

# **Analytical, combinatorial and topological properties of quadratic differentials and their applications**

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# Abstract

Quadratic differentials first appeared in 1930s in works of Teichmüller in connection with moduli problem for Riemann surfaces. Later it was revealed that quadratic differentials and their trajectories give solutions to extremal problems for moduli of families of curves and extremal partition of Riemann surfaces. Over the last decades there has been discovered a connection between quadratic differentials and extremal problems of different nature related to minimal surfaces, potential theory, approximation theory, mathematical physics.

In this work, we study quadratic differentials, discover new properties of certain types of quadratic differentials and explore the connection of quadratic differentials with other disciplines, such as discrete mathematics, topology and even applied mathematics. We apply the theory of quadratic differentials to solve problems of complex analysis and study analytical problems that are connected with quadratic differentials.

The dissertation is organized as follows. Chapter 1 contains an introduction to the notions and facts that are necessary for understanding the main results of this work. Section 1.1 gives a short introduction to quadratic differentials and their trajectory structure. It also recalls the definition of reduced moduli of digons. Section 1.2 describes the tools from discrete mathematics and topology that are used in this work. Chapter 2 gives a summary of the research papers that constitute the main scientific contributions of the thesis. Both Chapter 1 and Chapter 2 are followed by lists of references. The following papers are included in Chapter 3.

Paper A: A. Frolova, M. Levenshtein, D. Shoikhet, and A. Vasil'ev, **Boundary distortion estimates for holomorphic maps**, In *Complex Analysis and Operator Theory*, 8 (2013), no. 5, 1129–1149 .

Paper B: A. Frolova, A. Vasil'ev, **Combinatorial description of jumps in spectral networks**, accepted to *Proceedings AMS*.

Paper C: A. Frolova, D. Khavinson, and A. Vasil'ev, **Polynomial lemniscates and their fingerprints: from geometry to topology**, Submitted to *New Trends in Complex and Harmonic Analysis*.

Paper D: A. Frolova, M. Solberg, and A. Vasil'ev, **Pure braids and homotopy classes of closed loops**, paper in preparation.

In Paper A we apply extremal partitions and moduli of digons to obtain a boundary distortion estimate for a conformal self-mapping of the unit disk with two fixed points on the boundary. Paper B studies combinatorial structure of the family of quadratic differentials with a finite simple pole that possess a short trajectory. In Paper C we discover some properties of fingerprints of polynomial lemniscates and construct a non-

unitary operad that realizes a process of approximation of two dimensional shapes by polynomial lemniscates. In Paper D we define an action of a braid group on homotopy classes of closed curves on a punctured sphere, which is related to a problem of constructing an operad on quadratic differentials with several double poles.

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# Chapter 1

## Preliminaries

We begin the preliminaries chapter with a short introduction to the quadratic differentials and reduced moduli in Section 1.1. In Section 1.2 we give an introduction to some tools from discrete mathematics and topology that are used in Papers B, C and the paper in progress D. In particular, we introduce trees and operads in Subsection 1.2.2, Subsections 1.2.2 and 1.2.3 show how an associahedron, which is also called the Stasheff polytope, is realizable as triangulations of a regular polygon. Subsection 1.2.4 contains the definition of a braid group and an action of a braid group on a free group.

### 1.1 Quadratic differentials and moduli

#### 1.1.1 Quadratic differentials: definition, critical points and natural parameter

**Definition 1.** Let  $S$  be a Riemann surface with atlas  $\{(U_\alpha, \phi_\alpha)\}$ , where each  $\phi_\alpha$  is a holomorphic homeomorphism of an open set  $U_\alpha$  of  $S$  onto a neighbourhood of zero in the complex plane with a local parameter  $z_\alpha$ . For each chart  $(U_\alpha, \phi_\alpha)$  about an arbitrary point  $P$  of  $S$  we define a function element  $Q_{z_\alpha}(z_\alpha)$ . If  $(U_\beta, \phi_\beta)$  is any other chart about  $P$ , let the corresponding function elements satisfy the following invariance property:

$$Q_{z_\alpha}(z_\alpha) (dz_\alpha)^2 = Q_{z_\beta}(z_\beta) (dz_\beta)^2, \quad dz_\beta = \frac{dz_\beta}{dz_\alpha} dz_\alpha. \quad (1.1)$$

Such a collection of function elements forms a *quadratic differential* on  $S$ .

Relation (1.1) can be interpreted as the rule of change of coordinates. Keeping this in mind, we can fix one local parameter denoted by  $z$  and refer to the quadratic differential simply as  $Q(z) dz^2$ .

**Example 2.** Consider the quadratic differential  $Q(z) dz^2 = (z^2 - 1) dz^2$  on the Riemann sphere  $\hat{\mathbb{C}}$ . The function element  $z^2 - 1$  represents the quadratic differential in the finite complex plane. Let us find the form of  $Q(z) dz^2$  about infinity. Let  $w := \frac{1}{z}$ . The quadratic differential about infinity looks as follows:

$$Q(w) = Q(z) \left( \frac{dz}{dw} \right)^2 = \left( \frac{1}{w^2} - 1 \right) \frac{1}{w^4} = (1 - w)(w + 1) \frac{1}{w^6}.$$

Note that  $Q(z)$  has simple zeros at 1 and  $-1$ . The function  $Q(w)$  also has simple zeros at 1 and  $-1$ , together with a pole of order 6 at infinity. We say that the quadratic differential  $Q(z) dz^2$  has simple *zeros* at  $\pm 1$  and a *pole* of order 6 at infinity.

**Remark 1.** Further on we consider only quadratic differentials  $Q(z) dz^2$ , where  $Q(z)$  is a rational or a polynomial function.

**Definition 3.** Zeros and poles of a quadratic differential  $Q(z) dz^2$  are called *critical points* of  $Q(z) dz^2$ . Points that are not critical are called *regular*.

The following lemma (see [11]) shows that the notion of a zero or a pole of a quadratic differential is well-defined.

**Lemma 4.** *Let  $z$  and  $w$  be two local coordinates about a point  $P \in S$ ,  $z = 0$  and  $w = 0$  correspond to  $P$ , and let  $Q(z)$  and  $Q(w)$  be the function elements of a quadratic differential that correspond to  $z$  and  $w$ . If  $Q(z)$  has a zero (respectively, pole) of order  $n$  about  $z = 0$ , then  $Q(w)$  also has zero (respectively, pole) of order  $n$  about  $w = 0$ .*

Given a quadratic differential  $Q(z) dz^2$ , it is possible to choose a single valued branch of a square root  $Q^{1/2}$  in a small enough neighbourhood  $U$  of a regular point. Let us define a local parameter  $w$  by

$$w = \Phi(\zeta) = \int_{\zeta} \sqrt{Q} \quad (1.2)$$

and note that it is well-defined.

**Remark 2.** The integral in the latter formula is understood as an integral of  $\sqrt{Q(z)}$  along a piecewise smooth curve connecting an arbitrary point  $\zeta_0$  with a point  $\zeta$  in the  $z$ -plane.

Let us express  $Q(z) dz^2$  in terms of this new parameter.

$$Q(z) = Q(w) \left( \frac{dw}{dz} \right)^2 = Q(w) Q(z).$$

Therefore, in a small enough neighbourhood of a regular point the function element  $Q(w)$  is identically equal to 1 and we can write

$$Q(z) dz^2 = dw^2.$$

The parameter  $w$  is called the *natural parameter* near the regular point. It gives the simplest possible representation of the quadratic differential in the given neighbourhood. The natural parameter of  $Q(z) dz^2$  around a pole or a zero has to be chosen differently. For example, in a neighbourhood of a double zero  $P$  the quadratic differential can be represented as

$$Q(z) dz^2 = 4\zeta^2 d\zeta^2. \quad (1.3)$$

In a neighbourhood of a double zero function element  $Q(z)$  has form

$$Q(z) = z^2 (a_2 + a_3 z + \dots),$$

where  $(a_2 + a_3z + \dots)$  is a non-vanishing holomorphic function. In a small enough neighbourhood of the origin we can choose  $\sqrt{Q}$  to be

$$\sqrt{Q(z)} = z(b_1 + b_2z + \dots),$$

and integrate it term by term to obtain

$$z^2 \left( \frac{b_1}{2} + \frac{b_2}{3}z + \dots \right).$$

We denote by

$$\zeta = z \left( \frac{b_1}{2} + \frac{b_2}{3}z + \dots \right)^{\frac{1}{2}} = z(c_1 + c_2z + \dots),$$

where  $\zeta$  becomes a conformal change of variable. We define

$$w = \Phi(\zeta) := \zeta^2.$$

The latter implies that

$$dw^2 = 4\zeta^2 d\zeta^2.$$

We have illustrated a particular case of the following theorem.

**Theorem 3.** *Let  $Q(z) dz^2$  be a quadratic differential defined on a Riemann surface  $R$  and let  $P \in R$ .*

1. *If  $P$  is a regular point, then in a neighbourhood of  $P$  the quadratic differential can be represented as*

$$Q(z) dz^2 = d\zeta^2.$$

2. *If  $P$  is a zero of order  $n$ , then in a neighbourhood of  $P$  the quadratic differential can be represented as*

$$Q(z) dz^2 = \left( \frac{n+2}{2} \right)^2 \zeta^n d\zeta^2.$$

*The corresponding function  $\Phi(\zeta)$  has the form  $\zeta^{\frac{n+2}{2}}$ .*

3. *If  $P$  is a pole of odd order  $n$ , then in a neighbourhood of  $P$  the quadratic differential can be represented as*

$$Q(z) dz^2 = \left( \frac{2-n}{2} \right)^2 \zeta^{-n} d\zeta^2.$$

*The corresponding function  $\Phi(\zeta)$  has the form  $\zeta^{\frac{2-n}{2}}$ .*

4. *If  $P$  is a pole of second order, then in a neighbourhood of  $P$  the quadratic differential can be represented as*

$$Q(z) dz^2 = \frac{a-2}{\zeta^2} d\zeta^2,$$

where  $Q(z)$  has expansion  $Q(z) = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1z + \dots$  about  $z = 0$  that corresponds to  $P$ .  $Q(z)$  can be represented as  $z^{-2}f(z)$ , where  $f(z)$  is a non-vanishing holomorphic function.

The corresponding to  $Q(z)$  function  $\Phi(\zeta)$  has form  $b \log \zeta$ , where  $b$  is the constant coefficient of the Taylor expansion of  $f^{1/2}$ .

5. If  $P$  is a pole of even order  $n > 2$ , then in a neighbourhood of  $P$  the quadratic differential can be represented as

$$Q(z)dz^2 = \left( \frac{b}{\zeta} + \frac{n+2}{2}\zeta^{\frac{n}{2}} \right)^2 d\zeta^2,$$

where  $b$  is a coefficient of the logarithmic term  $b \log z$ , which appears after integrating the square root of the expansion of  $Q(z)$  in a punctured neighbourhood of  $P$ .

### 1.1.2 Trajectory structure of a quadratic differential

**Definition 5.** Let  $Q(z)dz^2$  be a quadratic differential on a Riemann surface  $S$ . A trajectory of  $Q(z)dz^2$  is a maximal smooth curve  $\gamma$  in  $S$ , such that  $Q(z)dz^2$  is real and positive along it. A maximal smooth curve  $\gamma$ , such that  $Q(z)dz^2 < 0$  along it, is called an *orthogonal trajectory* of the quadratic differential.

**Example 6.** Consider  $z^2 dz^2$  on  $\mathbb{C}$ . Origin is a double zero of this quadratic differential. It is easy to check that the curves  $z_1(t) = t$ ,  $0 < t < \infty$ ,  $z_2(t) = -t$ ,  $0 < t < \infty$ ,  $z_3(t) = it$ ,  $0 < t < \infty$ ,  $z_4(t) = -it$ ,  $0 < t < \infty$  are trajectories of  $z^2 dz^2$ .

A quadratic differential  $Q(z)dz^2$  induces a metric with a length element  $\sqrt{|Q(z)|}|dz|$ . The length of a curve  $g$  with respect to this metric is called the  $Q$ -length of  $g$ .

**Remark 4.** Let  $S$  be a Riemann surface and  $z$  be a local coordinate about  $P \in S$ . We can regard a conformal mapping  $w = f(z)$  as a conformal mapping from  $S$  to the  $w$ -plane.

Let  $P$  be a regular point of  $Q(z)dz^2$  and  $U$  be the maximal neighbourhood of  $P$  where  $w = \Phi(z) = \int_z \sqrt{Q}$  is defined.  $\Phi(z)$  maps  $U$  conformally onto a disk  $V$  in the  $w$ -plane, and  $P$  is mapped onto the origin. The intersection  $r$  of  $V$  and the real line is a horizontal line segment in the  $w$ -plane, thus the pre-image of  $r$  by  $\Phi(z)$  is a piece of a trajectory of  $Q(z)dz^2$  which goes through  $P$ . Note that  $\Phi^{-1}(z)$  is a conformal mapping of  $V$  onto  $U$ . Let us continue  $\Phi^{-1}(w)$  analytically along the real line. We obtain a chain  $C$  of maximal disks centered on the real line, such that  $\Phi^{-1}(w)$  is conformal on  $C$ . The intersection of  $C$  with the real line is a real interval, denote it by  $(a, b)$ . The image of  $(a, b)$  by  $\Phi^{-1}(w)$  is a trajectory  $\gamma$  of  $Q(z)dz^2$  going through  $P$ . Assume that the trajectory  $\gamma$  is a Jordan curve. Then the  $Q$ -length of  $\gamma$  is given by

$$|\gamma|_Q = \int_{\gamma} \sqrt{|Q(z)|}|dz| = \int_{(a,b)} |dw| = b - a.$$

The trajectory  $\gamma$  can be divided into two trajectory rays  $\gamma_a$  and  $\gamma_b$ , which are the pre-images by  $\Phi^{-1}$  of the intervals  $(a, 0]$  and  $[0, b)$  correspondingly. We define a *limit set*

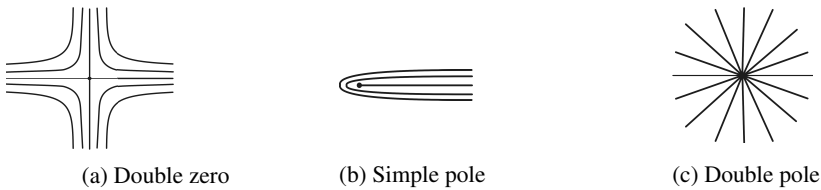


Figure 1.1: Local trajectory structure

of a trajectory ray  $\gamma_a$  as the set of all points  $P_1$  on  $R$ , such that there exists a sequence  $x_n$  of real numbers converging to  $a$  and the sequence  $\{\Phi^{-1}(x_n)\}$  converging to  $P_1$ . A limit set of a trajectory ray  $\gamma_b$  can be defined analogously. The limit set of a trajectory ray may be a zero or a pole of the quadratic differential. For example, a trajectory  $t$ ,  $0 < t < \infty$  of  $z^2 dz^2$  has trajectory rays with limit sets at the origin and infinity. A limit set of a trajectory ray sometimes may be a set with a non-empty interior, in this case the trajectory ray swipes out an entire domain. Such a trajectory ray is called a *recurrent* ray (See more details in [11], §10,11).

**Remark 5.** One can define *orthogonal trajectory rays* and give the  $Q$ -length of an orthogonal trajectory in an analogous manner.

Let  $P$  be a regular point of a quadratic differential  $Q(z) dz^2$  defined on  $S$ . According to Theorem 3, in a neighbourhood of  $P$  the following representation holds

$$Q(z) dz^2 = dw^2, \quad w = \Phi(z) = \int_z \sqrt{Q}.$$

Note that  $\Phi(z)$  defines a conformal change of variables. We note that  $dw^2 > 0$  along horizontal arcs in the  $w$ -plane. For example, we have  $dw^2 > 0$  along  $w(t) = t$ ,  $-\infty < t < \infty$ . The pre-image  $\gamma$  by conformal mapping  $\Phi(z)$  of a horizontal arc in the  $w$ -plane is a trajectory of the quadratic differential. We conclude that in a neighbourhood of a regular point trajectories of  $Q(z) dz^2$  look like horizontal arcs.

Let  $P$  be a double pole of  $Q(z) dz^2$ . We have shown that there exists a conformal change of variable  $\zeta(z)$ , such that the quadratic differential has form (1.3) with respect to the parameter  $\zeta$ . The  $\zeta$ -plane can be subdivided into 4 sectors  $k\frac{\pi}{2} \leq \arg \zeta \leq (k+1)\frac{\pi}{2}$ ,  $k = 0, 1, 2, 3$ , each of which is mapped by  $w = \Phi(\zeta) = \zeta^2$  onto a half-plane. We have that  $dw^2 > 0$  along horizontal arcs in the  $w$ -plane, the pre-images by  $\zeta^2$  of horizontal arcs in  $w$ -plane are shown on Figure 1.1 (a). This illustrates the local structure of trajectories of a quadratic differential around a double zero. Note that Figure 1.1 (a) demonstrates the trajectory structure of quadratic differential  $z^2 dz^2$  from Example 6 around the origin.

We have illustrated above two particular cases of the following theorem, which describes local trajectory structure of quadratic differentials (See more details in [12],[11]).

**Theorem 6** (Local trajectory structure of a quadratic differential). *Let  $Q(z) dz^2$  be a quadratic differential on Riemann surface  $S$  and  $P$  be a point on  $S$ .*

1. If  $P$  is a regular point of  $Q(z) dz^2$ , then in a small neighbourhood of  $P$  the trajectories are horizontal arcs.

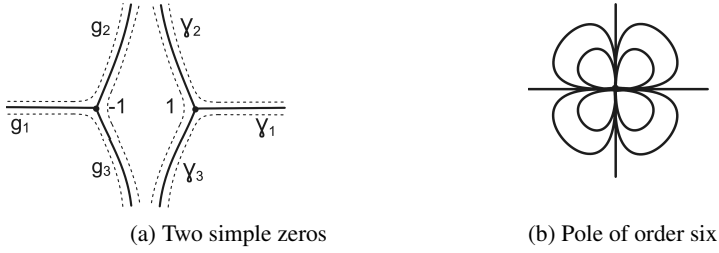


Figure 1.2: Trajectory structure of  $(z^2 - 1)dz^2$

2. If  $P$  is a zero of  $Q(z)dz^2$  of order  $n$ , then any trajectory entering a small neighbourhood  $U$  of  $P$  is a pre-image of a horizontal arc by the mapping  $\zeta^{\frac{n+2}{2}}$ . In particular, there are  $n+2$  trajectory rays that enter  $U$  and tend to  $P$  and divide  $U$  into  $n+2$  equal sectors. If  $P$  is a simple pole, then the situation is analogous to the one described above, with  $n$  being equal to  $-1$ .
3. If  $P$  is a double pole, then locally the trajectories around  $P$  are either concentric circles centered at  $P$ , or radial rays that tend to  $P$ , or spirals tending to  $P$ .
4. If  $P$  is a pole of order  $n$  larger than 2, then each trajectory ray that enters small neighbourhood  $U$  of  $P$ , tends to infinity in one of  $n-2$  distinguished directions, which divide  $U$  into  $n-2$  equal sectors. If a trajectory enters a sector, its trajectory rays tend to  $P$  in two directions. See Figure 1.2 (b).

Figure 1.1 illustrates trajectory structure of a quadratic differential around a double zero, simple pole and a double pole. Theorem 6 gives us a clear picture of what trajectories of a quadratic differential look like locally. To describe what happens with a trajectory outside a small neighbourhood of a point is generally a difficult task. Jenkins Main Structure Theorem (see [2]) sheds light on trajectory structure in large. We state this theorem after introducing certain types of domains that play an important role in describing global trajectory structure.

**Definition 7.** A circular domain  $D$  is a simply connected domain that contains a double pole  $P$  and is swept out by trajectories that separate  $P$  from the boundary of  $D$ . There is a mapping

$$g(z) = \begin{cases} e^{c \int_z \sqrt{Q}}, & z \neq P, c \neq 0; \\ 0, & z = P, \end{cases}$$

which maps  $D$  onto a disk centered at the origin.

A ring domain  $D$  is a doubly connected domain that contains no critical points and is swept out by trajectories that separate its two boundary components. There exists a map  $g(z) = \exp\{c \int_z \sqrt{Q}\}$ ,  $c \neq 0$ , which maps  $D$  onto an annulus.

A strip (half-plane) domain  $D$  is a simply connected domain that does not contain critical points and that is swept out by trajectories with both rays tending to poles of order at least two. There is a map  $g(z) = \int_z \sqrt{Q}$ , that maps  $D$  onto a horizontal strip (half-plane).

A dense structure is a domain that is swept out by a recurrent trajectory ray.

**Theorem 7** (Jenkins Main Structure Theorem, [2]). *Let  $Q(z) dz^2$  be a quadratic differential on a compact Riemann surface  $S$ , where  $Q(z)$  is a meromorphic function. Let  $\Psi$  be the union of all trajectories of  $Q(z) dz^2$  that contain zeros and simple poles in their closures and do not have recurrent rays. We denote by  $\bar{\Psi}$  the closure of  $\Psi$ .*

*Then  $S \setminus \bar{\Psi}$  is a union of non-overlapping domains of the following types: circular domains, ring domains, strip domains, half-plane domains, dense structures.*

**Remark 8.** Theorems 6 and 7 also hold for orthogonal trajectories.

**Example 8.** Consider  $z^2 dz^2$  on the Riemann sphere  $\hat{\mathbb{C}}$ . In this case  $\Psi$  is formed by the four trajectories described in Example 6. The closure of  $\Psi$  consists of  $z_1(t)$ ,  $z_2(t)$ ,  $z_3(t)$ ,  $z_4(t)$ , origin and infinity.  $S \setminus \bar{\Psi}$  is a union of four half-plane domains.

The quadratic differential  $(z^2 - 1) dz^2$  from Example 2 is defined on  $\hat{\mathbb{C}}$  and has simple zeros at  $z = 1$  and  $z = -1$  and a pole of order six at infinity.  $\Psi$  is formed by three trajectories  $\gamma_1, \gamma_2, \gamma_3$  that have  $z = 1$  and infinity in their closures, and three trajectories  $g_1, g_2, g_3$  that have  $z = -1$  and infinity in their closures. These are the trajectories that form three sectors in small neighbourhoods of  $z = 1$  and  $z = -1$  as described in Theorem 6, part 2; they are illustrated by bold lines on Figure 1.2 (a). Two of these trajectories, say  $\gamma_1$  and  $g_1$ , have an easy representation  $\gamma_1(t) = t$ ,  $1 < t < \infty$ ;  $g_1(t) = -t$ ,  $1 < t < \infty$ . On the other hand, these six trajectories  $\gamma_1, \gamma_2, \gamma_3, g_1, g_2, g_3$  tend to infinity in four distinguished directions as described in Theorem 6, part 4.  $\hat{\mathbb{C}} \setminus \bar{\Psi}$  consists of four half-plane domains and a strip domain. The strip domain is bounded by  $\gamma_2, \gamma_3, g_2, g_3$  and has  $1, -1$  and infinity on its boundary. Figure 1.2 (a) shows some of the trajectories that do not have trajectory rays that tend to simple zeros. The trajectory structure around infinity is illustrated on Figure 1.2 (b).

Let us consider orthogonal trajectories of the quadratic differentials discussed in this example. The quadratic differential  $z^2 dz^2$  has four orthogonal trajectories  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  that tend to the zero of the quadratic differential. We note that in a small neighbourhood of 0 the trajectories  $z_1, z_2, z_3, z_4$  form sectors  $k \frac{\pi}{2} \leq \arg \zeta \leq (k+1) \frac{\pi}{2}$ ,  $k = 0, 1, 2, 3$ . The orthogonal trajectories  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  divide the neighbourhood of the origin into sectors  $\frac{\pi}{4} + k \frac{\pi}{2} \leq \arg \zeta \leq \frac{\pi}{4} + (k+1) \frac{\pi}{2}$ ,  $k = 0, 1, 2, 3$ . On the other hand,  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  tend to the pole in four distinguished directions; locally around infinity the distinguished directions defined by trajectories and orthogonal trajectories of  $z^2 dz^2$  form eight equal sectors.

The quadratic differential  $Q(z) dz^2 = (z^2 - 1) dz^2$  has five orthogonal trajectories that tend to the zeros of  $(z^2 - 1) dz^2$ . One of them has an easy representation:  $w_1(t) = t$ ,  $-1 < t < 1$ . The  $Q$ -length of  $w_1$  is equal to 2; all the other orthogonal trajectories and trajectories of  $(z^2 - 1) dz^2$  have infinite  $Q$ -length. Orthogonal trajectory  $w_1$  has an orthogonal trajectory ray that tends to  $-1$ ; this orthogonal trajectory ray together with the two other orthogonal trajectory rays that tend to  $-1$  form three equal sectors locally around  $-1$ . Trajectory rays and orthogonal trajectory rays that tend to  $-1$  form 6 equal sectors locally about  $-1$ . The situation around the simple zero 1 is similar. There are four orthogonal trajectories that have an orthogonal trajectory ray that tends to either 1 or  $-1$  and an orthogonal trajectory ray that tends to infinity in one of four distinguished directions. The distinguished directions defined by trajectories and orthogonal trajectories form eight equal sectors locally around infinity.

Trajectories of quadratic differentials and, in particular, non-dense domains associated with them, give solutions to various mathematical problems. Some of them are

classical complex analysis problems (for example, extremal length problem, see [12]); some of them come from potential theory and approximation theory (see [5], [8], [9]).

### 1.1.3 Reduced modulus of a digon

Consider a simply connected domain  $Q$  bounded by a Jordan curve. Let us mark four points  $z_1, z_2, z_3, z_4$  on the boundary in a cyclic order. Let us fix the sequence  $z_1, z_2, z_3, z_4$ . Such a domain is called a quadrilateral and can be mapped conformally onto a rectangle  $R(r)$  with vertices at  $0, r, i+r, i$ , where  $r$  is a uniquely determined positive number. The mapping can be extended as a homeomorphism to the boundary. The fixed sequence  $z_1, z_2, z_3, z_4$  of vertices are mapped onto  $0, r, i+r, i$  respectively, (see Figure 1.3, left hand side). The number  $r$  is called the modulus of  $Q$ . A modulus of a quadrangle is a conformal invariant.

The same domain  $Q$  with the vertices  $z_1, z_2, z_3, z_4$ , but with fixed sequence  $z_2, z_3, z_4, z_1$  is mapped conformally onto  $R(\frac{1}{r})$ . Fixing the order of vertices in a sequence in this case is equivalent to choosing the "vertical" and "horizontal" sides of the quadrilateral. Let us summarize: the modulus of a simply connected domain  $Q$  with four vertices  $z_1, z_2, z_3, z_4$  is either a positive number  $r$  or  $1/r$ , depending on which sides we choose to be vertical.

A simply connected domain  $D$  bounded by a Jordan curve  $\partial D$  with marked vertices  $z_1$  and  $z_2$  on the boundary is called a *digon*. Let us require in addition that angles at the vertices  $z_1$  and  $z_2$  are defined; denote them by  $\varphi_1$  and  $\varphi_2$  respectively.

Let us define a reduced modulus of a digon. Let us choose two small positive numbers  $\varepsilon_1$  and  $\varepsilon_2$ . The circles  $|z - z_1| < \varepsilon_1$  and  $|z - z_2| < \varepsilon_2$  divide the interior of  $D$  into three domains: the intersections of the disks  $|z - z_1| < \varepsilon_1, |z - z_2| < \varepsilon_2$  with the interior of  $D$  and a quadrangle  $Q$ ; let arcs of  $\partial D$  be its horizontal sides.

Then the reduced modulus  $m(D, z_1, z_2)$  of the digon is defined by

$$m(D, z_1, z_2) = \lim_{\varepsilon_1, \varepsilon_2 \rightarrow 0} \left\{ \frac{1}{\varphi_1} \log \varepsilon_1 + \frac{1}{\varphi_2} \log \varepsilon_2 + m(Q) \right\}. \quad (1.4)$$

Assume that one of the vertices, say  $z_2$ , is infinity. In this case we can modify the definition as follows. We use the disk  $|z| > \varepsilon$  about infinity and replace the corresponding logarithmic term in the sum (1.4) with  $1/\varphi_2 \log 1/\varepsilon_2$ .

**Example 9.** Let  $\Gamma$  denote the real positive semi axis together with the origin and the infinity. Then  $\mathbb{C} \setminus \Gamma$  is a digon with vertices at 0 and infinity and angles  $2\pi$  at them. Let us show that  $m(D, 0, \infty) = 0$ . We consider a neighbourhood  $|z| < R_1$  about the origin and a neighbourhood  $|z| > R_2$  about infinity. The quadrangle with horizontal sides  $R_1 R_2$  is conformally mapped by  $\frac{1}{2\pi} \log z$  onto a rectangle with vertices  $\frac{1}{2\pi} \log R_1, \frac{1}{2\pi} \log R_2, \frac{1}{2\pi} \log R_2 + i, \frac{1}{2\pi} \log R_1 + i$ , (see Figure 1.3, right hand side). We obtain the following reduced modulus of  $Q$

$$m(Q) = \frac{1}{2\pi} \log \frac{R_2}{R_1}.$$

Thus the sum

$$\frac{1}{2\pi} \log R_1 + \frac{1}{2\pi} \log \frac{1}{R_2} + m(Q)$$

vanishes identically.



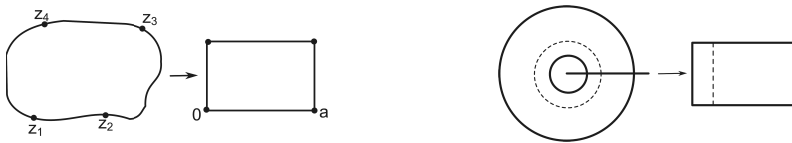


Figure 1.3: Mapping of a quadrangle

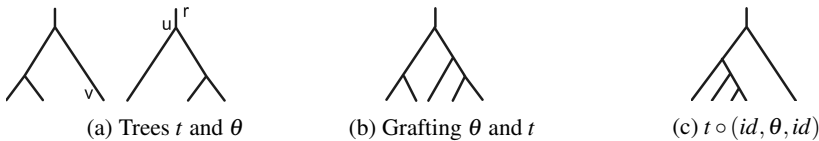


Figure 1.4: Composition of trees

## 1.2 Graphs, operads and polytopes

### 1.2.1 Trees and operads

Operads were defined first by J. Peter May in [6] in 1972, but appeared essentially even before that, for example in work by Boardman and Vogt [1].

**Definition 10.** An *operad* is a collection of sets  $\{O(n)\}$ ,  $n \in \mathbb{N}$ , together with an element  $id \in O(1)$  and a defined for each set of natural numbers  $k_1, \dots, k_n$  composition map

$$\begin{aligned} \circ : O(n) \times O(k_1) \times \dots \times O(k_n) &\longrightarrow O(k_1 + \dots + k_n), \\ (t, t_1, \dots, t_n) &\mapsto t \circ (t_1, \dots, t_n), \end{aligned}$$

which satisfies the following axioms:

1. (Identity axiom) For each  $t \in O(n)$

$$t \circ (id, \dots, id) = t,$$

$$id \circ t = t;$$

2. (Associativity axiom) For any  $t \in O(n)$ ,  $n$  elements  $t_1 \in O(k_1), \dots, t_n \in O(k_n)$ , and  $N = k_1 + k_2 + \dots + k_n$  elements  $\theta_1 \in O(j_1), \dots, \theta_{k_1} \in O(j_{k_1}), \theta_{k_1+1} \in O(j_{k_1+1}), \dots, \theta_N \in O(j_N)$  we have

$$\begin{aligned} t \circ (t_1 \circ (\theta_1, \dots, \theta_{k_1}), \dots, t_n \circ (\theta_{k_1+k_2+\dots+k_{n-1}+1}, \dots, \theta_{k_1+k_2+\dots+k_n})) = \\ (t \circ (t_1, \dots, t_n)) \circ (\theta_1, \dots, \theta_{k_1}, \dots, \theta_{k_1+k_2+\dots+k_{n-1}+1}, \dots, \theta_{k_1+k_2+\dots+k_n}). \end{aligned}$$

Let us give an example of an operad, namely an operad on trees. First we introduce some notions that are used in this example. A *tree* is a connected graph with  $n$  vertices and  $n - 1$  edges. A *degree* of a vertex  $v$  of a tree is the number of vertices adjacent to  $v$ . A *rooted tree*  $t$  is a tree with a distinguished vertex of degree 1, which is called the *root* of  $t$ ; all the other vertices of degree 1 are called *leaves* of  $t$ . A planar tree is a tree

together with an embedding into a plane. The Figure 1.4 (a) illustrates two different planar rooted trees. It is convenient to draw a planar rooted tree in such a way, that all the leaves lie on one horizontal line and the root is above the line. Then it is possible to order the leaves from left to right, in particular, we can number them according to this order.

Let us begin the construction of an operad on planar rooted trees. We define  $O(n)$  to be the set of planar rooted trees with  $n$  leaves. The identity element  $id$  is a tree that consists of one edge. For example, the trees on Figure 1.4 (a) are elements of  $O(3)$ .

The next step is to define an operation of composition, i.e. a composition map. Consider a planar rooted tree  $t$  with a distinguished leaf  $v$  and a planar rooted tree  $\theta$ , assume that the root  $r$  of  $\theta$  is adjacent to a vertex  $u$ . By grafting the root of  $\theta$  with the leaf  $v$  we understand deleting the edge  $ru$  and identifying the vertices  $v$  and  $u$ , i.e. gluing the tree  $\theta$  without the edge  $ru$  to the vertex  $v$ . The result of this process is illustrated on Figure 1.4(b). Given elements  $t \in O(n), t_1 \in O(k_1), \dots, t_n \in O(k_n)$ , we define the composition  $t \circ (t_1, \dots, t_n)$  by grafting the roots of  $t_1, \dots, t_n$  with the leaves  $1, \dots, n$  of  $t$ . The resulting tree is planar, it has  $k_1 + \dots + k_n$  leaves and has a root corresponding to the root of  $t$ . Figure 1.4 (b) illustrates in fact the composition of  $t$  with  $(id, id, \theta)$ . Figure 1.4 (c) shows  $t \circ (id, \theta, id)$ .

It is clear that both of the axioms of Definition 10 hold.

If we exclude the identity axiom from Definition 10, we obtain the definition of a *non-unitary operad*.

## 1.2.2 Triangulations of a convex polygon.

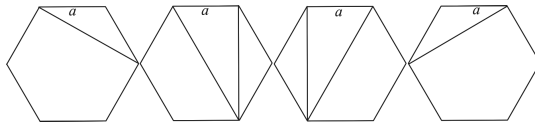
Consider a regular convex  $n$ -gon  $P$ . We call a set of non-intersecting diagonals of  $P$  a *triangulation* of  $P$ . A triangulation is *complete* if the number of diagonals is maximal, i.e. is equal to  $n - 3$ , and is *incomplete* otherwise. A triangulation divides the interior of  $P$  into convex polygons, which all are triangles if the triangulation is complete. It was proved by L. Euler that the number of distinct complete triangulations of a fixed regular convex  $n$ -gon  $P$  is equal to

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{k=0}^{n-1} C_k C_{n-1-k},$$

which is called the  $n$ -th Catalan number. Note that  $C_0 = 1$ .

**Example 11.** We denote by  $C_n$  number of complete triangulations of an  $n$ -gon. Let us count the number  $C_6$  of triangulations of a hexagon  $P_6$ .

Let us mark an edge of  $P_6$  as  $a$ . After a complete triangulation of  $P_6$ , the interior  $P_6$  is divided into triangles, and  $a$  is an edge of exactly one of them; denote this triangle by  $T$ . Note that  $T$  can be chosen in  $n - 2 = 4$  ways, see Figure 1.5. Once we have chosen  $T$ , we need to triangulate the rest of the interior of  $P_6$ . The sides of  $T$  divide the interior of  $P_6$  into a (possibly empty) polygon to the left from  $T$ , the triangle  $T$  and a (possibly empty) polygon to the right from  $T$ . The polygons to the left and to the right from  $T$  are to be triangulated. For example, the interior of the second from the left hexagon on Figure 1.5 has a 4-gon to the left from  $T$  and a triangle to the right from  $T$ . Therefore, the number of triangulations of this hexagon is equal to  $C_4 C_3$ . If we summarize all the 4 cases, we get a recursive formula for the 6-th Catalan number:

Figure 1.5: Triangulating a hexagon: choice of triangle  $T$ .

$$C_6 = C_5 + C_4C_3 + C_3C_4 + C_5 = \sum_{k=0}^5 C_k C_{5-k}.$$

### 1.2.3 Triangulations and the Stasheff polytope

In this subsection we describe combinatorial structure of the set  $\Sigma_n$  of all triangulations of a convex polygon. Complete triangulations of a convex regular  $n$ -gon correspond in fact to vertices of associahedron, which is sometimes also called Stasheff polytope. The Stasheff polytope was considered independently by J. Stasheff ([10]) and D. Tamari. The fact that  $\Sigma_n$  is realizable as a convex polytope dual to the associahedron was shown by M. Haiman and C. Lee ([4]). We start with a short introduction to polytopes, for more information see [13].

**Definition 12.** A convex polytope  $P$  in  $\mathbb{R}^n$  is a convex hull of a finite number of points in  $\mathbb{R}^n$ . We say that a hyperplane  $H \in \mathbb{R}^n$  supports  $P$  if  $H \cap P \neq \emptyset$  and  $P$  lies entirely in one of the closed half-spaces bounded by  $H$ ; the set  $H \cap P$  is called a *face* of  $P$ .

Every face  $F$  of a polytope  $P$  is a polytope of a possibly smaller dimension, every face of  $F$  is a face of  $P$ .

**Definition 13.** Faces of dimension 0 and 1 are called *vertices* and *edges* of  $P$  respectively. Faces of dimension  $n - 2$  and  $n - 1$  are called *subfacets* and *facets* of  $P$ .

Note that every point of the boundary of  $P$  belongs to its face.

**Definition 14.** A  $k$ -dimensional geometric simplex  $\sigma_k$  is a convex hull of  $k + 1$  points  $\{v_0, v_1, \dots, v_k\}$ , such that the vectors  $v_1 - v_0, \dots, v_k - v_0$  are linearly independent. Points  $\{v_0, v_1, \dots, v_k\}$  are called vertices of  $\sigma_k$ .

A  $k$ -dimensional geometric simplex  $\sigma_k$  is a  $k$ -dimensional convex polytope. Note that the notion of a geometric simplex is generalization of the notion of triangle to arbitrary dimensions: 1-simplex is a line segment, 2-simplex is a triangle, 3-simplex is a tetrahedron. Convex hull of any  $j$  vertices of a geometric simplex  $\sigma_k$  is a geometric simplex  $\sigma_{j-1}$ .

**Definition 15.** Let  $V$  be a finite set. A *simplicial complex*  $\Delta$  is a non-empty collection of subsets of  $V$  closed under the operation of taking a subset:

$$F \in \Delta, G \subset F \implies G \in \Delta.$$

An element  $F$  of a simplicial complex  $\Delta$  is called a *face* of  $\Delta$ . The dimension of  $F$  denoted by  $\dim F$  is defined as  $\text{card} F - 1$ , where  $\text{card} F$  means the cardinality of  $F$ .

Dimension of  $\Delta$  denoted by  $\dim\Delta$  is defined as maximal dimension of its faces. A face  $F \in \Delta$  is called a *vertex*, an *edge*, a *subfacet*, a *facet* of  $\Delta$  if dimension of  $F$  is  $0, 1, \dim\Delta - 1, \dim\Delta$  respectively.

**Definition 16.** [Isomorphism of simplicial complexes] Consider simplicial complexes  $\Delta$  and  $\Delta_1$ . Let  $V$  and  $V_1$  denote the set of vertices of  $\Delta$  and  $\Delta_1$  respectively. An *isomorphism* between  $\Delta$  and  $\Delta_1$  is a bijective map  $f: V \rightarrow V_1$ , such that  $\{v_{i_1}, \dots, v_{i_k}\} \in \Delta$  if and only if  $\{f(v_{i_1}), \dots, f(v_{i_k})\} \in \Delta_1$ . If such a map exists, we say that  $\Delta$  and  $\Delta_1$  are *isomorphic*.

**Definition 17.** [Boundary complex of a polytope] Suppose  $P$  is a convex polytope spanned by vectors  $\{v_1, \dots, v_j\}$ . The *boundary complex* of  $P$  is a simplicial complex  $\Delta'$ , which consists of all the proper subsets of  $\{v_1, \dots, v_j\}$ . There is a bijective correspondence between the faces of the boundary of  $P$  and the non-empty faces of  $\Delta'$ : the face  $\text{span}\{v_{k_1}, \dots, v_{k_{1+m}}\}$  of  $P$ ,  $m \leq j - 1$ , is assigned to the element  $\{v_{k_1}, \dots, v_{k_{1+m}}\}$  of  $\Delta'$ .

**Example 18.** A geometric simplex  $\sigma_k \subset \mathbb{R}^k$  is a convex polytope and thus generates a boundary complex. For example, a 2-simplex  $\sigma_2$  with vertices  $v_0, v_1$  and  $v_2$  has a boundary complex

$$\Delta' = \{\emptyset, \{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}.$$

The bijection between the faces of the boundary of  $\sigma_2$  and the non-empty faces of  $\Delta'$  is as follows. The vertices  $\{v_0\}, \{v_1\}, \{v_2\}$  of  $\Delta'$  correspond to the vertices of  $\sigma_2$ . The edges  $\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}$  of  $\Delta'$  as a simplicial complex correspond to the edges of  $\sigma_2$ .

The boundary complex  $\Delta'$  of a geometric simplex  $\sigma_k$  together with the set  $V'$  of vertices of  $\sigma_k$  is, of course, a simplicial complex as well. We can extend the bijection from above by assigning the interior of  $\sigma_k$  to  $V'$ . Such a simplicial complex  $\Delta''$  is called the *vertex scheme* of  $\sigma_k$ , for more details see [7].

**Definition 19.** (Stellar subdivision) Let  $\Delta$  be the boundary complex of a convex polytope  $P$ ,  $F$  be a non-empty face of  $\Delta$  (and, respectively,  $P$ ). Let us place a vertex  $v$  over the geometric center of  $F$ . Consider the convex hull  $P^*$  of  $v$  and  $P$ . The boundary complex  $\Delta^*$  of the complex polytope  $P^*$  is called the *stellar subdivision* of the face  $F$  with respect to  $v$  and is denoted by  $\text{st}(v, F)[\Delta]$ .

During the construction of  $P^*$  and  $\Delta^*$  we delete the faces that contain  $F$  and create new faces that contain  $v$ .

**Remark 9.** Note that a stellar subdivision of a vertex  $v_1 \in \Delta$  with respect to a vertex  $v$  is identical (isomorphic) to the original boundary complex.

**Example 20.** Consider the boundary complex

$$\Delta' = \{\emptyset, \{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}.$$

The stellar subdivision  $\text{st}(v, \{v_0\})[\Delta']$  of  $\{v_0\}$  with respect to a vertex  $v$  is (isomorphic to)  $\Delta'$ . The stellar subdivision  $\text{st}(v, \{v_0, v_1\})[\Delta']$  of the edge  $\{v_0, v_1\}$  with respect to a vertex  $v$  is a 4-gon with vertices  $\{v_0\}, \{v_1\}, \{v_2\}, \{v\}$ , which is obtained by placing  $v$  over the center of  $\{v_0, v_1\}$ , deleting the edge  $\{v_0, v_1\}$  and constructing the edges  $\{v_0, v\}$  and  $\{v_1, v\}$ .

Consider a regular convex polygon  $P_n$  with  $n$  vertices labelled 0 through  $n - 1$  in such a way, that vertex  $i$  is adjacent to the vertices  $i - 1$  and  $i + 1$ ,  $1 \leq i \leq n - 2$ . For a given pair of numbers  $\{i, j\}$ ,  $1 \leq i \leq j \leq n - 2$ , there is a diagonal  $(i - 1)(j + 1)$  connecting vertices  $i - 1$  and  $j + 1$ . Let us associate a set of consecutive numbers  $i, i + 1, \dots, j$  to this diagonal. In this way we generate all the diagonals of  $P_n$ . Thus we have a bijective correspondence between the set of diagonals of  $P_n$  and the set  $V^*$  of all collections of consecutive numbers  $\{i, i + 1, \dots, j\}$ ,  $1 \leq i \leq j \leq n - 2$ , except the maximal collection  $\{1, 2, \dots, n - 2\}$ .

**Example 21.** A pentagon  $P_5$  has vertices labelled 0 through 4. The set  $V^*$  for a polygon  $P_5$  has form  $V^* = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ . The corresponding diagonals are 02, 13, 24, 03, 14 respectively.

Denote by  $\Sigma_n$  the set of all collections of non-intersecting diagonals of  $P_n$  together with the empty set. If  $F \in \Sigma_n$ , then any subset  $G$  of  $F$  is again a collection of non-crossing diagonals or the empty set. Therefore,  $G \in \Sigma_n$  and  $\Sigma_n$  is a simplicial complex. The dimension of facets, and therefore of  $\Sigma_n$ , is  $n - 4$ , because the maximal number of non-intersecting diagonals of  $P_n$  is  $n - 3$ . The total number of vertices of  $\Sigma_n$ , which are just single diagonals, is  $\frac{n(n-3)}{2}$ . (To draw a diagonal we need two distinct non-adjacent vertices. The first one can be chosen in  $n$  ways, the second one in  $(n - 3)$  ways. The order of picking does not matter, thus we divide  $n(n - 3)$  by 2).

Consider an  $(n - 3)$ -dimensional geometric simplex  $\sigma_{n-3}$  with vertices labelled 1 through  $n - 2$ . Let  $\sigma_{n-3}$  have a boundary complex  $\Delta'$ , which consists of all the proper subsets of the vertex set  $\{1, \dots, n - 2\}$ . Then each element  $v^*$  of  $V^*$  corresponds to a face of  $\Delta'$ . Note that one-point elements of  $V^*$  are the vertices of  $\Delta'$ .

Let us label all the elements of  $V^*$  by  $v_1^*, \dots, v_m^*$ ,  $\text{card} V^* = m$ , in such a way that  $v_i^* \supset v_j^*$  implies  $i < j$ . The elements of  $V^*$  with highest cardinality will come first, the one-point elements will come last. We can assume that  $v_{m-n+2}^* = \{1\}, \dots, v_m^* = \{n - 2\}$ . Note that such a labelling is not unique. Each element  $v_i^* \in V^*$  is a face of  $\Delta'$ .

**Example 22.** When  $n = 5$  the polygon  $P_5$  is a pentagon,  $V^*$  has form  $V^* = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ , the corresponding geometric simplex is a triangle  $\sigma_2$ , its boundary complex has the form  $\Delta' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$ . Let us label the elements of  $V^*$  (faces of  $\Delta'$ ):  $v_1^* := \{1, 2\}$ ,  $v_2^* := \{2, 3\}$ ,  $v_3^* := \{1\}$ ,  $v_4^* := \{2\}$ ,  $v_5^* := \{3\}$ .

Consider  $\Delta'$ , a sequence  $\{v_i^*\}$  of faces of  $\Delta'$  corresponding to the elements of  $V^*$  and a sequence  $\{v_i\}$  of new vertices,  $1 \leq i \leq m$ . Let us construct a series of simplicial complexes  $\Delta_1, \Delta_2, \dots, \Delta_{m+1}$  in the following way.  $\Delta_1 := \Delta'$ ,  $\Delta_{i+1} := \text{st}(v_i, v_i^*)[\Delta_i]$  for  $1 < i \leq m$ . The vertices  $v_{m-n+2}^* = \{1\}, \dots, v_m^* = \{n - 2\}$  are to be subdivided last. By Remark 9 stellar subdivision of vertices gives the same simplicial complex, so we just label the new vertices as the old ones:  $v_{m-n+2} = \{1\}, \dots, v_m = \{n - 2\}$ .

**Remark 10.** Let us fix  $i$ ,  $1 \leq i \leq m$ . When we form  $\Delta_{i+1} := \text{st}(v_i, v_i^*)[\Delta_i]$  we remove only the faces of  $\Delta_i$  which contain  $v_i^*$ . All the faces  $v_j^*$ ,  $i < j \leq m$  do not contain  $v_i^*$  because of the ordering. Therefore,  $\Delta_j$  are all well defined. Thus the construction of the sequence  $\Delta_1, \Delta_2, \dots, \Delta_{m+1}$  is well defined.

There is one-to-one correspondence between the diagonals of  $P_n$  (that are on one hand the elements of  $V^*$  and on the other hand the vertices of  $\Sigma_n$ ) and the vertices of  $\Delta_{m+1}$ . Indeed, the set of vertices of  $\Delta_{m+1}$  consists of vertices  $1, \dots, n$  of  $\Delta'$  (that appeared as a result of stellar subdivision of one-point sets of  $V^*$ ) and "new" vertices that appeared as a result of stellar subdivision of the rest of  $V^*$ . So we have a bijective map  $f$  from the vertex set of  $\Delta_{m+1}$  to the vertex set of  $\Sigma_n$ . Our goal is to show that  $f$  is an isomorphism, which means that  $\Sigma_n$  is isomorphic to a boundary complex of a polytope.

**Example 23.** Consider  $\Delta'$  and sequence  $v_1^*, \dots, v_5^*$  from Example 22. Then  $\Delta_1$  is a stellar subdivision of the edge  $\{1, 2\}$ ; it is a 4-gon with vertices  $1, 2, 3$  and a "new" vertex which we label by  $12$ .  $\Delta_2$  is a pentagon with vertices  $1, 2, 3, 12$  and a "new" vertex which we label by  $23$ .  $\Delta_3, \Delta_4, \Delta_5$  are stellar subdivisions of vertices  $\{1\}, \{2\}, \{3\}$  and thus they all are identical to  $\Delta_2$ . The vertices  $1, 2, 3, 12, 23$  represent the diagonals of the pentagon  $P_5$ .

**Theorem 11** (Lee [4], 1989). *Let  $\Sigma_n$  be a simplicial complex, which consists of collections of non-intersecting diagonals of a convex  $n$ -gon  $P_n$ . Then there exists an isomorphism between  $\Sigma_n$  and a boundary complex of a convex  $(n-3)$ -dimensional polytope.*

**Remark 12.** The theorem shows that  $\Sigma_n$  is realizable as a boundary complex of a convex  $(n-3)$ -dimensional polytope  $P$ . The Stasheff polytope is a polytope dual to the polytope  $P$ . This illustrates the deep connection between the triangulations of a convex polygon and the Stasheff polytope.

*Proof.* Let us prove that if  $u$  and  $u_1$  are vertices of  $\Delta_{m+1}$  which correspond to intersecting diagonals of  $P_n$ , then  $\{u, u_1\}$  is not an edge of  $\Delta_{m+1}$ . Assume that  $u$  and  $u_1$  were constructed during a stellar subdivision of  $F = \{i, i+1, \dots, j\}$  and  $G = \{k, k+1, \dots, l\}$ . The diagonals intersect if the union  $E$  of the sets  $F$  and  $G$  consists of numbers that form a sequence of consecutive numbers. Without loss of generality we may assume that  $k \leq j+1, l > j$  and  $i < k$ ; thus the set  $E$  has form  $E = \{i, i+1, \dots, l\}$ .

We need the following lemma which can be found in [4].

**Lemma 24.** *Assume that  $\Delta_1, \dots, \Delta_{k+1}$  are boundary complexes,  $v_1, \dots, v_{k+1}$  are distinct vertices, and  $F_1, \dots, F_{k+1}$  are distinct faces, such that  $\Delta_{i+1} = \text{st}_{\Delta_i} F_i, 1 \leq i \leq k$ . Assume that there exist such numbers  $r, s, 1 \leq r < s \leq m$ , that  $F_s \in \Delta_r$  and  $F_r \cup F_s \notin \Delta_r$ . Then  $\{v_r, v_s\} \notin \Delta_{k+1}$ .*

If  $E = \{1, 2, \dots, n-2\}$ , then it is not a face of  $\Delta_1 = \Delta'$ , because  $\Delta'$  contains sets of cardinality not higher than  $n-3$ . It does not belong to any of  $\Delta_p, 1 \leq p \leq m+1$ . The stellar subdivision  $\text{st}(v_p^*, v_p)[\Delta_p]$  of  $v_p^*$  produces only faces containing  $v_p$ . If  $\{1, 2, \dots, n-2\}$  is produced during a subdivision, then it must have been a subdivision of a vertex  $\{1\}$  or  $\{2\}, \dots$ , or  $\{n-2\}$ . But during those subdivisions the complex stays the same, no faces are added. Therefore, by Lemma 24 the pair  $\{u, u_1\}$  does not belong to  $\Delta_{m+1}$ .

If  $E \neq \{1, 2, \dots, n-2\}$ , then  $E$  belongs to  $V^*$  and thus is a face of  $\Delta'$  which is supposed to be subdivided. Because it contains both  $F$  and  $G$ , it is subdivided before them. Without loss of generality let us assume that  $F$  is  $v_r^*$  and  $G$  is  $v_s^*, r < s$ . Because

$E$  is subdivided before both  $v_r^*$  and  $v_s^*$ , it belongs to neither  $\Delta_r$  nor  $\Delta_s$ . Therefore, by Lemma 24  $\{u, u_1\}$  does not belong to  $\Delta_{m+1}$ .

We have proved that if  $uv \in \Delta_{m+1}$ , then  $f(u)f(v) \in \Sigma_n$ . This implies that a face is mapped by  $f$  onto a face, and a facet - onto a facet.

Let us show that a face is mapped onto a face. Since all the edges of  $\Delta_{m+1}$  correspond to pairs of non-intersecting diagonals, we get that every face of  $\Delta_{m+1}$  corresponds to a set of non-crossing diagonals. Any face  $F$  of  $\Delta_{m+1}$  consists of a certain amount of vertices. An arbitrarily chosen pair of vertices in  $F$  constitutes an edge of  $\Delta_{m+1}$ , because  $\Delta_{m+1}$  is a complex. Every edge corresponds to a pair of non-intersecting diagonals, and therefore  $F$  consists of vertices representing mutually non-crossing diagonals.

Since we have a bijection between vertices of  $\Delta_{m+1}$  and  $\Sigma_n$ , a facet of  $\Delta_{m+1}$  corresponds to a facet of  $\Sigma_n$  under this bijection. This together with the last paragraph implies that a facet of  $\Delta_{m+1}$  corresponds to a complete triangulation of  $P_n$ , i.e. a facet of  $\Sigma_n$ .

Both  $\Sigma_n$  and  $\Delta_{m+1}$  satisfy the following two properties:

1. Each subfacet is contained in exactly two facets;
2. For each pair  $F, F^*$  of facets there exists a sequence  $F_1 := F, F_2, \dots, F_p := F^*$ , such that  $F_i$  and  $F_{i+1}$  have a common subfacet,  $1 \leq i \leq p - 1$ .

Let us note that a facet of  $\Sigma_n$  is a complete triangulation, a subfacet of  $\Sigma_n$  is a collection of  $n - 4$  diagonals, so that the interior of  $P_n$  is divided into triangles and exactly one 4-gon. Because there are only two ways to triangulate this 4-gon, there are only two facets which contain the subfacet.

The properties hold for  $\Delta_{m+1}$  because it corresponds to a boundary of a convex polytope.

The fact that the properties 1 and 2 hold for  $\Sigma_n$  and  $\Delta_{m+1}$  implies that  $f$  is an isomorphism.

Recall that if  $F$  is a facet of  $\Delta_{m+1}$ , then  $f(F)$  is a facet of  $\Sigma_n$ . Take an arbitrary facet  $G^*$  of  $\Sigma_n$ . Between  $f(F)$  and  $G^*$  there is a path  $G_1 := f(F), \dots, G_p := G^*$ , such that  $G_i$  and  $G_{i+1}$  share a subfacet,  $1 \leq i \leq p - 1$ . Let  $G_1 = f(F)$  and  $G_2$  share a subfacet  $v$ , then by property 1 above it is uniquely determined by these two facets. On the other hand,  $v$  is an image of a subfacet  $u \in F$ :  $v = f(u)$ . There is a unique facet  $F_2 \in \Delta_{m+1}$  which shares  $u$  with  $F$ . The image of  $F_2$  under  $f$  is a facet, it contains  $f(u) = v$  and it must be equal to  $G_2$ . So there is a facet  $F_2 \in \Delta_{m+1}$  corresponding to  $G_2 \in \Sigma_n$ . We continue this process going along the sequence  $G_i$  and find a facet of  $\Delta_{m+1}$  that corresponds to  $G^*$  under  $f$ . The correspondence  $f$  is a bijective correspondence between vertices of  $\Delta_{m+1}$  and  $\Sigma_n$  that preserves facets; thus  $f$  is an isomorphism.  $\square$

**Remark 13.** Associahedron (Stasheff polytope)  $K_n$  is the polytope dual to the polytope  $R$  whose boundary complex is  $\Delta_{m+1}$ .

### 1.2.4 Braid groups

The braid groups were constructed explicitly in 1925 by Emil Artin, but they appeared implicitly earlier in works of A. Hurwitz. Since then they have been actively studied

from both algebraic and topological point of view. Braid groups have applications in different areas, for example, in knot theory and in physics: they help to describe two-dimensional Hall systems. In this subsection we define and describe braid groups from algebraic and geometric point of view. We are particularly interested in action of a braid group on a free group, which we describe at the end of the section. Our general reference for the braid groups is [3].

### Algebraic definition of a braid group

We begin with an algebraic definition of a braid group  $\tilde{B}_n$ ,  $n \in \mathbb{N}$ .

**Definition 25.** Given a positive integer  $n$ , we define a braid group  $\tilde{B}_n$  as a group on  $n - 1$  generators satisfying the following relations:

$$\tilde{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle. \quad (1.5)$$

If  $n = 1$  the group  $\tilde{B}$  contains just one element - identity. If  $n \geq 3$  the braid group  $\tilde{B}_n$  is non-abelian.

The braid group  $\tilde{B}_n$  admits an elegant geometrical interpretation: it can be realized as a fundamental group of a configuration space. In what follows we construct the configuration space and the geometrical braid group.

### Braid group $B_n^*$ , geometric representation

We define the configuration space  $W_n$  as the space of  $n$ -tuples of pairwise distinct complex numbers:

$$W_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_k \neq z_j, k \neq j\}.$$

Topology on  $W_n$  is induced from a product topology. It is in fact a connected topological space.

**Definition 26.** A pure braid group  $B_n^*$  on  $n$  strings is the fundamental group of  $W_n$ :

$$B_n^* := \pi(W_n(\mathbb{C}), P), P \in W_n.$$

An element of  $B_n^*$  is called a *pure braid*.

An ordered tuple of  $n$  distinct points can be represented by  $n$  different points on one plane denoted by  $(1, \dots, n)$ . Imagine a Cartesian product of the plane and the interval  $[0, 1]$ . A continuous path starting and ending at  $(1, \dots, n)$  can be then visualized as  $n$  continuous paths in  $\mathbb{C} \times [0, 1]$  which do not cross; each path connects two identical points in the plane. A homotopy class of such paths forms a pure braid on  $n$  strings. We can view the pure braid as a collection of all continuous deformations of the  $n$  paths, such that there is no crossing and the ends of paths are fixed. This geometric illustration of a representative of a pure braid is shown on Figure 1.6 (a). Imagine in addition that the paths on Figure 1.6 (a) are physical elastic strings with fixed endpoints: we can stretch them back and forth, but they can not go through each other.

Let  $S_n$  be a group of permutations of  $n$  elements  $\{1, \dots, n\}$ . A permutation  $\sigma \in S_n$  can be viewed as a bijective mapping of the set  $\{1, \dots, n\}$  onto itself.





Figure 1.6: Braids

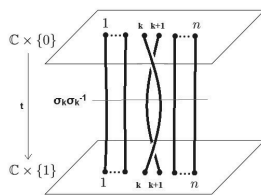


Figure 1.7: Visualization of a composition  $\sigma_k \circ \sigma_k^{-1}$ .

$S_n$  acts on the space  $W_n$  by permuting the coordinates:

$$\sigma(z_1, \dots, z_n) = (z_{\sigma(1)}, \dots, z_{\sigma(n)}), \sigma \in S_n.$$

We construct a set  $SW_n(\mathbb{C}) := W_n(\mathbb{C})/S_n$ , each element of  $SW_n$  is a collection of  $n$ -tuples which differ only by permutation of coordinates. This is a connected topological space with quotient topology induced from the space  $W_n$ .

**Definition 27.** A braid group  $B_n$  on  $n$  strings is the fundamental group of  $SW_n$ :

$$B_n := \pi(W_n(\mathbb{C}), P), P \in SW_n$$

An element of  $B_n$  is called a *braid*.

A braid can be viewed as continuous deformations of  $n$  continuous paths in  $\mathbb{C} \times [0, 1]$  which do not necessarily join two identical points on the plane, see Figure 1.6 (b).

It can be shown that the braid group  $B_n$  is generated by  $n - 1$  generators  $\sigma_1, \dots, \sigma_{n-1}$  which satisfy relation (1.5). This makes the algebraic group  $\hat{B}_n$  and the geometric group  $B_n$  isomorphic. The proof can be found in [3].

Let us give a geometric description of a generator  $\sigma_k$  of  $B_n$ ,  $1 \leq k \leq n - 1$ . The  $n$  paths representing  $\sigma_k$  can be pictured as follows: the path connecting points  $k$  and  $k + 1$  goes over the path connecting the points  $k + 1$  and  $k$ ; the rest of the paths are the identity paths. A representative of  $\sigma_1$  is in fact shown on Figure 1.6 (b). An identity of the braid group can be viewed as deformations of  $n$  identity paths. To obtain a representative of an inverse of a braid  $b$  we flip a representative of  $b$  vertically. To imagine a geometric representative of composition of two braids  $b_1$  and  $b_2$  we mount a representative of the braid  $b_1$  on top of a representative of the the braid  $b_2$  and glue the paths together. The process of obtaining a representative of  $\sigma_k \circ \sigma_k^{-1}$  is illustrated on Figure 1.7.

### Action of $B_n$ on a free group

In what follows we describe an action of  $B_n$  on a free group. We begin with recalling the definition of a group action.

**Definition 28.** An *action* of a group  $G$  on a set  $S$  assigns to each pair  $(g, s)$ ,  $g \in G, s \in S$ , an element  $t \in S$  (denote:  $g(s) = t$ ), such that:

1.  $e(s) = s \quad \forall s \in S$ ;
2.  $g \circ h(s) = g(h(s)) \quad \forall g, h \in G, \forall s \in S$ .

A *free group*  $F_n$  is a group on  $n$  generators without any relations.  $B_n$  is in fact a group of automorphisms of the free group on  $n$  generators.

For  $1 \leq k \leq n - 1$  we define an action of  $\sigma_k$  on the generators of  $F_n$  by

$$\sigma_k(x_j) = \begin{cases} x_{j+1}, & j = k, \\ x_j^{-1} x_{j-1} x_j, & j = k + 1, \\ x_j, & \text{otherwise.} \end{cases} \quad (1.6)$$

An action on generators of  $F_n$  determines an action on the entire group  $F_n$ : set  $\sigma_k(id) := id$ ,  $\sigma_k(xy) := \sigma_k(x)\sigma_k(y)$ ,  $\sigma_k(x^{-1}) := (\sigma_k(x))^{-1}$ , where  $x, y \in F_n$  and  $id$  is the group identity. In fact, we obtain that each  $\sigma_k$  is an automorphism of  $F_n$ . To show that  $\sigma_k$  is a bijective mapping we need to check that there is a mapping  $\sigma_k^{-1}$ , such that both  $\sigma_k \circ \sigma_k^{-1}$  and  $\sigma_k^{-1} \circ \sigma_k$  are identity mappings.

Let us define  $\sigma_k^{-1}$  as

$$\sigma_k^{-1}(x_j) = \begin{cases} x_j x_{j+1} x_j^{-1}, & j = k \\ x_{j-1}, & j = k + 1, \\ x_j, & \text{otherwise,} \end{cases}$$

and check that this is indeed the inverse that we are looking for. It is sufficient to verify the desired properties on generators of  $F_n$ .

$$\begin{aligned} \sigma_k^{-1}(\sigma_k(x_k)) &= \sigma_k^{-1}(x_{k+1}) = x_k, \\ \sigma_k^{-1}(\sigma_k(x_{k+1})) &= \sigma_k^{-1}(x_{k+1}^{-1} x_k x_{k+1}) = \sigma_k^{-1}(x_{k+1}^{-1}) \sigma_k^{-1}(x_k) \sigma_k^{-1}(x_{k+1}) = x_{k+1}, \\ \sigma_k(\sigma_k^{-1}(x_k)) &= \sigma_k(x_k x_{k+1} x_k^{-1}) = x_k, \\ \sigma_k(\sigma_k^{-1}(x_{k+1})) &= \sigma_k(x_k) = x_{k+1}. \end{aligned}$$

The desired properties clearly hold for  $x_j$ , where  $j \neq k, k + 1$ .

## Bibliography

- [1] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math., **347**, Springer-Verlag, (1973).
- [2] J. A. Jenkins, *Univalent functions and conformal mapping*. Ergebnisse der Mathematik und ihrer Grenzgebiete **18**, Springer-Verlag (1958).
- [3] Ch. Kassel and V. Turaev, *Braid groups*. Springer (2008).
- [4] C. W. Lee, *The associahedron and triangulations of the  $n$ -gon*. European J. Combin. **10** (1989), no. 6, 551–560.
- [5] M. J. Atia, A. Martínez-Finkelshtein, P. Martínez-González, and F. Thabet, *Quadratic differentials and asymptotics of Laguerre polynomials with varying complex parameters*. J. Math. Anal. Appl., **416** (2014), no. 1, 52–80.
- [6] J. P. May, *The geometry of iterated loop spaces*. Lecture Notes in Math. **271** (1972).
- [7] J. Munkres, *Elements of algebraic topology*. Addison-Wesley Publishing Company (1984).
- [8] B. Shapiro, K. Takemura, and M. Tater, *On spectral polynomials of the Heun equation II*. Commun. Math. Phys. **311** (2012), no. 2, 277–300.
- [9] H. Stahl, *Extremal domains associated with an analytic function. I, II*, Complex Var. Theory Appl. **4** (1985), no. 4, 311–324; 325–338.
- [10] J. D. Stasheff, *Associated fibre spaces*. Michigan Math. J. **15** (1968), 457–470.
- [11] K. Strebel, *Quadratic differentials*. Springer-Verlag (1984).
- [12] A. Vasil'ev, *Moduli of families of curves for conformal and quasiconformal mappings*. Lecture Notes in Mathematics, **1788**, Springer-Verlag (2002).
- [13] G. Ziegler, *Lectures on polytopes*. Springer-Verlag (1995).



# Chapter 2

## Main results

In this chapter we give an overview of new results introduced in the dissertation. In Sections 2.1, 2.2 and 2.3 we summarize Paper A, Paper B and Paper C respectively, as well as expand on our motivation for considering the corresponding problems. In Section 2.4 we give an introduction to our Paper in progress D, summarize our results and outline directions for further research.

### 2.1 Boundary distortion estimates for holomorphic maps

In Paper A we obtain Cowen-Pommerenke type estimates of angular derivatives of univalent self-maps of the unit disk. Such inequalities can be considered an extension of the Schwarz lemma; estimates of angular derivatives of holomorphic self-maps of the unit disk play an important role in dynamic systems theory.

Let  $\mathbb{D}$  denote the unit disk and  $\mathbb{T}$  denote the unit circle in the complex plane. Consider a univalent mapping  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ . A point  $\xi \in \mathbb{T}$  is called a boundary fixed point of  $\varphi$  if

$$\lim_{r \rightarrow 1^-} \varphi(r\xi) = \xi.$$

If  $\xi$  is a boundary fixed point of  $\varphi$ , we write simply  $\varphi(\xi) = \xi$ .

Let  $\theta \in (0, \frac{\pi}{2})$  and assume that  $\varphi(z)$  has boundary fixed points at  $e^{i\theta}$  and  $e^{-i\theta}$ , i.e.

$$\begin{aligned} \varphi(e^{i\theta}) &= e^{i\theta}, \\ \varphi(e^{-i\theta}) &= e^{-i\theta}. \end{aligned} \tag{2.1}$$

From the Julia-Wolff theory we obtain that equalities (2.1) imply that the limits

$$\begin{aligned} \varphi'(e^{i\theta}) &:= \lim_{z \rightarrow e^{i\theta}, z \in \Delta(e^{i\theta})} \frac{\varphi(z) - \varphi(e^{i\theta})}{z - e^{i\theta}}, \\ \varphi'(e^{-i\theta}) &:= \lim_{z \rightarrow e^{-i\theta}, z \in \Delta(e^{-i\theta})} \frac{\varphi(z) - \varphi(e^{-i\theta})}{z - e^{-i\theta}}. \end{aligned}$$

exist for any Stolz angle  $\Delta(e^{i\theta})$  (respectively,  $\Delta(e^{-i\theta})$ ) centred at  $e^{i\theta}$  (respectively,  $e^{-i\theta}$ ). We call  $\varphi'(e^{i\theta})$  and  $\varphi'(e^{-i\theta})$  the angular derivatives of  $\varphi(z)$  at  $e^{i\theta}$  and  $e^{-i\theta}$ . The Julia-Wolff theory implies that the angular derivatives in this case are real and positive or infinite, i.e.  $\varphi'(e^{\pm i\theta}) \in (0, \infty) \cup \{\infty\}$ . More details about boundary fixed points, angular derivatives and Julia-Wolff theory can be found in [12].

Among all the fixed points and all the boundary fixed points of  $\varphi(z)$  there exists only one point  $\tau$ , such that  $|\varphi'(\tau)| < 1$ . The following theorem is due to Cowen and Pommerenke [7].

**Theorem 14.** *Let  $\phi$  be a univalent self-mapping of the unit disk and let  $\phi$  have boundary fixed point at 1,  $\phi'(1) \leq 1$ ;  $\zeta_1, \dots, \zeta_n$  - other distinct boundary fixed points of  $\phi$ . Then*

$$\sum_{j=1}^n \frac{|1 - \zeta_j|^2}{\phi'(\zeta_j) - 1} \leq 2 \operatorname{Re} \left( \frac{1}{\phi(0)} - 1 \right). \quad (2.2)$$

Julia-Carathéodory-Wolff results give the following estimate.

**Theorem 15.** *Let  $\phi$  be a univalent self-mapping of the unit disk and let  $\phi$  have boundary fixed points at  $\zeta_1, \zeta_2$ ;  $\phi'(\zeta_1) \leq 1$ . Then*

$$\phi'(\zeta_1)\phi'(\zeta_2) \geq 1. \quad (2.3)$$

**Remark 16.** The notion of a boundary fixed point and an angular derivative can be defined for a class  $\operatorname{Hol}(\mathbb{D}, \mathbb{D})$  of holomorphic self-mappings of the unit disk. The last two theorems hold in fact for mappings in  $\operatorname{Hol}(\mathbb{D}, \mathbb{D})$ .

Inequality (2.3) holds in particular for the angular derivatives of the univalent mapping  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  with boundary fixed points at  $e^{\pm i\theta}$ ,  $\theta \in (0, \pi/2)$ , i.e.

$$\varphi'(e^{i\theta})\varphi'(e^{-i\theta}) \geq 1.$$

In Paper A we obtain an estimate for the product of angular derivatives  $\varphi'(e^{i\theta})$  and  $\varphi'(e^{-i\theta})$  from above with the value  $\varphi(0) \in \mathbb{D}$ , that refines inequality (2.3). Such an estimate is of Cowen-Pommerenke type, i.e. has similarities with inequality (2.2). Our method is based on reduced moduli of digons and the extremal partition problem.

To state the obtained result and sketch the proof let us recall the transition rule of the reduced moduli. The reduced modulus of a digon is not conformally invariant. If  $D$  is a digon with vertices  $z_1$  and  $z_2$  and angles  $\tau_1$  and  $\tau_2$ ,  $f$  is a conformal mapping of  $D$  onto a digon  $f(D)$  with vertices  $f(z_1)$  and  $f(z_2)$  and angles  $\psi_1$  and  $\psi_2$ , then we have

$$m(f(D), f(z_1), f(z_2)) = m(D, z_1, z_2) + \frac{1}{\psi_1} \log |f'(z_1)| + \frac{1}{\psi_2} \log |f'(z_2)|. \quad (2.4)$$

The transition rule involves the reduced modulus of the original domain and angular derivatives of the conformal mapping  $f$ .

We denote by  $D_{a_0}^1$  the digon  $\mathbb{D} \setminus [a_0, 1)$ ,  $a_0 \geq 0$ , with vertices at  $e^{i\theta}$  and  $e^{-i\theta}$  and with angles  $\pi$  at them. This type of digon and its reduced modulus is one of the key ingredients of the proof. Let us note that calculating the reduced modulus of a digon is not a trivial task; there are only few canonical digons with known reduced moduli. In Paper A we manage to find a canonical domain  $R$  with a known reduced modulus and a conformal mapping of  $D_{a_0}^1$  onto  $R$  and to find  $m(D_{a_0}^1)$  with help of transition rule (2.4). Namely, we choose the digon  $\mathbb{C} \setminus [0, \infty)$  with vertices at  $r > 0$  and angles  $\pi$  at them as the canonical digon. It is well-known that  $R$  has the reduced modulus  $2/\pi \log 4r$  (see

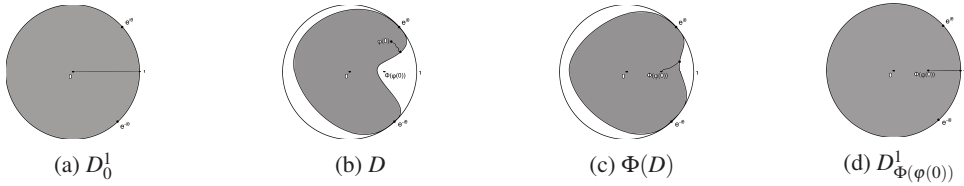


Figure 2.1: Dignons

[18] §2.4). Then we find a conformal mapping  $f$  of  $D_{a_0}^1$  onto  $R$  and obtain the reduced modulus of  $D_{a_0}^1$

$$m(D_{a_0}^1) = \frac{2}{\pi} \log \frac{4(1 - \cos \theta)}{\sin \theta} \frac{1 - 2a_0 \cos \theta + a_0^2}{(1 - a_0)^2}. \quad (2.5)$$

Consider a circle through the points  $e^{i\theta}$ ,  $e^{-i\theta}$  and the origin. We denote by  $\gamma_0$  the open arc of the circle that contains  $0$ ,  $e^{i\theta}$ ,  $e^{-i\theta}$ . We denote by  $\gamma_1$  the open straight line segment that connects  $e^{i\theta}$  and  $e^{-i\theta}$ . The unit disk is then divided into three domains: domain  $U_1$  bounded by  $\gamma_1$  and the arc of the unit circle that connects  $e^{i\theta}$ ,  $e^{-i\theta}$  and contains  $1$ ; domain  $U_2$  bounded by the arcs  $\gamma_0$  and  $\gamma_1$  and domain  $U_3$  bounded by  $\gamma_0$  and the arc of the unit circle that connects  $e^{i\theta}$ ,  $e^{-i\theta}$  and contains  $-1$ .

Let  $\varphi(0)$  be the image of the origin under the univalent self-mapping of the unit disk  $\varphi$  with fixed boundary points at  $e^{\pm i\theta}$ . Let  $\Phi$  be a Möbius mapping that fixes  $e^{i\theta}$  and  $e^{-i\theta}$  and moves the point  $\varphi(0)$  to the real line.  $\Phi(\varphi(0))$  is the point of intersection of the real line with the arc of a circle that goes through  $e^{i\theta}$ ,  $e^{-i\theta}$  and  $\varphi(0)$ . Assume that  $\varphi(0)$  lies in the union of  $U_1$ ,  $U_2$  and  $\gamma_1$ . In this case we have that  $\Phi(\varphi(0)) > 0$ .

Consider the digon  $D_0^1$ , its reduced modulus is given by formula (2.5) where  $a_0 = 0$ . Let  $D$  denote the image of  $D_0^1$  under the mapping  $\varphi(z)$ . The domain  $D$  is a digon with vertices at  $e^{i\theta}$  and  $e^{-i\theta}$  with angles  $\pi$  at them. Figures 2.1 (a) and 2.1 (b) illustrate the digons  $D_0^1$  and  $D = \varphi(D_0^1)$ . Using (2.4) we obtain the reduced modulus of  $D$

$$m(D) = m(D_0^1) + \frac{1}{\pi} \log \varphi'(e^{i\theta}) + \frac{1}{\pi} \log \varphi'(e^{-i\theta}).$$

The mapping  $\Phi$  maps the digon  $D$  onto a digon  $\Phi(D)$  with vertices at  $e^{i\theta}$  and  $e^{-i\theta}$  with angles  $\pi$  at them (see Figure 2.1 (c)). We obtain

$$m(\Phi(D)) = m(D_0^1) + \frac{1}{\pi} \log \varphi'(e^{i\theta}) \varphi'(e^{-i\theta}) + \frac{1}{\pi} \log \Phi'(e^{i\theta}) \Phi'(e^{-i\theta}). \quad (2.6)$$

The last term in the right hand side of (2.6) vanishes.

Denote by  $\mathcal{F}_{\Phi(\varphi(0))}^1$  the family of all digons  $\Delta$  in  $\mathbb{D} \setminus \{\Phi(\varphi(0))\}$  with vertices at  $e^{i\theta}$ ,  $e^{-i\theta}$  and angles  $\pi$  at them, such that any arc in  $\Delta$  that connects the vertices, is homotopic in  $\mathbb{D} \setminus \{\Phi(\varphi(0))\}$  to the arc of the unit circle which goes through  $e^{i\theta}$ ,  $e^{-i\theta}$  and  $-1$ . Then a general theorem ([9], [11], [18]) implies that

$$m(\Delta) \geq m(D^*), \quad (2.7)$$

where  $D^* := \mathbb{D} \setminus [\Phi(\varphi(0)), 1) = D_{\Phi(\varphi(0))}^1$  is a digon with vertices  $e^{i\theta}$ ,  $e^{-i\theta}$  and angles  $\pi$  at them. Figure 2.1 (d) illustrates the digon  $D^*$ . The domain  $D^*$  is a strip domain associated with trajectory structure of a quadratic differential.

The digon  $\Phi(D)$  belongs to the family  $\mathcal{F}_{\Phi(\varphi(0))}^1$ , so inequality (2.7) holds for  $\Delta = \Phi(D)$ , i.e.

$$m(\Phi(D)) \geq m(D^*) = m\left(D_{\Phi(\varphi(0))}^1\right). \quad (2.8)$$

The reduced modulus  $m\left(D_{\Phi(\varphi(0))}^1\right)$  is given by the formula (2.5), where  $a_0 = \Phi(\varphi(0))$ . Inequality (2.8) together with formula (2.6) gives us the estimate

$$\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\Phi(\varphi(0))\cos\theta + \Phi^2(\varphi(0))}{(1 - \Phi(\varphi(0)))^2}. \quad (2.9)$$

We have obtained an inequality that refines estimate (2.3) in the case when  $\varphi(0)$  lies to the right of  $\gamma_0$ .

Let us assume that  $\varphi(0)$  lies to the left of  $\gamma_0$ . In analogous manner we obtain the following estimate

$$\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\Phi(\varphi(0))\cos\theta + \Phi^2(\varphi(0))}{(1 + \Phi(\varphi(0)))^2}. \quad (2.10)$$

In the case when  $\varphi(0) \in \gamma_0$  we get that  $\Phi(\varphi(0)) = 0$ ,  $D^* = D_0^1$  and inequality (2.3) can not be refined.

Let us summarize these results in the following theorem

**Theorem 17.** *Let  $\varphi$  be a univalent self-mapping of the unit disk, such that  $\varphi(e^{\pm i\theta}) = e^{\pm i\theta}$ , where  $\theta \in (0, \pi/2)$ .*

1. *If  $\varphi(0)$  lies to the right of  $\gamma_0$ , then inequality (2.9) holds.*
2. *If  $\varphi(0)$  lies to the left of  $\gamma_0$ , then inequality (2.10) holds.*

*The estimates are sharp.*

A description of the extremal maps can be found in Paper A. In the cases when  $\varphi(0) \in U_1$  and  $\varphi(0) \in U_3$  one can improve inequalities (2.9) and (2.10). We prove the following result.

**Corollary 1.** *If  $\varphi$  is a univalent self-mapping of the unit disk, such that  $\varphi(e^{\pm i\theta}) = e^{\pm i\theta}$ , where  $\theta \in (0, \pi/2)$ , then the following sharp estimates hold:*

1. *If  $\varphi(0) \in U_1$ , then*

$$\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\operatorname{Re}(\varphi(0))\cos\theta + (\operatorname{Re}(\varphi(0)))^2}{(1 - \operatorname{Re}(\varphi(0)))^2};$$

2. *If  $\varphi(0) \in U_3$ , then*

$$\sqrt{\varphi'(e^{i\theta})\varphi'(e^{-i\theta})} \geq \frac{1 - 2\operatorname{Re}(\varphi(0))\cos\theta + (\operatorname{Re}(\varphi(0)))^2}{(1 + \operatorname{Re}(\varphi(0)))^2}.$$



**Remark 18.** There are different ways of obtaining inequalities that involve moduli or reduced moduli of domains, such as symmetrization, polarization and different extremal theorems (see [18]). Choosing the right tools and domains that have a modulus or reduced modulus that can be calculated, is a challenge. The right choice of domains and the extremal theorem, constructing an auxiliary function  $\Phi$  gave us a successful result that stands in a line with a myriad of classical inequalities that originate in the Schwarz lemma.

## 2.2 Combinatorial description of jumps in spectral networks defined by quadratic differentials

Consider quadratic differentials of the form

$$q(z) dz^2 = \frac{p_k(z)}{z} dz^2, \tag{2.11}$$

where  $p_k(z)$  is a polynomial of degree  $k$ , on  $\hat{\mathbb{C}}$ . In Paper B we describe the set  $\Lambda$  of quadratic differentials of form (2.11) that possess a short trajectory. In particular, we establish a weighted graph representation of quadratic differentials of form (2.11). Let us note that graphs in connection with quadratic differentials were used before, see for example [3],[16]. Further on, we characterize the graphs representing quadratic differentials of form (2.11) with a short trajectory. Finally, we describe the set  $\Lambda$  in terms of Stasheff fans, using the connection between weighted graphs and Stasheff fans given by Baryshnikov in [2]. Short trajectories of quadratic differentials are of particular interest; for example, they have applications in problems related to the potential and the approximation theories (e.g. [1],[15]), minimal surfaces (e.g. [4]), and mathematical physics.

A quadratic differential of type (2.11) has  $k$  zeros, 1 simple pole at the origin and a pole of order  $k + 3$  at infinity. There are  $k + 1$  distinguished directions in a neighbourhood of infinity, such that trajectory rays tend to the pole in those directions. One can associate strip domains and  $k + 1$  half-plane domains with its trajectory structure. Let  $q(z) dz^2$  be a quadratic differential of type (2.11). We associate a graph  $G_h$  (respectively,  $G_v$ ) with its trajectory structure (respectively, orthogonal trajectory structure). Let us describe the construction of  $G_h$  (respectively,  $G_v$ ). We start by constructing a regular  $(k + 3)$ -gon whose vertices represent the  $k + 1$  distinguished directions about infinity and edges represent the  $k + 1$  half-plane domains of the trajectory structure (respectively, orthogonal trajectory structure) of  $q(z) dz^2$ . In the geometric center of the polygon we place a vertex  $O$  that represents the simple pole of the quadratic differential. Recall that a strip domain  $S$  of the trajectory structure (respectively, orthogonal trajectory structure) of  $q(z) dz^2$  is swept out by trajectories (respectively, orthogonal trajectories) with trajectory rays (respectively, orthogonal trajectory rays) that tend to infinity in two distinguished directions. Assume that these directions are represented by vertices  $v_1$  and  $v_2$  of the  $(k + 3)$ -gon (that may coincide). Recall that  $S$  can be mapped conformally by  $w(z) = \int_z \sqrt{q}$  onto a strip  $a < \operatorname{Re} w < b$ . If the simple pole of  $q(z) dz^2$  does not lie on the boundary of  $S$ , we add an edge  $v_1 v_2$  with weight  $b - a$  to the graph. If the simple pole of  $q(z) dz^2$  belongs to the boundary of  $S$ , we add two edges  $v_1 O$  and

$v_2 O$  with weights  $b - a$  to the graph. When all the strip domains are marked on the graph, the construction of  $G_h$  (respectively,  $G_v$ ) is done, and the quadratic differential is represented by the pair  $(G_h, G_v)$ .

Furthermore we describe all admissible graphs, i.e. graphs that may represent the trajectory structure (respectively, orthogonal trajectory structure) of a quadratic differential of form (2.11). We show that for each pair of admissible graphs there is a quadratic differential that is represented by them. In this way we establish one-to-one correspondence between family (2.11) and the set of pairs of all admissible graphs.

A trajectory of  $q(z) dz^2$  with finite  $q$ -length and trajectory rays that tend to critical points of  $q(z) dz^2$  is called short. Using the established graph representation we describe quadratic differentials of form (2.11) that possess a short trajectory.

With each admissible graph  $G$  we associate a regular  $n$ -gon  $P$ , where  $n = k + 1$  or  $n = k + 2$ . The set of vertices of  $P$  is a subset of the set of vertices of  $G$  which correspond to the distinguished directions at infinity. The set  $E$  of weighted edges of  $G$  that connect the vertices of  $P$  produces a triangulation of  $P$ .

**Definition 29.** A regular convex polygon  $P$  together with a collection of non-intersecting weighted diagonals is called a *weighted diagram* based on  $P$ .

$P$  together with  $E$  forms a weighted diagram. In Paper B we establish the following fact.

**Proposition 1.** *Quadratic differential  $q(z) dz^2$  represented by  $G$  has a short trajectory if and only if the triangulation of  $P$  is incomplete.*

Let  $\Lambda \cong \mathbb{R}^{2k}$  be the set of all quadratic differentials in family (2.11) for a fixed  $k$ , let the elements of  $\Lambda$  with short trajectories (respectively, orthogonal trajectories) form set  $S \subset \Lambda$ .  $S$  can be divided into components  $S_h$  and  $S_v$ , where  $S_h$  (respectively,  $S_v$ ) is the set of quadratic differentials with a short trajectory (respectively, short orthogonal trajectory). In Paper B we prove the following result that describes the combinatorial structure of  $S$ .

**Theorem 19.** *The sets  $S_h$  and  $S_v$  have the following form*

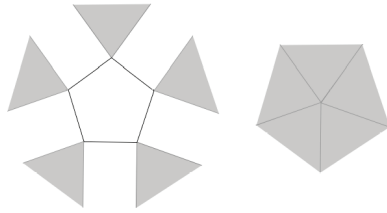
$$S_h = \left( \Sigma_k \times \mathbb{R}^{k+2} \right) \cup \left( \Sigma_{k+1} \times \mathbb{R}^{k+1} \right),$$

$$S_v = \left( \Sigma_k \times \mathbb{R}^{k+2} \right) \cup \left( \mathbb{R}^{k+1} \times \Sigma_{k+1} \right).$$

The main ingredients of the proof are the graph representation described above, Proposition 1 and a result from [2] by Baryshnikov, which we state later after introducing necessary terminology.

Given a regular convex  $n$ -gon  $P$ , a real-valued function  $f$  defined on its vertices  $v_1, \dots, v_n$  is called a *balanced weight* based on  $P$  if  $f(v_1) + \dots + f(v_n) = 0$  and the geometric center of masses is at the origin. A balanced weight  $f$  based on  $P$  is degenerate if there is a linear function  $L$ , such that  $L$  and  $f$  coincide at four vertices of  $P$  and  $L$  majorizes  $f$ . The balanced weights based on a  $n$ -gon  $P$  form a real vector space of dimension  $n - 3$ . Let  $C$  be a convex polytope in  $\mathbb{R}^d$ . It is well-known that a vector  $w \in \mathbb{R}^d$  defines a face  $F_w$  of  $C$  in the following way

$$F_w = \{x \in C : (x - y) \cdot w \geq 0 \ \forall y \in C\}.$$

Figure 2.2: Stasheff fan  $\Sigma_4$ 

The face  $F_w$  is an intersection of  $C$  with a hyperplane that has  $w$  as normal vector and goes through the point  $\operatorname{argmax}_{x \in C} x \cdot w$ .

Any face  $F$  of  $C$  defines a *normal cone*  $N(F)$  of  $C$  in the following way

$$N(F) = \{w \in \mathbb{R}^d : F = F_w\}.$$

The collection of all normal cones of  $C$  is called the *normal fan* of  $C$ . The normal fan  $\Sigma_p$  of a Stasheff polytope  $K_p$  is called the *Stasheff fan*. The Figure 2.2 illustrates the Stasheff fan  $\Sigma_4$ .

The following theorem was proved by Baryshnikov in [2]. In Paper B we complete some missing details of the proof.

**Theorem 20.** *There is a one-to-one correspondence between the balanced weights based on  $n$ -gon  $P$  and the weighted chord diagrams based on  $P$ . There is a one-to-one correspondence between the degenerate balanced weights based on  $n$ -gon and the weighted chord diagrams based on  $P$  with incomplete triangulations. The degenerate balanced weights based on  $P$  form a Stasheff fan  $\Sigma_{n-1}$ .*

Baryshnikov used this theorem to describe combinatorial structure of the Stokes sets for polynomials, and in particular, combinatorial structure of the quadratic differentials of the form  $p(z) dz^2$  with a short trajectory, where  $p(z)$  is a polynomial.

We considered a more difficult case when quadratic differentials have form (2.11). The trajectory structure of a quadratic differential of form (2.11) is more complicated than the trajectory structure of a quadratic differential of the form  $p(z) dz^2$ , as the presence of a simple pole changes significantly the configuration of strip domains. In order to use Theorem 20 for our case, we had to establish a connection between the quadratic differentials of type (2.11) with short trajectories and weighted diagrams, which was a challenging task. We managed to construct advanced graphs that represent quadratic differentials of form (2.11), to point out those graphs that correspond to quadratic differentials with a short trajectory and to link them to weighted diagrams, which allowed us to use Theorem 20 to establish a new result formulated in Theorem 19.

## 2.3 Polynomial lemniscates and their fingerprints: from geometry to topology

Let  $\Gamma$  be a smooth  $C^\infty$  closed Jordan curve in the complex plane  $\mathbb{C}$ . Let  $\hat{\mathbb{C}}$  denote the Riemann sphere, and let  $\Omega_+$  and  $\Omega_-$  denote the connected components of  $\hat{\mathbb{C}} \setminus \Gamma$ , where

$\infty \in \Omega_-$ . By the Riemann mapping theorem there exists a conformal mapping  $\phi_+$  of the unit disc  $\mathbb{D}$  onto  $\Omega_+$  and a conformal mapping  $\phi_-$  of  $\hat{\mathbb{C}} \setminus \mathbb{D}$  onto  $\Omega_-$ . The mappings are uniquely determined by a normalization and can be extended as homeomorphisms to the boundary. A *fingerprint* of  $\Gamma$  is a function  $k(\theta) : [0, 2\pi] \rightarrow [0, 2\pi]$  defined by

$$e^{ik(\theta)} = \phi_-^{-1} \circ \phi_+(e^{i\theta}).$$

The fingerprint is monotone, smooth and satisfies  $k(\theta + 2\pi) = k(\theta) + 2\pi$ .

The curve  $\Gamma$  can be considered as a 2D shape in the plane; the study of two dimensional shapes plays an important role in vision theory, in particular in shape recognition. Fingerprints are used to classify 2D shapes. Indeed, fingerprints are invariant under affine transformations.

Given  $R \in \mathbb{R}$ , a polynomial lemniscate  $\Gamma(R)$  is the level set  $|p(z)| = R$  of a polynomial  $p(z)$  of degree  $n$ . We say in this case that the polynomial lemniscate has degree  $n$ . The zeros of the polynomial are called the nodes of the lemniscate. We call a polynomial lemniscate  $\Gamma(1)$  *proper* if the domain  $\Omega^+$  bounded by  $\Gamma(1)$  is connected.

It was shown by P. Ebenfelt, D. Khavinson and H. S. Shapiro in [8] that a proper polynomial lemniscate  $\Gamma(1)$  has a rather simple fingerprint, it is given by the  $n$ -th root of a Blaschke product  $B_n(z)$  of degree  $n$ . Namely, the fingerprint  $k(\theta)$  of  $\Gamma(1)$  is given by

$$e^{ik(\theta)} = \sqrt[n]{B_n(e^{i\theta})} = \sqrt[n]{e^{i\alpha} \prod_{k=1}^n \frac{e^{i\theta} - a_k}{1 - e^{i\theta} \bar{a}_k}}, \quad (2.12)$$

where  $\alpha$  is a real number and  $a_k$ ,  $1 \leq k \leq n$ , are the preimages of the nodes of the lemniscate by  $\phi_+$ . On the other hand, the  $n$ -th root of a Blaschke product of degree  $n$  is realizable as a fingerprint of a polynomial lemniscate of degree  $n$ . In Paper C we study properties of fingerprints of polynomial lemniscates, in particular, properties related to their inflection points. We prove the following result:

**Theorem 21.** *The fingerprint of a proper lemniscate has an even number of inflection points, at least two and at most  $4n - 2$ .*

Let us sketch the proof of the theorem. First we obtain from (2.12) the following expressions for  $k'(\theta)$  and  $k''(\theta)$ :

$$k'(\theta) = \frac{1}{n} \operatorname{Re} \sum_{k=1}^n \frac{\zeta + a_k}{\zeta - a_k}, \quad \zeta = e^{i\theta};$$

$$k''(\theta) = -\frac{1}{2n} \sum_{k=1}^n \left( \frac{2i\zeta a_k}{(\zeta - a_k)^2} - \frac{2i\zeta \bar{a}_k}{(1 - \zeta \bar{a}_k)^2} \right), \quad \zeta = e^{i\theta}.$$

Then we note that as a real part of the function  $\sum_{k=1}^n \frac{\zeta + a_k}{\zeta - a_k}$ , the derivative  $k'(\theta)$  attains at least one maximum and minimum on  $[0, 2\pi)$ . The function  $k'(\theta)$  is periodic and hence its derivatives at the endpoints of the interval have the same sign, therefore  $k'(\theta)$  has an even number of critical points. The function

$$Z(\zeta) = \zeta \sum_{k=1}^n \left( \frac{2ia_k}{(\zeta - a_k)^2} - \frac{2i\bar{a}_k}{(1 - \zeta \bar{a}_k)^2} \right) \quad (2.13)$$

has degree  $4n$ , simple zeros at the origin and infinity, and satisfies  $\bar{Z}(\zeta) = Z(1/\bar{\zeta})$ . Therefore, number of zeros in  $\mathbb{D}$  and  $\hat{\mathbb{C}} \setminus \bar{\mathbb{D}}$  is equal to  $p + 2$  for some even  $p$ . The number of zeros on the unit circle (which are precisely the zeros of  $k''(\theta)$ ) is therefore smaller than  $4n - 2$ .

Let us note that the number  $4n - 2$  of inflection points is not necessarily achieved. We give explanation of this fact in the following theorem.

**Theorem 22.** *If all the zeros  $a_k$ ,  $1 \leq k \leq n$ , of the  $n$ -th Blaschke product lie on the same radius of  $\mathbb{D}$ , then the number of inflection points of the fingerprint  $k(\theta)$  is at most  $4n - 4$ .*

*If  $n = 2$ , then the number of inflection points of the fingerprint  $k(\theta)$  is at most 4 for arbitrary position of the zeros of the Blaschke product.*

Let us give an example when  $n = 2$  and the upper bound 4 of number of the inflection points is attained. Consider the function (2.13) with  $a_1 = 1/2$  and  $a_2 = -1/2$ . It has zeros of order two at the origin and infinity and four zeros on the unit circle:  $-1, -i, 1, i$ .

In Paper C we also describe the geometric meaning of inflection points of a fingerprint of a smooth Jordan curve. It is formulated in the following theorem.

**Theorem 23.** *The inflection points of the fingerprint  $k(\theta)$  divide the unit circle into  $m$  arcs  $\gamma_j = \{e^{i\theta} : \theta \in [\theta_j, \theta_{j+1}]\}$ ,  $\theta_{m+1} = \theta_1 + 2\pi$ , where  $j = 1, \dots, m$ , so that the ratio of the rates of change of the harmonic measures of the arc  $\alpha \in \Gamma$ ,  $\alpha = \{\phi_+(s) : s \in [\theta_1, \theta]\}$  with respect to  $(\Omega^+, 0)$  and  $(\Omega^-, \infty)$  respectively, alternates its monotonicity.*

In order to introduce the rest of results of Paper C, let us recall some terminology and statements from paper [5] by Catanese and Paluszny. A polynomial  $p(z)$  of degree  $n$  is *lemniscate-generic* if the zeros  $y_1, \dots, y_{n-1}$  of  $p'(z)$  (i.e. the critical points) are distinct, the critical values  $w_k = p(y_k)$ ,  $1 \leq k \leq n - 1$ , do not vanish and  $|w_j| < |w_k|$  whenever  $j < k$ . Each critical level set of  $p(z)$ , i.e. the set of  $z$ , such that  $|p(z)| = |w_k|$ ,  $1 \leq k \leq n - 1$ , is called a *big lemniscate* and consists of one "figure-eight" and possibly a number of circumferences. Only critical level sets contain "figure-eights". Let  $\Lambda_1$  and  $\Lambda_2$  denote the unions of big lemniscates of lemniscate-generic polynomials  $p_1(z)$  and  $p_2(z)$  of degree  $n$ . We say that  $p_1(z)$  and  $p_2(z)$  have the same *lemniscate configuration*  $(\Lambda, \mathbb{C})$  if there exists a homeomorphism  $h : \mathbb{C} \rightarrow \mathbb{C}$  that maps  $\Lambda_1$  onto  $\Lambda_2$ . Two polynomials with the same lemniscate configuration have a similar structure of lemniscates. If  $\mathcal{P}_n$  denotes the space of all polynomials of degree  $n$  and  $\mathcal{L}_n$  denotes the set of all lemniscate-generic polynomials of degree  $n$ , then  $\mathcal{L}_n$  is open and  $\mathcal{P}_n \setminus \mathcal{L}_n$  is a union of real hypersurfaces. There is a one-to-one correspondence between the connected components of  $\mathcal{L}_n$  and the lemniscate configurations that can be associated with lemniscate-generic polynomials of degree  $n$ . In particular, any two polynomials that belong to a connected component of  $\mathcal{L}_n$  have the same lemniscate configuration [5]. It was also shown ([5]) that there exists a bijection between the lemniscate configurations for lemniscate-generic polynomials of degree  $n$  and *central balanced trees of length  $n - 1$* . A central balanced tree of length  $n - 1$  has a distinguished vertex of degree 2 called the center (which represents the "figure-eight"  $|p(z)| = |w_{n-1}|$ ); it is at distance  $n - 1$  from the  $n$  leaves that represent the zeros of polynomial. There is exactly one vertex of degree three at distance  $j$ ,  $1 \leq j \leq n - 2$ , from the center (it represents a "figure-eight"). There are  $(n - 1)(n - 2)/2$  vertices of degree 2 that represent the circumferences of the lemniscate. Figure 2.3

illustrates examples of lemniscate configurations and corresponding central balanced trees of polynomials of degrees 2, 3 and 4.

By Hilbert's theorem, any 2D shape can be approximated by polynomial lemniscates. A better approximation algorithm was suggested by T. A. Rakcheeva in [13], [14]. At each step of the algorithm one obtains a lemniscate of a higher degree by replacing a simple zero  $z$  of the polynomial with a zero  $z^n$  of multiplicity  $n$ , which can be considered as  $n$  simple zeros that coincide, and moving the  $n$  simple zeros apart. At each step we obtain a lemniscate of a higher order that approximates the shape better. We consider the following process which is based on this idea. We start with a circle centered at  $z_1$ , plant a zero of multiplicity  $n_1$  at  $z_1$ , move the simple zeros apart. Then we pick another simple zero  $z_2$  and repeat the process. At each step we move the simple zeros apart so that there is no significant change in the structure of the lemniscates. We obtain a lemniscate-generic polynomial  $P(z)$  with a lemniscate configuration "inherited" from  $p(z)$ , i.e. the singular level sets outside a small neighbourhood  $U_k$  of the zero  $z_k$  are similar to the singular level sets of  $p(z)$ .

**Remark 24.** A significant change of lemniscate structure is a change of lemniscate configuration. An example of such change is illustrated on Figure 2.4. The lemniscates outside the shaded neighbourhood  $U_k$  belong to different lemniscate configurations.

The process described above can be realized as a composition of lemniscate configurations (and, respectively, central balanced trees). Figure 2.3 (b) shows the lemniscate configuration and the central balanced tree for a polynomial of the third degree. Figure 2.3 (c) shows the result of planting a double zero at the simple zero  $z$  and moving two simple zeros apart, which can be viewed as a composition of the lemniscate configurations (respectively, central balanced trees) on Figure 2.3 (a) and Figure 2.3 (b).

In Paper C we construct a non-unitary operad on a class of trees which contains central balanced trees. Thus the non-unitary operad realizes the composition of central balanced trees (respectively, lemniscate configurations).

Let us discuss now this process from analytical point of view. Given a polynomial  $p \in \mathcal{L}_n$ , consider its conjugacy class  $[p]$  with respect to precomposition from the right with an affine map and postcomposition from the left with multiplication by a complex constant. If  $p \in \mathcal{L}_n$ , then  $[p]$  is contained in one connected component of  $\mathcal{P}_n \setminus \mathcal{L}_n$ , i.e. has the same lemniscate configuration and central balanced tree of length  $n - 1$  [6]. Consider a simple zero  $z_k$  of  $p \in \mathcal{L}_n$  and another polynomial  $q \in \mathcal{L}_m \subset \mathcal{L}_n$ . The operation of composition of lemniscate configurations (and corresponding trees)  $[p] \circ_k [q]$  can be defined as follows. Let  $U_r(z_k)$  denote a disk of radius  $r$  centered at  $z_k$ , where  $r$  is, such that all the singular level sets of  $p(z)$  lie outside of  $U_r(z_k)$ . Let  $\tilde{q}$  be a representative of  $[q]$ , such that all big lemniscates of  $\tilde{q}$  are contained in  $U_r(z_k)$ . We define  $\tilde{p} = (z - z_k)^{-1} p(z) \tilde{q}(z)$ . If  $\tilde{p} \in \mathcal{L}_{n-1+m}$ , then  $[\tilde{p}] := [p] \circ_k [q]$ . We establish the following result.

**Remark 25.** Without loss of generality we can assume that  $z_k$  is at the origin.

**Theorem 26.** Given lemniscate generic polynomials  $p_n(z) = z \prod_{k=1}^{n-1} (z - z_k)$  and  $q_m(z) = z \prod_{j=1}^{m-1} (z - w_j)$  with

$$|w_j| < \varepsilon = \frac{m}{2(n-1) + m} \min_k |z_k|,$$

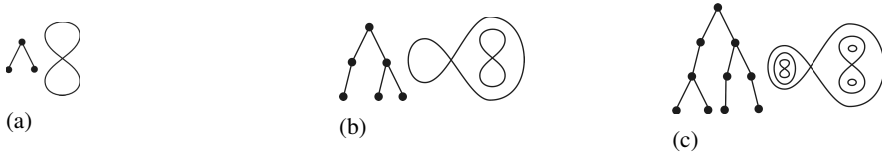


Figure 2.3: Composition of lemniscates



Figure 2.4: Different lemniscate configurations

the critical points of  $q_m$  and  $m - 1$  critical points of  $P(z) = z^{-1} q_m p_n$  ‘inherited’ from  $q_m$  lie within the same disk  $|z| < \varepsilon$ .

Namely, we show that  $P(z)$  and  $q_m(z)$  have the same number of critical points within the circle  $|z| < \varepsilon$ . We prove this using Rouchè’s theorem and the auxiliary result:

$$\left| \frac{p'(z)}{p(z)} \right| > \frac{n}{2}$$

on the unit circle if the zeros of polynomial  $p(z)$  lie in the unit disk, see Paper C.

On the other hand, we prove the following theorem.

**Theorem 27.** *Let a lemniscate generic polynomial  $p_n(z) = z \prod_{k=1}^{n-1} (z - z_k)$  belong to a connected component  $E' \subset \mathcal{L}_n$ , and let  $q_m(z) = z \prod_{j=1}^{m-1} (z - w_j)$  belong to a connected component  $E'' \subset \mathcal{L}_m$ . There is a small deformation of the polynomial  $z^{m-1} p_n(z)$ , such that the resulting polynomial  $P(z)$  belongs to a connected component  $E''' \subset \mathcal{L}_{n+m-1}$ , such that its projection to  $\mathcal{L}_n$  is from  $E'$  and its projection to  $\mathcal{L}_m$  is from  $E''$ .*

The theorem tells us that it is possible to find a polynomial  $\tilde{q}(z)$ , such that  $z^{-1} p(z) \tilde{q}(z)$  is lemniscate generic and has lemniscate configuration that represents the composition of lemniscate configuration of  $q(z)$  and  $p(z)$ . This is possible because of the conic-like structure of the sets  $\mathcal{L}_n$ ,  $\mathcal{L}_m$ , and  $\mathcal{L}_{n+m-1}$ .

Each polynomial  $p(z)$  gives rise to a quadratic differential, such that the polynomial lemniscates  $\Gamma(R)$  are trajectories of the quadratic differential. This subject is discussed in the next section.

## 2.4 Pure braids and homotopy classes of closed loops. Directions for further research

Let  $X_n$  denote the complex plane with  $n$  punctures  $z_1, \dots, z_n$ . Denote by  $G_{s,t}$  the set of homotopy classes generated by closed Jordan curves that separate  $z_s, z_t$  from the rest of the punctures. In Paper D we define an action of a braid group  $B_n$  on  $G_{s,t}$  and prove a number of properties of the action. The goal of Paper D is to give a braid  $b \in B_n$ ,

such that for arbitrary elements  $\Gamma_1, \Gamma_2 \in G_{s,t}$  we have  $b(\Gamma_1) = \Gamma_2$ . Paper D contains some results and work in progress, which is a natural continuation of Paper C. Let us outline the structure of this section. We begin with general motivation and describe the connection to the previous work. Then we give a brief introduction to the terms and constructions we use, state the obtained results and outline the further work on the project.

### Motivation

On the Riemann sphere  $S$  with  $p$  punctures we define a collection of piecewise smooth curves  $\gamma = (\gamma_1, \dots, \gamma_m)$  of two types I, II. The first one (I) consists of simple loops on  $S$  that are not freely homotopic pairwise, do not intersect and are homotopic neither to a point of  $S$  nor to a puncture. The second one (II) consists of loops homotopic to the punctures. Such a collection is called the *admissible system of curves*. A set  $\Gamma_j$  of piecewise smooth curves is said to be a *homotopy class* generated by  $\gamma_j$  from the admissible system  $(\gamma_1, \dots, \gamma_m)$ ,  $m \leq 2p - 3$ , if this set consists of all curves of type I or II that are freely homotopic to  $\gamma_j$  on  $S$ . A collection of the homotopy classes of curves  $\Gamma := (\Gamma_1, \dots, \Gamma_m)$  generated by the admissible system  $(\gamma_1, \dots, \gamma_m)$  on  $S$  is said to be the *free family of homotopy classes of curves*. Jenkins [10] proved that given a free family  $\Gamma$  of homotopy classes of curves and a non-zero vector  $\alpha := (\alpha_1, \dots, \alpha_m)$  with non-negative coordinates, there exists a unique quadratic differential  $q$ , such that the regular trajectories are of homotopy type  $\Gamma$  and the  $q$ -lengths of its trajectories are essentially  $\alpha$ . Namely, the domains of the trajectory structure of  $q$  are ring or circular domains  $D_k$ ,  $1 \leq k \leq m$ , that are swept out by closed trajectories of  $q$ -length  $\alpha_k$ . Note that there exist positive numbers  $\alpha_k^0$ , such that if  $0 \leq \alpha_k \leq \alpha_k^0$ , then the corresponding domain  $D_k$  degenerates. Let us summarize the result by Jenkins in the following lemma.

**Lemma 30.** *The following data defines a quadratic differential  $q(z) dz^2$  on the Riemann sphere uniquely:*

1.  $p$  punctures;
2. free family of homotopy classes of curves  $\Gamma = (\Gamma_1, \dots, \Gamma_m)$ ,  $m \leq 2p - 3$ ;
3. a set of lengths  $\alpha = (\alpha_1, \dots, \alpha_m)$ ,  $\alpha \neq \bar{0}$ .

*Any domain  $D$  of trajectory structure of  $q(z) dz^2$  is a circular or a ring domain swept out by regular trajectories that belong to a homotopy class  $\gamma_k$  and have length  $\alpha_k$  for some  $k$ ,  $1 \leq k \leq m$ .*

Let  $p(z)$  be a lemniscate-generic polynomial of degree  $n$ . The set of lemniscates associated with  $p$ , i.e. the level sets  $|p(z)| = R$ , is the set of trajectories of the quadratic differential

$$q(z) dz^2 = -\frac{1}{n^2} \left( \frac{p'(z)}{p(z)} \right)^2 dz^2. \quad (2.14)$$

We formulate our result in the following proposition.

**Proposition 2.** *For any polynomial  $p(z)$  of degree  $n$  there exists a quadratic differential of form (2.14), whose trajectories describe level curves of  $p(z)$ .*



## 2.4 Pure braids and homotopy classes of closed loops. Directions for further research 33

*Proof.* Let  $p(z)$  satisfy the assumption of the lemma and  $q(z)dz^2$  be a quadratic differential defined by (2.14). The equation (see [17])

$$q(z) \left( \frac{dz}{d\tau} \right)^2 = 1,$$

for trajectories of  $q(z)dz^2$  has the following form

$$-\frac{1}{n^2} \left( \frac{p'(z)}{p(z)} \right)^2 \left( \frac{dz}{d\tau} \right)^2 = 1.$$

The latter gives us

$$\frac{p'(z) dz}{p(z) d\tau} = \frac{d}{d\tau} \text{Log}[p(z(\tau))] = in$$

and

$$\text{Re} \left[ \frac{d}{d\tau} \left( \ln|p(z(\tau))| + i \arg p(z(\tau)) \right) \right] = 0.$$

Thus we obtain that  $|p(z(\tau))| = \text{const}$ . The trajectories of (2.14) are the level curves of  $p(z)$ . □

The "figure-eights" of the polynomial  $p(z)$  are the critical trajectories of  $q(z)dz^2$ . The domains of the trajectory structure of  $q(z)dz^2$  are ring and circular domains. All regular trajectories of  $q(z)dz^2$  are closed.

The quadratic differential satisfies  $n - 1$  length relations:

$$\begin{aligned} \alpha_1^0 &= \alpha_2^0 = \dots = \alpha_n^0 = \frac{2\pi}{n}, \\ \alpha^1 &= \alpha_1^0 + \alpha_2^0, \\ \dots &\dots \dots, \\ \alpha^{n-1} &= \sum_{k=1}^n \alpha_k^0 = 2\pi = \alpha^{n-2} + \alpha^{n-3} \quad \text{or} \quad = \alpha^{n-2} + \frac{2\pi}{n}, \end{aligned} \tag{2.15}$$

where  $\alpha_k^0$ ,  $1 \leq k \leq n$ ,  $\alpha^j$ ,  $1 \leq j \leq n - 1$  are the  $q$ -lengths of the regular trajectories of  $q(z)dz^2$ . The numbers  $\alpha_k^0$ ,  $1 \leq k \leq n$  are the  $q$ -lengths of the regular trajectories of  $q(z)dz^2$  that are homotopic to the zeros  $z_1, \dots, z_n$  of the polynomial  $p(z)$ . The number  $\alpha_2$  is the  $q$ -length of the trajectories that separate two zeros of  $p(z)$  from the rest of the zeros of  $p(z)$  and the infinity. The number  $\alpha^{n-1}$  is the  $q$ -length of the trajectories that separate  $z_1, \dots, z_n$  from infinity. Figure 2.6 illustrates the relation  $\alpha^1 = \alpha_2^0 + \alpha_3^0$ .

**Example 31.** Consider  $p(z) = \frac{z^3}{3} - z + 1$  and let  $z_1, z_2, z_3$  denote its zeros. Some of the level sets of  $p(z)$  are illustrated on Figure 2.5 (a). The set of polynomial lemniscates of  $p(z)$  is given by a quadratic differential  $q(z)dz^2$  of form (2.14). The quadratic differential satisfies the length relations

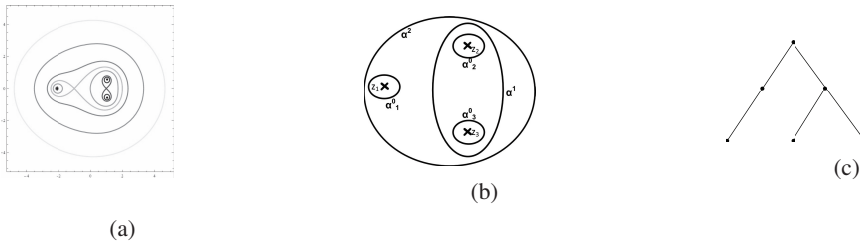


Figure 2.5: Lemniscates and free family of homotopy classes

$$\begin{aligned}
 \alpha_1^0 &= \alpha_2^0 = \alpha_3^0 = \frac{2\pi}{3}, \\
 \alpha^1 &= \alpha_2^0 + \alpha_3^0, \\
 \alpha^2 &= \alpha^1 + \alpha_1^0.
 \end{aligned}
 \tag{2.16}$$

Schematic example of trajectories of  $q(z) dz^2$  with  $q$ -lengths  $\alpha_1^0, \alpha_2^0, \alpha_3^0, \alpha^1, \alpha^2$  is shown on Figure 2.5 (b).

The quadratic differential  $q(z) dz^2$  is determined by the lemniscate-generic polynomial  $p(z)$ . With  $p(z)$  we associate a central balanced tree  $T$  of length  $n - 1$ . Recall that the leaves of the tree correspond to the zeros of  $p(z)$ . Let us mark the leaves by labels  $z_1, \dots, z_n$ . The vertices of degree 3 and the center correspond to the "figure-eights", which are components of the critical level sets of  $p(z)$ . The structure of the tree shows the position of the critical level sets relative to each other. As  $q(z) dz^2$  is uniquely determined by  $p(z)$ , let  $T$  serve as a tree representation of  $q(z) dz^2$ .

Let us consider regular trajectories  $\gamma_1, \dots, \gamma_{2n-1}$  of  $q(z) dz^2$  with lengths  $\alpha_k^0, 1 \leq k \leq n, \alpha^j, 1 \leq j \leq n - 1$ . They form an admissible system of  $2n - 1$  curves on the Riemann sphere with punctures at  $z_1, \dots, z_n$  and  $\infty$ . By Lemma 30 the free family of homotopy classes of curves generated by  $(\gamma_1, \dots, \gamma_{2n-1})$  together with the vector of the  $q$ -lengths of  $\gamma_1, \dots, \gamma_{2n-1}$  define a unique quadratic differential, that is the quadratic differential  $q(z) dz^2$ .

**Example 32.** The zeros  $z_1, z_2, z_3$  of  $p(z) = \frac{z^3}{3} - z + 1$ , the free family of homotopy classes of curves generated by the curves shown on Figure 2.5 (b) and relations (2.16) determine the quadratic differential (2.14) uniquely. The tree  $T$  for  $p(z)$  is illustrated on Figure 2.5 (c).

We obtain that on one hand,  $q(z) dz^2$  defined by (2.14) is determined uniquely by  $p(z)$  and can be represented by  $T$ . On the other hand,  $q(z) dz^2$  is determined uniquely by the points  $z_1, \dots, z_n$ , free family of homotopy classes of curves generated by the loops  $(\gamma_1, \dots, \gamma_{2n-1})$  and the vector  $(\alpha_0^1, \dots, \alpha_n^1, \alpha^1, \dots, \alpha^{n-1})$  defined by (2.15).

Let us fix the punctures  $z_1, \dots, z_n, \infty$  of the Riemann sphere, the relations (2.15) and change the free family of homotopy classes of curves. Each choice of a free family of homotopy classes of curves gives us a different quadratic differential with circular and ring domains of trajectory structure,  $n$  double poles at  $z_1, \dots, z_n, \infty, n - 1$  simple zeros and regular trajectories with lengths satisfying (2.15).

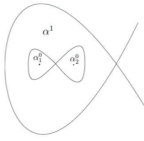
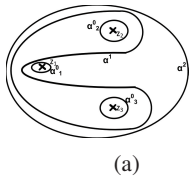
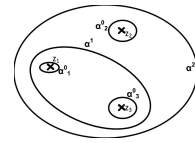


Figure 2.6: The relation  $\alpha^1 = \alpha_2^0 + \alpha_3^0$



(a)



(b)

Figure 2.7: Free families of homotopy classes

**Example 33.** The zeros  $z_1, z_2, z_3$  of  $p(z) = \frac{z^3}{3} - z + 1$ , relations (2.16) and each of the sets of loops shown on Figure 2.7 determine a unique quadratic differential.

Our goal is to extend the tree representation of  $p(z)$  to the whole family of different quadratic differentials obtained in this way and to extend the non-unitary operad constructed in Paper C to an operad in the category of quadratic differentials.

Our hypothesis is that the change of the homotopy classes can be realized through an action of the braid group  $B_n$  on the fundamental group of the complex plane with  $n$  punctures, which is a free group on  $n$  generators. In this case the vector of lengths defined by (2.15) together with the labelled central balanced tree of length  $n - 1$  and a braid gives us the desired representation of a quadratic differential and an extension of the non-unitary operad constructed in Paper C.

### Pure braids and homotopy classes of closed loops

Without loss of generality we can place  $n$  punctures at the points  $\{1, \dots, n\}$  on the real line and the base point  $P$  over the real line. The punctured complex plane is denoted by  $X_n$ . The fundamental group  $\pi(X_n, P)$  has  $n$  generators  $\gamma_1, \dots, \gamma_n$ , where  $\gamma_k$  is a homotopy class generated by a loop based at  $P$  that is homotopic to the puncture  $k$ ,  $1 \leq k \leq n$ , and is oriented counter-clockwise. The group  $\pi(X_n, P)$  is a free group on  $n$  generators.

Let  $\Gamma$  denote a free homotopy class of closed curves homotopic on  $X_n$  to a simple loop that separates the punctures  $s$  and  $t$  from the other  $n - 2$  punctures and is oriented counter-clockwise.

We can easily show that there exists an element  $\gamma_\Gamma \in \pi(X_n, P)$ , such that the loops  $\Gamma$  and the loops  $\gamma_\Gamma^{-1} \gamma_s \gamma_\Gamma \gamma_t$  are freely homotopic. Let us write in this case  $\Gamma =_f \gamma_\Gamma^{-1} \gamma_s \gamma_\Gamma \gamma_t$  meaning that the corresponding loops belong to the same free homotopy class.

Recall that the braid group  $B_n$  acts on the free group  $\pi(X_n, P)$  as follows. Let  $\sigma_k$ ,

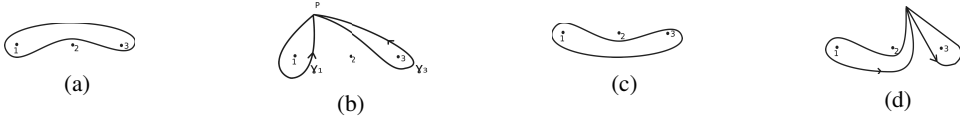


Figure 2.8: Homotopic loops

$1 \leq k \leq n-1$ , denote the  $k$ -th generator of  $B_n$ , then

$$\sigma_k(\gamma_j) = \begin{cases} \gamma_{j+1}, & j = k, \\ \gamma_j^{-1} \gamma_{j-1} \gamma_j, & j = k+1, \\ \gamma_j, & \text{otherwise.} \end{cases} \quad (2.17)$$

We have also

$$\sigma_k^{-1}(\gamma_j) = \begin{cases} \gamma_j \gamma_{j+1} x_j^{-1}, & j = k \\ \gamma_{j-1}, & j = k+1, \\ \gamma_j, & \text{otherwise.} \end{cases} \quad (2.18)$$

Let  $\Gamma$  be a free homotopy class generated by a simple loop and let  $\gamma \in \pi(X_n, P)$  be such that  $\Gamma =_f \gamma$ .

**Remark 28.** If  $\Gamma =_f \gamma$ , then  $\Gamma =_f \gamma^*$  for any  $\gamma^*$  in conjugacy class of  $\gamma$ .

The braid group  $B_n$  acts on  $\gamma$  through (2.17) and (2.18). Let  $[\gamma]$  denote the conjugacy class of  $\gamma$ . An element  $b$  of the braid group  $B_n$  acts on  $[\gamma]$  as follows

$$b([\gamma]) := [b(\gamma)].$$

We establish that the action is well-defined.

**Example 34.** Figure 2.8 (a) shows a closed Jordan curve that separates on  $X_3$  the punctures 1 and 3 from 2 and  $\infty$ . We can see that the curve is freely homotopic to a representative of the element  $\gamma_1 \gamma_3 \in \pi(X_n, P)$  shown of Figure 2.8 (b). The loops  $\gamma_1 \gamma_3$  are freely homotopic to the closed Jordan curve on 2.8 (a).

The braid  $\sigma_1^2$  acts on  $\gamma_1 \gamma_3$  in the following way.

$$\sigma_1^2(\gamma_1 \gamma_3) = \gamma_2^{-1} \gamma_1 \gamma_2 \gamma_3.$$

A representative of  $\sigma_1^2(\gamma_1 \gamma_3)$  is illustrated on Figure 2.8 (d). The loops  $\gamma_2^{-1} \gamma_1 \gamma_2 \gamma_3$  are freely homotopic to the closed Jordan curve shown on Figure 2.8 (c).

The pure braid group  $P_n$  on  $n$  strings is generated by the elements

$$A_{i,s} = \sigma_{s-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{s-1}^{-1},$$

where  $1 \leq i < s \leq n$ . Let us define  $A_{i,s}$  for  $1 \leq s < i \leq n$  in the following way

$$A_{i,s} = \sigma_s^{-1} \cdots \sigma_{i-2}^{-1} \sigma_{i-1}^2 \sigma_{i-2} \cdots \sigma_s.$$

We define in addition  $A_{s,s} = \text{id}$ . Note that  $A_{i,s} = A_{s,i}$ .

Let us fix  $s, 1 \leq s \leq n$ . The elements  $A_{i,s}, 1 \leq i \leq n$  generate a subgroup  $A_n^s$  of the pure braid group  $P_n$ . Let  $\Phi_s : \pi(X_n, P) \rightarrow A_n^s$  be a homomorphism defined by

$$\Phi_s(\gamma_i) = A_{i,s}, 1 \leq i \leq n.$$

We fix  $\gamma_s \gamma_t$  and define a function from  $A_n^s$  to  $\pi(X_n, P)$  by

$$a \mapsto a(\gamma_s \gamma_t) := a(\gamma_s) a(\gamma_t).$$

In Paper D we prove the following theorems.

**Theorem 29.** *Let  $a$  be an element of  $A_n^s$  and let  $\Gamma$  be a free homotopy class generated by a closed counter-clockwise oriented Jordan curve that separates on  $X_n$  the punctures  $s$  and  $t$  from the rest of the punctures. Let the loops  $\gamma \in \pi(X_n, P)$  be freely homotopic with the loops  $\Gamma$ . Then  $a(\gamma)$  is a homotopy class generated by a closed counter-clockwise oriented Jordan curve that separates on  $X_n$  the punctures  $s$  and  $t$  from the rest of the punctures.*

**Example 35.** There are many elements of  $\pi(X_n, P)$  that do not contain a closed Jordan curve. For example, any representative of  $\gamma_1 \gamma_1$  is not homotopic to any closed non-self-intersecting loop based at  $P$ . But even if we choose an element  $\gamma$  of  $\pi(X_n, P)$  which contains a non-self-intersecting loop based at  $P$ , the result of an action by an arbitrary braid  $b$  on  $\gamma$  does not necessarily contain a non-self-intersecting loop based at  $P$ .

Also, there are many braids that map a class of loops separating given punctures  $s$  and  $t$  from the other punctures onto a class of loops that separate two absolutely different punctures from the rest of the punctures. For example,

$$\sigma_2(\gamma_2 \gamma_4) = \gamma_3 \gamma_4.$$

Therefore the theorem above describes quite special class of braids.

**Theorem 30.** *Let  $s$  and  $t$  be fixed punctures of  $X_n$ . For any  $a \in A_n^s$  there exists  $\gamma \in \pi(X_n, P)$ , such that*

1. *there is a counter-clockwise oriented closed Jordan curve  $\Gamma$  that separates  $s$  and  $t$  from the rest of the punctures, such that loops  $\gamma$  are freely homotopic with  $\Gamma$ ;*
2. *the element  $\gamma$  has form  $\gamma = \gamma_\Gamma^{-1} \gamma_s \gamma_\Gamma \gamma_t$ , such that*

$$\Phi_s(\gamma_\Gamma) = a A_{t,s}^v$$

*for some integer  $v$ .*

Let us state the following hypothesis.

**Hypothesis**

Let  $\Gamma$  be a free homotopy class generated by a closed Jordan curve that separates on  $X_n$  the punctures  $s$  and  $t$  from the rest of the  $n - 2$  punctures. Let

$$\Gamma =_f \gamma_\Gamma^{-1} \gamma_s \gamma_\Gamma \gamma_t \in \pi(X_n, P)$$

for some  $\gamma_\Gamma \in \pi(X_n, P)$ . Then

$$\Phi_s(\gamma_\Gamma)(\gamma_s \gamma_t) = \gamma_\Gamma^{-1} \gamma_s \gamma_\Gamma \gamma_t$$

up to a cyclic permutation, i.e.  $\Phi_s(\gamma_\Gamma)(\gamma_s \gamma_t)$  and  $\gamma_\Gamma^{-1} \gamma_s \gamma_\Gamma \gamma_t$  belong to the same conjugacy class.

**Remark 31.** Let  $\Gamma$  and  $\Gamma^*$  be two free homotopy classes generated by non-self-intersecting loops that separate the fixed punctures  $s$  and  $t$  from the rest of the punctures. Let  $\gamma \in \pi(X_n, P)$  (respectively,  $\gamma^* \in \pi(X_n, P)$ ) be a class of loops based at  $P$  that are freely homotopic with loops  $\Gamma_1$  (respectively,  $\Gamma^*$ ). Then the hypothesis above gives us a braid  $b \in B_n$  (more precisely,  $b \in A_n^s$ ), such that  $b([\gamma]) = [\gamma^*]$ . In this way one can realize the change of homotopy classes of loops of this type by action of a braid group.

The proof of the hypothesis is in general completed.

### Future plans and open questions

Further research can be directed towards generalizing the results of this section to the case of homotopy classes of closed Jordan curves separating arbitrary number of punctures of  $X_n$  from the rest of the punctures. If this is possible, one could pose a question if a free family of homotopy classes can be mapped onto an arbitrary free family of homotopy classes by a single braid. The next possible step is to incorporate the obtained braids into the construction of an operad on quadratic differentials.

## Bibliography

- [1] M. J. Atia, A. Martínez-Finkelshtein, P. Martínez-González, and F. Thabet, *Quadratic differentials and asymptotics of Laguerre polynomials with varying complex parameters*. J. Math. Anal. Appl., **416** (2014), no. 1, 52–80.
- [2] Yu. Baryshnikov, *Bifurcation diagrams of quadratic differentials*. C. R. Acad. Sci. Paris Sér. I Math., **325** (1997), no. 1, 71–76.
- [3] A. Bogatyřev, *A combinatorial description of a moduli space of curves and of extremal polynomials*. Mat. Sb., **194** (2003), 27–48.
- [4] J. W. Bruce and D. B. O’Shea, *On binary differential equations and minimal surfaces*. Preprint, Liverpool, 1997.
- [5] F. Catanese and M. Paluszny, *Polynomial-lemniscates, trees and braids*. Topology **30** (1991), no. 4, 623–640.
- [6] F. Catanese and B. Wajnryb, *The fundamental group of generic polynomials*. Topology **30** (1991), no. 4, 641–651.
- [7] C. C. Cowen and Ch. Pommerenke, *Inequalities for the angular derivative of an analytic function in the unit disk*. J. Lond. Math. Soc. (2) **26** (1982), 271–289.
- [8] P. Ebenfelt, D. Khavinson, and H. S. Shapiro, *Two-dimensional shapes and lemniscates*. Complex analysis and dynamical systems IV. Part 1. Contemp. Math., 553, Amer. Math. Soc., Providence, RI, (2011), 45–59.
- [9] E. G. Emel’yanov, *On extremal partitioning problems*. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 154, 76–89 (1986) (English transl., J. Soviet Math. 43(4), 2558–2566 (1988))
- [10] J. A. Jenkins, *Univalent functions and conformal mapping*. Ergebnisse der Mathematik und ihrer Grenzgebiete **18**, Springer-Verlag (1958).
- [11] G. V. Kuz’mina, *On extremal properties of quadratic differentials with strip-like domains in their trajectory structure*. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 154, 110–129 (1986) (English transl., J. Soviet Math. 43(4), 2579–2591 (1988))
- [12] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*. Springer, Berlin (1992).

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- [13] T. A. Rakcheeva, *Multifocus lemniscates: approximation of curves*. Zh. Vychisl. Mat. Mat. Fiz. **50** (2010), no. 11, 2060–2072; translation in Comput. Math. Math. Phys. **51** (2011), no. 11, 1956–1967.
- [14] T. A. Rakcheeva, *Focal approximation on the complex plane*. Zh. Vychisl. Mat. Mat. Fiz. **51** (2011), no. 11, 1963–1972; translation in Comput. Math. Math. Phys. **51** (2011), no. 11, 1847–1855.
- [15] B. Shapiro, K. Takemura, and M. Tater, *On spectral polynomials of the Heun equation II*. Commun. Math. Phys. **311** (2012), no. 2, 277–300.
- [16] A. Yu. Solynin, *Quadratic differentials and weighted graphs on compact surfaces*. Analysis and Mathematical Physics, Trends Math., Birkhäuser, Basel, 2009, 473–505.
- [17] K. Strebel, *Quadratic differentials*. Springer-Verlag (1984).
- [18] A. Vasil'ev, *Moduli of families of curves for conformal and quasiconformal mappings*. Lecture Notes in Mathematics, **1788**, Springer-Verlag (2002).



## **Chapter 3**

### **Included papers**

