UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Master's Thesis in Algebraic Geometry

Theta–Regularity and Log–Canonical Threshold

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Introduction

An important invariant in algebraic geometry for projective spaces is the Castelnuovo–Mumford regularity index (CM–regularity). For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, this is defined as the smallest integer m, such that

$$H^i(\mathcal{F}(m-i)) = 0, \text{ for any } i > 0.$$

The index was first introduced by Guido Castelnuovo, and David Mumford was later a key contributor to the study of its properties. The main theorem of this theory says, amongst other things, that if $\mathcal{F}$ is m–regular, then $\mathcal{F}(m)$ is globally generated. Therefore the question of whether certain sheaves are generated by their global sections can be checked by the vanishing of cohomologies, which may be easier to compute.

If one changes the focus from $\mathbb{P}^n$ to principally polarized abelian varieties, Giuseppe Pareschi and Mihnea Popa introduced the Θ–regularity index in an article from 2003 [PP03]. This is defined using the language of derived categories, through the Fourier–Mukai Transform, $\hat{\mathcal{S}}(-)$. More precisely, a coherent sheaf $\mathcal{F}$ is said to be Mukai–regular if

$$\operatorname{codim}(\operatorname{Supp}(R^i(\hat{\mathcal{S}}(\mathcal{F}))) > i, \text{ for all } i > 0.$$

If Θ is a fixed polarization, then the Θ–regularity of $\mathcal{F}$ is defined to be the smallest integer m such that $\mathcal{F}((m-1)\Theta)$ is Mukai–regular. Pareschi and Popa go on to show that $\mathcal{F} \otimes \Theta$ is globally generated if $\mathcal{F}$ is Mukai–regular. This leads to a main theorem that mirrors the theorem of Mumford and Castelnuovo with (almost) identical numeric analogy.

The log-canonical threshold (lct) is a third invariant, whose definition holds for both projective spaces and abelian varieties, and is widely used in modern birational geometry. For a sheaf of ideals $\mathcal{I}$, this is defined as the lowest rational number c such that the multiplier ideal $\mathcal{J}(c \cdot \mathcal{I})$ is non–trivial. In [KP13], Alex Küronya and Norbert Pintye proved for $\mathbb{P}^n$ the following inequality relating the CM–regularity and the log–canonical threshold of an ideal sheaf $\mathcal{I}$:

$$1 \leq \operatorname{lct}(\mathcal{I})\operatorname{reg}(\mathcal{I}).$$

Given the similarities between Castelnuovo–Mumford–regularity and Θ–regularity, the goal of this thesis is to investigate the relation between the log–canonical threshold and the Θ–regularity of coherent sheaves on principally polarized abelian varieties. To this end, Chapter 1 will include the necessary background theory, including an introduction of multiplier ideals and log–canonical thresholds. In Chapter 2 we give a brief presentation of abelian varieties and then follow Shigeru Mukai's paper on the Fourier–Mukai Transform, and Pareschi and Popa's article on Θ–regularity. Chapter 3 will then conclude with the main result of this work, Theorem 3.1.6, which is an inequality that relates Θ–regularity and the log–canonical threshold in the following way:

Theorem. *Let (A,Θ) be a principally polarized abelian variety. For any coherent sheaf of ideals $\mathcal{I} \neq \mathcal{O}_A$, the following inequality holds:*

$$1 < \operatorname{lct}(\mathcal{I})(\Theta\text{–}\operatorname{reg}(\mathcal{I})).$$

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Notation and Conventions

This thesis will assume knowledge of terminology and basic results in algebraic geometry. We will for the most part follow the generally accepted notation used in [Har77]. For triangulated and abelian categories we have used the definitions of [Huy06]. Unless otherwise specified we also adopt the following conventions:

- All functors between abelian categories will be additive.
- We work throughout over the field of complex numbers, $k = \mathbb{C}$.
- A scheme X is in our context defined to be a smooth, separated algebraic scheme that is integral, projective and of finite type over $\mathbb{C}$. Thus it is in particular also an algebraic variety as defined in [Har77].
- A sheaf will be understood to be a sheaf of modules. $Mod(X)$ is then the category of sheaves on X, while $QCoh(X)$ and $Coh(X)$ are the categories of quasi–coherent and coherent sheaves, respectively. By a point of a scheme, we will more specifically mean a closed point.
- For sheaves $\mathcal{F}$ and $\mathcal{G}$ we will write $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ for the sheaf–hom, while $Hom(\mathcal{F}, \mathcal{G})$ will denote the $\mathbb{C}$–vector space.
- For clarity the i–th cohomology group of a complex $A^\bullet$ will be denoted as $\mathcal{H}^i(A^\bullet)$, while for a sheaf $\mathcal{F}$ on X the sheaf cohomology is written $H^i(X, \mathcal{F})$. When no confusion about the scheme in question can arise, we will also write the latter as $H^i(\mathcal{F})$. Lastly, we will denote the dimension $\dim_k(H^i(X, \mathcal{F}))$ as $h^i(X, \mathcal{F})$.

Contents

Chapter 1

Preliminary Ideas and Properties

The aim of this chapter is to review and state the theoretical background that will be needed for our context. It is not meant as a thorough introduction to any of the subjects presented, and we will mostly be content with referring to the appropriate literature for proofs.

Sections 1.1 and 1.2 will consist of basic theory in algebraic geometry, mostly focusing on results regarding cohomology and line bundles. Section 1.3 will introduce the bounded derived category of coherent sheaves and some associated functors. In section 1.4 we will present multiplier ideals, along with the associated vanishing theorem, Nadel Vanishing, and the rational invariant log–canonical threshold with its important geometrical interpretations. A short note on the Castelnuovo–Mumford regularity, including its relation to the log–canonical threshold is presented in section 1.5. This is mainly to motivate the work done in the second and third chapter.

1.1 General Theory in Algebraic Geometry

As previously noted, knowledge of terminology and basic results in algebraic geometry is assumed to be familiar to the reader. This section nevertheless is included to establish notation and serve as a quick reference to technical results that will be needed later. [Har77] is the main reference throughout this section and [Huy06] is the reference for the last part, regarding spectral sequences.

Recall that an object A in an abelian category $\mathcal{A}$ is said to be *injective* if the left exact contravariant functor $\mathrm{Hom}(-, A)$ is exact. An *injective resolution* of A is an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots, \; I^i \in \mathcal{A} \text{ injective for all } i \geq 0.$$

$\mathcal{A}$ is said to *contain enough injectives* if any object A admits an injective resolution. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a left exact functor and $\mathcal{A}$ has enough injectives, we can define the *right derived functors* R^iF, $i \geq 0$ as follows. For an object $A \in \mathcal{A}$ we fix an injective resolution $I^\bullet$ of A. Then $R^iF(A) = \mathcal{H}^i(F(I^\bullet))$ (which is independent of the chosen resolution). Now let X be a scheme and consider the category of sheaves of $\mathcal{O}_X$-modules, $Mod(X)$. In [Har77] III.Proposition 2.2. it is proved that this category has enough injectives. This allows for the following definition.

Definition 1.1.1. For $\mathcal{F} \in Mod(X)$, the functors $\mathrm{Ext}^i(\mathcal{F}, -)$ are defined as the right derived functors of $\mathrm{Hom}(\mathcal{F}, -)$.

It is immediate from the definition of injective resolutions, and the fact that $\mathrm{Hom}(\mathcal{F}, -)$ is left exact, that $\mathrm{Ext}^0(\mathcal{F}, -) = \mathrm{Hom}(\mathcal{F}, -)$. Some more properties of the Ext–functor are listed here, including the Serre Duality Theorem, which under certain conditions gives an isomorphism between the $\mathbb{C}$-vector space of the sheaf cohomology group H^i and the dual vector space of H^{n-i}.

Proposition 1.1.2. *Let $\mathcal{F}$ and $\mathcal{G}$ be any sheaves on X.*
a) *$Ext^i(\mathcal{O}_X, \mathcal{G}) \simeq H^i(X, \mathcal{G})$ for any $i \geq 0$.*
b) *Assume furthermore that $\mathcal{L}$ is a locally free sheaf on X. Then there is the following isomorphism:*

$$Ext^i(F \otimes \mathcal{L}, \mathcal{G}) \simeq Ext^i(F, \mathcal{L}^\vee \otimes \mathcal{G})$$

where $\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ is the dual of $\mathcal{L}$.

Proof. a) is [Har77] III.Proposition 6.3.c). Part b) is III.Proposition 6.7 in the same book. □

Theorem 1.1.3 (Serre Duality). *Let X be a scheme of dimension n, and $\mathcal{F}$ a coherent sheaf on X. Then for any $i \geq 0$ the following isomorphism holds*

$$Ext^i(\mathcal{F}, \omega_X) \simeq H^{n-i}(X, \mathcal{F})^\vee$$

where ω_X is the canonical sheaf and $H^{n-i}(X, \mathcal{F})^\vee$ is the dual cohomology group of $H^{n-i}(X, \mathcal{F})$.

Proof. [Har77] III Theorem 7.6 and III Corollary 7.12. □

Using Serre Duality we are able to show the following quick fact which will be of use to us in Chapter 2.

Corollary 1.1.4. *Let* $n = \dim X$, $\mathcal{F}$ *a coherent sheaf on* X *and* $Y = \mathrm{Supp}\mathcal{F}$. *Then* $Ext^i(F, \omega_X) = 0$ *if* $n - i > \dim Y$.

Proof. From the Serre Duality Theorem we have $Ext^i(\mathcal{F}, \omega_X) \simeq H^{n-i}(X, \mathcal{F})^\vee$. Now there is a canonical isomporhism $\mathcal{F} \to j_*(\mathcal{F}|_Y)$ where j denotes the inclusion $j : Y \to X$. So we have $H^{n-i}(X, \mathcal{F}) \simeq H^{n-i}(X, j_*(\mathcal{F}|_Y)) \simeq H^{n-i}(Y, \mathcal{F}|_Y)$ (by [Har77] III.Exercise 4.1) and the latter is trivial if $n - i > \dim Y$ by Grothendieck's Vanishing Theorem (ibid, III. Theorem 2.7). □

The Vanishing Theorem of Kodaira is also included here, which gives a useful condition for when the higher cohomology groups of line bundles are zero.

Theorem 1.1.5 (Kodaira Vanishing Theorem)**.** *If* X *is a projective nonsingular variety of dimension* n *and* $\mathcal{L}$ *an ample line bundle on* X, *then*

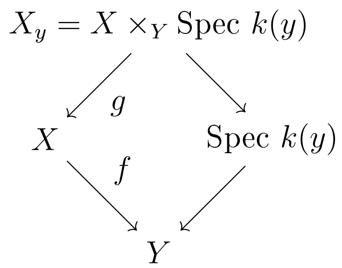
$$H^i(X, \omega_X \otimes \mathcal{L}) = 0 \text{ for } i > 0$$

By Serre Duality, an equivalent statement is

$$H^i(X, \mathcal{L}^{-1}) = 0 \text{ for } i < n$$

Proof. [Laz04a] Theorem 2.4.1. □

We fix some notation that will be useful to us. Recall that for schemes X, Y, a morphism $f : X \to Y$ and a point $y \in Y$, one writes X_y to mean the fibre of f over y. This is defined as $X \times_Y \mathrm{Spec}\ k(y)$, considered as a scheme over $\mathrm{Spec}\ k(y)$ and given by the fibre diagram



If $\mathcal{F}$ is a sheaf on X, one denotes the pullback $g^*\mathcal{F}$ on X_y as $\mathcal{F}_y$.

For a coherent sheaf $\mathcal{F}$ the *Euler characteristic* is defined as

$$\chi(\mathcal{F}) := \sum_{i\geq 0}^{\infty}(-1)^i \dim_k H^i(X,\mathcal{F})$$

It is a fact that the Euler characteristic is additive, i.e. if

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is a short exact sequence of coherent sheaves then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

The regularity condition we will be interested in the last part of this text will be related to sheaves of the form $R^i f_*(\mathcal{F})$. A very interesting question in this regard is how $R^i f_*(\mathcal{F})$ relates to the cohomology along the fibre $H^i(X_y, \mathcal{F}_y)$, and how this varies as a function of $y \in Y$. The following three results address exactly this question.

Theorem 1.1.6 (The Semicontinuity Theorem). *Let $f : X \to Y$ be a proper morphism of noetherian schemes, and $\mathcal{F}$ a coherent sheaf on X that is flat over Y. Then the following holds*

a) *For every $i \geq 0$, the function $\psi : Y \longrightarrow \mathbb{Z}$ given by*

$$y \longmapsto \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is upper semicontinous, i.e. $\psi^{-1}(-\infty, a)$ is open for any $a \in \mathbb{Z}$.

b) *The function $Y \longrightarrow \mathbb{Z}$, defined by*

$$y \longmapsto \chi(\mathcal{F}_y)$$

is locally constant on Y.

Proof. [Mum70] II.5. Corollary 1. □

Theorem 1.1.7 (Grauert's Theorem). *Consider $f : X \longrightarrow Y$ and $\mathcal{F}$ with the same assumptions as in the Semicontinuity Theorem above. If for some integer i the function $y \mapsto h^i(X_y, \mathcal{F}_y)$ is constant on Y, then $R^i f_*(\mathcal{F})$ is locally free on Y and for every y the natural map*

$$R^i f_*(\mathcal{F}) \otimes k(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$$

is an isomorphism.

Proof. [Har77] III. Corollary 12.9. □

Theorem 1.1.8 (Cohomology and Base Change)**.** *Let $f : X \to Y$, be a projective morphism of noetherian schemes, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Then for any point $y \in Y$ the following properties hold:*

a) *if the natural map*

$$\phi^i(y) : R^i f_*(\mathcal{F}) \otimes k(y) \to H^i(X_y, \mathcal{F}_y)$$

is surjective then it is an isomorphism.

b) *If $\phi^i(y)$ is surjective, the following are equivalent:*

(i) *$\phi^{i-1}(y)$ is also surjective.*

(ii) *$R^i f_*(\mathcal{F})$ is locally free in a neighborhood of y.*

Proof. [Har77] III. Theorem 12.11. □

Definition 1.1.9. A coherent sheaf $\mathcal{F}$ on an integral scheme X is *torsion–free* if for each $x \in X$ the multiplication map $s : \mathcal{F}_x \to \mathcal{F}_x$ is injective for all $s \in \mathcal{O}_{X,x} \setminus \{0\}$.

A useful property of torsion–free sheaves is that the natural restriction maps from the global sections to the stalks, $\mathcal{F}(X) \longrightarrow \mathcal{F}_x$, is injective.

Example 1.1.10. Let $\mathcal{F}$ be a coherent sheaf and $\mathcal{G}$ locally free of rank m. Then the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is torsion–free. Locally, this is the set of morphisms $\mathcal{F}_x \to \mathcal{G}_x \simeq \mathcal{O}_{X,x}^{\oplus m}$. Since X is integral, the ring $\mathcal{O}_{X,x}$ is an integral domain. So multiplying the direct sum with a non–zero element of $\mathcal{O}_{X,x}$ is non–zero, and hence the multiplication is injective.

The last concept to review in this section is that of spectral sequences. They are important tools for computing cohomology. As before we will not go into the technical details, but rather provide the necessary information needed for applications later on.

Definition 1.1.11. A *spectral sequence* over an abelian category $\mathcal{A}$ consists of a collection of objects

$$(E_r^{p,q}, E^n), \text{ for } n, p, q, r \in \mathbb{Z} \text{ and } r \geq 1$$

and morphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

such that the following conditions are satisfied

i) $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all p, q, r. In particular, this means that there are complexes $E_r^{p+\bullet r,q+\bullet(1-r)}$.

ii) There are isomorphisms $E_{r+1}^{p,q} \simeq \mathcal{H}^0(E_r^{p+\bullet r,q+\bullet(1-r)})$.

iii) For any pair (p, q) there exists an integer r_0 such that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for all $r \geq r_0$. Condition ii) then gives isomorphisms $E_r^{p,q} \simeq E_{r_0}^{p,q}$ for all $r \geq r_0$. This object is denoted $E_\infty^{p,q}$.

iv) There is a decreasing filtration

$$\cdots \subset F^{p+1}E^n \subset F^pE^n \subset \cdots \subset E^n$$

satisfying $\bigcap F^pE^n = 0$ and $\bigcup F^pE^n = E^n$. Lastly, there are isomorphisms

$$E_\infty^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}.$$

In applications it is often useful to view $E_r^{p,q}$ as objects converging, for increasing r, towards a certain quotient of the filtration on E^n, $n = p + q$. It is customary only to give the data associated to a fixed r-value, along with E^n, typically written

$$E_r^{p,q} \Rightarrow E^n.$$

As it is the converging that will be interesting to us, the fixed r–value need not be 1, but could also be a higher integer. An important class of examples for spectral sequences will be that of double complexes. We give the definition for them here.

Definition 1.1.12. A *double complex* $K^{\bullet,\bullet}$ consists of objects $K^{i,j}$ for $i, j \in \mathbb{Z}$, and morphisms

$$d_I^{i,j} : K^{i,j} \longrightarrow K^{i+1,j}$$
$$d_{II}^{i,j} : K^{i,j} \longrightarrow K^{i,j+1}$$

satisfying $d_I^2 = d_{II}^2 = d_Id_{II} + d_{II}d_I = 0$. This in particular makes $K^{i,\bullet}$ and $K^{\bullet,j}$ into complexes for any fixed i and j. The *total complex* $K^\bullet$ of the double complex $K^{\bullet,\bullet}$ is the complex $K^n = \bigoplus_{i+j=n} K^{i,j}$ with morphism $d = d_I + d_{II}$.

Proposition 1.1.13. *Let $K^{\bullet,\bullet}$ be a double complex such that for any integer n one has $K^{n-l,l} = 0$ for $|l| >> 0$. Then there is a spectral sequence*

$$E_2^{p,q} = H_{II}^pH_I^q(K^{\bullet,\bullet}) \Rightarrow H^{p+q}(K^\bullet)$$

with the filtration $F^lK^n = \bigoplus_{j \geq l} K^{n-j,j}$.

Proof. [Huy06] Proposition 2.64. □

1.2 Properties of Line Bundles and Divisors

The aim of this section is to introduce briefly the terminology and properties of divisors and line bundles that will be used throughout the rest of this text. We start by recalling some definitions and basic results regarding divisors. Then we introduce intersection numbers and further properties such as big and nef. We end by generalizing these definitions to $\mathbb{Q}$–divisors. The main references of the section will be [Har77], [Laz04a] and [Deb01]. Recall our usual convention that X is a smooth, projective algebraic variety.

Let $\mathcal{M}_X$ be the sheaf of rational functions on X. The structure sheaf $\mathcal{O}_X$ is a subsheaf of this, and hence there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups. A (integral) *Cartier divisor* on X is then a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We denote the group of all such divisors as

$$\mathrm{CDiv}(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$$

From the properties of quotient sheaves, a Cartier divisor can equally well be described as a collection $\{U_i, f_i\}$ where $\{U_i\}$ is an open cover of X and elements $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$ so that if $U_i \cap U_j \neq \emptyset$ then $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$.

Due to our assumptions on X, we may also define a (integral) *Weil divisor* to be a finite sum $\sum n_i Y_i$, where the coefficients are integers and Y_i is a *prime divisor*, i.e. a closed integral subscheme of X of codimension 1. If we denote the group of Weil divisors as $\mathrm{Div}(X)$, there is a group homomorphism

$$\begin{aligned} \mathrm{CDiv}(X) &\longrightarrow \mathrm{Div}(X) \\ D &\mapsto \sum ord_Y(D)Y \end{aligned}$$

Where $ord_Y(D)$ denotes the order of D along the prime divisor Y, and the sum is taken over all the prime divisors of X. In our setting the map will be an isomorphism by [Har77] Proposition III.6.11. Note that we will conventionally write the group operation for $\mathrm{CDiv}(X)$ additively (even though the group operation on $\mathcal{M}_X^*/\mathcal{O}_X^*$ is multiplicative), to preserve this relation with Weil divisors. We will also at times simply write "divisor" if there is no need to specify whether we are working with Weil or Cartier divisors.

A Cartier divisor is called *principal* if it lies in the image of the natural map

$$\Gamma(X, \mathcal{M}_X^*) \longrightarrow \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*)$$

Two divisors D_1, D_2 are linearly equivalent, written $D_1 \equiv_{lin} D_2$, if $D_1 - D_2$ is principal.

Recall that $\mathrm{Pic}(X)$ denotes the group of isomorphism classes of line bundles on X. It can also be expressed as the cohomology group $H^1(X, \mathcal{O}_X^*)$. Now consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 0$$

It induces a long exact sequence of cohomology groups

$$... \longrightarrow H^0(X, \mathcal{M}_X^*) \longrightarrow H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow ...$$

so in particular there is a homomorphism

$$\begin{aligned} \mathrm{CDiv}(X) &\longrightarrow \mathrm{Pic}(X) \\ D &\mapsto \mathcal{O}_X(D) \end{aligned}$$

with kernel $H^0(X, \mathcal{M}_X^*)$. In other words

$$\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2) \text{ if and only if } D_1 \equiv_{lin} D_2$$

As we assume X is integral, the homomorphism from $\mathrm{CDiv}(X)$ modulo linear equivalence is an isomorphism ([Har77] Proposition III.6.15). In light of this isomorphism, we will in the future say that a divisor D has a property defined for line bundles to mean that $\mathcal{O}_X(D)$ has said property, and vice versa.

A Cartier divisor D is *effective* if it can be represented by $\{(U_i, f_i)\}$ where $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. Equivalently, if the associated Weil divisor is written $\sum n_i Y_i$ with all $n_i \geq 0$. In this case D induces a sheaf of ideals $\mathcal{I}$ which is locally generated by the f_i's. There is furthermore an isomorphism $\mathcal{I} \simeq \mathcal{O}_X(-D)$ ([Har77] III Proposition 6.18).

Definition 1.2.1. The (complete) *linear series* of a divisor D, written $|D|$ or $|\mathcal{L}|$ for $\mathcal{L} \simeq \mathcal{O}_X(D)$, is defined as the set of all effective divisors linearly equivalent to D.

By [Har77] II.Proposition 7.7, $|D|$ is in a one–to–one correspondence with the set $(\Gamma(X, \mathcal{O}_X(D)) - \{0\})/k^*$. Hence $|D|$ has the structure of the set of closed points of some projective space over $\mathbb{C}$. One also defines the *base locus*, $\mathrm{Bs}(|D|) \subseteq X$ to be the set of points where all the sections of $\Gamma(X, \mathcal{O}_X(D))$ vanish. If there are no base points (i.e. $\mathrm{Bs}(|D|)$ is empty) then $|D|$ is called a *free linear series*.

We now want to introduce the intersection number for a collection of divisors. It will be defined through the Euler characteristic, following [Deb01]. If D is a divisor and $\mathcal{F}$ a sheaf then $\mathcal{F}(D)$ will denote the sheaf $\mathcal{F} \otimes \mathcal{O}_X(D)$. We start with the following result.

Theorem 1.2.2. *Let $D_1, ..., D_r$ be divisors and $\mathcal{F}$ a coherent sheaf on X. Then the function*

$$\mathbb{Z}^r \longrightarrow \mathbb{Z}$$
$$(m_1, ..., m_r) \mapsto \chi(X, \mathcal{F}(m_1 D_1 + ... + m_r D_r))$$

takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients having degree at most the dimension of Supp$\mathcal{F}$.

Proof. [Deb01] Theorem 1.5. □

In particular, if $\mathcal{F} = \mathcal{O}_X$, then $\chi(X, \mathcal{O}_X(m_1 D_1 + ... + m_r D_r))$ is a polynomial with rational coefficients with $\mathbb{Z}^r$ as domain. The degree is at most the dimension of X. Furthermore, it is shown in [Deb01] Proposition 1.8 that the coefficient of the term $m_1 \cdot ... \cdot m_r$ is an integer. This leads to the following definition.

Definition 1.2.3. Let $D_1, ..., D_r$ be divisors on X, with $r \geq$dim(X). The *intersection number*

$$(D_1 \cdot ... \cdot D_r)$$

is defined to be the coefficient of $m_1 \cdot ... \cdot m_r$ in the polynomial $\chi(X, m_1 D_1 + ... + m_r D_r)$.

If Y is a subscheme of X of dimension at most t, then we define the intersection number with respect to Y to be

$$(D_1 \cdot ... \cdot D_t \cdot Y) := (D_1|_Y \cdot ... \cdot D_t|_Y)$$

where $D_i|_Y$ is the restriction of the divisor D_i to Y.

Recall the following definitions of ample and very ample line bundles.

Definition 1.2.4. Let $\mathcal{L}$ be a line bundle on X.

i) $\mathcal{L}$ is *very ample* if there exists a closed embedding $i : X \longrightarrow \mathbb{P}^N$, for some integer N, where

$$\mathcal{L} \simeq i^* \mathcal{O}_{\mathbb{P}^N}(1)$$

ii) $\mathcal{L}$ is called *ample* if there is an integer $m > 0$ such that $\mathcal{L}^m$ is very ample.

As we are assuming all schemes are projective, there will always exist a very ample (and hence also ample) line bundle. Equivalence conditions for $\mathcal{L}$ to be ample is found in the following theorem.

Theorem 1.2.5 (Cartan–Serre–Grothendieck)**.** *Let $\mathcal{L}$ be a line bundle on X Then following conditions are then equivalent:*

i) $\mathcal{L}$ is ample.

ii) For any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer m_0 (depending on $\mathcal{F}$) such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0 \text{ for all } i > 0, \ m \geq m_0.$$

iii) Given any coherent sheaf $\mathcal{F}$ on X there is a positive integer m_1 (also depending on $\mathcal{F}$) such that $\mathcal{F} \otimes \mathcal{L}^m$ is globally generated for all $m \geq m_1$.

iv) There is a positive integer $m_{\mathcal{L}}$ such that $\mathcal{L}^m$ is very ample for all $m \geq m_{\mathcal{L}}$.

Proof. [Laz04a] Theorem 1.2.6 □

There is also a numerical way to view ample line bundles as seen by the following result.

Theorem 1.2.6 (The Nakai–Moishezon Criterion)**.** *A divisor D on the scheme X is ample if and only if for any integral subscheme Y of X one has*

$$(D \cdot ... \cdot D \cdot Y) > 0$$

where the intersection number is taken over dim(Y) copies of D.

Proof. [Deb01] Theorem 1.21. □

Based on this, there is the following weaker condition.

Definition 1.2.7. A divisor D on X is said to be *nef* (numerically effective) if for every integral subscheme Y of X one has

$$(D \cdot ... \cdot D \cdot Y) \geq 0$$

where the intersection number is taken over $\dim(Y)$ copies of D.

In other words, the definition is expanding the Nakai-Moishezon Criterion by allowing the intersection numbers to be equal to zero as well. Nefness can also be checked only on curves, i.e. D is nef if and only if it has a non–negative intersection number with every curve of X ([Deb01] Theorem 1.26). This shows that being nef is preserved by numerical equivalence, which we define as follows.

Definition 1.2.8. Two divisors D_1 and D_2 are said to be *numerically equivalent*, written $D_1 \equiv_{num} D_2$, if

$$(D_1 \cdot C) = (D_2 \cdot C)$$

for every curve $C \subseteq X$.

Now consider the following set associated to a line bundle $\mathcal{L}$ on X:

$$N(\mathcal{L}) = \{m \geq 0 \mid H^0(X, \mathcal{L}^m) \neq 0\}$$

Suppose $m \in N(\mathcal{L})$ and let $s_0, ...s_r$ be elements forming a basis for $H^0(X, \mathcal{L}^m)$. Denote $B = Bs(|\mathcal{L}^m|)$, then there is a map

$$\phi_m = \phi_{|\mathcal{L}^m|} : X \setminus B \longrightarrow \mathbb{P}^r$$
$$x \mapsto (s_0(x), ..., s_r(x))$$

Since B is closed, ϕ_m is a rational map $X \dashrightarrow \mathbb{P}^r$.

Definition 1.2.9. If $N(\mathcal{L}) \neq 0$ then the *Iitaka dimension* of a line bundle $\mathcal{L}$ on X is defined to be

$$\kappa(\mathcal{L}) = \max_{m \in N(\mathcal{L})}\{\dim\, \phi_m(X)\}$$

where $\phi_m(X)$ is the closure of the image of ϕ_m in $\mathbb{P}^r$. If $N(\mathcal{L}) = 0$, i.e. $H^0(X, \mathcal{L}^m) = 0$ for all $m > 0$, then one conventionally sets $\kappa(\mathcal{L}) = -\infty$.

Hence the Iitaka dimension of a line bundle is either $-\infty$ or $0 \leq \kappa(\mathcal{L}) \leq \dim\, X$.

Definition 1.2.10. $\mathcal{L}$ is said to be *big* if it has maximal Iitaka dimension, i.e. if $\kappa(\mathcal{L}) = \dim\, X$.

Example 1.2.11. An ample line bundle has a multiple which is very ample, and very ample bundles admit a set of global sections such that the associated map $X \dashrightarrow \mathbb{P}^n$ is an immersion. It follows that ample line bundles are big.

The rest of this section is devoted to $\mathbb{Q}$–divisors. As we will see shortly, this is a generalization of (Weil) divisors by allowing rational coefficients. We also remark on how to generalize all the properties seen so far to this setting.

Definition 1.2.12. A (Weil) $\mathbb{Q}$*–divisor* D on X is defined as an element of the $\mathbb{Q}$–vector space

$$\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Such an element can be represented as a finite sum

$$D = \sum a_i D_i$$

where $a_i \in \mathbb{Q}$ and D_i is a prime divisor.

The divisor is furthermore said to be *effective* if $a_i \geq 0$ for all i, and *integral* if $a_i \in \mathbb{Z}$ for all i. One can always go from a $\mathbb{Q}$–divisor to an integral divisor by considering its *round–down*. More specifically, this is the divisor

$$\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$$

where $\lfloor a_i \rfloor$ is the greatest integer $\leq a_i$.

The properties, operations and equivalences introduced previously for integral divisors extend easily to $\mathbb{Q}$–divisors in the natural way: one considers them for integral divisors and then extends by linearity. This is more concretely summed up in the following definition.

Definition 1.2.13. Let all divisors D_i denote $\mathbb{Q}$–divisors on X. Then we define the following:

i) If $Y \subseteq X$ is a subscheme of dimension k, then the $\mathbb{Q}$–valued intersection product

$$\mathrm{Div}_{\mathbb{Q}}(X) \times ... \times \mathrm{Div}_{\mathbb{Q}}(X) \longrightarrow \mathbb{Q}$$
$$(D_1, ..., D_k) \mapsto (D_1 \cdot ... \cdot D_k \cdot Y)$$

is defined via extension of scalars from the analogous map on $\mathrm{Div}(X)$. In particular, if r is an integer so that rD_i is an integral divisor for all i, then $(D_1 \cdot ... \cdot D_k \cdot Y) = \frac{1}{r}(rD_1 \cdot ... \cdot rD_k \cdot Y)$.

ii) D_1 and D_2 are numerically equivalent, $D_1 \equiv_{num,\mathbb{Q}} D_2$ (or $D_1 \equiv_{num} D_2$ if no confusion can arise) if

$$(D_1 \cdot C) = (D_2 \cdot C)$$

for every curve $C \subseteq X$, using the definition from i) on the intersection numbers.

iii) D_1 and D_2 are linearly equivalent, $D_1 \equiv_{lin,\mathbb{Q}} D_2$ if there is an integer r such that rD_1 and rD_2 are integral divisors that are linearly equivalent (as integral divisors).

iv) For a morphism $f : X' \longrightarrow X$ the pull–back f^*D is defined by performing the pullback on the prime divisors, f^*Y_i, and extending linearly. More specifically, f^*Y_i is defined by pulling back the local equations for Y_i. This is possible when f does not map X' into the support of Y_i ([Laz04a] pages 9–10).

v) D is ample if there is a positive integer $r > 0$ so that $r \cdot D$ is an ample integral divisor. Equivalently, if D satisfies the Nakai-Moishezon Criterion

$$(D^{\dim Y} \cdot Y) > 0$$

for every integral subscheme $Y \subseteq X$, with intersection numbers taken as in i).

Remark 1.2.14. It may happen that two different integral divisors become linearly equivalent in the sense of iii) if considered as $\mathbb{Q}$–divisors. For this reason we will rather be working with numerical equivalence for $\mathbb{Q}$–divisors, where this does not occur.

Since the definition of being nef (for integral divisors) only depends on numerical equivalence classes, the definition immediately generalizes to $\mathbb{Q}$–divisors, using the definition of intersection numbers as in i) in the definition above. We also define bigness for a $\mathbb{Q}$–divisor D if there is a positive integer $m > 0$ so that mD is integral and big. Clearly, an ample $\mathbb{Q}$–divisor D is nef and big. Indeed, by condition v) in the last definition, it satisfies the Nakai–Moishezon Criterion which is a stronger statement than nefness. Also, since there is a positive integer r so that rD is integral and ample, it is also big by Example 1.2.11.

Remark 1.2.15 (Convention). For the rest of this text we adapt the convention that *divisor* will always mean an integral divisor. The instances where $\mathbb{Q}$–divisors are used will be specified.

1.3 Derived Categories

So far we have only considered "usual" sheaves, forming the abelian categories of sheaves, $Mod(X)$, quasi–coherent sheaves, $Qcoh(X)$, and coherent sheaves $Coh(X)$. This section will be devoted to categories consisting of resolutions of sheaves. More precisely the goal will be to introduce the bounded derived category of coherent sheaves. We will also present the derived functors that are fundamental to the framework of the later sections. The main source of the section will be [Huy06].

To start off we fix some notation. $\mathcal{A}$ will, as before, be an abelian category. $Kom(\mathcal{A})$ denotes the category of complexes in $\mathcal{A}$. If $A^\bullet$ and $B^\bullet$ are objects in $Kom(\mathcal{A})$, we say that a morphism $f : A^\bullet \longrightarrow B^\bullet$ is a *quasi-isomorphism* (qis for short) if the map $\mathcal{H}^i(f) : \mathcal{H}^i(A^\bullet) \longrightarrow \mathcal{H}^i(B^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$. The idea behind the derived category is then to turn all the quasi-isomorphisms into isomorphisms. More formally, there is the following universal property.

Theorem 1.3.1. *For any abelian category $\mathcal{A}$ there exists a category $D(\mathcal{A})$ called the* derived category of $\mathcal{A}$, *along with a functor*

$$Q : Kom(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

satisfying the properties

i) *If $f : A^\bullet \longrightarrow B^\bullet$ is a quasi–isomorphism, then $Q(f)$ is an isomorphism in $D(\mathcal{A})$.*

ii) *If $F : Kom(\mathcal{A}) \longrightarrow D$ is another functor satisfying property i), then there exists a unique functor (up to isomorphisms) $G : D(\mathcal{A}) \longrightarrow D$ such that the following diagram commutes:*

$$\begin{array}{ccc} Kom(\mathcal{A}) & \xrightarrow{\quad Q \quad} & D(\mathcal{A}) \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & D & \end{array}$$

A proof of the theorem can be found in [Huy06] Theorem 2.10. In the context of this thesis we will be content with describing what the derived category looks like. A useful intermediate step when passing from $Kom(\mathcal{A})$ to $D(\mathcal{A})$ is the homotopy category of complexes. Recall that two complex morphisms $f, g : A^\bullet \longrightarrow B^\bullet$ are said to be *homotopically equivalent*, written $f \sim g$, if there is a collection of morphisms

$h^i : A^i \longrightarrow B^{i-1}$, $i \in \mathbb{Z}$, such that

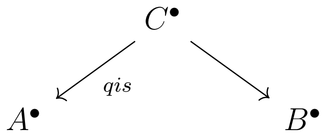
$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$$

Definition 1.3.2. The *homotopy category of complexes*, $K(\mathcal{A})$, is the category having the same objects as $Kom(\mathcal{A})$, $Ob(Kom(\mathcal{A})) = Ob(K(\mathcal{A})$, and morphisms $\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{Kom(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$, where $\sim$ denotes homotopy equivalence.

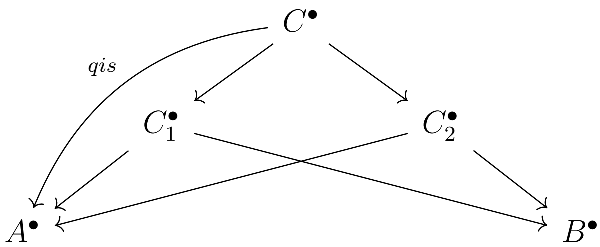
The following fact relates homotopy equivalence and quasi–isomorphisms, and suggests the usefulness of $K(\mathcal{A})$ in defining $D(\mathcal{A})$.

Lemma 1.3.3. *If $f, g : A^\bullet \longrightarrow B^\bullet$ and $f \sim g$ then $H^i(f) = H^i(g)$ for all i. In particular, if there also exists a morphism $h : B^\bullet \longrightarrow A^\bullet$ so that $h \circ f \sim id_A$ and $f \circ h \sim id_B$ then $A^\bullet$ and $B^\bullet$ are quasi-isomorphic.*

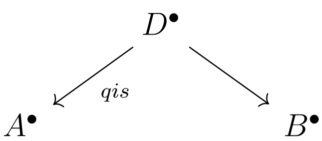
Everything is now properly set to define $D(\mathcal{A})$. For objects there is a particularly easy description, namely $Ob(D(\mathcal{A})) = Ob(K(\mathcal{A}) = Ob(Kom(\mathcal{A}))$. For $A^\bullet, B^\bullet \in D(\mathcal{A})$ the group of morphisms $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ are the equivalence classes of roofs

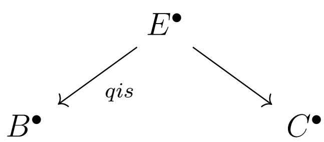


where $C^\bullet \longrightarrow A^\bullet$ is, as denoted, a quasi–isomorphism. Two such roofs, with $C_1^\bullet$ and $C_2^\bullet$ on top, are equivalent if one can write a third diagram of the following form that commutes in $K(\mathcal{A})$:

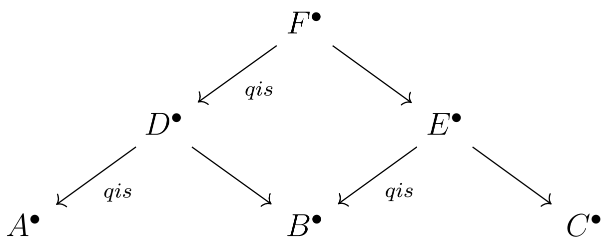


This in particular means that $C^\bullet \longrightarrow C_1^\bullet \longrightarrow A^\bullet$ is homotopy equivalent to $C^\bullet \longrightarrow C_2^\bullet \longrightarrow A^\bullet$. Composing the two morphisms





is done by giving a third commutative diagram (again commuting in $K(\mathcal{A})$) of the form



One of the reasons why we require commutativity of the diagrams in $K(\mathcal{A})$, as opposed to in $Kom(\mathcal{A})$, is that the constructed middle square in the latter diagram for composition of morphisms will only commute up to homotopy equivalence. See [Huy06] Proposition 2.16 for more details regarding this construction.

From the definition of $D(\mathcal{A})$, it is easy to see that one can identify objects of $\mathcal{A}$ with complexes in $D(\mathcal{A})$ that are concentrated in degree 0 (i.e. complexes $A^\bullet$ where $H^0(A^\bullet) = A$ and $H^i(A^\bullet) = 0$, $i \neq 0$). This identification makes $\mathcal{A}$ into a full subcategory of $D(\mathcal{A})$. One defines the *bounded derived category*, $D^b(\mathcal{A})$, to consist of only the bounded complexes in $D(\mathcal{A})$, which are complexes $A^\bullet$ where the cohomology groups $H^i(A^\bullet)$ are non–zero for only finitely many indices i. One defines $Kom^b(\mathcal{A})$ and $K^b(\mathcal{A})$ similarly. We will write $D^+(\mathcal{A})$ for the derived category where the cohomology of the complexes vanishes for $i << 0$ (similarly for $Kom^+(\mathcal{A})$ and $K^+(\mathcal{A})$).

Definition 1.3.4. Let $Coh(X)$ be the abelian category of coherent sheaves over the scheme X. Then the *bounded derived category of coherent sheaves* is $D^b(X) := D^b(Coh(X))$.

Interesting questions now arise concerning what functors one has in the derived category. An easy start is the *shift functor* that naturally appears in any category having complexes as objects. For any $n \in \mathbb{Z}$ this is the functor $[n] : D(\mathcal{A}) \longrightarrow D(\mathcal{A})$ that takes terms $A^i[n] = A^{i+n}$ and differentials $d^i_{A^\bullet[n]} = (-1)^n d^{i+n}_{A^\bullet}$. The shift functor is clearly an equivalence of categories, its inverse given by $[-n]$. Now consider a functor of abelian categories $F : \mathcal{A} \longrightarrow \mathcal{B}$. It will naturally induce a functor $K(F) : K(\mathcal{A}) \longrightarrow K(\mathcal{B})$ by applying F to each term in a given complex. The goal is to determine a functor $D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ that is as close as possible to F, and this will depend on the "degree" of exactness that F exhibits. If F is exact it will extend to a functor $D(\mathcal{A}) \longrightarrow D(\mathcal{B})$, again by applying F to each term in a complex. This is well defined as the exactness of F assures that the image of an acyclic complex (i.e. a complex quasi–isomorphic to 0) is still acyclic. If F is not exact the same approach cannot be applied, since the image of an acyclic complex would in general fail to be acyclic. This observation inspires the following definition left exact functors.

Definition 1.3.5. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor. A class of objects $\mathcal{I}_F \subset \mathcal{A}$, stable under finite sums, is said to be F–adapted if it satisfies the two following conditions:

i) If $A^\bullet \in K^+(\mathcal{A})$ is acyclic with $A^i \in \mathcal{I}_F$ for all i, then $F(A^\bullet)$ is acyclic.

ii) Any object in $\mathcal{A}$ can be embedded into an object of $\mathcal{I}_F$.

There is also the more general version, where the functor F needs only be given on the homotopy category.

Definition 1.3.6. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and suppose $F : K^+(\mathcal{A}) \longrightarrow K(\mathcal{B})$ is exact (as a functor of triangulated categories). Then a triangulated subcategory $\mathcal{R}_F \subset K^+(\mathcal{A})$ is said to be *F–adapted* if it satisfies the two conditions

i) If $A^\bullet \in \mathcal{R}_F$ is acyclic then $F(A^\bullet)$ is acyclic.

ii) Any object in $K^+(\mathcal{A})$ is quasi-isomorphic to an object in $\mathcal{R}_F$.

Intuitively speaking, condition ii) assures that we can replace any complex $A^\bullet$ with a complex in $\mathcal{R}_F$. Condition i) guarantees that F acts like an exact functor on this class of complexes. [Har66] I Theorem 5.1 formalizes and justifies this intuition. The class of injective sheaves in $Mod(X)$, discussed in the first section, is an example of a class adapted to all left exact functors.

Example 1.3.7. i) The idea in this example is to construct the tensor product as a left derived functor from the total complex of a double complex (as described in Definition 1.1.12). For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in Kom^b(X)$ we define the double complex consisting of objects $K^{p,q} = \mathcal{F}^p \otimes \mathcal{G}^q$ and morphisms $d_I = d_{\mathcal{F}} \otimes 1$ and $d_{II} = (-1)^{p+q} 1 \otimes d_{\mathcal{G}}$. The functor of interest is then defined as the associated total complex

$$\mathcal{F}^\bullet \otimes (-) : K^b(X) \longrightarrow K^b(X)$$

Where $(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^i := \bigoplus_{p+q=i} (\mathcal{F}^p \otimes \mathcal{G}^q)$ and the resulting complex has morphisms $d = d_{\mathcal{F}} \otimes 1 + (-1)^i \otimes d_{\mathcal{G}}$. To make it into a derived functor, we will show that the subcategory of bounded complexes of locally free sheaves is adapted to it (in the sense of Definition 1.3.6). Recall that any coherent sheaf on X admits a resolution of locally free sheaves, which is of finite lenght as we are assuming X to be smooth. One can therefore find a complex of locally free sheaves, $\epsilon^\bullet$, for any object in $K^b(X)$. Fix such a complex $\epsilon^\bullet$ that is also acyclic. The spectral sequence from Proposition 1.1.13 now reads

$$E_2^{p,q} = \mathcal{H}_I^p \mathcal{H}_{II}^q (K^{\bullet,\bullet}) \Rightarrow \mathcal{H}^{p+q}(\mathcal{F}^\bullet \otimes \epsilon^\bullet)$$

If we fix any i, giving the complex $K^{i,\bullet} = \mathcal{F}^i \otimes \mathcal{G}^\bullet$, then $\mathcal{H}^q_{II}(\mathcal{F}^i \otimes \epsilon^\bullet) = 0$, due to $\epsilon^\bullet$ being acyclic, and tensoring with locally free sheaves is an exact functor, so it commutes with cohomology. Since cohomology also commutes with direct sums, this implies that all $E_2^{p,q}$ are trivial and hence equal to the infinity–object $E_\infty^{p,q}$. Recall from Definition 1.1.11 of spectral sequences that we now have $0 = E_\infty^{p,q} \simeq F^pE^{p+q}/F^{p+1}E^{p+q}$. From $\cup F^pE^n = E^n = \mathcal{H}^n(\mathcal{F}^\bullet \otimes \epsilon^\bullet)$, and the fact that $\cap \mathcal{F}^pE^n = 0$, we then deduce that $\mathcal{H}^n(\mathcal{F}^\bullet \otimes \epsilon^\bullet)$ must be trivial, hence the complex $\mathcal{F}^\bullet \otimes \epsilon^\bullet$ is acyclic. The subcategory of bounded complexes of locally free sheaves is thus indeed adapted for $\mathcal{F}^\bullet \otimes (-)$ which means that it induces a well defined left derived functor $\mathcal{F}^\bullet \otimes^L (-) : D^b(X) \longrightarrow D^b(X)$.

ii) The pullback of a morphism $f : X \longrightarrow Y$ is, on coherent sheaves, a functor $f^* : Coh(Y) \longrightarrow Coh(X)$ that is the composition of the functors $f^{-1} : Coh(Y) \to Coh(X)$ and $(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (-)) : Coh(X) \to Coh(X)$. By the construction in i) it therefore induces a left derived functor $D^b(Y) \longrightarrow D^b(X)$.

In the previous section we used that $Mod(X)$ has enough injectives. This fails to be the case for $Coh(X)$, but it can, however, be shown that $Qcoh(X)$ has enough injectives (see [Har66] I Theorem 7.18). One furthermore has the following important result.

Proposition 1.3.8. *For a noetherian scheme X the natural functor*

$$D^b(X) \longrightarrow D^b(Qcoh(X))$$

defines an equivalence of categories

$$D^b(X) \simeq D^b_{coh}(Qcoh(X)).$$

Here $D^b_{coh}(Qcoh(X))$ denotes the category of bounded complexes of quasi–coherent sheaves with coherent cohomology.

Proof. [Huy06] Proposition 3.5. □

The idea for left exact functors on coherent sheaves is then to pass to the larger category of complexes of quasi–coherent sheaves. To go on about this more rigorously, let $F : Qcoh(X) \longrightarrow \mathcal{A}$ be a left exact functor where X is a scheme and $\mathcal{A}$ an abelian category. We have the following diagram.

$$\begin{array}{ccccc} K^+(\mathcal{I}) & \hookrightarrow & K^+(Qcoh(X)) & \xrightarrow{K(F)} & K^+(\mathcal{A}) \\ & \overset{i}{\searrow} \quad \overset{i^{-1}}{\nwarrow} & \downarrow Q_X & & \downarrow Q_{\mathcal{A}} \\ & & D^+(Qcoh(X)) & & D^+(\mathcal{A}) \\ & & \uparrow j & & \\ & & D^+(X) & & \end{array}$$

Here j denote the natural inclusion and Q_X, $Q_{\mathcal{A}}$ denotes the functors from Theorem 1.3.1 (after passing to the homotopy category). $\mathcal{I}$ is the class of injective objects of $Qcoh(X)$ and there is the natural functor i obtained from passing through Q_X. This is an equivalence of categories (c.f. [Huy06] Proposition 2.40) where i^{-1} takes a complex in $D^+(Qcoh(X))$ to a quasi–isomorphic complex consisting of injective objects. So for a left exact functor $F : Qcoh(X) \longrightarrow Qcoh(Y)$ one defines the associated right derived functor of coherent sheaves

$$RF := Q_{\mathcal{A}} \circ K(F) \circ i^{-1} \circ j : D^+(X) \longrightarrow D^+(\mathcal{A}). \tag{1.1}$$

For an object $A^\bullet \in D^+(\mathcal{A})$ one writes $R^iF(A^\bullet) := \mathcal{H}^i(RF(A^\bullet))$.

This will be the case for f_*, the direct image of a morphism $f : X \longrightarrow Y$. Since f_* is left exact, (1.1) guarantees the existence of a right derived functor $Rf_* : D^+(Qcoh(X)) \longrightarrow D^+(Qcoh(Y))$. For $i >$ dim$X$ and $\mathcal{F}$ any coherent sheaf, $R^if_*\mathcal{F}$ is trivial ([Har77] III Proposition 8.1 and Grothendiecks Vanishing Theorem). Corollary 2.68 in [Huy06] then assures that

$$Rf_* : D^+(Qcoh(X)) \longrightarrow D^+(Qcoh(Y))$$

induces a functor

$$Rf_* : D^b(Qcoh(X)) \longrightarrow D^b(Qcoh(Y)).$$

In our setting $R^if_*\mathcal{F}$ is coherent for all integers i when $\mathcal{F}$ is coherent (c.f. [Har77] III Theorem 8.8), so we can apply Proposition 1.3.8 to obtain a functor:

$$Rf_* : D^b(X) \longrightarrow D^b_{coh}(Qcoh(Y)) \simeq D^b(Y).$$

By combining the derived functors seen so far, we can state a general version of the Projection Formula and Flat Base Change.

Proposition 1.3.9 (Projection Formula). *Let $f : X \longrightarrow Y$ be a proper morphism of projective schemes. Then for objects $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{G}^\bullet \in D^b(Y)$ there is a natural isomorphism*

$$Rf_*(\mathcal{F}^\bullet) \otimes^L \mathcal{G}^\bullet \simeq Rf_*(\mathcal{F}^\bullet \otimes^L Lf^*\mathcal{G}^\bullet)$$

Proof. [Har66] II Proposition 5.6. □

Proposition 1.3.10 (Flat Base Change)**.** *Consider the following fibre product diagram*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{v} & Y \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ X & \xrightarrow{u} & Z \end{array}$$

where $u : X \longrightarrow Z$ is a flat morphism and $f : Y \longrightarrow Z$ is a morphism of finite type. Then there is a functorial isomorphism

$$u^* Rf_* \mathcal{F}^\bullet \simeq Rg_* v^* \mathcal{F}^\bullet$$

for any $\mathcal{F}^\bullet \in D(QCoh(Y))$.

Proof. [Har66] II Proposition 5.12. □

A "derived" version of the Ext–functor is given by the following example.

Example 1.3.11. The functor

$$Hom^\bullet(\mathcal{F}^\bullet, (-)) : K^b(Qcoh(X)) \longrightarrow K^+(Ab)$$

where Ab denotes the category of abelian groups, is defined by the complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ having terms

$$\mathrm{Hom}^i(A^\bullet, B^\bullet) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}(A^k, B^{k+i})$$

and morphisms

$$d(f) = d_B \circ f - (-1)^i f \circ d_A$$

It is noted in [Huy06] Remark 2.57 that the full triangulated subcategory of complexes of injectives is adapted to this functor. Thus one obtains a right functor

$$Hom^\bullet(\mathcal{F}^\bullet, (-)) : D^b(X) \longrightarrow D^+(Ab)$$

by (1.1). For any $\mathcal{F}^\bullet \in D^b(X)$ the i–th *Ext–functor* is then defined as $\mathrm{Ext}^i(\mathcal{F}^\bullet, (-)) := R^i Hom^\bullet(\mathcal{F}^\bullet, (-))$.

A particularly useful relation for the Ext–functors is the following

Proposition 1.3.12. *For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b(X)$ there is a natural isomorphism*

$$\mathrm{Ext}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i])$$

Proof. This is found in [Huy06] Remark 2.57 along with Proposition 1.3.8 (from this text). □

We will end this section with the following spectral sequences, which in many cases can greatly simplify calculations of the Ext–functors.

Proposition 1.3.13. *For $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet \in D^b(X)$, there are spectral spectral sequences:*

i) $E_2^{p,q} = \operatorname{Ext}^p(H^{-q}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$

ii) $E_2^{p,q} = \operatorname{Ext}^p(\mathcal{F}^\bullet, H^q(\mathcal{G}^\bullet)) \Rightarrow \operatorname{Ext}^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$

Proof. See [Huy06] Examples 2.70. □

1.4 Multiplier Ideals

Compared to the abstract language of derived categories, this section will be of a more geometric nature. Multiplier ideals can be defined for three objects; $\mathbb{Q}$–divisors, ideal sheaves and linear series. The theories for these different settings will essentially be equivalent, but there are slight variations. As all three settings will be useful to us, we intend to introduce the needed theory for them all. We will however be content with giving references to proofs for only one of the settings. The first step will be to look at *simple normal crossing*, a condition on the type of singularities a divisor can exhibit. We shall see that any of the three objects can be brought to a state of simple normal crossing through birational maps called *log–resolutions*. One then uses these log–resolutions to associate *multiplier ideal sheaves*. These multiplier ideals contain subtle information regarding the singularities of the underlying object, and we shall primarily study this through an associated numerical invariant named *log–canonical threshold*. The main reference will be [Laz04b].

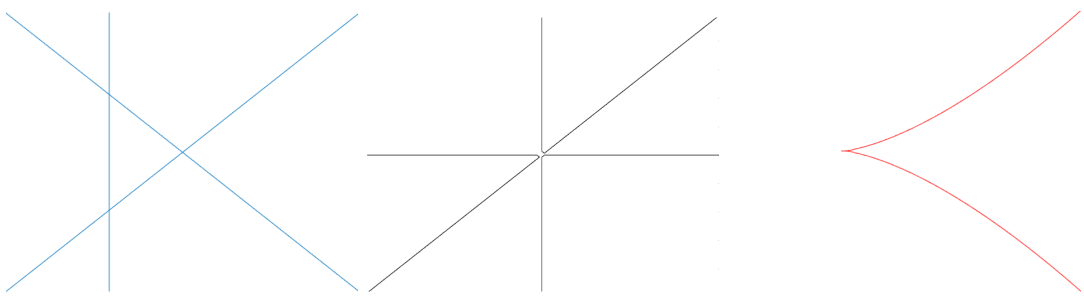
We once more recall the convention that X will always be a smooth, projective algebraic variety.

Definition 1.4.1. An integral effective divisor $D = \sum D_i$ is said to be a *simple normal crossing* (SNC) divisor if D_i is smooth for all i, and D can be written in the neighbourhood of any point as an equation in local analytic coordinates of the form

$$z_1 z_2 ... z_m = 0$$

for an integer $m \leq \dim X$. If $D = \sum a_i D_i$ is a $\mathbb{Q}$–divisor, with rational coefficients a_i, then it is said to have *simple normal crossing support* if $\sum D_i$ is a SNC divisor.

Example 1.4.2. Consider the following divisors in $\mathbb{A}^2_{\mathbb{C}} = \text{Spec } \mathbb{C}[x_1, x_2]$.



(a) (b) (c)

a) shows a SNC divisor whereas b) is not; the singularity needs three local analytic coordinates to be described. The cusp in c) is also not a SNC divisor, failing two conditions. The irreducible component is itself singular, and the singularity has a local description as a double point.

The intuition behind a SNC-divisor, as anecdotally seen in the example, is that intersections amongst the components of the divisor occur in a "transverse" manner. Thus it encompasses a class of divisor whose singularities are easy to understand. If we are concerned with a divisor that is not SNC, one can pull it back using a finite succession of blow–up maps until the divisor has this property. This will be the idea behind log–resolutions. We briefly describe the blow–up morphism, along with some of its properties through the following theorem.

Theorem 1.4.3. *If X is a variety and $Y \subset X$ a smooth, closed subvariety, the* blow–up of X along Y, *$\phi : Bl_Y(X) \longrightarrow X$, exists and has the following properties:*

i) $Bl_Y(X)$ is a variety. If X is furthermore projective, then so is $Bl_Y(X)$ and ϕ is hence a projective morphism.

ii) The inverse image $\phi^{-1}(Y)$ is a locally principal closed subvariety (i.e. it corresponds to an effecitve Cartier divisor) in $Bl_Y(X)$. This will be called the exceptional divisor, Except*(ϕ).*

iii) ϕ is universal with respect to property ii). I.e. if there is another morphism $f : W \longrightarrow X$ where $f^{-1}(Y)$ is a locally principal closed subvariety, then there exists a unique morphism $g : W \longrightarrow Bl_Y(X)$ so that $f = \phi \circ g$.

iv) The restricted morphism $\phi : Bl_Y(X) \setminus \phi^{-1}(Y) \longrightarrow X$ is an isomorphism. In particular, this makes ϕ a birational map.

Proof. Property i) is shown in [Har77] II Proposition 7.16. The rest can be found in [EH01] Theorem IV-23. □

The usefulness of blow–ups can be illustrated by inspecting how the troublesome divisors from Example 1.4.2 behave under the blow–up of the singular point. To this end, recall from [Har77] I.4 that if we denote the affine coordinates in $\mathbb{A}^n_{\mathbb{C}}$ as $x_1, ..., x_n$, and the homogeneous coordinates of $\mathbb{P}^{n-1}_{\mathbb{C}}$ as $y_1, ..., y_n$, then the blow–up of $\mathbb{A}^n_{\mathbb{C}}$ at the origin is given by

$$Bl_O(\mathbb{A}^n_{\mathbb{C}}) = \{x_i y_j = x_j y_i | i, j = 1, ..., n\} \subset \mathbb{A}^n_{\mathbb{C}} \times \mathbb{P}^{n-1}_{\mathbb{C}} \tag{1.2}$$

where ϕ is simply the projection onto $\mathbb{A}^n_{\mathbb{C}}$.

Example 1.4.4. We may treat Example 1.4.2 (b) as three lines passing through the origin. Any such line can be parametrized as $\mathcal{L} = (a_1 t, a_2 t)$ for some $a_1, a_2 \in k$, at

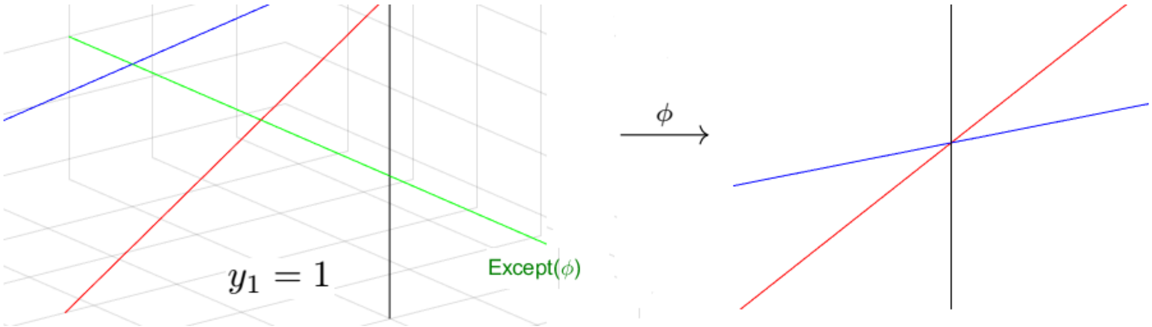


Figure 1.2: Blowing up lines of an affine space along the origin.

least one non–zero, and $t \in \mathbb{A}^1_{\mathbb{C}}$. If ϕ is the projection from (1.2), we consider its pre–image of $\mathcal{L}$ outside the origin. If, say, $a_1 \neq 0$ then we have $a_1 t y_2 = a_2 t y_1$, or $y_2 = y_1(a_2/a_1)$. Fixing the homogeneous coordinate $y_1 = a_1$, we get $y_2 = a_2$. Thus we have a description of $\phi^{-1}(\mathcal{L} \setminus O)$ that also gives the closure $\overline{\phi^{-1}(\mathcal{L} \setminus O)}$ in $Bl_O(\mathbb{A}^n_{\mathbb{C}})$. Namely, $\phi^{-1}(\mathcal{L}) = \overline{\phi^{-1}(\mathcal{L} \setminus O)} = (a_1 t, a_2 t) \times (a_1, a_2)$. The result is that $Bl_O(\mathbb{A}^n_{\mathbb{C}})$ separates all the lines at the origin of $\mathbb{A}^n_{\mathbb{C}}$ along the exceptional divisor, Except(ϕ)$= \mathbb{P}^1_{\mathbb{C}}$, and we obtain a SNC–divisor of our original three lines. This is all shown in Figure 1.2.

Example 1.4.5. Example 1.4.2 (c) shows the cubic cuspidal curve $V(x_2^2 - x_1^3) \subset \mathbb{A}^2_{\mathbb{C}}$. Again we inspect this under the blow–up at the origin. More specifically, we consider the pre–image of the curve in the two standard affine open sets of $Bl_O(\mathbb{A}^2_{\mathbb{C}})$.

For $y_1 = 1$ we obtain $y_2^2 x_1^2 = x_2^2 = x_1^3$. This has solutions $x_1^2 = 0$ and $y_2^2 = x_1$. Note that the point $x_1 = y_2 = 0$ contains a singularity for the solution where the double line and parabola meet. The solution fails to be a SNC–divisor at this singularity.

For $y_2 = 1$ one obtains the equalities $x_2^2 = x_1^3 = y_1^3 x_2^3$, with the solutions $x_2^2 = 0$ or $y_1^3 x_2 = 1$. There are no singularities to worry about here.

So after one blow–up we see that we are left with one singularity of a different type than the initial singularity. One aspires to blow up one more time in the neighbourhood $y_1 = 1$. In the interest of readability, we perform the change of variables $z_1 = x_1$, $z_2 = y_2$ and treat z_1, z_2 as affine coordinates in an affine plane $\mathbb{A}^2_{\mathbb{C}}$. We once more apply Equation (1.2) to this affine plane, and let w_1, w_2 denote the new homogeneous coordinates for this blow–up. Again we inspect the standard affine neighbourhoods. One easily checks that the pre–image of $V(z_1^2, z_1 - z_2^2)$ behaves like a SNC–divisor in the open set $w_1 = 1$. For $w_2 = 1$, one obtains $z_2^2 w_1^2 = z_1^2 = 0$, which has both the z_2–and w_1–axis as solutions, and $z_2 w_1 = z_1 = z_2^2$, which further gives as solution the line

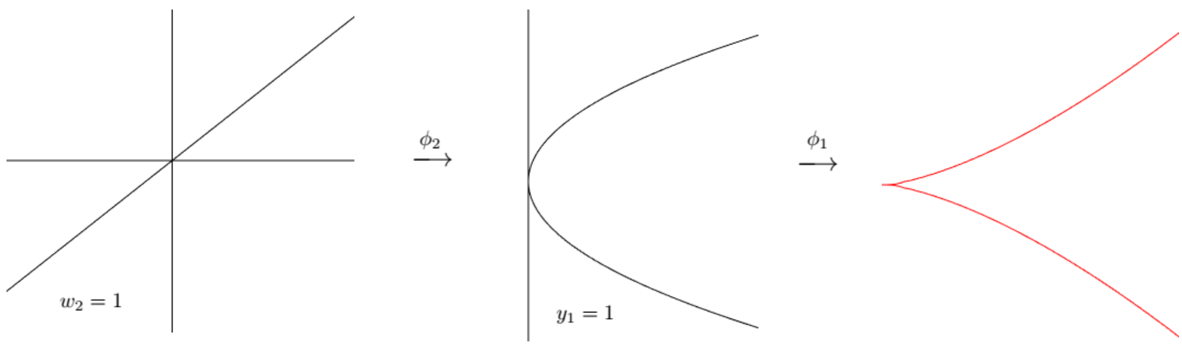


Figure 1.3: Blowing up a cubic cuspidal curve along the origin.

$w_1 = z_2$. In total we are left with three lines passing through the origin. But this is the case of Example 1.4.4, and we saw there that blowing up one more time will yield a SNC–divisor. This is illustrated in Figure 1.3.

Recall that if $f : \widetilde{X} \longrightarrow X$ is a morpism of varieties, and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals, then $f^{-1}\mathcal{I}$ is a sheaf of ideals in $f^{-1}\mathcal{O}_X$. Furthermore the associated sheaf morphism $f^{\#}$ extends to a natural map $f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_{\widetilde{X}}$. The *inverse image ideal sheaf*, $f^{-1}\mathcal{I}\cdot\mathcal{O}_{\widetilde{X}}$, is then defined to be the ideal sheaf generated by the image of $f^{-1}\mathcal{I}$ under this natural map. We are now in a position to state the following important Theorem on the resolution of singularities. It is due to Heisuke Hironaka.

Theorem 1.4.6. *Consider an irreducible complex algebraic variety X, and a non–zero coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. Then there is a projective birational morphism*

$$\mu : \widetilde{X} \longrightarrow X$$

such that $\widetilde{X}$ is smooth and Except*(μ) (i.e. the set of points where μ fails to be biregular) is a divisor. The inverse image ideal sheaf can be written $\mu^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$ where F is an effective divisor such that $F + Except(\mu)$ has simple normal crossing support.*

Furthermore this μ can be obtained as the composition of a finite amount of blow–up maps.

Proof. This is Main Theorem II of [Hir64]. □

The map μ from the theorem will be called a *log–resolution* of $\mathcal{I}$. Similarly, there exist log–resolutions for $\mathbb{Q}$–divisors and linear series, we will provide the definitions of them here.

Definition 1.4.7 (Log–resolutions for $\mathbb{Q}$–divisors and linear series)**.** a) Let D be a $\mathbb{Q}$–divisor on X. A *log–resolution* of D is a projective birational map $\mu : \widetilde{X} \longrightarrow X$, where $\widetilde{X}$ is a smooth variety, such that the divisor $\mu^*D + \text{Except}(\mu)$ has SNC support.

b) Consider an integral divisor L on X, and suppose $W \subseteq H^0(X, \mathcal{O}_X(L))$ is a non–zero finite–dimensional space of sections. A *log–resolution* of the linear series $|W|$ is then defined to be a projective birational map $\mu : \widetilde{X} \longrightarrow X$, again with $\widetilde{X}$ smooth, such that

$$\mu^*|W| = |K| + E$$

where $E + \text{Except}(\mu)$ is a divisor with SNC support. Furthermore we require

$$K \subseteq H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(\mu^*L - F))$$

to define a free linear series.

If K_X denotes the canonical divisor of X, the following relation holds for the canonical bundles of a log–resolution.

Proposition 1.4.8. *Consider a log–resolution $\mu : \widetilde{X} \longrightarrow X$ for some sheaf of ideals $\mathcal{I}$ on a smooth variety X. Then the following equality holds*

$$\mu_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \mathcal{O}_X(K_X)$$

Proof. [Laz04a] Corollary 4.1.4. □

The *relative canonical divisor* of $\widetilde{X}$ over X is defined as

$$K_{\widetilde{X}/X} := K_{\widetilde{X}} - \mu^*K_X$$

From the definition of canonical divisors this will naturally be an effective divisor, and it is supported on the exceptional locus of μ, as the blow–up map μ is an isomorphism outside of this. From the Projection Formula, Proposition 1.3.9, and the last Proposition we immediately check that

$$\mu_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X}) = \mu_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \otimes \mathcal{O}_X(-K_X) = \mathcal{O}_X \tag{1.3}$$

Definition 1.4.9 (Multiplier ideal sheaves)**.** For this definition we consider a fixed log–resolution $\mu : \widetilde{X} \longrightarrow X$ for either a non–zero ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, an effective $\mathbb{Q}$–divisor D, or a non–empty linear series $|W| \subseteq |L|$, respectively. $c > 0$ will denote any positive, rational number.

a) Let F be an effective integral divisor such that $\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-F)$. The *multiplier ideal sheaf* associated to $\mathcal{I}$ and c is defined as

$$\mathcal{J}(c \cdot \mathcal{I}) = \mu_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor c \cdot F \rfloor)$$

b) The *multiplier ideal sheaf* of the effective $\mathbb{Q}$–divisor D is

$$\mathcal{J}(D) = \mu_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor \mu^* D \rfloor)$$

c) Let $\mu|W| = |K| + E$ be the equation considered in Definition 1.4.7 for the log–resolution of $|W|$. Then the *multiplier ideal* of $|W|$ and c is

$$\mathcal{J}(c \cdot |W|) = \mu_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor c \cdot E \rfloor)$$

The definition just given would not make much sense unless the multiplier ideals are independent of the chosen log–resolution. This is exactly the case, as shown in [Laz04b] Theorem 9.2.18. A multiplier ideal is indeed, as its name suggests, a sheaf of ideals. One way to see this is to consider the short exact sequence obtained from an effective integral divisor, D, on $\widetilde{X}$:

$$0 \to \mathcal{O}_{\widetilde{X}}(-D) \to \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_D \to 0$$

and tensor with the line bundle $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X})$

$$0 \to \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - D) \to \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X}) \to \mathcal{O}_D \otimes \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X}) \to 0$$

lastly, applying the push-forward of μ yields the left–exact sequence

$$0 \to \mathcal{J}(D) \to \mathcal{O}_X$$

Through the latter injection, we may for any open set $\mathcal{U} \subseteq X$ realize $\mathcal{J}(D)(\mathcal{U})$ as a sub–$\mathcal{O}_X(\mathcal{U})$–module and hence an ideal of the ring $\mathcal{O}_X(\mathcal{U})$.

Example 1.4.10. Suppose D is a $\mathbb{Q}$–divisor having normal crossing support. Then we may choose $\mu = id_X$ which renders

$$\mathcal{J}(D) = \mathcal{O}_X(-\lfloor D \rfloor)$$

by the Projection Formula and Equation (1.3).

The next example is based on [Laz04b] Proposition 9.2.31.

Example 1.4.11. Consider a $\mathbb{Q}$–divisor of the form $D + A$ where D is a $\mathbb{Q}$–divisor and A an integral divisor. For any log–resolution $\mu : \widetilde{X} \longrightarrow X$ of D note that $\lfloor \mu^*(D + A) \rfloor = \lfloor \mu^*(D) \rfloor + \mu^*(A)$. From this, and the Projection Formula, we get the following isomorphisms

$$\mu_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor \mu^*(D + A) \rfloor) = \mu_*(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor \mu^* D \rfloor) \otimes \mathcal{O}_{\widetilde{X}}(\mu^*(-A)))$$

$$\simeq \mu_*(\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}/X} - \lfloor \mu^* D \rfloor)) \otimes \mathcal{O}_X(-A)$$
$$= \mathcal{J}(D) \otimes \mathcal{O}_X(-A)$$

And so $\mathcal{J}(D + A) = \mathcal{J}(D) \otimes \mathcal{O}_X(-A)$.

A particular case of this is when we are only considering an integral divisor A. In this case $\mathcal{J}(A) = \mathcal{O}_X(-A)$.

An important observation regarding multiplier ideals is that choosing a small enough rational number c will make the $\lfloor c \cdot F \rfloor$–part of $\mathcal{J}(c \cdot \mathcal{I})$ vanish. Equation (1.3) then assures us that $\mathcal{J}(c \cdot \mathcal{I})$ is trivial. Similar observations hold for $\mathbb{Q}$–divisors and linear series, which motivates the following definition.

Definition 1.4.12. For a non–zero sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ the *log–canonical threshold* of $\mathcal{I}$ is defined as

$$\operatorname{lct}(\mathcal{I}) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot \mathcal{I}) \neq \mathcal{O}_X\}$$

The *log–canonical threshold* of a $\mathbb{Q}$–divisor D is similarly defined, i.e.

$$\operatorname{lct}(D) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot D) \neq \mathcal{O}_X\}$$

And for a linear series $|W|$:

$$\operatorname{lct}(|W|) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot |W|) \neq \mathcal{O}_X\}$$

The log–canonical threshold is indeed a rational number, and the infima in the definition are actually a minima, which is shown in Example 9.3.16 of [Laz04b]. A $\mathbb{Q}$–divisor D furthermore said to be *log-canonical* if $\operatorname{lct}(D) \geq 1$. An equivalent way of phrasing this is that $\mathcal{J}(X, (1-\epsilon)D) = \mathcal{O}_X$ for any rational number $0 < \epsilon < 1$. Equality of the log–canonical threshold then holds in particular if D is a non–trivial, integral divisor as $\mathcal{J}(X, D) = \mathcal{O}_X(-D) \neq \mathcal{O}_X$ in this case by Example 1.4.11. For an effective $\mathbb{Q}$–divisor D on X, and an integer $k \geq 0$, one defines the *multiplicity locus* as:

$$\Sigma_k(D) = \{x \in X \mid \operatorname{mult}_x(D) \geq k\}.$$

The following result gives a condition on the multiplicity locus when D is log–canonical.

Proposition 1.4.13. *Let D be an effective $\mathbb{Q}$–divisor on X, and $k \geq 0$. If D is log–canonical, then every component of $\Sigma_k(D)$ has codimension $\geq k$ in X.*

Proof. [Laz04b] Example 9.3.10. □

Lastly, we state the following vanishing theorem for multiplier ideals, which will be crucial for our work in the very last chapter. We give the result adapted to ideal sheaves and linear series, as these will be the needed cases for us.

Theorem 1.4.14 (Nadel Vanishing). *Consider a smooth projective variety X. Let $c > 0$ be a rational number, while L and E are integral divisors on X such that $L - c \cdot E$ is big and nef.*

i) Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals such that $\mathcal{I} \otimes \mathcal{O}_X(E)$ is globally generated. Then there is the following vanishing of higher cohomologies

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(c \cdot \mathcal{I})) = 0 \quad \text{for} \quad i > 0.$$

ii) If $|W| \subseteq |E|$ is any linear series, then

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(c \cdot |W|)) = 0 \quad \text{for} \quad i > 0.$$

Proof. [Laz04b] Corollary 9.4.15. □

1.5 Casteluovo–Mumford Regularity and Log–Canonical Threshold

In this section we briefly present the Castelnuovo–Mumford regularity, and the main theorem of the associated theory. We then state a result due to Alex Küronya and Norbert Pintye that relates this regularity and the log–canonical threshold of an ideal sheaf. Although we will not be concerned with Castelnuovo–Mumford regularity in the later parts of this thesis, the section is meant to motivate and give some background for the later work. The Theta–regularity for abelian varieties that will be developed in section 2.4 is strikingly similar to that of the Castelnuovo–Mumford regularity. The similarity of these two theories will be crucial when we, motivated by Pintye and Küronya's ideas, will prove an inequality relating the Theta–regularity and log–canonical thresholds of ideal sheaves in Chapter 3. The main reference for this section is [Laz04a] and we will work with the projective space over $\mathbb{C}$, $\mathbb{P} = \mathbb{P}^n_{\mathbb{C}}$ for some dimension n.

Definition 1.5.1. A coherent sheaf $\mathcal{F}$ on a projective space $\mathbb{P}$ is called *Castelnuovo–Mumford m–regular* if

$$H^i(\mathbb{P}, \mathcal{F}(m-i)) = 0 \text{ for all } i > 0$$

Recall th Cartan–Serre–Grothendieck Theorem 1.2.5, which states that if $\mathcal{L}$ is ample then for any coherent sheaf $\mathcal{F}$ there is an integer n_0 such that for any $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated and the higher cohomology groups vanish. The following theorem suggests that Castelnuovo–Mumford regularity gives a quantitative measure for when these effects take place.

Theorem 1.5.2 (Mumford's Theorem)**.** *Let $\mathcal{F}$ be a (Castelnuovo–Mumford) m–regular sheaf on $\mathbb{P}$. Then the following properties hold for any $k \geq 0$:*
1) $\mathcal{F}(m+k)$ is globally generated.
2) $\mathcal{F}$ is $(m+k)$–regular.
3) The multiplication maps

$$H^0(\mathbb{P}, \mathcal{F}(m)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^0(\mathbb{P}, \mathcal{F}(m+k))$$

are surjective.

Proof. [Laz04a] Theorem 1.8.3. □

From 2) in the Theorem one sees that further twisting an m–regular sheaf will result in a sheaf still satisfying the regularity condition. Hence for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ we define the *Casteluovo–Mumford regularity*, $\mathrm{reg}(\mathcal{F})$, to be the lowest integer m for which $\mathcal{F}$ is m–regular. This may take the value of $-\infty$ if $\mathcal{F}$ is m–regular for all $m < 0$, as is the case if $\mathcal{F}$ is only supported on a finite set.

Theorem 1.5.3 (Castelnuovo–Mumford Regularity and Log–Canonical Threshold)**.** *If $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}}$ is a non–zero coherent sheaf of ideals on $\mathbb{P}$, then the following inequality holds.*

$$1 \leq \mathrm{lct}(\mathcal{I})\mathrm{reg}(\mathcal{I})$$

Proof. [KP13] Theorem 5. □

Chapter 2

Regularity on Abelian Varieties

We now turn our attention to abelian varieties. Motivated by the Castelnuovo–Mumford regularity for projective spaces, as presented in section 1.5, the goal will be to develop a similar regularity type here. We start in section 2.1 by introducing abelian varieties, along with presenting some fundamental properties such as the dual abelian variety. Section 2.2 is devoted to the Fourier–Mukai transform, and we shall see that it constitutes an equivalence of derived categories. This equivalence forms the basis for Mukai–regularity, and in section 2.3 we present the geometric consequences of this regularity condition. The table is then set for presenting the theta–regularity in section 2.4.

2.1 Abelian Varieties

This section is devoted to the definition and properties of abelian varieties that will be useful to us for the rest of the text. Intuitively speaking, these are varieties satisfying certain nice properties, including exhibiting a group structure. We shall present important isomorphisms of line bundles, such as the *See–Saw Principle* and the *Theorem of the Square*, as well as the *dual abelian variety*. The section ends with defining principal polarizations and theta divisors, which are the needed building blocks for the theta–regularity that will be presented in section 2.3. The main references will be David Mumford's treatment of the subject, [Mum70], as well as James S. Milne's notes [Mil08].

Definition 2.1.1. An *abelian variety* A over $\mathbb{C}$ is a complete algebraic variety over $\mathbb{C}$

along with a group structure. More specifically, there is a regular map:

$$m : A \times_k A \longrightarrow A$$

which is associative, i.e. $m(-, m(-,-)) = m(m(-,-),-)$. There is also an identity element, $0_A \in A$, which is a point.

$$0_A : \mathrm{Spec} k \longrightarrow A$$

satisfying $m(-, 0_A) = m(0_A, -) = id_A$ Lastly, there will also be an inverse map

$$(-1)_A : A \longrightarrow A$$

having the property that $m(a, (-1)_A(a)) = 0_A$. We will in the future write m additively, and omit the A–subscript when no confusion can arise, i.e. for points $a, b \in A$ we write $m(a, b) = a + b$ and $(-1)_A(a) = -a$.

The group operation makes it possible to define, for any closed point $a \in A$, the *translation by* a to be the isomorphism

$$\begin{aligned} t_a : &A \to A \\ &x \mapsto x + a \end{aligned}$$

where the inverse morphism is t_{-a}. The map consisting of multiplication by a fixed integer n will be given as

$$\begin{aligned} n_A : &A \to A \\ &a \mapsto \underbrace{a + a + ... + a}_{\text{n times}} \end{aligned}$$

The following proposition summarizes some fundamental properties of abelian varieties.

Proposition 2.1.2. *For any abelian variety A the following holds:*

i) The group operation on A is commutative.

ii) A is projective.

iii) The canonical sheaf of A is trivial, i.e. $\omega_A \simeq \mathcal{O}_A$.

iv) The map n_A *is surjective for any non–zero integer* n.

Proof. i) and ii) can be found in Corollary I 1.4 and Theorem I 6.4, respectively, of [Mil08].

iii) At [Mum70] page 4 it is noted that Ω^P_A is a globally free sheaf of $\mathcal{O}_A$–modules. From this it follows that $\omega_A = \Omega^g_A \simeq \mathcal{O}_A$.

iv) is given in [Mum70] II.4.Question 4 (iv). □

Perhaps the most famous class of examples of abelian varieties are those of dimension 1, namely the *elliptic curves.* These are by definition the non–singular projective curves of genus 1. The group operation for elliptic curves has a particularly nice geometric description, that will be presented in the following example.

Example 2.1.3 (Group law of an elliptic curve)**.** Let E be an elliptic curve and consider a distinguished point $P_0 \in E$. We let $[P]$ denote the divisor associated to the point P. The first step to analyze the group structure will be to embed E in $\mathbb{P}^2$ by the linear system $|3P_0|$. We have inclusions of vector spaces:

$$k \simeq H^0(\mathcal{O}_E) \subseteq H^0(\mathcal{O}_E([P_0])) \subseteq H^0(\mathcal{O}_E([2P_0])) \subseteq \cdots$$

and the Riemann-Roch Theorem ([Har77] IV Theorem 1.3) for genus 1 curves says that

$$h^0(\mathcal{O}_E([mP_0])) = deg([mP_0]) = m$$

Therefore we may choose an element x so that 1 and x form a basis for the 2-dimensional vector space $H^0(\mathcal{O}_E([2P_0]))$. Extend this to a basis 1, x, y of $H^0(\mathcal{O}_E([3P_0]))$. Thus the seven elements 1, x, y, xy, x^2, x^3 and y^2 are in the 6–dimensional vector space $H^0(\mathcal{O}_E([6P_0]))$, so there is a linear relation among them. As x and y have poles at P_0 of order 2 and 3 respectively, then x^3 and y^2 are the functions with pole at P_0 of order 6, and will have non–zero coefficients. Replacing x and y by suitable scalar multiples so that these coefficients are 1, we obtain the following relation

$$y^2 + a_1xy + a_2y = x^3 + a_3x^2 + a_4x + a_5$$

for suitable $a_i \in k$.

To aesthetically enhance this equation somewhat, we start by completing the square on the left hand side with the new variable $Y = y + \frac{1}{2}(a_1x + a_2)$. We obtain

$$Y^2 = x^3 + x^2(a_3 + \frac{a_1^2}{4}) + x(a_4 + \frac{a_1a_2}{2}) + c$$

for some constant $c \in k$. Introducing yet another variable $X = x + \frac{1}{3}(a_3 + \frac{1}{4}a_1^2)$ we end up with an equation

$$Y^2 = X^3 + aX + b \tag{2.1}$$

for suitable $a, b \in k$.

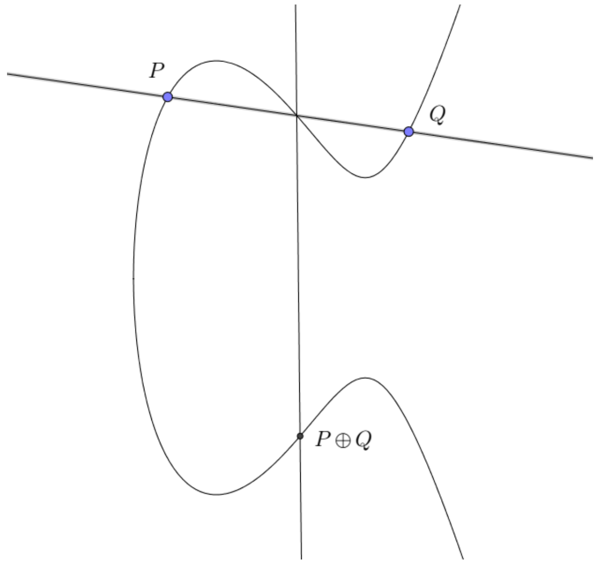


Figure 2.1: Group operation on an elliptic curve.

The functions X and Y are used to define a rational map

$$\begin{aligned} E &\longrightarrow \mathbb{P}^2 \\ e &\mapsto [X(e) : Y(e) : 1], \quad e \neq P_0 \\ P_0 &\mapsto [0 : 1 : 0] \end{aligned}$$

This gives an embedding of E into the projective plane whose image in the affine neighbourhood $Z = 1$ is the curve given by Equation (2.1), along with the *point at infinity* $[0 : 1 : 0]$. With the embedding done, we turn our attention towards defining the group structure on E. In the interest of keeping confusion at a minimum, we let $+, -$ denote the addition of divisors, while $\oplus$ denotes the group operation for the rest of this example. Consider the map

$$\begin{aligned} E &\longrightarrow \mathrm{Pic}^0(E) \\ P &\mapsto [P - P_0] \end{aligned}$$

where $\mathrm{Pic}^0(E)$ is the subgroup of $\mathrm{Pic}(E)$ consisting of (integral) divisors of degree 0. Example IV.1.3.7 in [Har77] shows that this is a bijection, so we define the group operation of E by the operation of the group $\mathrm{Pic}^0(E)$. In particular this makes P_0 the 0 element, and $P \oplus Q = T$ if and only if $[P] + [Q] \equiv_{lin} [T] + [P_0]$. The elliptic curve has been embedded in $\mathbb{P}^2$, so by Bézout's Theorem ([Har77] Corollary I.7.8) all lines will intersect E in three (not necessarily distinct) points. This means that the line $Z = 0$

intersects E in $[3P_0]$. Any two lines in the projective plane are linearly equivalent, so if P, Q and R are collinear points, we have $[P]+[Q]+[R] \equiv_{lin} [3P_0]$ and so trivial under the group law. If there is another line intersecting E in P_0, R and T, we would have $[P]+[Q]+[R] \equiv_{lin} [R]+[T]+[P_0]$, so in particular $P \oplus Q = T$ in the group operation (as exemplified in Figure 2.1).

In the above example we defined $\mathrm{Pic}^0(E)$, for elliptic curves E, as the group of divisors of degree 0. In order to generalize the notion of $\mathrm{Pic}^0(A)$ for higher dimensional abelian varieties, it is first useful to introduce the Theorem of the Square. We shall later see (in Example 2.1.12) why the two definitions are equal for curves.

Theorem 2.1.4 (Theorem of the Square)**.** *Let A be an abelian variety. For a line bundle $\mathcal{L}$ and closed points $a, b \in A$ there is an isomorphism*

$$t_{a+b}^*\mathcal{L} \otimes \mathcal{L} \simeq t_a^*\mathcal{L} \otimes t_b^*\mathcal{L}.$$

Proof. [Mum70] II.6.Corollary 4. □

For any line bundle $\mathcal{L}$ we define the map

$$\lambda_{\mathcal{L}} : A \longrightarrow \mathrm{Pic}(A)$$
$$a \mapsto t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

The isomorphism in the Theorem of the Square (twisted by $\mathcal{L}^{-2}$) shows that $\lambda_{\mathcal{L}}$ is a group homomorphism, if we view A as a group. We also denote its kernel $K(\mathcal{L}) = ker\lambda_{\mathcal{L}}$.

Definition 2.1.5. One defines $\mathrm{Pic}^0(A)$ to be the set of isomorphism classes of line bundles $[\alpha]$ on A where for a representative α of the isomorphism class, we have $\lambda_\alpha(a) \simeq \mathcal{O}_A$ for every point $a \in A$. We will in the future adopt the convention that writing $\alpha \in \mathrm{Pic}^0(A)$ means that the isomorphism class of α lies in $\mathrm{Pic}^0(A)$.

It is immediate that $K(\alpha) = A$, or $t_a^*\alpha \simeq \alpha$ for all $a \in A$, are equivalent conditions for α to be in $\mathrm{Pic}^0(A)$. From the basic identities $t_a^*(\mathcal{L}_1 \otimes \mathcal{L}_2) \simeq t_a^*\mathcal{L}_1 \otimes t_a^*\mathcal{L}_2$ and $t_a^*\mathcal{O}_A \simeq \mathcal{O}_A$, it is also easy to deduce that $\mathrm{Pic}^0(A)$ is a subgroup of Pic(A). The next proposition concerns the vanishing of cohomology for elements in $\mathrm{Pic}^0(A)$.

Proposition 2.1.6. *If α is a non-trivial element of $\mathrm{Pic}^0(A)$, then $H^i(A, \alpha) = 0$ for all integers i.*

Proof. [Mum70] II.8.vii). □

A line bundle $\mathcal{L}$ is said to be *non–degenerate* if $K(\mathcal{L})$ is finite. The next proposition tells how this relates to ample line bundles under certain conditions. The following theorem, sometimes known as *Mumford's Vanishing Theorem*, says that most of the cohomologies of non–degenerate line bundles vanish.

Proposition 2.1.7. *Let $\mathcal{L}$ be an effective line bundle on A, i.e. $\mathcal{L} \simeq \mathcal{O}_A(D)$ for an effective divisor D. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}$ is non–degenerate.*

Proof. [Mum70] II.6.Application 1. □

Theorem 2.1.8. *Let $\mathcal{L}$ be a non–degenerate line bundle on an abelian variety A of dimension g. Then there exists a unique integer $0 \leq i(\mathcal{L}) \leq g$ so that $H^k(A, \mathcal{L}) = 0$ for any $k \neq i(\mathcal{L})$, and $H^{i(\mathcal{L})}(A, \mathcal{L}) \neq 0$. Furthermore, for any positive integer m, $i(\mathcal{L}) = i(\mathcal{L}^m)$.*

Proof. This is "The Vanishing Theorem" in [Mum70] III.16, along with the corollary in the same section. □

Example 2.1.9. If $\mathcal{L}$ is ample, then $H^i(A, \mathcal{L})$ is non–zero only for $i = 0$. This is a direct consequence of the Kodaira Vanishing Theorem 1.1.5, along with the fact that abelian varieties have trivial canonical bundle.

Lemma 2.1.10. *If $\mathcal{L}$ is a line bundle, then $t_a^*\mathcal{L} \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(A)$ for any $a \in A$. In particular, $Im(\lambda_{\mathcal{L}}) \subset \mathrm{Pic}^0(A)$.*

Proof. If a and b are any points of A, we apply the Theorem of the Square to get the isomorphism:

$$\begin{aligned} t_b^*(t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}) &\simeq t_{b+a}^*\mathcal{L} \otimes t_b^*\mathcal{L}^{-1} \\ &\simeq t_b^*\mathcal{L} \otimes t_a^*\mathcal{L} \otimes \mathcal{L}^{-1} \otimes t_b^*\mathcal{L}^{-1} \\ &\simeq t_a^*\mathcal{L} \otimes \mathcal{L}^{-1} \end{aligned}$$

□

Due to the lemma we will from now on treat $\lambda_{\mathcal{L}}$ as a homomorphism $A \longrightarrow \mathrm{Pic}^0(A)$. The next goal will be to give the group $\mathrm{Pic}^0(A)$ the structure of an abelian variety, and the following theorem is key to achieve this.

Theorem 2.1.11. *For an ample line bundle $\mathcal{L}$ and any element $\alpha \in \operatorname{Pic}^0(A)$, there exists a point $a \in A$ such that*

$$\alpha \simeq t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

hence the map $\lambda_{\mathcal{L}} : A \longrightarrow \operatorname{Pic}^0(A)$ is surjective.

Proof. [Mum70] II.8.Theorem 1. □

Example 2.1.12. Back in Example 2.1.3 we adopted the ad hoc definition of $\operatorname{Pic}^0(E)$, for an elliptic curve E, to be all divisors of degree 0. We now give a justification of this. Recall the setting of the aforementioned example, E an elliptic curve with distinguished point P_0. If P is any other point on E, then $[P]$ will be a divisor of degree 1, and hence ample ([Har77] IV Corollary 3.3). Now for any $e \in E$, since t_e^* is pulling back by the translation of e, and $t_e^{-1} = t_{-e}$, we have

$$t_e^*\mathcal{O}_E([P]) \simeq \mathcal{O}_E(t_{-e}([P])) = \mathcal{O}_E([P-e])$$

On the other hand note that P, $-P$ and P_0 is collinear, and so are $P-e$, $-P$ and e, hence they are linear equivalent as noted in Example 2.1.3. Upon rearrangement one has

$$[P_0] - [e] \equiv_{lin} [P-e] - [P].$$

Added together, this gives

$$\begin{aligned}\lambda_{\mathcal{O}_E([P])}(e) &= t_e^*\mathcal{O}_E([P]) \otimes \mathcal{O}_E([P])^{-1} \\ &\simeq \mathcal{O}_E([P-e]) \otimes \mathcal{O}_E([-P]) \\ &\simeq \mathcal{O}_E([P_0] - [e])\end{aligned}$$

As P is ample, the surjectivity from Theorem 2.1.11 asserts that $\operatorname{Pic}^0(E)$ is fully described by degree 0 divisors. To see that it includes all of them, assume $D = \sum n_i[D_i]$ is any divisor of deg 0, with $D_i \in E$. Then $\sum n_i = 0$ and so

$$\begin{aligned}&\mathcal{O}_E(\sum n_i[D_i]) = \mathcal{O}_E(\sum n_i[D_i] - \sum n_i[P_0]) \\ &\simeq \mathcal{O}_E(n_1[D_1] - n_1[P_0]) \otimes \mathcal{O}_E(n_2[D_2] - n_2[P_0]) \otimes \cdots \otimes \mathcal{O}_E(n_k[D_k] - n_k[P_0])\end{aligned}$$

Here $\mathcal{O}_E(n_i[D_i] - n_i[P_0])$ is the n_i–fold tensor product of $\mathcal{O}_E([D_i] - [P_0]) \in \operatorname{Pic}^0(E)$. Since the group $\operatorname{Pic}^0(E)$ is closed under tensor products, we have shown that consists exactly of the divisors of degree 0.

This might be a good place to include the See–Saw Principle. It states that isomorphisms of line bundles on a product of two abelian varieties can be inspected by checking if the isomorphism holds when restricting to points on one of the varieties, and doing the same for a single point on the other variety. The result is an extremely useful practical tool that we will employ several times later in this text. We first fix some notation; let $\mathcal{F}$ be a sheaf on the product $X \times Y$ and $i : \{x\} \times Y \longrightarrow X \times Y$ the natural inclusion for a point $x \in X$. Then $\mathcal{F}|_{\{x\}\times Y} = i^*\mathcal{F}$, a sheaf on $\{x\} \times Y \simeq Y$. When no confusion can arise, we will also denote this $\mathcal{F}_x$.

Theorem 2.1.13 (See–Saw Principle)**.** *Let $A \times B$ be a product of abelian varieties and $\mathcal{L}$, $\mathcal{M}$ line bundles on this product. If $\mathcal{L}_a \simeq \mathcal{M}_a$ for all points $a \in A$ and furthermore $\mathcal{L}_b \simeq \mathcal{M}_b$ for one $b \in B$ then $\mathcal{L} \simeq \mathcal{M}$.*

Proof. [Mil08] Corollary 5.18. □

We follow up with some useful observations concerning elements in $\mathrm{Pic}^0(A)$ that is easily checked by the See–Saw Principle.

Corollary 2.1.14. *Consider the product $A \times A$ with the usual projections p_1, p_2. Then the following condition holds:*

$$\alpha \in \mathrm{Pic}^0(A) \text{ if and only if } m^*\alpha \simeq p_1^*\alpha \otimes p_2^*\alpha.$$

Proof. When restricted to $\{0\} \times A$, both m and p_2 are the identity morphism, while p_1 will be the constant map to 0. Hence $(m^*\alpha \otimes p_1^*\alpha^{-1} \otimes p_2^*\alpha^{-1})|_{\{0\}\times A}$ will be trivial. On the other hand, restricting to $A \times \{a\}$ will make p_2 a constant map, p_1 the identity map, and $m = t_a$, the translation by a. Thus the isomorphism

$$(m^*\alpha \otimes p_1^*\alpha^{-1} \otimes p_2^*\alpha^{-1})|_{A\times\{a\}} \simeq t_a^*\alpha \otimes \alpha^{-1}$$

where $t_a^*\alpha \otimes \alpha^{-1}$ is trivial if and only if $\alpha \in \mathrm{Pic}^0(A)$. In this case the See–Saw Principle guarantees the asserted isomorphism $m^*\alpha \simeq p_1^*\alpha \otimes p_2^*\alpha$. □

Corollary 2.1.15. *Let $\alpha \in \mathrm{Pic}^0(A)$ and consider a scheme X with morphisms $f, g : X \longrightarrow A$. Then*

$$(f + g)^*\alpha \simeq f^*\alpha \otimes g^*\alpha.$$

Proof. Consider the isomorphism in the previous corollary, and pull back both sides using $(f, g) : X \longrightarrow A \times A$. On one side we get $(f, g)^*m^*\alpha = (f + g)^*\alpha$. On the other side we have $(f, g)^*(p_1^*\alpha \otimes p_2^*\alpha) = f^*\alpha \otimes g^*\alpha$, as claimed. □

Corollary 2.1.16. *For $\alpha \in \mathrm{Pic}^0(A)$ and any integer n, $n_A^*\alpha \simeq \alpha^n$.*

Proof. We start by showing $(-1)_A^*\alpha \simeq \alpha^{-1}$. Consider the diagram for points $a, x \in A$:

$$\begin{array}{ccc} A & \xrightarrow{-1} & A \\ a & & -a \\ \downarrow{\scriptstyle t_{-x}} & & \downarrow{\scriptstyle t_x} \\ A & \xrightarrow{-1} & A \\ a - x & & -a + x \end{array}$$

which shows $t_x \circ (-1) = (-1) \circ t_{-x}$. Let $\mathcal{L}$ be an ample line bundle on A, and define $\mathcal{M} := \mathcal{L} \otimes (-1)^*\mathcal{L}$. Then $\mathcal{M}$ is ample and symmetric, i.e. $(-1)^*\mathcal{M} \simeq \mathcal{M}$. This in particular also means that $(-1)^*\mathcal{M}^{-1} \simeq \mathcal{M}^{-1}$ since:

$$(-1)^*\mathcal{M}^{-1} \otimes \mathcal{M} \simeq (-1)^*(\mathcal{M}^{-1} \otimes \mathcal{M}) \simeq \mathcal{O}_A.$$

By Theorem 2.1.11 there is a point $x \in A$ such that $\alpha \simeq t_x^*\mathcal{M} \otimes \mathcal{M}^{-1} = \lambda_{\mathcal{M}}(x)$. Combining all this we get the isomorphisms;

$$\begin{aligned} (-1)^*\alpha &\simeq (-1)^*(t_x^*\mathcal{M} \otimes \mathcal{M}^{-1}) \\ &\simeq (-1)^*t_x^*\mathcal{M} \otimes (-1)^*\mathcal{M}^{-1} \\ &\simeq t_{-x}^*(-1)^*\mathcal{M} \otimes \mathcal{M}^{-1} \\ &\simeq \lambda_{\mathcal{M}}(-x). \end{aligned}$$

Then, since $\lambda_{\mathcal{M}}$ is a group homomorphsim, we get

$$\alpha^{-1} \simeq \lambda_{\mathcal{M}}(x)^{-1} \simeq \lambda_{\mathcal{M}}(-x) \simeq (-1)^*\alpha.$$

The case for general n now follows by induction using the relation in Corollary 2.1.15 for the maps $(1)_A$ and $(-1)_A$. □

An immediate consequence of Theorem 2.1.11 is that if $\mathcal{L}$ is ample then it induces the isomorphism of groups $A/K(\mathcal{L}) \simeq \mathrm{Pic}^0(A)$. The quotient $A/K(\mathcal{L})$ can be given the structure of an abelian variety (see [Mum70] II.7.Theorem 4), which we will call the *dual abelian variety*, $A^\vee$. The dual abelian variety satisfies the following universal property.

Theorem 2.1.17. *An abelian variety A determines a pair $(A^\vee,\mathcal{P})$ where $A^\vee$ is the dual abelian variety of A and $\mathcal{P}$ is a line bundle on $A\times A^\vee$ named the* Poincaré sheaf, *such that*

a) $\mathcal{P}|_{A\times\{b\}}\in \mathrm{Pic}^0(A)$ *for all* $b\in A^\vee$.

b) $\mathcal{P}|_{\{0\}\times A^\vee}$ *is trivial.*

Furthermore, if $(T,\mathcal{L})$, where T is a variety over $\mathbb{C}$ and $\mathcal{L}$ a line bundle on $A\times T$, is another pair satisfying conditions a) and b), then there exists a unique regular map $\gamma : T\longrightarrow A^\vee$ such that $(1\times\gamma)^*\mathcal{P}\simeq\mathcal{L}$.

Proof. [Mil08] I.8. □

Remark 2.1.18. Consider any variety T and a morphism $\gamma : T\longrightarrow A^\vee$. The commutativity of the square

$$\begin{array}{ccc} A\times\{t\} & \xrightarrow{\;i_t\;} & A\times T \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1\times\gamma} \\ A\times\{\gamma(t)\} & \xrightarrow{\;i_{\gamma(t)}\;} & A\times A^\vee \end{array}$$

and the fact that $A\times\{t\}\simeq A\simeq A\times\{\gamma(t)\}$, implies the isomorphism

$$((1\times\gamma)^*\mathcal{P})|_{A\times\{t\}} = i_t^*(1\times\gamma)^*\mathcal{P}\simeq i_{\gamma(t)}^*\mathcal{P}=\mathcal{P}|_{A\times\{\gamma(t)\}}$$

which is in $\mathrm{Pic}^0(A)$ by property a) of the Poincaré bundle. Similarly, the square

$$\begin{array}{ccc} \{0\}\times T & \xrightarrow{\;j_T\;} & A\times T \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle 1\times\gamma} \\ \{0\}\times A^\vee & \xrightarrow{\;j_{A^\vee}\;} & A\times A^\vee \end{array}$$

gives the isomorphisms

$$((1\times\gamma)^*\mathcal{P})|_{\{0\}\times T} = (j_T)^*(1\times\gamma)^*\mathcal{P}\simeq \gamma^*(j_{A^\vee})^*\mathcal{P}=\gamma^*(\mathcal{P}|_{\{0\}\times A^\vee})$$

which is trivial by property b). To conclude, we have shown that there is a map

$$Hom(T,A^\vee)\longrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of line bundles on} \\ A\times T \text{ satisfying a) and b)} \end{array} \right\}$$
$$\gamma\longmapsto (1\times\gamma)^*\mathcal{P}$$

which is moreover a one–to–one correspondence due to the universal property in Theorem 2.1.17. In particular, setting T to be a one–point scheme, $\{*\}$, we get

$$Hom(\{*\}, A^\vee) \simeq A^\vee \simeq \mathrm{Pic}^0(A).$$

It follows that every element of $\mathrm{Pic}^0(A)$ is uniquely represented in the family

$$\{\mathcal{P}_{a'} \mid a' \in A^\vee\}.$$

Definition 2.1.19. A map $\lambda : A \longrightarrow A^\vee$ is called a *polarization* if there is an ample line bundle $\mathcal{L}$ such that $\lambda = \lambda_{\mathcal{L}}$. λ is furthermore said to be *principally polarized* if $h^0(A, \mathcal{L}) = 1$. An abelian variety admitting such a polarization is called a *principally polarized abelian variety*, and is usually given as (A, λ) for λ a fixed principal polarization.

Example 2.1.20. Let P be any point on an elliptic curve E. In Example 2.1.12 we saw that

$$\lambda_{\mathcal{O}_E([P])}(e) \simeq \mathcal{O}_E([P_0] - [e])$$

This map is clearly both injective and independent of the choice of point P. One concludes that any divisor of degree 1 determines the same principal polarization on an elliptic curve.

Note that while every abelian variety has a polarization (indeed, by virtue of being projective, we know that A has a very ample line bundle), not every abelian variety is principally polarized. In Example 2.1.20 we saw that the ample line bundle representing that principal polarization was not unique; in fact any translate of it would result in another representative for the polarization. This is true also in general, as shown in the following results.

Proposition 2.1.21. *Let $\mathcal{L}$ and $\mathcal{L}'$ be ample line bundles on A. Then the following are equivalent:*

i) $\lambda_{\mathcal{L}} = \lambda_{\mathcal{L}'}$

ii) $\mathcal{L}^{-1} \otimes \mathcal{L}' \in \mathrm{Pic}^0(A)$

If this holds then $\mathcal{L}$ and $\mathcal{L}'$ differ by a translation, i.e. $\mathcal{L}' \simeq t_{a_0}^*\mathcal{L}$ for some $a_0 \in A$.

Proof. The equality

$$\lambda_{\mathcal{L}'\otimes\mathcal{L}}(a) = t_a^*\mathcal{L}' \otimes t_a^*\mathcal{L}^{-1} \otimes (\mathcal{L}')^{-1} \otimes \mathcal{L}$$

$$= \lambda_{\mathcal{L}'}(a) \otimes (\lambda_{\mathcal{L}}(a))^{-1} = \mathcal{O}_A$$

holds true for any $a \in A$ if and only if $\mathcal{L}' \otimes \mathcal{L}^{-1} \in \mathrm{Pic}^0(A)$, by definition. For the second statement, Theorem 2.1.11 implies that there is an element $a_0 \in \mathrm{Pic}^0(A)$ so that $\mathcal{L}' \otimes \mathcal{L}^{-1} \simeq t_{a_0}^* \mathcal{L} \otimes \mathcal{L}^{-1}$, which immediately gives the desired isomorphism $\mathcal{L}' \simeq t_{a_0}^* \mathcal{L}$. □

Lemma 2.1.22. *Consider a line bundle $\mathcal{L}$ on A, along with any element $\alpha \in \mathrm{Pic}^0(A)$. The following statements are true.*

i) If $\mathcal{L}$ is ample then $\mathcal{L} \otimes \alpha$ is ample as well.

ii) $\chi(\mathcal{L}) = \chi(\mathcal{L} \otimes \alpha)$. If $\mathcal{L}$ is furthermore ample, then $h^0(A, \mathcal{L}) = h^0(A, \mathcal{L} \otimes \alpha) \neq 0$.

Proof. *i*) Since $\mathcal{L}$ is ample there is, by Theorem 2.1.11, a point $a \in A$ such that

$$\mathcal{L} \otimes \alpha \simeq \mathcal{L} \otimes t_a^* \mathcal{L} \otimes \mathcal{L}^{-1} \simeq t_a^* \mathcal{L}$$

This is the pull–back of an ample line bundle by an isomorphism, which is ample.

We want to prove the first statement of *ii*) by using the Semicontinuity Theorem 1.1.6 on the natural projection

$$X := A \times A^\vee \xrightarrow{\quad p_2 \quad} A^\vee$$

and line bundle $\mathcal{F} := p_1^* \mathcal{L} \otimes \mathcal{P}$ on X. For any point $b \in A^\vee$, we have $\mathcal{F}_b = (p_1^* \mathcal{L} \otimes \mathcal{P})|_{A \times \{b\}} = \mathcal{L} \otimes \mathcal{P}_b$ on $X_b \simeq A$. As $A^\vee$ is connected, and the family $\{\mathcal{P}_b \mid b \in A^\vee\}$ associates to the whole of $\mathrm{Pic}^0(A)$ as noted in Remark 2.1.18, $\chi(\mathcal{L} \otimes \alpha)$ is constant for any $\alpha \in \mathrm{Pic}^0(A)$. As this includes $\mathcal{O}_A$, we get $\chi(\mathcal{L}) = \chi(\mathcal{L} \otimes \alpha)$. The second part of *ii*) now immediately follows as

$$h^0(A, \mathcal{L}) = \chi(\mathcal{L}) = \chi(\mathcal{L} \otimes \alpha) = h^0(A, \mathcal{L} \otimes \alpha)$$

by Example 2.1.9. □

Although a principal polarization has many associated line bundles, there is one that is of particular interest to us. This line bundle is *symmetric*, meaning that $(-1)_A^* \mathcal{L} \simeq \mathcal{L}$. The existence of such a line bundle is handled in the following proposition.

Proposition 2.1.23. *Consider a principally polarized abelian variety (A, λ) where λ is a fixed principal polarization. Then there exists a symmetric line bundle $\mathcal{L}$ such that $\lambda_{\mathcal{L}} = \lambda$.*

Proof. Let $\mathcal{M}$ be any ample line bundle representing λ, and assume it is not symmetric. First note that the following square commutes for any point $a \in A$

$$\begin{array}{ccc} A & \xrightarrow{t_a} & A \\ x & & x+a \\ \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle -1} \\ A & \xrightarrow{t_{-a}} & A \\ -x & & -x-a \end{array}$$

so $(t_a)^*(-1)^* \simeq (-1)^*(t_{-a})^*$. This gives

$$\begin{aligned} \lambda_{(-1)^*\mathcal{M}}(a) &= t_a^*(-1)^*\mathcal{M} \otimes (-1)^*\mathcal{M} \\ &\simeq (-1)^*(t_{-a}^*\mathcal{M} \otimes \mathcal{M}) \\ &\simeq (-1)^*\lambda_{\mathcal{M}}(-a) \simeq (-1)^*\lambda_{\mathcal{M}}(a)^{-1} \\ &\simeq \lambda_{\mathcal{M}}(a) && \text{(Corollary 2.1.16)} \end{aligned}$$

where we have used $\lambda_{\mathcal{M}}(-a) \simeq \lambda_{\mathcal{M}}(a)^{-1}$ due to the fact that $\lambda_{\mathcal{M}}$ is a group homomorphism. Hence $\lambda_{(-1)^*\mathcal{M}} \simeq \lambda_{\mathcal{M}}$ and so we may write $(-1)^*\mathcal{M} \otimes \mathcal{M}^{-1} := \alpha_0$ for some α_0 in $\mathrm{Pic}^0(A)$ by Proposition 2.1.21. Now define $\mathcal{L} := \mathcal{M} \otimes \beta$, where $\beta \in \mathrm{Pic}^0(A)$ is chosen such that $\beta^2 = \alpha_0$ (which is possible due to Proposition 2.1.2). From construction, $\mathcal{L} \otimes \mathcal{M}^{-1}$ is in $\mathrm{Pic}^0(A)$, so $\mathcal{L}$ and $\mathcal{M}$ determine the same polarization. Furthermore, $\mathcal{L}$ is ample and $h^0(A, \mathcal{L}) = h^0(A, \mathcal{M}) = 1$ by Lemma 2.1.22. Lastly, we check that

$$\begin{aligned} (-1)^*\mathcal{L} \otimes \mathcal{L}^{-1} &= (-1)^*\mathcal{M} \otimes (-1)^*\beta \otimes \mathcal{M}^{-1} \otimes \beta^{-1} \\ &\simeq (-1)^*\mathcal{M} \otimes \mathcal{M}^{-1} \otimes \beta^{-2} \\ &= (-1)^*\mathcal{M} \otimes \mathcal{M}^{-1} \otimes \alpha_0^{-1} \\ &\simeq \mathcal{O}_A \end{aligned}$$

Which shows that $\mathcal{L}$ is indeed symmetric as claimed. □

Definition 2.1.24. Let λ be a principal polarization on A. Any divisor corresponding to a line bundle $\mathcal{M}$, where $\lambda_{\mathcal{M}} = \lambda$, is called a *theta divisor*. If $\mathcal{L}$ is the symmetric line bundle of λ then a divisor Θ such that $\mathcal{L} \simeq \mathcal{O}_A(\Theta)$ is called *the symmetric theta divisor*. In the future we will refer to this simply as *the* theta divisor.

2.2 Fourier–Mukai Transforms

Last section introduced the basic theory of abelian varieties and section 1.3 introduced derived categories. In this section we will follow the first part of a paper by Shigeru Mukai, [Muk81], that combines both these topics. To be more specific, we aim to introduce the Fourier–Mukai transform, a functor between the bounded derived categories of two schemes. The main result, Theorem 2.2.5, states that if we focus on the product consisting of an abelian variety and its dual, $A \times A^\vee$, this functor becomes an equivalence. This fact will be fundamental when developing the theory of Mukai–regularity in the next chapter.

Definition 2.2.1. Let $X \times Y$ be a product of schemes with the usual projections p_1 and p_2.

$$X \xleftarrow{\quad p_1 \quad} \begin{matrix} X \times Y \\ \mathcal{F} \end{matrix} \xrightarrow{\quad p_2 \quad} Y$$

For any $\mathcal{F} \in D^b(X \times Y)$, the *Fourier–Mukai transform of* $\mathcal{F}$ is the functor:

$$\begin{aligned} R\mathcal{S}_{X\to Y,\mathcal{F}} : D^b(X) &\longrightarrow D^b(Y) \\ \mathcal{G} &\mapsto Rp_{2*}(p_1^*(\mathcal{G}) \otimes_L \mathcal{F}) \end{aligned}$$

A functor in the opposite direction is similarly defined by swapping the projection maps:

$$\begin{aligned} R\mathcal{S}_{Y\to X,\mathcal{F}} : D^b(Y) &\longrightarrow D^b(X) \\ \mathcal{H} &\mapsto Rp_{1*}(p_2^*(\mathcal{H}) \otimes_L \mathcal{F}) \end{aligned}$$

Remark 2.2.2. i) By a slight abuse of notation, we will write $R\mathcal{S}_{X\to Y,F}(-) = p_{2*}(p_1^*(-) \otimes \mathcal{F})$, instead of the derived versions of the functors. Note that since projections are flat, p_1^* really does denote the usual pull–back.

ii) The Fourier–Mukai transform is exact as a functor of triangulated categories, and hence commutes with the shift functor.

For the rest of this section we will frequently apply the Projection Formula and Flat Base Change. They were stated in Propositions 1.3.9 and 1.3.10, respectively.

Example 2.2.3. Let $f : X \to Y$ be a morphism of schemes. We define its graph, Γ_f, to be the morphism $(id_X \times f) : X \to X \times Y$, in particular having the property $p_1 \circ \Gamma_f = id_X$ and $p_2 \circ \Gamma_f = f$. Γ_f is a closed immersion with structure sheaf $\mathcal{O}_{\Gamma_f} = \Gamma_{f*}\mathcal{O}_X$.

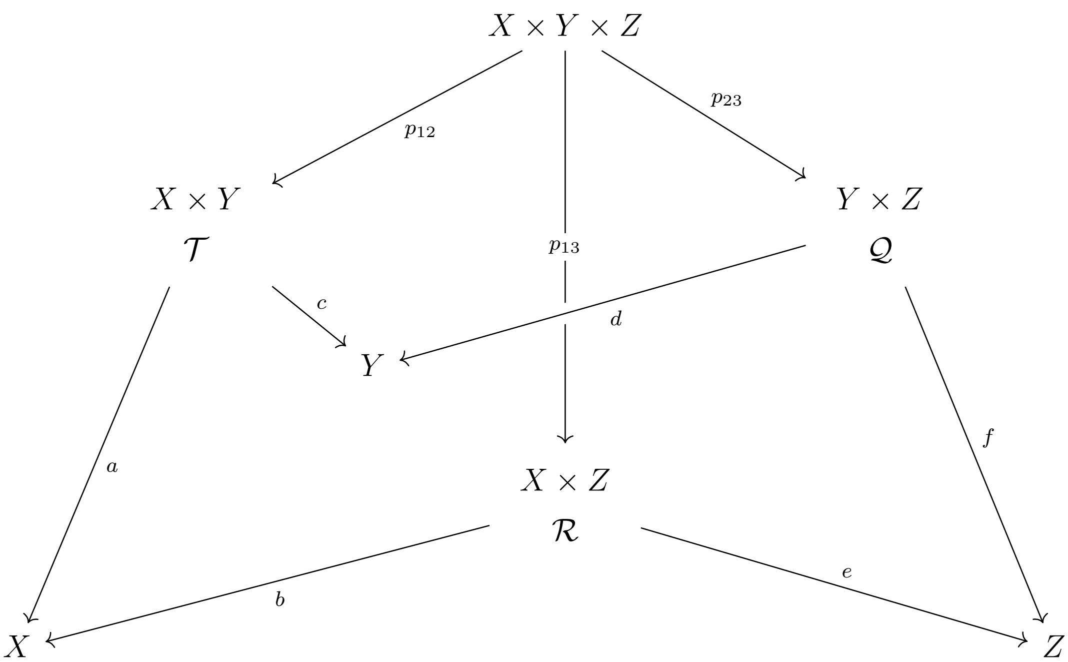
Using these properties, we calculate

$$\begin{aligned} R\mathcal{S}_{X\to Y,\mathcal{O}_{\Gamma_f}}(-) &= p_{2*}(\Gamma_{f*}\mathcal{O}_X \otimes {p_1}^*(-)) \\ &\simeq p_{2*}(\Gamma_{f*}(\mathcal{O}_X \otimes (\Gamma_f)^* {p_1}^*(-))) \qquad \text{(Projection Formula)} \\ &\simeq (p_2 \circ \Gamma_f)_*(p_1 \circ \Gamma_f)^*(-) \\ &\simeq f_*(-) \end{aligned}$$

Similarly, we can calculate the functor in the other direction

$$\begin{aligned} R\mathcal{S}_{Y\to X,\mathcal{O}_{\Gamma_f}}(-) &= p_{1*}(\Gamma_{f*}\mathcal{O}_X \otimes {p_2}^*(-)) \\ &\simeq p_{1*}(\Gamma_{f*}(\mathcal{O}_X \otimes (\Gamma_f)^* {p_2}^*(-))) \qquad \text{(Projection Formula)} \\ &\simeq (p_1 \circ \Gamma_f)_*(p_2 \circ \Gamma_f)^*(-) \\ &\simeq f^*(-) \end{aligned}$$

The composition of two Fourier–Mukai transforms is again a Fourier–Mukai transform. The following proposition gives an explicit description of this process. Start by considering schemes X, Y and Z, along with elements $\mathcal{T} \in D^b(X \times Y)$ and $\mathcal{Q} \in D^b(Y \times Z)$, in the following set-up:



where all maps are the natural projections and

$$\mathcal{R} = p_{13,*}(p_{12}^*\mathcal{T} \otimes p_{23}^*\mathcal{Q}) \in D^b(X \times Z).$$

Proposition 2.2.4 ([Muk81] Proposition 1.3). *The composition* $R\mathcal{S}_{Y\to Z,\mathcal{Q}} \circ R\mathcal{S}_{X\to Y,\mathcal{T}}(-)$ *is isomorphic to* $R\mathcal{S}_{X\to Z,\mathcal{R}}(-)$*, where all the objects are given as in the diagram above.*

Proof. For any element $\mathcal{E}^\bullet \in D^b(X \times Y)$ we have the isomorphisms

$$\begin{aligned}
R\mathcal{S}_{Y\to Z,\mathcal{Q}}(R\mathcal{S}_{X\to Y,\mathcal{T}}(\mathcal{E}^\bullet) &= f_*(d^*c_*(a^*\mathcal{E}^\bullet \otimes \mathcal{T}) \otimes \mathcal{Q}) \\
&\simeq f_*(p_{23,*}p_{12}^*(a^*\mathcal{E}^\bullet \otimes \mathcal{T}) \otimes \mathcal{Q}) && \text{(Flat Base Change)} \\
&\simeq f_*p_{23,*}(p_{12}^*(a^*\mathcal{E}^\bullet \otimes \mathcal{T}) \otimes p_{23}^*\mathcal{Q}) && \text{(Projection Formula)} \\
&\simeq e_*p_{13,*}(p_{12}^*a^*\mathcal{E}^\bullet \otimes p_{12}^*\mathcal{T} \otimes p_{23}^*\mathcal{Q}) && (f \circ p_{23} = e \circ p_{13}) \\
&\simeq e_*p_{13,*}(p_{13}^*b^*\mathcal{E}^\bullet \otimes p_{12}^*\mathcal{T} \otimes p_{23}^*\mathcal{Q}) && (a \circ p_{12} = b \circ p_{13}) \\
&\simeq e_*(b^*\mathcal{E}^\bullet \otimes p_{13,*}(p_{12}^*\mathcal{T} \otimes p_{23}^*\mathcal{Q})) && \text{(Projection Formula)} \\
&= e_*(b^*\mathcal{E}^\bullet \otimes \mathcal{R}) = R\mathcal{S}_{X\to Z,\mathcal{R}}
\end{aligned}$$

□

We now focus on the product $A \times A^\vee$, with A an abelian variety and $A^\vee$ its dual, and $\mathcal{P}$ the associated Poincaré bundle. g will denote the dimension of A. To shorten notation we will write $R\mathcal{S} = R\mathcal{S}_{A^\vee\to A,\mathcal{P}}$ and $R\hat{\mathcal{S}} = R\mathcal{S}_{A\to A^\vee,\mathcal{P}}$. p_{ij} will denote the projection from the i–th and j–th component of $A \times A \times A^\vee$. Projections with only one index, p_i, will as usual denote a projection from $A \times A^\vee$.

Theorem 2.2.5 ([Muk81] Theorem 2.2). *Let A be an abelian variety of dimension g. Then the following compositions are isomorphisms of functors:*

$$(1) R\mathcal{S} \circ R\hat{\mathcal{S}} \simeq (-1_A)^*[-g]$$

$$(2) R\hat{\mathcal{S}} \circ R\mathcal{S} \simeq (-1_{A^\vee})^*[-g]$$

In particular, this means that $R\mathcal{S}$ is an equivalence of the categories $D(A)$ and $D(A^\vee)$, its quasi–inverse is given by $(-1_{A^\vee})^* \circ R\hat{\mathcal{S}}[g]$.

Before giving the proof we need the two following lemmas.

Lemma 2.2.6. *With p_{ij} as described above, there is an isomorphism*

$$p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P} \simeq (m \times 1)^*\mathcal{P}$$

Proof. We want to show the isomorphism by the See–Saw Principle, where both sides are line bundles on $A \times A \times A^\vee$. At the point $(0,0) \in A \times A$ the maps m, p_1 and p_2 are all trivial, so there are isomorphisms

$$(p_{13}^*\mathcal{P})|_{(0,0)\times A^\vee} \simeq (p_{23}^*\mathcal{P})|_{(0,0)\times A^\vee} \simeq ((m \times 1)^*\mathcal{P})|_{(0,0)\times A^\vee} \simeq \mathcal{P}|_{0\times A^\vee}$$

and the latter is trivial by the universal property of the Poincaré bundle.

On the other hand, fix any point $b \in \mathrm{Pic}^0(A)$. For $n = 1, 2$, and i_b, j_b the natural inclusion maps, the commutativity of the squares

$$\begin{array}{ccc} A \times A \times \{b\} & \xrightarrow{\ i_b\ } & A \times A \times A^\vee \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_{n3}} \\ A \times \{b\} & \xrightarrow{\ j_b\ } & A \times A^\vee \end{array} \qquad \begin{array}{ccc} A \times A \times \{b\} & \xrightarrow{\ i_b\ } & A \times A \times A^\vee \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m\times 1} \\ A \times \{b\} & \xrightarrow{\ j_b\ } & A \times A^\vee \end{array}$$

ensures that showing an isomorphism

$$(p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P})|_{A\times A\times\{b\}} \simeq ((m \times 1)^*\mathcal{P})|_{A\times A\times\{b\}}$$

is equivalent to showing an isomorphism

$$p_1^*\mathcal{P}_b \otimes p_2^*\mathcal{P}_b \simeq m^*\mathcal{P}_b$$

and the latter isomorphism was shown to be true in Corollary 2.1.14. Hence the isomorphism in the statement is shown by the See–Saw Principle. □

Lemma 2.2.7.

$$R^i p_{1,*}\mathcal{P} \simeq \begin{cases} 0 & \text{if } i \neq g \\ k(0) & \text{if } i = g \end{cases}$$

We may therefore treat $Rp_{1,*}\mathcal{P}$ as the one term complex consisting of the skyscraper sheaf at 0, shifted g places to the right, i.e. $Rp_{1,*}\mathcal{P} \simeq k(0)[-g]$.

Proof. This is shown in the course of the proof of the theorem in [Mum70] III.13. □

Proof of Theorem 2.2.5. We start by showing the last statement from the assumption that the isomorphisms (1) and (2) are true. By shifting and applying the functor $(-1_{A^\vee})^*$ to both sides of equation (2), one immediately sees $(-1_{A^\vee})^* \circ R\hat{\mathcal{S}}[g] \circ R\mathcal{S} \simeq id_{A^\vee}$. To check the other way, notice that (2) also implies $(-1_{A^\vee})^* \simeq R\hat{\mathcal{S}} \circ R\mathcal{S}[g]$. From this, and equation (1), we get

$$R\mathcal{S} \circ (-1_{A^\vee})^* \circ R\hat{\mathcal{S}}[g] \simeq R\mathcal{S} \circ R\hat{\mathcal{S}} \circ R\mathcal{S}[g] \circ R\hat{\mathcal{S}}[g]$$

$$\simeq (-1_A)^*[-g] \circ (-1_A)^*[-g][2g]$$
$$\simeq id_A$$

This shows that $R\mathcal{S}$ is indeed an equivalence as claimed.

Now to show isomorphism (1); start by noting that from Proposition 2.2.4 $R\mathcal{S} \circ R\hat{\mathcal{S}} \simeq R\mathcal{S}_{A\to A,H}$ where $H = Rp_{12,*}(p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P})$ (note also the change of indices compared to what used in the proposition, as we are now working with $A \times A \times A^\vee$). By Lemma 2.2.6 $H \simeq Rp_{12,*}(m \times 1)^*\mathcal{P}$. Furthermore, applying Flat Base Change to the diagram

$$\begin{array}{ccc} A \times A \times A^\vee & \xrightarrow{p_{12}} & A \times A \\ \Big\downarrow{\scriptstyle m\times 1} & & \Big\downarrow{\scriptstyle m} \\ A \times A^\vee & \xrightarrow{p_1} & A \end{array}$$

gives $Rp_{12,*}(m\times 1)^*\mathcal{P} \simeq m^*Rp_{1,*}\mathcal{P} \simeq m^*k(0)[-g]$, where the latter isomorphism comes from the conclusion of Lemma 2.2.7. Now consider the "mirrored diagonal" subscheme of $A \times A$, which as a set is defined as $\Delta^- := \{(a, -a)|a \in A\}$. This consists of the set of points where $m = 0$, and it follows that $m^*k(0) = (m|_{\Delta^-})^*(k(0))$. But Δ^- gives the same closed subscheme as the graph $\Gamma_{(-1_A)}$, so $m^*k(0) \simeq \mathcal{O}_{\Gamma_{(-1_A)}}$. In summary, $H \simeq \mathcal{O}_{\Gamma_{(-1)_A}}[-g]$ and so, as was seen in Example 2.2.3, $R\mathcal{S}_{A\to A,H} \simeq (-1_A)_*[-g] \simeq (-1_A)^*[-g]$. Here the last isomorphism holds as (-1_A) is an isomorphism, being its own inverse.

By the symmetry of the problem, a similar line of arguments will show (2). $\square$

2.3 Mukai–Regularity

In this section we aim to introduce the notion of Mukai–regularity, which is a condition on the Fourier–Mukai transform of coherent sheaves on abelian varieties. In addition to being a precursor to Θ–regularity, we will see that Mukai–regularity in itself has serious geometric consequences, most notably seen in 2.3.19. The main reference of this section is [PP03], and several of the proofs given will follow the arguments of Pareschi and Popa. These instances will be sufficiently marked.

We begin by fixing some notation. A and $A^\vee$ will as before denote an abelian variety over $\mathbb{C}$, and its dual. $\mathcal{P}$ is the Poincaré bundle of A. If $\mathcal{F}$ is a coherent sheaf we will write $R\hat{\mathcal{S}}(\mathcal{F}) := R\mathcal{S}_{A\to A^\vee,\mathcal{P}}(\mathcal{F})$ for its Fourier–Mukai transform. $R^i\hat{\mathcal{S}}(\mathcal{F})$ will denote the i–th cohomology.

Definition 2.3.1. Let $\mathcal{F}$ be a coherent sheaf and $i \geq 0$. Then the set

$$V^i(\mathcal{F}) = \{\xi \in \mathrm{Pic}^0(A) \mid h^i(\mathcal{F} \otimes P_\xi) \neq 0\}$$

is called the i–th *cohomological support locus* of $\mathcal{F}$.

Lemma 2.3.2. *For any i the cohomological support loci of $\mathcal{F}$ is Zariski–closed in* $\mathrm{Pic}^0(A)$.

Proof. Consider the Semicontinuity Theorem 1.1.6 for the projection $p_2 : A \times A^\vee \longrightarrow A^\vee$ and sheaf $p_1^*(\mathcal{F}) \otimes \mathcal{P}$. For any $\xi \in A^\vee$, we have $A_\xi \simeq A$ and $(p_1^*(\mathcal{F}) \otimes \mathcal{P})_\xi \simeq \mathcal{F} \otimes \mathcal{P}_\xi$, so the theorem states that for any $i \geq 0$, the set $\psi^{-1}(-\infty, 1) = \{\xi \in A^\vee \mid h^i(A, \mathcal{F} \otimes \mathcal{P}_\xi) < 1\}$ is open, and this is exactly the complement of $V^i(\mathcal{F})$. □

We are now in a position to give the definition of Mukai–regularity, stated as two equivalent conditions. It was initially defined as condition i), using the Fourier–Mukai regularity, with the condition on the cohomological support loci assumed to be a stronger condition. It was later realized that they are, in fact, equivalent.

Proposition-Definition 2.3.3. *For a coherent sheaf $\mathcal{F}$, the following are equivalent:*

i) codim$(\mathrm{Supp}(R^i\hat{\mathcal{S}}(\mathcal{F}))) > i$, *for all* $0 < i \leq g$.

ii) codim$(V^i(\mathcal{F})) > i$, *for all* $0 < i \leq g$.

A sheaf satisfying this is called Mukai–regular, *or simply* M–regular.

Recall that the dimension of the empty set is defined to be -1. So in particular if $\mathcal{F}$ is M–regular, then $\mathrm{Supp}(R^g\hat{\mathcal{S}}(\mathcal{F})) = V^g(\mathcal{F}) = \emptyset$.

Proof. We start with *ii*) $\Rightarrow$ *i*). The idea is to utilize the Cohomology and Base Change Theorem 1.1.8, again for the projection $p_2 : A \times A^\vee \longrightarrow A^\vee$ and sheaf $p_1^*\mathcal{F} \otimes \mathcal{P}$. Then the domain of the map $\phi^i(\xi)$ in the theorem is $R^i\hat{\mathcal{S}}(\mathcal{F}) \otimes k(\xi)$ for any $\xi \in A^\vee$. Note that $\alpha \notin V^i(A, \mathcal{F})$ if and only if $h^i(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$. So pick such an α, then the map

$$\phi^i(\alpha) : R^i\hat{\mathcal{S}}(\mathcal{F}) \otimes k(\alpha) \longrightarrow H^i(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$$

is trivial, hence surjective and an isomorphism by Theorem 1.1.8. This implies $\alpha \notin \mathrm{Supp}(R^i\hat{\mathcal{S}})(\mathcal{F})$ and means, by complement, that we have the inclusion of sets $\mathrm{Supp}(R^i\hat{\mathcal{S}})(\mathcal{F}) \subseteq V^i(A, \mathcal{F})$.

For the other inclusion, suppose *i*) is satisfied whilst $codim(V^i(\mathcal{F})) \leq i$, for some $0 < i \leq g$. Let $j = max\{\, i \mid \; codim(V^i(A, \mathcal{F})) \leq i\}$, and choose $W \subseteq V^j(A, \mathcal{F})$, an irreducible component having $codim W \leq j$. Note that this implies $W \not\subseteq V^{j+1}(A, \mathcal{F})$, by our definition of j. This in turn means that $W \setminus (W \cap V^{j+1}(A, \mathcal{F}))$ is an open, non–empty set in W, so if ξ_0 denotes the generic point of W, it is not an element of $V^{j+1}(A, \mathcal{F})$. Then $H^{j+1}(\mathcal{F} \otimes \xi_0) = 0$ which means that $\phi^{j+1}(\xi_0)$ is surjective, hence an isomorphism, by the same line of reasoning as above. Now, part *b*) of Theorem 1.1.8 guarantees that:

$$\phi^j(\xi_0) : R^j\hat{\mathcal{S}}(\mathcal{F}) \otimes k(\xi_0) \to H^j(\mathcal{F} \otimes \xi_0) \neq 0$$

is surjective, so we have $\xi_0 \in \mathrm{Supp}(R^j\hat{\mathcal{S}}(\mathcal{F}))$. Since ξ_0 is the generic point of W, and the support is closed, we get the inclusion:

$$\overline{\xi_0^W} = W \subseteq \mathrm{Supp}(R^j\hat{\mathcal{S}}(\mathcal{F}))$$

Where $\overline{\xi_0^W}$ denotes the closure of ξ_0 in W. But then we must have $\mathrm{codim}(\mathrm{Supp}(R^j\hat{\mathcal{S}}(\mathcal{F}))) \leq j$, which contradicts our initial assumptions. $\square$

Recall from Theorem 2.1.8 that if a line bundle $\mathcal{L}$ on A is non–degenerate then its cohomologies vanish for all degrees, save for an integer $0 \leq i \leq g$, depending on $\mathcal{L}$. This inspires the following definition.

Definition 2.3.4. i) A coherent sheaf $\mathcal{F}$ on A is said to satisfy the *index theorem* (I.T.) of index i if

$$H^j(\mathcal{F} \otimes \alpha) = 0 \text{ for any } \alpha \in \mathrm{Pic}^0(A) \text{ and } i \neq j.$$

ii) $\mathcal{F}$ is said to satisfy the *weak index theorem* (W.I.T.) of index i if

$$R^j\mathcal{S}(\mathcal{F}) = 0 \text{ for all } j \neq i.$$

As the names suggests, a sheaf $\mathcal{F}$ satisfying I.T. of index i will also satisfy W.I.T. of the same index, as we will show in the next proposition. We then go on to show a partial converse for index 0.

Proposition 2.3.5. *If $\mathcal{F}$ is a sheaf satisfying I.T. of index i, then it also satisfies W.I.T. of index i. Furthermore, $R^i\hat{\mathcal{S}}(\mathcal{F})$ is locally free.*

Proof. The idea is to argue using the Cohomology and Base Change Theorem 1.1.8 in a similar way as was done in Proposition–Definition 2.3.3. Indeed, using the same setting, with $p_2 : A \times A^\vee \longrightarrow A^\vee$, sheaf $p_1^*\mathcal{F} \otimes \mathcal{P}$ and $j \neq i$, we have $H^j(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$ for every $\alpha \in A^\vee$. The isomorphism from the theorem then implies that $R^j\hat{\mathcal{S}}(\mathcal{F})$ is trivial. For the second statement, we know that $R^{i+1}\hat{\mathcal{S}}(\mathcal{F})$ is locally free, so ϕ^i is surjective by statement b) in Theorem 1.1.8. ϕ^{i-1} is also sujective, so another use of statement b) implies exactly that $R^i\hat{\mathcal{S}}(\mathcal{F})$ is locally free. □

Proposition 2.3.6. *If $\mathcal{F}$ is W.I.T. of index 0, then it is also I.T. of index 0.*

Proof. We intend to prove this using induction on part b) of the Cohomology and Base Change Theorem. Fix any point $\alpha \in A^\vee$ and use the same setting as done in the proof of Proposition 2.3.5, rendering the map

$$\phi^j(\alpha) : R^j\hat{\mathcal{S}}(\mathcal{F}) \otimes k(\alpha) \longrightarrow H^i(A, \mathcal{F} \otimes \mathcal{P}_\alpha).$$

If we choose $j = g+1$ then $H^{g+1}(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$ by Grothendiecks Vanishing Theorem, so $\phi^{g+1}(\alpha)$ is trivially surjective. Since $R^g\hat{\mathcal{S}}(\mathcal{F})$ is 0 (and hence trivially locally free), $\phi^g(\alpha)$ is surjective, implying $H^g(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$. We continue this argument all the way down to and including $j = 1$. □

Example 2.3.7. i) A coherent sheaf $\mathcal{F}$ satisfying W.I.T. of index $i = 0$ is M–regular. Clearly, as $R^j\hat{\mathcal{S}}(\mathcal{F})$ vanishes for all $j > 0$, the supports are empty here.
ii) From Example 2.1.9 and Lemma 2.1.22 we see that ample line bundles are I.T. of index 0. They will therefore provide important examples of M–regular sheaves.

We now give the main technical result for the theory of Mukai–regularity. It will be essential for us in order to derive statements relating M–regularity and global generation.

Theorem 2.3.8 ([PP03], Theorem 2.5)**.** *Let $\mathcal{F}$ be an M–regular sheaf on A, and H a locally free sheaf on A satisfying I.T. with index 0. Then, for any non–empty open set*

$U \subseteq A^{\vee}$, *the sum of multiplication maps on global sections*

$$\mathcal{M}_U : \bigoplus_{\xi \in U} H^0(A, \mathcal{F} \otimes P_\xi) \otimes H^0(A, H \otimes P_\xi^{\vee}) \stackrel{\oplus m_\xi}{\longrightarrow} H^0(A, \mathcal{F} \otimes H)$$

is surjective.

The theorem will be proved using the next three lemmas. For these results we let assumptions and notation be as in the previously stated Theorem.

Lemma 2.3.9. *The map*

$$\mathcal{M}_U : \bigoplus_{\xi \in U} H^0(A, \mathcal{F} \otimes P_\xi) \otimes H^0(A, H \otimes P_\xi^{\vee}) \stackrel{\oplus m_\xi}{\longrightarrow} H^0(A, \mathcal{F} \otimes H)$$

is surjective if and only if the co–multiplication map

$$Ext^g(\mathcal{F}, H^{\vee}) \to \prod_{\xi \in U} Hom(H^0(\mathcal{F} \otimes P_\xi), H^g(H^{\vee} \otimes P_\xi)) \tag{2.2}$$

is injective.

Proof. Recall that for a sheaf $\mathcal{G}$, we may view $H^0(A, \mathcal{G})$ as a $\mathbb{C}$-vector space, and consider its dual $H^0(A, \mathcal{G})^{\vee} = Hom(H^0(A, \mathcal{G}), k)$. This induces a dual map

$$H^0(A, \mathcal{F} \otimes H)^{\vee} \to \prod_{\xi \in U} H^0(A, \mathcal{F} \otimes P_\xi)^{\vee} \otimes H^0(A, H \otimes P_\xi^{\vee})^{\vee} \tag{2.3}$$

where a morphism $H^0(A, \mathcal{F} \otimes H) \to k$ is sent to a morphism $H^0(A, \mathcal{F} \otimes P_\xi) \otimes H^0(A, H \otimes P_\xi^{\vee}) \to k$ by pre–composing with the multiplication map m_ξ.

Claim 1. *The multiplication map $\mathcal{M}_U$ is surjective if and only if the dual map (2.3) is injective.*

Consider elements ϕ, $\psi \in H^0(A, \mathcal{F} \otimes H)^{\vee}$. It is clear that if $\mathcal{M}_U$ is surjective, then $\phi \circ \mathcal{M}_U = \psi \circ \mathcal{M}_U$ implies $\phi = \psi$. To see the other direction, assume that the dual map is injective, while $\mathcal{M}_U$ is not surjective. Choose a basis $< x_1, x_2, ...x_k >$ for the image of $\mathcal{M}_U$, and complete this to a basis $< x_1, ...x_k, y_{k+1}, ...y_m >$ for $H^0(A, \mathcal{F} \otimes H)$. Let ϕ be the projection of the element $k+1$ in this basis, i.e. $\phi(\sum(\alpha_i x_i) + \sum(\beta_j y_j)) = \beta_{k+1}$, and ψ be the trivial map. Then $\phi \circ \mathcal{M}_U = \psi \circ \mathcal{M}_U = 0$ which would contradict the injectivity of the dual map. This proves the claim.

We are left with showing the relation between the comultiplication (2.2) and the dual map (2.3). At the domain, we have $H^0(A, \mathcal{F} \otimes H)^{\vee} \simeq Ext^g(\mathcal{F} \otimes H, \mathcal{O}_A) \simeq$

$Ext^g(\mathcal{F}, H^\vee)$ by Serre Duality. For the codomain, we again use Serre Duality to note that

$$H^0(A, H \otimes P_\xi^\vee)^\vee \simeq Ext^g(H \otimes P_\xi^\vee, \mathcal{O}_A) \simeq Ext^g(\mathcal{O}_A, H^\vee \otimes P_\xi) \simeq H^g(A, H^\vee \otimes P_\xi).$$

This gives

$$\begin{aligned} H^0(A, \mathcal{F} \otimes P_\xi)^\vee \otimes H^0(A, H \otimes P_\xi^\vee)^\vee &\simeq Hom(H^0(A, \mathcal{F} \otimes P_\xi), k) \otimes H^g(A, H^\vee \otimes P_\xi) \\ &\simeq Hom(H^0(\mathcal{F} \otimes P_\xi), H^g(H^\vee \otimes P_\xi)) \end{aligned}$$

where the latter relation is the tensor product isomorphism $Hom(M, R) \otimes_R N \simeq Hom(M, N)$ that holds for any R-modules M and N. □

It is convenient to introduce some notation here. We define $\widehat{H^\vee} = R^g\hat{\mathcal{S}}(H^\vee)$. By Serre Duality we have

$$H^i(A, H^\vee \otimes \alpha) \simeq Ext^i(H \otimes \alpha^\vee) \simeq H^{g-i}(A, H \otimes \alpha^\vee)^\vee$$

so the assumption that H satisfies I.T. of index 0, implies that $H^\vee$ satisfies I.T. of index g. This makes $\widehat{H^\vee}$ a locally free sheaf by Proposition 2.3.5. Note also that the one–term complex $\widehat{H^\vee}$ is quasi–isomorphic to $R\hat{\mathcal{S}}(H^\vee)[g]$, and they are therefore isomorphic in $D^b(A)$. We will use these facts throughout the proofs of the next lemmas.

Lemma 2.3.10. *There is a natural inclusion*

$$Hom_{D(A^\vee)}(R\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}) \longrightarrow Hom(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee})$$

Proof. From Proposition 1.3.13 we have the spectral sequence

$$E_2^{p,q} = Ext^p(R^{-q}\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}) \Rightarrow E^{p+q} = Ext^{p+q}_{D(A^\vee)}(R\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee})$$

We want to inspect $E_2^{p,q}$ for the values p and q satisfying $p+q=0$. Since Ext^p is trivial for negative values of p, we restrict ourselves to the case when $p \geq 0$ (and, consequently, $q \leq 0$). Consider $Ext^p(R^{-q}\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}) \simeq Ext^p(R^{-q}\hat{\mathcal{S}}(\mathcal{F}) \otimes (\widehat{H^\vee})^\vee, \mathcal{O}_{A^\vee})$ and note that $\mathrm{Supp}(R^{-q}\hat{\mathcal{S}}(\mathcal{F}) \otimes \widehat{H^\vee}) = \mathrm{Supp}(R^{-q}\hat{\mathcal{S}}(\mathcal{F}))$ since $\widehat{H^\vee}$ is locally free. By definition, the M–regularity of $\mathcal{F}$ ensures that $g - \dim \mathrm{Supp} R^{-q}\hat{\mathcal{S}}(\mathcal{F}) > -q$ or equivalently $q > \dim \mathrm{Supp} R^{-q}\hat{\mathcal{S}}(\mathcal{F}) - g$. This in turn implies

$$g + q > g + (\dim \mathrm{Supp} R^{-q}\hat{\mathcal{S}}(\mathcal{F}) - g) = \dim \mathrm{Supp} R^{-q}\hat{\mathcal{S}}(\mathcal{F})$$

whenever $q < 0$. In this case, Proposition 1.1.4 shows that the only non–trivial element $E_2^{p,q}$ where $p + q = 0$ is $E_2^{0,0}$. This in particular means that the only non–trivial object $E_\infty^{p,q}$, keeping the same restrictions on p and q, is $E_\infty^{0,0}$. Recall from part *iv*) of Definition 1.1.11 of spectral sequences that for any integers k, l there are isomorphisms $E_\infty^{k,l} \simeq F^k E^{k+l}/F^{k+1}E^{k+l}$, where F denotes the decreasing filtration of E^n. Due to $E_\infty^{0,0}$ being the only non–trivial infinity object at 0, the filtration only changes values at the step F^1E^0 to F^0E^0. Since we furthermore have the relations $\cap_p F^pE^n = 0$ and $\cup_p F^pE^n = E^n$, we deduce that $E_\infty^{0,0} \simeq E^n$. Then use the isomorphisms from condition *ii*) of the same definition to obtain a chain of inclusions

$$\cdots \hookrightarrow E_{r+1}^{0,0} \simeq H^0(E_r^{\bullet r, \bullet - \bullet r}) \hookrightarrow E_r^{0,0} \simeq H^0(E_{r-1}^{\bullet(r-1), \bullet - \bullet(r-1)}) \hookrightarrow \cdots$$

Composing these gives us the natural inclusion

$$Hom_{D(A^\vee)}(R\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}) \simeq E^0 \hookrightarrow E_2^{0,0} \simeq Hom(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee})$$

□

Lemma 2.3.11. *There is a natural map of $\mathcal{O}_{A^\vee}$-modules*

$$\phi : Ext^g(\mathcal{F}, H^\vee) \otimes \mathcal{O}_{A^\vee} \longrightarrow \mathcal{H}om(R^0\hat{S}(\mathcal{F}), \widehat{H^\vee})$$

where for general points $\xi \in A^\vee$, the induced map on the fiber

$$\phi(\xi) : Ext^g(\mathcal{F}, H^\vee) \longrightarrow \mathcal{H}om(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee})(\xi)$$

is the co–multiplication map (2.2) at ξ.

Proof. Start by defining the open set $\mathcal{U}_0 := A^\vee \setminus (\cup_{i=1}^g V^i(\mathcal{F})) \subset A^\vee$ (which is non–empty due to the M–regularity assumption on $\mathcal{F}$) and fix an element $\xi \in \mathcal{U}_0$. We argue using the following diagram:

$$\begin{array}{ccc}
Ext^g(\mathcal{F}, H^\vee) & \xrightarrow[\;i)\;]{\simeq} & H^0(\mathcal{F} \otimes H)^\vee \\
\downarrow {\scriptstyle \simeq\; iv)} & & \downarrow {\scriptstyle ii)} \\
Hom_{D(A)}(\mathcal{F}, H^\vee[g]) & & H^0(\mathcal{F} \otimes \mathcal{P}_\xi)^\vee \otimes H^0(H \otimes \mathcal{P}_\xi^\vee)^\vee \\
\downarrow {\scriptstyle \simeq\; v)} & & \downarrow {\scriptstyle \simeq\; iii)} \\
Hom_{D(A^\vee)}(R\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}) & & \\
\hookdownarrow {\scriptstyle vi)} & & \\
Hom(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}) & \xrightarrow{\;vii)\;} & Hom(H^0(\mathcal{F} \otimes \mathcal{P}_\xi), H^g(H^\vee \otimes \mathcal{P}_\xi))
\end{array}$$

Going right then down on the diagram (i.e. composing maps $i) - iii)$) gives the co–multiplication map on ξ, as in equation (2.2). This has been described in the proof of Lemma 2.3.9. The other maps are as follows:

iv) Follows from Proposition 1.3.12.

v) Is the Fourier–Mukai transform. This was proven to be an equivalence of categories in Theorem 2.2.5, so it is in particular fully faithful.

vi) Is the natural inclusion obtained in Lemma 2.3.10.

Composing the maps $iv) - vi)$ yields a map

$$\Phi : Ext^g(\mathcal{F}, H^\vee) \longrightarrow Hom(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee}). \tag{2.4}$$

Upon twisting with the trivial bundle this is easily extended to the map

$$\phi : Ext^g(\mathcal{F}, H^\vee) \otimes \mathcal{O}_{A^\vee} \longrightarrow \mathcal{H}om(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee})$$

as asserted in the first part of the statement of this lemma. Map *vii*) will then account for the second claim of the lemma, namely that the map ϕ on the fiber ξ is the co-multiplication map at ξ. Indeed, note that by the choice of $\mathcal{U}_0$, and Lemma 2.1.22, we have

$$h^0(\mathcal{F} \otimes \mathcal{P}_\xi) = \chi(\mathcal{F} \otimes \mathcal{P}_\xi) = \chi(\mathcal{F}) = h^0(\mathcal{F})$$

This makes $h^0(\mathcal{F} \otimes \mathcal{P}_\xi)$ constant for $\xi \in \mathcal{U}_0$, and hence for the projection map $p_2 : A \times \mathcal{U}_0 \longrightarrow \mathcal{U}_0$ and sheaf $(p_1^*\mathcal{F} \otimes \mathcal{P})|_{A\times\mathcal{U}_0}$ we can apply Grauert's Theorem 1.1.7, to guarantee that the natural map

$$R^0\hat{\mathcal{S}}(\mathcal{F}) \otimes k(\xi) \longrightarrow H^0(\mathcal{F} \otimes \mathcal{P}_\xi)$$

is an isomorphism. One the other hand, $\widehat{H^\vee} = R^g\hat{\mathcal{S}}(H^\vee)$ which we noted is I.T. of index g. Therefore $\chi(\widehat{H^\vee} \otimes \mathcal{P}_\xi) = h^0(\widehat{H^\vee} \otimes \mathcal{P}_\xi)$ for any ξ. Using Grauert's Theorem again gives an isomorphism

$$R^g\hat{\mathcal{S}}(H^\vee) \otimes k(\xi) \longrightarrow H^g(H^\vee \otimes \mathcal{P}_\xi).$$

Thus for any $\xi \in \mathcal{U}_0$ we obtain a diagram as above, which commutes since all involved maps are natural. □

Proof of Theorem 2.3.8. For an open set $\mathcal{U}' \subseteq A^\vee$, we show that the multiplication map is surjective for $\mathcal{U} := \mathcal{U}' \cap \mathcal{U}_0$, where $\mathcal{U}_0$ is the open set from the proof of Lemma 2.3.11. In light of the two Lemmas 2.3.9 and 2.3.11, the global multiplication map,

$\mathcal{M}_{\mathcal{U}}$, is surjective if and only if ϕ from the latter lemma is injective. $\widehat{H^\vee}$ is locally free, so by Example 1.1.10 the sheaf $\mathcal{H}om(R^0\hat{\mathcal{S}}(\mathcal{F}), \widehat{H^\vee})$ is torsion free. In the discussion prior to the same example, we see that the restriction maps to the stalks are then injective. This means that ϕ is injective if it is injective on the global sections (of the subvariety $\mathcal{U} \subset A^\vee$), which is the map (2.4) introduced as Φ in the proof of Lemma 2.3.11. But this map is indeed injective by Lemma 2.3.10, and the proof of the Theorem is complete. □

Corollary 2.3.12. *If $\mathcal{F}$ is M–regular, then $H^0(\mathcal{F} \otimes \alpha) \neq 0$ for any non-trivial $\alpha \in$ $\mathrm{Pic}^0(A)$.*

Proof. Suppose that there exists an element $\alpha_0 \in \mathrm{Pic}^0(A)$ such that $H^0(A, \mathcal{F} \otimes \alpha_0) = 0$. Then $V^0(\mathcal{F})$ is not the whole of $\mathrm{Pic}^0(A)$, so we have an open, non–empty set $U = \mathrm{Pic}^0(A) \setminus V^0(\mathcal{F})$. Now consider an ample line bundle $\mathcal{L}$, so there is an integer n such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated. Denote $H = \mathcal{L}^n$. This means that $H^0(A, \mathcal{F} \otimes H) \neq 0$, while H at the same time satisfies the condition of the theorem. The contradiction arises as $H^0(A, \mathcal{F} \otimes P_\xi) = 0$, for any $\xi \in U$ by construction, so the map $\mathcal{M}_U$ fails to be surjective. □

Proposition 2.3.13. *Let $\mathcal{F}$ and H be as in Theorem 2.3.8. Then there is an integer N and elements $\xi_1, ...\xi_N$ such that the finite sum of the multiplication maps*

$$\oplus m_{\xi_i} : \bigoplus_{i=1}^{N} H^0(A, \mathcal{F} \otimes P_{\xi_i}) \otimes H^0(A, H \otimes P_{\xi_i}^\vee) \to H^0(A, \mathcal{F} \otimes H)$$

is surjective.

Proof. [PP03] Corollary 2.8. □

In the same setting as Theorem 2.3.8, and indeed using similar arguments, we can show that an M–regular sheaf in a sense preserves the vanishing of higher cohomologies. This is made precise in the following result.

Proposition 2.3.14 ([PP03] Proposition 2.9)**.** *Let $\mathcal{F}$ and H be sheaves on A, where $\mathcal{F}$ is M–regular and H is locally free and satisfying I.T. of index 0. Then $\mathcal{F} \otimes H$ satisfies I.T. with index 0.*

Proof. Consider an element α in $A^\vee$. Then $H \otimes \alpha$ still satisfies I.T. of index 0, and is also still locally free, so we have

$$H^i(\mathcal{F} \otimes H \otimes \alpha) \simeq Ext^i((H \otimes \alpha)^\vee, \mathcal{F}) \simeq Hom_{D(A)}((H \otimes \alpha)^\vee, \mathcal{F}[i])$$

Recall the following notation from the proof of the previous Theorem, namely $(\widehat{H \otimes \alpha})^\vee = R^g\hat{\mathcal{S}}((H \otimes \alpha)^\vee)$. As before this is locally free and can be identified with the one–term complex $R\hat{\mathcal{S}}((H \otimes \alpha)^\vee)[g]$. Using once more that the Fourier–Mukai transform is fully faithful, we get the following isomorphisms:

$$\begin{aligned} Hom_{D(A)}((H \otimes \alpha)^\vee, \mathcal{F}[i]) &\simeq Hom_{D(A^\vee)}((\widehat{H \otimes \alpha})^\vee, R\hat{\mathcal{S}}(\mathcal{F})[i+g]) \\ &\simeq Ext^{i+g}((\widehat{H \otimes \alpha})^\vee, R\hat{\mathcal{S}}(\mathcal{F})) \end{aligned}$$

and we want to show that this latter Ext–group vanishes when $i > 0$. To this end we employ the second spectral sequence from Proposition 1.3.13

$$E_2^{k,l} = Ext^k_{D(A^\vee)}((\widehat{H \otimes \alpha})^\vee, R^l\hat{\mathcal{S}}(\mathcal{F})) \Rightarrow Ext^{k+l}_{D(A^\vee)}((\widehat{H \otimes \alpha})^\vee, R\hat{\mathcal{S}}(\mathcal{F})).$$

Note that

$$Ext^k((\widehat{H \otimes \alpha})^\vee, R^l\hat{\mathcal{S}}(\mathcal{F})) \simeq H^k(A^\vee, ((\widehat{H \otimes \alpha})^\vee)^\vee \otimes R^l\hat{\mathcal{S}}(\mathcal{F}))$$

and the latter is trivial when $k >$ dim Supp$(R^l\hat{\mathcal{S}}(\mathcal{F}))$, since $(\widehat{H \otimes \alpha})^\vee$ is locally free and by arguing in the same way as was done in the proof of Lemma 2.3.10. We are interested in the case where $k + l > g$, and here the M–regularity assumption on $\mathcal{F}$ implies that $k > g - l >$ dim Supp$(R^l\hat{\mathcal{S}}(\mathcal{F}))$ when l is positive (note that $l = 0$ is a trivial case as the cohomology is 0 when the index exceeds the dimension). Hence the spectral sequence vanishes at the E_2-level for $k+l > g$, so we have $E_\infty^{k,l} = 0$ here. This in turn implies

$$E^{i+g}_{D(A^\vee)}((\widehat{H \otimes \alpha})^\vee, R^l\hat{\mathcal{S}}(\mathcal{F})) = 0, \text{ when } i > 0$$

which is what we wanted to show. □

To obtain statements of global generation from Mukai–regularity, it will be useful to introduce a concept called *continuous global generation*. We will then show that M–regular sheaves satisfy this condition. The twist of a a sheaf with a line bundle, both satisfying this new notion, will furthermore be globally generated.

Definition 2.3.15. A coherent sheaf $\mathcal{F}$ on A is called *continuously globally generated* if the sum of evaluation maps:

$$\oplus ev_\alpha : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^\vee \longrightarrow \mathcal{F}$$

is surjective for any non–empty open subset $U \subset \mathrm{Pic}^0(A)$.

The following result mirrors Proposition 2.3.13 in the sense that continuously globally generated is equivalent to the existence of a finite number of general points in $A^\vee$ where the sum of evaluation maps is surjective.

Lemma 2.3.16. *A coherent sheaf $\mathcal{F}$ on A is continuously globally generated if and only if there is a finite number N of general line bundles in $Pic^0(A)$ such that the sum of evaluation maps*

$$\oplus ev_{\xi_i} : \bigoplus_{i=1}^{N} H^0(\mathcal{F} \otimes \xi_i) \otimes \xi_i^\vee \longrightarrow \mathcal{F}$$

is surjective.

Proof. Suppose the map $\bigoplus_{\alpha \in U} ev_\alpha$ is surjective. Then for elements $p \in A$, the induced map $\bigoplus ev_{\alpha,p}$ at the stalk generates $\mathcal{F}_p$ as an $\mathcal{O}_{A,p}$–module. Since $\mathcal{F}$ is coherent, and thus finitely generated at the stalks, only a finite amount of these maps are needed to generate $\mathcal{F}_p$. Since $\mathcal{F}$ is locally represented at the stalks, we may use the noetherian hypothesis on A to restrict ourselves to a finite number of these stalks, and hence only a finite number of evaluation maps are needed to obtain surjectivity. □

Remark 2.3.17. Note that for a line bundle $\mathcal{L}$ to be continuously globally generated means that the intersection of the divisors in $|\mathcal{L} \otimes \alpha|$ is empty when varied over an open set in $\text{Pic}^0(A)$ (or over finitely many general elements as noted in the lemma above). Indeed, if $\cap_{\alpha \in U} \text{Bs}(|\mathcal{L} \otimes \alpha|)$ was non–empty, then all the global sections of $H^0(A, \mathcal{L} \otimes \alpha)$ for every $\alpha \in \text{Pic}^0(A)$ would vanish here, and the evaluation maps would fail to be surjective.

Proposition 2.3.18 ([PP03] Proposition 2.13.)**.** *Any M–regular sheaf is continuously globally generated. In other words, if $\mathcal{F}$ is M–regular, then there is a positive integer N such that the sum of evaluation maps*

$$\bigoplus_{i=1}^{N} H^0(\mathcal{F} \otimes P_{\xi_i}) \otimes P_{\xi_i}^\vee \longrightarrow \mathcal{F}$$

is surjective for general elements $\xi_1, ..., \xi_N$ in $A^\vee$.

Proof. Let $\mathcal{L}$ be an ample line bundle on A and, as was done in Corollary 2.3.12, we define $H = \mathcal{L}^n$ so that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated. Consider the following commutative diagram obtained by alternating the evaluation and multiplication maps:

$$\begin{array}{ccc}
\oplus_{i=1}^{N} H^0(\mathcal{F}\otimes P_{\xi_i})\otimes H^0(H\otimes P_{\xi_i}^{\vee})\otimes \mathcal{O}_A & \longrightarrow & H^0(\mathcal{F}\otimes H)\otimes \mathcal{O}_A \\
\Big\downarrow {\scriptstyle ev_{H\otimes P_{\xi_i}^{\vee}}} & & \Big\downarrow {\scriptstyle ev_{\mathcal{F}\otimes H}} \\
\oplus_{i=1}^{N} H^0(\mathcal{F}\otimes P_{\xi_i})\otimes H\otimes P_{\xi_i}^{\vee} & \longrightarrow & \mathcal{F}\otimes H
\end{array}$$

If the elements $\xi_1, ...\xi_N$ are chosen as in Proposition 2.3.13, then the top horizontal map is surjective and the right vertical map is surjective by choice of H. It now follows that the bottom horizontal map is surjective. □

Theorem 2.3.19 ([PP03] Proposition 2.12 and Theorem 2.4)**.** *Consider a coherent sheaf $\mathcal{F}$ and a line bundle $\mathcal{L}$ on A. If both $\mathcal{F}$ and $\mathcal{L}$ are continuously globally generated, then $\mathcal{F}\otimes\mathcal{L}$ is globally generated.*

Along with Proposition 2.3.18 *this particularly means that the tensor product of an M–regular sheaf and an M–regular line bundle is globally generated.*

Proof. We start by considering a diagram similar to what was used in Proposition 2.3.18, by alternating evaluation and multiplication maps:

$$\begin{array}{ccc}
\oplus_{i=1}^{N} H^0(\mathcal{F}\otimes P_{\xi_i})\otimes H^0(\mathcal{L}\otimes P_{\xi_i}^{\vee})\otimes \mathcal{O}_A & \longrightarrow & H^0(\mathcal{F}\otimes \mathcal{L})\otimes \mathcal{O}_A \\
\Big\downarrow {\scriptstyle ev_{\mathcal{L}\otimes P_{\xi_i}^{\vee}}} & & \Big\downarrow {\scriptstyle ev_{\mathcal{F}\otimes \mathcal{L}}} \\
\oplus_{i=1}^{N} H^0(\mathcal{F}\otimes P_{\xi_i})\otimes \mathcal{L}\otimes P_{\xi_i}^{\vee} & \longrightarrow & \mathcal{F}\otimes \mathcal{L}
\end{array}$$

Here N is chosen such that the sum of evaluation maps for both $\mathcal{F}$ and $\mathcal{L}$ is surjective. Then the bottom horizontal map is surjective, so $\mathrm{coker}(ev_{\mathcal{F}\otimes\mathcal{L}})$ is contained in the intersection of base loci $\mathrm{Bs}(|\mathcal{L}\otimes P_{\xi_i}^{\vee}|)$. But this intersection is empty, as noted in Remark 2.3.17. Hence $ev_{\mathcal{F}\otimes\mathcal{L}}$ is also surjective, making $\mathcal{F}\otimes\mathcal{L}$ globally generated. □

2.4 Theta–Regularity

The Mukai–regularity presented in the last section might at first not seem too similar to the Castelnuovo–Mumford regularity for projective spaces that was presented in section 1.5. However, by considering a principally polarized abelian variety one obtains a regularity condition that can be seen as a true "abelian" version of the regularity of projective spaces. In particular, Theorem 2.4.3 shows clear similarities to the Castelnuovo–Mumford Theorem 1.5.2.

Recall from section 2.1 that a principally polarized abelian variety is a fixed pair (A, Θ) where Θ is a symmetric ample divisor, such that $h^0(A, \mathcal{O}_A(\Theta)) = 1$.

Definition 2.4.1. A coherent sheaf $\mathcal{F}$ on (A, Θ) is called *m–Θ–regular* if $\mathcal{F}((m-1)\Theta)$ is M–regular. If $m = 0$ the sheaf is simply called *Θ–regular*.

Recall from Theorem 1.2.5 than for an ample line bundle $\mathcal{L}$ and any coherent sheaf $\mathcal{F}$, the higher cohomology groups of $\mathcal{F} \otimes \mathcal{L}^m$ vanish for m sufficiently large. It follows in particular that $\mathcal{F}$ is always m–Θ–regular for a sufficiently large integer m.

Example 2.4.2. i) Consider any $\alpha \in \mathrm{Pic}^0(A)$. As Θ is ample, so is $\mathcal{O}_A(\Theta) \otimes \alpha$ by Lemma 2.1.22, and therefore M–regular. It follows that α is 2–Θ–regular.

ii) $\mathcal{O}_A(n\Theta)$ is $(-n+2)$–Θ–regular. Indeed, $\mathcal{O}_A(n\Theta) \otimes \mathcal{O}_A((-n+1)\Theta) = \mathcal{O}_A(\Theta)$, and hence ample and M–regular.

Theorem 2.4.3. ([PP03] Theorem 6.3) *Suppose $\mathcal{F}$ is a Θ–regular sheaf on (A, Θ). The following holds:*

(1) $\mathcal{F}$ is globally generated.

(2) $\mathcal{F}$ is m–Θ–regular for any $m \geq 1$.

(3) The multiplication map

$$H^0(\mathcal{F}(\Theta)) \otimes H^0(\mathcal{O}(k\Theta)) \longrightarrow H^0(\mathcal{F}((k+1)\Theta))$$

is surjective whenever $k \geq 2$.

Proof. (1) We may write $\mathcal{F} \simeq \mathcal{F}(-\Theta) \otimes \mathcal{O}(\Theta)$. $\mathcal{F}(-\Theta)$ is M-regular by assumption and $\mathcal{O}(\Theta)$ is an M-regular line bundle, so we get the result from the last statement of Theorem 2.3.19.

(2) When $\mathcal{F}$ is Θ–regular, checking for m–Θ–regularity involves twisting with m copies of $\mathcal{O}(\Theta)$ and these are I.T. of index 0, as they are ample. The result then follows immediately from Proposition 2.3.14.

(3) It is a consequence of Akira Ohbuchi's proof in ([Ohb88]) that there is an open set $U \in A^\vee$ such that the multiplication map

$$H^0(\mathcal{O}_A(2\Theta) \otimes P_\xi) \otimes H^0(\mathcal{O}_A(k\Theta)) \longrightarrow H^0(\mathcal{O}_A((2+k)\Theta) \otimes P_\xi)$$

is surjective for any integer $k \geq 2$ when $\xi \in U$. With this choice of U we consider the following commutative diagram obtained by alternating the order of multiplication maps:

$$\begin{array}{ccc}
\oplus_{\xi \in U} H^0(\mathcal{F}(-\Theta) \otimes P_\xi) \otimes H^0(\mathcal{O}_A(2\Theta) \otimes P_\xi^\vee) \otimes H^0(\mathcal{O}_A(k\Theta)) & \longrightarrow & H^0(\mathcal{F}(\Theta)) \otimes H^0(\mathcal{O}_A(k\Theta)) \\
\downarrow & & \downarrow \\
\oplus_{\xi \in U} H^0(\mathcal{F}(-\Theta) \otimes P_\xi) \otimes H^0(\mathcal{O}_A((k+2)\Theta) \otimes P_\xi^\vee) & \longrightarrow & H^0(\mathcal{F}((k+1)\Theta))
\end{array}$$

We now argue inductively, using Proposition 2.3.14, that $\mathcal{O}_A((k+2)\Theta)$ is I.T. of index 0. So the bottom horizontal map is surjective by Theorem 2.3.8. Since the left vertical map is also surjective by choice of U, the right vertical map must be surjective. □

Remark 2.4.4. The numerical analogy between this theorem and the Castelnuovo–Mumford Theorem 1.5.2 is identical, with the exception that the original result includes the case $k = 1$ for statement (3). To see that this fails for abelian varieties, consider an elliptic curve E and $\mathcal{F} = \mathcal{O}(\Theta)$. The Riemann–Roch Theorem says that the dimension of $H^0(\mathcal{O}(d\Theta))$ is equal to d. So if $k = 1$ then the left hand side of the multiplication map in (3) has dimension 2, whilst the right hand side has dimension 3. It follows that the map cannot be surjective.

In light of part (2) of the Main Theorem for Θ–regularity, it makes sense to speak of the lowest integer m for which $\mathcal{F}$ is m–Θ–regular. We therefore conclude this section with the following definition.

Definition 2.4.5. Let $\mathcal{F}$ be a coherent sheaf on a principally polarized abelian variety (A,Θ). The Θ–regularity of $\mathcal{F}$ is then defined to be

$$\Theta\text{–reg}(\mathcal{F}) = \inf\{m \in \mathbb{Z} | \mathcal{F} \text{ is } m\text{–}\Theta\text{–regular}\}$$

Chapter 3

Theta-Regularity and Log-Canonical Threshold

In the end of the first chapter we noted how Alex Küronya and Norbert Pintye related the Castelnuovo–Mumford regularity to the log–canonical threshold of ideal sheaves by an inequality. In chapter 2 we followed Giuseppe Pareschi and Mihnea Popa's development of theta–regularity which turned out to be an analogous regularity condition for abelian varieties. This immediately raises the question of whether a similar relation holds between the log–canonical threshold and the theta–regularity, and we will explore this in Theorem 3.1.6. Along the way we also state a lower bound for the theta–regularity of ideal sheaves, as well as an upper bound for the theta–regularity of multiplier ideals (Propositions 3.1.2 and 3.1.4, respectively). We end the chapter with a discussion on how the relation between theta–regularity and log–canonical thresholds can be used to obtain statements on singularities. This will culminate in a new interpretation of a proof due to Lawrence Ein and Robert Lazarsfeld, regarding a statement of singularities of pluri–theta divisors (Theorem 3.1.8).

We start with the following lemma for short exact sequences and M–regular sheaves.

Lemma 3.1.1. *Consider a short exact sequence of coherent sheaves:*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{L}$ is ample and $\mathcal{F}$ is M–regular. Then $\mathcal{G}$ is also M–regular.

Proof. Fix an integer $0 < i \leq g$ and and twist the sequence by any element $\alpha \notin V^{i+1}(\mathcal{F})$. Since α is locally free, this gives a new short exact sequence, which induces the long exact sequence of cohomology groups:

$$\cdots \longrightarrow H^j(\mathcal{F}\otimes\alpha) \longrightarrow H^j(\mathcal{L}\otimes\alpha) \longrightarrow H^j(\mathcal{G}\otimes\alpha) \longrightarrow H^{j+1}(\mathcal{F}\otimes\alpha) \longrightarrow \cdots$$

Now $\mathcal{L}\otimes\alpha$ is still ample by Lemma 2.1.22 and hence I.T. of index 0. Therefore the i–th cohomology group of $\mathcal{L}\otimes\alpha$ vanishes and we have an injection $H^i(\mathcal{G}\otimes\alpha) \hookrightarrow H^{i+1}(\mathcal{F}\otimes\alpha) = 0$, where the latter is 0 by choice of α. We therefore have $\alpha \notin V^i(\mathcal{G})$ and so $\mathrm{codim}V^i(\mathcal{G}) \geq \mathrm{codim}V^{i+1}(\mathcal{F}) > i+1$. □

The following result gives an important lower bound for the Θ–regularity of ideal sheaves.

Proposition 3.1.2. *Let $\mathcal{I} \neq \mathcal{O}_A$ be an ideal sheaf on A. Then $\mathcal{I}$ cannot be m–Θ–regular for $m < 3$. In other words, Θ–reg$(\mathcal{I}) \geq 3$.*

Proof. By Theorem 2.4.3 b), it suffices to prove the statement for the case $m = 2$. So assume for a contradiction that $\mathcal{I}$ is a 2–Θ–regular ideal sheaf, and consider the associated short exact sequence:

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{G} \longrightarrow 0$$

where $\mathcal{G} = \mathcal{O}_D$ for some effective divisor D such that $-D$ is associated to $\mathcal{I}$. Twisting this sequence with Θ will preserve the exactness

$$0 \longrightarrow \mathcal{I}\otimes\Theta \longrightarrow \Theta \longrightarrow \mathcal{G}\otimes\Theta \longrightarrow 0$$

and since $\mathcal{I}\otimes\Theta$ is M–regular by assumption, then so is $\mathcal{G}\otimes\Theta$ by the last lemma. By definition of a principal polarization, we have $h^0(A,\Theta) = 1$, and the higher cohomologies vanish as Θ is ample. From this, and the additive property of the Euler characteristic on exact sequences, we have

$$1 = \chi(\Theta) = \chi(\mathcal{I}\otimes\Theta) + \chi(\mathcal{G}\otimes\Theta) \tag{3.1}$$

Now let $\mathcal{U}_i = A^\vee \setminus V^i(\mathcal{I}\otimes\Theta)$, and choose an element $\alpha_0 \in \cap_{i>0}\mathcal{U}_i$. Such an element necessarily exists, as $\mathcal{U}_i$ is non–empty for $i > 0$ by the M–regularity assumption, and the intersection of non–empty open sets in the Zariski topology is non–empty. Lemma 2.1.22 implies $\chi(\mathcal{I}\otimes\Theta) = \chi(\mathcal{I}\otimes\Theta\otimes\alpha_0) = h^0(\mathcal{I}\otimes\Theta\otimes\alpha_0)$, which is positive by Corollary 2.3.12. The same line of argumentation also shows $\chi(\mathcal{G}\otimes\Theta) > 0$. But this contradicts Equation (3.1), as the right hand side must be greater than 1. Hence $\mathcal{I}$ cannot be 2–Θ–regular. □

Example 3.1.3. Since Θ is an effective divisor, $\mathcal{O}_A(-\Theta)$ is an ideal sheaf. We have already seen that $\mathcal{O}_A(-\Theta)$ is 3–Θ–regular, so the inequality in Proposition 3.1.2 is sharp.

We note the following fact which will be used throughout the rest of the chapter. Consider the linear system $|\mathcal{O}_A(\Theta) \otimes \alpha|$ for any $\alpha \in \mathrm{Pic}^0(A)$. This is non–empty as $h^0(\mathcal{O}_A(\Theta) \otimes \alpha) = h^0(\mathcal{O}_A(\Theta)) = 1$, hence the linear system has dimension 0. If L denotes the unique effective divisor here, we may write $L = \Theta + D$ where D is a (not necessarily effective) divisor representing α. Now consider $q\Theta + D$ for a positive rational number q. Then there is an integer p such that $pq\Theta$ is an ample, integral divisor. Since pD will represent an element in $\mathrm{Pic}^0(A)$, $pq\Theta + pD$ is an integral ample divisor by Lemma 2.1.22. In particular, this makes $q\Theta + D$ an ample $\mathbb{Q}$–divisor.

For the convenience of the reader, we briefly recall some notation from section 1.4. For a rational number $c > 0$ and a sheaf of ideals $\mathcal{I}$ on A, the multiplier ideal sheaf associated to c and $\mathcal{I}$ is denoted $\mathcal{J}(c \cdot \mathcal{I})$. The log–canonical threshold of $\mathcal{I}$, $\mathrm{lct}(\mathcal{I})$, is the smallest c such that $\mathcal{J}(c \cdot \mathcal{I}) \neq \mathcal{O}_A$. The following statement gives an upper bound on the Θ–regularity for multiplier ideals.

Proposition 3.1.4. *Let $\mathcal{I}$ be a non–trivial sheaf of ideals with $m = \Theta\text{–reg}(\mathcal{I})$. Let furthermore c be a positive, rational number. Then:*

i) $\Theta\text{–reg}(\mathcal{J}(c \cdot \mathcal{I})) \leq \min\{m, \lceil cm \rceil + 1\}$, *if* $0 < c < 1$.

ii) $\Theta\text{–reg}(\mathcal{J}(c \cdot \mathcal{I})) \leq \lfloor cm \rfloor - \lfloor c \rfloor + 2$, *if* $c \geq 1$.

Proof. Note that by Proposition 3.1.2 we have $m \geq 3$. For the rest of the proof we fix the divisor $\mathcal{A} = (m-1)\Theta$, so that $\mathcal{I} \otimes \mathcal{O}_A(\mathcal{A})$ is globally generated by Theorem 2.4.3 a).

In case i) we have $0 < c < 1$. Let D be a divisor representing an element α in $\mathrm{Pic}^0(A)$, choose $\mathcal{L} = (m-1)\Theta + D$ and observe that

$$(m-1) - c(m-1) = m - cm - 1 + c = (m-1)(1-c) > 0.$$

This makes the $\mathbb{Q}$–divisor

$$\mathcal{L} - c\mathcal{A} = (m-1)(1-c)\Theta + D$$

ample, as noted prior to the statement of this proposition. In particular $\mathcal{L} - c\mathcal{A}$ is also big and nef. Keeping in mind that ω_A is trivial, we can apply the Nadel Vanishing Theorem 1.4.14:

$$H^i(A, \mathcal{J}(c \cdot \mathcal{I}) \otimes \mathcal{O}_A((m-1)\Theta) \otimes \alpha) = 0, \text{ for all } i > 0$$

This shows that $\mathcal{J}(c\cdot\mathcal{I})\otimes\mathcal{O}_A((m-1)\Theta)$ is I.T. with index 0, which makes $\mathcal{J}(c\cdot\mathcal{I})$ m–Θ–regular. In a similar manner we will show that $\mathcal{J}(c\cdot\mathcal{I})$ is also $(\lceil cm\rceil+1)$–Θ–regular, making the lowest of the two an upper bound. Indeed, keeping $\mathcal{A}$ and D as above, we now redefine $\mathcal{L}=\lceil cm\rceil\Theta+D$. Observe that $\lceil cm\rceil-c(m-1)\geq c>0$, so the same line of argumentation as the one above shows that $\mathcal{J}(c\cdot\mathcal{I})$ is indeed $(\lceil cm\rceil+1)$–Θ–regular.

We will also show ii) by use of the Nadel Vanishing Theorem. $\mathcal{A}$ and D are kept as before. $\mathcal{L}$ is now chosen to be $(\lfloor cm\rfloor-\lfloor c\rfloor+1)\Theta+D$, and consider the sum $\lfloor cm\rfloor-\lfloor c\rfloor+1-cm+c$. If c is an integer, then this is equal to 1. Otherwise we have the inequality

$$\lfloor cm\rfloor-\lfloor c\rfloor+1-cm+c>c-\lfloor c\rfloor>0.$$

In both cases the sum is positive and $\mathcal{L}-c\mathcal{A}$ is hence an ample $\mathbb{Q}$–divisor. The result now follows from the same application of the Nadel Vanishing Theorem as in i). □

Remark 3.1.5. We can at this point make some deductions regarding the log–canonical threshold of $\mathcal{I}$. From Example 2.4.2 we know that $\mathcal{O}_A$ is 2–Θ–regular. Proposition 3.1.2 guarantees that this is the only ideal sheaf with this property. This fact is reflected in Proposition 3.1.4, where we see that for a sufficiently small c we have $\Theta\text{–reg}(\mathcal{J}(c\cdot\mathcal{I}))\leq\lceil cm\rceil+1=2$. In light of these observations, an equivalent definition of the log–canonical threshold of an ideal sheaf $\mathcal{I}$ on an abelian variety A is

$$\operatorname{lct}(\mathcal{I})=\min\{c\in\mathbb{Q}\mid\mathcal{J}(c\cdot\mathcal{I})\text{ is not 2-}\Theta\text{–regular}\}.$$

The following theorem is the main result relating the log-canonical threshold and the Θ–regularity of an ideal sheaf.

Theorem 3.1.6. *Let (A,Θ) be a principally polarized abelian variety. For any coherent sheaf of ideals $\mathcal{I}\neq\mathcal{O}_A$, the following inequality holds:*

$$1<\operatorname{lct}(\mathcal{I})(\Theta\text{–reg}(\mathcal{I})).$$

Proof. Set $c=\operatorname{lct}(\mathcal{I})$ and note that if $c\geq1$ this is obvious, as $\Theta\text{–reg}(\mathcal{I})$ is at least 3.

Otherwise, if $c<1$, then by definition $\mathcal{J}(c\cdot\mathcal{I})\neq\mathcal{O}_A$ so by Proposition 3.1.2 and Proposition 3.1.4 part i), we have

$$3\leq\Theta\text{–reg}(\mathcal{J}(c\cdot\mathcal{I}))\leq\lceil c(\Theta\text{–reg}(\mathcal{I}))\rceil+1<c(\Theta\text{–reg}(\mathcal{I}))+2$$

Subtracting 2 from the left– and right–hand side of the above inequality leaves us with the desired result. □

Remark 3.1.7. We once again emphasize the similarity between this result and that of Küronya and Pintye for the Castelnuovo–Mumford regularity, as stated in Theorem 1.5.3. The only difference is that equality does not hold for Θ–regularity, while there are examples of cases where equality holds for the Castelnuovo–Mumford case (see [KP13] Example 6).

We now turn our attention to another application of the theories of log–canonical thresholds and Θ–regularity. Historically, an interesting problem has been to understand what sort of singularities Theta–divisors can have. In Theorem 17.13 of [Kol95], János Kollár showed that Θ is log–canonical, which we recall from section 1.4 means that $\mathcal{J}(A, (1-\epsilon)\Theta) = \mathcal{O}_A$ for any rational number $0 < \epsilon < 1$. The result was later generalized to pluri–theta divisors by Ein and Lazarsfeld ([EL97] Proposition 3.5). We will show the latter result; the proof will start with an application of the Nadel Vanishing Theorem, in the same manner as was done by Ein and Lazarsfeld. Then Remark 3.1.5 will simplify the rest of the argument, compared to the original proof.

Theorem 3.1.8. *Let (A, Θ) be a principally polarized abelian variety and $m \geq 1$. If we fix any divisor $D \in |m\Theta|$, then $\frac{1}{m}D$ is log-canonical. By Proposition 1.4.13 this implies that every component of $\Sigma_{mk}(D)$ has codimension $\geq k$ in A.*

Proof. The first step is to apply the linear series version of the Nadel Vanishing Theorem 1.4.14, for the following setting; choose $\mathcal{L} = \Theta + F$, where F is a divisor associated to an element $\alpha \in \mathrm{Pic}^0(A)$. E is chosen to be $m\Theta$ and $c = \frac{1-\epsilon}{m}$ for a rational number $0 < \epsilon < 1$. Then

$$\mathcal{L} - cE = \Theta + F - \frac{(1-\epsilon)}{m} m\Theta = \epsilon\Theta + F$$

which is an ample $\mathbb{Q}$–divisor and hence nef and big. For $D \in |m\Theta|$ there are the vanishing of the higher cohomologies

$$H^i\left(A, \mathcal{O}_A(\Theta) \otimes \alpha \otimes \mathcal{J}\left(\frac{(1-\epsilon)}{m} D\right)\right) = 0 \text{ for } i > 0$$

which makes $\mathcal{O}_A(\Theta) \otimes \mathcal{J}\big(\frac{(1-\epsilon)}{m} D\big)$ I.T. of index 0. But this in turn means that $\mathcal{J}\big(\frac{(1-\epsilon)}{m} D\big)$ is 2-Θ-regular, and hence $\mathcal{J}\big(\frac{(1-\epsilon)}{m} D\big) = \mathcal{O}_A$ by Remark 3.1.5. Since this holds for any rational $0 < \epsilon < 1$, we conclude that $\frac{1}{m}D$ is log–canonical. □

Further Developments

We conclude with a brief discussion regarding possible further work related to the topics presented in this thesis. A natural question to ask would be whether the statement on singularities in Theorem 3.1.8 can be generalized to polarizations of higher degree. The obstructing element in applying similar arguments to the ones shown in Chapter 3 is Proposition 3.1.2, where we actively used that Θ was a principal polarization. To be more precise, let (A, ℓ) be a polarized abelian variety of degree $d > 1$, and $\mathcal{L}$ an ample line bundle representing ℓ. If one were to show the following statement:

$$\{\mathcal{I} \otimes \mathcal{L} \text{ is not Mukai–regular for any non–trivial sheaf of ideals } \mathcal{I}\} \qquad (*)$$

then one can prove statements analogous to Theorem 3.1.8. If we furthermore assume that (A, ℓ) is indecomposable, then $(*)$ can be proved for $d = 2$. This would not, however, give a new result as the singularities for this case has already been studied by Christopher D. Hacon ([Hac00] Theorem 4.1). It is not clear whether $(*)$ holds (possibly with further assumptions on (A, ℓ)) for degrees greater than 2.

Another option is to work over fields of positive characteristic. The Nadel Vanishing Theorem, which we have used extensively, requires the Kodaira Vanishing Theorem. While Kodaira Vanishing is known to fail in general for positive characteristic, it holds in certain cases, such as abelian surfaces. It could be interesting to examine to what extent the results shown here would work in these settings.

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