{x^2+y^2}, \\ \frac{dp}{dy} &= -x\sin(A)\frac{yC_N}{x^2+y^2} - \sin(A) - y\cos(A)\frac{yC_N}{x^2+y^2}. \end{aligned}$$

Similarly we find

$$\begin{aligned} \frac{dq}{dx} &= -y\sin(A)\frac{xC_N}{x^2+y^2} + \sin(A) + x\cos(A)\frac{xC_N}{x^2+y^2}, \\ \frac{dq}{dy} &= \cos(A) - y\sin(A)\frac{yC_N}{x^2+y^2} + x\cos(A)\frac{yC_N}{x^2+y^2}. \end{aligned}$$

We hence compute $J(g,z) = \det\begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} \\ \frac{dp}{dy} & \frac{dq}{dy} \end{vmatrix}$

$$= \det\begin{vmatrix} \cos(A) - x\sin(A)\frac{xC_N}{x^2+y^2} - y\cos(A)\frac{xC_N}{x^2+y^2} & -y\sin(A)\frac{xC_N}{x^2+y^2} + \sin(A) + x\cos(A)\frac{xC_N}{x^2+y^2} - \\ x\sin(A)\frac{yC_N}{x^2+y^2} - \sin(A) - y\cos(A)\frac{yC_N}{x^2+y^2} & \cos(A) - y\sin(A)\frac{yC_N}{x^2+y^2} + x\cos(A)\frac{yC_N}{x^2+y^2} \end{vmatrix}$$

$$= \left(\cos(A) - x\sin(A)\frac{xC_N}{x^2+y^2} - y\cos(A)\frac{xC_N}{x^2+y^2}\right)\left(\cos(A) - y\sin(A)\frac{yC_N}{x^2+y^2} + x\cos(A)\frac{yC_N}{x^2+y^2}\right) -$$

$$\left(\left(x\sin(A)\frac{yC_N}{x^2+y^2} - \sin(A) - y\cos(A)\frac{yC_N}{x^2+y^2}\right)\left(-y\sin(A)\frac{xC_N}{x^2+y^2} + \sin(A) + x\cos(A)\frac{xC_N}{x^2+y^2}\right)\right)$$

$$= \left(\cos^2(A) - y\cos(A)\sin(A)\frac{yC_N}{x^2+y^2} + x\cos^2(A)\frac{yC_N}{x^2+y^2} - \right.$$

$$x\sin(A)\cos(A)\frac{xC_N}{x^2+y^2} + xy\sin^2(A)\frac{xy(C_N)^2}{(x^2+y^2)^2} - x^2\sin(A)\cos(A)\frac{xy(C_N)^2}{(x^2+y^2)^2}$$

$$-y\cos^2(A)\tfrac{xC_N}{x^2+y^2}+y^2\cos(A)\sin(A)\tfrac{xy(C_N)^2}{(x^2+y^2)^2}-xy\cos^2(A)\tfrac{xy(C_N)^2}{x^2+y^2}\Bigg)-$$

$$\Bigg(xy\sin^2(A)\tfrac{xy(C_N)^2}{(x^2+y^2)^2}-x\sin^2(A)\tfrac{yC_N}{x^2+y^2}-x^2\sin(A)\cos(A)\tfrac{xy(C_N)^2}{(x^2+y^2)^2}+$$

$$y\sin^2(A)\tfrac{xC_N}{x^2+y^2}-\sin^2(A)-x\sin(A)\cos(A)\tfrac{xC_N}{x^2+y^2}+y^2\cos(A)\sin(A)\tfrac{xy(C_N)^2}{(x^2+y^2)^2}-$$

$$y\cos(A)\sin(A)\tfrac{yC_N}{x^2+y^2}-xy\cos^2(A)\tfrac{xy(C_N)^2}{x^2+y^2}\Bigg)$$

$$=\cos^2(A)+\sin^2(A)=1.$$

Chapter 3

Quadratic differentials and Extremal problem

3.1 Quadratic differential and Teichmüller map

The extremal problem for spiral stretch map is related to Teichmüller theory, even though the class of mappings is wider then just quasiconformal. We try to explain relation of the extremal problem to the Teichmüller approach. We begin from the giving necessary definitions.

Definition 3.1. [12]

A Hausdorff topological space R is called a **Riemann surface** if there are:

- a collection of open sets $U_\alpha \subset R$, where α ranges over some index set, which cover R (i.e. $R = \cup_\alpha U_\alpha$), and
- A homeomorphism $\psi_\alpha : U_\alpha \to \widetilde{U}_\alpha$ for each α, where $\widetilde{U}_\alpha$ is an open set in $\mathbb{C}$ and satisfies the property: for all α, β, the composition map $\psi_\alpha \circ \psi_\beta^{-1}$ is holomorphic on its domain of definition.

The maps ψ_α are called charts, and the entire collection of data $\left(U_\alpha, \widetilde{U}_\alpha, \psi_\alpha\right)$ is called an atlas.

Definition 3.2. [15] Let $p\colon E \to B$ be a continuous and surjective map, where E, B are topological spaces. If every point $b \in B$ has a neighborhood U such that the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E, and for each α, the restriction of p to V_α is a homeomorphism of V_α onto U, then p is called a **covering map**, and E is said to be a **covering space** of B.

Definition 3.3. Let R be a Riemann surface. Let $U_1, U_2 \in R$ with $U_1 \cap U_2 \neq \emptyset$. Let $\psi_1\colon U_1 \to \mathbb{C}$ and $\psi_2\colon U_2 \to \mathbb{C}$ be charts. Consider a conformal mapping $g\colon U_1 \to U_2$. If given functions $\varphi^*\colon U_1 \to \mathbb{C}$ and $\varphi\colon U_2 \to \mathbb{C}$ in variable $z = g(z^*)$ the following transformation law holds

$$\varphi^*(z^*) dz^{*2} = \varphi(z) dz^2, \quad z = g(z^*), \quad z^* \in U_1 \cap U_2, \tag{3.1}$$

then the function $\varphi(z)dz^2$ is called **quadratic differential**.

In other words a quadratic differential is a collection of the functions defined on the charts, which is invariant with respect to the change of variables on the charts.

Example 3.4. Let S be the Riemann sphere. We can consider the atlas $\{(U_1 = \mathbb{C}, \mathrm{Id}), (U_2 = \mathbb{C}^* \cup \{\infty\}, w = \frac{1}{z})\}$. We choose an arbitrary function $\varphi_1(z)$ which is defined on the whole plane and we can compute the function element $\varphi_2(w)$ defined on $\mathbb{C}^* \cup \{\infty\}$ by the transformation rule $\varphi_2(w)dw^2 = \varphi_1(z)dz^2$. We get

$$\begin{aligned} \varphi_2(w)dw^2 &= \varphi_1\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}dw\right)^2, \\ \varphi_2(w) &= \varphi_1\left(\frac{1}{w}\right)\frac{1}{w^4}. \end{aligned}$$

From now on we will only consider the Riemann surfaces which are domains in the complex plain.

Definition 3.5. Let D be a domain in $\mathbb{C}$. A maximal regular curve γ on D on which $\varphi(z)dz^2 > 0$ is called the **horizontal trajectory** or simply **trajectory** of φ. **Orthogonal trajectories** or **vertical trajectory** are the maximal regular curves a long which the quadratic differential satisfies $\varphi(z)dz^2 < 0$.

Definition 3.6. Let $\varphi(z)dz^2$ be a holomorphic quadratic differential on R. The parameter

$$\zeta = \int_{z_0}^{z} \sqrt{\varphi}dz,$$

where z_0 is an arbitrary point in R, is called the **natural parameter** for R associated with quadratic differential.

Remark 3.7. Notice that $d\zeta^2 = \varphi(z)dz^2$ for the natural parameter associated with φ. If a parametric curve $\gamma\colon I \to R$ is a horizontal trajectory of φ then $\varphi(\gamma(t))\gamma'(t) > 0$ and therefore in the ζ-plane, where ζ is a natural parameter, the curve $\gamma(t)$ is transformed into a horizontal line. Correspondingly the vertical trajectories (the curves $\gamma(t)$ where $\varphi(\gamma(t))\gamma'(t) < 0$) are mapped into the vertical lines.

Definition 3.8. A **length element** of the metric φ associated with a quadratic differential $\varphi(z)dz^2$ is a differential $ds_\varphi := \sqrt{|\varphi|}|dz|$.

Definition 3.9. The length of a piecewise differentiable curve γ with respect to this metric φ is called the $|\varphi|$**-length** of γ and is equal to $l_\gamma(\varphi) = \int_\gamma ds_\varphi$.

Remark 3.10. A non constant holomorphic quadratic differential $\varphi(z)dz^2$ carries several invariants (under conformal transformations). The first one is the area element

$$dA_\varphi = |\varphi(z)|dxdy = d\xi\, d\eta, \quad z = x + iy.$$

Here z is the local parameter on the Riemann surface R (annulus domain in our case) and $\zeta = \xi + i\eta$ is any natural parameter (the rectangular in ζ-plane). Away from the singularities of φ (which is always true in our case) in the terms of the natural parameter $\zeta = \xi + i\eta$, one has $ds_\varphi^2 = d\xi^2 + d\eta^2$, so local geodesics are just straight lines in the ζ-plane. In the Figure 3.1, the radial curves are the horizontal trajectories of the quadratic differential $\frac{dz^2}{z^2}$, and they are mapped to the horizontal line. The vertical trajectories are the circle and they are mapped to the vertical lines see Figure 3.1 .

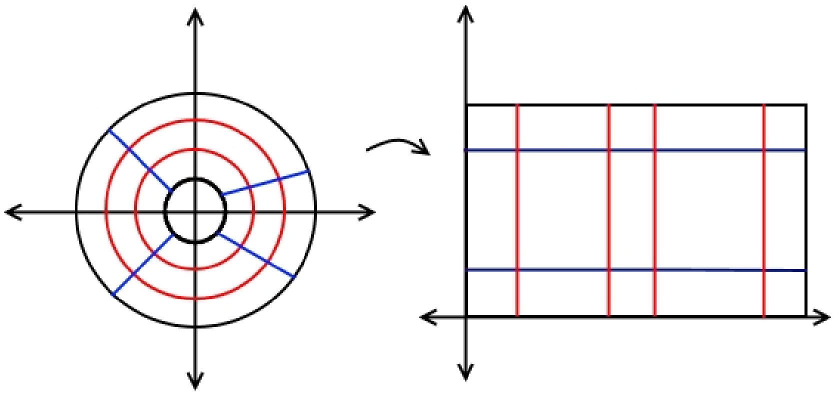


Figure 3.1

Definition 3.11. A quasiconformal mapping f is said to be a **Teichmüller map** if it satisfies the Beltrami equation $f_{\bar z} = \mu_f \, f_z$ with

$$\mu_f(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}, \tag{3.2}$$

where $\varphi(z)dz^2$ is a quadratic differential in D, and k is an arbitrary constant.

3.2 Spiral stretch map is a Teichmüller map

The quadratic differential related to the spiral and stretch map induces the length element given by $|\sqrt{\varphi}||dz| = \frac{k|dz|}{|z|}$. This length element is up to a constant defines the extremal metric in the module problem. Namely, $\rho_{0,N} = \frac{k|dz|}{|z|}$ is the metric such that $M_2(\Gamma_{0,N}) = \iint_{A(q,1)} \rho_{0,N}^2 \, d\mathcal{L}^2(z)$, from equation (2.24) We also observe that $\Gamma_{0,N} = f_N(\Gamma_0)$, i.e the extremal family of curves $\Gamma_{0,N}$ is the image of family of radial curves Γ_0 connecting the boundaries of the annulus domain. The family Γ_0 is given by the trajectories of the quadratic differential $\frac{dz^2}{z^2}$ as it will be shown after Proposition 3.12.

Proposition 3.12. *The spiral stretch map $f_N(z)$ defined in Definition 2.4 is a Teichmüller map.*

Proof. We compute the Beltrami coefficient $\mu_f(z) = \frac{f_{\bar z}}{f_z}$ with respect to the spiral-stretch map $f_N(z)$, as below. We have

$$f_N(z) = z(z\bar z)^{\frac{k_1 - 1}{2}} e^{ik_2 \log(z\bar z)^{1/2}}$$

We compute

$$\frac{\partial f_N}{\partial z} = |z|^{k_1-1}\, e^{i\frac{k_2}{2}\log(z\bar{z})}\,[\frac{1}{2}(k_1+1+ik_2)]. \tag{3.3}$$

and

$$\frac{\partial f_N}{\partial \bar{z}} = |z|^{k_1-1}\, \frac{z}{\bar{z}}\, e^{i\frac{k_2}{2}\log(z\bar{z})}\,[\frac{1}{2}(k_1-1+ik_2)]. \tag{3.4}$$

We obtain

$$\mu_f(z) = \frac{f_{\bar{z}}}{f_z} = \frac{\frac{z}{\bar{z}}(k_1-1+ik_2)}{(k_1+1+ik_2)} = \frac{z}{\bar{z}}\Big[\frac{k_1-1+ik_2}{k_1+1+ik_2}\Big] = \frac{|z|^2}{\bar{z}^2}\Big[\frac{k_1-1+ik_2}{k_1+1+ik_2}\Big]. \tag{3.5}$$

We write the Beltrami coefficient as $\mu_f(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}$, where

$$k = \sqrt{\frac{(k_1-1)^2+k_2^2}{(k_1+1)^2+k_2^2}}, \quad \text{and} \quad \varphi(z)dz^2 = \frac{dz^2}{z^2} \quad \text{is a quadratic differential.}$$

Hence we conclude that the map f_N is the Teichmüller map with respect to $\varphi(z)dz^2 = \frac{1}{z^2}dz^2$. □

Let us calculate the trajectories and the orthogonal trajectories of $\frac{dz^2}{z^2}$.
Let $z = re^{i\theta}$. If we fix θ then $dz = e^{i\theta}dr$ and

$$\frac{dz^2}{z^2} = \frac{e^{2i\theta}dr^2}{r^2e^{2i\theta}} = \frac{dr^2}{r^2} > 0.$$

Therefore the trajectories of $\frac{dz^2}{z^2}$ are the radial lines (in red colour). If we fix r then $dz = rie^{i\theta}d\theta$ and

$$\frac{dz^2}{z^2} = \frac{-r^2e^{2i\theta}d\theta^2}{r^2e^{2i\theta}} = -d\theta^2 < 0.$$

Therefore, the orthogonal trajectories are circles (in blue colour). See Figure 3.2.

Recall that according to the Teichnüller theory any Beltrami coefficient with $\|\mu_\infty\| \leq 1$ is in one to one correspondence with the quasiconformal maps f^μ satisfying some boundary conditions (normalised maps) see Theorem 1.9. The class of quasiconformal maps satisfying the same boundary conditions defines a Riemannian surface and the quasiconformal map minimising the norm of the Beltrami coefficient $\|\mu_\infty\|$ (or the distortion coefficient $k_f(z)$) is related to the quadratic differentials in the sense that $\mu_f = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}$, where $\varphi(z)dz^2$ is a holomorphic differential. Thus the extremal maps defining the Riemann surfaces are the Teichmüller maps.

In this work we considered two extremal problems for the Rieman surfaces, defined by the annulus domain: one in the class of the quasiconformal maps and one in the class of the mappings with finite distortion. It is interesting that even if the class of the mappings with finite distortion is bigger then the class of the quasiconformal mappings, the extremal maps, that are spiral stretch maps f_N are still the Teichmuller maps defining the different Riemann surfaces, based on the same annulus domain, but having different covering maps.

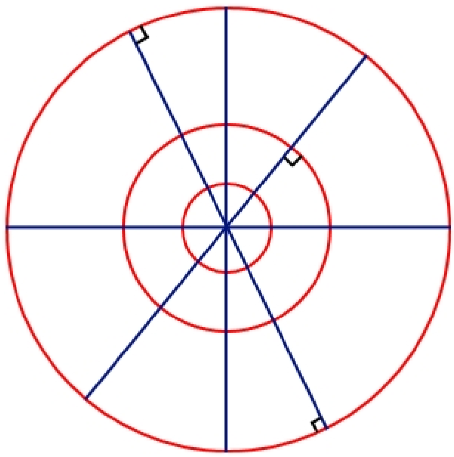


Figure 3.2

Remark 3.13. Notice also the following fact. The quadratic differential $\frac{dz^2}{z^2}$ is actually the area element in the mean distortion functional of Definition 2.6. It is also the area element in the functionals of the estimate (2.7). The linear element $k\frac{|dz|}{|z|}$ (up to the constant) is the extremal metric ρ_0 in the Examples 1.33 and the extremal metric $\rho_{0,N}$ in the inequality (2.20) of Example 2.7.

Chapter 4

Rodin's theorem

In Chapter 2, we have explained the method to calculate the extremal function in the module problem for spiral stretch map f_N. Namely the function

$$\rho_{0,N}(z) = -\frac{1}{\log q\sqrt{1+C_N^2}}\frac{1}{|z|},$$

is such that

$$M_2(\Gamma_{0,N}) = \int_{A(q,1)} \rho_{0,N}^2(z)\, d\mathcal{L}^2(z),$$

and

$$M_2(f_N(\Gamma_{0,N})) = \int_{A(q,1)} k(z, f_N)\rho_{0,N}^2(z)\, d\mathcal{L}^2(z).$$

The problem of finding the extremal metric in the module problem goes back to a result by Rodin. We state Rodin's theorem.

Theorem 4.1. *[7, Rodin's Theorem] Let f be a sufficiently smooth, orientation preserving homeomorphism of $Q_{1b} = \{(x,y) \in \mathbb{C} \mid 0 \le x \le 1, 0 \le y \le b\}$ onto a region $Q \in \mathbb{R}^2$, such that the Jacobi matrix exists and its determinant J_f is positive. Let Γ_0 be the family of vertical intervals $v_x(t) = \{(x,t)\colon t \in [0,b];\ x \in [0,1] \text{ is fixed}\}$, and let $c_x(t) = f(v_x(t)) \in Q$. Thus, the image of Γ_0 is $f(\Gamma_0) = \{c_x\colon [0,b] \to Q,\ x \in [0,1]\}$.*

Let

$$\ell(x) = \int_0^b \frac{|\dot{c}_x|^2}{J_f}\, dt, \quad x \in [0,1],$$

where $\dot{c}_x = \frac{\partial}{\partial t} c_x(t)$. Then

$$\rho_0(y) = \frac{1}{\ell(x)}\left(\frac{|\dot{c}_x|}{J_f}\right) \circ f^{-1}(y), \quad (x,t) \in Q_{1b}, \quad y = f(x,t) \in Q, \tag{4.1}$$

is the extremal function for the 2-module of the family $f(\Gamma_0)$ and

$$M_2(f(\Gamma_0)) = \int_Q \rho_0^2(y)\, dy = \int_0^1 \ell^{-1} dx. \tag{4.2}$$

We illustrate Theorem 4.1 on the example of the shear map.

4.1 Rodin theorem for shear map

Definition 4.2. Let D be a rectangle, $f\colon D \to D_1 = f(D)$ a map such that

$$f(z) = (x + y\cot\alpha, y) = z - i\frac{z-\bar{z}}{2}\cot\alpha,\ \alpha \in \left(0, \frac{\pi}{2}\right]. \tag{4.3}$$

The image $D_1 = f(D)$ is a parallelogram, and the map f is often called **shear map**. We apply Theorem 4.1 to the shear map f. We recall that $f(z) = (x + y\cot\alpha, y) = x + y\cot\alpha + iy$, we denote $p = Ref(z) = x + y\cot\alpha$ and $q = Imf(z) = y$. The Jacobian of f is given by

$$J_f = J(f,z) = \det\begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} \\ \frac{dp}{dy} & \frac{dq}{dy} \end{vmatrix} = \det\begin{vmatrix} 1 & 0 \\ \cot\alpha & 1 \end{vmatrix} = 1.$$

Let Γ_0 be the family of vertical straight lines parametrized by $\gamma_x(t) = x + it$, where $0 \le x \le \frac{2\pi}{\log R}$ is fixed, and $0 \le t \le 1$. Then f maps a curve Γ_D into Γ_{D_1} with $c_x(t) = f(x+it) = x + t\cot\alpha + it$, see Figure 4.2. Therefore,

$$\dot{c}_x = \frac{\partial}{\partial t}(x + t\cot\alpha + it) = \cot\alpha + i,$$

which implies

$$|\dot{c}_x| = \sqrt{1 + (\cot\alpha)^2}.$$

Hence,

$$\ell(x) = \int_0^1 \frac{|\dot{c}_x|^2}{J_f} = \int_0^1 1 + (\cot\alpha)^2\, dt = 1 + (\cot\alpha)^2.$$

We can also find

$$\rho_0(y) = \frac{1}{\ell(x)}\left(\frac{|\dot{c}_x|}{J_f}\right) \circ f^{-1}(y)$$

where $\frac{1}{\ell(x)}\left(\frac{|\dot{c}_x|}{J_f}\right)$ is constant. So we get

$$\rho_0(y) = \frac{1}{\ell(x)}\left(\frac{|\dot{c}_x|}{J_f}\right) = \frac{1}{1+(\cot\alpha)^2}\left(\frac{\sqrt{1+(\cot\alpha)^2}}{1}\right) \tag{4.4}$$

$$= \frac{1}{\sqrt{1+(\cot\alpha)^2}}. \tag{4.5}$$

Hence, we get

$$M_2(f(\Gamma_0)) = \int_D \rho_0^2(y)\, dy = \int_0^{\frac{2\pi}{\log R}} \ell^{-1} dx = \int_0^{\frac{2\pi}{\log R}} \frac{1}{1+(\cot\alpha)^2}\, dx = \sin^2\alpha \frac{2\pi}{\log R}.$$

We also observe that the shear map f is a Teichmüller map. Indeed, since

$$f_z = 1 - \frac{i}{2}\cot\alpha, \quad \text{and} \quad f_{\bar{z}} = \frac{i}{2}\cot\alpha,$$

the Beltrami coefficient is

$$\mu(z) = i\frac{\cot\alpha}{2 - i\cot\alpha}.$$

We can rewrite $\mu(z)$ as $\mu_f(z) = k\frac{\overline{\varphi(z)}}{|\varphi(z)|}$, where

$$k = \frac{\cos\alpha}{\sqrt{4 - 3\cos^2\alpha}}, \quad \theta = -\arctan(2\tan\alpha)$$

and hence $\varphi(z)dz^2 = dz^2$ is a quadratic differential. Hence we conclude that f is the Teichmüller map with the $\varphi(z)dz^2 = dz^2$.
Now we calculate the trajectories for shear map.

Let $z = x+iy$, if y is fixed then $dz = dx$, hence $dz^2 = dx^2 > 0$. Therefore trajectories are horizontal lines (in blue colour). Instead if x is fixed then $dz = idy$ hence $dz^2 = -dy^2 < 0$. Therefore orthogonal trajectories are vertical lines (in red colour), see Figure 4.1.

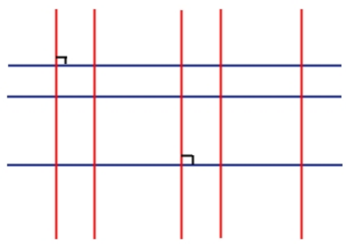


Figure 4.1

4.2 Rodin's theorem for the spiral stretch map

We apply the analogous of Rodin's theorem to annulus domain, see [17]

Proposition 4.3. *[17] Consider a smooth homeomophic function $f\colon A(1,R) \to f(A(1,R))$. If Γ_0 is the family of radial curves connecting boundaries of an annulus $A(1,R)$, then*

$$M_2(f(\Gamma_0)) = \int_0^{2\pi} \frac{1}{\int_1^R \frac{D_{f,\theta}}{r} dr} d\theta,$$

and the extremal function is given by

$$\rho_0 \circ f = \frac{D_{f,\theta}}{r\,|f_r| \int_1^R \frac{D_{f,\theta}}{r}\, dr},$$

where $f_r = e^{i\theta}(f_z + e^{-2i\theta} f_{\bar{z}})$.

Proposition 4.4. *[17] Consider a smooth homeorphic function $f\colon A(1,R) \to f(A(1,R))$. If Γ_0' is a family of a logarithmic spiral curves connecting boundaries of an annulus $A(1,R)$ such that $h(re^{i\theta}) = re^{i(-\beta\log r + \theta)}$, then*

$$M_2(f(\Gamma_0')) = \int_0^{2\pi} \frac{1}{\int_1^R (1+\beta^2)\frac{D_{f,\theta_0}}{r} dr} d\theta,$$

where $\theta_0 = -\beta \log r + \theta - \arctan \beta$.
If we consider a function $g(re^{i\theta}) = f(re^{i(-\beta \log r + \theta)})$ *then*

$$M_2(f(\Gamma_0')) = \int_0^{2\pi} \frac{1}{\int_1^R \frac{|g_r|^2}{J_g} dr} d\theta, \tag{4.6}$$

since $|g_r|^2 = (1+\beta^2)|f_z + f_{\bar{z}} e^{-2i\theta_0}|^2$, *and* $J_g = rJ_f$. *The extremal function is given by*

$$\rho_0 \circ h = \frac{D_{h,\theta}}{r\,|h_r| \int_1^R \frac{D_{h,\theta}}{r}\, dr}.$$

Lemma 4.5. *We show that the Rodin's type theorems will recuperate result for* $\rho_{0,N}$ *and* $\Gamma_{0,N}$ *in the extremal problem for the spiral stretch map.*

Consider the function $g(re^{i\theta}) = f_N \circ \gamma_\Phi^{C_N}(r)$, *where* $\gamma_\Phi^{C_N}(r) = re^{i(C_N \log r + \theta)}$. *By equation* (4.6), *we get*

$$\begin{aligned} M_2(f_N(\Gamma_0')) &= \int_0^{2\pi} \frac{1}{\int_q^1 \frac{|g_r|^2}{J_g} dr} d\theta \\ &= \int_0^{2\pi} \frac{1}{\int_q^1 \frac{(1+C_N^2)|f_{N_z} + f_{N_{\bar{z}}} e^{-2i\theta_0}|^2}{rJ_{f_N}} dr} d\theta \\ &= \int_0^{2\pi} \frac{1}{(1+C_N^2) \int_q^1 D_{f_N,\theta_0} \frac{dr}{r}} d\theta \end{aligned} \tag{4.7}$$

where

$$D_{f_N,\theta_0} = \frac{|f_{N_{\theta_0}}|^2}{J_{f_N}} = \frac{|f_{N_z} + f_{N_{\bar{z}}} e^{-2i\theta_0}|^2}{J_{f_N}} = \frac{|1 + e^{-2i\theta_0}\mu_f|^2}{1 - |\mu_f|^2} \tag{4.8}$$

is a directional dilatation of f_N, *and* $f_{N_{\theta_0}}$ *is directional derivative of* f_N *in direction* θ_0, *where* $\theta_0 = C_N \log r + \theta + \arctan C_N$. *From equations* (3.3) *and* (3.4) *it follows that*

$$\frac{\partial f_N}{\partial z} = |z|^{k_1 - 1}\, e^{i\frac{k_2}{2}\log(z\bar{z})}\, [\frac{1}{2}(k_1 + 1 + ik_2)],$$

$$\frac{\partial f_N}{\partial \bar{z}} = |z|^{k_1 - 1}\, \frac{z}{\bar{z}}\, e^{i\frac{k_2}{2}\log(z\bar{z})}\, [\frac{1}{2}(k_1 - 1 + ik_2)].$$

From equation (3.5) *we have*

$$\mu_f = \frac{f_{N_{\bar{z}}}}{f_{N_z}} = \frac{\frac{z}{\bar{z}}(k_1 - 1 + ik_2)}{(k_1 + 1 + ik_2)} = \frac{z}{\bar{z}}\Big[\frac{k_1 - 1 + ik_2}{k_1 + 1 + ik_2}\Big],$$

which implies

$$|\mu(z)| = \frac{|f_{N_{\bar{z}}}|}{|f_{N_z}|} = \frac{(k_1 - 1 + ik_2)}{(k_1 + 1 + ik_2)}.$$

Therefore, the direction dilatationD_{f_N,θ_0} *becomes*

$$D_{f_N,\theta_0} = \frac{|f_{N_{\theta_0}}|^2}{J_{f_N}} = \frac{|f_{N_z} + f_{N_{\bar{z}}} e^{-2i\theta_0}|^2}{J_{f_N}} = \frac{|1 + e^{-2i\theta_0}\mu_f|^2}{1 - |\mu_f|^2}. \tag{4.9}$$

First we calculate $1+e^{-2i\theta_0}\mu_f$ *on* $z = re^{i(C_N \log r+\theta)}$:

$$\begin{aligned} 1+e^{-2i\theta_0}\mu_f &= 1+e^{-2i\theta_0}\frac{z}{\bar{z}}\frac{k_1-1+ik_2}{k_1+1+ik_2} \\ &= 1+e^{-2i\theta_0}e^{2i(C_N\log r+\theta)}\frac{k_1-1+ik_2}{k_1+1+ik_2} \\ &= 1+\frac{k_1-1+ik_2}{k_1+1+ik_2}e^{2i(-C_N\log r-\theta-\arctan C_N+C_N\log r+\theta)} \\ &= 1+\frac{k_1-1+ik_2}{k_1+1+ik_2}e^{2i\arctan C_N}, \end{aligned} \tag{4.10}$$

where

$$e^{2i\arctan C_N} = \cos(2\arctan C_N) + i\sin(2\arctan C_N) = \frac{1-C_N^2+i2C_N}{1+C_N^2}, \tag{4.11}$$

by trigonometric property

$$\cos^2 x = \frac{1}{1+\tan^2 x} \quad \textit{and} \quad \cos 2x = 2\cos^2 x - 1 = \frac{2}{1+\tan^2 x} - 1 = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

Therefore

$$\cos 2\arctan C_N = \frac{1-\tan^2(\arctan C_N)}{1+\tan^2(\arctan C_N)} = \frac{1-C_N^2}{1+C_N^2},$$

From the property

$$\sin^2 2x = 1-\cos^2 2x = 1-\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right) = \frac{4\tan^2 x}{(1+\tan^2 x)^2}$$

we also obtain,

$$\sin 2\arctan C_N = \frac{2C_N}{1+C_N^2}.$$

In order to simplify we rationalise the term $\frac{k_1-1+ik_2}{k_1+1+ik_2}$ *and get*

$$\begin{aligned} \left[\frac{k_1-1+ik_2}{k_1+1+ik_2}\right]\left[\frac{k_1+1-ik_2}{k_1+1-ik_2}\right] &= \frac{(k_1-1)(k_1+1)+ik_2(k_1+1)-ik_2(k_1-1)-i^2k_2^2}{(k_1+1)^2+k_2^2} \\ &= \frac{k_1^2+k_2^2-1+2ik_2}{(k_1+1)^2+k_2^2} \\ &= \frac{P}{Q}, \end{aligned} \tag{4.12}$$

where we denote $P = k_1^2+k_2^2-1+2ik_2$ *and* $Q = (k_1+1)^2+k_2^2$. *If we denote* $B = (k_1-1)^2+k_2^2$ *then* $|P|^2 = QB$. *We can also express* C_N *in terms of* P *as*

$$C_N = \frac{-ReP-|P|}{ImP}.$$

Now we calculate $e^{2i\arctan C_N} = \frac{1-C_N^2+i2C_N}{1+C_N^2}$ *in terms of P. In particular*

$$\begin{aligned}
1+C_N^2 &= 1+\frac{(ReP)^2+|P|^2+2ReP|P|}{(ImP)^2}, \\
&= \frac{(ImP)^2+(ReP)^2+|P|^2+2ReP|P|}{(ImP)^2}, \\
&= \frac{2|P|^2+2ReP|P|^2}{(ImP)^2}. \\
1-C_N^2 &= 1-\frac{(ReP)^2+|P|^2+2ReP|P|}{(ImP)^2} = \frac{(ImP)^2-(ReP)^2-|P|^2-2ReP|P|}{(ImP)^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1-C_N^2}{1+C_N^2} &= \frac{(ImP)^2-(ReP)^2-|P|^2-2ReP|P|^2}{2|P|^2+2ReP|P|} \\
&= \frac{(ImP)^2+(ReP)^2-(ReP)^2-(ReP)^2-|P|^2-2ReP|P|}{2|P|^2+2ReP|P|} \\
&= \frac{-2(ReP)^2-2ReP|P|}{2|P|^2+2ReP|P|} \\
&= \frac{-(ReP)^2-ReP|P|}{|P|^2+ReP|P|},
\end{aligned}$$

and

$$\begin{aligned}
\frac{2C_N}{1+C_N^2} &= \frac{\frac{-2ReP-2|P|}{ImP}}{\frac{2|P|^2+2ReP|P|}{(ImP)^2}} \\
&= \frac{(-ReP-|P|)ImP}{|P|^2+ReP|P|},
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1-C_N^2+i2C_N}{1+C_N^2} &= \frac{-(ReP)^2-ReP|P|^2+i(-(ReP)(ImP)-|P|(ImP))}{|P|(|P|+ReP)} \\
&= \frac{-ReP(ReP+|P|+iImP)-i|P|ImP}{|P|(|P|+ReP)} \\
&= \frac{-ReP(P+|P|)-i|P|ImP}{|P|(|P|+ReP)} \\
&= \frac{-ReP(P)-ReP|P|-i|P|ImP}{|P|(|P|+ReP)} \\
&= \frac{-ReP(P)-|P|(ReP+iImP)}{|P|(|P|+ReP)} \\
&= \frac{-P(ReP+|P|)}{|P|(|P|+ReP)} \\
&= \frac{-P}{|P|}. \qquad (4.13)
\end{aligned}$$

Therefore, by using equations (4.12), (4.13), *and* (4.9) , *we deduce*

$$\begin{aligned} D_{f_N,\theta_0} &= \frac{|1+e^{-2i\theta_0}\mu_f|^2}{1-|\mu_f|^2} = \frac{\left|1+\frac{k_1-1+ik_2}{k_1+1+ik_2}\left(\frac{1-C_N^2+i2C_N}{1+C_N^2}\right)\right|^2}{1-\left|\frac{k_1-1+ik_2}{k_1+1+ik_2}\right|^2} \\ &= \frac{\left|1-\frac{P^2}{Q|P|}\right|^2}{1-\left|\frac{P}{Q}\right|^2} = \frac{\frac{|Q|P|-P^2|^2}{|Q|^2|P|^2}}{\frac{|Q|^2-|P|^2}{|Q|^2}} \\ &= \frac{|Q|P|-P^2|^2}{|P|^2(|Q|^2-|P|^2)} \end{aligned} \tag{4.14}$$

If we solve $|Q|^2-|P|^2$ *we get*

$$|Q|^2-|P|^2 = \left((k_1+1)^2+k_2^2\right)^2 - \left((k_1^2+k_2^2-1)^2+4k_2^2\right) = 4k_1Q.$$

and

$$\begin{aligned} |Q|P|-P^2|^2 &= Q^2|P|^2+|P^2|^2-2Re(Q|P|\bar{P}^2) \\ &= Q^2((ReP)^2+(ImP)^2)+|(ReP+iImP)^2|^2-2Q|P|Re(\bar{P}^2) \\ &= Q^2((ReP)^2+(ImP)^2)+|(ReP)^2-(ImP)^2+2iRePImP|^2 \\ &\quad -2Q|P|\left((ReP)^2+(ImP)^2\right) \\ &= Q^2((ReP)^2+(ImP)^2)+\left((ReP)^2-(ImP)^2\right)^2+4(ReP)^2(ImP)^2-2Q|P|(ReP)^2 \\ &\quad -2Q|P|(ImP)^2 \\ &= Q^2((ReP)^2+(ImP)^2)+(ReP)^4+(ImP)^4-2(ReP)^2(ImP)^2+4(ReP)^2(ImP)^2 \\ &\quad -2Q|P|(ReP)^2-2Q|P|(ImP)^2 \\ &= Q^2(ReP)^2+Q^2(ImP)^2+(ReP)^2(ReP)^2+(ImP)^2(ImP)^2 \\ &\quad +(ReP)^2(ImP)^2+(ReP)^2(ImP)^2-2Q|P|(ReP)^2-2Q|P|(ImP)^2 \\ &= (ReP)^2\left(Q^2+(ReP)^2+(ImP)^2-2Q|P|\right) \\ &\quad +(ImP)^2\left(Q^2+(ReP)^2+(ImP)^2-2Q|P|\right) \\ &= (ReP)^2\left(Q^2+|P|^2-2Q|P|\right)+(ImP)^2\left(Q^2+|P|^2-2Q|P|\right) \\ &= (ReP)^2\left(Q^2+QB-2Q|P|\right)+(ImP)^2\left(Q^2+QB-2Q|P|\right) \\ &= \left(Q^2+QB-2Q|P|\right)(ReP)^2+(ImP)^2 \\ &= (Q+B-2|P|)\,Q|P|^2, \end{aligned}$$

Hence the equation (4.14) *becomes*

$$D_{f_N,\theta_0} = \frac{|Q|P|-P^2|^2}{|P|^2(|Q|^2-|P|^2)} = \frac{(Q+B-2|P|)\,Q|P|^2}{4k_1(Q)|P|^2} = \frac{Q+B-2|P|}{4k_1}. \tag{4.15}$$

We now compute

$$\begin{aligned}
\frac{1}{k(z,f_N)} &= \frac{|k_1+1+ik_2|-|k_1-1+ik_2|}{|k_1+1+ik_2|+|k_1-1+ik_2|} \\
&= \left(\frac{|k_1+1+ik_2|-|k_1-1+ik_2|}{|k_1+1+ik_2|+|k_1-1+ik_2|}\right)\left(\frac{|k_1+1+ik_2|-|k_1-1+ik_2|}{|k_1+1+ik_2|-|k_1-1+ik_2|}\right) \\
&= \frac{(k_1+1)^2+k_2^2+(k_1-1)^2+k_2^2-2\sqrt{\left((k_1+1)^2+k_2^2\right)(k_1-1)^2+k_2^2}}{(k_1+1)^2+k_2^2-(k_1-1)^2-k_2^2} \\
&= \frac{Q+B-2|P|}{4k_1}. \qquad (4.16)
\end{aligned}$$

Hence from equations (4.15) *and* (4.16) *we get*

$$D_{f_N,\theta_0} = \frac{1}{k(z,f_N)}.$$

We put this value in (4.7) *and obtain*

$$\begin{aligned}
M_2(f_N(\Gamma_0')) &= \int_0^{2\pi} \frac{1}{(1+C_N^2)\int_q^1 D_{f_N,\theta_0}\frac{dr}{r}}\,d\theta \\
&= \int_0^{2\pi} \frac{1}{(1+C_N^2)\int_q^1 \frac{1}{k(z,f_N)}\frac{dr}{r}}\,d\theta \\
&= \int_0^{2\pi} \frac{k(z,f_N)}{(1+C_N^2)\int_q^1 \frac{dr}{r}}\,d\theta \\
&= \int_0^{2\pi} \frac{k(z,f_N)}{(1+C_N^2)(-\log q)}\,d\theta \\
&= \frac{2\pi\, k(z,f_N)}{(1+C_N^2)(-\log q)} \\
&= \frac{2\pi\, k(z,f_N)}{(1+C_N^2)(-\log q)}\,\frac{(-\log q)}{(-\log q)} \\
&= \frac{1}{(1+C_N^2)(-\log q)^2}\int_{A(q,1)} k(z,f_N)\frac{1}{|z|^2}d\mathcal{L}^2(z)). \qquad (4.17)
\end{aligned}$$

The extremal function given by

$$\begin{aligned}
\rho_0\circ\gamma_\Phi^{C_N}(r) &= \frac{D_{f_N,\theta_0}}{r\left|\frac{\partial\gamma_\Phi^{C_N}(r)}{\partial r}\right|\int_q^1 \frac{D_{f_N,\theta_0}}{r}\,dr} \\
&= \frac{\frac{1}{k(z,f_N}}{r\left|\frac{\partial\gamma_\Phi^{C_N}(r)}{\partial r}\right|\int_q^1 \frac{1}{r\,k(z,f_N)}}\,dr \\
&= \frac{1}{r(-\log q)\sqrt{1+C_N^2}} \qquad (4.18) \\
&= \frac{1}{|z|(-\log q)\sqrt{1+C_N^2}}.
\end{aligned}$$

4.3 Relation between the shear map and spiral map

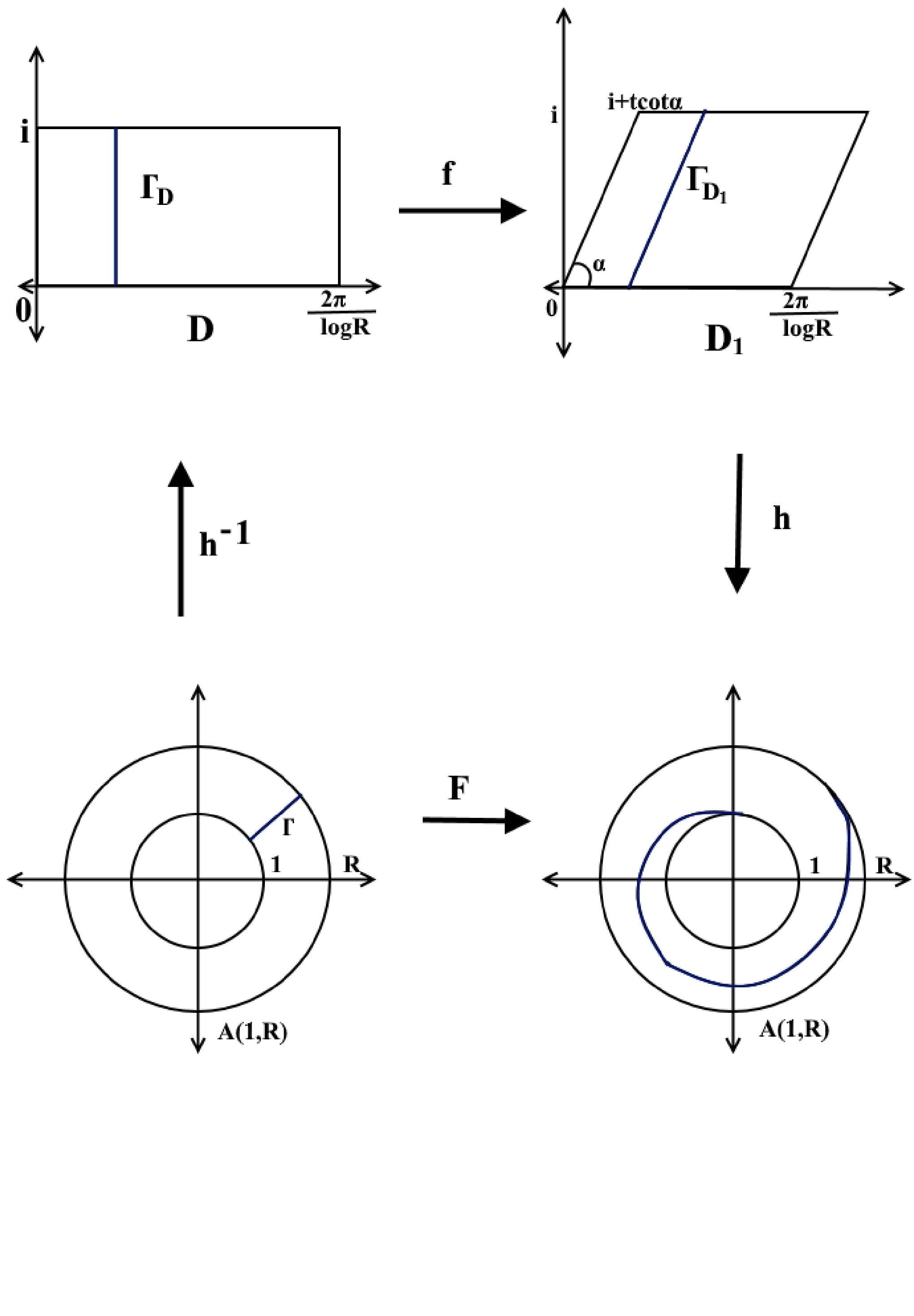
Definition 4.6. Let $h\colon D \to A(1,R)$ be a map such that

$$h(z) = e^{(iz+1)\log R} \tag{4.19}$$

From Definition 4.6, the inverse of h is :

$$h^{-1}(w) = \frac{1}{i}\left(\frac{\log w}{\log R} - 1\right). \tag{4.20}$$

The aim is to find the image of radial curves Γ in $A(1,R)$, under the map F, where $F = h \circ f \circ h^{-1}$ is the composition of the functions f, h defined in Definitions 4.2, 4.6 respectively, see Figure 4.2. We proceed by steps.



i
Γ_D
0
D
$\frac{2\pi}{\log R}$
f
i
i+tcotα
Γ_{D_1}
α
0
D_1
$\frac{2\pi}{\log R}$
h^{-1}
h
Γ
1
R
A(1,R)
F
1
R
A(1,R)

Figure 4.2

Step 1 In Figure 4.2 we can observe that if Γ is a radial curve joining the boundaries in annulus $A(1, R)$, and if we take $\Gamma = w = re^{it}$ to be a parametrization of Γ, where $0 \leq t \leq 2\pi,\ 1 \leq r \leq R$, then we obtain

$$h^{-1}(w) = \frac{1}{i}\left(\frac{\log r + it}{\log R} - 1\right).$$

If we impose the condition $r = 1$, then we get
$h^{-1}(w) = i$ for $t = 0$ and

$$h^{-1}(w) = \frac{2\pi}{\log R} + i,$$

for $t = 2\pi$. Similarly, if we impose the condition $r = R$, then we get
$h^{-1}(w) = 0$, for $t = 0$ and

$$h^{-1}(w) = \frac{2\pi}{\log R},$$

for $t = 2\pi$. This means under the inverse function $h^{-1}(w)$ the radial curve Γ are maped to the vertical curve Γ_D connecting the horizontal sides of the rectangle D, see Figure 4.2.

Step 2 We consider Γ_D be a vertical curve connecting horizontal sides in D as shown in Figure 4.2. We parametrize it as $\Gamma_D = a + it$, where $0 \leq a \leq \frac{2\pi}{\log R}$ is fixed, $0 \leq t \leq 1$. Then f maps a curve Γ_D into Γ_{D_1} as $f(a + it) = a + t\cot\alpha + it$, see Figure 4.2.

Step 3 Let $\Gamma_{D_1} = a + t(\cot\alpha + i)$ be a parametrization of curve Γ_{D_1}, where Γ_{D_1} is a curve connecting horizontal sides in D_1, as shown in Figure 4.2 and a, t are as *Step 2*. Then h acts as follows:

$$h(a + t(\cot\alpha + i)) = e^{[i(a+t(\cot\alpha+i))+1]\log R} = Re^{-t\log R + i(a+t\cot\alpha)\log R}.$$

This implies that

$$|h(z)| = Re^{-t\log R} = R^{1-t} \quad \text{and} \quad arg(h(z)) = a\log R + t\cot\alpha\log R.$$

In conclusion we obtain that the image of family of radial curves in $A(1, R)$, is logarithmic spirals with winding number 0 under the composition of function $F = h \circ f \circ h^{-1}$, shown in Figure 4.2. But if we imposed that the range of $\cot\alpha$ is $\frac{2\pi}{\log R} \leq \cot\alpha < 2\frac{2\pi}{\log R}$ then we get the same image as before, but now we obtain the winding number 1. In general if N is a winding number then range of $\cot\alpha$ becomes

$$N\frac{2\pi}{\log R} \leq \cot\alpha < (N+1)\frac{2\pi}{\log R} \quad \text{where} \quad N \in \mathbb{N}.$$

Bibliography

[1] L. Alhfors, *Lectures on quasiconformal mappings.* Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10 D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London 1966, 146 pp.

[2] L. Alhfors, A. Beurling, *Conformal invariants and function-theoretic null-sets.* Acta Math. **83**, (1950), 101–129.

[3] F. P. Gardiner, *Teichmüller theory and quadratic differentials.* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987. 236 pp.

[4] H. Grötzsch. *Über die Verzerrung bei schlichten nichtkonformenAbbildungen und über eine damit zusammenhängende Erweiterung des Picardschen Satzes.* Ber. Verh, Sächs. Akad. Wiss. Leipzig, **80**, (1928), 503-507.

[5] Strebel, K.*Quadratic Differentials.* Springer Verlag, Berlin - Heidelberg - New York - Tokyo 1984, XII, 184 S., 74 Abb., DM 98

[6] Kari Astala, Tadeusz Iwaniec, Gaven J. Martin, and Jani Onninen, *Extremal mappings of finite distortion,* Proc. London Math. Soc. (3) 91 (2005), no. 3, 655–702.

[7] B. Rodin, *The method of extremal length,* Bull. Amer. Math. Soc. 80 (1974), no. 4, 587–606

[8] O. Lehto, K. I. Virtanen, *Quasikonforme Abbildungen.* (German) Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Bercksichtigung der Anwendungsgebiete, Band Springer-Verlag, Berlin-New York 1965, 269 pp.

[9] Jussi, Väisälä . *Lectures on n-dimensional quasiconformal mappings,* Lecture Notes in Mathe-matics, Vol. 229, Springer-Verlag, Berlin, 1971.

[10] Olli Martio, Vladimir Ryazanov, Uri Srebro, and Eduard Yakubov. *On Q-homeomorphisms,* Ann. Acad. Sci. Fenn. Math. 30 (2005), no. 1, 49–69.

[11] Balogh, Zoltán M.(CH-BERN); Fässler, Katrin(CH-BERN); Platis, Ioannis D.(GR-CRET) Modulus of curve families and extremality of spiral-stretch maps. (English summary) J. Anal. Math. 113 (2011), 265–291.

[12] S. K. Donaldson *Riemann Surfaces*, 2004. (Online: wwwf.imperial.ac.uk/~skdona/RSPREF.PDF)

[13] Teichmüller mapping. *Encyclopedia of Mathematics.* (Online:http://www.encyclopediaofmath.org/index.php?title=Teichm%C3%BCller_mapping&oldid=23544)

[14] Nathan Mercer, Rich Stankewitz *Introduction to Quasiconformal Mappings in the plane with an application to quasiconformal surgery*, (Online:rstankewitz.iweb.bsu.edu/QCMapsIntro.pdf)

[15] James Munkres, *Topology*, 2nd edition, Prentice Hall, Inc. 2000. (see page 336).

[16] Belinskii, P. P. *General properties of quasiconformal mappings,* Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk, 1974. 98 pp. (errata insert). (Reviewer: R. Kuhnau) 30A60

[17] Brakalova, M., Markina, I. and Vasil'ev, A.,*Extremal functions for modules of systems of measures*, (Online:arXiv:1409.1626), [math.CA], 2014.

[18] Krushkal, S. L., Apanasov, B. N., and Guseveski, N. A. *Kleinian groups and uniformization in examples and problems,* Providence, R.I., 1986. American Mathematical Society.

[19] Astala, Kari(FIN-HELS); Iwaniec, Tadeusz(1-SRCS); Martin, Gaven(NZ-MASS2) *Elliptic partial differential equations and quasiconformal mappings in the plane* , Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009, ISBN: 978-0-691-13777-3.