An Effective Quadrilateral Mesh Adaptation

Sanjay Kumar Khattri

Department of Mathematics, The University of Bergen, Norway

sanjay@mi.uib.no

http://www.mi.uib.no/∼sanjay

Abstract: Accuracy of a simulation is strongly depend on the grid quality. Here, quality means orthogonality at the boundaries and quasi-orthogonality within the critical regions, smoothness, bounded aspect ratios and solution adaptive behaviour. Instead of wasting computational resources by refining part of the domain where the solution shows little variation. It is desired to concentrate grid points and cells in the part of the domain where solution shows strong gradients or variations. We present a simple, effective and computationally efficient approach for quadrilateral mesh adaptation. Several numerical examples are presented for supporting our claim.

Keywords: Quadrilateral mesh; Area functional; Adaptive function; Jacobian; Partial Differential Equations.

MSC2000: 65L50;65N50;65M50.

1 Introduction

Adaptive grids are desired for solving partial differential equations (PDEs) [2; 6]. There are various techniques for generating adaptive quadrilateral meshes. For example, solution of coupled elliptic system [1; 13], minimization of a functional or variational approach (see [8; 9; 12, and references therein]), etc. In this article, we are presenting a simple and effective technique for generating adaptive quadrilateral meshes. Several numerical examples are presented for validating our approach. We are extending the area functional for generating adaptive meshes. For a detailed description of the area functional, we refer the interested reader to References [10; 11; 13; 14].

Let us define some quantities of interest. Figure 1 shows a quadrilateral cell. In this figure, $\mathbf{g}_1$ and $\mathbf{g}_2$ are the two co-variant vectors at the node o. Other interesting quantities such as the Jacobian matrix ($\boldsymbol{J}$) and the $\boldsymbol{g}$-tensor at the node o and for the given cell can be defined from these two vectors. As can be seen in the Figure 1, the columns of the Jacobian matrix are the two co-variant vectors.

An outline of the article is as follows. In the Section 2, a discrete functional for quadrilateral mesh adaptation is presented. Section 3 presents several numerical examples. Finally, Section 4 concludes the article.

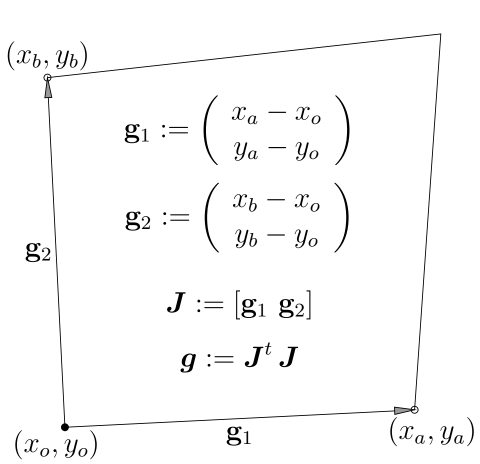


Figure 1: Quantities of interest for a quadrilateral cell.

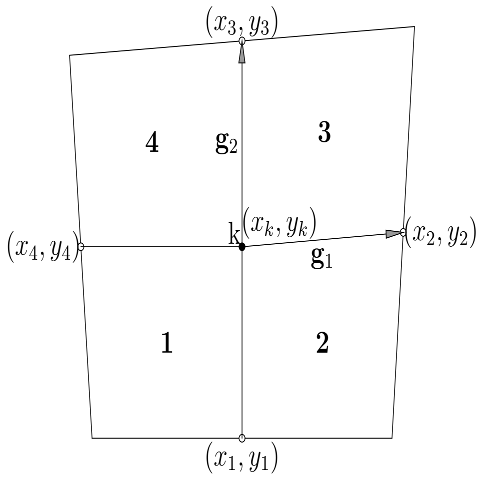


Figure 2: 2D Structured mesh.

Table 1: Jacobian matrix at the node k for the four surrounding cells (see Figure 2).

$\boldsymbol{J}(k_1) = \begin{bmatrix} (x_4 - x_k) & (x_1 - x_k) \\ (y_4 - y_k) & (y_1 - y_k) \end{bmatrix}$	$\boldsymbol{J}(k_2) = \begin{bmatrix} (x_2 - x_k) & (x_1 - x_k) \\ (y_2 - y_k) & (y_1 - y_k) \end{bmatrix}$
$\boldsymbol{J}(k_3) = \begin{bmatrix} (x_2 - x_k) & (x_3 - x_k) \\ (y_2 - y_k) & (y_3 - y_k) \end{bmatrix}$	$\boldsymbol{J}(k_4) = \begin{bmatrix} (x_4 - x_k) & (x_3 - x_k) \\ (y_4 - y_k) & (y_3 - y_k) \end{bmatrix}$

2 Area functional for mesh adaptation

The first of study of the area functional was done by Castillo and Steinberg [10; 11]. As per authors information, area functionals have not been used for generating adaptive mesh.

Let a quadrilateral mesh be consist of n internal nodes, and each of these node is surrounded by four quadrilaterals (mesh can also be unstructured). According to [10; 11; 14] the Area functional is given as

$$\mathcal{F}(\mathbf{x}, \mathbf{y}) \stackrel{\mathbf{def}}{=} \sum_{k=1}^{n} \left[\sum_{i=1}^{4} |\boldsymbol{J}(k_i)|^2 \right]. \tag{1}$$

Here, $\boldsymbol{J}(k_i)$ is the Jacobian matrix at the node k and for the quadrilateral cell i, and $|\boldsymbol{J}(k_i)|$ is the determinant of the Jacobian matrix. The Jacobain is a measure of area of the quadrilateral cell. Thus, the optimization of the area functional tries to produce grids with least variation in cell areas [14]. Figure 2 shows a 2×2 mesh. The internal node k is surrounder by four cells. The Jacobian matrices for the four cells are given in the Table 1. The article [14] presents some general properties of the area functional.

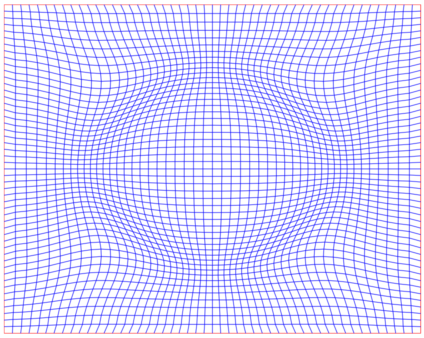


Figure 3: $\eta = 1.0$.

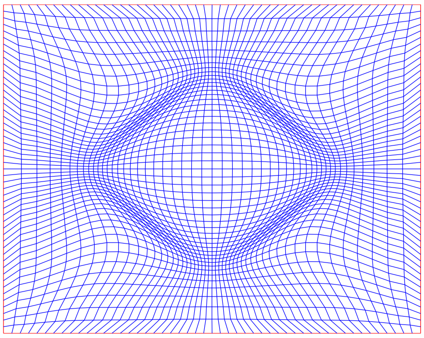


Figure 4: $\eta = 5.0$.

For grid adaptation, the authors propose the following form of the area functional

$$\mathcal{F}(\mathbf{x}, \mathbf{y}) := \sum_{k=1}^{I} \left[\sum_{i=1}^{4} |\boldsymbol{J}(k_i)|^2 \, \Phi(k_i) \right]. \tag{2}$$

Here, Φ is called the adaptive function, and $\Phi(k_i)$ is the value of the adaptive function at the center of the cell i surrounding the node k. It is assumed that Φ is greater than zero. The functional can be optimized by algorithms such as the Newton [2]. Optimization of (2) will equi-distribute the product of area of each cell and adaptive function. Roughly speaking, the cells with larger value of Φ will have smaller area. If Φ is same for each cell than the optimization of (2) will try to generate cells of equal area.

Some of the properties of the functional $\mathcal{F}(\mathbf{x}, \mathbf{y})$ are: the critical point of the functional is a grid for which the product of cell area and the adaptive function is same for every cell, and the Hessian is semipositive definite.

3 Numerical Examples

3.1 Example: 1

Let the adaptive functions be given as

$$\Phi(x, y) = 1.0 + \eta \,\mathrm{sech}\left[20\,(x-0.5)^2 + 20\,(y-0.5)^2 - 1.8\right] \tag{3}$$

$$\Phi(x, y) = 1.0 + 1.0\,\mathrm{sech}\left[\alpha\,(x+y-1)^2\right] \tag{4}$$

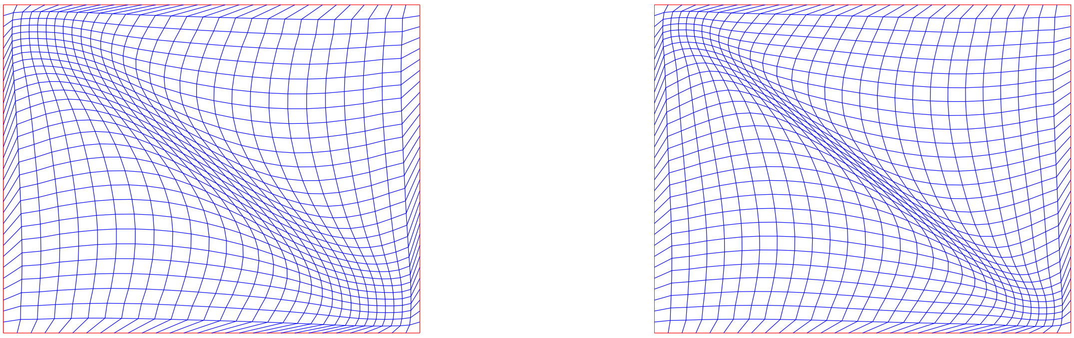
$$\Phi(x, y) = 5.0 + \kappa\,\left[\sin(2\,\pi\,x)\,\sin(2\,\pi\,y)\right] \tag{5}$$

$$\Phi(x, y) = 5.0 + \beta\,\left|\sin(2\,\pi\,x)\,\cos(2\,\pi\,y)\right| \tag{6}$$

$$\Phi(x, y) = 5.0 + 200.0\,\left|\sin(\pi\,x)\,\sin(2\,\pi\,y)\right| \tag{7}$$

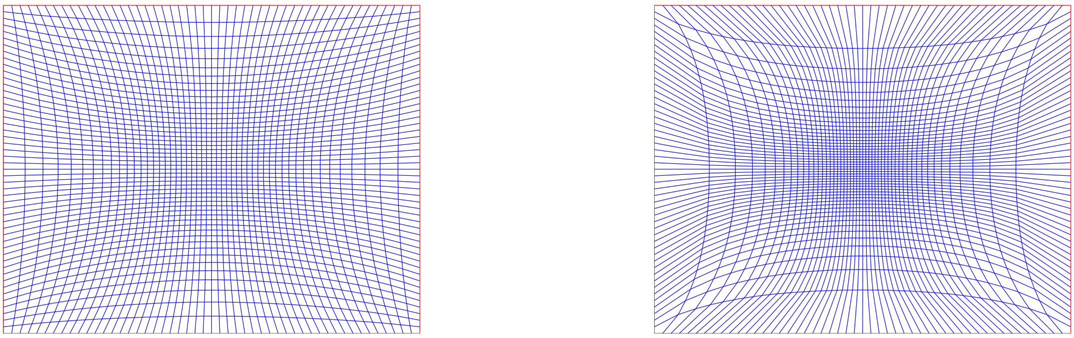
$$\Phi(x, y) = 1.0 + \tanh\left[(10\,(x-0.5)^2 + 50\,(y-0.5)^2 - 1.875)\right] \tag{8}$$

For different value of η, the adapted meshes are shown in Figures 3 and 4. Figures 5(a) and 5(b) reports the outcome of our experiments for different values of the parameter α. Figures 6(a) and 6(b) are the adaptive meshes for $\kappa = 20.0$ and $\kappa = 200.0$ respectively.



(a) $\alpha = 20.0$ (b) $\alpha = 50.0$.

Figure 5: Adaptive function is given by the equation (4).



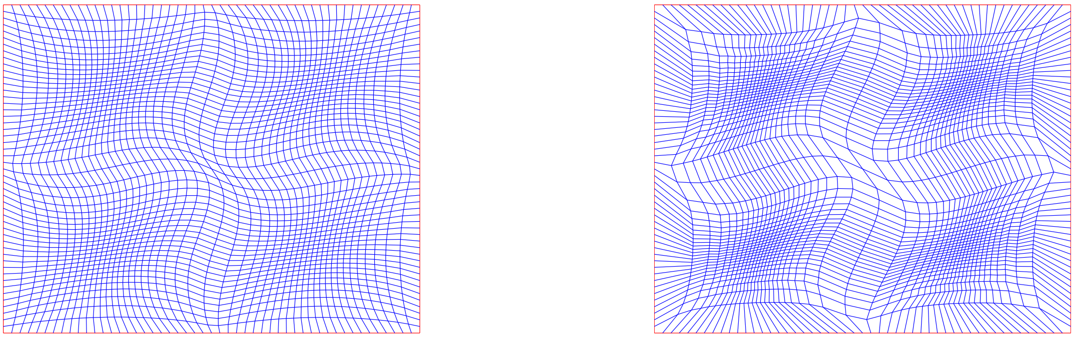
(a) $\kappa = 20.0$ (b) $\kappa = 200.0$.

Figure 6: Adaptive function is given by the equation (5).

For $\beta = 20.0$ and $\beta = 50.0$ the adaptive meshes are given in the Figures 7(a) and 7(b) respectively. Figure 8 is the adapted by the equation (7) and Figure 9 is adapted by the equation (8).

3.2 Example: 2

We are solving the Poisson problem $-\operatorname{div}(\operatorname{grad} u) = f(x, y)$ on an adaptive and on a uniform mesh by the method of Finite Volumes [3; 4; 5; 7, and references theirin]. Our domain is $\Omega = [0, 1]\times[0, 1]$. Let the exact solution be $u(x, y) = \exp\left[-100\left((x-0.5)^2 + (y-0.5)^2\right)\right]$. The solution inside the domain is enforced by the Dirichlet boundary condition and source term. Table 2 reports the errors in the L_2 and L_∞ norms on the adapted (see Figure 11) and uniform mesh (see Figure 10). It can be seen in the table that error (in the L_2 and L_∞ norms) on the adaptive mesh is substantially smaller.



(a) $\beta = 20.0$ (b) $\beta = 200.0$.

Figure 7: Adaptive function is given by the equation (6).

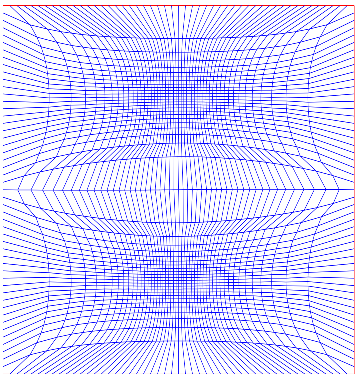


Figure 8: Adapted Functional is give by the equation 7.

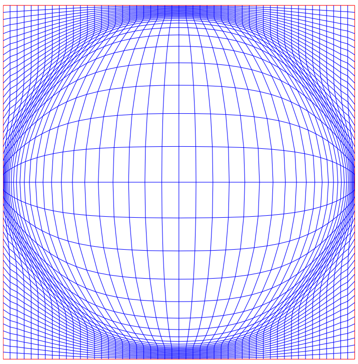


Figure 9: Adapted Functional is give by the equation 8.

Table 2: Error on the adaptive (see Figure 11) and uniform meshes (see Figure 10).

Mesh	$\|\mathbf{p}-\mathbf{p}_h\|_{L_2}$	$\|\mathbf{p}-\mathbf{p}_h\|_{L_\infty}$
Uniform	**0.003**	**0.03**
Adaptive	**0.0009**	**0.008**

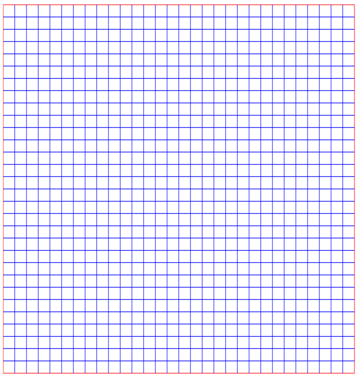


Figure 10: Initial grid for the Example 3.2.

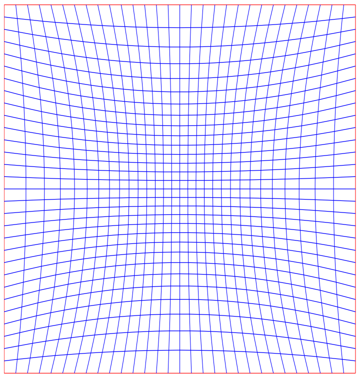


Figure 11: Adapted grid for the Example 3.2. Grid is generated by the equation 5 with $\kappa = 10.0$.

4 Conclusions

It is not always feasible to blindly refine the mesh in the hope of capturing the physics because of the computational resources. It is desired to adapt the grid to the requirement of the underlying problem. In this article, a simple and robust technique for generating adaptive quadrilateral meshes is presented. We have presented various examples for generating adaptive meshes. It is shown that the error on the adaptive mesh is small. The approach can be useful for solving evolutionary problems (parabolic and hyperbolic equations) on adaptive meshes.

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